What is a Spinor?

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(CNRS-Institut des Hautes Études Scientifiques)

Edinburgh
11 January, 2021

Dedicated to Sir Michael ATIYAH
for the inspiration and with admiration

Dedication

It a great honour for me to be given the opportunity to deliver the inaugural "Atiyah Lecture" as I owe to **Sir Michael** a lot as a mathematician, but also more generally as a scientist active in the international scientific community in the many capacities **he** held.

I could interact with **him** over a number of years and through numerous encounters, sometimes in conferences, sometimes in **his** various capacities such as Master of Trinity College or President of the Royal Society.

The very last one was in September 2018 at the Heidelberg Laureate Forum. I remember vividly sitting next to **him** for a dinner where I did not have many opportunities to say anything...

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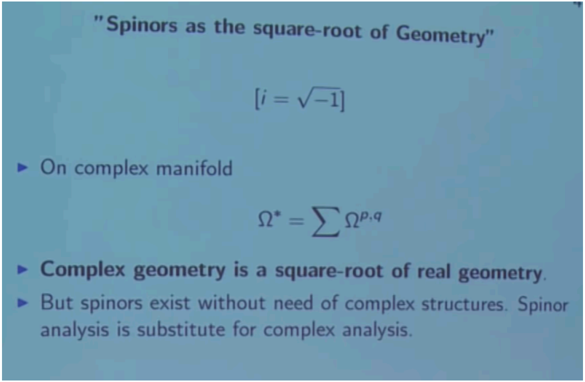

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- I asked Sir Michael about it, and he confessed that he did not know how to deal with it;
- the article by Thomas FRIEDRICH *Der erste Eigenwert des Dirac-Operators einer kompakten Riemannschen Mannigfaltigkeit nichtnegativer Skalarkrümmung* made me acquainted with the geometric content of the Dirac operator;
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- Basic Facts on Spinors and Dirac Operators
- Spinors in General Relativity
- Varying the Metric
- Killing Spinors
- Perspectives

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Section 1

Basics on Spinors and Dirac Operators

Spinors

To any n-dimensional vector space V endowed with a symmetric bilinear form g one attaches its Clifford algebra

$$Cl(V, g) = \otimes V / \langle x \otimes x + g(x, x)1 \rangle \ .$$

Of course the Clifford algebra $(V, 0)$ coincides with the exterior algebra ΛV. In the sequel **we will only consider non-degenerate bilinear forms** g. If we work over $\mathbb{C}$, they are all equivalent. Over $\mathbb{R}$, they are classified by their signature.

Clifford algebras over $\mathbb{C}$ have a 2-fold periodicity:

- if $n = 2m$, $Cl(V, g)$ is a simple algebra, hence $Cl(V, g) = \mathrm{End}(\Sigma_g V)$;
- if $n = 2m + 1$, the $Cl(V, g) = \mathrm{End}(\Sigma_g V) \oplus \mathrm{End}(\tilde{\Sigma}_g V)$.

$\Sigma_g V$ and $\tilde{\Sigma}_g V$ are the spaces of *spinors*; they are 2^m-dimensional.

Clifford algebras over $\mathbb{R}$ have an 8-fold periodicity but we will stick to complex spinors except when we discuss **Lorentzian metrics**.

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Spinors (continued)

Spinors are *vectors in a fundamental representation space of the group $Spin_n$*, the universal cover of the group SO_n for $n \geq 3$.

A key ingredient in the theory is the exact sequence of groups

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow Spin_n \longrightarrow SO_n \longrightarrow 1 \ ,$$

which holds true for $n \geq 3$.

The link with Clifford algebras goes as follows:

- the group $Spin(V, g)$ is realised as the multiplicative subgroup of the Clifford algebra stabilising the image of V inside $Cl(V, g)$ and satisfying a certain normalisation condition;
- $Spin(V, g)$ acts irreducibly on Σ_g;
- through the adjoint representation, $Spin(V, g)$ acts on V;
- a key property of spinors in even dimensions is their *chirality*, i.e. the fact that the volume element acts as an involution and gives rise to the decomposition $\Sigma_g = \Sigma_g^+ \oplus \Sigma_g^-$.

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Spinors are *vectors in a fundamental representation space of the group* $Spin_n$, the universal cover of the group SO_n for $n \geq 3$.

A key ingredient in the theory is the exact sequence of groups

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which holds true for $n \geq 3$.

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If one insists in not specifying a metric, this means working with another exact sequence of groups

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Some general comments on spinors as geometric objects:

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- this probably prevented them from playing a central role in Mathematics immediately;
- in his book *Leçons sur la théorie des spineurs* published in 1937 (and translated into English only in 1966), Élie CARTAN is concerned with a geometric view on them;
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- he continues: *"The very notion of a 'spinorial object' is somewhat confusing and non-intuitive, and some people prefer to resort to a purely (Clifford) algebraic approach to their study. This certainly has its advantages... but I feel that it is important also not to lose sight of the geometry..."*
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I end this review of visions about spinors by quotes from H. Blaine LAWSON and Marie-Louise MICHELSOHN, authors of the very influential book *Spin Geometry* published in 1989:

- *"One of the great insights of Atiyah and Singer was their construction of the Dirac operator on any spin manifold."*
- *"This was the key to their proof of the Index Theorem. It also related the Riemannian geometry of a spin manifold in very subtle ways to the global topology."*
- *"How else do we think about spinors? At any given point, they are not all the same, as is true of differential forms. There is a $Spin_n$-invariant flag $S_0 \subset S_1 \subset S_2 \subset \cdots$ where S_0 are the pure spinors. For n even, the projectivisation of S_0 corresponds to the orthogonal almost complex structures on the tangent space."*
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Bringing Spinors to General Manifolds

The purpose is to carry these constructions over to a Riemannian manifold (M, g), not a trivial matter. Here are some signs:

- the fact that spinors make sense only after a metric has been chosen was recognised by Hermann WEYL in a 1929 article and prompted Élie CARTAN to state at the end of his book: *"With the geometric sense we have given to the word "spinor" it is impossible to introduce fields of spinors into the classical Riemannian technique..."* meaning by that one cannot use general changes of coordinates when dealing with spinors;
- this was in a sense misinterpreted, e.g.,Leopold INFELD and Bartel VAN DER WAERDEN proposed in 1933 that, in a space-time with a general metric, spinors should be related to a background Minkowski metric;
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- in his article on bandors, Yuval NE'EMAN takes the pain of observing what it means in terms of changes of coordinates, addressing the question alluded to by Élie CARTAN.

Spinor Fields

Going to manifolds, the solution comes naturally in the language of principal and associated bundles.

Lifting usual constructions from the bundle of orthonormal bases to that of spinorial bases can be done if and only if $w_2(M) = 0$.

The set up goes as follows:

- choosing a $Spin_n$-principal bundle Γ covering the SO_n bundle of oriented orthonormal frames determines the *spin structure*.
- the *bundle of spinors* associated to the spin structure is the associated bundle $\Sigma_g M = \Gamma \times_{\mathrm{Spin}_n} \Sigma_n$.
- a *spinor field* is of course a section of the bundle $\Sigma_g M \longrightarrow M$.
- the bundle in Clifford algebras $Cl_g(M) \longrightarrow M$ acts on the spinor bundle via pointwise Clifford multiplication.
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Natural Operators on Spinor Fields

On the space of spinor fields only two first order differential operators are universally defined, and combinations thereof:

- $T^*M \otimes \Sigma_\gamma M$ decomposes into exactly two invariant subspaces: a copy of $\Sigma_\gamma M$ and another space $\Sigma_\gamma^{3/2} M$;
- the *Dirac operator* $\mathcal{D}$ maps spinor fields to spinor fields, and is defined, for a spinor field ψ, by

$$\mathcal{D}\psi = \sum_{i=1}^{n} e_i . D_{e_i} \psi \ ,$$

  where (e_i) denotes an orthonormal basis of the tangent space;
- the *Penrose twistor operator* $\mathcal{P}$, which maps spinor fields to sections of the bundle $T^*M \otimes \Sigma_\gamma M$, is defined as follows:

$$(\mathcal{P}\psi)(X) = D_X \psi + \frac{1}{n} X . \mathcal{D}\psi \ ,$$

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Dirac Operators

Here are the important properties of the Dirac operator:

- it is *a square root of the Laplace-Beltrami operator*, hence an elliptic operator in a Riemannian setting;
- its principal symbol is given by Clifford multiplication;
- it is self-adjoint;
- in even dimensions, it exchanges chirality, i.e. it maps positive spinor fields to negative ones and vice versa; indeed chirality is preserved by the covariant derivative, and Clifford product by a vector changes chirality; this is key to relate the Dirac operator to topology via the Atiyah-Singer Index Theorem;
- for a spinor field ψ, the *Schrödinger-Lichnerowicz formula* holds

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3. Spinors in General Relativity

Lorentzian Metrics

Both Special Relativity and General Relativity rely on the use of a non-degenerate metric g with signature $(-,+,+,+)$:

- it was in the setting of the Minkowski space-time $(\mathbb{R}^4, m)$ with $m = -c\, dt^2 + dx^2 + dy^2 + dz^2$ with coordinates $(t, x.y.z)$ on $\mathbb{R}^4$, that Paul-Adrien-Maurice DIRAC wanted to formulate a first-order operator invariant under the group preserving m whose square would be the Klein-Gordon operator;
- this led him to look for matrices satisfying the fundamental identity defining the Clifford algebra for m, and hence to realise that wave functions could not just be functions, but elements in a vector space on which these matrices operate!
- this brought radically new objects purely in Theoretical Physics from the consideration of a necessary symmetry;
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Further Considerations on General Relativity

In the 1930s serious considerations were given to the incorporation of spinors in the formulation of the theory of General Relativity:

- Paul-Adrien-Maurice DIRAC extended the formulation in Minkowski space to a number of models of space-time, De Sitter and anti-De Sitter spaces for example;
- in 1933 Erwin SCHRÖDINGER published an article entitled *Dirac equation in the gravitational field* in which he generalises the Dirac equation to a curved space-time, something later rediscovered by André LICHNEROWICZ as mentioned earlier, hence the name 'Schrödinger-Lichnerowicz' given to the key identity for the square of the Dirac operator;
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- the notion of a 'spin connection' was debated in the physics community around this time with contributions from Hermann WEYL and Vladimir FOCK;
- Élie CARTAN makes explicit references to such considerations towards the end of his book.

Further Considerations on General Relativity

In 1960 Roger PENROSE published a key article entitled *A Spinor Approach to General Relativity* in which he systematically considers all aspects of the theory:

- he in particular discusses the representation in spinor terms of the various irreducible components of the Riemann curvature tensor, the Weyl conformal part corresponding to a fully symmetric traceless 4-spinor χ and the Ricci traceless tensor to another 4-spinor Φ with specific symmetries: he quotes a 1959 article by Louis WITTEN for the algebraic background;
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4. Varying the Metric

Dealing with the Problem

The question of varying the metric has several sides to it.

The first one is algebraic.

The second one is geometric as one has to move to manifolds and the Riemannian context.

I will be following the point of view taken in the article *Spineurs, opérateurs de Dirac et variations de métriques*, which I wrote with Paul GAUDUCHON and which appeared in Communications in Mathematical Physics in 1992.

Another more elegant approach was proposed in 2003 by Christian BÄR, Paul GAUDUCHON and Andrei MOROIANU in the article entitled *Generalized Cylinders in Semi-Riemannian and Spin Geometry*, which allows them to deal with Lorentzian metrics and also to prove some important geometric theorems.

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Correspondence between Bases

Let $g,\ h \in \mathcal{M}et V$, where V is an n-dimensional vector space. The first step is to establish a natural correspondence between g-orthonormal and h-orthonormal bases:

- it will be useful to think of a g-orthonormal basis b as a linear invertible map from $\mathbb{R}^n$ to V so that $(b^{-1})^*(e) = g$, where e denotes the standard scalar product on $\mathbb{R}^n$;
- the other g-orthonormal bases can be obtained from b as $b \circ O$ where $O \in O_n$, the group of e-isometries;
- an explicit correspondence can be built using a square root of the linear map $H_g = g^{-1}.h$;
- To get to the spinorial setting, one needs to exploit the geometry of the bundle $Inv(\mathbb{R}^n, V) \longrightarrow \mathcal{M}et\ V$ and take a horizontal lift along a curve from g to h;
- this path can then be lifted to the spinorial bases for spinorial metrics γ and η sitting above g and h.

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The Conformal Situation

Among the simplest changes of metrics one of course has the conformal ones:

- two metrics g and $\tilde{g}$ are conformally related if there exists a positive function a such that $\tilde{g} = a^2 g$;
- in such a situation to any g-orthonormal basis $(e_i)_{i=1,\dots,n}$ one can associate the $\tilde{g}$-orthonormal basis namely $(\tilde{e}_i)_{i=1,\dots,n}$ defined by $\tilde{e}_i = a^{-1} e_i$;
- it is therefore easy to create a map between spinorial bases for two conformally related metrics;
- by using appropriate weights, it is possible to construct spinors attached to a conformal class of metrics, a construction due to Nigel HITCHIN;
- under a conformal change the Dirac operator's behaviour is:

$$a^{\frac{n+1}{2}} \tilde{D}(\tilde{\psi}) = \widetilde{D\left(a^{\frac{n-1}{2}} \psi\right)},$$

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Comparing Spinors and Dirac Operators

One can map spinorial bases for two spinorial metrics γ and η subordinated to the same spin structure still denoted by c^{η}_{γ}:

- the Spin_n-principal bundles associated with γ and η being mapped to one another in a Spin_n-equivariant way, this gives rise to a map still denoted c^{η}_{γ} between the bundles $\Sigma^{\gamma} M \longrightarrow M$ and $\Sigma^{\eta} M \longrightarrow M$;
- there is one case which allows an easier description, namely when the two metrics are *conformally related*, i.e. $h = a^2 g$. Then c^h_g and c^{η}_{γ} are just multiplication by a^{-1};
- of course γ and η have different covariant derivatives as they come from the D^g and D^h;
- the proper formula to compare the Dirac operators for the spinorial metrics γ and η introduces a transmuted Dirac operator ${}^{\gamma}\mathcal{D}^{\eta}$ defined as

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Varying the Metric: the Dirac Operator

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Theorem (JPB-P. GAUDUCHON)

The infinitesimal variation of the Dirac operator at a spinorial metric γ for an infinitesimal deformation k of the metric g is given for a γ-spinor field ψ by the formula

$$\left(\frac{d}{dt}{}^{\gamma}\mathcal{D}^{\gamma_t}{}_{|t=0}\right)\psi = \frac{1}{2}\sum_{i=1}^{n} e_i \cdot_\gamma D^{\gamma}_{K_g(e_i)}\psi + \frac{1}{4}(\delta^g k + d\,\mathrm{Trace}_g k)\cdot_\gamma \psi\ .$$

The formula allows also to study the variation of the eigenvalues of the Dirac operator, something of obvious geometric interest.

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The formula allows also to study the variation of the eigenvalues of the Dirac operator, something of obvious geometric interest.

Varying the Metric: the Dirac Operator

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Varying the Metric: Eigenvalues of $\mathcal{D}$

We give the infinitesimal variation of the eigenvalues of $\mathcal{D}^\gamma$ in the case where M is compact and the eigenvalue λ simple.

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where Q_ψ is the symmetric covariant 2-tensor field defined as

$$Q_\psi(X, Y) = \frac{1}{2}\mathcal{R}e\big((X \cdot_\gamma D^\gamma_Y \psi, \psi) + (Y \cdot_\gamma D^\gamma_X \psi, \psi)\big)$$

for ψ a unit vector in the eigenspace for the eigenvalue λ.

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Comments on the Variation of the Eigenvalues

Several comments on this formula are in order:

- eigenvalues are of course invariant when the metric is replaced by a diffeomorphic image of it; this is reflected in the formula in the fact that the symmetric 2 -tensor field Q_ψ is divergence free provided ψ is an eigenspinor;
- the reason is that Q_ψ must be orthogonal to the tangent subspace at g to the space of metrics consisting of Lie derivatives of g with respect to vector fields; hence integrating by parts, $\delta_g Q_\psi$ must be orthogonal to all vector fields;
- one easily sees that $\mathrm{Trace}_g Q_\psi$ is exactly $\lambda\,||\psi||^2$, hence is non-negative as soon as $\lambda \neq 0$;
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5. Killing Spinors

Killing Spinors

Definition (R. PENROSE)

A Killing spinor ψ *is a spinor field lying in the kernel of* $\mathcal{P}$ *and an eigenspinor for* $\mathcal{D}$. *Its characteristic equation is, for some* $\lambda \in \mathbb{C}$,

$$\forall X \in TM,\ D_X\psi + \frac{1}{n}\lambda X.\psi = 0 \ .$$

Here are some key properties:

- the 1-form ξ_ψ defined on $X \in TM$ by $\xi_\psi(X) = (X.\psi, \psi)$ is dual to a *Killing vector field*, i.e. an infinitesimal isometry;
- other components of $\psi \otimes \bar{\psi}$ also satisfy interesting conditions;
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Killing Spinors (continued)

It is useful to discuss cases according to the eigenvalue λ:

- if $\lambda = 0$, then the Killing spinor is parallel, and hence the metric has *reduced holonomy*;
- if $\lambda \in i\,\mathbb{R}^*$, then M is non compact;
- if $\lambda \in \mathbb{R}^*$, then the Ricci curvature is uniformly positive, and by Myers' Theorem, M is compact if the metric g is complete.

The key construction, due to Christian BÄR, goes as follows:

- construct the cone $CM = M \times \mathbb{R}^{+*}$ over M and endow it with the cone metric $\bar{g} = dr^2 + r^2\, g$;
- then, through an identification of an action of the group $Spin_{n+1}$ within the Clifford algebra $Cl_g(M)$, map spinor fields on M into spinor fields on CM;
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- then, through an identification of an action of the group $Spin_{n+1}$ within the Clifford algebra $Cl_g(M)$, map spinor fields on M into spinor fields on CM;
- through this identification, Killing spinors on M are mapped to parallel fields on CM.

Killing Spinors (continued)

It is useful to discuss cases according to the eigenvalue λ:

- if $\lambda = 0$, then the Killing spinor is parallel, and hence the metric has *reduced holonomy*;
- if $\lambda \in i\,\mathbb{R}^*$, then M is non compact;
- if $\lambda \in \mathbb{R}^*$, then the Ricci curvature is uniformly positive, and by Myers' Theorem, M is compact if the metric g is complete.

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Manifolds with Killing Spinors

Many examples are connected to holonomy considerations:

- manifolds with parallel spinors have reduced holonomy; a classification is due to Nigel HITCHIN: SU_m, G_2 and $Spin_7$;
- manifolds with imaginary Killing spinors (actually the case if the manifold is complete and non compact) have been classified by Helga BAUM: hyperbolic spaces or special warped products of $\mathbb{R}$ with a manifold with a parallel spinor;
- the classification for real Killing spinors is a bit more involved;
- for $2 \leq n$ the standard sphere has only one spin structure; for the standard metric, the bundle of spinors is trivialized by Killing spinors that are induced by parallel spinors in $\mathbb{R}^{n+1}$;
- in even dimensions other than 6, only standard spheres carry Killing spinors; in dimension 6, one also finds the manifolds endowed with a nearly Kähler non Kähler metric.

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Other examples are interesting from a geometric or even a physical point of view:

- in dimension 5, Thomas FRIEDRICH and Ines KATH proved that one finds only manifolds with an Einstein-Sasaki metric; this was generalized by BÄR in all dimensions $4q + 1$;
- in dimensions $4q + 3$ (for $2 \leq q$), BÄR showed that one has to add the Sasaki 3-manifolds, as shown by Andrei MOROIANU.
- in the remaining dimension 7, the extra family to add corresponds to manifolds for which the cone built over them carries a metric with $Spin_7$ holonomy.
- the so-called *squashed 7-sphere* is a very interesting metric on the sphere S^7 as one can come to it from a purely Riemannian point of view (an exotic Einstein metric on the sphere besides the standard one) or from a supergravity point of view; it corresponds to squeezing appropriately the S^3-fibres of the fibration over S^4.

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6. A more Global Perspective

Using Spinors in Geometry

Spinor fields proved to be subtle objects that play an absolutely fundamental role in present models in Theoretical Physics.

Thanks to the work of Atiyah and Singer, they gained an important role in Mathematics and many developments are now available:

- using the fact that the $\hat{A}$-genus is the index of the Dirac operator operating on chiral spinors in dimension $4k$ André LICHNEROWICZ showed, using the Schrödinger-Lichnerowicz formula, that compact spin manifolds with a non-vanishing $\hat{A}$-genus do not admit metrics with positive scalar curvature;
- this has been sophisticated in many ways, starting with work by Misha GROMOV and H. Blaine LAWSON on appropriately twisted Dirac operators after some geometric constructions;
- they obtain the non-existence of metrics with positive scalar curvature on tori, a result that was obtained initially with a restriction on the dimension by Richard SCHOEN and Shing Tung YAU using minimal surfaces techniques.

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Using Spinors in Geometry (cont.)

Many more results have been obtained:

- by using more subtle topological invariants than the $\hat{A}$-genus, Nigel HITCHIN could show that, for $n \geq 9$, a number of exotic spheres do not admit metrics with positive scalar curvature;
- from further works by several people, it is possible to decide when Riemannian metrics with positive scalar curvature exist;
- we already touched on the links of special spinor fields with special geometries. In a series of remarkable papers, Reese HARVEY showed that many calibrations could be defined by the square of some special spinors;
- Edward WITTEN provided a proof of the *Positive Mass Conjecture* using spinors and the fact that the mass appears in the Schrödinger-Lichnerowicz formula as boundary-at-infinity contribution; recently Marc HERZLICH showed that such a feature is not specific to spinors but at the expense that other terms besides the scalar curvature appear.

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Towards a true Spinorial Geometry?

In my opinion, the role of spinors to deal with Riemannian questions has not yet reached its full potentiel. We may be missing a geometric picture for them. Hence we are back to the question *"What is a spinor?"* Can there be more to come? Some hints:

- as the existence of a spin structure is very constraining, one should investigate the broader class of manifolds admitting a spin_c-structure, that includes all complex manifolds;
- a link to the Ricci curvature, as to the one that appeared in presence of a Killing spinor should be considered;
- generalisations of Killing spinors have been considered by Andrei MOROIANU and others who studied the equation satisfied by the restriction of a parallel spinor to a hypersurface that involves its second fundamental form;
- mathematicians focused their attention on spinor fields of spin $\frac{1}{2}$; spinors with spin $\frac{3}{2}$ should be investigated with the Rarita-Schwinger operator that acts on them.

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