BORDISM AND COBORDISM

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Introduction. In (10), (11) Wall determined the structure of the cobordism ring introduced by Thom in (9). Among Wall's results is a certain exact sequence relating the oriented and unoriented cobordism groups. There is also another exact sequence, due to Rohlin (5), (6) and Dold (3) which is closely connected with that of Wall. These exact sequences are established by *ad hoc* methods. The purpose of this paper is to show that both these sequences are 'cohomology-type' exact sequences arising in the well-known way from mappings into a universal space. The appropriate 'cohomology' theory is constructed by taking as universal space the Thom complex $MSO(n)$, for n large. This gives rise to (oriented) *cobordism groups* $MSO^*(X)$ *of a space* X.

We consider also a 'singular homology' theory based on differentiable manifolds, and we define in this way the (oriented) *bordism groups* $MSO_*(X)$ *of a space* X. The justification for these definitions is that the main theorem of Thom (9) then gives a 'Poincaré duality' isomorphism $MSO^*(X) \cong MSO_*(X)$ for a compact oriented differentiable manifold X. The usual generalizations to manifolds which are open and not necessarily orientable also hold (Theorem (3·6)).

The unoriented corbordism group $\mathcal{N}_k$ of Thom enters the picture via the isomorphism (4·1)

$$\mathcal{N}_k \cong MSO^{2n-k}(P_{2n}) \quad (n \text{ large}),$$

where P_{2n} is real projective $2n$-space. The exact sequence of Wall is then just the cobordism sequence for the triad $P_{2n}, P_{2n-1}, P_{2n-2}$, while the exact sequence of Rohlin–Dold is essentially the corbordism sequence of the pair P_{2n}, P_{2n-2}. A simple geometrical construction shows that this latter sequence splits canonically (4·4). This fact can then be used to shorten Wall's proof (11) of the non-existence of elements of order 4 in the oriented cobordism groups Ω_k.

In addition to the applications given here the bordism–cobordism theory has close connexions with the methods and results of (2). These aspects of cobordism and in particular the question of multiplicative structures will be treated elsewhere.

The general ideas developed here on bordism and cobordism were the result of discussions with J. Milnor.

1. *Cobordism groups.* We recall some standard definitions and results of homotopy theory (cf. (7)). If X, Y are spaces with base points x_0, y_0 we denote by $[X, Y]$ the set of homotopy classes of maps $(X, x_0) \to (Y, y_0)$. We have the suspension sequence

$$[X, Y] \to [SX, SY] \to \dots \to [S^nX, S^nY] \to \dots, \tag{1}$$

in which all terms after the first two are Abelian groups, the maps being then group homomorphisms. Moreover we have an isomorphism

$$[S^nX, S^nY] \to [S^{n+1}X, S^{n+1}Y] \tag{2}$$

if $n+2(\text{connectivity } Y) \geqslant \dim X$, provided X is, say, a finite CW-complex. In this case the direct limit of the sequence (1), i.e. the common value of the groups $[S^n X, S^n Y]$ for large n is denoted by $\{X, Y\}$ and called the group of S-maps of X into Y.

We recall now the definition of the space $MSO(n)$ given by Thom in (9). Let $BSO(n)$ be the classifying space of $SO(n)$, let A denote the universal unit ball bundle over $BSO(n)$, $\dot{A}$ the universal unit sphere bundle. Then $MSO(n)$ is defined to be $A/\dot{A}$, the space obtained by collapsing $\dot{A}$ to a point; it has a canonical base point, namely, the image of $\dot{A}$. It is shown in (9) that the natural map

$$S\{MSO(n)\} \to MSO(n+1) \tag{3}$$

induces isomorphism of the homotopy groups π_{n+r} provided n is large ($n > 2r$). It follows that, if X is a finite CW-complex with base point, the mapping

$$[X, S\{MSO(n)\}] \to [X, MSO(n+1)] \tag{4}$$

will be bijective if n is large.

Let X be a finite CW-complex, Y a subcomplex, then from (1) and (3) we have a map

$$[S^{n-k}(X/Y), MSO(n)] \to [S^{n+1-k}(X/Y), MSO(n+1)]. \tag{5}$$

We define the *oriented cobordism group of dimension k of the pair* (X, Y) to be

$$MSO^k(X, Y) = \lim_{n\to\infty} [S^{n-k}(X/Y), MSO(n)],$$

the limit being taken with respect to the maps (5). We note that $MSO^k(X, Y)$ is defined for all integers k, positive or negative.

In view of (2) and (4) and the fact that $MSO(n)$ is $(n-1)$-connected it follows that, for large n, (5) is also an isomorphism, and so for large n

$$MSO^k(X, Y) \cong [S^{n-k}(X/Y), MSO(n)] \cong \{S^{n-k}(X/Y), MSO(n)\}. \tag{6}$$

The absolute cobordism groups are defined by

$$MSO^k(X) = MSO^k(X, \emptyset),$$

where $\emptyset$ is the empty set and $X/\emptyset$ is interpreted as the disjoint union of X and a base point. In view of (6), and the fact that the group of S-maps satisfies exactness (7) we deduce

PROPOSITION (1·1). *Let X be a finite* CW-*complex, Y a subcomplex of X, Z a subcomplex of Y. Then we have an exact sequence*

$$\ldots \to MSO^k(X, Y) \to MSO^k(X, Z) \to MSO^k(Y, Z) \to MSO^{k+1}(X, Y) \to \ldots.$$

From their definition the cobordism groups are invariants of S-type, and satisfy the excision axiom. Thus they satisfy all the axioms of a cohomology theory except the dimension axiom.

PROPOSITION (1·2). $MSO^k(X, Y) = 0$ *for* $k > \dim X$.

Proof. For $k > \dim X$ we have $\dim S^{n-k}(X/Y) \leqslant n-1$. But $MSO(n)$ is $(n-1)$-connected, and so $MSO^k(X, Y) = 0$.

If we take X to be a point then $MSO^{-k}(X) \cong \pi_{n-k}\{MSO(n)\}$ for n large. From the main theorem of Thom (9) it follows that $MSO^{-k}(\text{point}) \cong \Omega_k$, the cobordism group of dimension k in the sense of Thom (9).

Replacing $SO(n)$ by any of the other classical groups: $O(n)$, $\mathrm{Spin}(n)$, $U(n)$, $\mathrm{Sp}(n)$ we may give corresponding definitions. In particular $MO^{-k}(\text{point}) \cong \mathcal{N}_k$, the unoriented cobordism group of dimension k in the sense of Thom (9).

2. *Bordism groups.* In this section all spaces will belong to some fixed category $\mathscr{A}$ satisfying the conditions of the classification theorem for fibre-bundles (8). For example, $\mathscr{A}$ could be the category of countable finite-dimensional CW-complexes.

We shall consider the category $\mathscr{B}$ of pairs (X, α) where $X \in \mathscr{A}$ and α is a principal $\mathbf{Z}_2$-bundle over X, i.e. a double covering of X. Maps and homotopies in $\mathscr{B}$ will mean bundle maps and bundle homotopies in the sense of (8). Thus a map $F: (Y, \beta) \to (X, \alpha)$ consists of a map $f: Y \to X$ together with an isomorphism $\beta \cong f^*\alpha$.

Let $\mathscr{M}_k$ be the subset of $\mathscr{B}$ consisting of pairs (M, τ) with M a compact differentiable† manifold (with boundary) of dimension k, and τ the orientation bundle of M (cf. (1)). Let $\mathscr{M}_k^0$ be the subset of $\mathscr{M}_k$ for which M is closed.‡ If M is a manifold with boundary N we may identify the orientation bundle of N with the restriction to N of the orientation bundle of M. Then we have a boundary map $d: \mathscr{M}_k \to \mathscr{M}_{k-1}^0$.

Let $(X, \alpha) \in \mathscr{B}$ and define $C_k(X, \alpha)$ to be the set of all pairs $\{(M, \tau), F\}$ with $(M, \tau) \in \mathscr{M}_k^0$ and F a map $(M, \tau) \to (X, \alpha)$. We define an equivalence relation on $C_k(X, \alpha)$ as follows. $\{(M, \tau), F\} \sim \{(M', \tau'), F'\}$ if there exists $(N, \sigma) \in \mathscr{M}_{k+1}$ and a map $G: (N, \sigma) \to (X, \alpha)$ such that

(i) $dN = M + M'$ (disjoint sum),

(ii) $G \mid M = F$, $G \mid M' = F'$.

The set of equivalence classes will be denoted by $MSO_k(X, \alpha)$. The disjoint sum induces on it an Abelian group structure. We shall call this group the *k-dimensional oriented bordism group of X with coefficients in α*.

If α is the trivial bundle $X \times \mathbf{Z}_2$ we write simply $MSO_k(X)$. Elements of $MSO_k(X)$ are then represented by maps $f: M \to X$ where M is oriented.

From the definition it is clear that $MSO_k(X, \alpha)$ is a covariant functor of (X, α). Moreover, we have

LEMMA (2·1). *Homotopic maps $(Y, \beta) \to (X, \alpha)$ induce the same homomorphism $MSO_k(Y, \beta) \to MSO_k(X, \alpha)$.*

Proof. By definition ((8), § 11·2) a homotopy $F_t: (Y, \beta) \to (X, \alpha)$ is a map

$$\Phi: (Y \times I, \bar{\beta}) \to (X, \alpha),$$

where I is the unit interval and $\bar{\beta}$ is the bundle induced on $Y \times I$ from β by the projection $Y \times I \to Y$. An element of $MSO_k(Y, \beta)$ is represented by a map

$$G: (M, \tau) \to (Y, \beta),$$

† Differentiable will mean C^∞-differentiable.

‡ A closed manifold will mean a compact manifold without boundary.

with M a closed manifold and τ its orientation bundle. Then we have a map

$$\Phi \circ (G \times 1)\colon\ (M \times I, \tilde{\tau}) \to (X, \alpha).$$

Restricted to $M \times 0$ this gives $F_0 G$, and restricted to $M \times 1$ it gives $F_1 G$. But

$$F_i G\colon\ (M, \tau) \to (X, \alpha) \quad (i = 1, 2)$$

are representatives for the induced elements of $MSO_k(X, \alpha)$. This completes the proof.

The unoriented bordism groups $MO_k(X)$ are defined similarly except that no orientation bundles are introduced, and we work in the original category $\mathscr{A}$. Unitary bordism groups $MU_k(X)$ may be defined using generalized almost complex manifolds in the sense of Milnor (4). One can also introduce groups $M\,\mathrm{Sp}_k(X)$, $M\,\mathrm{Spin}_k(X)$.

From the definitions it is clear that

$$MSO_k(\text{point}) \cong \Omega_k, \quad MO_k(\text{point}) \cong \mathscr{N}_k.$$

Let P_n denote n-dimensional real projective space, ξ the double covering $S^n \to P_n$, where S^n is the n-sphere. Then ξ is an $(n-1)$-universal $\mathbf{Z}_2$-bundle. Hence we have (cf. (8), proof of § 19·3)

LEMMA (2·2). *Let* $\dim X \leqslant n-2$, *then there is a unique homotopy class*

$$(X, \alpha) \to (P_n, \xi).$$

PROPOSITION (2·3). *For large n we have a natural isomorphism*

$$MSO_k(P_n, \xi) \cong \mathscr{N}_k.$$

Proof. If $F\colon (M, \tau) \to (P_n, \xi)$ represents an element λ of $MSO_k(P_n, \xi)$, then M represents an element $\theta(\lambda)$ of $\mathscr{N}_k$. From the definitions it is clear that $\theta(\lambda)$ depends only on λ and not on the choice of representative. Conversely, given a manifold M representing an element μ of $\mathscr{N}_k$, there is by (2·2) a map $(M, \tau) \to (P_n, \xi)$ unique up to homotopy. By (2·1) this defines a unique element $\phi(\mu)$ of $MSO_k(P_n, \xi)$. Moreover, applying (2·2) to manifolds of dimension $k+1$, it follows that $\phi(\mu)$ depends only on μ. Clearly ϕ and θ are inverses of each other, and since they are homomorphisms the proposition is established,

3. *The Poincaré–Thom duality.* Let X, Y be differentiable manifolds with orientation bundles ξ, η, respectively. Then a map $f\colon Y \to X$ is *orientable* if $f^*\xi \cong \eta$. An *oriented* map $Y \to X$ is a map f together with a given isomorphism $f^*\xi \cong \eta$, i.e. a map $(Y, \eta) \to (X, \xi)$ in the sense of § 2. In particlar a differentiable embedding $f\colon Y \to X$ is oriented if and only if the normal bundle of $f(Y)$ in X is oriented.

In (9) Thom introduced the notion of L-equivalence for submanifolds of a manifold. The definition is as follows. Two closed differentiable submanifolds Y_0, Y_1 (of a differentiable manifold X without boundary) with oriented normal bundles are L-equivalent if there exists a compact differentiable submanifold Z of $X \times I$ with oriented normal bundle such that Z intersects $X \times 0$, $X \times 1$ transversally in $Y_0 \times 0$, $Y_1 \times 1$, induces there the given normal orientations and has no other boundary. The set of L-equivalence classes of X of dimension k is denoted by $L_k(X)$. In the stable range, i.e. for $2k < \dim X$, $L_k(X)$ is an Abelian group, the addition being induced by the disjoint sum.

If Y is a closed k-dimensional differentiable submanifold of X with oriented normal bundle then, as remarked above, this defines an oriented map i: $Y \to X$ and thus an element $\psi(Y)$ of $MSO_k(X, \xi)$, where ξ is the orientation bundle of X. If Y_0 and Y_1 are L-equivalent then the map $Z \to X$ (with Z as above) induced by the projection $X \times I \to X$ is oriented and is the map which shows that $\psi(Y_0) = \psi(Y_1)$. Hence we have a natural map

$$\psi: L_k(X) \to MSO_k(X, \xi).$$

In the stable range this is a group homomorphism.

Before proceeding to examine ψ more closely we digress to give a lemma on manifolds with boundary which is an elementary consequence of the general theorems of Whitney (12).

Lemma (3·1). *Let X, Y be differentiable manifolds with boundaries X_0, Y_0, respectively, let Y be compact and suppose* $\dim X > 2 \dim Y$. *Let f: $(Y, Y_0) \to (X, X_0)$ be a continuous map such that f: $Y_0 \to X_0$ is a differentiable embedding. Then there exists a differentiable embedding g: $Y \to X$ such that*

(i) *g is homotopic to f,*

(ii) $g(Y - Y_0) \subset X - X_0$,

(iii) *$g = f$ on Y_0,*

(iv) *$g(Y)$ meets X_0 transversally.*

Proof. We can find a neighbourhood Y_1 of Y_0 in Y diffeomorphic to $Y \times I$, with Y_0 corresponding to $Y_0 \times 0$. Let X_1 be defined similarly. Now define g_1: $Y_1 \to X_1$ by $g_1(y, t) = (f(y), t), y \in Y_0, t \in I$. Since $\overline{X - X_1}$, $\overline{Y - Y_1}$ are homeomorphic to X, Y, respectively, the existence of f can be used to extend g_1 to a continuous map $Y \to X$. Now we can apply Whitney's extension theorem (Theorem 5 of (12)) to the closed submanifold $Y_1 - Y_0$ of $Y - Y_0$, and we deduce that $g_1 | Y_1 - Y_0$ can be extended to a differentiable embedding G: $Y - Y_0 \to X - X_0$. The maps G, g_1 together define a differentiable embedding g: $Y \to X$ with the required properties.

Now we can prove

Proposition (3·2). *Let X be a differentiable manifold (without boundary) of dimension n with orientation bundle ξ. Then, in the stable range* $(2k < n)$,

$$\psi: L_k(X) \to MSO_k(X, \xi)$$

is an isomorphism.

Proof. Let f: $Y \to X$ be an oriented map representing an element λ of $MSO_k(X, \xi)$. Then by Theorem 1 of (12) f is homotopic to a differentiable embedding g. The orientation of f induces an orientation of g and with this orientation g: $Y \to X$ is another representative for λ (by (2·1)). This proves that ψ is surjective. Suppose now that $\mu_0, \mu_1 \in L_k(X)$ are such that $\psi(\mu_0) = \psi(\mu_1)$, and let μ_0, μ_1 be represented by the oriented embeddings i_0: $Y_0 \to X$, i_1: $Y_1 \to X$. Then there exists a compact differentiable manifold Z with boundary $Y_0 + Y_1$ and an oriented map f: $Z \to X$ with $f | Y_0 = i_0$, $f | Y_1 = i_1$. Let h: $Z \to I$ be a continuous map of Z into the unit interval with $h^{-1}(0) = Y_0$, $h^{-1}(1) = Y_1$, and define F: $Z \to X \times I$ by $F(z) = \{f(z), h(z)\}$. We are now in a position to apply (3·1) and we deduce the existence of a differentiable embedding $g\colon Z \to X \times I$ having properties (i)–(iv). The orientation of f induces an orientation of F and hence of g (by (i)). Hence Y_0 and Y_1 are L-equivalent, i.e. $\mu_0 = \mu_1$. Thus ψ is injective and the proof is completed.

An immediate consequence of (3·2) is

COROLLARY (3·3). *Let X be a differentiable manifold (without boundary) with orientation bundle ξ, and let $\bar{\xi}$ denote the bundle on $X \times \mathbf{R}^n$ induced from ξ. Then, for large n,*

$$\psi\colon\ L_k(X \times \mathbf{R}^n) \to MSO_k(X \times \mathbf{R}^n, \bar{\xi})$$

is an isomorphism.

In view of (2·1) we have $MSO_k(X \times \mathbf{R}^n, \bar{\xi}) \cong MSO_k(X, \xi)$ and so from (3·3) we deduce

COROLLARY (3·4). *Let X, ξ be as in (3·3). Then for large n we have a canonical isomorphism*

$$L_k(X \times \mathbf{R}^n) \cong MSO_k(X, \xi).$$

The main theorem of Thom (9) may be stated in the following form

THEOREM (3·5). *Let X be a finite* CW*-complex, Y a subcomplex and let $X - Y$ be a differentiable manifold (without boundary) of dimension m. Then we have a canonical isomorphism*

$$L_k(X - Y) \cong [X/Y, MSO(m-k)].$$

We can now translate (3·4) into a Poincaré duality theorem for bordism and cobordism.

THEOREM (3·6). *Let X be a finite* CW*-complex, Y a subcomplex and let $X - Y$ be a differentiable manifold (without boundary) of dimension m, with orientation bundle τ. Then we have a canonical isomorphism*

$$MSO^k(X, Y) \cong MSO_{m-k}(X - Y, \tau).$$

Proof. By (6) of § 1 we have, for large n,

$$MSO^k(X, Y) \cong [S^{n-k}(X/Y), MSO(n)].$$

Now $S^{n-k}(X/Y) = X \times S^{n-k}/Y \times S^{n-k} \cup X \times a$, where $a \in S^{n-k}$ is a base point. Hence by (3·5) we have an isomorphism

$$[S^{n-k}(X/Y), MSO(n)] \cong L_{m-k}\{(X - Y) \times (S^{n-k} - a)\}.$$

Identifying $\mathbf{R}^{n-k}$ with $S^{n-k} - a$ and using (3·4) we obtain finally

$$MSO^k(X, Y) \cong MSO_{m-k}(X - Y, \tau)$$

as required.

Remark. Theorem (3·6) is the purely 'additive' Poincaré duality. There is, however, a multiplicative form, with cap products, quite analogous to the ordinary Poincaré duality. One has first to show that $MSO^*(X)$ is a graded ring and that $MSO_*(X)$ is a graded module over $MSO^*(X)$. These structures are induced from the natural maps:

$$MSO(n) \wedge MSO(m) \to MSO(m+n).$$

Alternatively they may be defined using products and intersections of manifolds. For our present purposes the additive theory is sufficient. We will return to the multiplicative structure on a future occasion.

4. *The exact sequences.* First we establish the following

Proposition (4·1). *For large n we have a canonical isomorphism*

$$\mathcal{N}_k \cong MSO^{2n-k}(P_{2n}).$$

Proof. For large n we have

$$\begin{aligned}\mathcal{N}_k &\cong MSO_k(P_{2n}, \xi) \quad \text{by} \quad (2{\cdot}3)\\ &\cong MSO^{2n-k}(P_{2n}) \quad \text{by} \quad (3{\cdot}6),\end{aligned}$$

since ξ is the orientation bundle of P_{2n}.

We now define $\mathscr{W}_k = MSO^{2-k}(P_2, P_0)$. Since $P_{2n}/P_{2n-2} = S^{2n-2}(P_2/P_0)$ we have

$$\mathscr{W}_k \cong MSO^{2n-k}(P_{2n}, P_{2n-2}). \tag{7}$$

Applying the cobordism exact sequence (1·1) to the triad P_2, P_1, P_0, and observing that $P_2/P_1 = S^2$, $P_1/P_0 = S^1$ we deduce

Theorem (4·2). *We have an exact sequence*

$$\ldots \to \mathscr{W}_{k+1} \to \Omega_k \to \Omega_k \to \mathscr{W}_k \to \ldots.$$

Applying (1·1) to the triad P_{2n}, P_{2n-2}, $\emptyset$, and using (4·1) and (7), we obtain

Theorem (4·3). *We have an exact sequence*

$$\ldots \to \mathscr{W}_k \to \mathcal{N}_k \to \mathcal{N}_{k-2} \to \ldots.$$

(4·2) is the exact sequence of Wall (10), (11) though we have not yet identified our $\mathscr{W}_k$ with his. This identification arises from a further examination of (4·3). In fact we shall establish

Theorem (4·4). *There is a splitting homomorphism θ: $\mathcal{N}_{k-2} \to \mathcal{N}_k$ of the exact sequence (4·3). If Y is a representative manifold for $\lambda \in \mathcal{N}_{k-2}$, a representative of $\theta(\lambda)$ is the P_2-bundle over Y associated to the vector bundle $L \oplus 1 \oplus 1$, where L is the line-bundle defined by $w_1(Y)$, and 1 is the trivial line-bundle.*

Remark. This result has also been found by Dold (3), though his proof is on different lines.

Assuming for the moment that (4·4) has been proved, we deduce from (4·3) the exact sequence

$$0 \to \mathscr{W}_k \to \mathcal{N}_k \to \mathcal{N}_{k-2} \to 0. \tag{8}$$

Bearing in mind the definition of $\mathcal{N}_k \to \mathcal{N}_{k-2}$ this enables us to identify $\mathscr{W}_k$ with the subgroup of $\mathcal{N}_k$ represented by manifolds Y whose characteristic map $Y \to P_{2n}$ factors through $Y \to P_{2n} - P_{2n-2}$. Since $P_{2n} - P_{2n-2}$ retracts onto P_1 it follows that $\mathscr{W}_k$ is represented by manifolds Y with $w_1(Y)$ the image of an integral class. This is the definition of Wall. The geometrical interpretation of the homomorphisms in (4·2), i.e. their identification with the homomorphisms in Wall's sequence, presents no difficulties.

From (8) we deduce at once

$$\dim \mathscr{W}_k = \dim \mathscr{N}_k - \dim \mathscr{N}_{k-2}, \tag{9}$$

the dimensions being over $\mathbf{Z}_2$. This formula is an essential step in Wall's proof (11) that Ω_k has no elements of order 4 and Wall establishes it by a calculation with the Steenrod algebra.

The exact sequence of Rohlin (5), (6) and Dold (3) is obtained by putting (8) and (4·2) together. It can also be obtained directly as the cobordism sequence of the pair P_{2n}, P_{2n-1}; in order to determine $MSO^k(P_{2n-1})$ we use the pair P_{2n-1}, P_{2n-2} and observe that their cobordism sequence splits (in consequence of (4·4)).

It remains now to prove (4·4). First we establish an auxiliary lemma

LEMMA (4·5). *Let L denote the real line-bundle over P_{2n-2} associated to the principal $O(1)$-bundle $S^{2n-2} \to P_{2n-2}$. Let Q_{2n} denote the bundle over P_{2n-2} with P_2 as fibre associated to the vector bundle $L \oplus 1 \oplus 1$, where 1 denotes the trivial line-bundle. Let P_{2n-2} and P_1 be embedded in P_{2n} with $P_{2n-2} \cap P_1 = \emptyset$. Then there is an orientable map σ: $Q_{2n} \to P_{2n}$ such that*

(i) *σ maps each fibre of Q_{2n} linearly onto a P_2 in P_{2n},*

(ii) *σ is a diffeomorphism of $Q_{2n} - \sigma^{-1}(P_1)$ onto $P_{2n} - P_1$.*

Proof. Let $\mathbf{R}^{2n+1} = \mathbf{R}^{2n-1} \oplus \mathbf{R}^2$ be the decomposition corresponding to the subspaces P_{2n-2} and P_1 of P_{2n}. The fibre L_x of L at a point $x \in P_{2n-2}$ may be identified with the 1-dimensional vector space of $\mathbf{R}^{2n-1}$ corresponding to x. Hence the fibre of Q_{2n} at x may be identified with xP_1, the P_2 containing x and P_1. Thus Q_{2n} may be defined as the subspace of $P_{2n-2} \times P_{2n}$ consisting of pairs (x, y) with $y \in xP_1$. The projection $P_{2n-2} \times P_{2n} \to P_{2n}$ then induces a map σ: $Q_{2n} \to P_{2n}$ which has properties (i) and (ii). It remains to check that σ is orientable, i.e. that $\sigma^* w_1(P_{2n}) = w_1(Q_{2n})$. Now

$$H^1(Q_{2n}, \mathbf{Z}_2) \cong H^1(P_2, \mathbf{Z}_2) \oplus H^1(P_{2n-2}, \mathbf{Z}_2),$$

and so it is sufficient to check that $\sigma^* w_1(P_{2n}) - w_1(Q_{2n})$ restricts to zero

(*a*) on a fibre P_2,

(*b*) on the cross-section $\sigma^{-1}(P_{2n-2})$.

Now (*b*) follows at once from property (ii) of σ. As for (*a*), it follows from property (i) of σ and the fact that the normal bundles of P_2 in Q_{2n} and P_{2n} are both oriented.

Remark. Q_{2n} is just obtained from P_{2n} by 'blowing up' P_1 as in complex algebraic geometry.

We now make a choice of orientation for σ. In view of property (ii) of σ there is actually a preferred choice. Let f: $Y \to P_{2n-2}$ be an oriented map representing $\lambda \in MSO_{k-2}(P_{2n-2}, \xi)$. By (2·1) we may assume f differentiable. Let $Z = f^{-1}(Q_{2n})$ be the induced P_2-bundle over Y. Then we have an oriented map F: $Z \to Q_{2n}$ induced by f, and hence an oriented map $g = \sigma F$: $Z \to P_{2n}$. It is clear that the element of $MSO_k(P_{2n}, \xi)$ defined by g depends only on λ. Thus we have defined a homomorphism

$$MSO_{k-2}(P_{2n-2}, \xi) \to MSO_k(P_{2n}, \xi)$$

and so by (2·3) a homomorphism

$$\theta\colon \mathscr{N}_{k-2} \to \mathscr{N}_k,$$

and this is the one described in (4·4). It remains to show that this does in fact give a splitting of (4·3).

An element λ of $\mathcal{N}_{k-2} = MSO^{2n-k}(P_{2n-2})$ may be represented by a submanifold Y of $P_{2n-2} \times \mathbf{R}^N$ with oriented normal bundle (3·5). Let f: $Y \to P_{2n-2}$, h: $Y \to \mathbf{R}^N$ be induced by the projections, and let g: $Z \to P_{2n}$ be as constructed above from f. Let π: $Z \to Y$ be the bundle map and define g': $Z \to P_{2n} \times \mathbf{R}^N$ by $g'(z) = (g(z), h\pi(z))$. Then g' is an embedding in a neighbourhood of $P_{2n-2} \times \mathbf{R}^N$ and $g'(Z)$ intersects $P_{2n-2} \times \mathbf{R}^N$ transversally in Y. By Theorem 5 of (12) we can approximate g' by an embedding g'' with $g' = g''$ in a neighbourhood of $P_{2n-2} \times \mathbf{R}^N$. Then $g''(Z)$ is a submanifold of $P_{2n} \times \mathbf{R}^N$ with oriented normal bundle. It represents the element

$$\theta(\lambda) \in \mathcal{N}_k = MSO^{2n-k}(P_{2n})$$

and it intersects $P_{2n-2} \times \mathbf{R}^N$ transversally in Y, inducing on Y its normal orientation. Hence $r\theta(\lambda) = \lambda$, where r: $MSO^{2n-k}(P_{2n}) \to MSO^{2n-k}(P_{2n-2})$ is the restriction homomorphism (cf. (9)). This completes the proof of (4·4).

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