Kähler–Einstein Fano 3-folds

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Abstract

We classify Kähler–Einstein smooth Fano threefolds of Picard rank at least two.

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1 Introduction

All varieties are defined over $\mathbb{C}$. A smooth projective variety is Fano if its anticanonical divisor is ample. The classification of smooth Fano threefolds by Iskovskikh, Mori and Mukai gives 105 deformation families; we use the usual numbering, whose first integer is the Picard rank. Determining their K-polystable members is a finer problem, related to the existence of Kähler–Einstein metrics and not constant in every deformation family. We consider the following families; see [3]. Each family is treated in a separate part.

1. In Family 1.9, $X \subset \mathbb{P}^{11}$ is a codimension-two linear section of the five-dimensional adjoint G_2-variety in $\mathbb{P}^{13}$. It is a prime Fano threefold of genus ten, with $\mathrm{Pic}(X) = \mathbb{Z}[-K_X]$ and $(-K_X)^3 = 18$. We prove K-stability for every smooth member; the final assembly is in Section **??**.

2. In Family 1.10, X is the zero locus of three sections of $\bigwedge^2 \mathcal{U}^\vee$ on $\mathrm{Gr}(3,7)$, where $\mathcal{U}$ is the tautological rank-three bundle. It is a prime Fano threefold of genus twelve with $(-K_X)^3 = 22$. Part **??** contains a symmetric counterexample to the unrestricted anticanonical-singularity formulation of Donaldson's conjecture in [3, Conjecture 7.1], and partial classification results, including local estimates from ordinary point and line blow-ups, a nodal reduction for terminal Gorenstein degenerations, an exclusion of a nonterminal boundary family, strict monomial-valuation estimates, and an exact ribbon description of the section ring under the transverse width-two hypotheses.

3. In Family 2.5, X is the blow-up of a smooth cubic threefold $V \subset \mathbb{P}^4$ along a smooth plane cubic $C = V \cap \mathbb{P}^2$. The hyperplanes containing this plane give a cubic-surface fibration $X \to \mathbb{P}^1$, and $(-K_X)^3 = 12$.

4. In Family 2.9, X is the blow-up of $\mathbb{P}^3$ along a smooth curve of degree seven and genus five whose ideal sheaf is generated by three cubics. The cubic net gives a conic bundle $X \to \mathbb{P}^2$, and $(-K_X)^3 = 16$.

5. In Family 2.10, X is the blow-up of a smooth complete intersection of two quadrics $V \subset \mathbb{P}^5$ along the smooth elliptic quartic $C = V \cap \mathbb{P}^3$. The hyperplanes containing this linear subspace give a quartic del Pezzo fibration $X \to \mathbb{P}^1$, and $(-K_X)^3 = 16$.

6. In Family 2.11, X is the blow-up of a smooth cubic threefold $V \subset \mathbb{P}^4$ along a line ℓ. Projection from the line gives a conic bundle $X \to \mathbb{P}^2$, and $(-K_X)^3 = 18$. Both possible normal bundles, $\mathcal{O}_\ell^{\oplus 2}$ and $\mathcal{O}_\ell(1) \oplus \mathcal{O}_\ell(-1)$, are included.

7. In Family 2.12, X is a smooth complete intersection of three divisors of bidegree $(1,1)$ in $\mathbb{P}^3 \times \mathbb{P}^3$. Each projection blows up a smooth nonhyperelliptic sextic curve of genus three, and $(-K_X)^3 = 20$.

8. In Family 2.13, X is the blow-up of a smooth quadric $Q \subset \mathbb{P}^4$ along a smooth sextic curve of genus two cut out on Q by quadrics. The quadrics give a conic bundle $X \to \mathbb{P}^2$, and $(-K_X)^3 = 20$.

9. In Family 2.14, X is the blow-up of the quintic del Pezzo threefold $V_5 \subset \mathbb{P}^6$ along the smooth elliptic quintic cut out by two hyperplanes. Their pencil gives a quintic del Pezzo fibration $X \to \mathbb{P}^1$, and $(-K_X)^3 = 20$.

10. In Family 2.16, X is the blow-up of a smooth intersection of two quadrics $V_4 \subset \mathbb{P}^5$ along a smooth conic. Projection from the plane of the conic gives a conic bundle $X \to \mathbb{P}^2$, and $(-K_X)^3 = 22$.

11. In Family 2.17, X is the blow-up of $\mathbb{P}^3$ along a smooth elliptic quintic. Its second contraction is the blow-up of a smooth quadric along a smooth elliptic normal quintic, and $(-K_X)^3 = 24$. Write E, E' for the exceptional divisors over $\mathbb{P}^3$ and the quadric, respectively. For a point $p \in \mathrm{Sing}(E \cap E')$, let ℓ'_p be the fibre of E' through p, and put $\Sigma_X = \{p \in \mathrm{Sing}(E \cap E') : E|_{\ell'_p} = 3p\}$.

12. In Family 2.20, X is the blowup of V_5 along a smooth twisted cubic. Projection from its span gives a conic bundle over $\mathbb{P}^2$, and $(-K_X)^3 = 26$. Its K-stable, K-polystable and strictly K-semistable members are classified in Part XI.

13. In Family 3.2, X is a divisor of class $2\xi + \mathcal{O}(2,3)$ in $\mathbb{P}_{\mathbb{P}^1\times\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-1,-1)^{\oplus 2})$, where ξ is the tautological class and its restriction to X is irreducible. The projection is a conic bundle with cubic and sextic del Pezzo pencils, and $(-K_X)^3 = 14$.

14. In Family 3.5, X is the blow-up of $\mathbb{P}^1 \times \mathbb{P}^2$ along the graph of a degree-five morphism $\eta : \mathbb{P}^1 \to \mathbb{P}^1$, with the source embedded as a smooth conic in $\mathbb{P}^2$. Here $(-K_X)^3 = 20$. The stability of its smooth members is classified below.

15. In Family 3.6, X is the blow-up of $\mathbb{P}^3$ along a smooth elliptic quartic and a disjoint line. The plane and quadric pencils give del Pezzo fibrations of degrees five and six, and their product is a conic bundle over $\mathbb{P}^1 \times \mathbb{P}^1$. Here $(-K_X)^3 = 22$.

16. In Family 3.7, X is the blow-up of a smooth divisor $V \subset \mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(1,1)$ along a smooth complete intersection of two members of $|\mathcal{O}_V(1,1)|$. The resulting pencil consists of sextic del Pezzo surfaces, and $(-K_X)^3 = 24$.

17. In Family 3.8, X is the blow-up of $\mathbb{P}^1 \times \mathbb{P}^2$ along the graph of a degree-four morphism $\mathbb{P}^1 \to \mathbb{P}^1$, with its source embedded as a smooth conic in $\mathbb{P}^2$. Equivalently, the centre is the intersection of divisors of bidegrees $(0,2)$ and $(1,2)$. Here $(-K_X)^3 = 24$. Its K-polystable members are classified in Part XVI.

Theorem 1.1. *Every smooth Fano threefold in Families 1.9, 2.5, 2.9, 2.10, 2.11, 2.12, 2.13, 2.14, 2.16, 2.17, 3.2, 3.6 and 3.7 is K-stable.*

Every smooth member X of Families 2.20, 3.5 and 3.8 is K-semistable. Their K-stable and K-polystable members are characterized as follows.

1. *In Family 2.20, let $U = \langle f, g\rangle$ be the pencil of binary quartics corresponding to the twisted cubic, as in Section 69, and let $w_U = f_x g_y - f_y g_x$ be its Wronskian sextic. Then X is K-stable if and only if every root of w_U has multiplicity at most two. It is K-polystable if and only if this holds or w_U has two distinct triple roots.*

2. *In Families 3.5 and 3.8, let $\eta : \mathbb{P}^1 \to \mathbb{P}^1$ be the morphism whose graph is the blow-up centre, of degree five and four, respectively. Then X is K-stable if and only if η has no totally ramified point. It is K-polystable if and only if η has either no totally ramified point or two such points.*

In each of these three families there is, up to isomorphism, exactly one K-polystable member which is not K-stable. The strictly K-semistable members are precisely those with exactly one triple root in (1)*, or exactly one totally ramified point in* (2)*.*

Proof. The universal K-stability assertions for Families 1.9, 2.11, 2.13, 2.16, 2.17, 3.2, 3.6 and 3.7 are respectively Theorems **??**, 23.1, 38.1, 53.1, 61.1, 83.1, 103.1 and 108.1. For Family 2.5, combine Lemma 7.1 with Theorems 7.2 and 7.3. Family 2.9 follows from Proposition 10.2. The proofs for Families 2.10, 2.12 and 2.14 are completed in Sections 21, 37 and 52, respectively.

The three classifications are Theorems 69.1, 101.1 and 128.1. □

1.1 Description of the results

Every smooth Fano threefold in Family 1.9 is K-stable. We prove $\delta_p(X) \geq 6/5$ away from all lines and $\delta_p(X) \geq 72/71$ at every point on an ordinary line, and exclude all curve centres. The remaining points lie on special lines. At a marked point with nonzero quadratic coefficient μ, an ordinary point extraction gives $\delta_p(X) \geq 9/8$. When $\mu = 0$, the weight-$(1, 2, 2)$ extraction has expected order $287/72$. A surface estimate treats one sector of valuations. For the other sector, we blow up the strict transform of the line and apply adjunction with both exceptional divisors. A normal-jet bound makes the boundary effective, and the resulting inequality $A_X(v) - (288/245)S_X(v) \geq 11v(a)/35 > 0$ completes the proof. These point extractions require no threefold Zariski decomposition.

We prove K-stability of every smooth Fano threefold in Mori–Mukai Families 2.5, 2.9, 2.10, 2.11, 2.12, 2.13, 2.14, 2.16, 2.17, 3.2, 3.6 and 3.7. We obtain $\delta(X) \geq 96/95$ in Family 2.9, $\delta(X) \geq 320/319$ in Family 2.13, and $\delta(X) \geq 176/171$ in Family 2.16. The proofs use Abban–Zhuang refinement on del Pezzo fibres, cubic surfaces, and exceptional surfaces of point or curve extractions. We compute expected vanishing orders from filtered section rings or bound them by explicit spaces containing the refined sections. Each family is treated in a separate part, after Part I, which gives the definitions, adjunction tools and worked examples. In Family 2.11 we treat both normal-bundle types of the blown-up line. Weak cubic sections handle the special reduced fibre. At the remaining smooth tangencies, a weight-$(1, 1, 2)$ point extraction and a valuation centred on the ramification ruling give the exact local threshold $216/167$. Double conic fibres have local threshold at least $144/127$. For Family 2.12, the published reduction to points of the intersection of the two exceptional divisors is completed using ordinary point blow-ups and a weight-$(2, 1, 1)$ extraction. Exact initial rings and monomial moment certificates control all tangent configurations. The weighted calculations identify the actual refined section spaces and do not require a Zariski decomposition on an extracted threefold. For Family 2.13, quartic del Pezzo sections treat reduced conic fibres. Blowing up the node on such a section gives the bound $160/159$. For double fibres, the actual weighted refinement consists of ideal-power series for a length-two subscheme of $\mathbb{P}^1 \times \mathbb{P}^1$. A degeneration preserving the different, together with an explicit correction for the dimensions of these series, gives the bound $320/319$. For Family 2.14, the quintic del Pezzo pencil reduces the proof to four types of singular points and to lines on singular fibres. We treat both normal-bundle types of lines and both positions of their intersection with the elliptic quintic. The refined threshold can equal one in the A_3 case; a positive discrepancy term completes that case. For Family 2.16, the known reduced-fibre estimates leave double lines. Their weighted extraction has expected order $57/22$. The refined series consists of powers of the ideal of one point on the different, and a degeneration preserving that different gives the local bound $88/85$ on every double fibre. An independent GIT proof for Family 2.16 computes the CM class $6\zeta + 20\xi + 66\eta$, proves stability of every smooth pair by the Hilbert–Mumford criterion, and uses relative reconstruction and CM-degree minimization to obtain K-stability.

For Family 2.17, two line extractions reduce the problem to a finite triple-contact locus. A weight-$(1, 1, 2)$ point extraction at every remaining point has expected order $29/8$. Its actual refined section ring is identified by an integral initial divisor in a symplectic flag variety. A second degeneration gives complete binary spaces on $\mathbb{P}(1, 1, 2)$, whose moments yield $\delta_p(X) \geq 64/59$. This completes the proof for all smooth members, without a threefold Zariski decomposition on the weighted point blow-up. For Families 3.6 and 3.2, weighted curve extractions above double conics have expected orders $243/88$ and $135/56$, respectively. Exact section spaces give refined thresholds at least one, with the different retained. The positive discrepancy remainders complete both proofs, including the special double-conic configurations. Every smooth member of Family 3.5 is K-semistable. It is K-stable exactly when its associated degree-five map has no totally ramified point; the K-polystable members are also classified.

For Family 3.7, sextic fibres and quintic sections reduce the proof to finitely many point configurations. Exact section algebras on ordinary and weighted point blow-ups complete the proof for every smooth member.

For Family 2.20 we identify every nonclosed smooth GIT orbit by a pencil of binary quartics and its Wronskian sextic. These give a one-parameter collection of strictly K-semistable threefolds with a common K-polystable smooth limit. Abban–Zhuang estimates exclude every divisor centre and every point on a smooth conic-bundle fibre. Exact singular-fibre surface calculations isolate the remaining valuative problem. At every double fibre the weighted extraction has expected order 297/104 and discrepancy three. We determine its complete refined series by multiplication and a dimension count. Ordinary and weighted surface blowups prove $\delta_p(X) \geq 104/99$ along every double fibre of a GIT-stable member. For reduced fibres, two flops give the expected order 43/26 for the extraction of a component. Its complete refinement yields the same local bound along ordinary components when the fibre node is outside the original exceptional divisor. Negative-section flags and a retained discrepancy treat special-line components. The possible nonpositive-beta centres on a GIT-stable member reduce to at most three fibre nodes on the original exceptional divisor. Their ordinary point extractions have $S_X(J) = 11/4 < 3 = A_X(J)$. We identify their complete initial section algebras in every degree. Surface flags, including an extra fixed line in the degenerate-conic case, give $\delta_p(X) \geq 104/103$ at all remaining nodes. This proves that a smooth member of Family 2.20 is K-stable exactly when its quartic pencil is GIT-stable, and K-polystable exactly when that pencil is GIT-polystable.

For Family 3.8, a smooth member is determined by a basepoint-free pencil of binary quartics. It is K-stable exactly when the associated degree-four morphism has no totally ramified point. There is one K-polystable member which is not K-stable; all other smooth members are strictly K-semistable and are parametrized by $\mathbb{P}(2,3)$. The proof treats every singular del Pezzo fibre by explicit surface flags and point or curve extractions. The actual refined sections, or complete containing series with the same asymptotic dimension, replace Zariski decompositions on the extracted threefolds.

For Family 1.10, we construct a symmetric counterexample to the unrestricted anticanonical-singularity formulation of Donaldson's conjecture, stated as Conjecture 7.1 in *The Calabi Problem for Fano Threefolds*. A small deformation of the Mukai–Umemura threefold in the stable binary-octic direction $uv(u^6 + v^6)$ is K-stable with finite automorphism group and has two points on no line exchanged by an $\mathfrak{S}_3$-action. Each point has an anticanonical divisor with exact germ $y^2 = x^3 + t^4x$; the involution exchanges the two divisors. We compute a central pencil, impose $j = 1728$ at a simple root, and prove equivariant persistence. The additional hypothesis that a bad point lies on a line still implies failure of K-stability. We identify the exact bad-point-on-line locus with the locus of members having a special line meeting no other line, and with the complement of affine GIT-stable Mukai nets. In particular this locus is closed, and every member on it has $\delta(X) = 1$. The line-incidence and closed-net revisions are therefore equivalent. We prove the revised criterion near Mukai–Umemura; the global implication from GIT stability to K-stability remains open. We also obtain the following partial classification results. Nonclosed smooth Mukai-net orbits yield strictly K-semistable members with the prescribed anticanonical singularity. The ordinary point blow-up gives $\delta_p(X) \geq 264/259$ at every point on no line, including all nonordinary conic flops. We give sharper bounds according to the singularities of the associated cubic surface, and exhibit a smooth pointed member with one double conic parameter. For ordinary lines we classify all resolving-surface configurations and compute their fibre flags. In particular, $\delta_p(X) \geq 88/83$ if exactly one line passes through p and it is ordinary. The same bound holds along lines with smooth or nodal associated quintic surface and in further configurations treated by a point extraction on the exceptional surface. By changing the chosen line, the surface estimates reduce the points away from special lines to two explicit incidence configurations. A new

point-jet estimate settles the directed three-line cycle for all divisorial valuations, giving $\delta_p(X) \geq 270336/269791$. A weighted extraction also gives a strict bound along the special lines satisfying the explicit hypotheses of Definition **??**. This includes all ramification cases on their exceptional surfaces. The proof retains actual fixed parts, differents and total-pullback corrections throughout. We also prove that every special test configuration with factorial terminal central fibre has zero Donaldson–Futaki invariant. Every terminal Gorenstein limit is nodal, with divisor-class rank one plus its number of nodes. We exclude all four one-nodal families as smooth test-configuration limits. An explicit fixed-point obstruction excludes the nonterminal boundary family obtained from smooth bidegree-$(4,1)$ curves on a quadric in $\mathbb{P}^3$. For the two exceptional ordinary-line configurations we obtain $A_X/S_X \geq 132/125$ and $A_X/S_X \geq 264/229$, respectively, for valuations monomial in coordinates straightening the lines. Under the transverse width-two hypotheses, the full anticanonical ring is described by powers of the ideal of a rational ribbon on a smooth quadric. The special-line links have four conductor configurations. Double conductor contact on $\mathbb{F}_1$ now has $\delta_p(X) \geq 88/83$ along the entire special line, with the infinitely near conditions retained. On the GIT-stable locus, the remaining local cases are the mutually width-three ordinary pair and the distinguished point in the $\mathbb{F}_3$ width-two case. These results do not complete the global classification for Family 1.10, which is not included in the Main Theorem.

The three classifications confirm the conjectures in [3, Sections 7.2, 7.5 and 7.6].

We also construct a counterexample to Donaldson's conjecture in *The Calabi Problem for Fano Threefolds* [3, Conjecture 7.1]. It concerns smooth prime Fano threefolds of genus twelve and predicts K-polystability from the absence of certain singular anticanonical divisors, apart from the members admitting a torus action. Our example contradicts the necessary direction of that prediction.

Theorem 1.2. *There exists a smooth K-stable threefold X in Family 1.10 with finite automorphism group and an anticanonical divisor having the analytic singularity $y^2 = x^3 + t^4x$. In particular, Donaldson's conjecture in [3, Conjecture 7.1] is false as stated.*

Proof. Theorem **??** constructs such threefolds as small nonzero deformations of the Mukai–Umemura threefold in the binary-octic direction $uv(u^6 + v^6)$. Their automorphism groups are finite, so the torus alternative in the conjecture does not apply. □

The construction has an $\mathfrak{S}_3$-action exchanging two points on no line and two anticanonical divisors with the prescribed germs. A single such divisor already contradicts the conjecture. At these points the space of sections of weighted order at least six has dimension two, whereas the torus degeneration giving the known obstruction has dimension eight. Thus the analytic equation does not determine the global anticanonical filtration. Subsection **??** discusses possible revisions. Theorem **??** identifies the exact bad-point-on-line locus with the complement of affine GIT-stable Mukai nets and proves that it is closed. Every member of this locus has $\delta(X) = 1$. The line-incidence and closed-net conjectures are equivalent, and both hold near Mukai–Umemura. The global implication from a smooth GIT-stable net to a K-stable threefold remains unproved.

The valuative approach of Fujita and Li replaces test configurations by inequalities between log discrepancies and expected vanishing orders [18, 22]. The stability threshold, introduced through basis divisors and developed systematically by Blum–Jonsson [6], packages these inequalities in the invariant δ. Its relation to K-stability and the existence of Kähler–Einstein metrics is described in [21]. Abban and Zhuang [4] developed refinement by admissible flags, allowing threefold estimates to be reduced to series on surfaces and curves. We use this method throughout.

Table 1 records the seventeen families for which a complete classification of K-polystable smooth members remained unresolved before the results of this paper, with the qualification

concerning Family 3.5 below. Theorem 1.1 summarizes our results. The entry for Family 1.10 records partial results only; its global classification remains open. The geometric descriptions follow the classification in [3, Section 6]. All varieties and blow-up centres in the table are smooth, and the usual Fano conditions on the centres are understood.

In the table, $V_3 \subset \mathbb{P}^4$ is a cubic, $V_4 \subset \mathbb{P}^5$ is an intersection of two quadrics, $V_5 \subset \mathbb{P}^6$ is a three-dimensional linear section of $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$, and $Q \subset \mathbb{P}^4$ is a quadric. For Family 1.10, $\mathcal{U}$ is the tautological rank-three subbundle on $\mathrm{Gr}(3,7)$. For Family 3.2, put

$$\pi : \mathcal{B} = \mathbb{P}_{\mathbb{P}^1\times\mathbb{P}^1}\bigl(\mathcal{O} \oplus \mathcal{O}(-1,-1)^{\oplus 2}\bigr) \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1, \qquad \xi = \mathcal{O}_{\mathcal{B}}(1),$$

using the quotient convention, and let D be the unique divisor in $|\xi|$.

Table 1: Families with an incomplete classification before the present results.

Family	Geometric description	This paper
1.9	Codimension-two linear section of the five-dimensional adjoint G_2-variety in $\mathbb{P}^{13}$.	Part ??
1.10	Zero locus of a section of $(\bigwedge^2 \mathcal{U}^\vee)^{\oplus 3}$ on $\mathrm{Gr}(3,7)$.	Part ?? (partial)
2.5	Blow-up of V_3 along a plane cubic.	Part II
2.9	Blow-up of $\mathbb{P}^3$ along a degree-seven, genus-five curve whose ideal sheaf is generated by three cubics.	Part III
2.10	Blow-up of V_4 along an elliptic quartic cut out by a $\mathbb{P}^3$.	Part IV
2.11	Blow-up of V_3 along a line.	Part V
2.12	Complete intersection of three $(1,1)$-divisors in $\mathbb{P}^3 \times \mathbb{P}^3$.	Part VI
2.13	Blow-up of Q along a genus-two sextic.	Part VII
2.14	Blow-up of V_5 along an elliptic quintic cut out by two hyperplanes.	Part VIII
2.16	Blow-up of V_4 along a conic.	Part IX
2.17	Blow-up of Q along an elliptic quintic.	Part X
2.20	Blow-up of V_5 along a twisted cubic.	Part XI
3.2	A divisor $X \in \lvert 2\xi + \pi^*\mathcal{O}(2,3)\rvert$ in $\mathcal{B}$, with $X \cap D$ irreducible.	Part XII
3.5*	Blow-up of $\mathbb{P}^1 \times \mathbb{P}^2$ along a $(5,2)$-curve embedded by projection to $\mathbb{P}^2$.	Part XIII
3.6	Blow-up of $\mathbb{P}^3$ along a disjoint line and elliptic quartic.	Part XIV
3.7	Blow-up of a $(1,1)$-divisor $W \subset \mathbb{P}^2 \times \mathbb{P}^2$ along the intersection of two members of $\lvert\mathcal{O}_W(1,1)\rvert$.	Part XV
3.8	Blow-up of $\mathbb{P}^1 \times \mathbb{P}^2$ along the intersection of $(0,2)$- and $(1,2)$-divisors.	Part XVI

*Family 3.5 is retained in this historical list; its smooth classification is proved in Part XIII. Denisova–Papazachariou's announced manuscript is listed under "In Preparation", with a PDF available upon request, on Papazachariou's research page, consulted on 30 September 2026 [15]. We have not used that manuscript as a theorem. Part XIII proves K-semistability for every smooth member and K-stability whenever there is no totally ramified point, including all mixed fibres.

Abban–Zhuang prove K-stability for smooth prime Fano threefolds of degree at most 16 in [44]; their argument does not cover degree 18. The published proof of [23, Lemma 4.2] treats Family 1.9 under a fixed-point-free abelian symmetry hypothesis. Its multiplier-ideal argument excludes curve centres of degree at least two without using that symmetry. Xu–Zhuang still list the universal K-semistability assertion for Family 1.9 in [51, Problem 10], in the revision dated 1 June 2026.

Part ?? proves K-stability for every smooth member of Family 1.9. The exceptional divisor of an ordinary line blow-up has expected order $101/72$; the weighted extraction over a special

line has expected order 43/18. Surface flags and changes of projection line treat every point on an ordinary line. The special-line analysis reduces to marked points on higher-width special lines. At those points, two-divisor adjunction retains the positive term $11v(G)/18$ for valuations whose centres on the smooth model are contained in G.

An ordinary point extraction gives $\delta_p(X) \geq 9/8$ when the quadratic coefficient μ is nonzero. If $\mu = 0$, the actual initial sections of the weight-$(1, 2, 2)$ extraction split into binary spaces. Their fixed orders and full Hilbert mass give $S_X(F) = 287/72$ and a refined bound $144/119$ on $C \neq 0$. The last valuations have centre $P = [1 : 0 : 0]$ on this extraction. The ordinary blowup of P on F has refined moment $115/72$, and its point moments on the required affine chart are at most $1/3$. To lift these surface estimates to the threefold, we blow up the strict transform of the original line. The normal bundle gives

$$\operatorname{ord}_F(s) \leq m + 4 \operatorname{ord}_L(s) \qquad (0 \neq s \in H^0(X, mH)).$$

This inequality controls the extra boundary on the surface and allows both applications of inversion of adjunction to use effective boundaries. Retaining the threefold discrepancy gives $A_X(v) - (288/245)S_X(v) \geq 11v(a)/35 > 0$. No Zariski decomposition on either extracted threefold enters this final calculation.

For Family 2.5, Cheltsov–Denisova–Fujita [8] proved K-stability under the assumption that the cubic-surface fibration has no singular point of type D_5 or E_6. Their local exclusions reduce the general case to these two singularities. Part II supplies the bounds $96/95$ and $16/15$ for the corresponding local thresholds and completes their reduction.

For Family 2.9, the argument in [3, Section 5.2] proves K-stability of a particular member using a cyclic group of order five and hence gives K-stability for a general member. Part III treats every smooth member. Flags on smooth cubic surfaces handle reduced conic fibres away from a distinguished exceptional fibre. At its remaining smooth-fibre points, blowing up the point on the cubic surface gives $\delta_p(X) \geq 6912/5335$. We also give an independent proof using the exceptional plane of a threefold point blow-up. Double conic fibres require a weighted curve extraction; its exceptional surface is $\mathbb{P}^1 \times \mathbb{P}^1$ with different one half of a smooth $(2, 1)$-curve. The boundary is retained in the surface calculation.

For Family 2.10, Kaloghiros and Petracci proved K-stability of a general member [20, Theorem 1.3], and the equivariant results of [23] include special members. Part IV removes the generality and symmetry assumptions. Fibre estimates and nodal auxiliary sections treat the global reduction. Weighted point extractions treat D_4 and D_5 fibre singularities, while ordinary point blow-ups treat smooth points on the special lines of singular fibres. The latter computation requires an additional anticanonical section whose order exceeds the orders of its displayed summands.

The new exceptional-surface estimates do not require a complete Zariski decomposition on the extracted threefold. In Families 2.5 and 2.10, we identify the actual section filtrations, including cancellations, by regularity, localization and saturation. Weighted Hilbert functions and polynomial intervals then give the expected orders. In the exceptional-surface arguments for Family 2.9, explicit necessary vanishing conditions give spaces containing the refined sections. Their order sums provide upper bounds with the original normalization. We retain the volume calculations on the original threefolds, the Zariski decompositions on auxiliary surfaces, and the complete transverse-position calculation in Part III. Thus the absence of a full decomposition on an extracted threefold is justified by a description or comparison of section spaces, not assumed.

For Family 2.11, Part V treats both kinds of lines. The original exceptional divisor suffices for first-kind lines. For second-kind lines, its refined threshold is only $24/29$, so another argument is needed. A weak cubic section gives $\delta_p(X) \geq 36/35$ on the special reduced fibre. After the curve centres and transverse points have been excluded, the remaining points lie

on two ramification rulings over smooth conic fibres. A weight-$(1,1,2)$ blow-up at each such point has expected order $26/9$ and refined surface threshold at least $216/167$. We calculate the latter from complete binary spaces with explicit fixed factors. A valuation centred on the ramification ruling has discrepancy three and expected order $167/72$, giving the matching upper bound. Thus the local threshold at each of these tangency points is exactly $216/167$; see Corollary 29.9. For double fibres, the two possible expected orders are $91/36$ and $169/72$, according to whether the reduced fibre lies in the original exceptional divisor; both refined surface thresholds are at least $144/127$. None of these weighted extractions requires a threefold Zariski decomposition.

For Family 2.12, Cheltsov–Li–Ma'u–Pinardin [1] reduced every possible nonpositive-beta valuation to a closed point of $E \cap E'$ and proved stability for several special members. Part VI completes the point analysis. Smoothness of the determinant quartic forces the coprimality needed for an exact ordinary initial ring. Three cases on its exceptional plane are treated by explicit monomial moment certificates. In one of them the refined threshold equals one, but the extraction discrepancy retains a strictly positive term. Proportional residual directions require weights $(2,1,1)$; the expected order is $13/4$ and the refined threshold is at least $40/37$. We compute both ordinary and weighted expected orders in a separate Hilbert-weight lemma. These computations, too, require no Zariski decomposition on an extracted threefold.

For Family 2.13, the symmetric construction in [3, Section 5.5] proves K-stability of a particular member and hence of a general member. The smooth-conic estimate in the proof of [24, Lemma 5.6] gives $\delta_p(X) \geq 80/77$. Part VII treats all remaining fibres. The ordinary blow-up of a node on a quartic del Pezzo section improves the polarized surface estimate to $\delta_p(S, -K_S + tF) \geq 4/(3+2t)$. For a double conic, the blow-up of its scheme-theoretic fibre ideal is a weight-$(1,2)$ curve extraction. Its expected vanishing is $5/2$, and its exceptional pair is $(\mathbb{P}^1 \times \mathbb{P}^1, \frac{1}{2}D)$ with D a smooth $(2,1)$-curve. We identify the actual weighted initial ring and its refined ideal-power spaces before estimating their thresholds. A degeneration keeps D fixed. The limiting spaces can be smaller than the natural monomial spaces containing them; the resulting dimension correction is essential for the strict inequality. This weighted argument uses no Zariski decomposition on the extracted threefold.

For Family 2.14, the symmetric construction in [3, Section 4.3] proves K-stability of a particular member, and hence of a general member. Part VIII treats every smooth member. Every pencil fibre is a normal quintic del Pezzo surface with Du Val singularities. The exact fibre refinement reduces possible nonpositive-beta centres to certain lines and singular points. Ordinary line blow-ups and weight-$(1,2)$ line extractions treat the two normal-bundle types. The latter require separate calculations at the intersection with the original exceptional divisor. Point extractions treat A_1, A_2, A_3, A_4. Prime Pfaffian initial rings and full-volume section certificates account for all cancellations. A restriction lemma retains the zeros on a second divisor, giving the complementary point bounds. At an A_3 point, the first extraction has expected order $71/16$ and discrepancy five; its positive margin completes the argument even when the refined threshold equals one. None of these extractions requires a threefold Zariski decomposition.

For Family 2.16, Cheltsov–Hwang–Küronya–Laface–Mangolte–Massarenti–Park–Zhao [28] prove strict local estimates at every point on a reduced conic fibre. Their main theorem consequently proves K-stability when the discriminant is smooth, and more generally under an explicit equivariant condition. Kaloghiros–Liu–Petracci–Zhao [34, Appendix A, Theorem A.1] describe the geometry of all K-semistable degenerations of this family; that result does not by itself prove stability of every smooth member. Part IX treats the remaining double fibres. Blowing up their fibre ideals gives a plt exceptional surface $F \cong \mathbb{P}^1 \times \mathbb{P}^1$ with different $D/2$ of bidegree $(2,1)$. A prime initial complete intersection identifies the actual refined spaces as $H^0(\mathcal{I}_P^k(m-\epsilon, r))$ for one point $P \in D$. Their dimensions give $S_X(F) = 57/22$. Moving P to a ramification point while preserving D gives the exact monomial spaces used for the

bound 88/85. The reduced-fibre estimates then give $\delta(X) \geq 176/171$. This proof also uses no Zariski decomposition on the weighted extracted threefold. A second proof in Sections 57–60 uses GIT. We compute the CM polarization on the incidence space of a conic and a pencil of quadrics and prove a strict Hilbert–Mumford inequality for every smooth pair. The relative reconstruction of [34, Proposition A.13], together with CM-degree minimization, then gives K-stability. This argument is independent of the local threshold estimates and does not assume the conjectural description of the entire K-moduli space as a GIT quotient.

For Family 2.17, Cheltsov–Pokora [40] reduce possible nonpositive-beta centres to points of $E \cap E'$. Part X uses the two contracted lines to reduce further to Σ_X. The remaining triple-contact points are treated by a weight-$(1,1,2)$ blow-up. The exact initial section ring gives $S_X(G) = 29/8$, and a second degeneration gives refined threshold at least $64/59$. The exact Hilbert count ensures that no higher-order sections are lost. This completes K-stability for every smooth member. No Zariski decomposition on this weighted point blow-up is required.

For Family 3.6, Cheltsov [42] proves K-stability when the conic bundle has reduced fibres. Part XIV treats the omitted double conics. The blow-up of their fibre ideals gives a weighted curve extraction of discrepancy three and expected order $243/88$. We compute its actual refined spaces, retain the different $D/2$, and prove that the refined threshold is at least one. The positive remainder $21/88$ then proves strict K-stability. The argument includes the case where the line passes through a quadric-cone vertex and the corresponding sextic fibre is nonnormal. As in the preceding weighted arguments, no threefold Zariski decomposition is used at this extraction.

For Family 3.2, Belousov–Loginov [43] prove K-stability when the cubic fibre containing each double conic has only A_1 singularities along it. We use their corrected third version: its reduced-conic estimate is non-strict, and we retain the positive remainder in the first Abban–Zhuang step. Part XII treats the omitted double conics. Their weighted extraction has expected order $135/56$. An exact dimension count identifies its refined sections with monomial spaces. A second degeneration, with the different retained, gives refined threshold at least one. The finite-level equality at one weighted surface point is exact; the threefold discrepancy remainder $33/56$ gives strict K-stability. No Zariski decomposition on the weighted threefold is used.

For Family 3.5, Denisova [14] proves K-stability when all ramification is simple. The totally ramified cases are described in [3, Section 7.5]. Part XIII gives their explicit degeneration and its Abban–Zhuang input. It also computes $S_X(G) = (91r + 25)/80$ for the weight-$(1, 2, r-2)$ extraction at a fibre singularity of ramification index $r = 3, 4, 5$. These first moments do not by themselves prove the required exceptional-surface inequalities. For $r = 3$, an exact normal basis and an explicit surface log resolution now give $\delta(G; W^G) \geq 160/151$ and $\delta_p(X) \geq 160/151$. For $r = 4$, a conic refinement with its different gives $\delta(G, \Delta; W^G) \geq 80/77$ and the exact local threshold $\delta_p(X) = 400/389$. Explicit line refinements cover the other centres outside the joining line of a mixed fibre. Ordinary point extractions then give $240/221$ at its A_1 endpoint and $32/29$ at its interior points. Their exact initial algebras and conic or cubic pencil resolutions complete the proof for every smooth member without a totally ramified point. The fibre-refinement calculation at an A_2 point is audited separately: the displayed surface volumes in [14] give $91/80$, rather than the stated $83/80$. The general semistability assertion for Family 3.5 and its complete smooth classification are included in the main theorem.

Section 102 applies the CM and GIT calculations in Liu's moduli-continuity approach. The graph parameter space has anticanonical CM line $\mathcal{O}(22)$, and a smooth graph is GIT-stable exactly when its degree-five map has no totally ramified point. Liu–Zhao's volume theorem applies to K-semistable limits of volume 20; the anticanonical K3 lattice on a smooth member is $U \oplus \langle -28 \rangle$. The independent local proof above establishes the smooth classification. Reconstructing contractions on arbitrary limits remains a question for the K-moduli compactification.

For Family 3.7, [3, Section 4.3] treats symmetric examples, while [23, Lemma 4.3] imposes an additional group-action hypothesis. Part XV treats every such centre. The matrix pencil has only smooth fibres and fibres with one A_1 or A_2 singularity. Fibre refinements reduce the problem to the two lines through these singularities. Quintic del Pezzo sections treat their open parts. Ordinary and weighted point extractions treat the singular points, the intersections with the original exceptional divisor, and the remaining smooth corner. Their actual initial section spaces are proved complete by exact Hilbert counts. No Zariski decomposition on an extracted threefold is used.

For Family 2.20, [3, Section 7.2] gives GIT-semistable smooth parameters whose orbits are not closed and whose associated threefolds are strictly K-semistable. Its Proposition 5.43 proves K-polystability of the unique member with infinite automorphism group. Part XI rewrites the parameter classification intrinsically using binary quartics and a sextic Wronskian, constructs the smooth special degenerations, and gives Abban–Zhuang reductions to singular conic fibres. The complete weighted refinement treats all double fibres on the GIT-stable locus. Component extractions treat reduced fibres away from finitely many nodes on E. The full point refinements at these nodes give $\delta_p(X) \geq 104/103$ and prove the GIT-stable converse. This completes the smooth classification.

For Family 3.8, [3, Section 7.6] formulates the classification in terms of the degree-four cover and proves the total ramification cases. Part XVI completes the K-stable case. The quartic pencil can have fibres of types $(2,1,1)$, $(2,2)$ or $(3,1)$, and each type is included. Exact initial algebras treat the A_1 and A_2 points. Weighted curve extractions, surface flags and a marked specialization treat the exceptional rulings and the joining lines, including their endpoints. The final criterion is stated in Section 128.

Part I collects the Abban–Zhuang theory, the shared definitions and computational tools, and examples illustrating their later applications. The thirteen universal K-stability assertions of Theorem 1.1 are proved in Parts **??**, II, III, IV, V, VI, VII, VIII, IX, X, XII, XIV and XV, respectively. The complete classification for Family 2.20 is proved in Part XI and summarized in Section 82. The classifications for Families 3.5 and 3.8 are treated in Parts XIII and XVI, respectively. The local theorems in those parts record the stronger estimates used in the proofs. In Part **??** we write $H = -K_X$ and use L for a line. In the other parts we write $L = -K_X$; the remaining geometric notation is introduced separately in each part. In particular, the letters denoting fibre and exceptional classes are local to that part. Every local stability threshold takes the infimum over divisors whose centres contain the specified point, including positive-dimensional centres.

Part I
Abban–Zhuang theory

This part collects the definitions, adjunction inequalities and computational tools used throughout the manuscript. We state the hypotheses on the actual section spaces, explain the normalization of their expected vanishing orders, and retain fixed divisors and differents in every restriction. The general refinement theorem is due to Abban and Zhuang [4]; its surface applications are developed in [3, Section 1.7], and the rational-divisor formulation used here is given in [17]. The examples below are taken from the later proofs, where their geometric inputs and full calculations are established.

2 Stability thresholds and refinement

2.1 Discrepancies, alpha and delta

Let Y be a normal projective variety and let Δ be an effective rational divisor such that $K_Y + \Delta$ is $\mathbb{Q}$-Cartier. A *prime divisor over* Y is a prime divisor F on a normal variety Y' with a projective birational morphism $f : Y' \to Y$. Its centre is $c_Y(F) = f(F)$, and its log discrepancy is

$$A_{Y,\Delta}(F) = 1 + \operatorname{coeff}_F(K_{Y'} - f^*(K_Y + \Delta)).$$

One may compute this on a resolution; it is independent of the chosen model. The pair is *log canonical*, respectively *klt*, if every such discrepancy is nonnegative, respectively positive. We write $A_Y(F)$ when $\Delta = 0$. A divisorial valuation is a positive multiple of ord_F; discrepancy and all expected vanishing orders are homogeneous in that multiple.

For an effective $\mathbb{Q}$-Cartier rational divisor D on a klt pair, its log canonical threshold is

$$\operatorname{lct}(Y, \Delta; D) = \sup\{c \geq 0 : (Y, \Delta + cD) \text{ is log canonical}\} = \inf_F \frac{A_{Y,\Delta}(F)}{\operatorname{ord}_F(D)},$$

where a zero denominator is interpreted as $+\infty$. If M is an ample rational divisor, we define

$$\alpha(Y, \Delta; M) = \inf_{D \geq 0,\ D \sim_{\mathbb{Q}} M} \operatorname{lct}(Y, \Delta; D).$$

The local alpha invariant is defined by requiring log canonicity at the specified point. This is the algebraic alpha invariant; for the anticanonical polarization it corresponds to Tian's alpha invariant [6]. For $a > 0$, the inequality $\alpha(Y, \Delta; M) \geq a$ means that $v(D) \leq A_{Y,\Delta}(v)/a$ for every such D and every divisorial valuation v. When Y is Fano, $\alpha(Y)$ means $\alpha(Y, 0; -K_Y)$.

For a big rational Cartier divisor M on an n-fold, define its volume by

$$\operatorname{vol}(M) = \limsup_{m \to \infty} \frac{n! h^0(Y, mM)}{m^n},$$

where m is divisible enough to make mM integral. Volume extends continuously to real divisor classes. Choose a smooth projective model $f : Y' \to Y$ carrying a given prime divisor F, so the volume expressions below involve Cartier real divisors. Set

$$S_M(F) = \frac{1}{\operatorname{vol}(M)} \int_0^\infty \operatorname{vol}(f^*M - tF)\, dt, \qquad \delta(Y, \Delta; M) = \inf_F \frac{A_{Y,\Delta}(F)}{S_M(F)}.$$

For an irreducible subvariety $Z \subset Y$, the local invariant $\delta_Z(Y, \Delta; M)$ restricts the infimum to F with $Z \subset c_Y(F)$. In particular, $\delta_P(Y, \Delta; M)$ tests every centre containing P, including curves and divisors through P. It does not test only divisors whose centre equals P. The corresponding

local log canonical threshold requires log canonicity at the generic point of Z. For a smooth Fano threefold, write $S_X(F) = S_{-K_X}(F)$, $\delta(X) = \delta(X, 0; -K_X)$ and $\delta_P(X) = \delta_P(X, 0; -K_X)$.

We use K-stability through its equivalent valuative formulation: X is K-stable if $\beta_X(F) = A_X(F) - S_X(F) > 0$ for every prime divisor F over X. K-semistability is characterized by the corresponding non-strict inequalities. The inequality $\delta(X) > 1$ characterizes uniform K-stability. For a smooth Fano variety, the equivalence between K-stability and uniform K-stability follows from [21]; in particular K-stability is equivalent to $\delta(X) > 1$ in our setting. These characterizations, and their relation to the definition using test configurations, are supplied by [18, 22, 6, 21]; they are external inputs here.

Lemma 2.1 (Expected vanishing from an adapted basis)**.** *Let M be ample and Cartier and put $N_m = h^0(Y, mM)$. Filter $H^0(Y, mM)$ by the subspaces $\mathcal{F}^j = \{s : \mathrm{ord}_F(s) \geq j\}$. If $\mathcal{B}_m$ is a compatible basis, then*

$$S_M(F) = \lim_{m\to\infty} \frac{1}{mN_m} \sum_{s\in\mathcal{B}_m} \mathrm{ord}_F(s).$$

Proof. Compatibility means that each filtered subspace is spanned by the basis vectors it contains. Therefore the sum of the basis orders equals $\sum_{j\geq 1} \dim \mathcal{F}^j H^0(Y, mM)$. On a model carrying F, the summand is $h^0(Y', mf^*M - jF)$. Divide by mN_m and regard the sum as a Riemann sum with $t = j/m$. The asymptotic formula for sections, together with the linear bound on the filtration and the standard convergence theorem for filtered series [6], gives the volume integral above. The denominator uses $N_m = \mathrm{vol}(M)m^n/n! + O(m^{n-1})$. □

Example 2.2. For $Y = \mathbb{P}^1$, $M = \mathcal{O}_{\mathbb{P}^1}(d)$ and $F = Q$ a point, an adapted basis of $H^0(Y, mM)$ has orders $0, 1, \ldots, md$. Its average order divided by m is $d/2$. Equivalently, $S_M(Q) = d^{-1}\int_0^d (d - t)\, dt = d/2$. Thus $\delta(\mathbb{P}^1) = 1$ when $M = -K_{\mathbb{P}^1}$.

2.2 The section spaces and their normalization

The auxiliary spaces in an adjunction argument are usually not complete linear systems. We specify them before discussing their invariants. Fix rational Cartier divisors $L_0, \ldots, L_r$ on a normal projective variety T. A multigraded linear series V consists of finite-dimensional subspaces

$$V_{m,\mathbf{a}} \subset H^0(T, mL_0 + a_1L_1 + \cdots + a_rL_r), \qquad (m, \mathbf{a}) \in \mathbb{N} \times \mathbb{N}^r,$$

with $V_{0,\mathbf{0}} = \mathbb{C}$ and products contained in the space whose index is the sum of the indices. A divisor representing each L_i is fixed. Thus, if s is represented by a rational function in the indicated space, its effective zero divisor is $mL_0 + \sum a_iL_i + \mathrm{div}(s)$. For a rational divisor D, the notation $H^0(T, D)$ means $H^0(T, \mathcal{O}_T(\lfloor D \rfloor))$, but its section divisors retain the fractional part of D.

The support is the closed convex cone generated by indices of nonzero spaces. We require its slice $m = 1$ to be bounded. We say that V contains an ample series if the support has nonempty interior, every integral index in its interior has nonzero sufficiently large multiples, and there is an interior integral index $\boldsymbol{\alpha}$ for which the associated divisor is $A + E$, with A ample rational Cartier and E effective rational Cartier, such that multiplication by the section of kE embeds $H^0(T, kA)$ in $V_{k\boldsymbol{\alpha}}$ for all sufficiently divisible k. These are the hypotheses on series used in this part. We work in the lattice generated by the grading; when necessary a finite-index Cartier sublattice is chosen explicitly.

For a fixed first degree m, set $N_m(V) = \sum_{\mathbf{a}} \dim V_{m,\mathbf{a}}$. Bounded support makes the sum finite. An m-basis divisor is

$$D_m = \frac{1}{mN_m(V)} \sum_{\mathbf{a}} \sum_{s\in\mathcal{B}_{m,\mathbf{a}}} \mathrm{div}_0(s),$$

where $\mathcal{B}_{m,\mathbf{a}}$ is a basis of $V_{m,\mathbf{a}}$. The denominator is the dimension of the whole degree-m series, multiplied by m. It is not the dimension of one summand. For a divisorial valuation v, define $S_m(V;v) = \sup_{D_m} v(D_m)$, and define $\delta_m(T,\Delta;V)$ as the infimum of $\mathrm{lct}(T,\Delta;D_m)$ over these divisors. The local version $\delta_{Z,m}$ uses the generic point of the irreducible subvariety Z. Zero-dimensional summands contribute nothing.

The filtration of a summand by v is $\mathcal{F}_v^\lambda V_{m,\mathbf{a}} = \{s : v(\mathrm{div}_0(s)) \geq \lambda\}$. It has finitely many jumps. A compatible basis is a basis for which every filtered subspace is spanned by a subset. Its orders maximize the sum of orders among all bases. Thus it realizes the supremum in $S_m(V;v)$. This is the same linear-algebra argument used in Lemma 2.1.

Lemma 2.3 (Two compatible filtrations)**.** *Two finite, exhaustive and separated decreasing filtrations of a finite-dimensional vector space admit a common compatible basis.*

Proof. Number the distinct steps of the filtrations F^i and G^j. In each intersection $F^i \cap G^j$, choose a complement to $(F^{i+1} \cap G^j) + (F^i \cap G^{j+1})$, and choose a basis of that complement. Descending induction shows that the chosen vectors with indices at least (i,j) span $F^i \cap G^j$. The dimension identity for a sum of two subspaces shows, by the same induction, that their number equals the dimension of the intersection. They are therefore independent. Taking the first intersection gives the required basis. This also proves the version for any filtrations with finitely many distinct steps; see [4, Lemma 3.1]. □

In particular, when estimating one chosen valuation, we may require its basis to be compatible also with a divisor used for restriction. More generally, a flag defines one lexicographically ordered filtration. Apply the lemma to that single filtration and the valuation filtration. No assertion that three arbitrary filtrations admit a simultaneous compatible basis is needed.

Proposition 2.4 (Passage to the limit)**.** *Let (T,Δ) be klt, and let V have bounded support and contain an ample series. Then $S(V;v) = \lim_m S_m(V;v)$ exists for every divisorial valuation v. Moreover,*

$$\lim_m \delta_{Z,m}(T,\Delta;V) = \delta_Z(T,\Delta;V) = \inf_{Z \subset c_T(v)} \frac{A_{T,\Delta}(v)}{S(V;v)}.$$

The same assertion without the centre condition gives the global invariant. The infimum can be taken either over prime divisors or over nontrivial real valuations of finite log discrepancy. For every $\epsilon > 0$ there is m_0, independent of v, such that

$$S_m(V;v) \leq (1+\epsilon) S(V;v) \qquad (m \geq m_0)$$

for all valuations of finite log discrepancy and all degrees in which the series is nonzero.

Proof. The convergence and its uniform upper estimate are the multigraded convergence theorem [4, Lemma 2.21 and its proof]. Its hypotheses apply to the actual series V, not just to the divisor classes containing its summands. For completeness, the implication for thresholds follows directly from the uniform estimate. At finite degree, exchanging the two infima over basis divisors and valuations gives $\delta_{Z,m} = \inf_{Z \subset c_T(v)} A_{T,\Delta}(v)/S_m(V;v)$. The uniform estimate gives $\delta_{Z,m} \geq \delta_Z/(1+\epsilon)$. Conversely, fixing any one divisorial valuation in this infimum and using convergence of its expected order gives $\limsup_m \delta_{Z,m} \leq A_{T,\Delta}(v)/S(V;v)$. Take the infimum and then let ϵ decrease to zero. The global argument is identical. □

If $V'_{m,\mathbf{a}} = V_{km,k\mathbf{a}}$, with every grading coordinate multiplied by the same positive integer k, then

$$S(V';v) = kS(V;v), \qquad \delta_Z(T,\Delta;V') = \frac{1}{k}\delta_Z(T,\Delta;V).$$

This is the Cartier Veronese comparison [4, Lemma 2.24]. It permits one to establish a rational-divisor statement on a Cartier sublattice and then return to the original normalization.

Selecting only some residue classes while retaining the original denominator does not give this comparison. In explicit Hilbert sums below we keep all nonzero summands; alternatively both numerator and denominator must be changed according to a specified Veronese series.

Lemma 2.5 (Comparison of filtered moments)**.** *For each m, let $U_m \subset V_m$ be finite-dimensional vector spaces with the filtration induced by the same order function. Suppose their weights lie in $[0, Cm]$, with C independent of m. Let $w(U_m), w(V_m)$ be the sums of the weights of compatible bases, and let $d_m = \dim V_m - \dim U_m$. Then $0 \le w(V_m) - w(U_m) \le Cmd_m$. The same assertion holds after summing over the degree-m summands. If $\dim V_m$ has order m^n and $d_m = o(m^n)$, the normalized moments differ by $o(1)$, also when each is normalized by its own dimension. The estimate persists after an associated graded construction provided the subsequent filtration on the smaller space is induced from the larger one.*

Proof. Equip V_m/U_m with its quotient filtration. At every step the induced, ambient and quotient filtered pieces form an exact sequence. Their weight sums are additive, so $w(V_m) - w(U_m) = w(V_m/U_m)$ is the sum of d_m numbers in $[0, Cm]$. Division by $m \dim V_m$ proves the assertion with a common denominator. Changing the denominator to $m \dim U_m$ changes the normalized moment by at most $Cd_m/\dim V_m$. Associated graded dimensions preserve the deficit in the exact sequence, so the same argument applies to any subsequent induced filtration. Uniform bounds on C and d_m give a uniform error in any additional parameter, such as a point on a fixed curve. □

2.3 Actual restriction and the different

Let F be a prime divisor on a normal model $\pi : Y \to T$. Suppose either that π is the identity, or that its only exceptional divisor is F, with F rational Cartier and $-F$ relatively ample. Starting with Cartier grading divisors, put $D_{m,\mathbf{a}} = \pi^*(mL_0 + \sum a_i L_i)$ and define the refinement by

$$\begin{aligned} W_{m,\mathbf{a},j} &= \frac{\mathcal{F}_F^j V_{m,\mathbf{a}}}{\mathcal{F}_F^{j+1} V_{m,\mathbf{a}}} \\ &= \operatorname{im}\left(V_{m,\mathbf{a}} \cap H^0(Y, D_{m,\mathbf{a}} - jF) \longrightarrow H^0(F, (D_{m,\mathbf{a}} - jF)|_F) \right). \end{aligned}$$

Here a section is first divided by its prescribed exceptional factor and then restricted. The displayed image is the definition. Replacing it by the complete space on the right requires a proof of surjectivity, or an asymptotic comparison such as Lemma 2.5. In particular, vanishing conditions at infinitely near points cannot be dropped merely because the ambient divisor class is known. Telescoping dimensions gives $N_m(W) = N_m(V)$.

For a klt pair (T, Δ), write Δ_Y for the strict transform of Δ, omitting F if F is a component on T. We call F of plt type if the above model exists and $(Y, \Delta_Y + F)$ is purely log terminal near F. A divisor already on T is included by taking the identity model and imposing the same plt condition. In particular, F is normal. Adjunction defines its different by

$$(K_Y + \Delta_Y + F)|_F = K_F + \Delta_F.$$

The pair (F, Δ_F) is klt. On a smooth model transverse to F this is ordinary restriction of the other boundary components. On a quotient chart, the generic ramification indices contribute to Δ_F; they must be retained even when the coarse variety F is smooth.

If F is not Cartier, the restriction in the refinement uses the rational divisor class furnished by [4, Lemma 2.7], whose sheaf is the quotient of the two adjacent divisorial sheaves. Thus the geometric quotient, its fractional fixed divisor, and its multiplication agree. Cartier powers can be used to compute section divisors, but they do not erase the other graded summands. For rational grading divisors one may first use a Cartier Veronese and then the comparison

above. Under these conventions the refinements have bounded support and contain ample series whenever the original series does; this is [4, Example 2.15]. Repeated Cartier restrictions are covered by [4, Example 2.16].

For an extraction of a point P of a smooth threefold X, with $L = -K_X$, the notation simplifies to

$$W_{m,j} = \mathcal{F}_G^j H^0(X, mL)/\mathcal{F}_G^{j+1} H^0(X, mL).$$

Since $\sigma^* L|_G$ is trivial, the section class on G is $-jG|_G$. For a curve $B \subset G$, its further refinement is $W^B_{m,j,q} = \operatorname{gr}^q_B W_{m,j}$. Again this means restriction of the actual filtered subspace after removing qB. The dimensions of all pieces still sum to $h^0(X, mL)$.

2.4 Adjunction inequalities and strictness

Theorem 2.6 (The centre form of Abban–Zhuang adjunction)**.** *Let (T, Δ) be klt, let V have bounded support and contain an ample series, and let F be a divisor of plt type with model $\pi : Y \to T$. Let W be its refinement and Δ_F its different. Fix an irreducible subvariety $Z \subset T$ meeting $c_T(F)$ and an irreducible component Z_0 of their intersection. Put*

$$b = \inf_{Z' \subset F,\ \pi(Z') = Z_0} \delta_{Z'}(F, \Delta_F; W).$$

If $Z \subset c_T(F)$, then

$$\delta_Z(T, \Delta; V) \geq \min\left\{ \frac{A_{T,\Delta}(F)}{S(V; F)}, b \right\}.$$

If $Z \not\subset c_T(F)$, then $\delta_Z(T, \Delta; V) \geq b$. If $\delta_Z(T, \Delta; V) = b < A_{T,\Delta}(F)/S(V; F)$ and a valuation v computes this threshold, then $c_Y(v) \not\subset F$. For any component U of $c_Y(v) \cap F$ whose image contains Z_0, there is a valuation on F centred at U computing $\delta_{Z'}(F, \Delta_F; W) = b$ for every $Z' \subset U$ with $\pi(Z') = Z_0$.

Proof. This is [4, Theorem 3.3]. We give the basis-divisor argument, including the term useful for strict inequalities. Choose a basis compatible with F and with a chosen valuation v satisfying $Z \subset c_T(v)$. Its divisor satisfies $\pi^* D_m = S_m(V; F)F + \Gamma_m$, where F is not a component of Γ_m and $\Gamma_m|_F$ is an m-basis divisor of W. If λ_m is at most all the required local thresholds of this restricted divisor, inversion of adjunction gives log canonicity of $(Y, \Delta_Y + F + \lambda_m \Gamma_m)$ at the relevant centres meeting F. Comparing its discrepancies with those of (T, Δ) yields

$$A_{T,\Delta}(v) \geq \lambda_m S_m(V; v) + (A_{T,\Delta}(F) - \lambda_m S_m(V; F))v(F). \tag{2.1}$$

When $Z \subset c_T(F)$, also choose $\lambda_m \leq A_{T,\Delta}(F)/S_m(V; F)$. When $Z \not\subset c_T(F)$, the centre of v on Y is not contained in F, so $v(F) = 0$ and this extra restriction is unnecessary. Proposition 2.4, with its uniform estimate, lets λ_m tend to the stated minimum or to b, respectively. This proves the inequalities and the limiting form of (2.1). Under the equality hypothesis the coefficient multiplying $v(F)$ is positive, so $v(F) = 0$. The existence of the computing valuation on F is the equality clause of [4, Theorem 3.3], proved there by restricting the valuation ideals of v and applying inversion of adjunction to the resulting graded sequence. □

In particular, if the relevant refined thresholds are at least λ and $A_{T,\Delta}(F) \geq \lambda S(V; F)$, then

$$A_{T,\Delta}(v) - \lambda S(V; v) \geq (A_{T,\Delta}(F) - \lambda S(V; F))v(F). \tag{2.2}$$

This is useful even when the refined threshold equals one. If $\lambda = 1$, the coefficient on the right is positive, and $c_Y(v) \subset F$, then $v(F) > 0$ and $A_{T,\Delta}(v) > S(V; v)$. This is a strict inequality for that valuation; it does not by itself give a uniform positive lower bound as v varies. For a point extraction, valuations centred at the extracted point have $v(F) > 0$. Valuations centred on a curve or divisor through that point have $v(F) = 0$ and require their own estimate. This distinction is used in the final assembly of the Family 1.9 proof.

Proposition 2.7 (Abban–Zhuang refinement)**.** *Let X be a smooth Fano threefold, and suppose that $\sigma : \widehat{X} \to X$ extracts a plt-type divisor G over a point P. Let W be the refinement of the anticanonical series. Then*

$$\delta_P(X) \geq \min\left\{\frac{A_X(G)}{S_X(G)}, \delta(G, \Delta_G; W)\right\}. \tag{2.3}$$

Suppose that $B \subset G$ is rational Cartier, $B \not\subset \operatorname{Supp}\Delta_G$, and $(G, \Delta_G + B)$ is plt along B. Put $(K_G + \Delta_G + B)|_B = K_B + \Delta_B$. For every point $Q \in B$,

$$\delta_Q(G, \Delta_G; W) \geq \min\left\{\frac{1}{S(W; B)}, \frac{1 - \operatorname{coeff}_Q \Delta_B}{S(W^B; Q)}\right\}. \tag{2.4}$$

For a plt-type extraction of a divisor K over a point R of a klt surface pair (T, Δ), with any series satisfying the preceding hypotheses, the corresponding formula is

$$\delta_R(T, \Delta; W) \geq \min\left\{\frac{A_{T,\Delta}(K)}{S(W; K)}, \inf_{Q \in K} \frac{1 - \operatorname{coeff}_Q \Delta_K}{S(W^K; Q)}\right\}. \tag{2.5}$$

In all three formulas the series are the actual refinements with the original total-degree normalization, and they have bounded support and contain ample series.

Proof. Apply Theorem 2.6. For the first formula every centre on G maps to P, so the second term is the global threshold on G. For the second formula $A_{G,\Delta_G}(B) = 1$, and the normal curve B is smooth. Its only divisorial valuation centred at Q, up to scaling, is ord_Q; its discrepancy is $1 - \operatorname{coeff}_Q \Delta_B$. For the third formula every point of the exceptional curve must be considered. This proves the three assertions. The rational-divisor formulation is also given in [17, Theorem 3.20]. □

For a smooth surface $S \subset X$ and a smooth curve $C \subset S$ through p, repeated restriction gives the familiar bound by the minimum of the three ratios attached to S, C and p. The first denominator is $S_X(S)$, the second is the expected order of the actual restricted series along C, and the third is the expected point order of its further restriction. When a quotient singularity occurs, replace the numerator one at p by its discrepancy for the different. A longer Cartier flag is handled in precisely the same way by its lexicographic filtration and repeated inversion of adjunction; see [4, Theorem 3.5]. The surface intersection formulas in the next section compute these denominators.

If the curve used for refinement already has coefficient $b < 1$ in the surface boundary, write $\Delta = bB + \Delta'$ with $B \not\subset \operatorname{Supp}\Delta'$. The first numerator is then $1 - b$, and the different is defined from $(G, B + \Delta')$, not from $(G, (1 + b)B + \Delta')$. This is the divisor-on-the-variety case of Theorem 2.6.

Lemma 2.8 (A point that may depend on the basis)**.** *Let C be an irreducible curve on a smooth variety X, let $S \subset X$ be a smooth divisor not containing C, and let $\Sigma \subset C \cap S$ be nonempty. Let V be a complete ample series. Suppose that a fixed flag filtration refines the order along S, and that, for every basis divisor compatible with this filtration, the decomposition $D_m = s_m S + D'_m$ has the following property: for some $p_m \in \Sigma$, the pair $(S, \lambda_m D'_m|_S)$ is log canonical at p_m, where $\lambda_m \geq 0$, $\lambda_m s_m \leq 1$ and $\lambda_m \to \lambda$. Then $\delta_C(X; V) \geq \lambda$.*

Proof. Fix v whose centre contains C, and choose a basis compatible with v and the single flag filtration. By inversion of adjunction, $(X, S + \lambda_m D'_m)$ is log canonical near its associated point p_m. Lowering the coefficient of S to $\lambda_m s_m$ proves the same for $(X, \lambda_m D_m)$. If $A_X(v) < \lambda_m v(D_m)$, its closed non-log-canonical locus contains $c_X(v)$ and therefore every point of C, a contradiction. Hence $A_X(v) \geq \lambda_m S_m(V; v)$. Take the limit. No point of Σ has to work for every basis divisor. This is the finite-level form of the flag argument used in Lemma **??**. □

Example 2.9 (An ordinary point blow-up). Let $T = \mathbb{P}^2$, $M = \mathcal{O}_{\mathbb{P}^2}(d)$, with d a positive integer, and blow up p with exceptional curve E. In affine coordinates centred at p, homogeneous polynomials of degree md have arbitrary initial homogeneous forms of degree j for every $0 \le j \le md$. Hence the actual refined space is $W_{m,j} = H^0(E, \mathcal{O}_E(j))$, of dimension $j+1$. Summing over j recovers $N_m = \binom{md+2}{2}$. It follows that

$$S_M(E) = \lim_m \frac{\sum_{j=0}^{md} j(j+1)}{mN_m} = \frac{2d}{3}, \qquad S(W;Q) = \lim_m \frac{\sum_{j=0}^{md} j(j+1)/2}{mN_m} = \frac{d}{3}$$

for every $Q \in E$. Since $A_T(E) = 2$ and the different is zero, Proposition 2.7 gives $\delta_p(T;M) \ge 3/d$. A line through p has discrepancy one and expected order $d/3$, so equality holds. This example exhibits the two different discrepancies, two and one, in a point blow-up, and explains why an average over one fixed j would have the wrong normalization.

2.5 Comparison with a complete series

There is a useful special case in which an entire multigraded calculation reduces to one complete linear series. For each summand write its linear system as a fixed divisor $F_{m,\mathbf{a}}$ plus a system with no fixed divisorial component. Its average fixed divisor is the sum of $F_{m,\mathbf{a}}$, weighted by $\dim V_{m,\mathbf{a}}/(mN_m(V))$. Suppose its limit $F(V)$ is supported on finitely many prime divisors. Fix a big line bundle H. We say that the moving series are asymptotically complete multiples of H if, for every rational interior index $\boldsymbol{\gamma}$ on the support slice, their moving spaces lie in complete systems of divisors numerically equivalent to $\ell_{m,\boldsymbol{\gamma}} H$, with $\ell_{m,\boldsymbol{\gamma}}$ positive integral, and their dimension divided by $h^0(T, \ell_{m,\boldsymbol{\gamma}} H)$ tends to one along sufficiently divisible m. These are the conditions called almost completeness in [4, Definition 2.27]; the actual moving spaces, not just their numerical classes, enter the condition. The limiting divisors may have real coefficients. Log discrepancies, log canonicity and the valuative definition of the threshold are used for these real boundaries by the same formulas.

Proposition 2.10. *Under these conditions, and assuming bounded support and the ample-series condition, the average moving class is μH for a nonnegative real number μ, the divisor $F(V)$ is real Cartier, and*

$$S(V;v) = \mu S_H(v) + v(F(V))$$

for every divisorial valuation v. Consequently, whenever $\Delta + \lambda F(V)$ is an effective klt boundary and $\mu > 0$, an estimate

$$\delta_Z(T, \Delta + \lambda F(V); \mu H) \ge \lambda$$

implies $\delta_Z(T, \Delta; V) \ge \lambda$.

Proof. The expected-order identity, including the existence of the average moving class, is [4, Lemma 2.29]. Fixed divisors contribute their actual valuation; asymptotic completeness supplies the expected order of the complete moving series. One way to see why the dimension condition is relevant is Lemma 2.5: an asymptotically negligible loss of sections changes the normalized first moment by an asymptotically negligible amount when the orders have a linear bound. Integrating over the bounded support gives the stated coefficient μ. For the last assertion, the assumed inequality says $A_{T,\Delta}(v) - \lambda v(F(V)) \ge \lambda \mu S_H(v)$ for each tested valuation. Substitute the expected-order identity and take the infimum. □

Example 2.11 (Restriction to an anticanonical surface). Let X be a smooth Fano threefold with base point free $H = -K_X$, and let $S \in |H|$ be smooth. Its refined summands are the images of $H^0(X, (m-j)H)$ on S, for $0 \le j \le m$. They are complete for $m - j \gg 0$ by Serre

vanishing. Their asymptotic dimensions are $H^3(m-j)^2/2$, whereas $h^0(X,mH) \sim H^3m^3/6$. Thus their average moving class is

$$3\int_0^1 (1-u)^3\,du\; H|_S = \frac{3}{4}H|_S,$$

and $S(W;v) = \frac{3}{4}S_{H|_S}(v)$. This is the coefficient used in the final remark of Part **??**. The calculation does not assert that the resulting surface estimate is strong enough for K-stability; the choice of an auxiliary surface affects the strength of the bound.

2.6 An isolated point and the multiplier ideal

The following supplement is used when estimates have already excluded all positive-dimensional centres through a point. It is distinct from restriction to a surface.

Lemma 2.12. *Let T be smooth and projective, let D be effective rational Cartier, and suppose that $-(K_T + D)$ is ample. If (T, D) is klt in a punctured neighbourhood of p, then every divisorial valuation v centred at p satisfies*

$$A_T(v) \geq \frac{\nu(v)}{\nu(v)+1}v(D), \qquad \nu(v) = \frac{A_T(v)}{v(\mathfrak{m}_p)}.$$

Proof. This is the multiplier-ideal argument of [4, Lemma 4.24]. If (T, D) is klt at p, the assertion follows from $A_T(v) > v(D)$. Otherwise put $\mathcal{J} = \mathcal{J}(T, D)$. Nadel vanishing gives $H^1(T, \mathcal{J}) = 0$ and hence a surjection $H^0(T, \mathcal{O}_T) \to H^0(T, \mathcal{O}_T/\mathcal{J})$. The point p is an isolated component of its support, so its local quotient is a direct summand of these global sections. It has length at most one, and is nonzero. Therefore $\mathcal{J}_p = \mathfrak{m}_p$. The valuation inequality for a multiplier ideal gives $v(\mathfrak{m}_p) \geq v(D) - A_T(v)$. Substituting $v(\mathfrak{m}_p) = A_T(v)/\nu(v)$ gives the result. □

Corollary 2.13. *Let X be a smooth Fano threefold with base point free $H = -K_X$. Suppose that every valuation whose positive-dimensional centre contains p has $A_X(w) \geq \chi S_X(w)$, where $1 < \chi < 4/3$. Then, for every divisorial valuation v centred at p,*

$$\frac{A_X(v)}{S_X(v)} \geq \chi\frac{\nu(v)}{\nu(v)+1}, \qquad \nu(v) = \frac{A_X(v)}{v(\mathfrak{m}_p)}.$$

Proof. Choose a smooth $T \in |H|$ avoiding p and a basis compatible with T and v. Write $D_m = a_mT + \Gamma_m$, where $a_m = S_m(H;T) \to 1/4$ and $\Gamma_m \sim_{\mathbb{Q}} (1-a_m)H$ is effective. Fix a rational number $0 < \lambda < \chi$. The uniform estimate in Proposition 2.4 implies, for large m, that $(X, \lambda\Gamma_m)$ is klt at the generic point of every positive-dimensional centre containing p. Its non-klt locus is closed, so it is klt in a punctured neighbourhood of p. Moreover $-(K_X + \lambda\Gamma_m)$ is ample because $\lambda(1-a_m) < 1$. Apply Lemma 2.12. Since T avoids p, one has $v(\Gamma_m) = v(D_m) = S_m(H;v)$. Let m tend to infinity and then let λ increase to χ. This is the argument used in Lemma **??**. □

2.7 Fractional section divisors

Equalities of rational divisor classes do not identify the actual boundary or the zero divisor of a specified section. The following convention is used on every weighted exceptional surface in the manuscript.

Lemma 2.14 (Divisors of weighted forms)**.** *Let $T = \operatorname{Proj}\mathbb{C}[x,y,z]$ for positive primitive weights, and let h be a nonzero homogeneous form of weight j. Choose r for which $\mathcal{O}_T(rj)$ is*

invertible, and define $D(h) = r^{-1}\operatorname{div}_0(h^r)$. *This effective rational divisor is independent of* r *and additive under multiplication. Forms* h_1, h_2 *of the same weight satisfy* $D(h_1) - D(h_2) = \operatorname{div}(h_1/h_2)$. *If a prime curve* B *is not a component of* $D(h)$, *restriction to* B *is computed by restricting a Cartier power and dividing by that power.*

Proof. Pass to a common multiple of two possible choices of r. Both give the same divisor after multiplication by that common multiple; this proves independence and additivity. The ratio of two forms of equal weight is a rational function on T, and the same Cartier-power computation proves the difference formula. The final assertion is the definition of restriction of an effective rational Cartier divisor not containing B. Removing the exceptional factor in a weighted blow-up gives exactly these divisors. □

On a fixed weight space, the valuation of a section is therefore an ordinary valuation of its rational representative plus a fixed shift. Compatible bases remain legitimate. In Family 2.5, type D_5, the relation $z^2 = W$ gives $D(z) = \frac{1}{2}(W = 0)$ on the coarse surface. In type E_6, the relations $y^3 = W$ and $z^2 = V$ give $D(y) = \frac{1}{3}(W = 0)$ and $D(z) = \frac{1}{2}(V = 0)$. Here y, z are weighted sections; they are not rational functions on the coarse surface. These fractional factors enter the point orders of the refined curve series.

For clarity, representatives of the positive normal class $-G|_G$ in these two cases are $D_1 = \frac{1}{2}(W = 0) - (V = 0) \sim_{\mathbb{Q}} H_G/2$ and $D_1 = \frac{1}{3}(W = 0) - \frac{1}{2}(V = 0) \sim_{\mathbb{Q}} H_G/6$, respectively. The semi-invariants $\theta = z/y$ and $\theta = y/z$ have native weight one. Thus a form h of native weight j defines a rational function h/θ^j on the coarse surface and satisfies $D(h) = jD_1 + \operatorname{div}(h/\theta^j)$. This realizes all native weights in the same rational-divisor convention. The divisor D_1 specifies a section space and need not be effective. The zero divisor $D(h)$ is effective, whereas the different is a separate boundary determined by adjunction.

3 Elementary estimates for refined series

Lemma 3.1 (Complete series on a rational curve)**.** *Let a space of sections on* $\mathbb{P}^1$ *have a fixed effective rational divisor* F *and complete moving part* $H^0(\mathbb{P}^1, \mathcal{O}(\ell))$. *Its dimension is* $\ell + 1$, *and its mean order at a point* Q *is* $\operatorname{coeff}_Q F + \ell/2$.

Proof. The complete moving series has a basis of orders $0, 1, \dots, \ell$ at Q. Restoring the fixed factor adds $\operatorname{coeff}_Q F$ to every order. □

Lemma 3.2 (Toric alpha invariant)**.** *Let* $T = \mathbb{P}(a_0, a_1, a_2)$ *be well formed,* $H_T = \mathcal{O}_T(1)$, *and* $\Delta = \sum c_i D_i$, *where* D_i *are the coordinate divisors and* $0 \le c_i < 1$. *Then* $\alpha(T, \Delta; H_T) = \min_i a_i(1 - c_i)$. *For* $T = \mathbb{P}(2, 1, 3)_{U,V,W}$ *and* $\Delta = \frac{1}{2}(W = 0)$, *the local alpha invariant of* H_T *is at least* $3/2$ *on* $V \neq 0$.

Proof. Take $D \ge 0$ with $D \sim_{\mathbb{Q}} H_T$ and clear denominators. A generic one-parameter subgroup of the torus degenerates its defining polynomial to a monomial. The resulting divisor is $D_0 = \sum d_i D_i$, where $d_i \ge 0$ and $\sum a_i d_i = 1$. If $\lambda \le \min_i a_i(1 - c_i)$, then $c_i + \lambda d_i \le 1$ for every i. The toric pair $(T, \Delta + \lambda D_0)$ is log canonical, so lower semicontinuity of log canonical thresholds gives the same conclusion for D. This proves the lower bound. Taking $D = D_i/a_i$ gives the reverse inequality at the generic point of D_i.

On $V = 1$, sections of kH_T are polynomials in U, W of weighted degree at most k, for weights $(2, 3)$. At a point of $W = 0$, translate U to put the point at the origin. This preserves the degree bound and the boundary. Monomial degeneration there gives threshold at least $\min\{2, 3/2\}/k$. Away from $W = 0$, both coordinates may be translated and the same argument without boundary gives at least $2/k$. This proves the local assertion. The semicontinuity argument applies to the germ at the fixed origin; translations do not increase weighted degree. □

Lemma 3.3 (Residual-degree estimate)**.** *Let (G, Δ_G) be a projective klt surface, let H_G be ample and $\mathbb{Q}$-Cartier, and suppose $\mathrm{Cl}(G)_{\mathbb{Q}} = \mathbb{Q}[H_G]$. Let $B \sim_{\mathbb{Q}} b_0 H_G$ be a prime curve. Assume that sections of $W_{m,j}$ have class $d_{m,j} H_G$ and that their orders along B are integral. Set $h_{m,j,q} = \dim \mathrm{gr}_B^q W_{m,j}$ and suppose the limit*

$$R = \lim_{m\to\infty} \frac{1}{mN_m} \sum_{j,q} h_{m,j,q}(d_{m,j} - b_0 q)$$

exists. If $\alpha(G, \Delta_G; H_G) \geq a > 0$, then $S(W; v) \leq (R/a) A_{G,\Delta_G}(v)$ for every divisorial valuation whose centre is not contained in B. The same conclusion holds for a chosen valuation if the local alpha bound a holds at some point of its centre.

Proof. Choose a basis in every $W_{m,j}$ compatible with v and ord_B, using Lemma 2.3. A basis section with order q along B has divisor $qB + D'$, where $D' \geq 0$ and $D' \sim_{\mathbb{Q}} (d_{m,j} - b_0 q) H_G$. Ampleness implies that the residual coefficient is nonnegative; if it is zero, $D' = 0$. Since the centre of v is not contained in B, one has $v(B) = 0$. The global or local alpha bound gives $v(s) = v(D') \leq (d_{m,j} - b_0 q) A_{G,\Delta_G}(v)/a$. Summing and dividing by mN_m proves the claim in the limit. Compatibility with v computes the left side, while compatibility with B fixes the multiplicities $h_{m,j,q}$. □

Example 3.4. On $\mathbb{P}(1,1,2)$ with $\Delta = \frac{1}{2}(V = 0) + \frac{2}{3}(W = 0)$, Lemma 3.2 gives alpha equal to $1/2$. If $B \sim 2H_G$, a section of class dH_G divisible by B^q has residual class $(d - 2q)H_G$. Away from B, its valuation is therefore at most $2(d - 2q)A_{G,\Delta}(v)$. Section 9 averages exactly these residual coefficients.

4 Surface calculations and their normalization

The refinement theorem concerns spaces of sections, including their fixed divisors. A numerical equivalence of divisors does not identify such spaces. This distinction matters when a restriction map is not onto, when points become infinitely near, and when a surface is replaced by its resolution. We give the formulas used below in a form which records these issues. They are the surface calculations underlying [3, Section 1.7].

4.1 A family of complete surface series

Let L be ample on a threefold X, put $V = L^3$, and let $W_{m,j}$ be an actual refinement of $H^0(X, mL)$. Thus $N_m = \sum_j \dim W_{m,j} = Vm^3/6 + O(m^2)$. Suppose that on a smooth projective surface T, after pulling back the sections, its pieces are complete moving series with their fixed factors:

$$W_{m,j} = s_{mJ_t} H^0(T, mP_t), \qquad t = j/m.$$

Here P_t is nef, J_t is an actual effective divisor, and both are rational on rational parameters. The parameter belongs to a bounded union of intervals. Divisibility and bounded rounding are understood. We require the uniform section estimates used in taking the following Riemann sums; they hold in our applications because the divisors range over finitely many rational polyhedral chambers with the section descriptions proved in the corresponding part. It also suffices to have inclusions in the displayed spaces whose total dimension deficit is $o(m^3)$, with linearly bounded orders, by Lemma 2.5. Equal total leading dimensions are useful only after those inclusions have been proved.

For a big nef divisor P, write $S_P(v) = (P^2)^{-1} \int_0^\infty \mathrm{vol}(P - xv)\, dx$, with the usual birational interpretation of the integrand. Products $P_t^2 S_{P_t}(v)$ at zero-volume endpoints are interpreted

by limits. Then

$$\frac{3}{V}\int P_t^2\,dt = 1, \qquad S(W;v) = \frac{3}{V}\int P_t^2\,(v(J_t) + S_{P_t}(v))\,dt. \tag{4.1}$$

Indeed, a piece has $m^2P_t^2/2 + o(m^2)$ sections. Its fixed factor adds $mv(J_t)$ to each order, while its complete moving series has mean order $mS_{P_t}(v) + o(m)$. There are order m pieces; dividing their sums by mN_m gives the coefficient $3/V$. This proof also shows why a fixed divisor cannot simply be discarded.

In particular, suppose at a chosen point of T that $S_{P_t}(v) \le b(t)A_{T,\Delta}(v)$ and $v(J_t) \le e(t)A_{T,\Delta}(v)$ for every valuation under consideration. Then

$$S(W;v) \le \frac{3}{V}\left(\int P_t^2(b(t) + e(t))\,dt\right) A_{T,\Delta}(v). \tag{4.2}$$

Thus the relevant bound is the dimension-weighted average of the surface bounds. It need not equal their worst value. If the spaces only embed in complete spaces and the leading dimensions differ, the same argument gives an upper bound with the original N_m; one must not normalize by the larger dimension.

4.2 A curve on the surface and a point on that curve

Suppose first that $C \subset T$ is a smooth rational curve. Put $c_t = \operatorname{ord}_C J_t$ and take the Zariski decomposition

$$P_t - uC = Q_{t,u} + N_{t,u}, \qquad 0 \le u \le \tau_t,$$

where $Q_{t,u}$ is nef and $N_{t,u}$ is effective with negative definite intersection matrix on its nonempty support. The number τ_t is the pseudoeffective threshold. Its coefficients satisfy $Q_{t,u} \cdot N_{t,u} = 0$. Assume that C is not a component of $N_{t,u}$ in the interior of the range and that the restricted moving series has, asymptotically, the complete series of degree $m\,Q_{t,u} \cdot C$. Set

$$d(t,u) = Q_{t,u} \cdot C, \qquad F_{t,u} = (J_t - c_tC + N_{t,u})|_C.$$

The fixed divisor in this last expression is an actual divisor on C, not its degree. Under the section and convergence hypotheses above, one obtains

$$S(W;C) = \frac{3}{V}\int \left(c_tP_t^2 + \int_0^{\tau_t} Q_{t,u}^2\,du\right) dt, \tag{4.3}$$

$$S(W^C;q) = \frac{3}{V}\iint d(t,u)^2\,du\,dt + \frac{6}{V}\iint d(t,u)\operatorname{ord}_q F_{t,u}\,du\,dt. \tag{4.4}$$

For the first identity sum the dimensions of the filtered surface spaces, or integrate their volumes. For the second, the curve piece has $md(t,u) + O(1)$ sections and mean order $m\operatorname{ord}_q F_{t,u} + md(t,u)/2 + O(1)$ by Lemma 3.1. Multiplying and summing proves the formula. There are two filtration indices, so order m^2 curve pieces; the numerator has order m^4, as does mN_m.

If only containment in these complete curve spaces is known, their weight sums give upper bounds with the same normalization. To claim the equalities, it is necessary to prove the stated section identification, an appropriate almost-completeness statement, or an $o(m^3)$ total dimension deficit. Checking the intersection numbers alone does not establish any of them.

To apply surface adjunction we require $(T, \Delta + C)$ to be plt near the centre, as in Proposition 2.7. Its estimate divides the two moments by different discrepancies. When C is absent from Δ, the first numerator is $A_{T,\Delta}(C) = 1$. At q, the numerator is $1 - \operatorname{coeff}_q \Delta_C$, where $(K_T + \Delta + C)|_C = K_C + \Delta_C$. If $\Delta = dB$, with B smooth and transverse to C at q, this is $1 - d$, not one. If the surface is a quotient, the different includes the quotient contribution as well. The boundary and the fixed divisor $F_{t,u}$ have different roles and neither is absorbed into the other.

4.3 Total transforms when the flag is defined downstairs

Suppose the series was pulled back by $r : T \to T_0$ and the curve used for refinement is a rational Cartier curve $B \subset T_0$. Write $r^*B = C + Z$, where C is its strict transform. The first moment is still (4.3), since $\mathrm{ord}_B = \mathrm{ord}_C$. The fixed divisor for the *downstairs* restriction, however, is

$$F^{\mathrm{down}}_{t,u} = (J_t - c_t C + N_{t,u} - (c_t + u)Z)|_C. \tag{4.5}$$

To see this, divide a section of order $m(c_t + u)$ along B by the same power of a local equation of B, before restricting. On T that equation has divisor $C + Z$. Subtracting only C would leave the extra factor $m(c_t + u)Z$. Formula (4.5) removes precisely that factor. It is to be used in the geometric identification of the restricted rational sections, including their fractional factors.

For example, if a smooth point $p \in B$ is blown up, then $r^*B = C + E$ and $C \cap E$ is one point. Omitting the correction would increase the refined point moment there by

$$\frac{6}{V} \iint d(t,u)(c_t + u)\, du\, dt = S(W; B).$$

The equality follows either from the dimension sum for the B-orders, or by integration of $-\partial_u Q^2_{t,u} = 2d(t,u)$ on the chambers. This is the correction used in the infinitely near configurations of Family 1.9 and in Example **??**. It is substantial even when the numerical class of the moving part is unchanged.

4.4 Blowing up a point of the surface

Let p be a smooth point of a smooth surface S, let D be an ample divisor, and let $\sigma : T \to S$ blow up p, with exceptional curve E. Write $\sigma^*D - uE = P_u + N_u$. The preceding calculation with just one filtration parameter gives

$$S_D(E) = \frac{1}{D^2} \int P_u^2\, du, \qquad S(W^E; q) = \frac{1}{D^2} \int (P_u \cdot E)^2\, du + \frac{2}{D^2} \int (P_u \cdot E)\, \mathrm{ord}_q(N_u|_E)\, du. \tag{4.6}$$

For zero boundary, $A_S(E) = 2$ and the different on E is zero. Consequently

$$\delta_p(S; D) \geq \min\left\{ \frac{2}{S_D(E)}, \inf_{q \in E} \frac{1}{S(W^E; q)} \right\}.$$

The infimum must include every point of E, including intersections with all negative curves occurring in N_u. Computing only the first ratio proves no lower bound for $\delta_p(S; D)$. If a boundary Δ is present, the first numerator becomes $2 - \mathrm{mult}_p \Delta$. Provided $(T, E + \sigma_*^{-1}\Delta)$ is plt, the boundary for the second step is $(\sigma_*^{-1}\Delta)|_E$, with the corresponding point numerators. If this pair is not plt, a different extraction or a further calculation is needed.

An alternative is to refine the full family W by this exceptional curve immediately, using (4.3) and (4.4). This often improves taking the worst polarized-surface bound. In either approach this operation is a blow-up of a surface; it does not require a Zariski decomposition on the threefold blown up at p.

Example 4.1 (The node of a reduced conic in Family 2.13). The geometry and the complete restricted series are proved in Lemmas 39.4 and 40.6. Here S is a smooth quartic del Pezzo surface, $F = Z + Z^\dagger$ is a reducible conic fibre, $p = Z \cap Z^\dagger$, and $D_t = -K_S + tF$, with $0 \leq t \leq 1$. Blow up p and call the exceptional curve E. The strict transforms of $Z, Z^\dagger$ and the complementary conic G are pairwise disjoint, of squares $-2, -2, -1$, and meet E at three distinct points. Put $c = 3 + 2t$. For $0 \leq u \leq c$, the negative parts are respectively

$$0, \quad \frac{u-1}{2}(\widetilde{Z} + \widetilde{Z}^\dagger), \quad \frac{u-1}{2}(\widetilde{Z} + \widetilde{Z}^\dagger) + (u - 2 - 2t)\widetilde{G}$$

on $[0,1]$, $[1,2+2t]$, and $[2+2t,c]$. Squaring the positive parts gives $4+4t-u^2$, $5+4t-2u$, and $(c-u)^2$; their intersections with E are $u, 1, c-u$. Substitution into (4.6) gives

$$S_{D_t}(E) = \frac{3+2t}{2}, \qquad \max_{q\in E} S(W^E;q) = \frac{12t^2+30t+17}{24(1+t)} < \frac{3+2t}{4}.$$

The maximum occurs in either line direction, where the fixed negative part contributes. Thus $\delta_p(S;D_t) \geq 4/(3+2t)$. At $t=1$, the two relevant moments are $5/2$ and $59/48$, so the resulting surface bound is $4/5$.

This number below one does not obstruct the threefold argument. The actual surface refinement has a fixed smooth curve of mean coefficient $1/80$, and its dimension-weighted average gives

$$\frac{S(W^S;v)}{A_S(v)} \leq \frac{1}{80} + \frac{3}{20}\int_0^1 (1+t)(3+2t)\,dt + \frac{9}{20}\int_1^{3/2} (3-2u)^3\,du = \frac{159}{160}.$$

Together with $S_X(S) = 41/80$, adjunction proves $\delta_p(X) \geq \min\{80/41, 160/159\} = 160/159$. This illustrates both the point blow-up on the auxiliary surface and the averaging formula, including its fixed-curve contribution.

5 Identifying the section spaces and taking limits

Lemma 5.1 (Identifying a geometric filtration)**.** *Let P be a polynomial ring with an auxiliary grading whose homogeneous components are finite dimensional. Give its variables integer weights, let $f_1,\dots,f_s$ be homogeneous for the auxiliary grading, and let g_i be the lowest-weight form of f_i. If $g_1,\dots,g_s$ form a regular sequence, the associated graded ring for the induced quotient filtration is $P/(g_1,\dots,g_s)$.*

Suppose that the assigned weights are the geometric orders of the generators. Assume that $P/(g_i)$ injects into its localization at homogeneous elements of weight zero that are units at the point. If the localized map to the geometric associated graded ring is injective, the quotient filtration equals the geometric filtration on each fixed auxiliary degree.

Proof. Filter the Koszul complex of the f_i, shifting the filtration on its generators by the weights of the g_i. Its associated graded complex is the Koszul complex of the g_i, which is exact in positive homological degrees by regularity. In each auxiliary degree the filtration is finite, so exactness of the associated graded complex makes the filtered complex strict. Its zeroth associated graded quotient is therefore $P/(g_i)$.

The assigned geometric orders give a natural map from this ring to the geometric associated graded ring. It becomes injective after the stated localization. Since the source has no torsion with respect to the elements inverted, it is already injective before localization. Thus a nonzero initial form cannot disappear on the local chart, and no additional cancellation can increase the order of a section. □

Lemma 5.2 (Weighted quotient geometry)**.** *For primitive positive weights (w_1,w_2,w_3) at a smooth point, the weighted extraction is of plt type and $A_X(G) = w_1+w_2+w_3$. The coefficient of the different along $x_k=0$ is $1-1/\gcd(w_i,w_j)$, where $\{i,j,k\}=\{1,2,3\}$.*

Proof. On the ith chart use the cover $x_i = s^{w_i}$, $x_j = s^{w_j}y_j$, $x_k = s^{w_k}y_k$, with cyclic action of weights $(1,-w_j,-w_k)$. Primitivity rules out divisorial ramification on the threefold cover. Thus the smooth plt pair with boundary $s=0$ descends. The Jacobian has order $w_1+w_2+w_3-1$, giving the discrepancy. Restriction to the exceptional surface has generic ramification index $\gcd(w_i,w_j)$ along the indicated coordinate curve, giving the different. Relative ampleness follows from the weighted Proj construction. □

Lemma 5.3 (A monomial degeneration bound)**.** *Let (T, Δ) be a projective toric surface with torus-invariant boundary, and let W be a multigraded series with bounded support. Suppose that, under a one-parameter subgroup of its torus, each finite-dimensional summand has a monomial initial space. Write the normalized sum of the zero divisors of all these initial monomials as*

$$D_m^0 = \sum_i c_{i,m} D_i, \qquad c_i = \lim_{m\to\infty} c_{i,m},$$

where D_i are the invariant prime divisors. The normalization is mN_m, as before. Then

$$\delta(T, \Delta; W) \geq \min_{c_i>0} \frac{1 - \operatorname{coeff}_{D_i} \Delta}{c_i}.$$

The same conclusion holds for a succession of torus degenerations.

Proof. Fix m and a divisorial valuation v. First perturb the one-parameter subgroup. Only finitely many characters occur in the relevant spaces, so choose an integral torus weight ρ distinguishing them and replace the original weight λ by $N\lambda + \rho$, with N sufficiently large. The λ-initial spaces are already monomial, so this refinement does not change them, and their monomials now have distinct weights. Now choose a basis of each summand compatible with v and with this perturbed weight filtration, using [4, Lemma 3.1]. Its initial terms are distinct and form precisely the monomial basis of the initial space. The perturbation must precede the choice of basis: a basis compatible with the old weights need not be compatible with the new ones.

The resulting basis divisor degenerates to D_m^0. A toric pair is log canonical if its coefficients along all invariant divisors are at most one: the discrepancies of further toric divisors are nonnegative linear combinations of the corresponding discrepancies, and a toric resolution checks log canonicity. Hence $(T, \Delta + \lambda_m D_m^0)$ is log canonical for $\lambda_m = \min_{c_{i,m}>0}(1 - \operatorname{coeff}_{D_i} \Delta)/c_{i,m}$. All nonzero fibres of the degeneration are isomorphic as pairs, since the torus preserves Δ. Lower semicontinuity of log canonical thresholds therefore gives the same bound for the original basis divisor. Its basis is compatible with v, so $A_{T,\Delta}(v) \geq \lambda_m S_m(W; v)$. The bound is independent of v. Passing to the limit and taking the infimum over v proves the assertion. For successive degenerations combine the successive weights with sufficiently dominant integral coefficients at the fixed finite level, and then choose the compatible bases.

For rational divisor classes on the coarse weighted planes, use the effective rational zero divisors as in Proposition 2.7. Their bounded fractional coordinate factors are retained in D_m^0; they contribute $O(1/m)$ to its normalized coefficients. Clearing Cartier indices does not remove the other residue classes or change N_m. □

5.1 Hilbert series with a weight variable

Once an assigned grading has been identified with the geometric filtration, its Hilbert series computes the expected order directly. If $h_{m,j}$ is the dimension in degree m and weight j, set $H(z, q) = \sum h_{m,j} z^m q^j$. Coefficientwise differentiation gives

$$N_m = [z^m]H(z, 1), \qquad \sum_j j h_{m,j} = [z^m]\, q \frac{\partial H}{\partial q}\bigg|_{q=1}.$$

Dividing the second quantity by mN_m and taking its limit gives S, by the adapted-basis formula. For a polynomial ring with generator degrees and weights (d_i, w_i), divided by a homogeneous regular sequence with degrees and weights (e_k, r_k), the series is

$$H(z, q) = \frac{\prod_k (1 - z^{e_k} q^{r_k})}{\prod_i (1 - z^{d_i} q^{w_i})}.$$

This follows from the weighted Koszul resolution. A multigrading is handled by replacing z^{d_i} by the corresponding monomial and then extracting the required diagonal coefficients. The weights of the equations contribute to the numerator and cannot be omitted.

For example, the diagonal ring of three equations of bidegree $(1,1)$ in $\mathbb{P}^3 \times \mathbb{P}^3$, used in Family 2.12, has $N_m = (2m+1)(5m^2+5m+3)/3$. If the eight coordinate weights sum to A and the three equation weights sum to B, logarithmic differentiation and extraction of the diagonal give leading weight coefficient $35A/24 - 5B/6$. Therefore its limiting mean is $7A/16 - B/4$. Lemma 33.5 records the full polynomial calculation. It computes a geometric expected order only after the identification of the actual initial ring; an arbitrary weighted degeneration of the equations would not do so.

5.2 Bounding an ordinary point order by a flag

For a flag $p \in C \subset S \subset X$ smooth at p, choose regular local parameters x, y, z with $S = (x = 0)$ and $C = (x = y = 0)$. For a section s in a local trivialization, let i be its order along S; let j be the order along C of $(s/x^i)|_S$; and let k be the order at p of the remaining nonzero section on C after its y^j factor is removed. These three nonnegative integers are successive, defined vanishing orders. The formal expansion of s contains the term $x^i y^j z^k$ with nonzero coefficient, so $\operatorname{mult}_p(s) \le i + j + k$.

On the finite-dimensional space $H^0(X, mL)$, lexicographic order on these triples defines one filtration, with finitely many steps. Choose a common basis for it and ordinary multiplicity at p. Let a_m, b_m, c_m be the sums of its three respective orders, each divided by mN_m. If their limits are a, b, c, then the ordinary exceptional divisor of $\operatorname{Bl}_p X$ satisfies $S_L(E) \le a + b + c$. There are exactly two filtrations in this argument; one is the entire successive flag filtration. It is not an assertion about simultaneous splitting of three unrelated filtrations. Lemma 14.3 applies this observation to the cubic-surface flags.

5.3 Containing spaces and the dimensions of the actual spaces

Lemma 5.4 (Keeping the missing dimensions)**.** *Let $U \subset V$ have the induced filtration, and suppose all weights of V lie in $[a, b]$. If w denotes the sum of weights of an adapted basis and $d = \dim V - \dim U$, then*

$$w(V) - bd \le w(U) \le w(V) - ad.$$

The same statement holds for a direct sum of pieces, with the corresponding a, b, d in each piece.

Proof. The quotient filtration on V/U has d weights in $[a, b]$. Additivity of the weight sums gives $w(V) - w(U) = w(V/U) \in [ad, bd]$. □

If the containing space is larger by $o(m^3)$ and its weights are $O(m)$, the discrepancy in normalized moments tends to zero. If its extra dimension has order m^3, it can change the leading moment. The last lemma gives a correction even if the smaller space is not spanned by monomials: the induced filtration, or an invariant subspace of a diagonal action, suffices. Its dimension is always the dimension of the actual flat limit, not the dimension of the complete linear system on the limiting scheme.

There are two different ways in which we certify that a calculation uses the whole space. For equality in every degree we construct a map from a candidate initial algebra onto the relevant initial algebra, or a space of genuine initial sections inside it, and compare their Hilbert functions. An equality of dimensions then proves equality of spaces. For asymptotic moments it suffices that the total dimension deficit is $o(m^3)$, provided all weights are linearly bounded. The latter statement does not assert an equality of graded algebras or surjectivity of each restriction map. These two arguments occur respectively in Lemma 94.2 and Lemma **??**.

5.4 Certifying all moments from actual leading terms

The following algebraic observation is useful when cancellations produce additional leading terms beyond those of the chosen generators. A valuation with one-dimensional leaves admits an adapted basis of each finite-dimensional space whose value vectors are pairwise distinct. Here "one-dimensional leaves" means that each nonzero associated graded piece for this valuation has dimension one. Orders are additive under multiplication.

Lemma 5.5. *Let $A = \bigoplus_{m\geq 0} A_m$ be a domain and suppose $\dim A_m = cm^n + O(m^{n-1})$. A valuation with one-dimensional leaves takes its values on $A_m \setminus \{0\}$ in a rank-n lattice Λ, and all these values have norm at most Cm. Suppose finitely many actual sections have degree and value pairs (m_i, v_i), with $m_i > 0$, whose generated group is $\mathbb{Z} \times \Lambda$. Set $P = \operatorname{conv}\{v_i/m_i\}$. Give a fundamental parallelepiped of Λ volume one. If the volume of P in this measure is c, then*

$$\lim_{m\to\infty} \frac{1}{m \dim A_m} \sum_{v\in\nu(A_m\setminus\{0\})} v = \frac{1}{c}\int_P x\,dx.$$

Proof. Let Γ be the semigroup generated by the displayed degree and value pairs. Its saturation is the lattice points of its rational cone. This saturation is finite as a Γ-module. Choose finitely many module representatives, write each as a difference of two elements of Γ, and add the negative parts. The resulting $\eta \in \Gamma$ satisfies $\eta + \overline{\Gamma} \subset \Gamma$. Therefore the lattice points missing from a degree-m slice lie in bounded-width boundary layers. Their number is $O(m^{n-1})$. Lattice-point counting gives $\#\Gamma_m = cm^n + O(m^{n-1})$ and $\sum_{v\in\Gamma_m} v = m^{n+1}\int_P x\,dx + O(m^n)$. Products of the certified sections show that these vectors all occur in A_m. The full space has only $O(m^{n-1})$ more values; each has norm $O(m)$. Their contribution to the sum is $O(m^n)$. Division by $m \dim A_m$ proves the claim. □

The group condition includes the degree coordinate; full real span is insufficient. An index in the value lattice changes the volume. Negative value coordinates cause no problem if their absolute values are bounded by Cm. We also sometimes use an increasing sequence of polytopes from certified sections, each satisfying the same full-group condition, whose volumes tend to c. Fix one polytope first, bound the contribution of all remaining values by C times its missing volume, let m tend to infinity, and only then enlarge the polytope. This gives the same conclusion. These are the precise dimension arguments used for the ordinary and weighted point extractions in Families 2.12 and 2.14; see Lemma 34.2. A list of leading monomials without this check need not determine an expected order.

5.5 Degenerating the series while preserving the boundary

The monomial lemma above uses an action preserving the boundary. More generally, let a one-parameter group act on a projective pair (T, Δ) and preserve Δ. Suppose its action linearizes the line bundles of the finite-dimensional pieces of a series. Fix a valuation v and a degree m. In each piece choose a basis compatible with the v-filtration and with the filtration of this action. After rescaling the basis vectors by their weights, their limits are independent and form a basis of the Grassmannian limit of the piece. Thus the total basis divisor D_m specializes to a basis divisor D_m^0 of spaces of exactly the same dimensions. For rational section classes, use a common Cartier multiple in forming these divisors and retain all fractional fixed factors.

If $(T, \Delta + \lambda_m D_m^0)$ is log canonical, lower semicontinuity of log canonical thresholds gives log canonicity for the nonzero fibres. These fibres are isomorphic as pairs because the action preserves Δ. Consequently

$$\lambda_m S_m(W; v) = \lambda_m v(D_m) \leq A_{T,\Delta}(v).$$

A bound for the limiting divisors independent of v, followed by the limit in m, bounds the stability threshold. This argument does not require the limiting spaces to be complete or monomial. Repeated weights allow non-monomial eigenvectors. A containing monomial space can be used to estimate their orders, but its extra dimensions must be treated as in Lemma 5.4. Moving the sections by an action which does not preserve the boundary instead degenerates a different pair and does not prove the asserted bound for (T, Δ).

A frequent exceptional pair in this manuscript is $(\mathbb{P}^1 \times \mathbb{P}^1, D/2)$, where D is a smooth curve of bidegree $(2,1)$. Coordinates put it in the form $u = s^2$ on an affine chart, and the action $(s,u) \mapsto (\lambda s, \lambda^2 u)$ preserves it. Its invariant prime curves are the four coordinate fibres and the closures of $u = cs^2$, $c \neq 0$. The latter curves meet at the two ramification points of D over the second factor. Away from those points the invariant boundary has simple normal crossings.

Lemma 5.6. *At the origin of the smooth affine plane, an effective divisor*

$$B = a(s=0) + b(u=0) + \sum_{i=1}^{r} c_i (u = \gamma_i s^2)$$

with distinct nonzero γ_i gives a log canonical pair if and only if

$$a \le 1, \quad b \le 1, \quad c_i \le 1, \qquad a + 2b + 2\sum_i c_i \le 3.$$

Proof. Blow up the origin, then blow up the point where the strict transforms of $u = 0$ and the parabolas meet the first exceptional curve. The resulting boundary has simple normal crossings. The two exceptional valuations have weights $(1,1)$ and $(1,2)$ on (s,u) and discrepancies two and three. Their inequalities are $a + b + \sum c_i \le 2$ and $a + 2b + 2\sum c_i \le 3$. The first follows from the second and $a \le 1$, by taking their average. Together with the prime coefficients at most one these are all inequalities on this resolution, proving both directions. □

For $D/2$, the weighted valuation $(1,2)$ therefore has discrepancy $3 - \frac{1}{2} \cdot 2 = 2$. The first ordinary blow-up alone does not separate the curves tangent to D, which is why the second blow-up or the corresponding weight-$(1,2)$ extraction is useful. This elementary resolution explains the boundary calculation used in Families 2.9, 2.13, 2.16 and 3.8.

Example 5.7 (Missing sections in Family 2.13). For a double conic in Family 2.13 the actual spaces on the exceptional product surface are ideal-power series for a length-two subscheme of D; their identification is proved in Lemma 41.4. A boundary-preserving specialization sends that scheme to (u, s^2), except for the separately treated pair of ramification points. In degree m, a limiting piece embeds in the monomial space

$$\langle s^i u^l : 0 \le i \le m - \epsilon,\ 0 \le l \le r, i + 2l \ge 2\max(0, r-m) \rangle, \qquad \epsilon \in \{0,1\}.$$

Put $\rho = r/m$. The monomial space has limiting polygon $0 \le a \le 1$, $0 \le b \le \rho$, $a + 2b \ge 2\max(0, \rho - 1)$. Its total dimension, normalized by the actual N_m, tends to $41/40$, not one. Its normalized $(1,2)$-weight sum is $163/80$, which is larger than the discrepancy two just computed.

For $3/2 < \rho \le 2$, the excess area over the actual dimension is $(\rho - 3/2)^2$, and every containing monomial has weight at least $2m(\rho - 1)$. Lemma 5.4 subtracts

$$\frac{3}{5} \int_{3/2}^{2} 2(\rho - 1)(\rho - 3/2)^2 \, d\rho = \frac{7}{160}.$$

The actual weight sum is consequently at most $163/80 - 7/160 = 319/160 < 2$. At the opposite ramification point it is at most $253/160$. The four coordinate coefficients are at

most $17/32, 79/160, 241/320, 87/160$, and every $(2,1)$-curve coefficient is at most $81/320$; these integrals are displayed in Lemma 41.7. At $\lambda = 320/319$, the largest coordinate coefficient is at most $241/319 < 1$ and the coefficient on D is at most $1/2 + 81/319 < 1$. The weighted inequalities of Lemma 5.6 hold as well. Using slightly smaller λ at finite degree and then passing to the limit gives $\delta(F, D/2; W) \geq 320/319$ in this case. The pair of ramification points gives the stronger bound $8/7$. Thus the strict estimate uses both the different and the number of sections missing from the containing space.

6 Keeping information from more than one divisor

The minimum in the adjunction inequality gives a convenient threshold bound, but it discards part of the discrepancy. The finite-degree inequality proved above retains that part. Two further forms used in this manuscript are recorded here.

6.1 Zeros away from the chosen point of a curve

Lemma 6.1. *Let W be a multigraded series on a normal projective surface T, with bounded support and with all expected-order limits appearing below finite and existent. Suppose the numerical classes of its basis divisors tend to D_W. Let B, C be distinct rational Cartier prime curves and let $q \in C \setminus B$. Assume restriction to C defines the refinement under consideration, and C is smooth near q. Then*

$$S(W^C; q) \leq (D_W - S(W; B)B - S(W; C)C) \cdot C.$$

The class inside the parentheses is a limit of effective classes. A known lower bound for $S(W; B)$ may replace it in the right side.

Proof. First take the associated graded for the C-filtration. On each piece, the original subspaces with B-order at least j induce a filtration. Taking the C-associated graded preserves their total dimensions, hence preserves their total B-weight. Choose a basis of each piece compatible with this induced filtration and with the order at q. Lift it with the indicated C- and B-orders. Once its C factor is removed, a lift with B-order j still has the factor $(B|_C)^j$, all of whose zeros are away from q. The order at q is at most the degree remaining after these zeros are removed. Sum and divide by mN_m. The lifts also show that removal of both factors before restriction leaves an effective divisor. Passing to the limit proves both assertions. For the last statement, $B \cdot C \geq 0$, so decreasing the subtracted coefficient only weakens the upper bound. □

The argument uses two filtrations on each already restricted space; it does not assume that three arbitrary filtrations admit one compatible basis. This estimate is useful on the ruled exceptional surfaces in Family 2.14, where a large known mean on one section forces a smaller residual degree at the chosen point of a ruling; see Lemma 43.6.

6.2 Two effective adjunctions on a smooth model

Here is a general calculation which explains the smaller boundary appearing in Family 1.9. Work locally on a smooth birational model $\pi : Z \to X$ with smooth exceptional divisors E, G, meeting transversely in C. Suppose, in the neighbourhood being used,

$$K_Z = \pi^* K_X + (A_E - 1)E + (A_G - 1)G.$$

For basis divisors adapted to the valuation being tested, write $\pi^* D_m = D'_m + b_m E + a_m G$. Suppose that the actual refinement on G has basis divisor

$$W_m = D'_m|_G + (b_m - \rho a_m)C, \qquad b_m \geq \rho a_m,$$

where $\rho \geq 0$. For example, this occurs when the pullback of the weighted exceptional divisor to Z is $G + \rho E$.

Assume $\lambda_m > 0$ is chosen so that $(G, \theta C + \lambda_m W_m)$ is log canonical near $q \in C$, and

$$0 \leq \rho\lambda_m a_m - (A_E - 1) \leq \theta, \qquad \lambda_m b_m - (A_E - 1) \geq 0.$$

After decreasing the coefficient of C, adjunction and inversion of adjunction give log canonicity of the effective pair

$$(Z, G + \lambda_m D'_m + (\lambda_m b_m - (A_E - 1))E)$$

near q. Comparing its boundary with the pullback of $K_X + \lambda_m D_m$ yields

$$A_X(v) - \lambda_m v(D_m) \geq (A_G - \lambda_m a_m) v(G) \tag{6.1}$$

for every valuation whose centre there is contained in G. Indeed the restricted boundary is exactly $\lambda_m W_m + (\rho\lambda_m a_m - (A_E - 1))C$; the only remaining difference of coefficients is on G. Taking limits is legitimate for a basis adapted to v, because $v(D_m) = S_m(v)$. A finite-degree threshold slightly below its limit supplies the required λ_m; one must not replace asymptotic log canonicity by finite-degree log canonicity without this step.

Example 6.2 (The special line in Family 1.9). In Lemma **??** the smooth model has $A_E = 2$, $A_G = 3$, $\rho = 1/2$ and

$$a_m \longrightarrow \frac{43}{18}, \qquad \frac{a_m}{2} \leq b_m \leq S_m(E), \qquad S_m(E) \longrightarrow \frac{101}{72}.$$

The surface calculation proves $\delta_q(G, 7C/36; W) \geq 1$. Notice that $(43/18)/2 - 1 = 7/36$. Taking $\lambda_m \to 1$ with $\lambda_m a_m \leq 43/18$ makes the preceding boundaries effective for large m and gives

$$A_X(v) - S_X(v) \geq \left(3 - \frac{43}{18}\right) v(G) = \frac{11}{18} v(G) > 0$$

when the centre is contained in G. The usual different of the weighted extraction is still $C/2$. The coefficient $7/36$ is an effective boundary on this smooth two-divisor model obtained by the displayed calculation, not a replacement value for that different.

Sometimes $\lambda_m b_m - (A_E - 1)$ is negative. Ordinary inversion of adjunction for effective boundaries does not justify keeping that coefficient. In that situation we use its nonnegative part and verify the new restricted coefficient. The local identity in Lemma **??** makes this explicit. There F' and E are smooth and transverse, W'_m is effective and does not contain J, and the identities are

$$K_Z = \pi^* K_X + 4F' + E, \quad \pi^* D_m = D'_m + a_m F' + e_m E, \quad D'_m|_{F'} = W'_m + (d_m - e_m) J,$$

where $J = E|_{F'}$ and $d_m \geq e_m$. Use the effective boundary $B_m = \lambda D'_m + (\lambda e_m - 1)_+ E$. Its restriction is

$$B_m|_{F'} = \lambda W'_m + (\lambda d_m - \min\{\lambda e_m, 1\}) J.$$

If the last coefficient belongs to $[0, 1]$ and $\lambda W'_m|_J$ has every coefficient at most one near the centre in question, inversion of adjunction on F' and then on Z gives

$$A_X(v) - \lambda v(D_m) \geq (5 - \lambda a_m) v(F') + (1 - \lambda e_m)_+ v(E).$$

Both inversions have effective boundary. The numerical input for that application is $a_m \to 287/72$, $d_m \to 115/72$, $e_m \geq (a_m - 1)/4$, and point orders on J at most $1/3 + o(1)$. For $1 < \lambda < 288/245$ these verify the stated coefficient conditions. Indeed, when $\lambda e_m \leq 1$ the coefficient is at most $\lambda(d_m - (a_m - 1)/4) \to 245\lambda/288 < 1$; otherwise it tends to $115\lambda/72 - 1 < 1$. The limiting coefficient on F' at $\lambda = 288/245$ is $5 - (288/245)(287/72) = 11/35 > 0$. The local restriction identity and its section-theoretic origin are proved in the cited lemma.

Example 6.3 (A refined threshold equal to one). For one of the ordinary point configurations of Family 2.12, the refined series on $G = \mathbb{P}^2$ has a monomial limit with mean exponents $1, 53/80, 47/80$. Its surface threshold is at least one; the first mean prevents that calculation from giving a larger bound. Nevertheless $A_X(G) = 3$ and $S_X(G) = 9/4$. The finite-degree adjunction inequality retains

$$A_X(v) - S_X(v) \geq \left(3 - \frac{9}{4}\right) v(G) = \frac{3}{4} v(G) > 0$$

for every divisor centred at the extracted point. The actual initial algebra and the moment calculation are proved in Lemma 35.4. A surface bound equal to one is therefore compatible with strict K-stability on the threefold; the positive remaining discrepancy is part of the proof.

Part II
Family 2.5: cubic-surface fibrations

7 Geometry and the two remaining local cases

Let $V \subset \mathbb{P}^4$ be a smooth cubic threefold, let $\Pi \subset \mathbb{P}^4$ be a plane such that $C = V \cap \Pi$ is a smooth cubic curve, and let $\pi : X = \mathrm{Bl}_C(V) \to V$ be the blow-up. Write E_0 for its exceptional divisor and $H = \pi^*\mathcal{O}_V(1)$. Define $L = -K_X = 2H - E_0$ and $A = H - E_0$. The pencil of hyperplanes through Π defines a morphism $\varphi : X \to \mathbb{P}^1$ whose fibres have class A. These are exactly the smooth threefolds in Family 2.5. A fibre is a normal cubic surface, identified with the corresponding hyperplane section of V.

The following result on K-stability of smooth threefolds in Family 2.5 was shown in [8].

Lemma 7.1 ([8, Section 2])**.** *If X is not K-stable, there exist a prime divisor F over X and a point P of its centre such that $A_X(F) \leq S_X(F)$, $P \notin E_0$, and the fibre through P has a singularity of type D_5 or E_6 at P.*

Proof. By the valuative criterion, there exists a prime divisor F over X with $\beta_X(F) \leq 0$. Then [8, Lemma 2.2 and preceding paragraph] show that the centre of F on X is not a surface and does not meet E_0. Let P be a point of the centre of F on X. Then the fiber of π over $\pi(P)$ has a Du Val singularity of type D_4, D_5, or E_6 at P by [8, paragraph before Lemma 2.8] and [8, Lemma 2.10]. Finally, [8, paragraph before Lemmas 2.11] and [8, Lemmas 2.11, 2.12, and 2.13] show that P is not a D_4 singularity (note that on [8, page 19] there is a typo: $S(W^A_{\bullet,\bullet}; L_1) < \frac{4}{3}$ should instead read $S(W^A_{\bullet,\bullet}; L_1) > \frac{3}{4}$). □

In this section, we prove the following two results; together with Lemma 7.1, these will establish K-stability of X.

Theorem 7.2. *Suppose that $P \in X \setminus E_0$ is a singular point of type D_5 of a fibre of φ. Then $\delta_P(X) \geq 96/95$.*

Theorem 7.3. *Suppose that $P \in X \setminus E_0$ is a singular point of type E_6 of a fibre of φ. Then $\delta_P(X) \geq 16/15$.*

The main computational point deserves emphasis. *We do not compute a Zariski decomposition of $\sigma^*L - uG$ on the weighted blow-up $\sigma : \widehat{X} \to X$.* Instead, we identify the actual filtration of $H^0(X, mL)$ by ord_G, including all cancellations, and compute its associated graded ring. Its weighted Hilbert function gives $S_X(G)$. A second, principal filtration gives a series on a rational curve; its complete polynomial intervals give all the point terms in the Abban–Zhuang inequality. An alpha estimate deals with centres away from that curve.

This should be distinguished from a claim that Zariski decompositions play no role anywhere in the argument. We retain the published reduction, whose proofs use them. For example, [8, Lemma 2.1] decomposes $L - uA$ on the original threefold. What is avoided here is any *new threefold Zariski-decomposition calculation in the two missing local cases.* We neither determine the birational chambers of $\widehat{X}$ nor assume that it is a Mori dream space. The replacement is an exact description of the refined section spaces, not merely a count of unconstrained local monomials.

7.1 Why no Zariski decomposition is required

In geometric volume calculations one may write $\sigma^*L - uG = P(u) + N(u)$ on suitable birational models, compute the volume from the positive part, and restrict positive and fixed parts

to obtain the refinement. In dimension three, making this explicit can require the chamber structure and changes of model. This is a method of *computing* the input to Proposition 2.7; it is not a hypothesis of that proposition.

Here the computation proceeds entirely in the section ring. First, a global bigrading records exactly which polynomials represent anticanonical sections. Second, a weight filtration is proved to coincide with ord_G. Third, the tangent curve is cut out by a prime homogeneous element, and saturation identifies its powers with the second geometric filtration. The resulting finite-dimensional spaces determine both expected vanishing and fixed factors on the curve. The alpha estimate then controls all centres outside the curve. We use the general refinement theorem, without a Mori dream space assumption on $\widehat{X}$.

Thus no Zariski decomposition on the weighted threefold blow-up is computed or needed in either local proof. There is still a global issue to solve: a local weighted polynomial need not come from an anticanonical section. The degree inequalities and the injectivity and saturation arguments below are precisely what solve that issue. Omitting them would invalidate the replacement of volumes by monomial counting.

8 The D5 point and the cubic section ring

8.1 The anticanonical sections

We first make constructions common to both cases. The E_6 statements in this subsection will be used in Section 9.

Lemma 8.1. *The divisor $L = 2H - E_0$ is ample, $L^3 = 12$, and $|A| = |H - E_0|$ gives a fibration in normal cubic surfaces.*

Proof. The blow-up resolves the pencil generated by the two equations of Π. Its graph is a closed subvariety of $V \times \mathbb{P}^1$, and $H + A$ is the restriction of $\mathcal{O}_V(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1)$, hence is ample. The blow-up formula gives $-K_X = 2H - E_0 = H + A$. Since $N_{C/V} = \mathcal{O}_C(1)^{\oplus 2}$, the intersection numbers are $H^3 = 3$, $H^2E_0 = 0$, $HE_0^2 = -3$, and $E_0^3 = -6$. Expanding $(2H - E_0)^3$ gives 12.

For normality, the Gauss map of V is defined by its five partial derivatives, which have no common zero on V. Its pullback of the hyperplane class is $\mathcal{O}_V(2)$, an ample class. It is therefore finite: a positive-dimensional fibre would contain a curve on which this class has degree zero. The singular locus of a hyperplane section is contained in a fibre of the Gauss map, so it is finite. A reducible or nonreduced cubic surface in $\mathbb{P}^3$ has positive-dimensional singular locus; hence our section is integral. As a hypersurface with only isolated singularities, it is normal. The curve C is Cartier on each such section, so its blow-up is the same surface; this identifies it with the fibre of φ. □

Lemma 8.2 (Coordinates at the two singularities)**.** *Suppose that $P \notin E_0$ is a D_5 or E_6 point of a fibre. There are coordinates $[x_0 : x : y : z : t]$ on $\mathbb{P}^4$ in which $P = [1 : 0 : 0 : 0 : 0]$, the fibre is $t = 0$, and its equation is respectively*

$$x_0x^2 + xy^2 + yz^2 = 0, \qquad x_0x^2 + xz^2 + y^3 = 0.$$

Moreover, $\Pi = (\ell = t = 0)$ with $\ell = x_0 + \lambda_1 x + \lambda_2 y + \lambda_3 z$. After replacing x_0 by ℓ and rescaling t, the cubic equation $\Phi(x, y, z, \ell, t) = 0$ of V has coefficient one at $t\ell^2$.

Proof. We derive the surface equations directly. At a Du Val point of type D_5 or E_6, the local analytic equations can be written $x^2 + yz^2 + y^4 = 0$ and $x^2 + y^3 + z^4 = 0$, respectively. In particular, the quadratic tangent cone has rank one. Put $P = [1 : 0 : 0 : 0]$ on its hyperplane

and write its cubic equation as $x_0x^2 + f_3(x,y,z) = 0$. A linear change of x_0 removes the terms of f_3 divisible by x^2. Thus the equation is

$$x_0x^2 + x(\alpha y^2 + \beta yz + \gamma z^2) + c_3(y,z) = 0.$$

The restriction of the cubic jet to the kernel of the quadratic tangent cone is invariant, up to a linear change and a scalar, under analytic coordinate changes: the contribution of a quadratic coordinate change to the cubic jet is divisible by x. The analytic equations just displayed therefore allow us to take $c_3 = yz^2$ in type D_5, and $c_3 = y^3$ in type E_6.

In the first case take new coordinates $Y = y + \gamma x$, $Z = z + (\beta/2)x$, removing the resulting x^2-multiples by changes of x_0. After renaming Y, Z as y, z, the equation becomes $x_0x^2 + \alpha xy^2 + yz^2 = 0$. If $\alpha = 0$, the line $x = z = 0$ is singular, contrary to the isolated singularity at P. Rescaling gives the asserted D_5 form.

In the second case $\gamma \neq 0$, since otherwise $x = y = 0$ is a singular line. Taking the new coordinate $Z = z + \beta y/(2\gamma)$ diagonalizes the binary quadratic form. Taking a further new coordinate $Y = y + \alpha' x/3$, where α' is its remaining y^2 coefficient, removes the xy^2 term; the new x^2-multiples are again absorbed in x_0. Rescaling z gives $x_0x^2 + xz^2 + y^3 = 0$. These equations agree with [7] and with entries 17° and 14° of [9, Table 3].

Extend these coordinates by the fibre equation $t = 0$. Since Π avoids P, its second equation has nonzero x_0 coefficient and has the stated form ℓ. Replacing x_0 by ℓ adds only x^2 times a linear form in x, y, z. All derivatives along $t = 0$ vanish at P, so smoothness of V forces the $t\ell^2$ coefficient to be nonzero; rescale t to make it one. □

Example 8.3. On $x_0 = 1$, completing the square gives

$$x^2 + xy^2 + yz^2 = (x + y^2/2)^2 + yz^2 - y^4/4, \quad x^2 + xz^2 + y^3 = (x + z^2/2)^2 + y^3 - z^4/4.$$

These are the local D_5 and E_6 equations. Each occurs on a smooth threefold in our family: take the corresponding homogeneous cubic $F_0(x_0, x, y, z)$ and set $V = (F_0 + tQ = 0)$ for a general quadratic form Q with $Q(P) \neq 0$. Along $(F_0 = t = 0)$ away from P, smoothness follows from that of the cubic surface there; at P it follows from $Q(P) \neq 0$; away from the base surface it follows from Bertini. A general plane in $t = 0$ avoids P and cuts out a smooth plane cubic. Thus the two local cases do not impose an artificial degeneration of the threefold.

Let $\mathcal{P} = \mathbb{C}[x, y, z, a, b, e]$ and put

$$R = \mathcal{P}/(\Phi(x, y, z, ea, eb)), \qquad \deg(x, y, z) = (1, 0), \quad \deg(a, b) = (1, -1), \quad \deg e = (0, 1). \tag{8.1}$$

The bidegree of an element of R will indicate the coefficients of H and E_0, respectively. The variable e is the exceptional section; the equations $\ell = ea$, $t = eb$ are the usual extended Rees presentation of the blow-up.

Lemma 8.4 (Global section spaces)**.** *For every $m \geq 0$, $H^0(X, mL) = R_{(2m,-m)}$. On the chart $a = e = 1$, its monomials are represented by*

$$x^i y^j z^k b^q, \qquad i + j + k \leq m, \qquad i + j + k + q \leq 2m. \tag{8.2}$$

Their homogeneous lifts are $x^i y^j z^k b^q a^{2m-i-j-k-q} e^{m-i-j-k}$.

Proof. The ideal $\mathcal{I}_C = (\ell, t)$ is a complete-intersection ideal on V. For $m \geq 1$ its power has the resolution

$$0 \longrightarrow \mathcal{O}_V(-m-1)^{\oplus m} \longrightarrow \mathcal{O}_V(-m)^{\oplus(m+1)} \longrightarrow \mathcal{I}_C^m \longrightarrow 0.$$

After twisting by $2m$, the first cohomology of the left term vanishes: this follows from the hypersurface sequence for $V \subset \mathbb{P}^4$ and the vanishing of intermediate cohomology of line bundles

on $\mathbb{P}^4$. Hence $H^0(V, \mathcal{I}_C^m(2m))$ is generated by $\ell^{m-q}t^q$ times homogeneous degree-m polynomials on V.

Adjoining $a = \ell/e$, $b = t/e$, and e to the homogeneous coordinate ring and eliminating ℓ, t gives (8.1). There are no further relations: after inverting e, the quotient becomes the coordinate ring of V with $e^{\pm 1}$ adjoined. The displayed relation is irreducible after localization and is not divisible by e, so its principal ideal is already saturated with respect to e. The generated degree piece is therefore $H^0(V, \mathcal{I}_C^m(2m)) = H^0(X, mL)$. The case $m = 0$ is immediate. Solving the two degree equations gives the exponents of a, e and proves (8.2). □

8.2 The two filtrations

On the chart $\ell \neq 0$, the ratios $x/\ell, y/\ell, z/\ell$ are regular parameters on V at P, by the last assertion of Lemma 8.2. We use the following primitive weights:

	x	y	z	b	d
D_5	4	2	3	8	8
E_6	6	4	3	12	12

$$\mathrm{wt}(a) = \mathrm{wt}(e) = 0. \tag{8.3}$$

Here d is the weight of the initial cubic relation. Let $\sigma : \widehat{X} \to X$ be the weighted blow-up, defined as the normalized relative Proj of the algebra of monomial ideals of weight at least j. Its exceptional divisor G defines the monomial valuation with the indicated weights in the three regular parameters. Primitivity ensures this is the normalization ord_G, rather than a multiple of it.

We recall the section spaces that will be identified below. For $m, j \geq 0$, put

$$\mathcal{F}_G^j H^0(X, mL) = \{s \in H^0(X, mL) : \mathrm{ord}_G(\sigma^* s) \geq j\}, \qquad W_{m,j} = \frac{\mathcal{F}_G^j H^0(X, mL)}{\mathcal{F}_G^{j+1} H^0(X, mL)}.$$

Here the order of a section means the order of its zero divisor. Thus m is the anticanonical degree and j is its order along G; the latter is measured with the primitive weights in (8.3). A class in $W_{m,j}$ is represented on G by dividing the pullback section by its exceptional factor of order j and restricting to G. Equivalently, it is the weighted initial form of the section. These are the initial forms of actual global sections of $\mathcal{O}_X(mL)$, so their space is not assumed to be the complete space of weighted forms of weight j. Since σ contracts G to P, their rational divisor class on G is $-jG|_G$. We write $W = \bigoplus_{m,j} W_{m,j}$ and retain the normalization

$$N_m = \sum_j \dim W_{m,j} = h^0(X, mL), \qquad S_m(G) = \frac{1}{mN_m} \sum_j j \dim W_{m,j}.$$

In particular, passing to a weight summand does not replace the denominator by its own dimension.

The lowest-weight parts of the relation $\Phi(x, y, z, ea, eb)$ in (8.1) are

$$\Psi_D = eax^2 + xy^2 + yz^2 + e^3a^2b, \qquad \Psi_E = eax^2 + xz^2 + y^3 + e^3a^2b. \tag{8.4}$$

Indeed, replacing x_0 by ℓ adds terms x^2 times positive-weight variables. Every term containing t, except $t\ell^2$, also has an additional positive-weight variable. Thus all remaining terms have strictly greater weight.

Lemma 8.5 (Identification of the first filtration)**.** *The refinement $\bigoplus_j W_{m,j}$ is the bidegree $(2m, -m)$ piece of $R_0 = \mathcal{P}/(\Psi)$, graded by the weights in (8.3), with $\Psi = \Psi_D$ or Ψ_E as in (8.4) as appropriate.*

Proof. Each Ψ is primitive and linear in b: its coefficient e^3a^2 is coprime to its constant term. Hence R_0 is a domain. For a principal ideal in a polynomial ring, the initial ideal for a weight filtration is generated by the initial form of the generator, because initial forms multiply. Consequently R_0 is the weight-associated graded ring of R. In bidegree $(2m, -m)$ the polynomial filtration is finite: (8.2) bounds every monomial weight by $2dm$. Thus passage from the graded quotient to a basis of the original section space involves no completion or infinite reduction.

It remains to identify this algebraic filtration with the geometric valuation. The completed local ring at P is $\mathbb{C}[[x/\ell, y/\ell, z/\ell]]$. The formal implicit-function theorem, with coefficient one at t/ℓ, gives $t/\ell = -g +$ terms of weight greater than d; this also follows recursively in the positive weighted filtration. Hence the weights assigned to all generators agree with their actual valuation orders. Localize at ae and set $x' = x/(ea)$, $y' = y/(ea)$, $z' = z/(ea)$ and $b' = b/a$. Then

$$R_0[(ae)^{-1}] = \frac{\mathbb{C}[x', y', z', b']}{(b' + g(x', y', z'))}[a^{\pm 1}, e^{\pm 1}],$$

where $g = x'^2 + x'y'^2 + y'z'^2$ or $g = x'^2 + x'z'^2 + y'^3$. The degrees of a, e form a basis of $\mathbb{Z}^2$, so a fixed bidegree has a common Laurent factor in them. After removing it, setting $a = e = 1$ is injective on that homogeneous component. Localization itself is injective because R_0 is a domain. Finally, eliminating b' gives $\mathbb{C}[x', y', z']$ with the stated native weights, which is the associated graded ring of the monomial valuation in the regular parameters. Thus no nonzero global initial form disappears after passage to the local valuation. This proves the assertion, including possible cancellation among global sections. □

Let $B \subset G$ be the curve defined by the initial form of b. Geometrically it is the weighted tangent curve of the singular cubic fibre. The second refinement is

$$W^B_{m,j,q} = \frac{\{s \in W_{m,j} : \operatorname{ord}_B(s) \geq q\}}{\{s \in W_{m,j} : \operatorname{ord}_B(s) \geq q+1\}}, \qquad q \geq 0.$$

It is realized on B after removing the factor of order q along B. The next lemma proves that this geometric order is exactly divisibility by b in the homogeneous coordinate ring. In particular, $\sum_{j,q} \dim W^B_{m,j,q} = N_m$, and the normalization of all second-refinement averages is again mN_m.

Lemma 8.6 (Identification of the second filtration)**.** *The curve B is reduced and prime, and refinement by ord_B is the b-adic filtration on the homogeneous pieces of R_0.*

Proof. The quotient $R_0/(b)$ is a domain: its relation is primitive and linear in a, with coefficient ex^2 coprime to $xy^2 + yz^2$ or $xz^2 + y^3$. Thus (b) is prime and contains neither a nor e. For every $q \geq 0$,

$$(b^q) : (ae)^\infty = (b^q). \tag{8.5}$$

In fact, if $(ae)^N h \in (b^q)$, reduction modulo b gives $h = bh_1$; cancellation in the domain and induction give $h \in (b^q)$.

Now we prove that for $s \in W_{m,j}$, $\operatorname{ord}_B(s) \geq q$ if and only if the corresponding element of R_0 is in the ideal (b^q). After localizing at ae, we have $b' = b/a = -g(x', y', z')$. Viewing g as a monic quadratic in x', the calculation in Example 8.3 shows that its discriminant is not a square. Thus g is irreducible, and the cone $(g = 0)$ is reduced and irreducible. This cone is not contained in any coordinate hyperplane, so all three coordinates are nonzero on a dense open subset. Since the original weight triple given in (8.3), either $(4, 2, 3)$ or $(6, 4, 3)$, has greatest common divisor one, the weighted projective quotient has trivial stabilizer on this open subset. It follows that the vanishing order along B equals the vanishing order along $(g = 0)$ on the cone.

Fix $s \in W_{m,j}$, and let $f \in R_0$ be its homogeneous representative under Lemma 8.5. After localizing at ae and removing the common Laurent factor in a, e, write its representative as $\bar{f}(x', y', z')$. The preceding geometric identification gives

$$\mathrm{ord}_B(s) \geq q \quad \Longleftrightarrow \quad g^q \mid \bar{f} \quad \Longleftrightarrow \quad f \in (b^q)R_0[(ae)^{-1}].$$

For the last equivalence, we use $b = -ag$ and the invertibility of a. Clearing denominators, the last condition means that $(ae)^N f \in (b^q)$ for some $N \geq 0$. Equation (8.5) then gives $f \in (b^q)$. Consequently,

$$\{s \in W_{m,j} : \mathrm{ord}_B(s) \geq q\} \simeq (b^q) \cap (R_0)_{(2m,-m),j},$$

where the subscripts on the right specify the bidegree and weight. Thus the geometric filtration by vanishing along B agrees with the b-adic filtration on each homogeneous component. □

Lemma 8.7 (Simultaneous standard monomials)**.** *A basis of the first refinement compatible with the second is given by (8.2), subject to $j = 0$ or $k \leq 1$ in type D_5, and to $j \leq 2$ in type E_6.*

Proof. In type D_5 take lexicographic order $z > y > x > b > a > e$, so Ψ_D has leading term yz^2. In type E_6 take $y > x > z > b > a > e$, so Ψ_E has leading term y^3. Standard monomials are the monomials in $\mathcal{P}$ that are not divisible by this leading coefficient. They respect both bidegree and valuation weight. Since the leading term of Ψ is monic and contains no b, the quotient $R_0 = \mathcal{P}/(\Psi)$ is a free $\mathbb{C}[b]$-module on the b-free standard monomials. The resulting basis is adapted to the filtration by powers of b, and hence, by Lemma 8.6, to the filtration by vanishing order along B. □

Example 8.8. In the D_5 case, the equation on the local chart begins $b + x^2 + xy^2 + yz^2 = 0$ in weight eight. Hence this sum has valuation greater than eight in $\mathcal{O}_{X,P}$, although each summand has weight eight. Passing to $\mathcal{P}/(\Psi_D)$ imposes exactly this initial relation. Counting free local monomials without imposing it would count a nonzero initial form where none exists. Lemma 8.5 proves that all such cancellations, including those among global sections, are accounted for.

8.3 The separate computation of the expected vanishing orders

Lemma 8.9 (Weighted Hilbert function and $S_X(G)$)**.** *In either case D_5 or E_6, put $w = w_x + w_y + w_z$ and use d from (8.3). Then $N_m = h^0(X, mL) = 2m^3 + 3m^2 + 3m + 1$, and the sum of the orders of a G-adapted basis is*

$$M_m(G) = w\left(\frac{5}{8}m^4 + \frac{1}{4}m^3 + \frac{7}{8}m^2 + \frac{1}{4}m\right) + d\left(\frac{7}{6}m^4 + \frac{7}{3}m^3 + \frac{11}{6}m^2 + \frac{2}{3}m\right).$$

In particular,

$$S_X(G) = \frac{5w}{16} + \frac{7d}{12}, \qquad S_X(G_D) = \frac{359}{48}, \quad S_X(G_E) = \frac{177}{16}. \tag{8.6}$$

Proof. For the polynomial ring $\mathcal{P}$, let $D_m(s) = \dim \mathcal{P}_{(s,-m)}$ and let $T_m(s)$ be the sum of the weights of all monomials of bidegree $(s, -m)$ in $\mathcal{P}$. If h is the total exponent of a, b, then the exponent of e is $h - m$, and the total exponent of x, y, z is $s - h$. For fixed h, there are $h + 1$ choices in a, b and $\binom{s-h+2}{2}$ choices in x, y, z. Among the monomials in $\mathcal{P}_{(s,-m)}$ whose exponents of a and b sum to h, the mean weight is $w(s - h)/3 + dh/2$, by symmetry. Thus

$$D_m(s) = \sum_{h=m}^{s} (h+1)\binom{s-h+2}{2}, \tag{8.7}$$

$$T_m(s) = \sum_{h=m}^{s} (h+1)\binom{s-h+2}{2}\left(\frac{w}{3}(s-h) + \frac{d}{2}h\right). \tag{8.8}$$

In the above, empty sums are zero. The relation Ψ has bidegree $(3,0)$ and weight d, and multiplication by Ψ is injective on $\mathcal{P}$, so the homogeneous exact sequence

$$0 \to \mathcal{P}_{(s-3,-m)} \xrightarrow{\cdot\Psi} \mathcal{P}_{(s,-m)} \to (R_0)_{(s,-m)} \to 0$$

for $s = 2m$ and the isomorphism $(R_0)_{(2m,-m)} \cong \bigoplus_j W_{m,j}$ of Lemma 8.5 give

$$\begin{aligned} N_m &= D_m(2m) - D_m(2m-3), \\ M_m(G) &= T_m(2m) - T_m(2m-3) - dD_m(2m-3). \end{aligned} \tag{8.9}$$

For the second equality above, multiplication by Ψ sends the monomial basis of $\mathcal{P}_{(2m-3,-m)}$ to a basis of its image, increasing the weight of each basis element by d. The total weight of this image is therefore $T_m(2m-3) + dD_m(2m-3)$. Subtracting this quantity from $T_m(2m)$ gives the total weight of the quotient, which is $M_m(G)$ by Lemma 8.5.

To compute N_m and $M_m(G)$, we evaluate the terms on the right-hand side of (8.9). For $r = s - m \geq 0$, the binomial convolution identities give

$$\begin{aligned} D_m(s) &= m\binom{r+3}{3} + \binom{r+4}{4}, \\ T_m(s) &= w\left(m\binom{r+3}{4} + \binom{r+4}{5}\right) \\ &\quad + d\left(\binom{m+1}{2}\binom{r+3}{3} + (m+1)\binom{r+3}{4} + \binom{r+3}{5}\right). \end{aligned}$$

For example, the w-coefficient follows from $\frac{s-h}{3}\binom{s-h+2}{2} = \binom{s-h+2}{3}$; the d-coefficient follows by writing $h = m + k$ and expanding $\binom{m+k+1}{2} = \binom{m+1}{2} + (m+1)k + \binom{k}{2}$. These identities also follow by multiplying the elementary generating functions $(1-z)^{-j}$.

To obtain the formulas for N_m and $M_m(G)$, we substitute the above expressions in (8.9) and expand; for $m < 3$ use the empty-sum convention directly. For (8.6), by Lemma 2.1 and the formulas above,

$$S_X(G) = \lim_{m\to\infty} \frac{M_m(G)}{mN_m} = \frac{1}{2}\left(\frac{5w}{8} + \frac{7d}{6}\right) = \frac{5w}{16} + \frac{7d}{12},$$

where we use $mN_m = 2m^4 + O(m^3)$. Substituting $(w,d) = (9,8)$ and $(w,d) = (13,12)$ gives the expressions for $S_X(G_D)$ and $S_X(G_E)$. □

More explicitly, the two cases of $M_m(G)$ in Lemma 8.9 are

$$\begin{aligned} M_m(G_D) &= \frac{359m^4 + 502m^3 + 541m^2 + 182m}{24}, \\ M_m(G_E) &= \frac{177m^4 + 250m^3 + 267m^2 + 90m}{8}. \end{aligned}$$

Lemma 8.10 (The average order along the tangent curve)**.** *In both cases, $S(W;B) = 11/16$. In degree m, choose a B-adapted basis of $W_{m,J}$; then the sum of the orders along B of all these basis elements is*

$$K_m = \frac{11m^4 + 14m^3 + 17m^2 + 6m}{8}.$$

Proof. By Lemmas 8.6 and 8.7, the standard monomials give a basis compatible with vanishing along B, and for each the exponent of b is equal to its order along B. Thus K_m is the sum of these exponents.

Let $B_m(s)$ denote the sum of the b-exponents of all monomials in $\mathcal{P}_{(s,-m)}$. Grouping them by their total exponent h in a, b gives

$$B_m(s) = \frac{1}{2}\sum_{h=m}^{s} h(h+1)\binom{s-h+2}{2},$$

with empty sums understood to be zero. Indeed, for each choice of the exponents of x, y, z, the exponent of b ranges from 0 to h.

The monomials excluded from the standard basis are precisely the multiples of the leading monomial of Ψ. This leading monomial has bidegree $(3, 0)$ and contains no b, so multiplication by it preserves the b-exponent. So $K_m = B_m(2m) - B_m(2m-3)$, and using the above expression for $B_m(s)$ this gives the formula for K_m in the lemma statement. Finally, since $N_m = 2m^3 + O(m^2)$ by Lemma 8.9, the above expression for K_m gives $S(W; B) = \lim_{m\to\infty} \frac{K_m}{mN_m} = \frac{11}{16}$. □

Example 8.11. For $m = 1$ a basis in either case is $1, b, b^2, x, xb, y, yb, z, zb$, with the homogeneous lifts prescribed by Lemma 8.4. There are nine sections. Their native weights sum to 66 in type D_5 and to 98 in type E_6, in agreement with Lemma 8.9. Their exponents of b sum to 6, in agreement with Lemma 8.10. These are finite-degree checks; S is obtained only after taking the asymptotic average.

Lemma 8.12 (Weighted quotient charts)**.** *Let G be the exceptional divisor of the weighted blow-up of a smooth threefold point with positive primitive weights (w_1, w_2, w_3). The extraction is of plt type and $A_X(G) = w_1 + w_2 + w_3$. The generic coefficient of the different along the coordinate curve $x_k = 0$ is $1 - 1/r_k$, where $r_k = \gcd(w_i, w_j)$ and $\{i, j, k\} = \{1, 2, 3\}$.*

Proof. On the chart indexed by i, use the finite cover with coordinates (u, v_j, v_k) and map $x_i = u^{w_i}$, $x_j = u^{w_j}v_j$, $x_k = u^{w_k}v_k$. The cyclic group μ_{w_i} acts with weights $(1, -w_j, -w_k)$, and G is the image of $u = 0$. Primitivity implies that this cover has no divisorial ramification: a nontrivial element fixing the generic point of $u = 0$ would have order dividing all three weights. No other divisor has generic stabilizer either. Thus the smooth pair with boundary $u = 0$ descends to a plt pair, by the discrepancy formula for finite maps. Relative ampleness of $-G$ follows from the weighted Proj construction.

The pulled-back volume form has factor $u^{w_1+w_2+w_3-1}$ times a nonzero constant, so its discrepancy along G is $w_1 + w_2 + w_3 - 1$. This proves the formula for the log discrepancy. At a general point of $G \cap (x_k = 0)$, the stabilizer has order r_k, and on the transverse cover it acts with weights $(1, -w_k)$. Since $\gcd(r_k, w_k) = 1$, restriction to $u = 0$ has ramification index r_k along the coordinate curve. Adjunction consequently contributes $1 - 1/r_k$ to the different. The calculation is unchanged after an étale pullback from regular local coordinates. □

8.4 The exceptional surface in type D_5

For the remainder of Section 8, we assume the singularity is of type D_5.

Lemma 8.13. *The weighted blow-up is of plt type, $A_X(G_D) = 9$, and its coarse exceptional surface is $G_D = \mathbb{P}(2, 1, 3)$ with coordinates $U = x$, $V = y$, $W = z^2$. Writing $H_G = \mathcal{O}_{G_D}(1)$, one has*

$$-G_D|_{G_D} = \tfrac{1}{2}H_G, \quad \Delta_G = \tfrac{1}{2}(W = 0), \quad B = (U^2 + UV^2 + VW = 0) \sim 4H_G.$$

Moreover, $\alpha(G_D, \Delta_G; H_G) = 1$.

Proof. Lemma 8.12 gives the plt property and log discrepancy $4 + 2 + 3 = 9$. For the description of G_D, consider $\mathbb{C}[x, y, z]$ with weights $(4, 2, 3)$ as given in (8.3); its second Veronese subring is $\mathbb{C}[x, y, z^2]$, with regraded weights $(2, 1, 3)$. To compute $-G_D|_{G_D}$, write D_y for

the strict transform of $(y = 0)$; then $\sigma^*(y = 0) = D_y + 2G_D$. Restriction to G_D gives $-2G_D|_{G_D} \sim_{\mathbb{Q}} D_y|_{G_D} = (V = 0) \sim_{\mathbb{Q}} H_G$, so $-G_D|_{G_D} \sim_{\mathbb{Q}} \frac{1}{2}H_G$.

At the generic point of $z = 0$, the quotient index is $\gcd(4, 2) = 2$; the other two coordinate curves have index one. Adjunction therefore gives $\Delta_G = \frac{1}{2}(W = 0)$. Substituting the coarse coordinates in the initial form of b gives the equation of B. The formulas satisfy $-(K_{G_D}+\Delta_G) = \frac{9}{2}H_G = 9(-G_D|_{G_D})$. Finally, Lemma 3.2 gives $\alpha(G_D, \Delta_G; H_G) = \min\{2, 1, 3/2\} = 1$. □

Lemma 8.14 (The tangent curve and its different)**.** *The curve B is isomorphic to $\mathbb{P}^1$. On $V \neq 0$, a coordinate is $r = U/V^2$, and $W/V^3 = -r(r + 1)$. Denote the points $r = 0, -1, \infty$ by Q_0, Q_{-1}, Q_∞. Then $(G_D, \Delta_G + B)$ is plt along B and*

$$\Delta_B = \tfrac{1}{2}Q_0 + \tfrac{1}{2}Q_{-1} + \tfrac{2}{3}Q_\infty.$$

The coordinate sections restrict to rational divisors

$$(V|_B) = \tfrac{2}{3}Q_\infty, \qquad (U|_B) = Q_0 + \tfrac{1}{3}Q_\infty, \qquad (W|_B) = Q_0 + Q_{-1}.$$

Proof. On $V = 1$, the equation solves for W and identifies this open set of B with the affine r-line. The only remaining point is $[0 : 0 : 1]$. Its index-three cover has coordinates (u, v) with μ_3-weights $(2, 1)$, and the curve is $u^2 + uv^2 + v = 0$. It is smooth, with parameter u and $v = -u^2 + O(u^5)$. The quotient has parameter u^3, so it too is smooth. Moreover, $r = u/v^2$ has a simple pole on the quotient. This proves $B \cong \mathbb{P}^1$.

The boundary $(W = 0)$ meets B transversely at Q_0, Q_{-1} and misses Q_∞. The smooth transverse charts and the smooth index-three cover prove plt. Adjunction contributes $1/2$ at the first two points and $1 - 1/3$ at the last. On the cover, V has order two and U has order one at infinity; division by three gives their stated rational orders. Their remaining orders follow on the affine chart, where $U/V^2 = r$ and $W/V^3 = -r(r + 1)$. □

Lemma 8.15 (Residual degree in type D_5)**.** *The residual-degree average in Lemma 3.3 is $R_D = 95/96$.*

Proof. A section in $W_{m,j}$ has class $jH_G/2$, and removing B^q subtracts $4qH_G$. Consequently

$$R_D = \tfrac{1}{2}S_X(G_D) - 4S(W; B) = \frac{359}{96} - 4 \cdot \frac{11}{16} = \frac{95}{96},$$

by Lemmas 8.9 and 8.10. Both averages use the same denominator mN_m. □

8.5 The exact series on the curve

Fix m, q and put $M = \min\{m, 2m - q\}$. The exponent of z has a fixed parity $\eta \in \{0, 1\}$ in every native weight summand. After removing $b^q z^\eta$, set $T = M - \eta$. The remaining standard monomials are

$$U^i V^j W^k, \qquad 2i + j + 3k = n, \quad i + j + 2k \leq T, \quad j = 0 \text{ or } k = 0. \tag{8.10}$$

Their original valuation weight is $2n + 3\eta + 8q$. A summand is empty if $T < 0$; otherwise $0 \leq n \leq 2T$. On $B \cap (V \neq 0)$, division by V^n turns these sections into r^i for $k = 0$ and, up to sign, $r^{i+k}(r + 1)^k$ for $j = 0$.

Lemma 8.16 (The polynomial interval)**.** *Their span is $r^a\langle 1, r, \dots, r^{D-a}\rangle$, where*

$$a = \max\{0, n - T\}, \qquad D = \min\{T, \lfloor 2n/3 \rfloor\}.$$

It has dimension $D - a + 1$. Before restoring z^η, the fixed orders at Q_0, Q_∞ are a and $2n/3 - D$, and there are no other fixed points.

Proof. For $k = 0$, the allowed powers are r^i for $a \le i \le \lfloor n/2 \rfloor$. For $j = 0$, one has $i = (n-3k)/2$ and

$$0 \le k \le \min\{\lfloor n/3 \rfloor, 2T - n\}, \qquad k \equiv n \pmod 2.$$

The polynomial is $r^{(n-k)/2}(r+1)^k$. It is divisible by r^a and has leading degree $(n+k)/2$. The positive allowed values of k give exactly the consecutive leading degrees from $\lfloor n/2 \rfloor + 1$ through D. Together with the first group, they give a triangular basis of the claimed interval.

The common factor r^a gives the fixed order at zero. A polynomial of degree d represents $V^n f(r)$ and has order $2n/3 - d$ at infinity, by Lemma 8.14. Taking the largest possible degree gives the fixed order there. After removing these two factors, the interval is a complete series of degree $D - a$, so it has no further fixed points. □

Example 8.17. Take $m = 6$, $q = 0$, $\eta = 0$ and $n = 8$. Then $T = 6$, $a = 2$, $D = 5$. The standard sections $U^2V^4, U^3V^2, U^4, UW^2$, divided by V^8, become $r^2, r^3, r^4, r^3(r+1)^2$. Since the last polynomial is $r^5 + 2r^4 + r^3$, their span is exactly $\langle r^2, r^3, r^4, r^5 \rangle$. The fixed orders are 2 at Q_0 and 1/3 at Q_∞, and the moving degree is three. Thus the mean orders are 7/2 at Q_0, 11/6 at Q_∞, and 3/2 at every other point. The fractional order at infinity is part of the divisor on the coarse curve and must be retained.

Lemma 8.18 (Finite-degree point orders)**.** *For the summand indexed by (m, q, η, n) above, the mean orders of point-adapted bases are*

Q	*mean order*
Q_0	$(a + D)/2 + \eta/2$
Q_{-1}	$(D - a)/2 + \eta/2$
Q_∞	$2n/3 - (a + D)/2$
other points	$(D - a)/2$.

Its residual class on G_D is $(n + 3\eta/2)H_G$.

Proof. Apply Lemma 3.1 to the fixed factors and the complete moving interval in Lemma 8.16. The residue factor z^η is interpreted by Lemma 2.14. Since its square is W^η, it adds $\eta/2$ at Q_0, Q_{-1} and nothing at infinity. Its class is $3\eta H_G/2$, proving the last assertion. As a degree check, the fixed orders sum to $a + (2n/3 - D) + \eta$ and the moving degree is $D - a$. Their total is $2n/3 + \eta = (n + 3\eta/2)\deg(H_G|_B)$, since $\deg(H_G|_B) = 2/3$. □

Lemma 8.19 (Positivity of the refined series in type D_5)**.** *The series W and W^B have bounded support and contain ample series. Hence Proposition 2.7 applies to both refinements.*

Proof. We verify all the positivity conditions, including eventual nonvanishing on every integral interior ray. Write J for the native weight and put $M = \min\{m, 2m - q\}$. The support cone of the second refinement is

$$\mathcal{C}_D = \{(m, J, q) : m \ge 0,\ 0 \le q \le 2m,\ 8q \le J \le 8q + 4M\}. \tag{8.11}$$

Every standard monomial satisfies these inequalities. Conversely, fix an integral point (m_0, J_0, q_0) in the interior of this cone, and set $M_0 = \min\{m_0, 2m_0 - q_0\}$. For its kth multiple choose $\eta_k \in \{0, 1\}$ with $\eta_k \equiv kJ_0 \pmod 2$, and put

$$n_k = \frac{kJ_0 - 8kq_0 - 3\eta_k}{2}, \qquad T_k = kM_0 - \eta_k.$$

The strict inequalities $0 < J_0 - 8q_0 < 4M_0$ imply that both n_k and $2T_k - n_k$ grow linearly. In Lemma 8.16, $D - a$ therefore grows linearly as well. The summand is nonzero for every sufficiently large k, with no divisibility restriction on k. The same argument applies to rational

interior rays after rescaling. Such rays are dense in the cone, proving equality in (8.11) and the required nonvanishing assertion.

The projection to (m, J) is $0 \le J \le 16m$. If $0 < J_0 < 16m_0$, choose a rational q_0 just below $J_0/8$, so that (m_0, J_0, q_0) is strictly inside (8.11). For each k approximate kq_0 by an integer. The resulting bounded errors, together with the parity correction just used, do not affect eventual nonvanishing. This proves the condition on every integral interior ray for the first refinement too. Both support slices are bounded and have nonempty interior.

For the complete ample subseries on G_D, take $(m, J) = (12k, 12k)$. Every native weighted polynomial of weight $12k$ has ordinary degree at most $6k \le m$, so it comes from a section with no b factor in its expression. These give all of $H^0(G_D, \mathcal{O}_{G_D}(6k))$, and $6H_G$ is ample Cartier. On B, take $(m, J, q) = (12k, 14k, k)$: then $\eta = 0$, $n = 3k$, $a = 0$, $D = 2k$, and the fixed order at infinity is zero. The series is exactly $H^0(B, \mathcal{O}_B(2k))$. Both rays are interior. Finally, Lemmas 8.13 and 8.14 verify the plt conditions. This proves every hypothesis of Proposition 2.7 directly, in agreement with the general refinement statement of [4, Example 2.15]. □

Lemma 8.20 (Limiting point orders in type D_5)**.** *One has*

$$S(W^B; Q_0) = \frac{5}{12}, \qquad S(W^B; Q_\infty) = \frac{35}{144}, \qquad S(W^B; Q) = \frac{65}{288} \quad (Q \neq Q_0, Q_\infty).$$

Proof. We give the Riemann sums explicitly. Rescale q, n by m and write $M(q) = \min\{1, 2-q\}$ for $0 \le q \le 2$. The parity correction in T and the rounding in D are bounded. The limiting dimension divided by m is $\ell = D - a$, with

n	$[0, M]$	$[M, 3M/2]$	$[3M/2, 2M]$
a	0	$n - M$	$n - M$
D	$2n/3$	$2n/3$	M
ℓ	$2n/3$	$M - n/3$	$2M - n.$

The dimension integral is $I_D = \int_0^2 \int_0^{2M(q)} \ell \, dn \, dq = \frac{3}{4} \int_0^2 M(q)^2 dq = 1$. The two parity classes therefore give $N_m = 2I_D m^3 + O(m^2)$, consistent with Lemma 8.9. Their common factor two cancels in every normalized average.

Set $f_0 = (a + D)/2$, $f_\infty = 2n/3 - (a + D)/2$, and $f_* = (D - a)/2$. These are the limiting mean orders from Lemma 8.18. Direct integration on the three intervals above gives

f	n	f_0	f_∞	f_*
$\int_0^{2M} \ell f \, dn$	$19M^3/24$	$M^3/3$	$7M^3/36$	$13M^3/72.$

For instance, twice the f_* integral is

$$\int_0^M (2n/3)^2 dn + \int_M^{3M/2} (M - n/3)^2 dn + \int_{3M/2}^{2M} (2M - n)^2 dn = \frac{13M^3}{36}.$$

Since $\int_0^2 M(q)^3 dq = 5/4$, the last three columns, divided by $I_D = 1$, give $5/12$, $35/144$, and $65/288$. The first gives $95/96$, agreeing with Lemma 8.15. Similarly, the mean of q is $\frac{3}{4} \int_0^2 qM(q)^2 dq = 11/16$, agreeing with Lemma 8.10.

For completeness, there are $O(m^2)$ summands, each of dimension $O(m)$ and with orders $O(m)$. Bounded changes in endpoints or fixed orders change the total weight by $O(m^3)$, whereas mN_m has order m^4. Thus all residue and rounding corrections disappear in the limit. The finite formulas distinguish only the three named points, so the error estimate is uniform in the point of B. □

8.6 The D5 estimate

Lemma 8.21. *One has* $\delta(G_D, \Delta_G; W) \geq 96/95$.

Proof. For valuations whose centres are not contained in B, Lemmas 3.3, 8.13, and 8.15 give $S(W; v) \leq (95/96)A_{G_D,\Delta_G}(v)$. For centres on B, apply the second refinement, justified by Lemma 8.19. The curve term is $1/S(W; B) = 16/11$. The point terms are

Q	$A_{B,\Delta_B}(Q)$	$S(W^B; Q)$	$A_{B,\Delta_B}(Q)/S(W^B; Q)$
Q_0	$1/2$	$5/12$	$6/5$
Q_{-1}	$1/2$	$65/288$	$144/65$
Q_∞	$1/3$	$35/144$	$48/35$
other points	1	$65/288$	$288/65$.

They give a bound at least $6/5 > 96/95$. This covers the valuation of B itself and all valuations centred at points of B, and hence all remaining centres. □

Proof of Theorem 7.2. The discrepancy is 9 by Lemma 8.13, and $S_X(G_D) = 359/48$ by Lemma 8.9. The first refinement and Lemma 8.21 give

$$\delta_P(X) \geq \min\{432/359, 96/95\} = 96/95.$$

□

9 The E6 point

We retain the global section ring and the two filtration identifications from Section 8, now with the E_6 weights. In particular, Lemmas 8.9 and 8.10 already give $S_X(G_E) = 177/16$ and $S(W; B) = 11/16$.

9.1 The exceptional surface and its tangent curve

Lemma 9.1. *The weighted blow-up is of plt type,* $A_X(G_E) = 13$*, and its coarse exceptional surface is* $G_E = \mathbb{P}(1, 1, 2)$ *with coordinates* $U = x$, $V = z^2$, $W = y^3$. *For* $H_G = \mathcal{O}_{G_E}(1)$ *one has*

$$-G_E|_{G_E} = \tfrac{1}{6}H_G, \quad \Delta_G = \tfrac{1}{2}(V = 0) + \tfrac{2}{3}(W = 0), \quad B = (U^2 + UV + W = 0) \sim 2H_G.$$

Moreover, $\alpha(G_E, \Delta_G; H_G) = 1/2$.

Proof. Lemma 8.12 applies with weights $(6, 4, 3)$ and gives discrepancy $6 + 4 + 3 = 13$. A monomial of weight divisible by six has exponent of y divisible by three and exponent of z divisible by two. Thus the sixth Veronese is $\mathbb{C}[x, z^2, y^3]$ with regraded weights $(1, 1, 2)$; it gives the coarse surface and the normal class $H_G/6$.

The generic quotient indices are two along $z = 0$ and three along $y = 0$, and one along $x = 0$. They give the stated different. The equation of B follows from the initial relation. As a check, $-(K_{G_E} + \Delta_G) = 13H_G/6 = 13(-G_E|_{G_E})$. Finally, Lemma 3.2 gives $\min\{1, 1/2, 2/3\} = 1/2$. □

Lemma 9.2. *The curve* B *is a smooth rational curve avoiding the singular vertex of* G_E*. It is parametrized by* $[U : V]$*, with* $W = -U(U + V)$*. Let* Q_U, Q_{U+V}, Q_V *denote its points* $U = 0$, $U + V = 0$, $V = 0$*. Then* $(G_E, \Delta_G + B)$ *is plt along* B *and*

$$\Delta_B = \tfrac{2}{3}Q_U + \tfrac{2}{3}Q_{U+V} + \tfrac{1}{2}Q_V.$$

Proof. At the vertex $[0:0:1]$, the equation of B is nonzero. Away from that point it solves for W and gives the claimed isomorphism with $\mathbb{P}^1$. The three marked points are distinct. The boundary $(W=0)$ cuts out $Q_U + Q_{U+V}$ transversely, and $(V=0)$ cuts out Q_V transversely. Thus near B the pair is a smooth surface with a smooth coefficient-one curve and transverse boundary curves of coefficients less than one. It is plt, and ordinary adjunction gives the different. □

Lemma 9.3 (Residual degree in type E_6)**.** *The residual-degree average is* $R_E = 15/32$.

Proof. The class of a section of $W_{m,j}$ is $jH_G/6$, and removal of B^q subtracts $2qH_G$. Therefore

$$R_E = \tfrac{1}{6}S_X(G_E) - 2S(W;B) = \frac{177}{96} - 2\cdot\frac{11}{16} = \frac{15}{32}.$$

□

9.2 The exact curve series and its averages

Lemma 9.4 (Finite-degree series in type E_6)**.** *Fix m, q and set $M = \min\{m, 2m-q\}$. Every standard monomial is uniquely written*

$$y^\epsilon z^\eta U^i V^j b^q, \qquad 0\le\epsilon\le 2, \quad 0\le\eta\le 1, \quad i+j=n.$$

Its native weight is $6n+4\epsilon+3\eta+12q$. Put $T = M-\epsilon-\eta$. After removing b^q and the residue factor $y^\epsilon z^\eta$, its weight summand restricts to

$$\langle U^{n-j}V^j : 0\le j\le \ell\rangle = U^{n-\ell}H^0(\mathbb{P}^1,\mathcal{O}(\ell)), \qquad \ell = \min\{n, T-n\}, \quad 0\le n\le T. \tag{9.1}$$

It is empty if $T<0$. For a nonempty summand the dimension is $\ell+1$, and its mean point orders, with the residue factor restored, are

Q	Q_U	Q_{U+V}	Q_V	*other points*
mean	$n-\ell/2+\epsilon/3$	$\ell/2+\epsilon/3$	$\ell/2+\eta/2$	$\ell/2$.

The residual divisor class on G_E is $(n+2\epsilon/3+\eta/2)H_G$.

Proof. The standard-monomial condition is that the exponent of y is at most two; splitting the exponent of z into its parity and its even part gives the unique expression. The six residues $4\epsilon+3\eta$ are distinct modulo six. Thus ϵ,η are fixed in each summand of given native weight and given q.

The ordinary-degree inequality from Lemma 8.4 is $i+2j\le M-\epsilon-\eta = T$. Since $i+j=n$, it is equivalent to $0\le j\le\min\{n, T-n\}$. This proves (9.1). Its fixed part is $(n-\ell)Q_U$, and the remaining series is complete. Lemma 3.1 gives the means before restoring the residue factor. Using Lemma 2.14 and $y^3 = W = -U(U+V)$, $z^2 = V$, that factor contributes $\epsilon/3$ at Q_U, Q_{U+V} and $\eta/2$ at Q_V. Its class is $(2\epsilon/3+\eta/2)H_G$, proving the last assertion. The fixed degree plus the moving degree is $(n-\ell)+2\epsilon/3+\eta/2+\ell = n+2\epsilon/3+\eta/2$, as required by $\deg(H_G|_B)=1$. □

Example 9.5. Take $m=5$, $q=0$, $\epsilon=\eta=0$, $n=3$. Then $T=5$ and $\ell=2$. The space is $\langle U^3, U^2V, UV^2\rangle = UH^0(\mathbb{P}^1,\mathcal{O}(2))$. Its three orders at Q_U are $1,2,3$, with mean two. At any other point an adapted basis has orders $0,1,2$, with mean one. Thus the extra vanishing at Q_U comes from the explicit fixed factor, rather than from a separate decomposition on the threefold.

Lemma 9.6 (Positivity in type E_6)**.** *The two refined series have bounded support and contain ample series, so both inequalities of Proposition 2.7 apply.*

Proof. Use the native weight J and put $M = \min\{m, 2m - q\}$. Here the second support cone is

$$\mathcal{C}_E = \{(m, J, q) : m \geq 0,\ 0 \leq q \leq 2m,\ 12q \leq J \leq 12q + 6M\}. \tag{9.2}$$

Containment follows from the standard monomials. To prove equality and eventual nonvanishing, take an integral interior point (m_0, J_0, q_0) and write $M_0 = \min\{m_0, 2m_0 - q_0\}$. For each k, choose the unique pair $0 \leq \epsilon_k \leq 2$, $0 \leq \eta_k \leq 1$ such that $4\epsilon_k + 3\eta_k \equiv kJ_0 \pmod 6$. Set

$$n_k = \frac{kJ_0 - 12kq_0 - 4\epsilon_k - 3\eta_k}{6}, \qquad T_k = kM_0 - \epsilon_k - \eta_k.$$

The strict inequalities $0 < J_0 - 12q_0 < 6M_0$ imply that n_k and $T_k - n_k$ both grow linearly. Hence the interval in Lemma 9.4 is nonempty for every sufficiently large k. Density of rational interior rays proves (9.2).

The first support cone is its projection $0 \leq J \leq 24m$. For any interior ray in this projection, choose q_0 just below $J_0/12$ to lift it to an interior ray of (9.2), and approximate kq_0 by integers. The bounded rounding and residue errors preserve eventual nonvanishing, just as above. Thus both series have bounded support with nonempty interior and are eventually nonzero on every integral interior ray.

For an ample subseries on G_E, take $(m, J) = (12k, 12k)$. Every weighted polynomial of weight $12k$ has ordinary degree at most $4k \leq m$ and is a global initial form. These constitute $H^0(G_E, \mathcal{O}_{G_E}(2k))$, where $2H_G$ is ample Cartier. For the curve, take $(m, J, q) = (12k, 24k, k)$; then $\epsilon = \eta = 0$, $n = \ell = 2k$, and the series is exactly $H^0(B, \mathcal{O}_B(2k))$. Both rays are interior. Together with the plt assertions in Lemmas 9.1 and 9.2, this verifies all the hypotheses of Proposition 2.7. □

Lemma 9.7 (Limiting point orders in type E_6)**.** *One has $S(W^B; Q_U) = 5/16$ and $S(W^B; Q) = 5/32$ for every $Q \neq Q_U$.*

Proof. Rescale q, n by m and set $M(q) = \min\{1, 2 - q\}$. The limiting dimension function is $\ell(n, q) = \min\{n, M(q) - n\}$ for $0 \leq n \leq M(q)$. The normalizing integral is

$$I_E = \int_0^2 \int_0^{M(q)} \ell \, dn \, dq = \frac{1}{4} \int_0^2 M(q)^2 dq = \frac{1}{3}.$$

There are six residue classes, so the total dimension is $6I_E m^3 + O(m^2) = 2m^3 + O(m^2)$. The common factor six cancels in normalized means. The limiting mean order is $n - \ell/2$ at Q_U and $\ell/2$ at every other point. Splitting the inner integrals at $n = M/2$ gives

$$\int_0^M n\ell \, dn = \frac{M^3}{8}, \qquad \int_0^M \ell \left(n - \frac{\ell}{2}\right) dn = \frac{M^3}{12}, \qquad \int_0^M \frac{\ell^2}{2} \, dn = \frac{M^3}{24}.$$

For example, the middle integral is $\int_0^{M/2} n^2/2 \, dn + \int_{M/2}^M (M - n)(3n - M)/2 \, dn = M^3/12$. Multiplying the last two values by $\int_0^2 M(q)^3 dq = 5/4$ and dividing by $I_E = 1/3$ gives $5/16$ and $5/32$. The first gives the residual average $15/32$, agreeing with Lemma 9.3. Similarly, $S(W; B) = I_E^{-1} \frac{1}{4} \int_0^2 qM(q)^2 dq = 11/16$.

All residue corrections in Lemma 9.4 are bounded, uniformly in Q. Their total contribution is $O(m^3)$ by the same dimension estimate as in the proof of Lemma 8.20, so they disappear after division by mN_m. □

9.3 The E6 estimate

Lemma 9.8. *One has $\delta(G_E, \Delta_G; W) \geq 16/15$.*

Proof. For centres not contained in B, the residual-degree estimate and Lemmas 9.1 and 9.3 give $S(W;v) \le 2(15/32)A_{G_E,\Delta_G}(v) = (15/16)A_{G_E,\Delta_G}(v)$. For centres on B, apply the second refinement using Lemma 9.6. The curve term is $16/11$, and the point terms are

Q	$A_{B,\Delta_B}(Q)$	$S(W^B;Q)$	$A_{B,\Delta_B}(Q)/S(W^B;Q)$
Q_U	$1/3$	$5/16$	$16/15$
Q_{U+V}	$1/3$	$5/32$	$32/15$
Q_V	$1/2$	$5/32$	$16/5$
other points	1	$5/32$	$32/5$.

The smallest bound is $16/15$, which proves the assertion for every divisorial valuation on the surface. □

Proof of Theorem 7.3. The first refinement, Lemma 9.8, and the separate calculation of $S_X(G_E)$ give

$$\delta_P(X) \ge \min\left\{\frac{13}{177/16}, \frac{16}{15}\right\} = \min\left\{\frac{208}{177}, \frac{16}{15}\right\} = \frac{16}{15}.$$

□

Remark 9.9. The two independent routes to the first expected vanishing agree: $2(95/96) + 8(11/16) = 359/48$ and $6(15/32) + 12(11/16) = 177/16$. The left sides use the curve-series distributions; Lemma 8.9 uses only the weighted Hilbert function and the weight of the relation. This agreement checks the normalization of the native valuations against that of the coarse exceptional surfaces. Throughout both proofs, all new expected vanishing orders and fixed factors were obtained from actual graded section spaces, without a Zariski decomposition on the weighted threefold blow-ups.

Proof of Theorem 1.1 for Family 2.5. Suppose that X is not K-stable and choose F, P as in Lemma 7.1. Then $\delta_P(X) \le A_X(F)/S_X(F) \le 1$. This contradicts Theorem 7.2 or Theorem 7.3. □

Part III
Family 2.9: conic bundles

10 Geometry and the local estimates

Let $C \subset \mathbb{P}^3$ be a smooth curve of degree 7 and genus 5, scheme-theoretically cut out by three cubics, and let $\pi : X = \mathrm{Bl}_C \mathbb{P}^3 \to \mathbb{P}^3$. These are the smooth members of Mori–Mukai Family 2.9; see [3, Sections 5.2 and 6]. Write E for the exceptional divisor and $H = \pi^*\mathcal{O}_{\mathbb{P}^3}(1)$. Set

$$B = 3H - E, \qquad L = -K_X = 4H - E = H + B.$$

The net $|B|$ gives the conic bundle $\varphi : X \to \mathbb{P}^2$. One has $L^3 = 16$.

Theorem 10.1. *For every smooth member of Family 2.9,*

$$S_X(H) = \frac{25}{64}, \qquad S_X(B) = \frac{91}{192}, \qquad S_X(E) = \frac{65}{432}.$$

Every prime divisor D on X satisfies $\beta_X(D) > 0$.

There is a point $p_0 \in C$ determined by C itself: it is the common zero of the three linear forms in the linear syzygy between its cubic generators. The syzygy is unique up to a scalar, and the span of its three coefficients is independent of the generators, as proved in Lemma 10.4. This is why we call p_0 distinguished. Put $f_0 = \pi^{-1}(p_0) \cong \mathbb{P}^1$. Let $\Sigma \subset \mathbb{P}^2$ be the finite set of points whose conic fibres are nonreduced, and set

$$T = f_0 \cup \bigcup_{y \in \Sigma} \varphi^{-1}(y)_{\mathrm{red}}.$$

Proposition 10.2. *If P belongs to a reduced singular conic fibre, then $\delta_P(X) \geq 96/95$. If $P \notin f_0$ belongs to a smooth conic fibre, then $\delta_P(X) \geq 96/91$. If $P \in f_0$ belongs to a smooth conic fibre, then $\delta_P(X) \geq 512/453$. Consequently, every point on a reduced conic fibre satisfies $\delta_P(X) \geq 96/95$. If P belongs to a double conic fibre, then $\delta_P(X) \geq 128/123$. These estimates prove the Family 2.9 assertion of Theorem 1.1. At a point of f_0 on a smooth conic fibre, Proposition 12.4 gives the stronger bound $\delta_P(X) \geq 6912/5335$.*

The proof proceeds as follows. Section 12 treats every reduced conic fibre, using a curve flag or the blow-up of a point on a smooth cubic surface. Sections 14–17 give an independent proof at the smooth-fibre points of f_0 by a threefold point extraction; the complete transverse-position decomposition is included as an additional calculation. Section 18 treats double conic fibres by a weighted curve extraction and completes the proof. In the two exceptional-surface arguments, necessary vanishing conditions suffice to bound the refined series. We do not assume completeness of those series, or a threefold Zariski decomposition that has not been proved.

10.1 The two contractions and the distinguished point

Lemma 10.3. *A general cubic surface S_0 containing C is smooth. For each fixed smooth such surface, writing $A = -K_{S_0}$, there is a unique line $\ell = \ell_{S_0} \subset S_0$ such that*

$$C \sim 2A + \ell, \qquad A^2 = 3, \quad A \cdot \ell = 1, \quad \ell^2 = -1.$$

The strict transform $S \subset X$ is isomorphic to S_0. Identifying A and ℓ with their pullbacks to S and putting $C_S = (\pi|_S)^{-1}(C)$, one has

$$H|_S = A, \qquad E|_S = C_S \sim 2A + \ell, \qquad B|_S = F := A - \ell.$$

The pencil $|F|$ is the conic bundle on S obtained by restricting φ. The line ℓ_{S_0} varies with S_0; their images in $\mathbb{P}^3$ all pass through the fixed point p_0, as shown in the proof of Lemma 10.6.

Proof. The cubic equations generate the ideal sheaf of C. At a point of C, their differentials span its two-dimensional conormal space. The cubics singular at that point form one point of the parameter plane. Varying the point of C gives a subset of dimension at most one. Bertini applies away from C, so a general cubic in the net is smooth.

Adjunction on S_0 gives $C^2 = 2g(C) - 2 + A \cdot C = 15$. Thus $D = C - 2A$ satisfies $A \cdot D = 1$ and $D^2 = -1$. Riemann–Roch gives $\chi(\mathcal{O}_{S_0}(D)) = 1$, while $h^2(D) = h^0(-A - D) = 0$ by ampleness of A. Hence D is effective. Its A-degree is one, so it is a line. Two distinct lines have nonnegative intersection and cannot both represent a class of square -1, proving uniqueness.

The blow-up of the Cartier divisor $C \subset S_0$ is S_0 itself. The restriction formulas follow. The planes through ℓ cut out ℓ plus a residual conic, giving the base point free pencil $|A - \ell|$. □

Lemma 10.4 (The linear syzygy)**.** *There are independent linear forms l_0, l_1, l_2 and quadratic forms q_0, q_1, q_2 such that the cubic generators are the components of*

$$f = l \times q, \qquad \begin{pmatrix} l_0 & q_0 \\ l_1 & q_1 \\ l_2 & q_2 \end{pmatrix}.$$

The ideal has resolution

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-4) \oplus \mathcal{O}_{\mathbb{P}^3}(-5) \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-3)^{\oplus 3} \longrightarrow \mathcal{I}_C \longrightarrow 0.$$

The space of linear syzygies is one dimensional. The three-dimensional subspace

$$U_C = \langle l_0, l_1, l_2 \rangle \subset H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$$

depends only on C, and not on the choice of cubic generators or of their syzygy. Its common zero is a single point $p_0 \in C$, and $q(p_0) \neq 0$.

Proof. We include the algebra behind this familiar determinantal description. On a smooth cubic from Lemma 10.3, the hypersurface sequence reduces the vanishing of $H^1(\mathcal{I}_C(n))$ to that of $H^1(S_0, tA - \ell)$ for every integer t. For $t \geq 0$, use $0 \to \mathcal{O}(tA - \ell) \to \mathcal{O}(tA) \to \mathcal{O}_\ell(t) \to 0$; the restriction of degree-t polynomials to a line is surjective and $H^1(tA) = 0$. For $t < 0$, Serre duality reduces the assertion to $H^1(kA + \ell) = 0$, $k = -t - 1 \geq 0$, and the sequence with quotient $\mathcal{O}_\ell(k - 1)$ gives this vanishing. Thus C is arithmetically Cohen–Macaulay.

For completeness, the three given cubic equations generate the ideal sheaf, so their kernel $\mathcal{K}$ is locally free of rank two. The preceding vanishing gives $H^2(\mathcal{K}(n)) = 0$ for every n. Since $\det \mathcal{K} = \mathcal{O}(-9)$, rank-two duality and Serre duality give $H^1(\mathcal{K}(n)) = 0$ as well. Horrocks' splitting criterion [19] therefore writes $\mathcal{K} = \mathcal{O}(-a) \oplus \mathcal{O}(-b)$. There is no constant relation among the three cubics, and $a + b = 9$, so $\{a, b\} = \{4, 5\}$. The Hilbert–Burch theorem [16] now gives the displayed matrix and its minors.

If the linear column had rank two, coordinates would make it $(x, y, 0)$. Its minors would be $yq_2, -xq_2, xq_1 - yq_0$. Unless $q_2 = 0$, irreducibility and nonplanarity of C would force $C \subset (q_2 = 0)$. But C lies on no quadric: on S_0, such a quadric would give an effective divisor $2A - C \sim -\ell$. If $q_2 = 0$, the minors do not have codimension two. Rank at most one similarly forces a planar component or the wrong codimension. Thus the linear forms are independent. They vanish at a point p_0, which belongs to the determinantal curve. If also $q(p_0) = 0$, all the minors have zero differential there, contradicting smoothness of C.

Twisting the resolution by 4 identifies the space of linear syzygies with $H^0(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(-1)) = \mathbb{C}$, so its nonzero element is unique up to a scalar. To check independence of the cubic generators,

regard f and l as columns, so that the relation is ${}^{\mathrm{t}}lf = 0$. If $f' = Mf$ for $M \in \mathrm{GL}_3(\mathbb{C})$, its linear syzygy is a nonzero scalar multiple of ${}^{-\mathrm{t}}l$. Invertible recombination and multiplication by a nonzero scalar preserve the span of the three coefficients. Hence U_C and its common zero p_0 are intrinsic to C. In particular, their construction does not involve the choice of a cubic surface through C. □

Lemma 10.5 (The incidence model)**.** *The morphism (π, φ) identifies X with the complete intersection*

$$\sum_{i=0}^{2} l_i(x)y_i = 0, \qquad \sum_{i=0}^{2} q_i(x)y_i = 0 \quad \text{in } \mathbb{P}^3_x \times \mathbb{P}^2_y. \tag{10.1}$$

The divisors H, B are the restrictions of $\mathcal{O}(1,0), \mathcal{O}(0,1)$, respectively. The first equation defines a plane bundle over $\mathbb{P}^2_y$, and φ is a flat family of nonzero plane conics in this bundle.

Proof. Off C, the two equations specify the point $y = l \times q$. Along C, their coefficient matrix has rank one: rank zero would make the differentials of all its minors vanish, contradicting smoothness of C. Near any point of C, a unit matrix entry allows row and column operations reducing the incidence equations to the usual equation for the blow-up of a two-generated ideal defining a smooth codimension-two centre. Thus (10.1) is the graph of the cubic net, including over C. The restriction formulas follow.

For each y, the linear form $l_y = \sum l_i y_i$ is nonzero because the l_i are independent. Its zero plane is denoted by Π_y. The quadratic $q_y = \sum q_i y_i$ does not vanish identically on Π_y. Otherwise, after an invertible linear recombination of the three rows, $l_y = l_0$ and $q_y = q_0 = l_0 r$ for a linear form r. On Π_y, two minors then vanish identically and the remaining one is a cubic. Their common zero scheme contains a plane curve, contradicting the fact that C is irreducible and nonplanar. Hence the second equation cuts out a nonzero conic on every plane fibre. It is a relative effective Cartier divisor of degree two in the plane bundle, and its equation remains a nonzerodivisor on every fibre. This proves flatness. □

Lemma 10.6 (Smooth members of a constrained pencil)**.** *Let $y \in \mathbb{P}^2$ have reduced conic fibre. A general line $M \subset \mathbb{P}^2$ through y has the following properties: $S = \varphi^{-1}(M)$ is a smooth cubic surface under π, and the image $\pi(\ell)$ of the line in Lemma 10.3 passes through p_0. If $P \in \varphi^{-1}(y)$ and $\pi(P) \neq p_0$, M can be chosen so that $P \notin \ell$.*

Proof. Write a cubic of the net as $S_a = (a \cdot f = 0)$, where $a \in \mathbb{P}^2$ is a coefficient vector. The condition that it contains the conic over y is $a \cdot y = 0$, a line in the parameter plane. For $c \in C \setminus \{p_0\}$, the relation $q(c) \parallel l(c)$ and the rank-two Jacobian of C show that the unique cubic singular at c has coefficient $a = l(c)$. At p_0 its coefficient is $q(p_0)$. This defines the extension of the projection $c \mapsto l(c)$. Its image is not a line, since otherwise C would lie in a plane through p_0. Hence only finitely many members of our parameter line are singular along C.

On X, Bertini makes a general member of the pencil smooth away from its base fibre. Along a smooth point of that fibre, φ is smooth. If the fibre is a union of two lines, it has one node; locally the conic equation has a nonzero derivative in a base direction there, since X is smooth. Choosing M transverse to that direction makes its inverse image smooth at the node. Thus a general S is smooth, and the preceding paragraph ensures that its cubic image is smooth along C too.

For $a = [a_0 : a_1 : a_2]$, let $\ell_a \subset \mathbb{P}^3$ be the line defined by $l_i(x)a_j - l_j(x)a_i = 0$ for $0 \leq i < j \leq 2$. Independence of the l_i shows that this is a line through p_0. Its equations imply $a \cdot (l \times q) = 0$, so it lies in S_a. If S_a is smooth, this line cannot meet C away from p_0, for such an intersection would give $a = l(c)$ and make S_a singular at c. At p_0 it is transverse to C: the tangent direction of C in the l-coordinates is $q(p_0)$, and $a \not\parallel q(p_0)$. Its intersection with C is therefore one. It is the line $\ell = \ell_{S_a}$ of Lemma 10.3, since any other line has intersection at

least two with $2A + \ell$. Distinct coefficient vectors $a \in \mathbb{P}^2$ give distinct lines ℓ_a: projection from p_0 sends $\ell_a \setminus \{p_0\}$ to a. Thus the line depends on the chosen cubic S_a, whereas their common point p_0 depends only on C.

If $P \notin E$ and $\pi(P) \neq p_0$, at most one coefficient a, namely $a = l(\pi(P))$, makes ℓ pass through P. Avoid that coefficient. If $P \in E$ lies above $c \neq p_0$, the line ℓ avoids P automatically, as it meets C only at p_0. □

Lemma 10.7. *The set Σ of nonreduced conic fibres is finite.*

Proof. A nonreduced plane conic is a double line. If the rank-one locus of the conic equation contained a curve in the base, work at its general point and write the equation as $x^2 + uq(y, z) +$ higher terms in the transverse base parameter u. Differentiation in the direction tangent to the rank-one locus vanishes along $x = u = 0$. The binary quadratic q has a zero on the double line. At that zero the other base derivative vanishes as well, making the total space singular. This contradicts smoothness of X. The rank-one locus is closed and hence is finite on the projective surface $\mathbb{P}^2$. □

11 Expected vanishing orders on the threefold

We use the invariants defined in Section 2, with $L^3 = 16$. The distinction between divisors on X and divisors over X is retained throughout.

Lemma 11.1. *One has*

$$H^3 = 1, \quad H^2E = 0, \quad HE^2 = -7, \quad E^3 = -36, \qquad H^2B = 3, \quad HB^2 = 2, \quad B^3 = 0.$$

The nef cone is generated by H, B, and the effective cone by E, B.

Proof. The blow-up formulas give the first three intersections. The degree of $N_{C/\mathbb{P}^3}$ is $4 \deg C + 2g(C) - 2 = 36$, giving $E^3 = -36$. Substituting $B = 3H - E$ gives the remaining intersections and $(H + B)^3 = 16$.

Both H and B are nef. If f is a fibre of $E \to C$, then $Hf = 0$ and $Bf = 1$. If z is a general conic fibre, then $Hz = 2$ and $Bz = 0$. As the Picard rank is two, these tests give the asserted nef cone.

Let $D \neq E$ be a prime divisor. It is the strict transform of a surface of degree $a \geq 1$ with multiplicity $b \geq 0$ along C, so $D \sim aH - bE$. Restrict to a general smooth cubic S not contained in D. The restriction is effective, and its intersection with the nef conic class F is $2a - 6b$. Thus $a \geq 3b$. It follows that $D = (a - 3b)H + bB$, with nonnegative coefficients. Since $H = (E + B)/3$, all effective classes lie in the cone generated by E, B, and both generators are effective. □

Lemma 11.2 (The three volume integrals)**.** *The volume functions needed in Theorem 10.1 are*

$$\begin{aligned}
\operatorname{vol}(L - uH) &= (1-u)^3 + 9(1-u)^2 + 6(1-u), && 0 \leq u \leq 1,\\
\operatorname{vol}(L - uE) &= 16 - 60u + 24u^2 + 36u^3, && 0 \leq u \leq \tfrac{1}{3},\\
\operatorname{vol}(L - uB) &= \begin{cases} 1 + 9(1-u) + 6(1-u)^2, & 0 \leq u \leq 1,\\ (4-3u)^3, & 1 \leq u \leq \frac{4}{3}. \end{cases}
\end{aligned}$$

They vanish beyond the displayed intervals.

Proof. The effective cone gives the thresholds $1, 1/3, 4/3$, respectively. Up to the first two thresholds the divisors are nef:

$$L - uH = (1-u)H + B, \qquad L - uE = (1-3u)H + (1+u)B.$$

Likewise $L - uB = H + (1-u)B$ is nef for $0 \le u \le 1$. The cube of $aH + bB$ is $a^3 + 9a^2b + 6ab^2$.

For $1 \le u \le 4/3$, write

$$L - uB = (4 - 3u)H + (u - 1)E.$$

The E term is fixed: restriction to a general fibre of $E \to C$ forces its coefficient, and the remaining complete series is pulled back from $\mathbb{P}^3$. Equivalently, $\pi_* \mathcal{O}_X(kE) = \mathcal{O}_{\mathbb{P}^3}$ for $k \ge 0$. The volume is therefore $(4 - 3u)^3$. □

Proof of Theorem 10.1. Integration of Lemma 11.2, with denominator 16, gives

$$S_X(H) = \frac{1}{16}\left(\frac{1}{4} + 3 + 3\right) = \frac{25}{64}, \quad S_X(B) = \frac{1}{16}\left(\frac{15}{2} + \frac{1}{12}\right) = \frac{91}{192},$$

$$S_X(E) = \frac{1}{16}\left(\frac{16}{3} - \frac{10}{3} + \frac{8}{27} + \frac{1}{9}\right) = \frac{65}{432}.$$

If $D \neq E$ is prime, put $k = a - 3b$ as in Lemma 11.1. The integers k, b are nonnegative and not both zero. If $k \ge 1$, then $D - H$ is effective and $S_X(D) \le S_X(H)$. If $k = 0$, then $b \ge 1$, $D - B$ is effective and $S_X(D) \le S_X(B)$. These inequalities follow from monotonicity of volume. Since every prime divisor on the smooth threefold has discrepancy one, all these beta invariants are positive, including that of E. □

12 Refining to a cubic surface and then to a curve

Choose a smooth cubic $S \in |B|$ as above and put $C_S = E|_S$. This is a curve on $S \subset X$; the isomorphism $\pi|_S : S \to S_0$ maps it isomorphically to the original curve $C \subset \mathbb{P}^3$. All fixed divisors and local intersections in this section are taken on S. The refinement of the anticanonical series is $W_{m,j} = \mathrm{gr}_S^j H^0(X, mL)$. Its dimension in fixed degree, summed over j, is $h^0(X, mL) = 16m^3/6 + O(m^2)$. Dividing a section by its vanishing factor along S and restricting to S realizes this series geometrically.

For $u = j/m$, its moving and fixed classes on S are

u	D_u	$n_u C_S$
$[0, 1]$	$A + (1-u)F$	0
$[1, 4/3]$	$(4 - 3u)A$	$(u-1)C_S$.

(12.1)

On rational interior rays, these are asymptotically complete series with the stated fixed factor. For $u < 1$, $L - uB$ is ample, and Serre vanishing makes restriction to S surjective in large divisible degree. For $u > 1$, remove the fixed exceptional divisor and use surjectivity of restriction of polynomials in $\mathbb{P}^3$ to its cubic surface. Endpoint and rounding terms do not affect the limits.

Let $Z \subset S$ be a smooth rational curve that is not a component of C_S. This condition holds for the conics and lines used below, since C_S is the smooth irreducible curve of genus 5. Denote by $P_{u,v} + N_{u,v}$ the surface Zariski decomposition of $D_u - vZ$, and by $\tau(u)$ its pseudo-effective threshold. Thus $P_{u,v}$ is nef, $N_{u,v}$ is effective with negative definite intersection matrix, and $P_{u,v}$ has zero intersection with every component of $N_{u,v}$. Put $d(u, v) = P_{u,v} \cdot Z$.

Lemma 12.1 (The flag inequality and its normalization)**.** *For $P \in Z$, set*

$$\mathcal{S}_Z = \frac{3}{16} \int_0^{4/3} \int_0^{\tau(u)} P_{u,v}^2 \, dv \, du, \tag{12.2}$$

$$\mathcal{T}_P = \frac{3}{16} \int_0^{4/3} \int_0^{\tau(u)} d(u,v)^2 \, dv \, du + \frac{6}{16} \int_0^{4/3} \int_0^{\tau(u)} d(u,v) \, \mathrm{ord}_P\big((n_u C_S + N_{u,v})|_Z\big) \, dv \, du. \tag{12.3}$$

For the decompositions used below,

$$\delta_P(X) \geq \min\left\{\frac{192}{91}, \frac{1}{\mathcal{S}_Z}, \frac{1}{\mathcal{T}_P}\right\}.$$

Proof. This is the smooth surface and smooth curve case of Abban–Zhuang refinement [4, Theorem 3.3]; the surface-volume formulation is given in [17, Theorems 4.8 and 4.17], with the numbering of the cited arXiv version. The different on Z is zero. Thus the three discrepancies in the flag are one. The first expected vanishing is $S_X(S) = 91/192$.

We explain the constants to prevent a normalization error. For fixed u, the surface series has asymptotic dimension $m^2D_u^2/2$. This gives the coefficient $3/16$ in (12.2). After the second refinement, a weight summand has dimension $md(u,v) + O(1)$. On $Z \cong \mathbb{P}^1$ its moving point orders have mean $md(u,v)/2 + O(1)$; a fixed divisor adds its local coefficient to that mean. Dividing by $mh^0(X, mL) = 16m^4/6 + O(m^3)$ gives coefficients $3/16$ for the square term and $6/16$ for the fixed term. In particular, the square term must not be divided by two again.

The explicit decompositions below have bounded support. At an interior rational point of a chamber the positive part is big and nef, its restricted degree is positive, and the restricted series has dimension $md + O(1)$. Thus interior rays are eventually nonzero. Complete ample subseries occur, for example, at $u = 1/2$ and sufficiently small positive v, where the positive part is ample and restriction is surjective in large degree. These verify the positivity assumptions of refinement. Alternatively they follow from [4, Example 2.15]. The bounded errors and finitely many linear chamber walls disappear in the normalized integrals. □

In both calculations below, the contribution of the original fixed curve C_S to $\mathcal{T}_P$ is

$$\frac{3}{16}\int_1^{4/3}(u-1)\big((4-3u)A\big)^2 du \cdot (C_S \cdot Z)_P = \frac{(C_S \cdot Z)_P}{192}. \tag{12.4}$$

Indeed, $\int_0^{\tau(u)} d(u,v)\,dv = D_u^2/2$, by differentiating the volume, and $\int_1^{4/3}(u-1)(4-3u)^2du = 1/108$. Here $(C_S \cdot Z)_P$ denotes the local intersection multiplicity, and is zero when $P \notin C_S$.

12.1 A smooth conic fibre

Lemma 12.2 (Decomposition along a smooth fibre)**.** *Suppose $Z \in |F|$ is smooth. For $0 \leq u \leq 1$, put $s = 1 - u$. The decomposition of $A + sF - vF$ is*

v	$P_{u,v}$	$N_{u,v}$
$[0, s + 1/2]$	$A + (s - v)F$	0
$[s + 1/2, s + 1]$	$(1 + s - v)(A + \ell)$	$(2v - 2s - 1)\ell$.

The threshold is $s + 1$. For $1 \leq u \leq 4/3$, put $a = 4 - 3u$ and scale the $s = 0$ decomposition by a, with $v = at$.

Proof. The classes F and $A+\ell$ are nef. For the latter, intersection with ℓ is zero and intersection with any other irreducible curve is positive. For $0 \leq t \leq 1/2$, $A - tF = (1 - 2t)A + t(A + \ell)$ is nef. For $1/2 \leq t \leq 1$,

$$A - tF = (1 - t)(A + \ell) + (2t - 1)\ell,$$

which has the required orthogonality and negative part. For $t > 1$ its intersection with the nef class $A + \ell$ is negative, so it is not pseudo-effective. Substitute $t = v - s$ and include the initial interval $v \leq s$. Scaling proves the last assertion. □

Lemma 12.3 (The smooth-fibre integrals)**.** *If $P \in Z$ and Z is a smooth conic fibre, put $r = (\ell \cdot Z)_P$ and $\mu = (C_S \cdot Z)_P$. Then*

$$\mathcal{S}_Z = \frac{247}{384}, \qquad \mathcal{T}_P = \frac{176 + 13r + \mu}{192}.$$

In particular, if $P \notin \ell$, then $\delta_P(X) \geq 96/91$. If $r = 1$ and $\mu \leq 2$, then $\delta_P(X) \geq 192/191$.

Proof. For the first range of u, the integrals from Lemma 12.2 are

$$\begin{aligned}\int P_{u,v}^2 dv &= \int_0^s (3 + 4(s-v))dv + \int_0^{1/2} (3-4t)dt + \int_{1/2}^1 4(1-t)^2 dt \\ &= 3s + 2s^2 + \frac{7}{6}, \\ \int d(u,v)^2 dv &= \int_0^{s+1/2} 4\, dv + \int_{1/2}^1 16(1-t)^2 dt = 4s + \frac{8}{3}, \\ \int d(u,v)\operatorname{coeff}_\ell N_{u,v}\, dv &= \int_{1/2}^1 4(1-t)(2t-1)dt = \frac{1}{6}.\end{aligned}$$

For the second range these three expressions become, respectively, $7a^3/6$, $8a^3/3$, and $a^3/6$. Since $\int_1^{4/3} a^3 du = 1/12$, we obtain

$$\mathcal{S}_Z = \frac{3}{16}\left(\frac{10}{3} + \frac{7}{72}\right) = \frac{247}{384}, \qquad \frac{3}{16}\int\!\!\int d^2 = \frac{3}{16}\left(\frac{14}{3} + \frac{2}{9}\right) = \frac{11}{12},$$

$$\frac{6}{16}\int\!\!\int d \operatorname{coeff}_\ell N = \frac{6}{16}\left(\frac{1}{6} + \frac{1}{72}\right) = \frac{13}{192}.$$

Adding (12.4) proves the formula. As $C_S \cdot Z = (2A + \ell)F = 6$, one has $\mu \leq 6$. When $r = 0$, the point term is at most $91/96$, and the other two ratios are larger than $96/91$. When $r = 1$, $\mu \leq 2$, the point term is at most $191/192$ and the other ratios are again larger. □

12.2 Blowing up the point on the cubic surface

The points of f_0 on smooth conic fibres can also be treated by blowing up the point on the cubic surface. This gives a stronger bound and completes the smooth-fibre case without a threefold point extraction. We retain the latter calculations below as separate applications of refinement to an exceptional plane.

Proposition 12.4. *If $p \in f_0$ lies on a smooth conic fibre, then*

$$\delta_p(X) \geq \frac{6912}{5335} > 1.$$

Proof. Put $Z = \varphi^{-1}(\varphi(p))$. Choose a smooth cubic surface $S \in |B|$ containing Z as in Lemma 10.6. Its line ℓ passes through p, and we may choose S so that ℓ and Z are transverse there. Indeed, under $S \cong S_0$, the point p is p_0, the curve Z has a fixed smooth conic image, and the direction of ℓ varies in the pencil of lines through p_0 in the plane of that conic. At most one direction is tangent to Z, and only finitely many choices give a singular cubic. Write $C_S = E|_S$. This is a smooth curve, and $C_S \cdot \ell = 1$, so C_S and ℓ are transverse at p. We make no transversality assumption between C_S and Z.

Let $\sigma : \widehat{S} \to S$ be the blow-up of p, with exceptional curve J. Pullbacks of divisor classes on S will be written without σ^*. Put $R = \ell - J$ and $\widehat{Z} = F - J$. These are the strict transforms of ℓ and Z, respectively, and

$$R^2 = -2, \qquad \widehat{Z}^2 = -1, \qquad R \cdot \widehat{Z} = R \cdot J = \widehat{Z} \cdot J = 1.$$

The points $q_R = R \cap J$ and $q_Z = \widehat{Z} \cap J$ are distinct. Also $\sigma^* C_S = \widehat{C}_S + J$, where $\widehat{C}_S$ meets J transversely at a point $q_C \neq q_R$. The equality $q_C = q_Z$ is allowed; it occurs when C_S is tangent to Z at p.

For $0 \le u \le 1$, put $s = 1-u$. We compute the Zariski decomposition $D_{s,t} = A + sF - tJ = P_{s,t} + N_{s,t}$. Its threshold is $2+s$, and its negative part is as follows:

s	t	$N_{s,t}$
$[0, \frac{1}{2}]$	$[0, 1+2s]$	0
$[0, \frac{1}{2}]$	$[1+2s, (5+2s)/3]$	$\frac{t-1-2s}{2} R$
$[0, \frac{1}{2}]$	$[(5+2s)/3, 2+s]$	$(2t-3-2s)R + (3t-5-2s)\widehat{Z}$
$[\frac{1}{2}, 1]$	$[0, 2]$	0
$[\frac{1}{2}, 1]$	$[2, s+\frac{3}{2}]$	$(t-2)\widehat{Z}$
$[\frac{1}{2}, 1]$	$[s+\frac{3}{2}, 2+s]$	$(2t-3-2s)R + (3t-5-2s)\widehat{Z}$.

Here $P_{s,t} = D_{s,t} - N_{s,t}$. We verify nefness directly. Since the plane containing ℓ and Z cuts out $\ell + Z \sim A$, any irreducible curve $\Gamma \neq \ell, Z$ on S satisfies $A \cdot \Gamma \ge 2\,\mathrm{mult}_p\,\Gamma$, and $F \cdot \Gamma \ge 0$. These inequalities prove nefness in the rows with $N = 0$. In the rows with $N = aR$, write $P = (1-a)A + (s+a)F - (t-a)J$ and use $a \ge 0$, $1-a \ge 0$ and $2-t-a \ge 0$. In the rows with $N = (t-2)\widehat{Z}$, one has $P = A + (s+2-t)F - 2J$, which is nef by the same inequalities. In the remaining rows,

$$P_{s,t} = (2+s-t)Q, \qquad Q = 2A + F - 4J.$$

The same argument shows that Q is nef on the strict transform of any $\Gamma \neq \ell, Z$, while $Q \cdot R = Q \cdot \widehat{Z} = 0$ and $Q \cdot J = 4$ and $Q^2 = 4$. The intersections with $R, \widehat{Z}, J$ in the other rows are nonnegative as well. The supports of the displayed negative parts have negative definite intersection matrices, and direct intersection gives the required orthogonality. Finally, $Q \cdot D_{s,t} = 4(2+s-t)$ excludes pseudo-effectivity for $t > 2+s$; at $t = 2+s$ one has $D_{s,t} = R + (1+s)\widehat{Z}$. This proves both the decompositions and the thresholds.

Put $d_{s,t} = P_{s,t} \cdot J$ and write $N_{s,t} = a_{s,t}R + b_{s,t}\widehat{Z}$, so $d_{s,t} = t - a_{s,t} - b_{s,t}$. Integration of the displayed linear expressions gives

	$\int P_{s,t}^2\,dt$	$\int d_{s,t}^2\,dt$	$\int 2d_{s,t}a_{s,t}\,dt$	$\int 2d_{s,t}b_{s,t}\,dt$
integrated over $0 \le s \le 1$	$1105/144$	$241/72$	$10/27$	$263/432$
$s = 0$	$32/9$	$13/9$	$14/27$	$4/27$.

In every entry the inner integral is over $0 \le t \le 2+s$. For $1 \le u \le 4/3$, put $a = 4-3u$. The moving class is aA, so the $s=0$ calculation scales by a^3; its integral over this range is $1/12$ times the second row. Thus, with the normalization $3/16$ of Lemma 12.1, the moving contributions to the expected order along J, the final square term on J, and the two fixed point terms at q_R, q_Z are respectively

$$\frac{3443}{2304}, \qquad \frac{749}{1152}, \qquad \frac{67}{864}, \qquad \frac{805}{6912}.$$

The original fixed part $(u-1)C_S$ for $u \ge 1$ must also be included. Its exceptional multiplicity adds

$$\frac{3}{16}\int_1^{4/3} (u-1)\big((4-3u)A\big)^2\,du = \frac{1}{192}$$

to the expected order along J. Its strict transform adds the same quantity at q_C on J, and adds nothing at the other points. This uses $\widehat{C}_S|_J = q_C$, so it remains valid for every possible contact order between C_S and Z.

Let W be the surface refinement of the anticanonical series along S, and let W^J be its further refinement along J. The surface refinement formulas (4.3) and (4.4), applied on $\widehat{S}$ along J with the actual pulled-back fixed divisors, give

$$S(W;J) = \frac{3455}{2304}, \qquad S(W^J;q) = \frac{749}{1152} + \frac{67}{864}\mathbf{1}_{q=q_R} + \frac{805}{6912}\mathbf{1}_{q=q_Z} + \frac{1}{192}\mathbf{1}_{q=q_C} \le \frac{5335}{6912}.$$

Here $A_S(J) = 2$ and the different on $J \cong \mathbb{P}^1$ is zero. The original surface term is $S_X(S) = 91/192$. Applying refinement first along S and then along J gives

$$\delta_p(X) \geq \min\left\{\frac{192}{91}, \frac{4608}{3455}, \frac{6912}{5335}\right\} = \frac{6912}{5335}.$$

The restricted series have bounded support and contain ample subseries by Lemma 12.1; these properties persist under the point extraction. On the interiors of the displayed chambers, surface Zariski decomposition and restriction to J compute the leading dimensions of the actual refined spaces. The finitely many walls and degree-rounding errors do not affect the integrals. Thus the calculation applies to the original anticanonical series. □

12.3 A reduced singular conic fibre

Suppose the fibre is $Z + Z'$, where Z, Z' are its two line components on the smooth cubic surface. They satisfy

$$Z^2 = (Z')^2 = -1, \quad Z \cdot Z' = 1, \quad F \cdot Z = F \cdot Z' = 0, \quad \ell \cdot Z = \ell \cdot Z' = 1.$$

Both components are smooth, including at their transverse intersection.

Lemma 12.5 (Decomposition along a line component)**.** *For $0 \leq u \leq 1$, put $s = 1 - u$. The decomposition of $A + sF - vZ$ is*

v	$P_{u,v}$	$N_{u,v}$
$[0,1]$	$A + sF - vZ$	0
$[1, 1+s]$	$(A - Z) + (1 + s - v)F$	$(v-1)Z'$.

The threshold is $1+s$. For $1 \leq u \leq 4/3$, with $a = 4-3u$, the class $aA - vZ$ is nef for $0 \leq v \leq a$, and a is its threshold.

Proof. For $v \leq 1$, write the divisor as $(1-v)A + v(A - Z) + sF$. The three classes are nef; $A - Z$ is the conic pencil obtained from planes through Z. In the second interval the displayed positive part is nef and orthogonal to Z', so it is a Zariski decomposition. The intersection of the original divisor with $A - Z$ is $2(1 + s - v)$, proving that the threshold cannot be larger. Scaling the case $s = 0$ proves the last assertion. □

Lemma 12.6 (The line-component integrals)**.** *For $P \in Z$, let $\epsilon = 1$ when $P = Z \cap Z'$ and $\epsilon = 0$ otherwise, and put $\mu = (C_S \cdot Z)_P$. Then*

$$\mathcal{S}_Z = \frac{161}{192}, \qquad \mathcal{T}_P = \frac{163 + 24\epsilon + \mu}{192}.$$

Consequently $\delta_P(X) \geq 96/95$.

Proof. For $0 \leq v \leq 1$, the volume is $3 + 4s - 2v - v^2$ and $d(u,v) = 1 + v$. For $1 \leq v \leq 1 + s$, the volume is $4(1 + s - v)$ and $d(u,v) = 2$. Hence

$$\int P_{u,v}^2 dv = \frac{5}{3} + 4s + 2s^2,$$
$$\int d(u,v)^2 dv = \frac{7}{3} + 4s,$$
$$\int d(u,v)\,\mathrm{coeff}_{Z'} N_{u,v} dv = s^2.$$

For the second range of u, the corresponding quantities are $5a^3/3$, $7a^3/3$, and zero. Therefore

$$\mathcal{S}_Z = \frac{3}{16}\left(\frac{13}{3}+\frac{5}{36}\right) = \frac{161}{192}, \qquad \frac{3}{16}\int\int d^2 = \frac{3}{16}\left(\frac{13}{3}+\frac{7}{36}\right) = \frac{163}{192},$$

while the fixed Z' contribution is $(6/16)\int_0^1 s^2 ds = 1/8$. Add (12.4). Finally $C_S \cdot Z = 2A \cdot Z + \ell \cdot Z = 3$, so $\mu \le 3$ and $\mathcal{T}_P \le 190/192 = 95/96$. Both $192/91$ and $192/161$ exceed $96/95$, which proves the assertion by Lemma 12.1. □

The initial local estimates. For a point of a reduced fibre, choose S by Lemma 10.6. If the fibre is singular, choose a component Z containing the point and apply Lemma 12.6. If it is smooth and the point is outside f_0, choose S so that its distinguished line avoids the point and apply Lemma 12.3. If the smooth-fibre point belongs to f_0, apply Proposition 12.4.

Now let F be a divisor over X with $\beta_X(F) \le 0$. Every point P of its centre satisfies $\delta_P(X) \le A_X(F)/S_X(F) \le 1$. These three estimates exclude every point on a reduced conic fibre. Thus the centre is contained in the union of the reduced supports of the double conic fibres. Their estimate is proved in Theorem 18.10. Theorem 17.7 gives an independent, weaker estimate at the smooth-fibre points of f_0 by a threefold point extraction. □

13 The limitations of the fibre curve flag

The curve flag $S \supset Z \supset P$ used in Lemma 12.3 need not give a strict bound at a point of ℓ. Proposition 12.4 resolves this issue by blowing up the point on S. For comparison, we explain the limitation of the original curve flag and then give a separate proof using the exceptional plane of a threefold point blow-up. Double conic fibres require the different construction in Section 18.

Example 13.1 (A flag estimate that does not finish the proof). Suppose a smooth conic fibre Z on a chosen smooth cubic surface is tangent to ℓ at P, and $P \notin C_S$. Then $r = 2$, $\mu = 0$, and the exact point term of Lemma 12.3 is

$$\mathcal{T}_P = \frac{101}{96} > 1.$$

The resulting lower bound for $\delta_P(X)$ is only $96/101$. This does *not* prove instability: a lower bound below one says nothing of that kind. It shows precisely why this particular flag cannot complete a K-stability proof. Even a transverse intersection $r = 1$ gives $\mathcal{T}_P = 1$ when $\mu = 3$, and gives a value greater than one when $\mu > 3$.

For points above p_0, the avoidance argument in Lemma 10.6 is unavailable. Every smooth cubic through such a point has its distinguished line through that point: in local coordinates $l = (x, y, z)$ and $q(p_0) = (0, 0, 1)$, a point of f_0 has normal direction $[x : y] = [a : b]$. The coefficients of cubics through it satisfy $b\lambda_0 - a\lambda_1 = 0$, exactly the condition that the line $l \parallel \lambda$ have that same normal direction. The line therefore cannot be chosen to avoid p. Proposition 12.4 handles these points by blowing up p on S, retaining the tangent direction of $C_S = E|_S$ in the calculation on the exceptional curve. The exceptional-plane argument in Section 17 provides another proof.

At a double-line fibre, a general inverse image of a base line through its image need not be a smooth cubic surface. Locally a rank-one conic has equation

$$x^2 + uq_1(y, z) + vq_2(y, z) + \text{higher base-order terms} = 0.$$

Smoothness of X requires q_1, q_2 to have no common zero on the double line. On a base line $v = cu$, singularities of the surface occur at zeros of $q_1 + cq_2$, so the smooth-surface flag used above is not available there. We therefore use the weighted curve extraction in Section 18.

For comparison, the two original curve flags give the following conditional reduction.

Corollary 13.2 (A conditional reduction from the curve flags)**.** *If $\delta_P(X) > 1$ for every point $P \in T$, then X is K-stable.*

Proof. For any prime divisor F over X, choose a point P of its centre. If $P \notin T$, Lemmas 12.3 and 12.6 give $\delta_P(X) > 1$. If $P \in T$, use the hypothesis. In either case $A_X(F)/S_X(F) \geq \delta_P(X) > 1$. Apply the valuative criterion. □

The proof for reduced fibres now uses only the elementary fixed part on the original blow-up and Zariski decompositions on smooth cubic surfaces or their point blow-ups. The next sections give the alternative construction on the ordinary threefold blow-up of a point. Its induced series consists of leading homogeneous terms of anticanonical sections. We estimate its threshold in Section 17 using forced components on further blow-ups; the general higher-contact estimate does not require a complete threefold Zariski decomposition.

14 Refinement by the exceptional plane of a point blow-up

Fix $p \in X$, and write $\sigma : Y = \mathrm{Bl}_p X \to X$ and $G = \sigma^{-1}(p)$. Then $G \cong \mathbb{P}(T_pX) \cong \mathbb{P}^2$, $G|_G = \mathcal{O}_{\mathbb{P}^2}(-1)$ and $K_Y = \sigma^* K_X + 2G$. In particular, $A_X(G) = 3$. We reserve E for the exceptional divisor of $X \to \mathbb{P}^3$.

Put $R_m = H^0(X, mL)$ and $N_m = \dim R_m$. After trivializing L near p, define

$$\mathcal{F}^j R_m = H^0(X, mL \otimes \mathfrak{m}_p^j), \qquad W_{m,j} = \mathcal{F}^j R_m / \mathcal{F}^{j+1} R_m \ \subset \ H^0(G, \mathcal{O}_G(j)).$$

The inclusion sends a section to its homogeneous leading term of degree j at p. Equivalently, $W_{m,j}$ is the image of restriction from $H^0(Y, m\sigma^* L - jG)$ to G. In particular, it is the *image* of restriction, not a complete linear system by definition. We have $\sum_j \dim W_{m,j} = N_m$.

For a prime divisor Q over G, let $S_m(W; Q)$ be the largest value of

$$\frac{1}{mN_m} \sum_j \sum_{i=1}^{\dim W_{m,j}} \mathrm{ord}_Q(s_{j,i}),$$

where $\{s_{j,i}\}_i$ is a basis of $W_{m,j}$, and where orders mean orders of the corresponding zero divisors on G. Write $S(W; Q)$ for its asymptotic limit and define

$$\delta(G; W) = \inf_Q \frac{A_G(Q)}{S(W; Q)}.$$

This is the stability threshold of a multigraded series on $\mathbb{P}^2$; it is not automatically the threshold of a complete linear system on $\mathbb{P}^2$.

Lemma 14.1 (Abban–Zhuang at a point)**.** *For every $p \in X$,*

$$\delta_p(X) \geq \min\left\{ \frac{3}{S_X(G)},\ \delta(G; W) \right\}, \qquad S_X(G) = \frac{1}{16} \int_0^\infty \mathrm{vol}(\sigma^* L - tG)\, dt. \tag{14.1}$$

Moreover, putting $\lambda = \min\{3/S_X(G), \delta(G; W)\}$, every divisorial valuation v centred at p satisfies

$$A_X(v) \geq \lambda S_X(v) + (3 - \lambda S_X(G)) v(\mathfrak{m}_p). \tag{14.2}$$

Consequently, if $S_X(G) < 3$ and $\delta(G; W) \geq 1$, then $\beta_X(v) > 0$ for every such v.

Proof. The blow-up is a plt extraction and the different on G is zero. The refinement has bounded support, since the order-at-p filtration of an ample section ring is linearly bounded, and it contains an ample series by the complete jets in Lemma 14.2. Apply [4, Theorem 3.3]

to G and $\{p\}$. Every centre on G maps to p, so the second term in that theorem is precisely $\delta(G;W)$. This gives (14.1).

Here is also the discrepancy calculation underlying the assertion. For an m-basis divisor D_m compatible with ord_G, write

$$\sigma^*D_m = S_m(G)G + \Gamma_m, \qquad S_m(G) = \frac{\sum_j j \dim W_{m,j}}{mN_m}.$$

Then G is not a component of Γ_m, and $\Gamma_m|_G$ is a basis divisor of W. The coefficient of G in the boundary of the pullback of (X, cD_m) is $cS_m(G) - 2$. It is at most one when $cS_m(G) \le 3$. Inversion of adjunction therefore gives the stated minimum, after passage to the limit. Using a basis compatible with both ord_G and v retains the additional discrepancy term. This is [4, equation (3.6)]; since $\mathfrak{m}_p\mathcal{O}_Y = \mathcal{O}_Y(-G)$, it gives (14.2). If the two hypotheses in the last assertion hold, use the same inequality with $\lambda = 1$. For a valuation centred at p, $v(\mathfrak{m}_p) > 0$, proving strict positivity. □

The last assertion concerns valuations whose centre *equals* p. It does not, by itself, prove $\delta_p(X) > 1$, since the definition of δ_p also includes positive-dimensional centres containing p.

Lemma 14.2 (The initial nef interval and the complete jets)**.** *The line bundle L is very ample. For every point p,*

$$W_{m,j} = H^0(\mathbb{P}^2, \mathcal{O}(j)) \quad (0 \le j \le m), \qquad \mathrm{vol}(\sigma^*L - tG) = 16 - t^3 \quad (0 \le t \le 1).$$

If p lies on a curve $\Lambda \cong \mathbb{P}^1$ with $L \cdot \Lambda = 1$, the nef threshold is exactly one. If $q = [T_p\Lambda] \in G$, then

$$W_{m,j} \subset H^0(G, \mathcal{O}_G(j) \otimes \mathcal{I}_q) \quad (j > m). \tag{14.3}$$

Such a line passes through every point of the set T in Proposition 10.2.

Proof. By Lemma 10.5, the morphism (π, φ) is a closed embedding in $\mathbb{P}^3 \times \mathbb{P}^2$. Hence $L = H + B$ is the restriction of the very ample bundle $\mathcal{O}(1,1)$. Choose a section nonzero at p and three sections vanishing at p whose linear terms are independent. Their degree-m products realize every homogeneous polynomial of degree $j \le m$ as a leading term. This proves the equality of jet spaces.

Hyperplanes through a point of an embedded variety generate its maximal ideal, so $\sigma^*L - G$ is globally generated. Thus $\sigma^*L - tG$ is nef for $0 \le t \le 1$. Its cube is $16 - t^3$, giving its volume. On the strict transform of Λ it has degree $1 - t$, proving the threshold assertion. For $j > m$, a section in $\mathcal{F}^j R_m$ restricts to zero on Λ: a nonzero section of $\mathcal{O}_{\mathbb{P}^1}(m)$ cannot vanish to order greater than m. Its degree-j leading term therefore vanishes at q, proving (14.3).

Every fibre of $E \to C$ has H-degree zero and B-degree one. The reduced support of a double conic fibre has H-degree one and B-degree zero. These give the required lines through points of T. □

Thus the part of the volume integral with $0 \le t \le 1$ is $63/64$. This is only part of $S_X(G)$; it must not be used as its full value. Likewise, completeness of the first jets does not justify completeness of the higher jets.

Lemma 14.3 (Bounding point multiplicities by a smooth flag)**.** *Let $p \in Z \subset S \subset X$ be a flag of subvarieties smooth at p, of codimensions $3, 2, 1$. Write $R_m = H^0(X, -mK_X)$ and $N_m = \dim R_m$. Choose regular formal parameters x, y, z at p such that $S = (x = 0)$ and $Z = (x = y = 0)$ locally, and trivialize $-K_X$ there. For a nonzero section $s \in R_m$, define $\nu(s) = (i, j, k)$ by first taking its order i along S, then the order j along Z of $(s/x^i)|_S$, and finally the order k at p of the remaining nonzero restriction to Z after division by y^j.*

Choose a basis $s_{m,1},\dots,s_{m,N_m}$ compatible with these lexicographically ordered values, and write $\nu(s_{m,r})=(i_{m,r},j_{m,r},k_{m,r})$. Suppose that the following limits exist:

$$a=\lim_m \frac{\sum_r i_{m,r}}{mN_m},\qquad b=\lim_m \frac{\sum_r j_{m,r}}{mN_m},\qquad c=\lim_m \frac{\sum_r k_{m,r}}{mN_m}.$$

They do not depend on the compatible basis: the number of its vectors with any prescribed value is the dimension of the corresponding graded quotient. If G is the exceptional divisor of $\mathrm{Bl}_p X$, then $S_X(G)\le a+b+c$. For the flags of Section 12, the limits are respectively $S_X(S),\mathcal{S}_Z,\mathcal{T}_p$, and hence

$$S_X(G)\le S_X(S)+\mathcal{S}_Z+\mathcal{T}_p.$$

Proof. If $\nu(s)=(i,j,k)$, the formal expansion of s has a nonzero $x^iy^jz^k$ coefficient. Its multiplicity at p is therefore at most $i+j+k$. The two-filtration lemma [4, Lemma 3.1] supplies a basis of R_m compatible both with multiplicity at p and with the flag values. Indeed, the finitely many flag values occurring in R_m, ordered lexicographically, define a finite filtration by subspaces. The mean multiplicity of this basis, divided by m, is $S_m(G)$. Summing the preceding inequality over the basis gives

$$S_m(G)\le \frac{1}{mN_m}\sum_{r=1}^{N_m}(i_{m,r}+j_{m,r}+k_{m,r}).$$

Passing to the limits proves the inequality. In the flags of Section 12, the surface and curve refinements give the three limits from their definitions and the integrals of Lemma 12.1. □

Proposition 14.4 (An explicit estimate for the point extraction)**.** *If $p\in f_0$ lies on a smooth conic fibre, then*

$$S_X(G)\le\frac{273}{128},\qquad \beta_X(G)\ge\frac{111}{128},\qquad \delta_p(X)\ge\min\left\{\frac{128}{91},\delta(G;W)\right\}. \tag{14.4}$$

In addition, the cubic-surface flag can be chosen to give $\delta_p(X)\ge 64/65$. Thus one may take the maximum of this numerical bound and the last bound in (14.4). For comparison, if the conic fibre through p is reduced and singular, then $S_X(G)\le 221/96$. If it is smooth and $p\notin f_0$, then $S_X(G)\le 793/384$.

Proof. First suppose $p\in f_0$ and put $Z=\varphi^{-1}(\varphi(p))$. Choose a smooth cubic S_a containing Z, as in Lemma 10.6. Under the identification $S_a\cong S_{a,0}\subset\mathbb{P}^3$, the point p is p_0. The line ℓ_a has direction a at p_0, and a varies on the parameter line of cubics containing Z. The image of Z is a fixed smooth plane conic through p_0. At most one direction a is its tangent direction at p_0. Avoid it, as well as the finitely many singular cubics. This makes $(\ell_a\cdot Z)_p=1$. Put $C_{S_a}=E|_{S_a}$ and $\mu=(C_{S_a}\cdot Z)_p$; since $C_{S_a}\cdot Z=6$, we have $\mu\le 6$.

Lemmas 14.3 and 12.3 now give

$$S_X(G)\le\frac{91}{192}+\frac{247}{384}+\frac{189+\mu}{192}=\frac{807+2\mu}{384}\le\frac{819}{384}=\frac{273}{128}.$$

Subtracting from $A_X(G)=3$ proves the beta bound, and (14.1) gives the last inequality in (14.4).

For this same choice of cubic surface, Lemma 12.1 gives

$$\delta_p(X)\ge\frac{192}{189+\mu}\ge\frac{192}{195}=\frac{64}{65}.$$

In particular, if $\mu\le 2$, this flag already proves $\delta_p(X)>1$.

For a reduced singular fibre, use a line component and Lemma 12.6. The sum is at most $(91+161+190)/192=221/96$. For a smooth fibre outside f_0, choose ℓ avoiding p. The sum is at most $91/192+247/384+182/192=793/384$. □

These are upper bounds for $S_X(G)$, not asserted exact values. Their proof uses no Zariski decomposition on Y. The surface calculations already made supply an upper bound for ordinary point multiplicity, and Abban–Zhuang refinement then passes to the actual jet series on G.

Remark 14.5 (Computing the remaining series without a threefold decomposition). There is a concrete algebraic description of every required jet space. The incidence model of Lemma 10.5 is a complete intersection of bidegrees $(1,1)$ and $(2,1)$. Its Koszul resolution gives

$$R_m = \bigl(\mathbb{C}[x_0,x_1,x_2,x_3,y_0,y_1,y_2]/(\textstyle\sum l_iy_i, \sum q_iy_i)\bigr)_{(m,m)}, \qquad N_m = \frac{8}{3}m^3 + 4m^2 + \frac{10}{3}m + 1.$$

Surjectivity onto $H^0(X,mL)$ follows from the same resolution and the vanishing of the relevant intermediate cohomology of line bundles on $\mathbb{P}^3 \times \mathbb{P}^2$.

On an affine chart at p, solve the two equations formally for two of the five affine coordinates, using the rank-two Jacobian. The other three are regular parameters. Evaluation of the representatives of R_m in these parameters gives the jet map

$$J_{m,j} : R_m \longrightarrow \mathcal{O}_{X,p}/\mathfrak{m}_p^j.$$

Writing $r_{m,j} = \operatorname{rank} J_{m,j}$, we obtain the exact formulas

$$\dim W_{m,j} = r_{m,j+1} - r_{m,j}, \qquad S_m(G) = \frac{1}{mN_m}\sum_{j\geq 1}(N_m - r_{m,j}).$$

The degree-j terms of elements of $\ker J_{m,j}$ also give the actual homogeneous polynomials spanning $W_{m,j}$. Thus jet computations can replace a threefold Zariski decomposition. However, finitely many computed degrees do not establish the necessary asymptotic description, and the ranks alone do not determine $\delta(G;W)$: the base ideals and their infinitely near behaviour matter.

Example 14.6 (The normalization on projective space). For $X = \mathbb{P}^3$ and $L = -K_X = \mathcal{O}(4)$, the same construction gives $W_{m,j} = H^0(\mathbb{P}^2, \mathcal{O}(j))$ for $0 \leq j \leq 4m$. Here

$$S_X(G) = \frac{1}{64}\int_0^4 (64 - t^3)\,dt = 3.$$

The weighted average degree of this complete series on G is three. Since $\delta(\mathbb{P}^2;\mathcal{O}(1)) = 3$, its threshold is $\delta(G;W) = 1$. The point extraction therefore gives $\delta_p(\mathbb{P}^3) = 1$, the reverse inequality being supplied by G. In Family 2.9, the base condition (14.3) prevents this completeness argument at the points of T.

To complete this alternative point-extraction proof at the points in (14.4), one must still prove $\delta(G;W) > 1$, or prove $\delta(G;W) \geq 1$ and separately treat positive-dimensional centres. The surface blow-up proof in Proposition 12.4 already gives the required local bound on X. For nonreduced conic fibres, Section 18 instead uses a weighted curve extraction. These assertions do not follow from the calculations of this section alone. Sections 15 and 16 improve the point-extraction term by genuine threefold decompositions, and Section 17 proves $\delta(G;W) > 1$ in the smooth-fibre case. Positivity of $\beta_X(G)$ alone would not imply positivity for every divisor lying above p.

15 Blowing up the first base curve

We now compute a genuine Zariski decomposition on a modification of the threefold. Throughout this section, $p \in f_0$ and the conic $Z = \varphi^{-1}(\varphi(p))$ is smooth. There is no transversality

assumption on $C \cap Z$. Put $D_t = \sigma^* L - tG$ on $Y = \mathrm{Bl}_p X$, and let f be the strict transform of f_0. In formulas on further models, H, B, G denote pullbacks; the centres below meet G properly, so its pullback is also its strict transform.

Lemma 15.1 (Two successive curve blow-ups)**.** *One has $N_{f/Y} = \mathcal{O}(-1) \oplus \mathcal{O}(-2)$. Blow up f, obtaining $\tau_1 : Y_1 \to Y$ with exceptional surface $R_1 \cong \mathbb{F}_1$. Its negative section s has $N_{s/Y_1} = \mathcal{O}(-1)^{\oplus 2}$. Blow up s, obtaining $\tau_2 : Y_2 \to Y_1$, with exceptional surface $T \cong \mathbb{P}^1 \times \mathbb{P}^1$. Write R for the strict transform of R_1 and $\rho = \tau_1\tau_2$. Then $R \cong \mathbb{F}_1$, and, writing e for its negative section and r for a ruling fibre,*

$$R|_R = -2e - 2r, \qquad T|_R = e, \qquad R|_T = \mathcal{O}(0,1), \qquad T|_T = \mathcal{O}(-1,-1).$$

The first factor of T maps isomorphically to $s \cong f$.

Proof. The fibre f_0 in E has normal bundle $\mathcal{O} \oplus \mathcal{O}(-1)$ in X: its normal bundle in E is trivial, $E \cdot f_0 = -1$, and the normal sequence splits. Blowing up one point of the curve twists both normal summands by $\mathcal{O}(-1)$. Thus $R_1 = \mathbb{P}(\mathcal{O}(-1) \oplus \mathcal{O}(-2))$. On R_1 its normal bundle has class $-e - 2r$. Restriction to $s = e$ has degree -1, as does N_{s/R_1}. The normal sequence splits, giving $N_{s/Y_1} = \mathcal{O}(-1)^{\oplus 2}$.

Blowing up a Cartier curve on R_1 leaves that surface unchanged and subtracts s from its normal bundle. This gives $R|_R = -2e - 2r$ and $T|_R = e$. The other two formulas follow from $T = \mathbb{P}(N_{s/Y_1})$: $T|_T = \mathcal{O}(-1,-1)$ and $\tau_2^* R_1|_T = \mathcal{O}(-1,0)$, so $R|_T = \mathcal{O}(0,1)$. □

Lemma 15.2 (A globally generated divisor)**.** *On Y_2, the divisor $\mathcal{A} = 2H - 2G - R - 2T$ is globally generated. The strict transform of Z is disjoint from $R \cup T$.*

Proof. Let $\Pi \subset \mathbb{P}^3$ be the plane of the conic $\pi(Z)$. It contains $T_{p_0}C$, and the normal direction represented by p lies in Π. Choose affine linear coordinates x, t, r at p_0 such that $\Pi = (x = 0)$, $T_{p_0}C = (x = r = 0)$, and the selected normal direction has $x/r = 0$. Locally C has equations $x = a(t)$, $r = b(t)$, with $a, b \in (t^2)$. A chart of X at p is

$$x = a(t) + \eta z, \qquad r = b(t) + \eta, \qquad p = (t, \eta, z) = (0,0,0).$$

Here $E = (\eta = 0)$ and $f_0 = (t = \eta = 0)$.

Consider the quadrics spanned by x times all linear forms, together with t^2, tr, r^2. Their base locus in $\mathbb{P}^3$ is just p_0. On the displayed chart their ideal is

$$(x, t^2, tr, r^2) = (\eta z, t^2, \eta t, \eta^2).$$

Blowing up p extracts a factor of order two. The only remaining base locus is in the z-chart, where the residual ideal is (η_1, t_1^2). Blowing up $f = (\eta_1 = t_1 = 0)$ extracts one copy of R_1 and leaves the ideal of its negative section. Blowing up that section resolves the ideal. The total exceptional factor is $2G + R + 2T$. The same computation along the rest of f_0 uses (η, t^2); there are no other base points. This proves global generation.

Finally, $\varphi|_{f_0}$ is an isomorphism to a line. Hence $T_p f_0$ differs from the tangent direction of the fibre Z. The two strict transforms are disjoint after blowing up p, and all subsequent centres lie over f. □

Proposition 15.3 (The first non-nef chamber)**.** *For $1 \le t \le 2$, put $a = (t-1)/2$. Then*

$$\rho^* D_t = P_t + N_t, \qquad N_t = aR + 2aT, \qquad P_t = H + B - tG - aR - 2aT \tag{15.1}$$

is a Zariski decomposition in the following explicit sense: P_t is nef and, for sufficiently divisible m, multiplication by the defining section of mN_t identifies $H^0(Y_2, mP_t)$ with $H^0(Y_2, m\rho^* D_t)$. Moreover,

$$\mathrm{vol}(D_t) = 16 - t^3 + \frac{(t-1)^3}{2} \qquad (1 \le t \le 2). \tag{15.2}$$

Proof. First prove that the displayed coefficients are forced. Let an effective divisor numerically equivalent to $\rho^* D_t$ have coefficients α, β along R, T, respectively, and remove those components. Its restrictions to R and T are effective and have classes

$$(2\alpha - \beta)e + (1 - t + 2\alpha)r, \qquad \mathcal{O}_T(1 - t + \beta, \beta - \alpha).$$

The effective cones of $\mathbb{F}_1$ and $\mathbb{P}^1 \times \mathbb{P}^1$ give $\alpha \geq (t-1)/2$ and $\beta \geq t - 1$. This proves the fixed-part assertion once nefness is established.

The divisor $P_1 = H + B - G$ is globally generated. At the other endpoint,

$$P_2 = \tfrac{1}{2}\mathcal{A} + (B - G).$$

The system $|B - G|$ has base locus exactly the strict transform of Z. Indeed, it comes from lines through $\varphi(p)$, and the fibre is smooth. It therefore has nonnegative intersection with every curve other than Z. Lemma 15.2 gives the same assertion for $\mathcal{A}$, and $\mathcal{A} \cdot Z = 2$, $(B-G)\cdot Z = -1$. Thus $P_2 \cdot Z = 0$ and P_2 is nef. Since $P_t = (2-t)P_1 + (t-1)P_2$, every P_t is nef.

For later use, $2P_2$ is in fact globally generated. Products from $|\mathcal{A}|$ and $|B - G|^2$ generate it off Z. To cover Z, choose a plane section $h \in H^0(X, H)$ whose restriction to Z vanishes exactly at $2p$. Its differential annihilates $T_p Z$. Since φ is smooth at p, a section of B vanishing at $\varphi(p)$ has the same differential, after choosing nonvanishing local frames. Subtracting the corresponding products gives a global section of L vanishing to order two at p whose strict transform does not vanish anywhere on Z. Its square, after removing $R + 2T$, supplies the missing section of $2P_2$.

For divisible m, the forced coefficients show that every section of $m\rho^* D_t$ is divisible by mN_t. Conversely, multiplying any section of mP_t gives a section of $m\rho^* D_t$. As P_t is nef, the volume is P_t^3.

Here is the cubic calculation. On the first blow-up, $D_t \cdot f = 1 - t$, $R_1^3 = 3$, and $(\tau_1^* D_t)R_1^2 = t - 1$. Subtracting aR_1 changes the cube by $3a^2(t-1) - 3a^3 = 3a^3$. Its intersection with s is $-a$. On blowing up s, subtracting aT changes the cube by $3a^3 - 2a^3 = a^3$. The total change is $4a^3 = (t-1)^3/2$, proving (15.2). □

Lemma 15.4 (The resolved jets on the exceptional plane)**.** *The strict transform $\widetilde{G} \subset Y_2$ is obtained from $G \cong \mathbb{P}^2$ by blowing up $q = [T_p f_0]$ and then a point q_1 on the first exceptional curve. Write h, e_1, e_2 for the total-transform plane and exceptional classes. Then*

$$P_t|_{\widetilde{G}} = th - \frac{t-1}{2}(e_1 + e_2), \qquad N_t|_{\widetilde{G}} = \frac{t-1}{2}(e_1 - e_2) + (t-1)e_2.$$

For $1 < t < 2$, the moving restricted series is asymptotically complete. In affine coordinates (u, v) at q, with q_1 specifying the direction $v = 0$, its pushforward consists asymptotically of degree-j forms satisfying the weighted base condition

$$\mathrm{wt}_{(1,2)}(u^i v^k) = i + 2k \geq j - m, \qquad j = mt. \tag{15.3}$$

Its restricted volume is $t^2 - (t-1)^2/2$.

Proof. The two threefold centres meet the transform of G transversely at q, q_1. On the final surface, $R|_{\widetilde{G}} = e_1 - e_2$ and $T|_{\widetilde{G}} = e_2$. This proves the class formulas. The exceptional valuations of the two surface blow-ups have weights $(1,1)$ and $(1,2)$ in (u, v). The required orders are at least $(j-m)/2$ and $j - m$, respectively; the latter implies the former. This gives (15.3).

We justify asymptotic completeness, including the birationality needed in passing to the image. For rational $1 < t < 2$, write $P_t = (2-t)P_1 + (t-1)P_2$. The divisors P_1 and $2P_2$ are globally generated by Proposition 15.3. The complete linear jets in Lemma 14.2 show that the

morphism of P_1 restricts on $\widetilde{G}$ to a map recovering its blow-down to $G \cong \mathbb{P}^2$. It is therefore birational on $\widetilde{G}$.

Choose an integer $d > 0$ with $dP_t = aP_1 + b(2P_2)$ for positive integers a, b. Products of endpoint sections give a subsystem of $|dP_t|$. On a dense open subset of $\widetilde{G}$, ratios of these products recover the ratios of the sections of aP_1. Hence the morphism of dP_t is also birational on $\widetilde{G}$. Take its Stein factorization $\psi : Y_2 \to V$, with $dP_t = \psi^* A_V$ for an ample line bundle A_V, and let $S_V = \psi(\widetilde{G})$ have its reduced structure. Restriction from $H^0(V, nA_V)$ to $H^0(S_V, nA_V|_{S_V})$ is surjective for large n. Since $\psi|_{\widetilde{G}}$ is birational, the cokernel of

$$\mathcal{O}_{S_V} \longrightarrow (\psi|_{\widetilde{G}})_* \mathcal{O}_{\widetilde{G}}$$

is supported in dimension at most one. Twisting by nA_V shows that the deficit of the restricted series inside $H^0(\widetilde{G}, ndP_t|_{\widetilde{G}})$ is $O(n)$. This is asymptotic completeness in a surface summand. Squaring the nef restricted class gives the stated volume, also equal to $-\frac{1}{3}\frac{d}{dt}\operatorname{vol}(D_t)$. □

Corollary 15.5 (A uniform improvement)**.** *For $p \in f_0$ on a smooth conic fibre,*

$$S_X(G) \le \frac{233}{112} < \frac{273}{128}, \qquad \beta_X(G) \ge \frac{103}{112}, \qquad \delta_p(X) \ge \min\left\{\frac{336}{233}, \delta(G; W)\right\}.$$

Proof. Put $V(t) = \operatorname{vol}(D_t)$. The proven chambers give

$$\int_0^2 V(t)\,dt = \frac{225}{8}, \qquad V(2) = \frac{17}{2}, \qquad V'_-(2) = -\frac{21}{2}.$$

The function $V^{1/3}$ is concave on the big interval; see [5, Section 2.4]. Its left tangent at 2 therefore gives

$$V(t) \le \frac{17}{2}\left(1 - \frac{7}{17}(t-2)\right)^3 \quad \text{while } D_t \text{ is big and } t \ge 2.$$

In particular, the big interval ends no later than $31/7$, and

$$\int_2^\infty V(t)\,dt \le \frac{289}{56}, \qquad S_X(G) \le \frac{1}{16}\left(\frac{225}{8} + \frac{289}{56}\right) = \frac{233}{112}.$$

The other assertions follow from $A_X(G) = 3$ and Lemma 14.1. This argument uses concavity only for the unresolved tail; it does not pretend that the first chamber continues past $t = 2$. □

16 Complete decomposition in transverse position

We can continue the calculation when

$$C \cap \Pi = 2p_0 + c_1 + \cdots + c_5, \quad c_1, \dots, c_5 \text{ distinct}, \quad c_i \notin T_{p_0} C. \tag{16.1}$$

The intersections at the c_i are simple. These are additional assumptions, not consequences of smoothness of X. In particular, this section does not address the higher-contact cases in which $(C \cdot Z)_{p_0} \ge 2$.

Lemma 16.1 (The six curves at the next wall)**.** *Under (16.1), let Γ_i be the strict transform in X of the line $p_0 c_i$. Each passes through p, has L-degree two and has $N_{\Gamma_i/X} = \mathcal{O}^{\oplus 2}$. On Y_2, the strict transforms of $Z, \Gamma_1, \dots, \Gamma_5$ are pairwise disjoint, disjoint from $R \cup T$, and have normal bundle $\mathcal{O}(-1)^{\oplus 2}$.*

Proof. Choose a smooth cubic containing Z. Its plane section by Π is $Z + \ell$, and $C \cdot \ell = p_0$. Thus $C \cap Z = p_0 + c_1 + \cdots + c_5$ scheme-theoretically. In particular, Z meets C transversely at all six points.

Since Π contains $T_{p_0}C$, every line in Π through p_0 other than $T_{p_0}C$ lifts to a curve through the selected point $p \in f_0$. The line p_0c_i meets C only at those two points, both transversely. An additional intersection would make its B-degree zero (or negative). Degree zero would put its lift in the smooth conic fibre through p, which cannot contain a line; negative degree contradicts nefness of B. Hence $H \cdot \Gamma_i = B \cdot \Gamma_i = 1$.

The normal bundle of a line in $\mathbb{P}^3$ is $\mathcal{O}(1)^{\oplus 2}$. Blowing up C makes one elementary modification at each of p_0, c_i. The two quotient directions are distinct: the plane spanned by the line and $T_{p_0}C$ is Π, whereas $T_{c_i}C$ is transverse to Π. Two modifications in distinct directions at distinct points yield $\mathcal{O}^{\oplus 2}$. Blowing up p then yields $\mathcal{O}(-1)^{\oplus 2}$. The same conclusion for Z follows from its being a smooth fibre of φ.

Distinct lines p_0c_i have distinct directions, since three collinear points would give the excluded trisecant through p. None is tangent to the smooth conic through both its endpoints. Blowing up C separates the line and conic at c_i, and blowing up p separates all their remaining intersections. Their tangent directions differ from T_pf_0, so they also avoid the first two centres. □

Blow up these six curves on Y_2, and denote the resulting smooth model by $\widehat{Y}$. Write $U_0, U_1, \ldots, U_5$ for the exceptional quadrics, with U_0 corresponding to Z. Let Q be the strict transform of the surface $\pi^{-1}(\Pi)$, and let $\eta : \widehat{Y} \to Y$ be the composite morphism. On this model,

$$Q = H - 2G - R - 2T - \sum_{i=0}^{5} U_i.$$

All six centres lie in the surface; blowing up a Cartier curve does not change its intrinsic surface.

Lemma 16.2 (The endpoint of the second chamber)**.** *The divisor*

$$P_{5/2} = H + B - \tfrac{5}{2}G - \tfrac{3}{4}R - \tfrac{3}{2}T - \tfrac{1}{2}\sum_{i=0}^{5} U_i$$

is nef on $\widehat{Y}$. The divisor $F = B - G - U_0$ is globally generated.

Proof. The pencil of base lines through $\varphi(p)$ has ideal $\mathcal{I}_Z$. Blowing up p and then the strict transform of Z principalizes this ideal, with total divisor $G + U_0$. The other centres are disjoint from Z. Hence F is globally generated.

For nefness, first observe the effective expression

$$P_{5/2} = Q + F + \tfrac{1}{2}G + \tfrac{1}{4}R + \tfrac{1}{2}T + \tfrac{3}{2}U_0 + \tfrac{1}{2}\sum_{i=1}^{5} U_i.$$

It gives nonnegative intersection with curves outside the displayed divisors. On R the restriction is zero, on T it is $\mathcal{O}(0, 3/4)$, and on each U_i it is $\mathcal{O}(0, 1/2)$, with the ruling conventions of the relevant blow-up. These restrictions are nef.

The transform of G is the plane blown up at q, q_1 and six further distinct points. Those eight centres, counting the infinitely near point, lie on the smooth conic cut out by the tangent cone of $\pi^{-1}(\Pi)$ at p. Smoothness of this conic follows from the contact order two in (16.1): its local equation is $\eta z + ct^2 = 0$, $c \neq 0$. Write $e_3, \ldots, e_8$ for the six new total-transform exceptional classes. The restriction is

$$\tfrac{5}{2}h - \tfrac{3}{4}(e_1 + e_2) - \tfrac{1}{2}\sum_{i=3}^{8} e_i.$$

Its intersection with the strict conic is $1/2$, and its intersections with the exceptional curves are nonnegative. Any other irreducible plane curve of degree d, with multiplicities m_i at the eight centres, satisfies $\sum m_i \le 2d$ by intersection with that conic. Its intersection with the displayed divisor is therefore at least $5d/2 - 3(\sum m_i)/4 \ge d$. This proves nefness on G.

Finally, the intrinsic surface Q is the plane blown up at p_0, the point infinitely near it in the direction $T_{p_0}C$, and the five points c_i. This follows directly by resolving the curvilinear ideal of length two at p_0; the two later curve blow-ups are along Cartier curves on this surface. Write $h, \epsilon_0, \epsilon_0', \epsilon_1, \dots, \epsilon_5$ for total-transform classes. The restrictions before the last six blow-ups are

$$H|_Q = h, \quad G|_Q = \epsilon_0 - \epsilon_0', \quad R|_Q = 0, \quad T|_Q = \epsilon_0', \quad B|_Q = 3h - \epsilon_0 - \epsilon_0' - \sum_{i=1}^{5} \epsilon_i.$$

Also $U_0|_Q = 2h - \epsilon_0 - \sum \epsilon_i$ and $U_i|_Q = h - \epsilon_0 - \epsilon_i$ for $i \ge 1$. Substitution gives $P_{5/2}|_Q = (h - \epsilon_0)/2$, a nef pencil class. All curves in the displayed support have now been checked. □

Proposition 16.3 (All chambers in transverse position)**.** *Assume* (16.1)*. On* $\widehat{Y}$*, let* $P_t = \eta^* D_t - N_t$*, where*

t	N_t
$[0,1]$	0
$[1,2]$	$\frac{t-1}{2}R + (t-1)T$
$[2,5/2]$	$\frac{t-1}{2}R + (t-1)T + (t-2)\sum_{i=0}^{5} U_i$
$[5/2,3]$	$\frac{t-1}{2}R + (t-1)T + (t-2)\sum_{i=0}^{5} U_i + (2t-5)(Q+U_0)$.

Then P_t *is nef and* N_t *is the fixed part in the section-space sense of Proposition 15.3. In particular,*

$$\mathrm{vol}(D_t) = \begin{cases} 16 - t^3, & 0 \le t \le 1, \\ 16 - t^3 + \frac{1}{2}(t-1)^3, & 1 \le t \le 2, \\ 16 - t^3 + \frac{1}{2}(t-1)^3 + 6(t-2)^3, & 2 \le t \le \frac{5}{2}, \\ 12(3-t)^2 - \frac{3}{2}(3-t)^3, & \frac{5}{2} \le t \le 3, \\ 0, & t \ge 3. \end{cases} \tag{16.2}$$

Consequently,

$$S_X(G) = \frac{503}{256}, \qquad \beta_X(G) = \frac{265}{256}, \qquad \delta_p(X) \ge \min\left\{\frac{768}{503}, \delta(G;W)\right\}.$$

Proof. The first two chambers were proved above. On each of the six curves, D_t has degree $2-t$ and the conormal bundle after the point blow-up is $\mathcal{O}(1)^{\oplus 2}$. Restriction to its exceptional quadric forces coefficient at least $t-2$ for $t \ge 2$. Thus the first three rows are forced. Nefness on $[2,5/2]$ follows by interpolation between the nef endpoint divisors P_2 and $P_{5/2}$.

For $t > 5/2$, subtract the already forced third-row part, obtaining a divisor with class P_t^{old}. Let a and b be the additional coefficients of Q and U_0 in any effective representative. On U_0, one has $P_t^{\mathrm{old}}|_{U_0} = \mathcal{O}(0, t-2)$ and $Q|_{U_0} = \mathcal{O}(1,1)$. Removing the two components and restricting to U_0 forces $b \ge a$. For a general ruling curve l of Q obtained from a line through p_0, the surface formulas in Lemma 16.2 give

$$P_t^{\mathrm{old}} \cdot l = 5 - 2t, \qquad Q \cdot l = -2, \qquad U_0 \cdot l = 1.$$

The remaining effective divisor does not contain Q, so $5 - 2t + 2a - b \ge 0$. Together these inequalities give $a \ge 2t - 5$ and $b \ge 2t - 5$. This proves exactly the extra fixed part $(2t-5)(Q+U_0)$. The positive parts on this interval are linear in t; at $t = 3$ they equal

$F = B - G - U_0$. Interpolation with Lemma 16.2 proves nefness. The fixed-component inequalities give equality of section spaces as before.

The first two cubes are known. Blowing up a $(-1,-1)$ curve on which a divisor has degree $-b$, and subtracting b times the exceptional quadric, adds b^3 to its cube. The six disjoint curves therefore add $6(t-2)^3$ in the third chamber.

For clarity, the last cube can also be computed without expanding all intersections on $\widehat{Y}$. Put $s = 3 - t$. After removing the fixed plane, the positive part has underlying pullback class $2sH + B - (1+3s)G$, first two exceptional coefficients $3s/2, 3s$, coefficient $1 - s$ along U_0, and coefficient s along each U_i, $i \geq 1$. The same blow-up formula gives

$$\begin{aligned} P_t^3 &= (2sH + B)^3 - (1+3s)^3 + \tfrac{1}{2}(3s)^3 + (1-s)^3 + 5s^3 \\ &= 12s^2 - \tfrac{3}{2}s^3. \end{aligned}$$

At $t = 3$ the volume is zero; monotonicity makes it zero thereafter. Finally, the four integrals in (16.2) are

$$\frac{63}{4}, \qquad \frac{99}{8}, \qquad \frac{363}{128}, \qquad \frac{61}{128}.$$

Their sum is $503/16$. Divide by $L^3 = 16$ and apply Lemma 14.1. □

Remark 16.4 (Scope of the transverse-position calculation). This calculation uses threefold Zariski decompositions obtained by blowing up base curves. Corollary 15.5 is uniform for $p \in f_0$ on a smooth conic fibre. Proposition 16.3 computes the entire volume function under the additional assumptions (16.1). These assumptions must not be suppressed: in higher-contact cases, bisecants can become tangent, their transforms can meet, and the six disjoint $(-1,-1)$ blow-ups are no longer justified. Double conic fibres are treated by the weighted curve extraction of Section 18.

These volume calculations improve the $3/S_X(G)$ term in the Abban–Zhuang minimum. They do not by themselves establish $\delta(G;W) > 1$. The next section supplies that further estimate, using only necessary vanishing conditions for the jet series. The nonreduced conic fibres are treated separately in Section 18.

17 A uniform bound on the exceptional plane

We now complete the alternative exceptional-plane proof at the smooth-fibre points of f_0, including arbitrary contact between C and the conic. Throughout this section, $p \in f_0$ and $Z = \varphi^{-1}(\varphi(p))$ is smooth. We retain the two blow-ups and the divisors R, T from Section 15. We do *not* assume (16.1).

Lemma 17.1 (A forced plane component without transverse-position assumptions)**.** *Blow up the strict transform of Z on Y_2, and write U for the exceptional divisor and Q for the strict transform of the plane surface $\pi^{-1}(\Pi)$. On this smooth threefold,*

$$U \cong \mathbb{P}^1 \times \mathbb{P}^1, \qquad Q = H - 2G - R - 2T - U.$$

For $t \geq 5/2$, every effective representative of the pullback of D_t contains

$$\frac{t-1}{2}R + (t-1)T + (t-2)U + (2t-5)(Q+U). \tag{17.1}$$

Moreover, $H^0(X, mL \otimes \mathfrak{m}_p^j) = 0$ for $j > 3m$, and the pseudo-effective threshold of G is exactly three.

Proof. The curve Z is a smooth fibre, so $N_{Z/X} \cong \mathcal{O}^{\oplus 2}$. After blowing up p, its normal bundle is $\mathcal{O}(-1)^{\oplus 2}$. It is disjoint from the centres producing R, T, by Lemma 15.2. Thus its blow-up has exceptional quadric U, with $U|_U = \mathcal{O}(-1,-1)$.

In the local coordinates of that lemma, the plane surface has equation

$$\eta z + a(t) = 0, \qquad a(t) \in (t^2).$$

It has multiplicity two at p. Its multiplicities along the subsequent centres give one copy of R and two of T. For example, after blowing up p, its equation in the z-chart is $\eta_1 + z^{-2}a(zt_1) = 0$; it contains f, and its transform after blowing up f contains the negative section. Along Z its generic multiplicity is one. These facts prove the class formula for Q, regardless of the order of $a(t)$.

The effectivity argument on R, T in Proposition 15.3 does not use $t \leq 2$: it forces coefficients at least $(t-1)/2$ and $t-1$ whenever $t \geq 1$. Remove these components. Since R, T are disjoint from U, restriction to U then has class $\mathcal{O}(2-t, 0)$. The normal bundle of U forces a further coefficient at least $t-2$ for $t \geq 2$. After removing it, call the remaining class M_t. We have

$$M_t|_U = \mathcal{O}(0, t-2), \qquad Q|_U = \mathcal{O}(1,1).$$

The latter equality follows from $(H - 2G - R - 2T) \cdot Z = 0$ before this last blow-up.

Let l be the strict transform of a general line in Π through p_0. It meets C only at p_0, transversely, and its lift passes through p. Its tangent at p differs from those of f_0 and Z. After the point blow-up it therefore avoids f and meets the strict conic once, at their other plane intersection. Consequently

$$Hl = 1, \quad Bl = 2, \quad Gl = 1, \quad Rl = Tl = 0, \quad Ul = 1,$$

$$Ql = -2, \qquad M_t l = 5 - 2t.$$

These general ruling curves cover Q. The calculation concerns a general member and does not require Q to be smooth everywhere.

Let γ, ε be the coefficients of Q, U in an effective representative of M_t. Remove them. Restriction to U, where the remaining divisor is effective, gives $\varepsilon \geq \gamma$. Intersection with a general ruling of Q gives

$$5 - 2t + 2\gamma - \varepsilon \geq 0.$$

Hence $\gamma \geq 2t - 5$ and $\varepsilon \geq 2t - 5$, proving (17.1). Notice that no bisecant curve has been blown up in this argument.

Apply this to a nonzero section of mL vanishing to order at least j at p. If $j \geq 5m/2$, its divisor on X contains the plane divisor, of class H, with coefficient at least $2j - 5m$. But $mL - cH$ can be effective only if $c \leq m$: in the effective-cone basis E, B its E-coefficient is $(m-c)/3$. Thus $2j - 5m \leq m$, or $j \leq 3m$. Finally, the plane section of H has order two at p, and a section of B vanishing simply at $\varphi(p)$ pulls back to order one, since φ is smooth there. Their product is a section of L of order three. This proves the asserted threshold. □

Remark 17.2 (What was, and was not, decomposed). The two initial blow-ups give a genuine Zariski decomposition for $1 \leq t \leq 2$. Beyond that interval, Lemma 17.1 gives *necessary fixed components* on a smooth modification. It does not assert that the remaining divisor is nef, or that all base curves have been resolved. Such assertions in higher contact would require additional work. The estimate below needs only inclusions of jet spaces, so these necessary components suffice. This is how we avoid a complete threefold Zariski decomposition in the higher-contact cases while still using the geometry of the blown-up base curves.

Corollary 17.3 (A further uniform improvement of the volume bound)**.** *For $p \in f_0$ on a smooth conic fibre,*

$$S_X(G) \le \frac{9441}{4624}, \qquad \frac{3}{S_X(G)} \ge \frac{4624}{3147}.$$

Proof. Use the concavity estimate in Corollary 15.5, but now the volume vanishes for $t \ge 3$ by Lemma 17.1. Thus

$$\int_2^3 \mathrm{vol}(D_t)\,dt \le \frac{289}{56}\left(1 - \left(\frac{10}{17}\right)^4\right) = \frac{10503}{2312}.$$

Adding $\int_0^2 \mathrm{vol}(D_t)\,dt = 225/8$ and dividing by 16 gives the result. □

Lemma 17.4 (Explicit spaces containing the jets)**.** *There are homogeneous coordinates $[u : v : w]$ on G such that $q = [T_p f_0] = [0 : 0 : 1]$, the infinitely near point q_1 has tangent direction $v = 0$, and the tangent conic of the plane surface is*

$$K = (vw + cu^2 = 0)$$

for some $c \in \mathbb{C}$, possibly zero. For integers $m, j \ge 0$, put

$$k_{m,j} = \max\{0, 2j - 5m\}, \quad d_{m,j} = j - 2k_{m,j}, \quad b_{m,j} = \max\{0, j - m - 2k_{m,j}\}.$$

For $0 \le j \le 3m$, let $U_{m,j}$ be the span of the monomials

$$u^i v^r w^{d_{m,j}-i-r}, \qquad i, r \ge 0, \quad i + r \le d_{m,j}, \quad i + 2r \ge b_{m,j}. \tag{17.2}$$

Then

$$W_{m,j} \subset K^{k_{m,j}} U_{m,j} \subset H^0(G, \mathcal{O}_G(j)), \qquad W_{m,j} = 0 \quad (j > 3m).$$

Here $K^{k_{m,j}}$ means multiplication by the corresponding power of its equation.

Proof. Take $u = t, v = \eta, w = z$ in the tangent space of the local coordinates above. The quadratic initial form of $\eta z + a(t)$ is $vw + cu^2$. Near q, its equation has weighted order two for weights $(1, 2)$ on (u, v). The second exceptional curve of the successive point blow-ups of G computes exactly this weighted order. Indeed, in the chart $v = uv_1$, the second blow-up at $u = v_1 = 0$ assigns orders one and two to u, v.

The forced coefficient $j - m$ along T gives weighted order at least $j - m$ for a degree-j leading form when $j \ge m$. Lemma 17.1 forces divisibility by $K^{k_{m,j}}$. Dividing by it leaves degree $d_{m,j}$ and weighted order at least $j - m - 2k_{m,j}$. A polynomial has that weighted order precisely when its monomials satisfy (17.2). For $j \le m$ there is no additional condition. The preceding lemma gives the last vanishing assertion.

No completeness of a restricted positive part is used: the displayed relation is an inclusion. In particular, collisions among other tangent directions do not invalidate it. □

Lemma 17.5 (Comparing with monomial spaces)**.** *Let $V = \bigoplus_{m,j} V_{m,j}$ be a bigraded linear series on $G \cong \mathbb{P}^2$, with bounded support and containing an ample series, in the sense of [4, Section 2.5]. These hypotheses hold for the refinement used here by [4, Example 2.15] and ensure the existence of the expected-vanishing limits. Suppose $V_{m,j} \subset K^{k_{m,j}} U_{m,j}$, where $k_{m,j} \ge 0$, each $U_{m,j}$ is a space of homogeneous forms of a single degree spanned by monomials in u, v, w, and $K = vw + cu^2$. For each m, assume that only finitely many $U_{m,j}$ are nonzero. Normalize basis divisors by mN_m, where $N_m = \sum_j \dim V_{m,j} > 0$. Define $C_{u,m}, C_{v,m}, C_{w,m}$ by summing the respective exponents of all monomials of all $U_{m,j}$ and dividing by mN_m. Put*

$$C_{K,m} = \frac{\sum_j k_{m,j} \dim U_{m,j}}{mN_m}, \qquad C_m = \max\{C_{u,m}, C_{v,m} + C_{K,m}, C_{w,m} + C_{K,m}\}.$$

Then, for every divisorial valuation ν over G,

$$S_m(V;\nu) \le C_m A_G(\nu).$$

If the four constants have limits, the reciprocal of the limiting maximum is a lower bound for $\delta(G;V)$, with the convention $1/0 = +\infty$.

Proof. All orders of zero divisors of nonzero sections are nonnegative. The dimension of each step of the valuation filtration of $K^{-k_{m,j}}V_{m,j}$ is at most the corresponding dimension for $U_{m,j}$. Summing the dimensions of these steps bounds the maximal sum of valuation orders. Also $\dim V_{m,j} \le \dim U_{m,j}$. It follows that $S_m(V;\nu)$ is at most $\nu(\mathcal{D}_m)$, where

$$\mathcal{D}_m = C_{K,m}K + \frac{1}{mN_m}\sum_{j,i}\operatorname{div}(s_{j,i})$$

and each $\{s_{j,i}\}_i$ is a valuation-compatible basis of $U_{m,j}$.

Choose an integer M larger than every degree under consideration. The one-parameter action

$$[u:v:w] \longmapsto [\lambda^M u:\lambda v:w]$$

has distinct weights on the monomials in each fixed degree. By the two-filtration lemma [4, Lemma 3.1], we can choose each basis simultaneously compatible with the valuation and this weight filtration. Its initial forms are then the distinct monomials spanning $U_{m,j}$. The initial form of K is vw. Hence $\mathcal{D}_m$ degenerates to

$$C_{u,m}(u=0) + (C_{v,m}+C_{K,m})(v=0) + (C_{w,m}+C_{K,m})(w=0).$$

If $C_m = 0$, all the sections involved are constant and the desired inequality is immediate. Otherwise, after division by C_m, the limit is a simple-normal-crossing boundary with all coefficients at most one, and is therefore log canonical. Lower semicontinuity of log canonical thresholds [11, Theorem 0.2] shows that $(G, \mathcal{D}_m/C_m)$ is log canonical as well: all nonzero fibres of the degeneration are projectively equivalent. Thus $\nu(\mathcal{D}_m) \le C_m A_G(\nu)$ for every ν, proving the assertion. Passing to the expected-vanishing limits gives the last claim. This also proves that the comparison controls arbitrary divisors over the plane, not just monomial valuations. □

Lemma 17.6 (The four integrals for the comparison)**.** *For the spaces of Lemma 17.4, the constants in Lemma 17.5 converge to*

$$C_u = \frac{387}{512}, \qquad C_v = \frac{801}{1024}, \qquad C_w = \frac{603}{1024}, \qquad C_K = \frac{105}{1024}.$$

Proof. Set $t = j/m$ and use the following piecewise linear functions:

t	$d(t)$	$a(t)$	$k(t)$
$[0,1]$	t	0	0
$[1,5/2]$	t	$(t-1)/2$	0
$[5/2,3]$	$10-3t$	$(9-3t)/2$	$2t-5.$

The normalized exponent region is

$$\Omega_t = \{(x,y) \in \mathbb{R}^2_{\ge 0} : x+y \le d(t),\ x+2y \ge 2a(t)\}.$$

In every row $d \geq 2a \geq 0$. Thus Ω_t is the triangle of side d with the corner triangle of vertices $(0,0),(2a,0),(0,a)$ removed. Its area and moments are

$$\int_{\Omega_t} 1 = \frac{d^2}{2} - a^2, \qquad \int_{\Omega_t} x = \frac{d^3}{6} - \frac{2a^3}{3},$$
$$\int_{\Omega_t} y = \frac{d^3}{6} - \frac{a^3}{3}, \qquad \int_{\Omega_t} (d-x-y) = \frac{d^3}{6} - da^2 + a^3.$$

For example, the removed triangle has area a^2 and centroid $(2a/3, a/3)$, giving its first two moments immediately.

Since $N_m = (8/3)m^3 + O(m^2)$, lattice summation and division by mN_m give the factor $3/8$, not $3/16$:

$$C_u = \frac{3}{8}\int_0^3 \left(\frac{d^3}{6} - \frac{2a^3}{3}\right) dt, \qquad C_v = \frac{3}{8}\int_0^3 \left(\frac{d^3}{6} - \frac{a^3}{3}\right) dt,$$
$$C_w = \frac{3}{8}\int_0^3 \left(\frac{d^3}{6} - da^2 + a^3\right) dt, \qquad C_K = \frac{3}{8}\int_{5/2}^3 k\left(\frac{d^2}{2} - a^2\right) dt.$$

The separate contributions, with the factor $3/8$ already included, are

t	C_u	C_v	C_w	C_K
$[0,1]$	$1/64$	$1/64$	$1/64$	0
$[1,5/2]$	$1137/2048$	$2355/4096$	$1761/4096$	0
$[5/2,3]$	$379/2048$	$785/4096$	$587/4096$	$105/1024.$

Adding the columns proves the formulas. Boundary lattice points and rounding contribute $O(m^3)$ to the numerator, hence vanish after division by mN_m. The larger spaces have total asymptotic dimension $(73/64)N_m$; they have deliberately not been renormalized to their own dimension. □

Theorem 17.7 (All smooth conic fibres on the distinguished fibre)**.** *For every $p \in f_0$ whose conic fibre is smooth,*

$$\delta(G;W) \geq \frac{512}{453}, \qquad \delta_p(X) \geq \frac{512}{453} > 1.$$

No transversality assumption on $C \cap \Pi$ or $C \cap Z$ is required.

Proof. The three coefficients after the monomial degeneration are

$$C_u = \frac{387}{512}, \qquad C_v + C_K = \frac{453}{512}, \qquad C_w + C_K = \frac{177}{256}.$$

Their maximum is $453/512$. Lemmas 17.4–17.6 give the first inequality. By Lemma 14.1 and Corollary 17.3,

$$\delta_p(X) \geq \min\left\{\frac{4624}{3147}, \frac{512}{453}\right\} = \frac{512}{453}.$$

For the same conclusion the earlier, weaker bound $S_X(G) \leq 273/128$ would already suffice. Thus this conclusion does not depend on the complete transverse-case decomposition in Section 16. □

Example 17.8 (Why higher contact is allowed). If $a(t) = ct^2 + O(t^3)$ with $c \neq 0$, the tangent conic $vw + cu^2 = 0$ is smooth. If $a(t) = ct^r + O(t^{r+1})$ with $r \geq 3$, its tangent conic is the reduced union $vw = 0$. In both cases its weighted order at q is two and the one-parameter degeneration in Lemma 17.5 has initial form vw. Thus the same numerical bound applies. These are local illustrations; the proof does not assume that the higher-contact case deforms with unchanged Zariski chambers.

Corollary 17.9 (A sharper bound in transverse position)**.** *Under* (16.1)*,*

$$\delta(G;W) \geq \frac{512}{431}, \qquad \delta_p(X) \geq \frac{512}{431}.$$

Proof. In the comparison lemma, the fixed-conic term may be normalized using the actual dimensions of $W_{m,j}$; only the moving part needs to be enlarged. Its limiting coefficient is

$$C_K^{\mathrm{actual}} = \frac{1}{16}\int_{5/2}^{3}(2t-5)(-V'(t))\,dt = \frac{1}{8}\int_{5/2}^{3} V(t)\,dt = \frac{61}{1024},$$

where $V(t) = \mathrm{vol}(D_t)$ is given by Proposition 16.3. Integration by parts explains the middle equality, and the last chamber integral is $61/128$. The other three constants remain valid upper bounds. Thus the maximum coefficient becomes

$$\max\left\{\frac{387}{512}, \frac{801+61}{1024}, \frac{603+61}{1024}\right\} = \frac{431}{512}.$$

The same degeneration proves the asserted bound for $\delta(G;W)$, and $3/S_X(G) = 768/503 > 512/431$ proves the local bound. □

Corollary 17.10 (Reduction to double conic fibres)**.** *Every point on a reduced conic fibre satisfies $\delta_p(X) \geq 96/95$. If the conic bundle has no nonreduced fibre, then $\delta(X) \geq 96/95$, and X is K-stable. In general, the centre of every divisor with nonpositive beta invariant is contained in the union of the reduced supports of the nonreduced conic fibres.*

Proof. Combine Proposition 12.4 with the smooth-fibre estimate away from f_0 and the reduced-singular-fibre estimate in Section 12. Alternatively, Theorem 17.7 gives a second sufficient bound at the smooth-fibre points of f_0. The least of the three bounds is $96/95$. A centre containing a point with local threshold greater than one cannot have nonpositive beta invariant. If all fibres are reduced, this also gives the stated global delta bound. □

Lemma 17.11 (The different normal bundle at a double conic fibre)**.** *Suppose $\varphi^{-1}(y) = 2\Lambda$, where $\Lambda \cong \mathbb{P}^1$ is its reduced support. Then*

$$L\cdot\Lambda = 1, \qquad N_{\Lambda/X} \cong \mathcal{O}(1)\oplus\mathcal{O}(-2).$$

For $p \in \Lambda$, let Λ_Y be its strict transform on $Y = \mathrm{Bl}_p X$. Then

$$N_{\Lambda_Y/Y} \cong \mathcal{O}\oplus\mathcal{O}(-3).$$

Blowing up Λ_Y gives an exceptional surface $R_\Lambda \cong \mathbb{F}_3$. If e is its negative section, then

$$(\tau^*D_t - \alpha R_\Lambda)\cdot e = 1 - t \quad \text{for every real } \alpha.$$

In particular, subtracting any multiple of this one exceptional divisor cannot produce a nef positive part for $t > 1$.

Proof. The reduced fibre is a line in the plane of the conic, so $H\Lambda = 1$ and $B\Lambda = 0$. Locally on the base the conic bundle lies in a trivial plane bundle. Along Λ, the normal sequence of its inclusion in that fourfold is the kernel of

$$\mathcal{O}(1)\oplus\mathcal{O}^{\oplus 2} \longrightarrow \mathcal{O}(2).$$

The map on $\mathcal{O}(1)$ is zero because the fibre equation is a square. The other two entries are binary quadrics q_1, q_2 with no common zero: a common zero would make the total threefold singular. Their kernel is $\mathcal{O}(-2)$. Thus the displayed kernel is $\mathcal{O}(1)\oplus\mathcal{O}(-2)$.

Blowing up a point of a smooth curve twists both normal summands by $\mathcal{O}(-1)$, giving the second formula. On the exceptional $\mathbb{F}_3$, with ruling class r, the normal divisor is $-e-3r$. The restriction of τ^*D_t is $(1-t)r$. As $e^2=-3$ and $r\cdot e=1$, the restriction of $\tau^*D_t-\alpha R_\Lambda$ has intersection

$$(\alpha e+(1-t+3\alpha)r)\cdot e=1-t.$$

This proves the obstruction to stopping after this blow-up. It gives the starting geometry for the remaining calculation, not a completed decomposition there. □

Remark 17.12 (Changing the order of extraction). The smooth-fibre cases, including high contact at the distinguished point, are settled by the surface blow-up in Proposition 12.4 and independently by the exceptional-plane refinement. Lemma 17.11 explains why a double conic fibre requires different geometry. We now extract its reduced curve with weights $(1,2)$ before making any point extraction, and apply Abban–Zhuang to the resulting exceptional surface.

18 Weighted extraction of a double conic fibre

Suppose $\varphi^{-1}(y)=2\Lambda$, where $\Lambda\cong\mathbb{P}^1$ is the reduced line. In this section we extract the curve before extracting any point. The scheme structure of the fibre specifies the weights and, crucially, the different on the exceptional surface.

Lemma 18.1 (The canonical weighted extraction). *Let $\mathcal{J}=\mathfrak{m}_y\mathcal{O}_X$ and let $\omega: X^\sharp=\mathrm{Bl}_{\mathcal{J}}X\to X$. This is a normal threefold, and its reduced exceptional divisor F is a plt-type divisor over X. Transversely to Λ, the morphism is a weighted blow-up with weights $(1,2)$. Moreover,*

$$F\cong\Lambda\times\mathbb{P}^1,\qquad \mathcal{J}\mathcal{O}_{X^\sharp}=\mathcal{O}_{X^\sharp}(-2F),\qquad A_X(F)=3.$$

Writing $h=\mathcal{O}_F(1,0)$ and $f=\mathcal{O}_F(0,1)$, one has

$$\omega^*L|_F=h,\qquad 2F|_F=-f,\qquad (K_{X^\sharp}+F)|_F=K_F+\tfrac{1}{2}D,$$

where D is a smooth curve of bidegree $(2,1)$, and $D\to\Lambda$ is an isomorphism.

Proof. Locally on the base the conic bundle is a quadratic hypersurface in a trivial plane bundle. Let x be a local equation of the line in its plane, and let u,v be parameters at y. Its equation, to the relevant weighted order, is

$$x^2+uq_1(s_0,s_1)+vq_2(s_0,s_1)+\text{terms of weighted order greater than two}=0,$$

where the weights of x,u,v are $1,2,2$. The binary quadrics q_1,q_2 have no common zero, since otherwise X would be singular at such a point of Λ. Where q_1 is a unit we can eliminate u and write $\mathcal{J}=(x^2,v)$. Thus the extraction has transverse weights $(1,2)$.

The two charts of the blow-up of (x^2,v) are the smooth chart $v=x^2r$ and the chart $x^2=vr$. The latter is an A_1 surface singularity times the parameter along Λ, hence is normal. The exceptional Cartier divisor has multiplicity two on its reduced support.

For a global description, use flatness of φ from Lemma 10.5. Formation of the Rees algebra, and hence of the blow-up, commutes with flat base change. Thus

$$X^\sharp=X\times_{\mathbb{P}^2}\mathrm{Bl}_y\mathbb{P}^2.$$

The exceptional scheme is $(2\Lambda)\times\mathbb{P}^1$, whose reduction is $F=\Lambda\times\mathbb{P}^1$, with second coordinate $[u:v]$. The pullback of the exceptional divisor of the blown-up base is $2F$. Restricting its normal bundle to F gives $2F|_F=-f$. The singular-locus curve of $X^\sharp$ on F is

$$D=(q_1(s_0,s_1)u+q_2(s_0,s_1)v=0)\subset F.$$

It is the graph of a degree-two morphism $\Lambda \to \mathbb{P}^1$, and is therefore smooth of bidegree $(2,1)$.

On the smooth chart, $dx \wedge dv = x^2 dx \wedge dr$, giving $K_{X^\sharp} = \omega^* K_X + 2F$ and $A_X(F) = 3$. On the other chart use the double cover $x = ab$, $v = a^2$, $r = b^2$, with $(a,b) \mapsto (-a,-b)$. This is an index cover for F, which has Cartier index two along the singular locus; the threefold $X^\sharp$ itself is Gorenstein there. The divisor F is the image of $a = 0$. The quotient is free in codimension one, the pair with this divisor is plt, and adjunction gives different $\frac{1}{2}D$. Here "plt" means purely log terminal; in particular, the exceptional divisor is normal and its adjoint pair is klt. Also $-F$ is relatively ample, since $-2F$ is the tautological divisor of the blow-up. This verifies the hypotheses for Abban–Zhuang refinement.

Finally, $L\Lambda = 1$ gives $\omega^* L|_F = h$. The adjunction identity can also be checked numerically:

$$-h + 3(-f/2) = K_F + \tfrac{1}{2}D = -2h - 2f + \tfrac{1}{2}(2h + f).$$

□

We spell out the invariants for this refinement, since the boundary is nonzero and F is not Cartier. Put $\Delta_F = D/2$ and

$$W^F_{m,j} = \operatorname{gr}^j_F H^0(X, mL), \qquad N_m = h^0(X, mL) = \sum_j \dim W^F_{m,j}.$$

A nonzero refined section $s \in W^F_{m,j}$ has an effective rational zero divisor $Z_F(s)$ on F: lift s to a section of mL of order j along F, remove jF from its pullback zero divisor, and take the reflexive restriction to F. This is the restriction of [4, Lemma 2.7], and it depends only on the refined section. Fractional coefficients are allowed; the odd-weight factor in Lemma 18.5 contributes $D/2$.

For a prime divisor Q over F, let

$$S_m(W^F; Q) = \sup_{\{s_{j,i}\}} \frac{1}{mN_m} \sum_j \sum_{i=1}^{\dim W^F_{m,j}} \operatorname{ord}_Q(Z_F(s_{j,i})),$$

where the supremum is over bases in each summand. Set $S(W^F; Q) = \lim_m S_m(W^F; Q)$ and

$$A_{F,\Delta_F}(Q) = A_F(Q) - \operatorname{ord}_Q(\Delta_F), \qquad \delta(F, \Delta_F; W^F) = \inf_Q \frac{A_{F,\Delta_F}(Q)}{S(W^F; Q)}.$$

Zero denominators are omitted. Bounded support and the ample-subseries property for this refinement, verified below, give the existence of the limits by [4, Sections 2.5–2.6]. Thus this invariant is computed from the actual refined spaces $W^F_{m,j}$ and the pair $(F, D/2)$. Replacing these spaces by all sections of their divisor classes, or omitting the boundary $D/2$, would change the invariant.

Remark 18.2 (Why the exceptional surface is a product). The weights belong to the filtration defined by the scheme-theoretic fibre, not just to a chosen splitting of $N_{\Lambda/X}$. The quadratic transition terms matter. Its even normal algebra is generated by the two base parameters, and its degree-two part fits into

$$0 \longrightarrow \mathcal{O}_\Lambda(-2) \xrightarrow{(q_1,q_2)} \mathcal{O}_\Lambda^{\oplus 2} \longrightarrow \mathcal{O}_\Lambda(2) \longrightarrow 0.$$

The first map represents the square of the weight-one coordinate. Taking the even Veronese algebra therefore gives the product $\Lambda \times \mathbb{P}^1$. This explains why the ordinary normal-bundle calculation in Lemma 17.11 must not be substituted for the weighted extraction.

Lemma 18.3 (The initial volume interval)**.** *Put $D_t^\sharp = \omega^* L - tF$. For $0 \le t \le 2$ this divisor is nef and*

$$\operatorname{vol}(D_t^\sharp) = 16 - \tfrac{3}{2}t^2.$$

Proof. The pencil of lines through y resolves on $X^\sharp$, so $\omega^* B - 2F$ is globally generated. Both $\omega^* H$ and $\omega^* B$ are nef, which proves nefness on the stated interval. On F the intersection formulas of Lemma 18.1 give

$$(\omega^* L)^2 F = 0, \qquad \omega^* L \cdot F^2 = -\tfrac{1}{2}, \qquad F^3 = 0.$$

Taking the cube proves the formula. Its contribution to $S_X(F)$ is $\frac{1}{16}\int_0^2 (16 - 3t^2/2)dt = 7/4$; this is not the whole expected vanishing order. □

Lemma 18.4 (The plane component and the threshold)**.** *Let Π be the plane of the double conic, and let Q be the strict transform of the plane divisor $\pi^{-1}(\Pi)$ on $X^\sharp$. Then*

$$Q = \omega^* H - 2F, \qquad R := Q|_F \in |\mathcal{O}_F(1,1)|.$$

For $t \ge 3$, every effective representative of $D_t^\sharp$ contains $(t-3)Q$. The pseudo-effective threshold of F is four. These assertions also hold when $\pi(\Lambda)$ passes through p_0, or when it has nontransverse intersection with C.

Proof. Choose base coordinates with $y = [1:0:0]$. In the graph description of X, the first equation is $l_0 y_0 + l_1 y_1 + l_2 y_2 = 0$, and $\Pi = (l_0 = 0)$. The order of l_0 along F is two, with initial form

$$l_1|_\Lambda\, u + l_2|_\Lambda\, v.$$

The two linear forms do not both vanish identically on Λ, because the common zero of l_0, l_1, l_2 is only p_0. This proves both class formulas. The curve R may be reducible; no smoothness assumption on it will be made.

Take a general line $g \subset \Pi$ through p_0, and let $\widetilde{g}$ be its lift. It meets C only at p_0, transversely, so $H\widetilde{g} = 1$ and $B\widetilde{g} = 2$. On g, the ratio $l_1 : l_2$ is constant. The graph equation therefore shows that $\varphi(\widetilde{g})$ is a line through y, and the map to that line has degree two. Thus the pullback of $\mathfrak{m}_y$ to $\widetilde{g} \cong \mathbb{P}^1$ has degree two. It is invertible, so the curve lifts to $X^\sharp$, with $2F\widetilde{g} = 2$. This argument applies whether or not Λ passes through p_0.

Consequently, $Q\widetilde{g} = -1$ and $D_t^\sharp \widetilde{g} = 3 - t$. These curves cover Q. Removing its coefficient c from any effective representative gives $3 - t + c \ge 0$, hence $c \ge t - 3$.

For a section of mL of order at least j along F, this forces a plane component of coefficient at least $j - 3m$. As before, $mL - cH$ is effective only if $c \le m$, by the effective cone in Lemma 11.1. Hence $j \le 4m$. Conversely, the plane section of H has order two, as does a general section of B through y. Their product has order four, proving the threshold assertion. □

Lemma 18.5 (The refined series, including odd weights)**.** *For the refinement W^F defined above, put*

$$\epsilon_j = j \bmod 2, \qquad k_{m,j} = \max\{0, j - 3m\},$$

$$a_{m,j} = m - k_{m,j} - \epsilon_j, \qquad b_{m,j} = \frac{j - \epsilon_j}{2} - k_{m,j}.$$

For $0 \le j \le 4m$, the zero divisor of every refined section has the form

$$\frac{\epsilon_j}{2} D + k_{m,j} R + \operatorname{div}(s), \qquad s \in U_{m,j} := H^0(F, \mathcal{O}_F(a_{m,j}, b_{m,j})). \tag{18.1}$$

More precisely, removing the displayed fixed factors identifies $W^F_{m,j}$ with a linear subspace of $U_{m,j}$. For $j > 4m$ the refined space is zero.

Proof. Define the valuation ideals $\mathcal{I}_j = \omega_* \mathcal{O}_{X^\sharp}(-jF)$, where the sheaves on $X^\sharp$ are divisorial sheaves. The weighted normal algebra has generators x, u, v of weights $1, 2, 2$ and initial relation $x^2 + q_1 u + q_2 v = 0$. Eliminating x^2 gives the following sheaves on Λ:

$$\mathcal{I}_{2r}/\mathcal{I}_{2r+1} \cong \mathcal{O}_\Lambda \otimes \operatorname{Sym}^r \mathbb{C}^2,$$
$$\mathcal{I}_{2r+1}/\mathcal{I}_{2r+2} \cong \mathcal{O}_\Lambda(-1) \otimes \operatorname{Sym}^r \mathbb{C}^2.$$

These descriptions glue globally. The two base parameters are global generators on a neighbourhood of y. The initial term of a change of the weight-one coordinate is multiplied by the transition function of the conormal line $\mathcal{O}_\Lambda(-1)$; all terms involving base parameters have higher weight. Thus the even part is generated by the base parameters, while the odd part has the additional conormal factor.

Restriction of sections gives an injection

$$W^F_{m,j} \hookrightarrow H^0(\Lambda, L^m|_\Lambda \otimes (\mathcal{I}_j/\mathcal{I}_{j+1})).$$

Since $L|_\Lambda = \mathcal{O}_\Lambda(1)$, the target is identified with $H^0(F, \mathcal{O}_F(m, r))$ for $j = 2r$, and with $H^0(F, \mathcal{O}_F(m-1, r))$ for $j = 2r+1$. In the odd case the effective rational zero divisor also contains $D/2$: the square of the odd factor has equation $q_1 u + q_2 v$, whose divisor is D. This is exactly the fractional term prescribed by reflexive restriction.

This is also the reflexive restriction convention of [4, Lemma 2.7 and Example 2.15]. It is essential to retain the half divisor for odd weights, rather than treating F as Cartier. Lemma 18.4 forces $k_{m,j}$ copies of Q; restricting and removing them gives exactly (18.1). Only an inclusion is claimed. Extra conditions caused by intersections with exceptional curves are harmless for the upper bounds below. □

Lemma 18.6 (Five explicit integrals)**.** *Define piecewise linear functions by*

t	$a(t)$	$b(t)$	$k(t)$
$[0,3]$	1	$t/2$	0
$[3,4]$	$4-t$	$3-t/2$	$t-3$.

The constants associated to the complete spaces $U_{m,j}$, normalized by mN_m with $N_m = h^0(X, mL)$, are

$$c_h = \frac{3}{16}\int_0^4 a^2 b\,dt = \frac{65}{128}, \qquad c_f = \frac{3}{16}\int_0^4 ab^2\,dt = \frac{151}{256},$$
$$c_R = \frac{3}{8}\int_3^4 kab\,dt = \frac{5}{64}, \qquad c_D = \frac{3}{8}\int_0^4 \int_0^{\min(a/2,b)} (a-2z)(b-z)\,dz\,dt = \frac{53}{256}.$$

In addition,

$$S_X(F) \le c_F := \frac{3}{8}\int_0^4 tab\,dt = \frac{161}{64} < 3.$$

Proof. For a complete series of bidegree (A, B), the maximal sum of orders along a fixed fibre of the first projection is $A(A+1)(B+1)/2$, and that along a fibre of the second projection is $B(A+1)(B+1)/2$. These are the sums of exponents of a monomial basis. For any curve of bidegree $(2, 1)$, the maximal sum of orders is

$$\sum_{r\ge 1} h^0(F, \mathcal{O}_F(A-2r, B-r)).$$

Since $N_m = (8/3)m^3 + O(m^2)$, summing over j and taking limits gives exactly the displayed integrals. The $O(1)$ changes in bidegrees for odd j do not affect the limits. The separate

contributions are

t	c_h	c_f	c_R	c_D	c_F
$[0,1]$	$3/64$	$1/64$	0	$3/256$	$1/16$
$[1,3]$	$3/8$	$13/32$	0	$5/32$	$13/8$
$[3,4]$	$11/128$	$43/256$	$5/64$	$5/128$	$53/64$.

For example, the upper limit in the c_D integral is $t/2$ on $[0,1]$, $1/2$ on $[1,3]$, and $(4-t)/2$ on $[3,4]$. Thus every integral is a polynomial integral on the listed intervals.

For the last assertion, Lemma 18.5 gives $\dim W^F_{m,j} \leq \dim U_{m,j}$. Therefore

$$S_m(F) = \frac{\sum_j j \dim W^F_{m,j}}{mN_m} \leq \frac{\sum_j j \dim U_{m,j}}{mN_m} \longrightarrow c_F.$$

The total dimension of the larger spaces has leading ratio $\frac{3}{8}\int_0^4 ab\,dt = 35/32$ to N_m. We continue to normalize by N_m, not by the dimension of these larger spaces. □

Lemma 18.7 (Invariant boundaries on the exceptional surface)**.** *Choose coordinates on F so that*

$$D = (s_0^2 u_1 - s_1^2 u_0 = 0).$$

Consider the action

$$([s_0 : s_1], [u_0 : u_1]) \longmapsto ([\lambda s_0 : s_1], [\lambda^2 u_0 : u_1]).$$

Its invariant prime curves are the four coordinate fibres and the curves

$$D_c = (s_0^2 u_1 - c s_1^2 u_0 = 0), \qquad c \in \mathbb{C}^*.$$

Let M be an effective rational divisor supported on finitely many of these curves. Let ν_0, ν_∞ be the valuations of weights $(1,2)$ at

$$o_0 = ([0:1],[0:1]), \qquad o_\infty = ([1:0],[1:0]).$$

If every prime coefficient of M is at most one and

$$\nu_0(M) \leq 3, \qquad \nu_\infty(M) \leq 3,$$

then (F, M) is log canonical.

Proof. The projection $D \to \mathbb{P}^1_{[u_0:u_1]}$ has degree two. Its two ramification points and two branch points are distinct in characteristic zero. Moving them to zero and infinity gives the stated equation. The invariant-curve list follows either from the invariant function $s_0^2 u_1/(s_1^2 u_0)$ or by factoring a weight-homogeneous bihomogeneous polynomial.

Outside o_0, o_∞, the listed curves have only simple normal crossings, so the coefficient condition suffices. At o_0, use coordinates (s,u). The curves through the point are $s=0$, $u=0$, and $u = s^2/c$. Let their coefficients be $a, b, c_1, \ldots, c_r$. The two successive ordinary point blow-ups resolve this collection. The exceptional valuations have weights $(1,1)$ and $(1,2)$ and log discrepancies two and three. The second required inequality is

$$a + 2b + 2\sum_i c_i \leq 3.$$

Together with $a \leq 1$, it implies $a + b + \sum_i c_i \leq 2$, which is the inequality for the first blow-up. After the second blow-up, the parabolas meet the exceptional curve at distinct points; all remaining intersections are simple normal crossings. This proves log canonicity at o_0. The coordinates at o_∞ give the identical calculation. □

Proposition 18.8 (The surface threshold with its different)**.** *For the refinement by F,*

$$\delta(F, \tfrac{1}{2}D; W^F) \geq \frac{128}{123} > 1.$$

Proof. We apply the surface point-extraction inequality to the actual series W^F, with boundary $D/2$. This is the form of Abban–Zhuang refinement explained in [3, Corollary 1.102 and Remark 1.113], with the boundary retained as in [4, Theorem 3.3]. For a plt extraction $\sigma : \widetilde{F} \to F$ at p, with exceptional curve T, write V^T for the refinement of W^F by T and Δ_T for the different of $(\widetilde{F}, T + \frac{1}{2}\widetilde{D})$. Then

$$\delta_p(F, \tfrac{1}{2}D; W^F) \geq \min\left\{ \frac{A_{F,\frac{1}{2}D}(T)}{S(W^F; T)}, \inf_{q \in T} \frac{1 - \operatorname{coeff}_q \Delta_T}{S(V^T; q)} \right\}.$$

All moments below are normalized by mN_m, where $N_m = h^0(X, mL)$, even when we enlarge the refined spaces to $U_{m,j}$. We first explain this comparison for a point extraction. Removing the factors in (18.1) gives an inclusion into $U_{m,j}$. After shifting the exceptional order by $k_{m,j} \operatorname{ord}_T(R) + (\epsilon_j/2) \operatorname{ord}_T(D)$, its associated graded spaces inject into those of $U_{m,j}$. A basis in each such space can be extended to a basis of the larger space. All orders of effective zero divisors, including their effective restrictions to T, are nonnegative. Thus the maximal sums of orders are bounded above by those for the complete spaces. The fixed factor $k_{m,j}R$ contributes its order along T to the first moment and the restriction of its strict transform to the point moments on T. Replacing the dimensions of the actual spaces by $\dim U_{m,j}$ bounds both contributions from above. The additional factor $\epsilon_j D/2$ contributes $O(m^{-1})$ to all the moments considered below. This follows because $\epsilon_j \leq 1$, while the sum of the dimensions of the $U_{m,j}$ is $O(m^3)$ and mN_m has order m^4.

Fix $p \in F$ and choose affine coordinates (s, u) from the two factors, both zero at p. A basis of $H^0(F, \mathcal{O}_F(A, B))$ is given locally by $s^i u^j$, with $0 \leq i \leq A$ and $0 \leq j \leq B$. First let σ be the ordinary blow-up of p. The order of this monomial along T is $i + j$. The two ruling directions give points $q_s, q_u \in T$, and its restriction after removing this exceptional order has orders i, j at these points. Therefore the complete-space contributions to the three moments are respectively $c_h + c_f$, c_h, and c_f.

At any other point $q \in T$, the degree-k initial space has dimension

$$n_{A,B}(k) = \max\{0, \min(A, k) - \max(0, k - B) + 1\}.$$

After removing its common zeros at q_s, q_u, it is the complete space of polynomials of degree $n_{A,B}(k) - 1$ on $T \cong \mathbb{P}^1$. Its maximal sum of orders at q is $n_{A,B}(k)(n_{A,B}(k) - 1)/2$. If a, b denote the functions in Lemma 18.6 and

$$d(t, v) = \max\{0, \min(a(t), v) - \max(0, v - b(t))\},$$

then the normalized limiting sum is

$$c_{\mathrm{pt}} = \frac{3}{8} \int_0^4 \int_0^{a(t)+b(t)} \frac{d(t,v)^2}{2}\, dv\, dt = \frac{3}{8} \int_0^4 \left(\frac{\min(a,b)^2 \max(a,b)}{2} - \frac{\min(a,b)^3}{6} \right) dt = \frac{43}{128}.$$

For completeness, the contributions to the last integral before multiplication by $3/8$ are $1/4$, $11/24$, and $3/16$ on $[0, 2]$, $[2, 3]$, and $[3, 4]$, respectively.

A local equation of R is $\alpha + \beta s + \gamma u + \eta su$. Hence $\operatorname{mult}_p R \leq 2$. If this multiplicity is two, the initial equation is a nonzero multiple of su, so its two tangent directions are distinct. If the multiplicity is one, its initial equation is linear. In either case the restriction of the strict transform of R to T has coefficient at most one at every point. This also covers reducible R: a

divisor of bidegree $(1,1)$ is either irreducible or the reduced union of one fibre of each ruling. Consequently

$$\begin{aligned} S(W^F;T) &\le c_h + c_f + 2c_R = \frac{321}{256}, \\ S(V^T;q_s) &\le c_h + c_R = \frac{75}{128}, \qquad S(V^T;q_u) \le c_f + c_R = \frac{171}{256}, \\ S(V^T;q) &\le c_{\mathrm{pt}} + c_R = \frac{53}{128} \quad (q \neq q_s, q_u). \end{aligned}$$

These estimates are uniform in p and q: the finite-dimensional formulas depend only on A, B, and the fixed-factor multiplicities are uniformly bounded as above.

If $p \notin D$, the different on T is zero and $A_{F,\frac{1}{2}D}(T) = 2$. The point-extraction inequality gives $\delta_p(F, \frac{1}{2}D; W^F) \ge 256/171$. Suppose next that $p \in D$ is not a ramification point of the degree-two projection $D \to \mathbb{P}^1_{[u_0:u_1]}$. Since the other projection restricts to an isomorphism on D, the tangent direction of D at p is neither ruling direction. Thus $\Delta_T = \frac{1}{2}q_D$, with $q_D \neq q_s, q_u$, and $A_{F,\frac{1}{2}D}(T) = 3/2$. It follows that

$$\delta_p(F, \tfrac{1}{2}D; W^F) \ge \min\left\{\frac{128}{107}, \frac{128}{75}, \frac{256}{171}, \frac{64}{53}\right\} = \frac{128}{107} > \frac{128}{123}.$$

It remains to consider the two ramification points. Choose product coordinates there so that $D = (u = s^2)$. Blow up p and then blow up the intersection of the first exceptional curve with the strict transform of D. Contracting the strict transform of the first exceptional curve, which has self-intersection -2, gives the weighted blow-up with weights $\mathrm{ord}_T(s) = 1$ and $\mathrm{ord}_T(u) = 2$. Its exceptional curve is $T = \mathbb{P}(1,2)$. The weighted blow-up has a quotient singularity of type $\frac{1}{2}(1,1)$ at $q_s = (s = 0)$ and is smooth elsewhere on T. The strict transform of D meets T transversally at the smooth point $q_D = (u/s^2 = 1)$, which differs from q_s and $q_u = (u = 0)$. Adjunction and the discrepancy formula therefore give

$$\Delta_T = \tfrac{1}{2}q_s + \tfrac{1}{2}q_D, \qquad A_{F,\frac{1}{2}D}(T) = 1 + 2 - \tfrac{1}{2}\cdot 2 = 2.$$

In particular, $(\widetilde{F}, T + \frac{1}{2}\widetilde{D})$ is plt, so the same point-extraction inequality applies.

For the complete bidegree spaces the exceptional order is now $i + 2j$, giving the moment $c_h + 2c_f$. The orders on T at q_s, q_u are $i/2, j$, respectively; the factor $1/2$ is the quotient-index contribution at q_s. At any other point, say $u/s^2 = c \neq 0$, order at least r is equivalent to divisibility of the weighted initial polynomial by $(u - cs^2)^r$. Summing over all weighted degrees, the dimension of these divisible polynomials is

$$(A - 2r + 1)_+(B - r + 1)_+, \qquad x_+ := \max\{x, 0\}.$$

Thus the maximal normalized sum of orders at this point is $c_D = 53/256$, by exactly the dimension sum in Lemma 18.6.

The four monomials $1, s, u, su$ in the equation of R have distinct weights $0, 1, 2, 3$. Its initial equation is therefore one monomial $s^i u^j$ with $i, j \in \{0, 1\}$. Its exceptional order is at most three, and the restriction of its strict transform to T has coefficients at most $1/2$ at q_s and one at q_u, with no other points in its support. Consequently

$$\begin{aligned} S(W^F;T) &\le c_h + 2c_f + 3c_R = \frac{123}{64}, \\ S(V^T;q_s) &\le \frac{c_h + c_R}{2} = \frac{75}{256}, \qquad S(V^T;q_u) \le c_f + c_R = \frac{171}{256}, \\ S(V^T;q) &\le c_D = \frac{53}{256} \quad (q \neq q_s, q_u). \end{aligned} \tag{18.2}$$

The point-extraction inequality now yields

$$\delta_p(F, \tfrac{1}{2}D; W^F) \geq \min\left\{\frac{128}{123}, \frac{128}{75}, \frac{256}{171}, \frac{128}{53}\right\} = \frac{128}{123}.$$

Every point of F belongs to one of the three cases. Taking the infimum proves the asserted global threshold bound, including valuations whose centres on F are curves. □

Example 18.9 (Why a second point blow-up is useful). At a ramification point the boundary is $\frac{1}{2}(u - s^2 = 0)$. After the ordinary blow-up, its strict transform meets the exceptional curve in the ruling direction q_u. The estimates above would then give only $(1/2)/(171/256) = 128/171$ for that point of the exceptional curve. Thus these estimates after one ordinary blow-up do not prove the required strict bound. Blowing up once more separates the boundary from that ruling direction. The resulting exceptional valuation has weights $(1, 2)$ and discrepancy two for $(F, D/2)$; its expected vanishing is at most $123/64$. This gives $128/123$. Both the half coefficient of the boundary and the half coefficient in the different at the quotient point are included in the calculation.

Theorem 18.10 (Points on a double conic fibre)**.** *For every point $p \in \Lambda$, where $\varphi^{-1}(y) = 2\Lambda$, one has*

$$\delta_p(X) \geq \frac{128}{123} > 1.$$

Proof. The extraction in Lemma 18.1 is of plt type, with centre Λ and discrepancy three. Abban–Zhuang refinement applies with the different $\frac{1}{2}D$. The local thresholds of the refined surface along centres mapping to p are bounded below by its global threshold. Thus

$$\delta_p(X) \geq \min\left\{\frac{3}{S_X(F)}, \delta(F, \tfrac{1}{2}D; W^F)\right\} \geq \min\left\{\frac{192}{161}, \frac{128}{123}\right\} = \frac{128}{123}.$$

The refined series has bounded support by Lemma 18.4; it contains an ample series, as in [4, Example 2.15]. Concretely, $\omega^*L - tF$ is ample for sufficiently small positive t, since L is ample and $-F$ is relatively ample. This also verifies the positivity hypothesis needed for refinement. □

Proof of Theorem 1.1 for Family 2.9. A conic fibre is smooth, a reduced union of two lines, or a double line. Sections 12 and 17 give $\delta_p(X) \geq 96/95$ at every point of a reduced fibre. Theorem 18.10 gives the stronger bound $128/123$ at every point of a double fibre. Therefore $\delta(X) \geq 96/95 > 1$, proving K-stability for every smooth member of Family 2.9. □

Remark 18.11 (The role of threefold Zariski decompositions). For the weighted curve extraction, we computed the nef interval $0 \leq t \leq 2$ and forced the plane component for $t \geq 3$. We did not compute a complete Zariski decomposition of $\omega^*L - tF$ for $2 < t < 4$. The spaces $U_{m,j}$ are explicit spaces containing the refined sections, not asserted complete descriptions of them. Keeping the original normalization makes their valuation sums valid upper bounds. The surface estimate with the correct different then supplies the Abban–Zhuang inequality. Thus a complete threefold decomposition is unnecessary for this proof, even though blowing up the scheme-theoretic base fibre is essential.

Part IV
Family 2.10: quartic del Pezzo fibrations

19 Geometry and the local estimates

Let $V \subset \mathbb{P}^5$ be a smooth complete intersection of two quadrics, and let C be its smooth intersection with a linear subspace $\mathbb{P}^3$. Put $X = \mathrm{Bl}_C(V)$, and denote the blowup by $\pi\colon X \to V$ and its exceptional divisor by E. These are the smooth Fano threefolds in Family 2.10 of the Mori–Mukai classification; see [3, §4.3]. The pencil of hyperplanes containing C gives a quartic del Pezzo fibration $f\colon X \to \mathbb{P}^1$. If $H = \pi^*\mathcal{O}_V(1)$ and F is the fibre class, then

$$F = H - E, \qquad L = -K_X = H + F = 2H - E, \qquad L^3 = 16.$$

Every fibre is normal and has at worst Du Val singularities. This geometric fact, which does not require a group action, is recalled in [23, proof of Lemma 4.3]. Each fibre is smooth along C: otherwise its intersection with the second hyperplane defining C would not be smooth.

A direct application of adjunction to a fibre does not settle all cases. Some singular quartic del Pezzo surfaces have lines on whose smooth loci the local stability threshold is less than one. We call them *special lines* in this paper. Their fibre types are

$$A_2 + A_1, \quad A_2 + 2A_1, \quad A_3 + A_1, \quad A_3 + 2A_1, \quad A_4, \quad D_5.$$

They are specified by equations in Lemma 22.17. The following estimates treat these lines and the singular points for which the nodal-section argument does not apply.

Theorem 19.1. *Let $p \in X$, and let F_0 be its fibre.*

1. *If p is a D_4 singular point of F_0, then $\delta_p(X) \geq 16/15$.*
2. *If p is a D_5 singular point of F_0, then $\delta_p(X) = 9/8$.*
3. *If p is smooth on F_0 and lies on a special line, then $\delta_p(X) \geq 32/27$. This includes the intersection with E.*
4. *The same bound $32/27$ holds at the A_2 point of a mixed A_2 fibre and at the A_3 point of a mixed A_3 fibre.*

Here δ_p takes the infimum over divisors whose centres *contain* p, not only those whose centres equal p. These local bounds therefore exclude the corresponding curve centres as well. The Family 2.10 case of Theorem 1.1 is proved in Section 21, using Theorem 19.1, which is proved in Section 22.

How we avoid Zariski decompositions on the extracted threefold. We identify the valuation filtration on every $H^0(X, mL)$ with a weight filtration of an explicit bigraded ring. Hilbert-weight calculations then give the expected orders, and further refinements are computed using polynomial spaces or monomial limits on the exceptional surfaces. In the ordinary point extraction along a special line, the seven original generators are insufficient. The additional section $r = \alpha au + bc$ has order three although its displayed summands have order two. Keeping this section and proving saturation gives $S_X(G) = 65/32$ and a refined threshold at least $32/27$. Merely assigning weights to the original generators would not prove either assertion. We do use a Zariski decomposition on an auxiliary *surface* in the nodal argument, but never on the threefold obtained by extracting the point.

In the D_4 standard monomials below, the factor z^η, $\eta = 0, 1$, records the two residue classes for passage to the coarse exceptional plane. In type D_5, the corresponding factor is v^η. Both classes are retained with their actual fractional zero divisors. Their fixed orders are bounded per section and hence contribute $O(1/m)$ to a normalized expected order. This is the convention of Proposition 2.7.

20 The section ring and its filtrations

20.1 Coordinates and the two quadric relations

Lemma 20.1. *At a D_4 or D_5 point p of a fibre, choose coordinates $[x_0 : u : v : z : w : t]$ so that $p = [1 : 0 : 0 : 0 : 0 : 0]$ and the fibre is $t = 0$. Its two equations can be taken to be*

	$Q_1\vert_{t=0}$	$Q_2\vert_{t=0}$
D_4	$x_0u + vw$	$z^2 + u(v + w)$
D_5	$x_0u + v^2 + zw$	$z^2 + uw.$

The centre is $(\ell = t = 0)$, with $\ell = x_0+$ a linear form in u, v, z, w. After rescaling, the coefficient of $t\ell$ in Q_2 is one. The three functions $v/\ell, z/\ell, w/\ell$ are regular parameters at p.

Proof. Quartic del Pezzo surfaces of types D_4 and D_5 each have a unique isomorphism class over $\mathbb{C}$; see [13, §3.4]. Their anticanonical equations are also displayed in [10, (1.4)–(1.5)]. Permuting variables and rescaling gives the equations above. An isomorphism of anticanonically embedded surfaces is induced by a linear change of coordinates.

Every fibre is smooth along C, since the two hyperplanes cutting out C have independent differentials on V there. Thus $p \notin C$, giving the stated form of ℓ. Along the fibre, the first quadric has differential du and the second has zero differential at p. Smoothness of V therefore forces the coefficient of $t\ell$ in the second quadric to be nonzero. The implicit-function theorem then eliminates u, t and gives the regular parameters. □

Example 20.2. On $x_0 = 1$, elimination of u gives $z^2 - vw(v+w)$ in type D_4, and $z^2 - v^2w - zw^2$ in type D_5. In the latter case replacing z by $z - w^2/2$ gives the usual equation $z^2 - v^2w - w^4/4$.

Put $\mathcal{P} = \mathbb{C}[u, v, z, w, a, b, e]$ and substitute $\ell = ea$, $t = eb$ in the two quadrics. Give the ring the bidegrees

$$\deg(u, v, z, w) = (1, 0), \quad \deg(a, b) = (1, -1), \quad \deg e = (0, 1).$$

The entries refer to H, E. Let $R = \mathcal{P}/(Q_1, Q_2)$ after substitution.

Lemma 20.3 (Actual global sections)**.** *One has $H^0(X, mL) = R_{(2m,-m)}$. Before imposing the relations, monomials on $a = e = 1$ are*

$$u^i v^j z^k w^l b^q, \qquad h = i + j + k + l \le m, \qquad h + q \le 2m.$$

Their homogeneous lifts multiply them by $a^{2m-h-q}e^{m-h}$.

Proof. The complete-intersection ideal $\mathcal{I}_C = (\ell, t)$ has the power resolution

$$0 \to \mathcal{O}_V(-m-1)^m \to \mathcal{O}_V(-m)^{m+1} \to \mathcal{I}_C^m \to 0.$$

After twisting by $2m$, intermediate cohomology vanishes because a complete intersection of two quadrics is arithmetically Cohen–Macaulay. Thus its global sections are generated by degree-m polynomials multiplied by the degree-m monomials in ℓ, t.

The extended Rees presentation above has no extra e-torsion. Indeed, modulo e the two equations define the affine cone over the complete-intersection curve C, so have height two. The two substituted equations form a regular sequence, and e is regular on their quotient. After inverting e this quotient is the coordinate ring of V with $e^{\pm1}$ adjoined. It is therefore the desired extended Rees ring. Taking the indicated component gives the assertion; solving the degree equations gives the monomial bounds. □

The finite-dimensionality assumption holds for our bidegrees even though some second coordinates are negative. Lemma 20.3 gives the degree bounds before any extra generator is added. If one adjoins a generator r of degree $L = (2, -1)$ and a monomial has exponent k in r, total exponent h in the four H-generators, and exponent q in b, its exponents in a, e are $2m - h - q - 2k, m - h - k$. Thus $h + k \leq m$ and $q \leq 2m - h - 2k$. All exponents in a fixed section degree are bounded by a constant times m.

Lemma 20.4 (The first and second filtrations)**.** *Use the following weights, with a, e of weight zero:*

	u	v	z	w	b	d_1	d_2
D_4	4	2	3	2	6	4	6
D_5	6	3	4	2	8	6	8.

They give the valuation of the weighted blowup in the three regular parameters. The first refinement is the $(2m, -m)$ component of $R_0 = \mathcal{P}/(\Psi_1, \Psi_2)$, where

D_4	$\Psi_1 = eau + vw$	$\Psi_2 = e^2ab + z^2 + u(v + w)$
D_5	$\Psi_1 = eau + v^2 + zw$	$\Psi_2 = e^2ab + z^2 + uw.$

Refinement by the tangent curve $B = (b = 0)$ is exactly the b-adic filtration. Simultaneous standard monomials are

	monomials on $a = e = 1$	*conditions*
D_4	$u^i v^j w^k z^\eta b^q$	$jk = 0, \quad \eta = 0, 1$
D_5	$u^i v^\eta z^j w^k b^q$	$ik = 0, \quad \eta = 0, 1,$

subject to the bounds of Lemma 20.3.

Proof. All omitted terms have larger weight: the changes replacing x_0 by ℓ add u times a positive-weight linear form to the first equation, and the only term in the second equation involving t with minimal weight is $t\ell$. The two initial equations form a regular sequence. Lemma 5.1 consequently identifies the polynomial weight-associated graded ring with the displayed complete intersection.

We check that this is also the geometric filtration, including cancellations. The equations obtained by setting $e = 0$ or $a = 0$ are

$$(vw, z^2 + u(v + w)) \quad \text{or} \quad (v^2 + zw, z^2 + uw),$$

both of height two. Since a complete intersection is Cohen–Macaulay, e, a are non-zero-divisors in R_0. Localizing at ae and setting $u' = u/(ea)$ and similarly for v, z, w, and $b' = b/a$, eliminates u', b' and gives a polynomial ring in v', z', w' with $a^{\pm 1}, e^{\pm 1}$ adjoined. Thus R_0 is a domain. The degrees of a, e form a lattice basis, so removing their common Laurent factor is injective on each fixed bidegree. This identifies its weight filtration with the local monomial valuation. In particular, a nonzero global initial form cannot disappear locally.

The same height-two argument applies after setting $b = 0$. After localization the remaining equation is the irreducible D_4 or D_5 tangent polynomial in the example above. Hence $R_0/(b)$ is a domain. It follows by induction that $(b^q) : (ae)^\infty = (b^q)$: reduce a putative divisibility relation modulo b, cancel b, and repeat. The same argument works for every multiplier outside (b). At the generic point of the tangent curve all native coordinates are nonzero and their weights have gcd one. Its geometric multiplicity is therefore one. These facts identify the geometric curve filtration with divisibility by b^q.

Finally choose leading monomials vw, z^2 in type D_4, and v^2, uw in type D_5. In each case these are relatively prime, so the two equations are a Gröbner basis. The standard monomials are as stated and form a free $\mathbb{C}[b]$-basis after allowing the other homogeneous variables. Thus they respect the second filtration too. □

20.2 A separate computation of $S_X(G)$

Lemma 20.5 (The weighted Hilbert calculation)**.** *Let $w_* = w_u + w_v + w_z + w_w$, let $d = w_b$, and let d_1, d_2 be the two relation weights. Then*

$$N_m = \frac{8}{3}m^3 + 4m^2 + \frac{10}{3}m + 1, \qquad S_X(G) = \frac{5}{16}w_* + \frac{11}{16}d - \frac{5}{32}(d_1 + d_2).$$

In particular, $S_X(G) = 6$ in type D_4, and $S_X(G) = 8$ in type D_5. In both cases $S(W;B) = 11/16$.

Proof. Let $D_m(s)$ be the number of monomials of $\mathcal{P}_{(s,-m)}$, and $T_m(s)$ their total weight. If h is the exponent sum in a, b, then

$$D_m(s) = \sum_{h=m}^{s} (h+1)\binom{s-h+3}{3}, \quad T_m(s) = \sum_{h=m}^{s} (h+1)\binom{s-h+3}{3}\left(\frac{w_*(s-h)}{4} + \frac{dh}{2}\right).$$

For $r = s - m \geq 0$, binomial convolution gives

$$\begin{aligned} D_m(s) &= m\binom{r+4}{4} + \binom{r+5}{5}, \\ T_m(s) &= w_*\left(m\binom{r+4}{5} + \binom{r+5}{6}\right) \\ &\quad + d\left[\binom{m+1}{2}\binom{r+4}{4} + (m+1)\binom{r+4}{5} + \binom{r+4}{6}\right]. \end{aligned}$$

Empty sums are zero. The two degree-two relations and their Koszul syzygy give

$$\begin{aligned} N_m &= D_m(2m) - 2D_m(2m-2) + D_m(2m-4), \\ M_m &= T_m(2m) - 2T_m(2m-2) + T_m(2m-4) \\ &\quad - (d_1+d_2)D_m(2m-2) + (d_1+d_2)D_m(2m-4). \end{aligned}$$

The last two terms account for the weights of the relations and syzygy themselves. Expansion yields

$$\begin{aligned} M_m = &\left(\frac{5w_*}{6} + \frac{11d}{6} - \frac{5(d_1+d_2)}{12}\right)m^4 \\ &+ \left(\frac{w_*}{3} + \frac{7d}{3} + \frac{d_1+d_2}{2}\right)m^3 \\ &+ \left(\frac{2w_*}{3} + \frac{13d}{6} - \frac{d_1+d_2}{12}\right)m^2 + \left(\frac{w_*}{6} + \frac{2d}{3}\right)m. \end{aligned}$$

Dividing by mN_m gives the formula. It yields 6 for $(w_*, d, d_1, d_2) = (11, 6, 4, 6)$ and 8 for $(15, 8, 6, 8)$.

For the b-order, the leading monomials of both relations have b-exponent zero. Their deletion and inclusion–exclusion therefore give the coefficient of d in the unshifted second difference of T_m. The total is

$$K_m = \frac{11}{6}m^4 + \frac{7}{3}m^3 + \frac{13}{6}m^2 + \frac{2}{3}m.$$

Lemma 20.4 makes this the actual curve-order sum. Its normalized limit is $11/16$. □

Example 20.6. For $m = 1$, the eleven sections are $1, b, b^2, u, ub, v, vb, z, zb, w, wb$, with their homogeneous lifts. Their total weights are 64 in type D_4 and 86 in type D_5, and their total b-order is 7. These agree with the finite formulas; the expected orders require the limit as $m \to \infty$.

20.3 Weights of all generators

Lemma 20.7 (Weights of the exceptional parameter)**.** *In the bigraded presentation of Lemma 20.3, allow all seven generators to have weights. Put w_* = the sum of the weights of the four H-generators. For a complete intersection with two relations of bidegree $2H$ and weights d_1, d_2, its normalized weight trace is*

$$\mathcal{T} = \frac{5}{16}w_* + \frac{11}{16}(w_a + w_b) + \frac{3}{8}w_e - \frac{5}{32}(d_1 + d_2). \tag{20.1}$$

This is a geometric expected order only after the relevant filtration has been identified.

Suppose a generator r of degree L and weight w_r is added, together with a relation of degree L and weight d_3. If the resulting three equations form a regular sequence, its trace is

$$\mathcal{T} = \frac{5}{16}w_* + \frac{11}{16}(w_a + w_b) + \frac{3}{8}w_e - \frac{5}{32}(d_1 + d_2) + \frac{1}{4}(w_r - d_3). \tag{20.2}$$

Proof. In the notation of Lemma 20.5, a monomial with exponent sum h in a, b has exponent $h - m$ in e. Thus the ambient total weight is

$$\sum_{h=m}^{s} (h+1)\binom{s-h+3}{3}\left(\frac{w_*(s-h)}{4} + \frac{(w_a+w_b)h}{2} + w_e(h-m)\right).$$

Apply the same Koszul second difference. The first two coefficients were computed in Lemma 20.5. The coefficient of w_e is $3/8$. One can also recover it without expansion: the two degree functions H and E give the constant weights $2m$ and $-m$ on the $(2m, -m)$ component. Substitution of the E-degrees, namely $w_a = w_b = -1$, $w_e = 1$ and $d_i = 0$, forces $-22/16 + 3/8 = -1$.

For the last assertion, take the logarithmic weight derivative of the Hilbert series. The new variable and relation contribute $(w_r - d_3)z^L/(1 - z^L)$ times the unweighted Hilbert series of R. Their normalized contribution is

$$(w_r - d_3)\lim_{m\to\infty} \frac{\sum_{k=1}^{m} N_{m-k}}{mN_m} = \frac{1}{4}(w_r - d_3).$$

Indeed, the Hilbert polynomial gives the exact identity $\sum_{k=1}^{m} N_{m-k} = (2m^4 + m^2)/3$. The ordinary Hilbert series remains unchanged: the factor $1 - z^L$ for the relation cancels that of the new variable. □

21 Global reduction and nodal auxiliary sections

We first record the fibre estimates and the argument at A-type singular points. The estimates at D-type points and at smooth points on the special lines are supplied by Theorem 19.1.

Theorem 21.1. *Let $p \in X \setminus E$, and let F_0 be its fibre. Suppose that there is a smooth hyperplane section $\overline{S} \subset V$ through $\pi(p)$ such that:*

1. *$\overline{S}$ contains no line through $\pi(p)$ and meets C transversally at four distinct points;*
2. *if S is its strict transform on X, then $B = F_0|_S$ is irreducible and has an ordinary node at p.*

Then every prime divisor Q over X with centre p satisfies

$$\beta_X(Q) \geq \frac{11}{16}\operatorname{ord}_Q(S) > 0.$$

For a prime Cartier divisor $T \subset X$, write

$$W^T_{m,j} = \operatorname{im}(H^0(X, mL - jT) \longrightarrow H^0(T, (mL - jT)|_T)).$$

Expected orders retain the normalization mN_m.

21.1 The fibre estimates

Lemma 21.2 (The strict adjunction inequality)**.** *Suppose that (X,T) is purely log terminal near p and that the different on T is zero. If the local threshold of W^T at p is at least 1 and $S_X(T)<1$, then*

$$\beta_X(Q)\geq (1-S_X(T))\,\mathrm{ord}_Q(T)$$

for every prime divisor Q whose centre contains p. In particular the inequality is strict when $c_X(Q)\subset T$.

Proof. This is [4, proof of Theorem 3.3, equation (3.6)], with $\lambda=1$. For completeness, choose an m-basis divisor compatible with the filtrations of both T and Q, and write it as $a_mT+D'_m$, where $a_m\to S_X(T)$. The restriction $D'_m|_T$ is a basis divisor of W^T. The threshold assumption and convergence for refined series provide numbers $\lambda_m\to 1$, with $\lambda_m>0$, such that $(T,\lambda_m D'_m|_T)$ is log canonical near p. Inversion of adjunction gives

$$A_X(Q)\geq \lambda_m\,\mathrm{ord}_Q(D'_m)+\mathrm{ord}_Q(T).$$

Compatibility with Q and passage to the limit give the assertion. A valuation has positive order on T when its centre is contained in T. □

Lemma 21.3. *One has $L^3=16$ and*

$$S_X(F)=\frac{11}{16},\qquad S_X(H)=\frac{5}{16},\qquad S_X(E)=\frac{3}{8}.$$

Proof. The intersection numbers are $H^3=H^2F=4$ and $HF^2=F^3=0$. For $0\leq t\leq 2$, the positive and fixed parts of $L-tF$ are

$$(P_t,N_t)=\begin{cases}(H+(1-t)F,\,0), & 0\leq t\leq 1,\\ ((2-t)H,\,(t-1)E), & 1\leq t\leq 2.\end{cases}$$

Indeed, H and F are nef. For $t>1$, restriction to an exceptional fibre of π forces multiplicity at least $t-1$ along E, and the remaining divisor is $(2-t)H$. For $t>2$ the intersection with H^2 is negative. Consequently

$$S_X(F)=\frac{1}{16}\left(\int_0^1(16-12t)\,dt+\int_1^2 4(2-t)^3\,dt\right)=\frac{11}{16}.$$

The divisor $L-uH=(1-u)H+F$ is nef precisely for $0\leq u\leq 1$ and is not pseudoeffective for $u>1$, since its intersection with HF is then negative. Similarly $L-uE=(1-u)H+(1+u)F$ is nef for $0\leq u\leq 1$ and is not pseudoeffective afterwards. Thus

$$S_X(H)=\frac{1}{16}\int_0^1 4(1-u)^2(4-u)\,du=\frac{5}{16},\qquad S_X(E)=\frac{1}{16}\int_0^1(16-24u+8u^3)\,du=\frac{3}{8}.$$

□

Lemma 21.4 (Refinement by a fibre)**.** *Let F_0 be a fibre with at most Du Val singularities and set $C_0=E|_{F_0}$. For every prime divisor Q over F_0,*

$$S(W^{F_0};Q)=\frac{1}{16}\,\mathrm{ord}_Q(C_0)+\frac{15}{16}S_{F_0}(Q).$$

Consequently, if $p\in F_0$ and $d_p=\delta_p(F_0)$, then

$$\delta_p(X)\geq\begin{cases}\min\{16/11,\,16d_p/15\}, & p\notin E,\\ \min\{16/11,\,16d_p/(d_p+15)\}, & p\in E.\end{cases}$$

If $d_p\geq 1$, every prime divisor over X whose centre contains p and is contained in F_0 has positive β-invariant.

Proof. The surface F_0 is isomorphic to its image on V: blowing up the Cartier curve C on that hyperplane section is an isomorphism. The curve C_0 is smooth, lies in the smooth locus, and is anticanonical on F_0. The decomposition in Lemma 21.3 restricts to $-K_{F_0}$ for $0 \le t \le 1$ and to $(2-t)(-K_{F_0})$ for $1 \le t \le 2$, with fixed part $(t-1)C_0$ in the latter interval.

These restrictions give complete moving systems for the asymptotic calculation. For rational $0 < t < 1$, the divisor $L - tF$ is ample; the restriction sequence and vanishing give surjectivity onto F_0 for sufficiently divisible large multiples. For $1 < t < 2$, remove the fixed multiple of E. The remaining sections come from multiples of H on V, and restriction to the hyperplane section is surjective since V is arithmetically Cohen–Macaulay. The endpoints have measure zero. Thus the refined volume formula gives

$$\begin{aligned} S(W^{F_0}; Q) &= \frac{3}{16}\int_1^2 4(2-t)^2(t-1)\,\mathrm{ord}_Q(C_0)\,dt \\ &\quad + \frac{3}{16}\left(1 + \int_1^2 (2-t)^3\,dt\right)\int_0^\infty \mathrm{vol}(-K_{F_0} - vQ)\,dv \\ &= \frac{1}{16}\,\mathrm{ord}_Q(C_0) + \frac{15}{16}S_{F_0}(Q). \end{aligned}$$

This is the same calculation as [8, Lemma 2.1]; its proof uses $L^3 = 4d$ and works here with $d = 4$. If the centre contains a point outside C_0, its order on C_0 is zero. Otherwise $\mathrm{ord}_Q(C_0) \le A_{F_0}(Q)$, since (F_0, C_0) is log canonical. Abban–Zhuang adjunction gives the displayed estimates. If $d_p \ge 1$, the refined threshold is at least 1, so Lemma 21.2, applied with $T = F_0$, gives $\beta_X(Q) \ge \frac{5}{16}\,\mathrm{ord}_Q(F_0) > 0$ for the indicated centres. □

Example 21.5 (Smooth fibres). A smooth quartic del Pezzo surface has $\delta = 4/3$; see [3, 12]. Thus $\delta_p(X) \ge 64/45$ if p lies on a smooth fibre and $p \notin E$, while $\delta_p(X) \ge 64/49$ if $p \in E$. In particular all centres meeting the locus of smooth fibres are excluded.

21.2 The nodal auxiliary section

We prove Theorem 21.1. Retain its notation. The morphism $S \to \overline{S}$ blows up four distinct points, with exceptional curves $e_1, \dots, e_4$. Write $h = H|_S$ and $b = F|_S = -K_S$. Thus $h^2 = hb = 4$, $b^2 = 0$ and $B \sim b$. For $a \in [0,1]$ put $D_a = ah + b$. These are the restrictions of $L - uH$ with $a = 1 - u$.

Let $\sigma\colon \widetilde{S} \to S$ be the blowup of p, with exceptional curve G. We use the same letters for pullbacks of h and b, and put $\widetilde{B} = b - 2G$. Then $G^2 = -1$, $\widetilde{B}^2 = -4$ and $G\widetilde{B} = 2$. The curve $\widetilde{B}$ is smooth and meets G transversally at two distinct points.

Lemma 21.6 (Nef class and Zariski decomposition)**.** *The class $T = h - 2G$ is nef. The pseudoeffective threshold of G with respect to $\sigma^* D_a$ is $2 + 2a$, and the Zariski decomposition of $\sigma^* D_a - vG$ is*

$$\begin{aligned} &0 \le v \le 2a: && P_{a,v} = ah + b - vG, && N_{a,v} = 0, \\ &2a \le v \le 2 + 2a: && P_{a,v} = aT + (1 + a - v/2)b, && N_{a,v} = (v/2 - a)\widetilde{B}. \end{aligned}$$

Proof. Blowing up $\pi(p)$ on the smooth quartic del Pezzo surface $\overline{S}$ gives a smooth cubic surface, since $\overline{S}$ contains no line through this point. Its exceptional curve is a line. Projection from this line gives a conic bundle with fibre class $h - 2G$. Pullback after the other four blowups proves that T is nef. The class b is also nef, being the pullback of the elliptic fibration on S.

For $0 \le v \le 2a$, the proposed positive part lies on the segment between the nef classes $ah + b$ and $aT + b$. On the second interval it is a nonnegative combination of T and b. The

negative part is supported on the curve $\widetilde{B}$ of square -4, and $P_{a,v}\widetilde{B} = 0$. These conditions prove that this is the Zariski decomposition. Finally

$$T \cdot (ah + b - vG) = 4a + 4 - 2v,$$

which is negative for $v > 2 + 2a$. This proves the claimed threshold. □

Lemma 21.7 (The three refined integrals)**.** *For the refinement of W^S by G, the expected order of G is at most 2. On G, the subsequent point terms are at most $5/8$ away from $G \cap \widetilde{B}$ and at most 1 at either point of $G \cap \widetilde{B}$. Consequently the local threshold of W^S at p is at least 1.*

Proof. The surface-volume and curve-restriction formulas of Abban–Zhuang theory, with the original normalization $L^3 = 16$, have factor $3/16$. Complete positive-part systems give upper bounds for the actual restricted systems, so the following upper bounds suffice without an assertion of completeness at boundary parameters. Lemma 21.6 gives

$$P_{a,v}^2 = \begin{cases} 4a^2 + 8a - v^2, & 0 \le v \le 2a, \\ 8a + 8a^2 - 4av, & 2a \le v \le 2 + 2a, \end{cases} \qquad P_{a,v} \cdot G = \begin{cases} v, & v \le 2a, \\ 2a, & v \ge 2a. \end{cases}$$

The expected order of G is therefore at most

$$\begin{aligned} &\frac{3}{16} \int_0^1 \left(\int_0^{2a} (4a^2 + 8a - v^2)\, dv + \int_{2a}^{2+2a} (8a + 8a^2 - 4av)\, dv \right) da \\ &\qquad = \frac{3}{16} \int_0^1 \left(\frac{16}{3} a^3 + 16a^2 + 8a \right) da = 2. \end{aligned}$$

The moving contribution to a point term on G is at most

$$\frac{3}{16} \int_0^1 \left(\int_0^{2a} v^2\, dv + \int_{2a}^{2+2a} 4a^2\, dv \right) da = \frac{3}{16} \int_0^1 \left(\frac{8}{3} a^3 + 8a^2 \right) da = \frac{5}{8}.$$

At a transverse intersection with $\widetilde{B}$, the fixed-part contribution is

$$\frac{3}{16} \int_0^1 \int_{2a}^{2+2a} 2(2a)(v/2 - a)\, dv\, da = \frac{3}{16} \int_0^1 4a\, da = \frac{3}{8}.$$

Thus the point term is at most $5/8 + 3/8 = 1$. Since $A_S(G) = 2$ and the different on G is zero, surface adjunction gives a local threshold at least $\min\{2/2, 1\} = 1$. □

Proof of Theorem 21.1. Apply Lemma 21.2 with $T = S$, using Lemma 21.7 and $S_X(S) = 5/16$. For a divisor centred at p one has $\operatorname{ord}_Q(S) > 0$, which proves the assertion. □

Remark 21.8. The equality $2/A_S(G) = 1$ is not discarded. Strictness comes from the term $(1 - S_X(S)) \operatorname{ord}_Q(S)$ on the threefold. The calculation uses a Zariski decomposition on the *surface* $\widetilde{S}$, together with the nef divisors $L - uH$ on X. It does not compute a Zariski decomposition on a threefold obtained by a weighted blowup.

21.3 Exclusion of the centres

Lemma 21.9. *If A is a quartic del Pezzo surface with Du Val singularities and $p \in A$ is smooth, then $\delta_p(A) \ge 1$ unless p lies on one of the special lines in Lemma 22.17. The exceptional possibilities are:*

singularity type of A	*threshold on the special line*
$A_2 + A_1$, $A_2 + 2A_1$	$6/7$
$A_3 + A_1$, $A_3 + 2A_1$	$3/4$
A_4	$24/29$
D_5	$3/4$.

Proof. This is the smooth-locus content of the complete quartic tables in [12, §6.2, Tables 21–35]. A smooth quartic del Pezzo surface has threshold $4/3$. For the singular surfaces, the values below one in their smooth loci are exactly those in the table. In particular, the smooth-locus bound for a D_4 surface is one, while that for a surface with a single A_2 or A_3 singularity is at least one. The equations and incidences of the special lines are given in Lemma 22.17. □

Lemma 21.10. *If p is an A_n singular point of a fibre, for $n \geq 1$, then a general hyperplane section $\overline{S} \subset V$ through $\pi(p)$ satisfies the hypotheses of Theorem 21.1.*

Proof. Every fibre is smooth along C, since the two hyperplanes cutting out C have independent differentials on V there. Thus $p \notin E$. A general hyperplane section through $\pi(p)$ is smooth and meets C transversally at four distinct points. There are only finitely many lines on V through $\pi(p)$. Indeed, their directions are cut out by two quadrics in the projectivized tangent space $\mathbb{P}^2$; a positive-dimensional intersection would give a plane or a quadric cone surface in V, or the two quadrics would coincide and give a quadric cone surface. Such a surface has degree 1 or 2, whereas $\mathrm{Pic}(V) = \mathbb{Z}[H]$ makes every effective divisor have degree divisible by 4. Hence $\overline{S}$ can be chosen to contain none of these lines.

The general hyperplane section of the integral fibre through its singular point is irreducible, by Bertini irreducibility. Here projection from that point has two-dimensional image: otherwise the fibre would be a cone with that vertex. Since the fibre is nondegenerate in $\mathbb{P}^4$, such a vertex has embedding dimension 4, whereas a hypersurface in the smooth threefold V has embedding dimension at most 3. Locally an A_n singularity has equation $xy + z^{n+1} = 0$. Restriction to a general smooth surface through the point has a nondegenerate quadratic term in two variables, and so gives an ordinary node. The section has arithmetic genus 1 and is irreducible; the node accounts for its entire arithmetic genus, so it has no other singularities. Its strict transform on S therefore has the required properties. □

Proof of Theorem 1.1 for Family 2.10. Let Q be a prime divisor over X, and put $Z = c_X(Q)$. If Z dominates $\mathbb{P}^1$, then Z meets a smooth fibre. Example 21.5 gives $\beta_X(Q) > 0$.

Otherwise Z is contained in a fibre F_0. Suppose first that Z meets its smooth locus, and choose a point p of this intersection. If p is on a special line, Theorem 19.1(3) gives $\delta_p(X) \geq 32/27 > 1$, so $\beta_X(Q) > 0$. If p is not on a special line, Lemma 21.9 gives $\delta_p(F_0) \geq 1$. The strict fibre estimate in Lemma 21.4 then gives

$$\beta_X(Q) \geq \frac{5}{16}\,\mathrm{ord}_Q(F_0) > 0,$$

because $Z \subset F_0$.

The remaining possibility is that Z is a singular point of F_0. Quartic Du Val del Pezzo surfaces have only A-type, D_4, or D_5 singularities. For an A-type point, Lemma 21.10 and Theorem 21.1 give $\beta_X(Q) > 0$. For a D_4 or D_5 point, apply Theorem 19.1(1)–(2). Thus every prime divisor over X has positive β_X, and the valuative criterion proves K-stability. □

22 The exceptional-surface estimates

22.1 The D4 point

Theorem 22.1. *If p is a D_4 singular point of a fibre, then $\delta_p(X) \geq 16/15$.*

Lemma 22.2. *The coarse exceptional surface is $G = \mathbb{P}(1,1,3)_{V,W,Z}$, where $V = v$, $W = w$, $Z = z^2$. If $M = \mathcal{O}_G(1)$, then*

$$-G|_G = \tfrac{1}{2}M, \quad A_X(G) = 7, \quad \Delta_G = \tfrac{1}{2}(Z = 0), \quad B = (Z - VW(V+W) = 0) \sim 3M.$$

One has $\alpha(G,\Delta_G;M)=1$. *The curve* B *is* $\mathbb{P}^1$ *with coordinates* $[V:W]$, *and* $\Delta_B=\frac12 Q_V+\frac12 Q_W+\frac12 Q_{V+W}$.

Proof. Take the second Veronese of the native ring of weights $(2,3,2)$. Lemma 5.2 gives the discrepancy and the boundary. Eliminating u gives the equation of B. It misses the vertex of G; projection to $[V:W]$ is an isomorphism. The boundary meets it transversally in the three distinct stated points, proving the plt assertion and different. The alpha computation follows from Lemma 3.2. □

Lemma 22.3 (The exact curve spaces and their point orders)**.** *After removing* $b^q z^\eta$, *set* $T=\min(m,2m-q)-\eta$ *and let* $2i+j+k=n$. *The restrictions of the standard monomials span*

$$V^aW^aH^0(\mathbb{P}^1,\mathcal{O}(n-2a)),\qquad a=\max(0,n-T),\quad 0\le n\le 2T.$$

Consequently

$$S(W^B;Q_V)=S(W^B;Q_W)=\frac{15}{32},\qquad S(W^B;Q)=\frac{5}{16}\quad (Q\neq Q_V,Q_W).$$

Proof. On B, $u=-VW$. Thus $u^iv^jw^k$ restricts, up to sign, to $V^{i+j}W^{i+k}$. The condition $jk=0$ says that i is the smaller of these two exponents. The ordinary-degree bound is $n-i\le T$. Therefore all and only the binary monomials with both exponents at least a occur. Their dimension is $n-2a+1$. The mean vanishing at either coordinate endpoint is $n/2$; at another point it is $(n-2a)/2$. Restoring z^η adds $\eta/2$ at the three boundary points, a bounded correction.

For completeness put $M(q)=\min(1,2-q)$, $0\le q\le 2$, after rescaling by m. With $a=\max(0,n-M)$ and $\lambda=n-2a$, the two intervals $[0,M]$, $[M,2M]$ give

$$\int_0^{2M}\lambda\,dn=M^2,\qquad \int_0^{2M}\lambda\frac{n}{2}\,dn=\frac{M^3}{2},\qquad \int_0^{2M}\frac{\lambda^2}{2}\,dn=\frac{M^3}{3}.$$

Since $\int_0^2 M^2dq=4/3$ and $\int_0^2 M^3dq=5/4$, division by $4/3$ gives the stated means. The two residue classes contribute equally to numerator and denominator, and give $N_m=(8/3)m^3+O(m^2)$. Endpoint and residue errors contribute $O(m^3)$ to order sums, so disappear after division by mN_m. □

Example 22.4. For $T=4,n=6$, the curve space is $V^2W^2H^0(\mathbb{P}^1,\mathcal{O}(2))$. Its dimension is three; its mean order at Q_V and Q_W is three, and at any other point it is one, before the bounded residue correction.

Proof of Theorem 22.1. The residual-degree average after removing B is

$$R=\tfrac12 S_X(G)-3S(W;B)=3-\frac{33}{16}=\frac{15}{16}.$$

Lemmas 3.3 and 22.2 give threshold at least $16/15$ for centres not contained in B. Along B, the curve term is $16/11$. The point terms are $16/15$ at Q_V,Q_W, $8/5$ at Q_{V+W}, and $16/5$ elsewhere. Thus $\delta(G,\Delta_G;W)\ge 16/15$.

The positivity conditions are elementary here. The support is described by $0\le q\le 2m$ and $0\le n\le 2\min(m,2m-q)$ up to bounded residue corrections. Every interior integral ray has linearly growing curve dimension. For W, sufficiently small positive native weight relative to m includes every weighted polynomial in v,z,w of that weight, and hence a complete ample series on a Cartier Veronese. For W^B, the interior ray $(m,q,n)=(6k,k,3k)$ gives the complete series of degree $3k$ with $\eta=0$. Bounded support and eventual nonvanishing on the other interior rays also follow from the refinement construction in [4, Example 2.15], since Lemma 20.4

identifies the actual refinement. Both residue classes are retained as in Proposition 2.7. The plt assertions were checked above. Proposition 2.7 now gives

$$\delta_p(X) \geq \min\{7/6, 16/15\} = 16/15.$$

□

22.2 The D5 point

Theorem 22.5. *If p is a D_5 singular point of a fibre, then $\delta_p(X) = 9/8$.*

22.2.1 The exceptional surface and the first obstruction

Lemma 22.6. *Here $G = \mathbb{P}(2,1,3)_{U,V,W}$, with $U = z$, $V = w$, $W = v^2$. For $M = \mathcal{O}_G(1)$,*

$$-G|_G = \tfrac{1}{2}M, \quad A_X(G) = 9, \quad \Delta_G = \tfrac{1}{2}(W = 0), \quad B = (U^2 - UV^2 - VW = 0) \sim 4M.$$

On $V = 1$, put $r = U$; then $W = r(r-1)$. Write Q_0, Q_1, Q_∞ for $r = 0, 1, \infty$. The pair $(G, \Delta_G + B)$ is plt along B, $B \cong \mathbb{P}^1$, and

$$\Delta_B = \tfrac{1}{2}Q_0 + \tfrac{1}{2}Q_1 + \tfrac{2}{3}Q_\infty,$$

$$(V|_B) = \tfrac{2}{3}Q_\infty, \quad (U|_B) = Q_0 + \tfrac{1}{3}Q_\infty, \quad (W|_B) = Q_0 + Q_1.$$

Proof. Take the second Veronese of the native weights $(3,4,2)$ and use Lemma 5.2. On $V = 1$ elimination gives $u = -W - U = -r^2$ and the displayed parameterization. The sole point outside this chart is $[0:0:1]$. Its index-three cover has coordinates (U, V) and equation $U^2 - UV^2 - V = 0$; it is smooth with $V = U^2 + O(U^5)$. The quotient parameter is U^3, so the quotient curve is smooth and r has a simple pole. The rational coordinate orders and the different follow from this chart and the two transverse boundary intersections in the affine chart. These charts also verify plt. □

Lemma 22.7 (The tangent-curve computation)**.** *One has*

$$S(W^B; Q_0) = \frac{5}{8}, \qquad S(W^B; Q) = \frac{5}{24} \quad (Q \neq Q_0).$$

Proof. Fix m, q, η, put $T = \min(m, 2m - q) - \eta$, and set $3i + 2j + k = n$ in the standard monomial $u^i v^\eta z^j w^k b^q$. After removing $b^q v^\eta$, dividing by V^n on B gives the powers r^{2i+j}, with $ik = 0$ and $i + j + k \leq T$. Their span is exactly

$$r^a\langle 1, r, \ldots, r^{D-a}\rangle, \quad a = \max(0, n - T), \quad D = \lfloor 2n/3 \rfloor, \quad 0 \leq n \leq 3T.$$

Indeed, the group $i = 0$ supplies consecutive exponents from $\max(0, n - T)$ to $\lfloor n/2 \rfloor$. In the group $k = 0$, increasing i by two and decreasing j by three increases $2i + j$ by one; this supplies the remaining consecutive exponents up to $\lfloor 2n/3 \rfloor$, subject to the same lower bound.

The mean orders before the residue correction are $(a + D)/2$ at zero, $2n/3 - (a + D)/2$ at infinity, and $(D - a)/2$ elsewhere. The factor v^η adds $\eta/2$ at Q_0, Q_1 only. Set $M = \min(1, 2 - q)$ in the limit. The dimension function is $\lambda = 2n/3$ for $0 \leq n \leq M$ and $\lambda = M - n/3$ for $M \leq n \leq 3M$. Direct integration gives

f	1	n	$(a + 2n/3)/2$	$(2n/3 - a)/2$
$\int_0^{3M} \lambda f \, dn$	M^2	$4M^3/3$	$2M^3/3$	$2M^3/9.$

The last column is both the mean at infinity and the mean at an ordinary point in the limit. Integrating in q and dividing by $4/3$ gives $5/8$ and $5/24$. The residue and rounding errors vanish as in the D_4 computation. □

Remark 22.8. The residual-degree average is

$$R = \tfrac{1}{2}S_X(G) - 4S(W;B) = 4 - \frac{11}{4} = \frac{5}{4}.$$

The global alpha invariant is one. Thus the direct residual estimate only gives $4/5$. Moreover, at Q_0 the tangent-curve estimate gives $(1/2)/(5/8) = 4/5$. Neither proves stability. This is the precise point at which a literal reuse of the Family 2.5 estimates fails. We treat Q_0 by an additional surface extraction, and cover the other centres separately.

22.2.2 The additional blowup on the exceptional surface

Let $\rho\colon \widehat{G} \to G$ blow up the smooth point Q_0, and let K be its exceptional curve. On $V = 1$ use coordinates U, W. Then $A_{G,\Delta_G}(K) = 3/2$, and the different on $K \cong \mathbb{P}^1$ is one half of the point corresponding to $W = 0$.

Lemma 22.9 (Identification of this refinement)**.** *For the refinement by K, the secondary orders of (u, v, z, w, b) are $(1, 1/2, 1, 0, 1)$. Its initial ring is*

$$R_{01} = \frac{\mathbb{C}[u, v, z, w, a, b, e]}{(eau + v^2 + zw,\ e^2ab + uw)}.$$

It is a domain and computes the actual point-order filtration of W at Q_0.

Proof. The second relation loses z^2, whose secondary order is two. The two equations are a regular sequence. Modulo e or a they become $(v^2 + zw, uw)$, which has height two. Hence a, e are regular. Localization at ae eliminates u, b and gives a polynomial ring in v, z, w, proving that the ring is a domain. It remains injective after localizing at w. On the chart $a = e = w = 1$, the equations are $u = -W - U$ and $b = W + U$. Their secondary orders are precisely those of the ordinary blowup of the origin in (U, W). The fixed bidegree and native weight remove the common unit factors before restriction. This proves the required identification, including cancellations. The half order of v means that its rational zero divisor is half of $(W = 0)$; it causes only bounded residue corrections below. □

Lemma 22.10 (Polynomial intervals on K)**.** *For the standard monomial $u^i v^\eta z^j w^k b^q$, put*

$$n = 3i + 2j + k + 4q, \qquad \kappa = i + j + q, \qquad l = i + q.$$

Its native weight is $2n + 3\eta$ and its order at K is $\kappa + \eta/2$. For fixed (m, η, n, κ), the possible integers l are precisely the interval $L_m \le l \le U_m$, where

$$L_m = \max\left(0, \left\lceil \frac{n - \kappa - m + \eta}{3} \right\rceil, \left\lceil \frac{n - \kappa - 2m + \eta}{2} \right\rceil\right),$$
$$U_m = \min(\kappa, n - 2\kappa, m - \eta + n - 3\kappa), \qquad \kappa \le 2m - \eta.$$

On K, write $r = U/W$. Apart from the residue factor, their restrictions span

$$r^{\kappa - U_m}(r + 1)^{L_m} H^0(\mathbb{P}^1, \mathcal{O}(U_m - L_m)).$$

Thus the mean orders at $r = 0, -1, \infty$ are respectively

$$\kappa - \frac{L_m + U_m}{2}, \quad \frac{L_m + U_m}{2}, \quad \frac{U_m - L_m}{2} + \frac{\eta}{2};$$

at another point the last term is $(U_m - L_m)/2$.

Proof. The leading monomials remain v^2, uw, so $ik = 0$. Given l, the equations imply

$$q = \min(l, n - l - 2\kappa), \quad i = l - q, \quad j = \kappa - l, \quad k = n - l - 2\kappa - q.$$

Nonnegativity and the two degree bounds are equivalent to the displayed interval and the bound on κ. On $V = 1$, the restriction is, up to sign, $(W + U)^{i+q}U^j W^{\eta/2}$. After taking its leading form on the blowup, division by $W^{\kappa+\eta/2}$ gives $(r + 1)^l r^{\kappa-l}$. Consecutive values of l form a basis of the displayed complete polynomial space. Its fixed factors and moving degree give the means. The factor $W^{\eta/2}$ adds only the stated order at infinity. □

Example 22.11. For $m = 6, \eta = 0, n = 9, \kappa = 3$, the interval is $0 \le l \le 3$. The four polynomials are $r^3, r^2(r + 1), r(r + 1)^2, (r + 1)^3$. They span the complete cubic series, so every point has mean order $3/2$.

Lemma 22.12 (The additional expected orders)**.** *One has*

$$S(W; K) = \frac{37}{32}, \qquad S(W^K; 0) = \frac{5}{16}, \quad S(W^K; -1) = \frac{27}{32}, \quad S(W^K; Q) = \frac{59}{288} \ (Q \neq 0, -1).$$

Consequently $\delta_{Q_0}(G, \Delta_G; W) \ge 32/27$.

Proof. We give the integrals to make the normalization explicit. Rescale (κ, n) by m and write them as (x, y). Put

$$L = \max(0, (y - x - 1)/3, (y - x - 2)/2), \quad U = \min(x, y - 2x, 1 + y - 3x), \quad \lambda = U - L.$$

Integrate over $\Omega = \{0 \le x \le 2,\ L \le U\}$. The five integrands for dimension, order at K, and the three point types are

$$\lambda, \quad \lambda x, \quad \lambda\left(x - \frac{L + U}{2}\right), \quad \lambda\frac{L + U}{2}, \quad \frac{\lambda^2}{2}.$$

For an explicit piecewise evaluation, number the three entries of each maximum or minimum by $0, 1, 2$. The nonempty regions, with their five integrals in the order just listed, are

(L, U)	I	S_K	S_0	S_{-1}	S_*
$(0, 0)$	$1/24$	$1/96$	$1/192$	$1/192$	$1/192$
$(0, 1)$	$1/24$	$1/48$	$1/64$	$1/192$	$1/192$
$(1, 0)$	$133/216$	$16/27$	$229/1728$	$265/576$	$229/1728$
$(1, 1)$	$7/72$	$23/288$	$79/1728$	$59/1728$	$37/1728$
$(1, 2)$	$7/27$	$13/36$	$4/27$	$23/108$	$1/18$
$(2, 0)$	$23/108$	$461/1296$	$101/2592$	$821/2592$	$101/2592$
$(2, 2)$	$7/108$	$157/1296$	$79/2592$	$235/2592$	$37/2592$
sum	$4/3$	$37/24$	$5/12$	$9/8$	$59/216$

Each region is cut out by linear inequalities, so these are integrals of quadratic polynomials over rational polygons. For example, region $(0, 0)$ is the triangle with vertices $(0, 0), (1/2, 3/2), (0, 1)$, where $L = 0, U = x$; it gives the first row directly. The defining max/min formulas specify every other region without a choice of subdivision.

Dividing the last four totals by $I = 4/3$ gives the assertion. The two residue classes again give $N_m \sim 2Im^3$ and cancel in the normalized means. All bounded rounding corrections vanish. At infinity the different has coefficient $1/2$, while it is zero elsewhere. The adjunction ratios are therefore

$$\frac{3/2}{37/32} = \frac{48}{37}, \quad \frac{1}{5/16} = \frac{16}{5}, \quad \frac{1}{27/32} = \frac{32}{27}, \quad \frac{1/2}{59/288} = \frac{144}{59}.$$

They give the claimed local threshold. □

22.2.3 The remaining centres on the exceptional surface

Let $A = (V = 0) \subset G$ and $Q_W = [1 : 0 : 0]$. Notice that $A \cap B = \{Q_\infty\}$.

Lemma 22.13 (The coordinate-curve refinement)**.** *One has $S(W; A) = 5/16$. The different on $A \cong \mathbb{P}^1$ is*

$$\Delta_A = \tfrac{3}{4}Q_W + \tfrac{2}{3}Q_\infty.$$

Moreover

$$S(W^A; Q_W) = \frac{5}{32}, \qquad S(W^A; Q_\infty) = \frac{11}{24}, \qquad S(W^A; Q) \le \frac{5}{32} \quad (Q \ne Q_W, Q_\infty).$$

In particular $\delta_Q(G, \Delta_G; W) \ge 8/5$ for $Q \in A \setminus \{Q_\infty\}$.

Proof. The quotient of R_0 by w has relations $eau + v^2$ and $e^2ab + z^2$. It is a domain: a, e are regular by the same height-two argument as before, and localization eliminates u, b. Thus powers of w are saturated and compute the geometric filtration by A. Its associated graded ring retains w as a free variable, and the two relations just displayed. The total w-order is the Hilbert-weight formula with $w_* = 1$ and all other weights zero, hence $5/16$.

After localization, this ring is monomial with $u = -v^2$ and $b = -z^2$. Give v order one, and hence u order two, keeping z, w, b at zero. Lemma 20.5, now applied to this multigraded ring, gives total average v-exponent

$$\frac{5}{16} \cdot 3 - \frac{5}{32} \cdot 2 = \frac{5}{8}.$$

Giving z order one and b order two similarly gives average z-exponent

$$\frac{5}{16} + \frac{11}{16} \cdot 2 - \frac{5}{32} \cdot 2 = \frac{11}{8}.$$

On the coarse curve $A = \mathbb{P}(2, 3)$, a section z has rational order $1/3$ at Q_∞, and v, whose square is W, has rational order $1/4$ at Q_W. These factors give $11/24$ and $5/32$.

Each fixed graded summand is a span of distinct monomials in a rational coordinate on A. At a nonzero finite point the vanishing sequence of s distinct monomials is $0, 1, \dots, s-1$: the derivative determinant is a nonzero Vandermonde determinant in their exponents. Its mean is no larger than the mean order at either endpoint, since those endpoint orders form a set of distinct nonnegative numbers with integral differences. Summing gives the last bound.

At Q_W the index-two chart has boundary coefficient $1/2$ on the transverse coordinate, so adjunction gives coefficient $1 - (1 - 1/2)/2 = 3/4$. At Q_∞ the index is three and the boundary misses the point, giving $2/3$. The smooth covers prove plt. The ratios away from Q_∞ are at least $\min\{16/5, (1/4)/(5/32)\} = 8/5$. □

Lemma 22.14 (Positivity of the refinements)**.** *The series W, W^B, W^K, W^A used in the D_5 proof have bounded support and contain ample series; the preceding adjunction applications are therefore valid.*

Proof. All generator exponents in a fixed degree are $O(m)$, so support is bounded. The interval descriptions show nonvanishing on every integral interior ray for W^B and W^K after the bounded parity corrections; their lengths grow linearly there. The first series contains every native weighted polynomial in v, z, w of weight $12k$ when $m = 12k$, giving the complete ample series $\mathcal{O}_G(6k)$. For W^B, take $(m, q, n, \eta) = (6k, k, 3k, 0)$; the interval is $0 \le \deg r \le 2k$, with no fixed rational order at infinity, so it is a complete series of degree $2k$. For W^K take $(m, n, \kappa, \eta) = (6k, 3k, k, 0)$; then $L_m = 0, U_m = k$ and the complete series has degree k.

For W^A, use $m = 24k$, order k along w, and remaining native weight $12k$. The monomials $v^{4i} z^{3(k-i)}$, $0 \le i \le k$, all satisfy the degree bounds, and restrict to the complete degree-k series

on the coarse curve. These displayed ample rays lie in the interiors of the respective support cones. The support and eventual nonvanishing for W^A also follow directly from the two linear degree bounds and the bounded residues modulo the quadric relations. More generally, [4, Example 2.15] gives the interior-ray condition for the actual refinements identified above. The rational restrictions and their residue classes are treated as in Proposition 2.7; passing to a Cartier Veronese for the ample ray does not discard the other summands in the expected-order sums. All the extractions and curve adjunctions are plt by the quotient-chart computations above. □

Proof of Theorem 22.5. Consider a divisorial valuation on (G, Δ_G). If its centre is not contained in $A \cup B$, Lemma 3.2 gives local alpha at least $3/2$ and Lemma 3.3 gives threshold at least $(3/2)/(5/4) = 6/5$.

For centres on B away from Q_0, Lemma 22.7 gives ratios $12/5$ at Q_1, $8/5$ at Q_∞, and $24/5$ elsewhere, together with the curve ratio $16/11$. At Q_0 use Lemma 22.12, obtaining $32/27$. For centres on A away from Q_∞ use Lemma 22.13, obtaining $8/5$; Q_∞ was already treated using B. This covers every centre, including the curves themselves. Hence

$$\delta(G, \Delta_G; W) \geq \min\{6/5, 16/11, 32/27, 8/5\} = \frac{32}{27}.$$

The positivity conditions are supplied by Lemma 22.14. Consequently

$$\delta_p(X) \geq \min\{9/8, 32/27\} = \frac{9}{8}.$$

The reverse inequality is supplied by G itself, since $A_X(G) = 9$ and $S_X(G) = 8$. This proves the equality. □

Remark 22.15. The additional extraction is an ordinary blowup of a point on the weighted exceptional *surface*. Its expected orders come from Lemmas 22.9–22.12. At no stage is a threefold Zariski decomposition on the weighted blowup assumed or used. The prime initial rings and the saturation arguments are what permit this replacement of the usual volume computation.

22.3 The higher singularities of the mixed A-type fibres

Proposition 22.16. *At the A_2 point of a fibre of type $A_2 + A_1$ or $A_2 + 2A_1$, and at the A_3 point of a fibre of type $A_3 + A_1$ or $A_3 + 2A_1$, one has*

$$\delta_p(X) \geq \frac{32}{27}.$$

In particular, the lines through these points cannot be centres of divisors with nonpositive β_X.

Proof. Use coordinates $[a_0 : z : x : y : w]$ on the fibre. The anticanonical normal forms are

$$\begin{array}{r|ll} A_2 + A_1 & a_0 z + x^2 - xy & yw + xz \\ A_2 + 2A_1 & a_0 z + x^2 & yw + xz \\ A_3 + A_1 & a_0 z + x^2 - xy & yw + z^2 \\ A_3 + 2A_1 & a_0 z + x^2 & yw + z^2. \end{array}$$

The first and third follow by linear changes from the equations in [13, §3.4]; the other two are the toric normal forms in the same classification. Their higher singular point is $[1 : 0 : 0 : 0 : 0]$. For example, on $a_0 = 1$, the second gives $yw - x^3$, and the fourth gives $yw + x^4$. In the two-node rows, the y- and w-coordinate points are both nodes. In each one-node row, only the w-coordinate point is a node; the y-coordinate point is smooth.

Write $\ell = a_0+$ a linear form in z, x, y, w, and extend the equations by t-multiples of linear forms. Smoothness of V forces a nonzero $t\ell$ coefficient in the second equation. Set $\ell = ea$, $t = eb$ as before. The native weights are

	z	x	y	w	b	d_1	d_2	$A_X(G)$	$S_X(G)$
A_2	4	2	3	3	6	4	6	8	101/16
A_3	2	1	2	2	4	2	4	5	4.

Here a, e have weight zero. The initial relations, after rescaling, are

$$eaz + x^2, \qquad \begin{cases} e^2ab + yw + xz, & A_2, \\ e^2ab + yw + z^2, & A_3. \end{cases}$$

Modulo e or a the two polynomials have height two; after inverting ae their quotient is a polynomial ring in x, y, w and the two units. Thus the domain argument of Lemma 20.4 identifies the actual valuation filtration. The discrepancies are the sums of the three local parameter weights, and formula (20.1) gives the displayed expected orders.

Next degenerate by the secondary weights $w_x = 1$, $w_z = 2$, all other weights zero. The initial relations become

$$eaz + x^2, \qquad e^2ab + yw. \tag{22.1}$$

Again they are a regular sequence, are regular with respect to a, e, and give a polynomial ring after localization. Thus this is the actual torus limit of every refined summand, including cancellations. It is monomial in x, y, w, with $z = -x^2$ and $b = -yw$ on the unit chart.

For A_2, the coarse exceptional surface is $\mathbb{P}(2,1,1)_{U,Y,W}$ with $U = x^3$, $Y = y$, $W = w$, and different $\frac{2}{3}(U = 0)$. For A_3, it is $\mathbb{P}^2_{U,Y,W}$ with $U = x^2$ and different $\frac{1}{2}(U = 0)$. In (22.1), the mean exponent of x is

$$\frac{5}{16}(1+2) - \frac{5}{32}\,2 = \frac{5}{8}.$$

The mean exponent of each of y, w is

$$\frac{5}{16} + \frac{11}{16} - \frac{5}{32} = \frac{27}{32}.$$

Consequently the coordinate divisor coefficients of the monomial basis limit are

	c_U	c_Y	c_W	$A(U)$	$A(Y)$	$A(W)$
A_2	5/24	27/32	27/32	1/3	1	1
A_3	5/16	27/32	27/32	1/2	1	1.

Lemma 5.3 gives $\delta(G, \Delta_G; W) \geq 32/27$. The first discrepancy ratios are 128/101 and 5/4, respectively, both larger than 32/27. Apply Proposition 2.7.

All these series have bounded support. To check positivity, note that on the unit chart the section space contains every monomial of total degree at most m in the three local parameters. For a range of small positive native degrees this contains the complete weighted homogeneous series on G, giving an interior ample ray after a fixed Veronese. Eventual nonvanishing on every other integral interior ray follows from [4, Example 2.15], applied to the actual point refinement. This verifies all the positivity hypotheses used in adjunction. □

22.4 An ordinary point extraction along every special line

The ordinary blowup is sufficient at the smooth points of all the special lines. Its section filtration requires one additional anticanonical section. We give both the coordinate argument and the saturation check, since omitting either would leave a gap in the expected-order computation.

Lemma 22.17 (The six local models)**.** *Let F_0 be a fibre, and let $\ell_0 \subset F_0$ be a line on whose smooth locus $\delta_p(F_0) < 1$. There are coordinates $[a_0 : u : z : v : w]$ on its anticanonical $\mathbb{P}^4$ for which $\ell_0 = (u = z = v = 0)$ and the equations are as follows:*

type of F_0	q_1	q_2
$A_2 + A_1$	$a_0 z + v^2 - vu$	$wu + vz$
$A_2 + 2A_1$	$a_0 z + v^2$	$wu + vz$
$A_3 + A_1$	$a_0 z + v^2 - vu$	$wu + z^2$
$A_3 + 2A_1$	$a_0 z + v^2$	$wu + z^2$
A_4	$a_0 u + wz + v^2$	$wu - vz$
D_5	$a_0 u + wz + v^2$	$wu + z^2$.

For either type with two nodes, the two special lines are interchanged by $u \leftrightarrow w$. At every smooth point of the indicated line, $w \neq 0$; in the first four rows also $a_0 \neq 0$.

Proof. These are linear changes of the anticanonical normal forms and the toric models in [13, §3.4]. The smooth-locus tables in [12, §6.2, Tables 27, 28, 31–33, 35] identify the lines in the statement. To specify the identification in the A_4 case, take $(x_0, x_1, x_2, x_3, x_4) = (u, w, z, v, a_0)$ in [13, §3.4]. Its curve E_5 is $(u = z = v = 0)$ and meets the second exceptional component of the A_4 chain. This is the line with smooth-locus threshold $24/29$. In the D_5 case it is the unique line. In the mixed cases the indicated lines join the higher singularity to a node.

The point $o = [1 : 0 : 0 : 0 : 0]$ is the higher singularity. The differential of q_2 vanishes at o. Along the displayed line, the differentials at a smooth point are independent precisely under the nonvanishing conditions stated. This also follows directly from the two equations. □

Lemma 22.18 (Coordinates off the exceptional divisor)**.** *Let $p \in \ell_0 \setminus E$ be smooth on F_0. After linear changes of the ambient coordinates, there are four H-coordinates u, z, v, c and two linear forms ℓ, t cutting out C such that $v/\ell, c/\ell, t/\ell$ are regular parameters at p. In the section ring, put $\ell = ea$, $t = eb$, and introduce*

$$r = \alpha au + bc \in H^0(X, L), \qquad \alpha \neq 0.$$

For the ordinary blowup of p, the generator orders are

	u	z	v	c	b	a	e	r
ord_G	2	2	1	1	1	0	0	3.

The three initial relations have the form

$$\Phi_1 = eaz + v^2 + eb(\gamma_1 c + \gamma_2 v + \gamma_3 eb), \tag{22.2}$$
$$\Phi_2 = er + \kappa vz + \lambda uc + \mu uv + \nu ebu, \tag{22.3}$$
$$\Phi_3 = \alpha au + bc, \tag{22.4}$$

where $\lambda \neq 0$. Their bidegrees are $2H, 2H, L$, and their ordinary orders are $2, 3, 2$, respectively. The constant κ is allowed to vanish.

Proof. Write the equations of V as $Q_i = q_i + tL_i$, with q_i from Lemma 22.17. Since $dq_2(o) = 0$ and V is smooth, the coefficient of ta_0 in Q_2 is nonzero. The curve C avoids o, so the coefficient of a_0 in its other linear equation ℓ is nonzero as well.

First replace u, z by themselves plus constant multiples of t, to remove the linear t-terms of the two equations at p. The second equation becomes

$$wu + \varepsilon vz + tc \quad \text{or} \quad wu + z^2 + tc,$$

where $\varepsilon \neq 0$ and c is a linear form vanishing at p. Its restriction to ℓ_0 has a nonzero a_0 coefficient. All new tv, tz, tu, t^2 terms are absorbed into tc. Next replace z by a nonzero multiple of $z + ku$, with k constant, to remove the linear u-term of Q_1 at p. In the first form of Q_2 this only adds an uv-term; in the second it adds uz, u^2 terms.

Here is the nonvanishing that will be essential. On ℓ_0, write

$$\ell = Aa_0 + Bw, \qquad c = Da_0 + E_0 w, \qquad A \neq 0,\ D \neq 0.$$

The restrictions of ℓ and c are independent because $\ell(p) \neq 0$ and c has a simple zero at p. Thus $DB - AE_0 \neq 0$, and

$$w = \frac{D}{DB - AE_0}\ell - \frac{A}{DB - AE_0}c \quad \text{on } \ell_0. \tag{22.5}$$

Both coefficients are nonzero. Globally we can therefore write

$$w = \alpha\ell + \lambda c + \text{a linear combination of } v, z, u, t, \qquad \alpha\lambda \neq 0.$$

The linear changes already made do not change this assertion on the line.

The linear parts of the two equations eliminate z, u. Hence v, c, t are regular parameters on the chart $\ell = 1$. The lowest ordinary parts of Q_1, Q_2 are

$$\ell z + v^2 + t(\gamma_1 c + \gamma_2 v + \gamma_3 t), \qquad \alpha\ell u + tc,$$

respectively. Terms in u, z multiplied by a function vanishing at p have order at least three. In particular u, z have order two. After substituting $\ell = ea$, $t = eb$, set $r = \alpha au + bc$. The second equation now reads

$$er + \kappa vz + \lambda uc + \mu uv + \nu ebu + \text{terms of ordinary order at least four} = 0.$$

Its cubic part is (22.3). The term λuc, together with $u = -bc/\alpha$ in degree two on the unit chart, contributes a nonzero multiple of bc^2. None of the other displayed terms contributes bc^2. Thus r has order exactly three. The defining relation $r - \alpha au - bc = 0$ has initial part (22.4). This proves the assertions about orders and initial equations. Equality with the full geometric filtration is established next. □

Lemma 22.19 (No further initial generators are missing)**.** *With the notation of Lemma 22.18, the associated graded ring of the ordinary point valuation on the bigraded section ring is*

$$R_p = \mathbb{C}[u, z, v, c, a, b, e, r]/(\Phi_1, \Phi_2, \Phi_3).$$

In particular its $(2m, -m)$ components give the actual refined series on $G \cong \mathbb{P}^2$.

Proof. The original presentation with the additional variable r has two quadric relations and the relation $r - \alpha au - bc$. We show that the three initial forms are a regular sequence and that a, e are non-zero-divisors in their quotient.

Modulo e, their ideal is

$$(v^2,\ \kappa vz + \lambda uc + \mu uv,\ \alpha au + bc).$$

Its height is three. Indeed, setting $v = 0$ leaves $uc = 0$ and $\alpha au + bc = 0$; every component has codimension two in the remaining variables because $\alpha\lambda \neq 0$.

Modulo a, the last equation is $bc = 0$. On the component $b = 0$, the first equation is $v^2 = 0$, and the second reduces, after $v = 0$, to $er + \lambda uc = 0$, which is a nonzero further equation. On the component $c = 0$, the first equation is

$$v^2 + \gamma_2 ebv + \gamma_3 e^2 b^2 = 0.$$

It is not divisible by e. The second equation has coefficient e in r, so it shares no factor with this quadratic. Thus the ideal modulo a also has height three.

After inverting ae, one eliminates u, z, r, in that order, and obtains a polynomial ring in the three parameters v, c, b, with $a^{\pm1}, e^{\pm1}$ adjoined. The original three-generator initial ideal therefore has height three; the preceding codimension checks rule out components on $a = 0$ or $e = 0$. It is a complete intersection, hence Cohen–Macaulay, and a, e are non-zero-divisors. Since its localization is a domain, it is a domain itself.

Lemma 5.1 identifies the polynomial-weight associated graded ring with this quotient. The orders in Lemma 22.18 give a map to the local point-associated graded ring. On inverting ae it is the polynomial identification just described. Since the quotient injects into this localization, no nonzero global initial form can vanish locally. Finally, the degrees of a, e form a lattice basis, so setting them equal to one is injective on any fixed bidegree after removing its common Laurent factor. This verifies the second assertion of Lemma 5.1, including all cancellations. □

Lemma 22.20 (The expected order of the point blowup)**.** *For the ordinary blowup $\sigma\colon Y \to X$ of the point in Lemma 22.18, with exceptional divisor G, one has*

$$A_X(G) = 3, \qquad S_X(G) = \frac{65}{32}.$$

Proof. The discrepancy is three because p is a smooth point of a threefold. Apply (20.2) to the actual filtration established in Lemma 22.19. The four H-weights sum to $2+2+1+1 = 6$, the weight of b is one, and the relation weights are $2, 3, 2$. The new L-generator has weight three. Therefore

$$S_X(G) = \frac{5}{16}\,6 + \frac{11}{16} - \frac{5}{32}(2+3) + \frac{1}{4}(3-2) = \frac{65}{32}.$$

The last term is required: using the seven old generators would not compute the actual expected order. □

Lemma 22.21 (A monomial limit on the exceptional plane)**.** *The refined series on $G = \mathbb{P}^2_{v,c,b}$ satisfies*

$$\delta(G; W) \geq \frac{32}{27}.$$

Proof. Apply the one-parameter subgroup with weights $(2, 0, 5)$ on (v, c, b). The corresponding secondary weights on the presentation of R_p are

	u	z	v	c	b	a	e	r
secondary weight	5	4	2	0	5	0	0	5.

The initial relations are

$$eaz + v^2 = 0, \qquad er + \lambda uc = 0, \qquad \alpha au + bc = 0. \tag{22.6}$$

In fact the two terms of the first equation have weight four, while the omitted ebc, ebv, e^2b^2 terms have weights $5, 7, 10$. In the second equation er, uc have weight five, whereas vz, uv, ebu have weights $6, 7, 10$. Thus no coefficient specialisation has been assumed.

Modulo e, the ideal in (22.6) is $(v^2, uc, \alpha au + bc)$, of height three. Modulo a it is $(v^2, er + \lambda uc, bc)$, again of height three. After inverting ae it is a polynomial ring in v, c, b. The same regularity and localization argument proves that this is the actual torus limit of each refined space. On the unit chart its generators are monomials:

$$u = -\frac{1}{\alpha}bc, \qquad z = -v^2, \qquad r = \frac{\lambda}{\alpha}bc^2.$$

We compute the three coordinate coefficients separately. For the exponent of v, give v, z weights $1, 2$ and all other variables weight zero. The relation weights are $(2, 0, 0)$, and (20.2) gives

$$c_v = \frac{5}{16}(1+2) - \frac{5}{32}\,2 = \frac{5}{8}.$$

For the exponent of c, give u, c, r weights $1, 1, 2$ and all other variables weight zero. The relation weights are $(0, 2, 1)$, so

$$c_c = \frac{5}{16}(1+1) - \frac{5}{32}\,2 + \frac{1}{4}(2-1) = \frac{9}{16}.$$

For the exponent of b, give u, b, r weight one and the other variables weight zero. The relation weights are $(0, 1, 1)$, giving

$$c_b = \frac{5}{16} + \frac{11}{16} - \frac{5}{32} + \frac{1}{4}(1-1) = \frac{27}{32}.$$

As a check, $c_v + c_c + c_b = 65/32 = S_X(G)$, since $-G|_G = \mathcal{O}_{\mathbb{P}^2}(1)$. The monomial basis divisor therefore tends to

$$\frac{5}{8}(v=0) + \frac{9}{16}(c=0) + \frac{27}{32}(b=0).$$

Its largest coefficient is $27/32$. Lemma 5.3 proves the assertion.

For bounded support, let h be the total exponent in the four H-generators, q the exponent in b, and k that in r. The remaining exponents are $2m-h-q-2k$ in a and $m-h-k$ in e. Their nonnegativity gives $h+k \le m$ and $q \le 2m-h-2k$, so every exponent and every filtration order is $O(m)$. To exhibit an interior ample ray, observe that all monomials in the regular parameters v, c, b of total degree at most m lift to sections of mL: for exponents i, j, k, the lift is $v^i c^j b^k a^{2m-i-j-k} e^{m-i-j}$. Thus $W_{m,J}$ contains the complete degree-J series on $\mathbb{P}^2$ for $0 \le J \le m$. Eventual nonvanishing on every integral interior ray follows from [4, Example 2.15], because Lemma 22.19 identifies the actual point refinement. This verifies the positivity conditions in Abban–Zhuang adjunction. □

Proposition 22.22. *At every smooth point $p \in \ell_0 \setminus E$ of any line in Lemma 22.17,*

$$\delta_p(X) \ge \frac{32}{27}.$$

Proof. The ordinary extraction has exceptional pair $(G, \Delta_G) = (\mathbb{P}^2, 0)$. Lemmas 22.20 and 22.21, followed by Proposition 2.7, give

$$\delta_p(X) \ge \min\left\{\frac{96}{65}, \frac{32}{27}\right\} = \frac{32}{27}.$$

□

Example 22.23 (The extra section at the first level). On the unit chart, an initial basis for $H^0(X, L)$ is obtained from

$$1, b, b^2, u, ub, z, zb, v, vb, c, cb$$

by replacing u with $r = \alpha u + bc$. In the final monomial limit, r is a nonzero multiple of bc^2. The total exponents of v, c, b are $6, 5, 9$, respectively, so the ordinary-order trace is 20. These are finite-level sums with $N_1 = 11$; their asymptotic means are $5/8, 9/16, 27/32$, not $6/11, 5/11, 9/11$.

Example 22.24 (The coordinates in a smooth threefold). Consider the threefold in Example 22.28, and write $(a_0,u,v,z,w,t)=(x_0,x_1,x_2,x_3,x_4,x_5)$. At $p=[0:0:0:0:1:0]$ take $c=a_0$ and $\ell=a_0+u+w$. Then $w=\ell-c-u$, so $\alpha=1$ and $\lambda=-1$. The two equations become

$$cu+v^2+(\ell-c-u)z+t^2=0, \qquad (\ell-c-u)u+z^2+tc=0.$$

With $r=au+bc$, their three initial relations are

$$eaz+v^2+e^2b^2=0, \qquad er-uc=0, \qquad au+bc=0.$$

In the monomial limit on $a=e=1$ they give $z=-v^2$, $u=-bc$, $r=-bc^2$. Thus the additional generator and the nonzero uc term occur in an explicit smooth example.

22.5 The intersections of the special lines with E

Lemma 22.25 (The point filtration on E)**.** *Let $p=\ell_0\cap E$ on the strict transform of any line in Lemma 22.17. There are local parameters v,e,b at p for which the ordinary point filtration on the section ring has generator orders $(u,z,v,w,a,b,e)=(3,2,1,0,0,1,1)$ and initial relations*

$$wz+v^2=0, \qquad wu+\kappa vz+e^2ab+Bebv=0.$$

Here a,w are units and κ may vanish. These relations compute the full geometric filtration.

Proof. Retain the fibre equations of Lemma 22.17. As in Lemma 22.18, shifts of u,z by constant multiples of t remove their linear t-terms at p and give

$$Q_2=wu+\varepsilon vz+tc \quad \text{or} \quad Q_2=wu+z^2+tc,$$

with $c|_{\ell_0}$ vanishing simply at p. The coefficient of a_0 in both $c|_{\ell_0}$ and $\ell|_{\ell_0}$ is nonzero. Since now $\ell(p)=c(p)=0$, their restrictions to the line are proportional. Therefore

$$c=\eta\ell+\eta_v v+\eta_z z+\eta_u u+\eta_t t, \qquad \eta\neq 0.$$

Use the chart $a=w=1$ of X. The functions v,e,b are regular parameters there. Rescale z to normalize its linear coefficient in the first equation, and rescale u and the second equation to make the coefficient η equal to one. The ordinary point blowup gives weights $(u,z,v,w,a,b,e)=(3,2,1,0,0,1,1)$, and initial equations

$$wz+v^2=0, \qquad wu+\kappa vz+e^2ab+Bebv=0, \tag{22.7}$$

where κ may vanish. Here $t=eb$ has order two. Thus the term z^2, when present, has order four, while vz has order three. This explains both cases of (22.7).

Modulo w, the ideal is $(v^2,\kappa vz+e^2ab+Bebv)$, of height two. Modulo a, it is $(wz+v^2,wu+\kappa vz+Bebv)$, also of height two. Its localization at wa eliminates z,u and is a polynomial ring in v,e,b. Hence w,a are regular and the quotient is a domain. Their bidegrees form a lattice basis; the localization argument identifies the geometric filtration. □

Lemma 22.26 (Expected orders on E)**.** *For the ordinary blowup of the point in Lemma 22.25, with exceptional plane G, one has*

$$A_X(G)=3, \qquad S_X(G)=\frac{69}{32}, \qquad \delta(G;W)\geq\frac{32}{27}.$$

Proof. The discrepancy is three. Formula (20.1) gives

$$S_X(G) = \frac{5}{16}(3+2+1) + \frac{11}{16} + \frac{3}{8} - \frac{5}{32}(2+3) = \frac{69}{32}.$$

Degenerate by secondary weights one on v, two on z, and zero on the other generators. The initial equations are

$$wz + v^2 = 0, \qquad wu + e^2ab = 0.$$

The same height checks modulo w, a and localization show that this is the actual torus limit. On $a = w = 1$, it is monomial with $z = -v^2$ and $u = -e^2b$. The three exponent means are, separately,

$$\begin{aligned} c_v &= \frac{5}{16}(1+2) - \frac{5}{32}\,2 = \frac{5}{8}, \\ c_e &= \frac{5}{16}\,2 + \frac{3}{8} - \frac{5}{32}\,2 = \frac{11}{16}, \\ c_b &= \frac{5}{16} + \frac{11}{16} - \frac{5}{32} = \frac{27}{32}. \end{aligned}$$

Their sum is $69/32$. Since $G = \mathbb{P}^2$ has zero different, Lemma 5.3 gives $\delta(G; W) \geq 32/27$.

Finally, $e^i b^j v^k$ lifts to a section of mL by multiplying by $a^{m+i-j} w^{m-i-k}$. This includes every monomial of total degree at most m, and hence gives an interior complete ample series on G. Bounded support follows from the bidegree bounds, and eventual nonvanishing on the other integral interior rays follows from [4, Example 2.15], applied to the refinement identified in Lemma 22.25. □

Proposition 22.27. *At the point $p = \ell_0 \cap E$ on the strict transform of any line in Lemma 22.17, one has $\delta_p(X) \geq 32/27$.*

Proof. Apply Proposition 2.7 and Lemma 22.26:

$$\delta_p(X) \geq \min\left\{\frac{3}{69/32}, \frac{32}{27}\right\} = \frac{32}{27}.$$

□

Proof of Theorem 19.1. Theorems 22.1 and 22.5 prove (1) and (2). Proposition 22.16 proves (4). Lemma 22.17 lists every special line. Propositions 22.22 and 22.27 give (3), respectively off and on E. Thus no smooth point on a special line is omitted. □

22.6 A smooth example with a D5 fibre

Example 22.28. In homogeneous coordinates $x_0, \dots, x_5$ on $\mathbb{P}^5$, set

$$V = \{q_1 = q_2 = 0\}, \qquad q_1 = x_0x_1 + x_2^2 + x_3x_4 + x_5^2, \quad q_2 = x_0x_5 + x_1x_4 + x_3^2.$$

Let $C = V \cap \{x_5 = x_0 + x_1 + x_4 = 0\}$. Then V and C are smooth. To check this directly, let M_i be the Hessian matrix of q_i. Their pencil satisfies

$$\det(sM_1 + tM_2) = 4s(s^5 + t^5).$$

This binary form has six distinct roots. On the subspace defining C, substitute $x_0 = -x_1 - x_4$ and $x_5 = 0$, and let M_i^C be the resulting four-by-four Hessian matrices. Then

$$\det(sM_1^C + tM_2^C) = -4s(t^3 - 2st^2 + s^2t - s^3).$$

The cubic $t^3 - 2t^2 + t - 1$ has discriminant -23, so this pencil also has distinct roots. The standard smoothness criterion for intersections of two quadrics proves the claims.

The hyperplane section $A = V \cap \{x_5 = 0\}$ has a unique singular point $p = [1 : 0 : 0 : 0 : 0 : 0]$. Indeed, on this hyperplane q_1 is nonsingular, and solving proportionality of the two gradients gives only p. On $x_0 = 1$ eliminate x_1 using q_1. The remaining equation is

$$x_3^2 - x_2^2 x_4 - x_3 x_4^2 = 0.$$

Putting $u = x_3 - x_4^2/2$, $v = x_2$ and $w = x_4$ gives

$$u^2 - v^2 w - \frac{1}{4} w^4 = 0,$$

which is a D_5 singularity. Since $p \notin C$, the corresponding fibre on $X = \mathrm{Bl}_C(V)$ retains this singularity. The blowup is smooth. Moreover, the graph embedding $X \hookrightarrow V \times \mathbb{P}^1$ identifies $H + F$ with the restriction of $\mathcal{O}(1, 1)$, so $-K_X = H + F$ is ample. This is therefore a smooth member of Family 2.10, not merely a singular degeneration.

At the singular point of this example the exact threshold is $9/8$. At every other point of the line

$$\ell_0 = \{x_1 = x_2 = x_3 = x_5 = 0\},$$

the fibre is smooth and the threshold on X is at least $32/27$. This includes its intersection $[1 : 0 : 0 : 0 : -1 : 0]$ with C, interpreted on the strict transform. Thus both the point extractions and the global theorem apply to a smooth threefold with this most singular quartic fibre.

Part V
Family 2.11: blow-ups of cubic threefolds along lines

23 Geometry and the family theorem

Let $V \subset \mathbb{P}^4$ be a smooth cubic threefold, let $\ell \subset V$ be a line, and let $\pi : X = \mathrm{Bl}_\ell V \to V$. Put $H = \pi^*\mathcal{O}_V(1)$, write E for the exceptional divisor, and set $B = H - E$ and $L = -K_X = H + B$. These are the members of Family 2.11. Projection from ℓ resolves to a conic bundle $f : X \to \mathbb{P}^2$, with $B = f^*\mathcal{O}_{\mathbb{P}^2}(1)$.

A line is of the *first kind* if $N_{\ell/V} \cong \mathcal{O}_\ell^{\oplus 2}$, and of the *second kind* if $N_{\ell/V} \cong \mathcal{O}_\ell(1) \oplus \mathcal{O}_\ell(-1)$. These are the only possibilities, as we check below. We prove the following results.

Theorem 23.1. *Every smooth member $X = \mathrm{Bl}_\ell V$ of Family 2.11 is K-stable. Every point p on a nonreduced conic fibre satisfies $\delta_p(X) \geq 144/127$. For a second-kind line, every point on the special reduced fibre has local threshold at least $36/35$, and every smooth tangency point described in Lemma 29.1 has local threshold exactly $216/167$.*

The two normal-bundle types require different arguments. For first-kind lines, refinement along E gives a strict beta inequality for every centre contained in E. For second-kind lines, we first treat the special reduced fibre and exclude all curve centres. The remaining point centres are tangencies on smooth conic fibres; a weighted point extraction treats them. We prove each local calculation separately and assemble the exclusions in Section 30. In particular, we do not use openness of K-stability to pass to second-kind limits.

The general-member result in [3, Section 5.3] uses a particular cubic with a dihedral group action. The list in [23] still includes Family 2.11 among the families without a complete classification of K-polystable smooth members. Our first-kind argument uses no automorphisms of X. The torus actions used later act on auxiliary exceptional surfaces and preserve their differents.

Lemma 23.2 (The plane-bundle model)**.** *Let $Y = \mathrm{Bl}_\ell \mathbb{P}^4$. Its Cox coordinates can be written as u, v, e, x_0, x_1, x_2, of degrees*

$$\deg u = \deg v = H, \qquad \deg e = E = H - B, \qquad \deg x_i = B.$$

The strict transform $X \subset Y$ has equation

$$G = u^2 l_0(x) + uv l_1(x) + v^2 l_2(x) + e\big(uq_0(x) + vq_1(x)\big) + e^2 c(x) = 0, \tag{23.1}$$

where l_i, q_i, c have degrees one, two and three in x, respectively. This equation has class $2H + B$. The map f is a flat family of nonzero plane conics. One has

$$H^3 = 3, \quad H^2B = 3, \quad HB^2 = 2, \quad B^3 = 0, \qquad L^3 = 18.$$

Proof. Choose coordinates $[u : v : z_0 : z_1 : z_2]$ with $\ell = (z_0 = z_1 = z_2 = 0)$. Substitute $z_i = ex_i$ in the cubic equation and divide by e. This gives (23.1). The map $Y \to \mathbb{P}^2_x$ is the projective plane bundle with relative coordinates $[u : v : e]$; the degree convention above accounts for the twisting of e.

No plane lies on a smooth cubic threefold: if a plane did lie on it, the two normal derivatives of its equation would be quadrics on that plane and would have a common zero, where the threefold would be singular. Consequently no conic equation in the plane bundle is identically

zero. A relative quadratic Cartier divisor whose equation is nonzero on every plane fibre is flat over the base. This proves the assertion about f.

The blow-up intersection formulas give $H^2E = 0$, $HE^2 = -1$ and $E^3 = -\deg N_{\ell/V} = 0$, since $\deg N_{\ell/V} = (-K_V)\ell - 2 = 0$. Substitute $B = H - E$. □

Lemma 23.3 (The two kinds of lines)**.** *In the two cases above, E is respectively $\mathbb{P}^1 \times \mathbb{P}^1$ and the Hirzebruch surface $\mathbb{F}_2$. If g denotes a fibre of $E \to \ell$, then*

E	$H\|_E$	$-E\|_E$
$\mathbb{P}^1 \times \mathbb{P}^1$	g	s
$\mathbb{F}_2$	g	$s+g$

where s is a section with $s^2 = 0$ in the first row and the negative section, with $s^2 = -2$, in the second. The three normal derivatives of the cubic along ℓ span all binary quadrics in the first case and a base point free pencil of binary quadrics in the second.

Proof. The normal sequence is

$$0 \longrightarrow N_{\ell/V} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 3} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(3) \longrightarrow 0.$$

Its last map is given by the three binary quadrics that are the normal derivatives along ℓ. They have no common zero. Write $N_{\ell/V} = \mathcal{O}(a) \oplus \mathcal{O}(-a)$, with $a \geq 0$. Its inclusion in $\mathcal{O}(1)^{\oplus 3}$ gives $a \leq 1$. Moreover, a copy of $\mathcal{O}(1)$ in its kernel is equivalent to a constant linear relation among the three quadrics. Their span cannot have dimension one, since a binary quadratic has a zero. This proves the last assertion and the dichotomy. Projectivizing the normal bundle gives the asserted ruled surfaces. The tautological normal bundle of E and $H|_E = \pi|_E^*\mathcal{O}_\ell(1)$ give the displayed classes; they also satisfy $(-E|_E)^2 = 0$ and $g(-E|_E) = 1$. □

Lemma 23.4 (Smooth cubic sections through a reduced fibre)**.** *If ℓ is of the first kind, every reduced conic fibre is contained in a surface $S \in |B|$ whose image under π is a smooth cubic surface and for which $\pi|_S$ is an isomorphism. If ℓ is of the second kind, the same holds except possibly for one base point $y_0 \in \mathbb{P}^2$. On such a surface put $A = H|_S$ and $c = E|_S$. Then*

$$A = -K_S, \quad c \text{ is a line}, \quad F = B|_S = A - c, \quad A^2 = 3, \quad Ac = 1, \quad c^2 = -1.$$

Proof. Hyperplanes containing ℓ are parametrized by a plane. Those singular on V at a specified point of ℓ are specified by the normal derivatives there. Their locus in the parameter plane is a smooth conic in the first-kind case and a line in the second-kind case, by Lemma 23.3. Hyperplanes containing the plane of a fixed conic fibre form a line in that parameter plane. It meets the former locus in finitely many points, unless in the second-kind case the two lines coincide. The latter event determines one point y_0 of the conic-bundle base.

A general inverse image of a base line through a reduced fibre is smooth on X. Bertini applies away from the fibre; at its smooth points the conic bundle is smooth. At the node of a reducible fibre, smoothness of the total space supplies a nonzero base derivative of the conic equation, and a general base line retains that derivative. Avoid also the finitely many hyperplanes singular along ℓ. The image surface is then smooth both along and away from ℓ. Blowing up the Cartier divisor ℓ on that surface is an isomorphism. Adjunction and restriction give the formulas. □

24 Three volume integrals and the reduced-fibre flags

Lemma 24.1. *The nef cone is generated by H, B, the effective cone is generated by E, B, and*

$$S_X(H) = \frac{3}{8}, \qquad S_X(B) = \frac{41}{72}, \qquad S_X(E) = \frac{5}{9}.$$

Every prime divisor on X has positive beta invariant.

Proof. A fibre of $E \to \ell$ and a general conic fibre test the two boundaries of the nef cone. For a prime divisor $D \neq E$, write $D \sim aH - bE$, with integers $a \geq 1$ and $b \geq 0$. A general smooth cubic section as in Lemma 23.4 is not a component of D. Intersect its effective restriction with the nef conic class F to obtain $2a - 2b \geq 0$. Thus $D = (a-b)H + bB$ has nonnegative integral coefficients. Since $H = E + B$, this also proves the assertion about the effective cone.

For $0 \leq t \leq 1$, the three nef divisors in the calculation have volumes

$$\begin{aligned} \operatorname{vol}(L - tH) &= 3(2-t)^3 - 3(2-t), \\ \operatorname{vol}(L - tB) &= 3 + 9(1-t) + 6(1-t)^2, \\ \operatorname{vol}(L - tE) &= 24 - 6(1+t)^2. \end{aligned}$$

The first and third volumes vanish for $t \geq 1$. For $1 \leq t \leq 2$, $L - tB = (2-t)H + (t-1)E$, with fixed part $(t-1)E$ and volume $3(2-t)^3$; beyond two the volume is zero. The fixed-part assertion follows by testing a general fibre of $E \to \ell$, or from $\pi_* \mathcal{O}_X(kE) = \mathcal{O}_V$ for $k \geq 0$. Hence the three integrals, divided by 18, are

$$\frac{27/4}{18} = \frac{3}{8}, \qquad \frac{19/2 + 3/4}{18} = \frac{41}{72}, \qquad \frac{10}{18} = \frac{5}{9}.$$

If $D \neq E$, either $D - H$ or $D - B$ is effective, by the integral expression above. Monotonicity of volume gives $S_X(D) \leq \max\{3/8, 41/72\} < 1 = A_X(D)$. The calculation for E gives the remaining assertion. □

Fix a smooth cubic $S \in |B|$ from Lemma 23.4. For $u = j/m$, its refined series has the following moving and fixed parts:

u	D_u	$N_u\|_S$
$[0,1]$	$A + (1-u)F$	0
$[1,2]$	$(2-u)A$	$(u-1)c.$

(24.1)

These are asymptotically complete. In the first open interval this follows from ampleness and Serre vanishing. In the second, remove the fixed multiple of E and restrict the complete series on V to its hyperplane section. Endpoint errors disappear in the limits.

For a smooth rational curve $Z \subset S$ different from c, write $D_u - vZ = P_{u,v} + N_{u,v}$ for its surface Zariski decomposition and put $d = P_{u,v}Z$. With $n_u = \max\{0, u-1\}$, the flag constants are

$$\mathcal{S}_Z = \frac{1}{6}\int\!\!\int P_{u,v}^2 \, dv\, du, \qquad \mathcal{T}_p = \frac{1}{6}\int\!\!\int d^2 \, dv\, du + \frac{1}{3}\int\!\!\int d \, \operatorname{ord}_p((n_u c + N_{u,v})|_Z) \, dv\, du. \tag{24.2}$$

The inner integral runs to the pseudo-effective threshold. The normalization is $h^0(X, mL) = 3m^3 + O(m^2)$. The proof of Lemma 12.1, with $L^3 = 18$, gives

$$\delta_p(X) \geq \min\left\{\frac{72}{41}, \frac{1}{\mathcal{S}_Z}, \frac{1}{\mathcal{T}_p}\right\}. \tag{24.3}$$

Only decompositions on the cubic surface and the elementary fixed part on X are used here.

Lemma 24.2 (The smooth-conic calculation)**.** *If Z is a smooth conic fibre and $p \in Z$, put $r = (cZ)_p$. Then*

$$\mathcal{S}_Z = \frac{29}{48}, \qquad \mathcal{T}_p = \frac{8+r}{9}.$$

In particular, if $p \notin E$, then $\delta_p(X) \geq 9/8$.

Proof. For $0 \le u \le 1$, put $s = 1 - u$. The decomposition is

v	$P_{u,v}$	$N_{u,v}$
$[0, s + 1/2]$	$A + (s - v)F$	0
$[s + 1/2, s + 1]$	$(1 + s - v)(A + c)$	$(2v - 2s - 1)c.$

For $1 \le u \le 2$, scale the $s = 0$ decomposition by $a = 2 - u$. Nefness, orthogonality and the threshold follow exactly as in Lemma 12.2: both F and $A + c$ are nef. The three inner integrals on the first interval are $3s + 2s^2 + 7/6$, $4s + 8/3$, and $1/6$, for P^2, d^2, and $d\,\mathrm{coeff}_c N$, respectively. On the second interval they are $7a^3/6$, $8a^3/3$, and $a^3/6$. Since $\int_1^2 a^3 du = 1/4$, these give

$$\mathcal{S}_Z = \frac{1}{6}\left(\frac{10}{3} + \frac{7}{24}\right) = \frac{29}{48}, \quad \frac{1}{6}\int\int d^2 = \frac{8}{9}, \quad \frac{1}{3}\int\int d\,\mathrm{coeff}_c N = \frac{5}{72}.$$

The original fixed part contributes

$$\frac{1}{6}\int_1^2 (u-1)\,3(2-u)^2 du \cdot r = \frac{r}{24}.$$

Adding the last two fixed contributions gives $r/9$. If $p \notin E$, then $r = 0$, and (24.3) proves the assertion. □

Lemma 24.3 (The reducible-conic calculation)**.** *Suppose the conic fibre is reduced and singular, and the smooth cubic of Lemma 23.4 is available. Then $\delta_p(X) \ge 36/35$ at every point of that fibre.*

Proof. Write the fibre as $Z + Z'$, choosing $p \in Z$. The two curves have square -1 and meet transversely once. For $0 \le u \le 1$, set $s = 1 - u$. The positive and negative parts of $A + sF - vZ$ are

v	$P_{u,v}$	$N_{u,v}$
$[0, 1]$	$A + sF - vZ$	0
$[1, 1 + s]$	$(A - Z) + (1 + s - v)F$	$(v - 1)Z'.$

The threshold is $1 + s$. The classes $A - Z, F$ are conic pencils and are nef, so the displayed decomposition has the required properties. For $1 \le u \le 2$, $aA - vZ$ is nef up to its threshold $v = a$, where $a = 2 - u$.

Put $\epsilon = 1$ if $p = Z \cap Z'$ and zero otherwise, and $\mu = (cZ)_p \le 1$. The inner integrals in the first interval are $5/3 + 4s + 2s^2$, $7/3 + 4s$, and s^2. In the second they are $5a^3/3$, $7a^3/3$, and zero. Consequently

$$\mathcal{S}_Z = \frac{1}{6}\left(\frac{13}{3} + \frac{5}{12}\right) = \frac{19}{24}, \qquad \mathcal{T}_p = \frac{59}{72} + \frac{\epsilon}{9} + \frac{\mu}{24} \le \frac{35}{36}.$$

The final term uses the same fixed-part integral as in Lemma 24.2. Apply (24.3). □

Example 24.4 (Why the point term must be retained). If a smooth conic is transverse to c at p, the flag gives $\mathcal{T}_p = 1$. If it is tangent, it gives $\mathcal{T}_p = 10/9$. Thus the cubic-surface flag alone gives no strict local threshold at these points of E. A lower bound $9/10$ does not indicate instability; it indicates the limitation of this flag.

25 The special reduced fibre for second-kind lines

Lemma 25.1 (The special reduced fibre for a second-kind line)**.** *Suppose that ℓ is of the second kind. Let y_0 be the exceptional point of Lemma 23.4. If $f^{-1}(y_0)$ is reduced, then*

$$\delta_p(X) \ge \frac{36}{35} > 1 \qquad \textit{for every } p \in f^{-1}(y_0).$$

Proof. The morphism $f|_E$ is given by a base point free subsystem of $|s+2g|$ on $E \cong \mathbb{F}_2$; it contracts the negative section s to y_0. Thus $f^{-1}(y_0) = s + Z$, where Z is another line. Choose a general base line through y_0 and let S be its inverse image. This surface is smooth: away from the fixed fibre this is Bertini, along its smooth locus it follows from smoothness of the conic bundle, and at its node a general base direction retains the nonzero derivative supplied by smoothness of X. The pencil induced on E has fixed component s and moving part a base point free pencil in $|2g|$. A general member of that pencil is $e_1 + e_2$, where e_1, e_2 are distinct ruling fibres. Put

$$A = H|_S = -K_S, \qquad F = B|_S = s + Z, \qquad R = e_1 + e_2, \qquad c = E|_S = s + R.$$

The line s is mapped isomorphically to ℓ, and Z is mapped to a line. Consequently $As = AZ = 1$, while $Ae_i = 0$. Adjunction on the smooth surface S gives

$$s^2 = Z^2 = -1, \quad e_i^2 = -2, \quad sZ = se_i = 1, \quad Ze_i = e_1e_2 = 0.$$

Here $sZ = 1$ also follows directly from the reduced nodal conic fibre, and the remaining intersections follow from the geometry of E and the equalities $Fe_i = 1$. In particular, the three points $s \cap Z, s \cap e_1, s \cap e_2$ are distinct. The divisors A, F are nef, and $A = c + F = 2s + Z + R$.

The refined surface series still has moving and fixed classes given by (24.1). Its asymptotic completeness for $0 < u < 1$ follows from ampleness of $L - uB$. For $1 < u < 2$, remove the fixed multiple of E. Restriction of $H^0(X, kH)$ to S is surjective for $k \gg 0$: its obstruction is $H^1(X, (k-1)H+E)$, which is $H^1(V, \mathcal{O}_V(k-1)) = 0$, since $\pi_*\mathcal{O}_X(E) = \mathcal{O}_V$ and $R^i\pi_*\mathcal{O}_X(E) = 0$ for $i > 0$. The latter statements follow on the blow-up fibres from $\mathcal{O}_X(E)|_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-1)$. Hence the surface flag formulas apply to this weak del Pezzo surface.

First take the flag curve Z. The decomposition in Lemma 24.3 applies verbatim with its second component equal to s. Indeed, $A - Z = 2s + R$ is nef: it is effective, has zero intersection with each component of its support, and has nonnegative intersection with every other irreducible curve. Moreover, $c|_Z$ is the single reduced point $s \cap Z$. Thus the constants are the same as in that lemma, and every point of Z has local threshold at least $36/35$.

It remains to use the flag curve s. For $0 \le u \le 1$, put $a = 1 - u$, so the moving class is $D_u = A + aF$. For the parameter v of this flag, its Zariski decompositions are

v	$P_{u,v}$	$N_{u,v}$
$[0, a]$	$A + aF - vs$	0
$[a, 1]$	$A + aF - vs - \frac{v-a}{2}R$	$\frac{v-a}{2}R$
$[1, 2+a]$	$(2 + a - v)(F + \frac{1}{2}R)$	$\frac{v-a}{2}R + (v-1)Z.$

To verify nefness, $A + aZ$ is nef for $0 \le a \le 1$: the only possible negative intersection of the effective summand aZ is with Z, where the full intersection is $1 - a$. The first two intervals therefore interpolate between nef endpoint classes. For the other endpoint use $M = F + R/2$, an effective rational divisor with $Ms = 1$, $MZ = Me_i = 0$, hence nef. The negative parts are supported on mutually disjoint negative curves and are orthogonal to their positive parts. Finally $(A + aF - vs)M = 2 + a - v$, so $2 + a$ is the pseudo-effective threshold.

Writing $d = P_{u,v}s$, the three intervals give

v	$P_{u,v}^2$	d
$[0, a]$	$3 + 4a - 2v - v^2$	$1 + v$
$[a, 1]$	$(1 + a)(3 + a - 2v)$	$1 + a$
$[1, 2+a]$	$(2 + a - v)^2$	$2 + a - v.$

The required inner integrals are

$$I(a) := \int P_{u,v}^2 \, dv = 2a^2 + 4a + \frac{7}{3},$$
$$J(a) := \int d^2 \, dv = -\frac{a^3}{3} + a^2 + 3a + \frac{4}{3},$$
$$K(a) := \int d \operatorname{coeff}_{e_i}(N_{u,v}) \, dv = \frac{(a+1)(a^2 - 4a + 7)}{12},$$
$$M(a) := \int d \operatorname{coeff}_Z(N_{u,v}) \, dv = \frac{(a+1)^3}{6}.$$

For $1 \le u \le 2$, scale the $a = 0$ decomposition by $b = 2 - u$; the four inner integrals are respectively $b^3 I(0)$, $b^3 J(0)$, $b^3 K(0)$, $b^3 M(0)$. The original fixed part $(u-1)c = (u-1)(s+R)$ shifts the order along s by $u - 1$. After removing this coefficient of s, the original fixed divisor that remains on the flag curve is $(u-1)R|_s$, not $(u-1)c|_s$. These terms contribute $1/24$ to the expected order along s and $1/24$ at each point $s \cap e_i$, respectively. Consequently

$$S(W^S; s) = \frac{1}{6}\left(\int_0^1 I(a) \, da + \frac{I(0)}{4}\right) + \frac{1}{24} = \frac{35}{36},$$
$$S(W^{S,s}; p) = \frac{41}{72} + \frac{11}{36}\,\mathbf{1}_{\{p \in s \cap R\}} + \frac{2}{9}\,\mathbf{1}_{\{p = s \cap Z\}} \le \frac{7}{8}.$$

For clarity, the first point coefficient is $\frac{1}{3}(\int_0^1 K(a)\,da + K(0)/4) + 1/24 = 11/36$, and the second is $\frac{1}{3}(\int_0^1 M(a)\,da + M(0)/4) = 2/9$. The point indicators cannot occur simultaneously. Abban–Zhuang adjunction now gives, for every $p \in s$,

$$\delta_p(X) \ge \min\left\{\frac{72}{41}, \frac{36}{35}, \frac{8}{7}\right\} = \frac{36}{35}.$$

This covers both components of the special fibre. □

Example 25.2 (The three types of points on the negative section). In the preceding calculation, a point of s outside $R \cup Z$ has point term $41/72$. At either point $s \cap e_i$ this becomes $7/8$, while at $s \cap Z$ it is $19/24$. These points are distinct, so the two fixed contributions are never added at the same point. In all three cases the larger flag constant is the curve term $35/36$.

26 Refinement by the exceptional divisor of the original blow-up

The refinement along E has asymptotically complete summands of classes

$$D_u = (L - uE)|_E, \qquad 0 \le u \le 1.$$

Indeed $L - uE = (1-u)H + (1+u)B$ is ample for $0 < u < 1$. The normalizing identity is $\int_0^1 D_u^2 du = 6$, so the surface-volume factor is again $1/6$.

Lemma 26.1 (First-kind lines). *If $N_{\ell/V} \cong \mathcal{O}^{\oplus 2}$, the refined series on E has stability threshold exactly one. Every prime divisor Q over X with $c_X(Q) \subset E$ satisfies*

$$\beta_X(Q) \ge \frac{4}{9} \operatorname{ord}_Q(E) > 0.$$

Proof. On $E = \mathbb{P}^1 \times \mathbb{P}^1$, one has $D_u = 2g + (1+u)s$. Each summand is complete on rational interior rays. For either of the two coordinate fibres of class g, the expected coefficient is

$$\frac{1}{6}\int_0^1 2^2(1+u)du = 1;$$

for either coordinate fibre of class s, it is

$$\frac{1}{6}\int_0^1 2(1+u)^2 du = \frac{7}{9}.$$

A common monomial degeneration of the complete bidegree series, as in Lemma 5.3, proves that its threshold is at least one. The divisor of class g proves equality. There is no different since both X and E are smooth. The strict adjunction inequality of Lemma 21.2, with $S_X(E) = 5/9$, gives the stated beta inequality. □

Lemma 26.2 (Second-kind lines)**.** *If $N_{\ell/V} \cong \mathcal{O}(1) \oplus \mathcal{O}(-1)$, then*

$$D_u = (1+u)s + (3+u)g, \quad S(W^E; s) = \frac{71}{72}, \quad S(W^E; g) = \frac{29}{24}, \quad \delta(E; W^E) = \frac{24}{29}.$$

Proof. Write $a = 1+u$ and $b = 3+u = a+2$. For subtraction of s, the divisor $(a-v)s + bg$ is nef for $0 \le v \le a$, which is its threshold. Thus

$$S(W^E; s) = \frac{1}{6}\int_0^1 \left(ba^2 - \frac{2}{3}a^3\right) du = \frac{71}{72}.$$

For subtraction of g, the divisor $as + (b-v)g$ is nef up to $v = b - 2a$. Thereafter its positive part is $(b-v)(s+2g)/2$, and the negative part is $(v - b + 2a)s/2$. The threshold is b. Hence

$$S(W^E; g) = \frac{1}{6}\int_0^1 \left[\int_0^{b-2a} (2ab - 2a^2 - 2av)dv + \int_{b-2a}^{b} \frac{(b-v)^2}{2} dv\right] du = \frac{29}{24}.$$

The other torus-invariant section has expected coefficient $41/72$, and the other invariant fibre has coefficient $29/24$. For example these numbers follow by summing distances to the four edges of the section polygon; the coefficients of the two section divisors sum to $\frac{1}{6}\int_0^1 aD_u^2 du = 14/9$. The toric degeneration lemma therefore gives the lower bound $24/29$, and a ruling fibre gives equality. This is the threshold of the actual asymptotically complete refined series, not an overestimate obtained by enlarging it. □

27 The weighted extraction of a double fibre

Let $f^{-1}(y) = 2\Lambda$. The construction of Lemma 18.1 applies to any smooth plane conic bundle and gives the normalized weighted extraction

$$\omega : X^\sharp = \mathrm{Bl}_{\mathfrak{m}_y \mathcal{O}_X} X \longrightarrow X.$$

Its exceptional divisor, denoted here by T, is of plt type. We recall the geometric data to fix all conventions.

Lemma 27.1. *The transverse weights are $(1,2)$, and*

$$T = \Lambda \times \mathbb{P}^1, \quad A_X(T) = 3, \quad \omega^* L|_T = h, \quad T|_T = -f_T/2, \quad \Delta_T = D/2,$$

where $h = \mathcal{O}_T(1,0)$, $f_T = \mathcal{O}_T(0,1)$, and D is a smooth $(2,1)$-curve. The notation f_T denotes a divisor class, not the conic-bundle morphism. The curve D is the graph of a degree-two map $\Lambda \to \mathbb{P}^1$.

Proof. In a local trivialization of the plane bundle, the equation has the form $x^2+\eta_1 Q_1(s_0,s_1)+\eta_2 Q_2(s_0,s_1)+$ higher weighted-order terms, with weights $1,2,2$. The quadrics Q_1, Q_2 have no common zero, by smoothness of X. Locally the base ideal is (x^2,η_2). Its blow-up has charts $\eta_2 = x^2 r$ and $x^2 = \eta_2 r$; the latter has an A_1 singularity transverse to the singular-locus curve. Flat base change from $\mathrm{Bl}_y \mathbb{P}^2$ shows that the reduced exceptional surface is the product, and that the pullback of the exceptional base curve is $2T$. The singular-locus curve is $D = (\eta_1 Q_1 + \eta_2 Q_2 = 0)$, giving the different $D/2$. The Jacobian on the first chart gives discrepancy three. Finally $L\Lambda = 1$, since L has degree two on the conic fibre. This proves all the formulas, including the case $\Lambda \subset E$. □

There are two different section filtrations: $\Lambda \not\subset E$ and $\Lambda \subset E$. The second must not be omitted. An example appears below.

Lemma 27.2 (Identifying the two filtrations)**.** *Choose $y = [1:0:0]$. If $\Lambda \not\subset E$, coordinates in (23.1) can be chosen so that its initial equation for ord_T is*

$$G_0 = x_0 u^2 + x_1 Q_1(v, e x_0) + x_2 Q_2(v, e x_0),$$

with weights 1 on u, 2 on x_1, x_2, and zero on v, e, x_0. If $\Lambda \subset E$, its initial equation is

$$G_0 = e^2 x_0^3 + x_1 Q_1(u,v) + x_2 Q_2(u,v),$$

with weights 1 on e, 2 on x_1, x_2, and zero on u, v, x_0. In either case Q_1, Q_2 are relatively prime binary quadrics. The actual associated graded anticanonical section spaces are the bidegree (m,m) pieces of the polynomial ring modulo G_0, for $m \geq 2$.

Proof. On the plane over y, the rank-one quadratic is the square of a linear form in $u, v, e x_0$. If its line is different from $e = 0$, a linear change in u, v, followed by addition of a multiple of $e x_0$, makes this form u. If its line is $e = 0$, normalize its nonzero coefficient to one. Taking initial terms gives the two equations. The two binary quadrics are the first base derivatives on Λ, so they have no common zero by smoothness of the total space.

For completeness, the section ring calculation is global. On the plane bundle $Y \to \mathbb{P}^2$, the pushforward of $\mathcal{O}_Y(pH + qB)$ for $p \geq 0$ is

$$\bigoplus_{a=0}^{p} \mathcal{O}_{\mathbb{P}^2}(q+a)^{\oplus(p-a+1)}.$$

Thus $H^1(Y, (m-2)H + (m-1)B) = 0$. Restriction from Y to the hypersurface of class $2H + B$ identifies $H^0(X, mL)$ with the bidegree (m,m) quotient by G.

The weighted hypersurface filtration has associated graded quotient by G_0. This is the one-equation case of Lemma 5.1. Each displayed G_0 is a primitive linear polynomial in x_1, x_2 over the ring of the other variables and is irreducible; the coprimality of the binary quadrics ensures primitivity. Hence this quotient is a domain, and it injects into its localization at x_0 and a nonzero homogeneous coordinate on Λ. On that chart the weighted normal equation is precisely the local equation in Lemma 27.1. Its associated graded ring is the geometric one. The two inverted weight-zero coordinates have bidegrees forming a lattice basis. In any fixed bidegree, their common Laurent factor is therefore uniquely determined, and removing it preserves injectivity. The injectivity criterion in Lemma 5.1 now proves equality of the global quotient and geometric filtrations. In particular, no unaccounted cancellation is used to assign a section its order. □

Lemma 27.3 (The two expected vanishing orders)**.** *One has*

$$S_X(T) = \begin{cases} 169/72, & \Lambda \not\subset E, \\ 91/36, & \Lambda \subset E. \end{cases}$$

Thus $A_X(T) - S_X(T)$ is respectively 47/72 and 17/36.

Proof. We give a separate Hilbert calculation. For the polynomial Cox ring, let $P(p,q)$ be the dimension of degree $pH+qB$, with $p,q\ge 0$. A monomial has e-exponent a, total u,v-degree $p-a$, and total x-degree $q+a$. Therefore

$$P(p,q)=\sum_{a=0}^{p}(p-a+1)\binom{q+a+2}{2}.$$

For $d=q+a$, the sum of the x_1,x_2 weights over all degree-d base monomials is $2d(d+1)(d+2)/3$. In the first case, the additional weight sum of the u,v monomials is $(p-a)(p-a+1)/2$. In the second case, each has the additional weight a. Consequently the polynomial weight sums are

$$U(p,q)=\sum_{a=0}^{p}\left[\frac{(p-a)(p-a+1)}{2}\binom{d+2}{2}+(p-a+1)\frac{2d(d+1)(d+2)}{3}\right],$$
$$V(p,q)=\sum_{a=0}^{p}(p-a+1)\left[a\binom{d+2}{2}+\frac{2d(d+1)(d+2)}{3}\right].$$

The relation has bidegree $(2,1)$ and weight two. Subtract its multiples, including their additional weight two. The quotient dimension is

$$N_m=P(m,m)-P(m-2,m-1)=\frac{(2m+1)(3m^2+3m+2)}{2}.$$

The two quotient weight sums are, respectively,

$$U(m,m)-U(m-2,m-1)-2P(m-2,m-1)=\frac{m(169m^3+230m^2+191m+58)}{24},$$
$$V(m,m)-V(m-2,m-1)-2P(m-2,m-1)=\frac{m(91m^3+128m^2+107m+34)}{12}.$$

Divide by mN_m and take the limit. Lemma 27.2 identifies these algebraic weights with the geometric valuation, so Lemma 2.1 applies. □

No Zariski decomposition on $X^\sharp$ is used in this calculation. The argument instead proves equality of the geometric filtration with a weighted hypersurface filtration, and then counts its sections. Computing only $(\omega^*L-tT)^3$ would not suffice: this divisor is visibly nef for $0\le t\le 2$, where its volume is $18-3t^2/2$, but the contribution of that interval to $S_X(T)$ is only $16/9$. Both values in Lemma 27.3 are larger.

28 The refined series on the weighted exceptional surface

Let $W_{m,j}=\mathrm{gr}_T^j H^0(X,mL)$, set $\epsilon=j \bmod 2$, $M=m-\epsilon$ and $J=(j-\epsilon)/2$, and denote by Q an equation of D on $T=\mathbb{P}^1\times\mathbb{P}^1$. Its bidegree is $(2,1)$. As in Part III, the rational zero divisor of an odd-weight section includes $D/2$. The normalization remains mN_m, with $N_m=3m^3+O(m^2)$.

Lemma 28.1 (A double fibre not contained in E)**.** *Suppose $\Lambda\not\subset E$. Let R be the ruling divisor on T lying over the single point $\Lambda\cap E$, and write r for its equation, of bidegree $(1,0)$. After removing the factor $\epsilon D/2$, the space $W_{m,j}$ is exactly*

$$I^n_{(M,J)},\qquad I=(r,Q),\qquad n=\max\{0,J-m\},$$

in the bihomogeneous polynomial ring of T. Here a graded ideal component is meant, not the complete space of sections of its saturated ideal sheaf.

Proof. In the first equation of Lemma 27.2, put $x_0 = 1$. On Λ, the restrictions of v, e give independent linear coordinates, with $e = r$. A monomial with e-exponent a and u-exponent $\epsilon + 2h$ has remaining v-degree $M - a - 2h$ and base weight degree $J - h$. Its existence before localization is exactly the condition

$$a, h \geq 0, \quad M - a - 2h \geq 0, \quad J - h \geq 0, \quad J - h \leq m + a.$$

Using $u^2 = -Q$ on this chart gives the forms $r^a v^{M-a-2h} Q^h$ times arbitrary binary base forms of degree $J - h$. The final inequality is $a + h \geq J - m$. Their span is precisely the indicated graded component of $(r, Q)^n$. Conversely every generator of that component satisfies these inequalities and lifts to a Cox monomial, using the nonnegative remaining power of x_0. Thus the assertion is an equality of spaces. An odd power of u contributes the rational factor $D/2$, since its square restricts to Q. □

Lemma 28.2 (A degeneration preserving the different)**.** *Coordinates on T can be chosen so that $Q = s_1^2 t_0 - s_0^2 t_1$. Under an action preserving D, the spaces in Lemma 28.1 have flat limits*

$$(s_0, s_1^2 t_0)^n_{(M,J)}.$$

Ignoring bounded parity errors, their normalized monomial region is

$$0 \leq t \leq 4, \quad \max\{0, t - 3\} \leq a \leq 1, \quad \max\{0, t/2 - 1 - a\} \leq b \leq t/2. \tag{28.1}$$

Here $t = j/m$, and a, b are the exponents of s_0, t_0, divided by m.

Proof. The map whose graph is D has degree two. Put its ramification and branch points at zero and infinity. The action $([s_0 : s_1], [t_0 : t_1]) \mapsto ([\lambda s_0 : s_1], [\lambda^2 t_0 : t_1])$ preserves D, and, after interchanging zero and infinity if needed, has the flat limit $R = (s_0 = 0)$. This degeneration of ideal components is flat, not merely an inclusion of limiting generators: r_λ, Q are a regular sequence for all parameters, including the limit. The powers of their ideal have generators of degrees $(n + h, h)$ for $0 \leq h \leq n$, and the consecutive syzygies have degrees $(n + h + 2, h + 1)$ for $0 \leq h < n$. The resulting two-term free resolution gives a constant Hilbert function in every bidegree. Thus the generator limit has the full required dimension. Finally $(s_0, Q) = (s_0, s_1^2 t_0)$.

A monomial $s_0^A s_1^{M-A} t_0^C t_1^{J-C}$ lies in this ideal power exactly when for some k one has $A \geq n - k$, $M - A \geq 2k$ and $C \geq k$. Equivalently,

$$A \geq \max\{0, 2n - M\}, \qquad C \geq \max\{0, n - A\}.$$

Divide by m. These are (28.1), up to $O(1/m)$ changes of the walls. The region is empty for $t > 4$. □

Lemma 28.3 (Integrals for the first double-fibre case)**.** *For the limiting monomial spaces, the four expected coordinate coefficients and the two ramification-point weight sums are*

	s_0	s_1	t_0	t_1	$(1,2)$ *at* o_0	$(1,2)$ *at* o_∞
coefficient	5/9	4/9	29/48	41/72	127/72	19/12.

The expected coefficient of any curve $D_c = (s_1^2 t_0 - c s_0^2 t_1 = 0)$, $c \neq 0$, is at most 2/9.

Proof. Let Ω be the region in (28.1) and define $\langle \phi \rangle = \frac{1}{3} \int_\Omega \phi(t, a, b)\, dt\, da\, db$. The factor $1/3$ comes from $N_m \sim 3m^3$. To make these polynomial integrations explicit, split the region into

t	a	b
$[0, 2]$	$[0, 1]$	$[0, t/2]$
$[2, 3]$	$[0, t/2 - 1]$	$[t/2 - 1 - a, t/2]$
$[2, 3]$	$[t/2 - 1, 1]$	$[0, t/2]$
$[3, 4]$	$[t - 3, t/2 - 1]$	$[t/2 - 1 - a, t/2]$
$[3, 4]$	$[t/2 - 1, 1]$	$[0, t/2]$.

Direct integration gives

$$\langle 1\rangle = 1, \quad \langle t\rangle = 169/72, \quad \langle a\rangle = 5/9, \quad \langle 1-a\rangle = 4/9, \quad \langle b\rangle = 29/48, \quad \langle t/2-b\rangle = 41/72.$$

These are the four divisor coefficients. The two weight sums are $\langle a+2b\rangle = 127/72$ and $\langle 1-a+2(t/2-b)\rangle = 19/12$. For instance, on the first row $\langle a\rangle = 1/6$ and $\langle b\rangle = 1/9$; the other four rows contribute $7/18$ and $71/144$, respectively.

To bound the order along D_c, take a monomial order in which its initial term is $s_1^2 t_0$. Choose a D_c-adapted basis and put its initial forms in echelon form for this monomial order. Their pivot monomials are distinct. If a form in one of the limiting monomial spaces is divisible by D_c^k, its initial term is divisible by $s_1^{2k}t_0^k$ and still belongs to that monomial space. Thus the sum of orders along D_c is bounded by one half of the sum of the s_1 exponents, with a bounded rounding error per section. Normalization gives at most $(4/9)/2 = 2/9$. □

Lemma 28.4 (A double fibre contained in E)**.** *If $\Lambda \subset E$, put*

$$h_{m,j} = \max\left\{0, \left\lceil \frac{J-m-\epsilon}{3} \right\rceil\right\}.$$

After removal of the factor $\epsilon D/2$, the refined space equals

$$Q^{h_{m,j}} H^0(T, \mathcal{O}_T(M-2h_{m,j}, J-h_{m,j})),$$

with the convention that it is zero if a degree is negative. Its limiting moving and fixed degrees are

t	$a(t)$	$b(t)$	$d(t)$
$[0,2]$	1	$t/2$	0
$[2,5]$	$(5-t)/3$	$(t+1)/3$	$(t-2)/6$,

where $d(t)D$ is fixed. The expected coordinate coefficients of the moving series are $3/8$ on either first-factor coordinate fibre and $41/72$ on either second-factor coordinate fibre. Its fixed coefficient of D is $1/8$, and its additional movable expected coefficient along a $(2,1)$-curve is $11/72$.

Proof. Use the second equation of Lemma 27.2. If the exponent of e is $\epsilon + 2h$, the remaining u,v-degree is $M-2h$ and the base weight degree is $J-h$. The unused x_0-exponent is nonnegative precisely when $J-h \le m+\epsilon+2h$. The least possible h is therefore $h_{m,j}$. For a fixed h all forms in the indicated bidegree occur. The subspaces for larger h are contained in the one for the least h, by multiplication by Q. This proves equality and the degree table.

The four required integrals, retaining normalization mN_m, are

$$c_h = \frac{1}{6}\int a^2 b\,dt = \frac{3}{8}, \qquad c_f = \frac{1}{6}\int ab^2\,dt = \frac{41}{72}, \qquad c_D = \frac{1}{3}\int dab\,dt = \frac{1}{8},$$

$$c_{\mathrm{mov},D} = \frac{1}{3}\int\int_0^{\min(a/2,b)} (a-2z)(b-z)\,dz\,dt = \frac{11}{72}.$$

For the last integral the upper limit is $t/2$ on $[0,1]$, $1/2$ on $[1,2]$, and $(5-t)/6$ on $[2,5]$. The contributions on these three intervals are respectively $1/96$, $7/144$, and $3/32$. For the first three integrals, the contributions from $[0,2]$ and $[2,5]$ are, respectively,

$$(1/6,\ 1/9,\ 0), \qquad (5/24,\ 11/24,\ 1/8).$$

These formulas follow by summing exponents in complete bidegree spaces and by summing dimensions after subtracting kD, as in Lemma 18.6. In addition $\frac{1}{3}\int ab\,dt = 1$ and $\frac{1}{3}\int tab\,dt = 91/36$, agreeing with the independent Hilbert calculation. □

Proposition 28.5 (The surface estimate in both cases)**.** *For either type of double fibre,*

$$\delta(T, D/2; W) \geq \frac{144}{127}.$$

Proof. We use the invariant-boundary criterion of Lemma 18.7. In the coordinates above, an invariant effective divisor is supported on the four coordinate fibres and finitely many curves D_c. It is log canonical if all prime coefficients are at most one and its weights at the two ramification points are at most three. This criterion is proved by two ordinary point blow-ups at each ramification point; the different contributes weight one there.

Fix a divisorial valuation on T. In each refined summand choose bases compatible with that valuation and with the filtration of the action preserving D, using Lemma 2.3. In the first case, Lemma 28.2 identifies their initial spaces exactly. Their initial bases are weight-homogeneous bases of those spaces, though individual forms need not be monomials. Their coordinate coefficients are bounded by the maxima in Lemma 28.3. Their D_c coefficients are at most $2/9 + o(1)$. Their two weight sums are $127/72 + o(1)$ and $19/12 + o(1)$: the sum of weights of a weight-compatible basis is independent of that basis. The parity factor contributes only $O(1/m)$.

For any fixed $0 < \lambda < 144/127$, all prime coefficients of the limiting pair with boundary $D/2 + \lambda$ times the normalized initial basis divisor are less than one for large m. Indeed the largest coordinate coefficient is at most $\lambda\, 29/48 < 1$, and that of D is at most $1/2 + \lambda\, 2/9 < 1$. At the two ramification points the weight is at most

$$1 + \lambda\, 127/72 + o(1) < 3.$$

The invariant-boundary criterion proves log canonicity of the limit. Semicontinuity, with D preserved throughout the degeneration, gives log canonicity of the original pair.

In the second case use the complete moving spaces and fixed factors from Lemma 28.4. The coordinate coefficients are at most $3/8$ and $41/72$, and the coefficient of D is at most $1/8 + 11/72 = 5/18$. At either ramification point the expected weight, including the fixed factor, is

$$c_h + 2c_f + 2c_D = \frac{3}{8} + \frac{82}{72} + \frac{1}{4} = \frac{127}{72}.$$

The same argument applies, since $\lambda\, 41/72 < 1$ and $1/2 + \lambda\, 5/18 < 1$. Compatibility with the original valuation converts log canonicity into its discrepancy bound. Taking limits and then letting λ increase to $144/127$ proves the proposition for every divisorial valuation. □

Corollary 28.6. *For every $p \in \Lambda$, one has $\delta_p(X) \geq 144/127$.*

Proof. The plt extraction of Lemma 27.1 has discrepancy three and different $D/2$. Abban–Zhuang refinement along its centre gives

$$\delta_p(X) \geq \min\{3/S_X(T), \delta(T, D/2; W)\}.$$

The two possible first entries are $216/169$ and $108/91$, both larger than $144/127$. Bounded support follows from the explicit refined series (with thresholds four and five). The ample-series hypothesis holds since L is ample and $-T$ is relatively ample; hence $\omega^*L - tT$ is ample for all sufficiently small positive rational t. Thus the refinement theorem applies. □

Example 28.7 (A double fibre inside the exceptional divisor). Take the Fermat cubic $V = (z_0^3 + z_1^3 + z_2^3 + z_3^3 + z_4^3 = 0)$ and the line $\ell = (z_1 = -z_0,\ z_3 = -z_2,\ z_4 = 0)$. Put $u = z_0$, $v = z_2$, $r = z_0 + z_1$, $s = z_2 + z_3$, and $t = z_4$. Its equation becomes

$$3u^2r - 3ur^2 + r^3 + 3v^2s - 3vs^2 + s^3 + t^3 = 0.$$

The normal derivatives along ℓ are $3u^2, 3v^2, 0$, so the line is of the second kind. The plane $r = s = 0$ cuts out 3ℓ. On the blow-up, substitute $(r, s, t) = e(x_1, x_2, x_0)$ and divide by e. Above $[x_0 : x_1 : x_2] = [1 : 0 : 0]$ the residual conic is $e^2 = 0$. Thus its reduced support lies in E. The second value $S_X(T) = 91/36$ and the second refined-series calculation are necessary even though V and X are smooth.

29 A weighted extraction at a ramification point

We now deal with the points where the conic is smooth but tangent to E. The calculation uses the associated graded section ring. In particular, it requires no three-dimensional Zariski decomposition on the weighted blow-up.

Lemma 29.1 (The two ramification rulings)**.** *Suppose that ℓ is of the second kind. Let $s \subset E$ be its negative section, and let $p \in E \setminus s$ lie on a smooth conic fibre. Choose a smooth cubic section $S \in |B|$ through that fibre as in Lemma 23.4, and put $c = E|_S$ and $F = f^{-1}(f(p))$. Then $(cF)_p$ is either one or two. The second possibility occurs precisely on two ruling fibres of $E \to \ell$. In this case the Cox coordinates can be chosen so that*

$$G = u^2x_1 + v^2x_2 + e\big(uQ_0(x) + vQ_1(x)\big) + e^2C(x) = 0,$$

$$\pi(p) = [u : v] = [0 : 1], \qquad f(p) = [x_0 : x_1 : x_2] = [0 : 1 : 0], \qquad Q_1(0, 1, 0) \neq 0.$$

Here Q_0, Q_1 are quadratic forms and C is a cubic form in the base coordinates.

Proof. The normal derivatives along ℓ span a base point free pencil of binary quadrics. It contains two distinct double members: equivalently, its associated degree-two map $\mathbb{P}^1 \to \mathbb{P}^1$ has two distinct ramification points. Sending their supports to $u = 0$ and $v = 0$, and changing the two generators of the pencil, gives the displayed form of G. The special base point is $y_0 = [1 : 0 : 0]$.

On E, the equation is $u^2x_1 + v^2x_2 = 0$. Away from s, the map $E \to \mathbb{P}^2$ is locally a double cover, ramified exactly on the two rulings $u = x_2 = 0$ and $v = x_1 = 0$. For instance, on the chart $v = x_1 = 1$ its expression is

$$(u, x_0) \longmapsto (x_0, x_2) = (x_0, -u^2).$$

Restricting this map to a general base line through $f(p)$ gives a smooth curve c through p. Its intersection with the fibre has multiplicity one away from those rulings and multiplicity two on them. This proves the assertion about $(cF)_p$; the same multiplicity holds for any smooth cubic section from the pencil.

In the ramified case interchange u, v and x_1, x_2 if necessary. Then $u = x_2 = 0$, $v \neq 0$, and $x_1 \neq 0$, the last assertion following from $p \notin s$. Replacing x_0 by $x_0 - ax_1$ gives the stated coordinates of p. On its plane conic fibre the derivatives with respect to u, v vanish at p, while the derivative with respect to e is $Q_1(0, 1, 0)$. Since that conic is smooth at p, this number is nonzero. □

Lemma 29.2 (The extraction and its expected order)**.** *Suppose that the equation of X is*

$$G = u^2x_1 + v^2x_2 + e(uQ_0 + vQ_1) + e^2C = 0,$$

where Q_0, Q_1, C have degrees $2, 2, 3$ in the base coordinates, respectively. Let p be given by $u = e = x_0 = x_2 = 0$ and $v = x_1 = 1$, and assume $c = Q_1(0, 1, 0) \neq 0$. Let $\rho : \widetilde{X} \to X$ be the weighted blow-up of p with weights $(1, 1, 2)$ in the regular parameters (u, x_0, e). Its exceptional divisor T satisfies

$$T \cong \mathbb{P}(1, 1, 2), \qquad A_X(T) = 4, \qquad \mathrm{Diff}_T(0) = 0, \qquad S_X(T) = \frac{26}{9}.$$

Proof. The implicit function theorem applies to x_2, because $\partial G/\partial x_2(p) = 1$. Its initial expression is $x_2 = -u^2 - ce$. The exceptional divisor and discrepancy are the usual ones for the weighted blow-up of a smooth threefold with weights $(1,1,2)$. The only nontrivial quotient chart has type $\frac{1}{2}(1,1,1)$; its restriction to T has type $\frac{1}{2}(1,1)$. The quotient is unramified in codimension one on T, so the different is zero. This extraction is plt.

Assign the six Cox variables the weights

$$\mathrm{wt}(u, v, e, x_0, x_1, x_2) = (1, 0, 2, 1, 0, 2).$$

The initial equation is

$$G_0 = v^2 x_2 + u^2 x_1 + cevx_1^2.$$

It is irreducible: viewed as a linear polynomial in x_2, its two coefficients v^2 and $x_1(u^2 + cevx_1)$ are coprime. Consequently the initial quotient is a domain, and localization at vx_1 creates no new relations. Moreover, the degrees of v and x_1 form a basis of the Cox grading lattice, so setting them equal to one is the local trivialization of each fixed bidegree. On that chart it is exactly the weighted associated graded ring in the three regular parameters above. Thus the Cox weight filtration coincides with the geometric T-valuation filtration on the section spaces. Here we use the section-ring presentation established earlier; the hypersurface exact sequence has no H^1 term in these bidegrees.

For completeness, write $P(p,q)$ for the dimension of the polynomial bidegree (p,q) piece, and $M(p,q)$ for its total weight. A monomial with e-exponent k has u,v-degree $p-k$ and x_0, x_1, x_2-degree $q+k$. Averaging the exponents gives

$$\begin{aligned} P(p,q) &= \sum_{k=0}^{p} (p-k+1)\binom{q+k+2}{2}, \\ M(p,q) &= \sum_{k=0}^{p} (p-k+1)\binom{q+k+2}{2}\left(\frac{p}{2} + q + \frac{5k}{2}\right). \end{aligned}$$

Since G has bidegree $(2,1)$ and initial weight two, the dimension N_m and total weight w_m of $H^0(X, mL)$ are

$$\begin{aligned} N_m &= P(m,m) - P(m-2,m-1) = \frac{(2m+1)(3m^2+3m+2)}{2}, \\ w_m &= M(m,m) - M(m-2,m-1) - 2P(m-2,m-1) \\ &= \frac{m(2m+1)(26m^2+21m+19)}{6}. \end{aligned}$$

These formulas hold for $m \geq 2$. Taking $\lim w_m/(mN_m)$ gives $26/9$. □

Lemma 29.3 (The exact refined spaces)**.** *Put $t = x_0$ on the chart used above and set $r = u^2$ and $Q = r + ce$. Let $W_{m,j}$ denote the refinement along T. Fix an exponent i of t, put $\epsilon \in \{0,1\}$ with $\epsilon \equiv j - i \pmod 2$, and write $h = (j - i - \epsilon)/2$ and $M = m - \epsilon$. The t^i summand is zero unless $h \geq 0$ and*

$$i + h \leq m + 2\min(h, M).$$

When it is nonzero, it is exactly

$$t^i u^\epsilon e^K Q^D \,\mathrm{Sym}^{h-K-D}\langle r, e\rangle, \qquad \begin{cases} K = \max\{0,\ i-m,\ \lceil (2h+2i-3m+\epsilon)/3 \rceil\}, \\ D = \max\{0, h-m+\epsilon\}. \end{cases}$$

Proof. A global monomial has the form $u^a v^b e^k x_0^i x_1^\nu x_2^d$, where $a+b+k=m$ and $i+\nu+d=m+k$. Its initial form on T, up to a nonzero constant, is $t^i u^a e^k Q^d$. Write $a=\epsilon+2a'$. Thus the binary forms that occur are precisely

$$r^{a'} e^k Q^d, \qquad a'+k+d=h, \quad 2a'+k\le M, \quad i+d\le m+k.$$

The least possible k is K, and the least possible d is D. Here is a dimension argument that also proves that there are no missing binary forms after these fixed factors are removed. For a fixed integer k, a feasible d exists exactly when

$$\max\left\{0, \left\lceil \frac{2h-M-k}{2} \right\rceil\right\} \le d \le \min\{h-k, k+m-i\}.$$

These intervals are nonempty for precisely the consecutive integers $K\le k\le h-D$. For each such k, choose one feasible d. The resulting form has exact order k along $e=0$, since r and Q are nonzero there. These $h-K-D+1$ forms are therefore linearly independent. Every generator is divisible by $e^K Q^D$, and the space of all binary forms of degree h with this divisibility has the same dimension. This proves the claimed equality. The nonemptiness condition follows also by taking $a'=0$ and $k=\min(h,M)$, which minimizes $d-k$ under the first constraint. □

Example 29.4 (A summand with a fixed factor). For $m=4$, $i=5$, $\epsilon=0$, and $h=1$, the only feasible triple is $(a',k,d)=(0,1,0)$. The refined summand is therefore spanned by $t^5 e$. In particular, the term $i-m$ in the formula for K is necessary. For $m=4$, $i=0$, $\epsilon=0$, and $h=2$, one obtains the complete binary space $\mathrm{Sym}^2\langle r,e\rangle$.

Lemma 29.5 (Surface orders and their integrals)**.** *On $T=\mathbb{P}(1,1,2)_{[u:t:e]}$, let $C_a=(e-au^2=0)$. For the refined series in Lemma 29.3,*

Z	$(u=0)$	$(t=0)$	$(e=0)$	$(Q=0)$	C_a $(a\neq 0,-1/c)$
$S(W;Z)$	$89/144$	$41/72$	$5/9$	$29/48$	$89/288.$

The expected $(1,2)$-weighted order at $o=[0:1:0]$, with local coordinates (u,e), is $167/72$.

Proof. For a summand in the preceding lemma put $n=h-K-D$. Complete binary forms of degree n have vanishing sequence $0,1,\dots,n$ at any point of their parameter line. The maximal average orders along the five curves in the table, computed using bases adapted to the respective vanishing filtrations, are consequently

$$\epsilon+n, \qquad i, \qquad K+\frac{n}{2}, \qquad D+\frac{n}{2}, \qquad \frac{n}{2},$$

respectively. These are upper bounds for the average orders of arbitrary bases. Every nonzero element of this summand has weighted order $\epsilon+2h$ at o.

Here are the integrals explicitly. Put $x=i/m$, $y=h/m$, and

$$\Omega=\{(x,y)\in\mathbb{R}^2_{\ge 0} : x+y\le 1+2\min(y,1)\},$$
$$k=\max\{0,x-1,(2x+2y-3)/3\}, \qquad d=\max\{0,y-1\}, \qquad z=y-k-d.$$

The two parity choices and $N_m=3m^3+O(m^2)$ give the normalized density $\frac{2}{3}z\,dx\,dy$. Hence the required quantities are

$$S(W;u=0)=\frac{2}{3}\int_\Omega z^2\,dx\,dy, \qquad S(W;t=0)=\frac{2}{3}\int_\Omega xz\,dx\,dy,$$
$$S(W;e=0)=\frac{2}{3}\int_\Omega z(k+z/2)\,dx\,dy, \qquad S(W;Q=0)=\frac{2}{3}\int_\Omega z(d+z/2)\,dx\,dy,$$
$$S(W;C_a)=\frac{1}{3}\int_\Omega z^2\,dx\,dy, \qquad S(W;\mathrm{wt}_{(1,2),o})=\frac{4}{3}\int_\Omega yz\,dx\,dy.$$

One may evaluate them on the following five polygons, on which k, d are linear. Vertices are listed in the (x, y) plane.

	vertices	k	d
P_1	$(0,0),(1,0),(1,1/2),(1/2,1),(0,1)$	0	0
P_2	$(1/2,1),(0,3/2),(0,1)$	0	$y-1$
P_3	$(1,0),(2,1),(1,1/2)$	$x-1$	0
P_4	$(2,1),(1/2,1),(1,1/2)$	$(2x+2y-3)/3$	0
P_5	$(2,1),(0,3),(0,3/2),(1/2,1)$	$(2x+2y-3)/3$	$y-1$

For further detail, their contributions to the displayed orders are

	u	t	e	Q	C_a	$\mathrm{wt}_{(1,2),o}$
P_1	$47/288$	$7/64$	$47/576$	$47/576$	$47/576$	$47/144$
P_2	$1/12$	$1/72$	$1/24$	$1/18$	$1/24$	$7/36$
P_3	$1/144$	$5/144$	$1/96$	$1/288$	$1/288$	$1/36$
P_4	$7/96$	$25/192$	$13/192$	$7/192$	$7/192$	$5/24$
P_5	$7/24$	$9/32$	$17/48$	$41/96$	$7/48$	$25/16$

Adding the columns gives the stated values. As two independent consistency checks, $\frac{2}{3}\int_\Omega z = 1$ and $\frac{2}{3}\int_\Omega (x+2y)z = 26/9$, the latter agreeing with Lemma 29.2. □

Lemma 29.6 (Invariant boundaries on the exceptional surface)**.** *Let*

$$\Delta = \alpha(t=0) + \beta(u=0) + \sum_a c_a C_a$$

be an effective divisor on $\mathbb{P}(1,1,2)$, with finitely many nonzero c_a. Then (T, Δ) is log canonical if and only if

$$\alpha \le 1, \qquad \beta \le 1, \qquad c_a \le 1 \text{ for every } a, \qquad \beta + 2\sum_a c_a \le 3.$$

Proof. Away from $o = [0:1:0]$ and the quotient point $[0:0:1]$, the displayed curves have only transverse intersections. At the quotient point the degree-two quotient cover has coordinates (u, t), with the two coordinate axes as boundary; the cover is unramified in codimension one. Thus the two coefficient inequalities suffice there.

At o the equations are $u = 0$ and $e - au^2 = 0$ in a smooth affine plane. Blow up the origin and then the common point of the strict transforms of the parabolas. The resulting exceptional coefficients in the log pullback are $\beta + \sum c_a - 1$ and $\beta + 2\sum c_a - 2$. The second blow-up separates the parabolas because their parameters a are distinct, and the total boundary then has simple normal crossings. The last inequality in the statement bounds the second coefficient by one. Together with $\beta \le 1$, it also implies $\beta + \sum c_a \le 2$, which bounds the first coefficient by one. Conversely all the inequalities are necessary, either along the prime curves or along the second exceptional divisor, whose valuation has weights $(1, 2)$ and discrepancy three. □

Lemma 29.7 (The local estimate at a smooth tangent conic)**.** *Under the hypotheses of Lemma 29.2,*

$$\delta(T; W) \ge \frac{216}{167}, \qquad \delta_p(X) \ge \frac{216}{167} > 1.$$

Proof. Consider the action scaling t and fixing u, e. The refined spaces are invariant under this action by Lemma 29.3. For an arbitrary divisorial valuation of T, choose in every refined summand a basis compatible both with its valuation filtration and with the weight filtration of this action. Lemma 2.3 applies. Degenerate these bases by the action. The initial bases

remain bases of the corresponding weight spaces, and their divisors are supported on $(t = 0)$, $(u = 0)$, and the curves C_a: indeed a homogeneous binary form in u^2, e factors into these curves and a power of u.

The maximal normalized coefficients of these prime curves are bounded by the table in Lemma 29.5, up to $O(1/m)$. For an arbitrary basis of a finite-dimensional series, its sum of vanishing orders is bounded above by the sum for a basis adapted to that vanishing filtration. The bounds in the table therefore apply to every initial basis. The weighted order at o is independent of the choice of weight-homogeneous bases and equals $167/72 + O(1/m)$. Fix $0 < \lambda < 216/167$. All the individual coefficients of the normalized initial basis boundary multiplied by λ are less than one for sufficiently large m, and its weighted order at o is less than

$$\lambda\frac{167}{72} + o(1) < 3.$$

Lemma 29.6 gives log canonicity of the limit. After clearing the Cartier index two, semicontinuity of log canonical thresholds on the fixed klt surface gives log canonicity before degeneration. Compatibility with the arbitrary starting valuation then proves its discrepancy bound. Taking the asymptotic limit and then letting λ increase to $216/167$ proves the surface estimate.

The extraction is plt, with zero different. Its refined series has bounded support $0 \le j \le 6m$ by its explicit description. It contains an ample series directly: for $0 \le j \le m$, every weighted monomial of degree j in u, t, e lifts to a global monomial, since its u-exponent plus its e-exponent is at most m, and its t-exponent is at most m plus its e-exponent. Hence $W_{m,j} = H^0(T, \mathcal{O}_T(j))$ in this range. Also $\rho^*L - \eta T$ is ample for sufficiently small positive rational η, since L is ample and $-T$ is relatively ample. Abban–Zhuang refinement therefore gives

$$\delta_p(X) \ge \min\left\{\frac{A_X(T)}{S_X(T)}, \delta(T; W)\right\} \ge \min\left\{\frac{18}{13}, \frac{216}{167}\right\} = \frac{216}{167}.$$

All orders used in this argument came from the associated graded ring and complete binary spaces. No Zariski decomposition on the weighted threefold $\tilde{X}$ was used. □

Lemma 29.8 (The expected order of a ramification-ruling valuation)**.** *Suppose that p is a smooth tangency point as in Lemma 29.1, and let $g \subset E$ be its ramification ruling. In the coordinates of that lemma, let ν_g be the divisorial valuation with transverse weights $(1, 2)$ in (u, e) at the generic point of g. Then*

$$c_X(\nu_g) = g, \qquad A_X(\nu_g) = 3, \qquad S_X(\nu_g) = \frac{167}{72}.$$

Proof. The ruling is $g = (u = e = x_2 = 0)$. On the chart $v = x_1 = 1$, put $t = x_0$. At the generic point of g, the functions u, e are regular parameters over $\mathbb{C}(t)$. Their monomial valuation of weights $(1, 2)$ is divisorial, with centre g and log discrepancy $1 + 2 = 3$.

Assign the Cox variables the weights

$$\mathrm{wt}(u, v, e, x_0, x_1, x_2) = (1, 0, 2, 0, 0, 2).$$

The initial equation is

$$G_0 = v^2x_2 + u^2x_1 + evQ_1(x_0, x_1, 0).$$

Its coefficients as a linear polynomial in x_2 are coprime, so the initial quotient is a domain and injects into its localization at vx_1. The degrees of v, x_1 form a basis of the Cox grading lattice; thus trivializing each fixed bidegree by these coordinates preserves injectivity. Setting $v = x_1 = 1$ eliminates x_2 and gives

$$\mathbb{C}[t, u, e, x_2]/(x_2 + u^2 + eQ_1(t, 1, 0)) \cong \mathbb{C}[t, u, e] \hookrightarrow \mathbb{C}(t)[u, e].$$

The last ring, graded by $\mathrm{wt}(u,e)=(1,2)$, is the associated graded ring at the generic point of g. Hence the Cox weight filtration is the geometric ν_g-filtration on every section space, including possible cancellations. The global section-ring presentation is the one established in Lemma 27.2.

For the polynomial Cox ring, let $P(p,q)$ and $M_g(p,q)$ denote the dimension and total weight of bidegree (p,q). A monomial with e-exponent k has u,v-degree $p-k$ and base degree $q+k$, so

$$P(p,q)=\sum_{k=0}^{p}(p-k+1)\binom{q+k+2}{2},$$
$$M_g(p,q)=\sum_{k=0}^{p}(p-k+1)\binom{q+k+2}{2}\left(\frac{p}{2}+\frac{2q}{3}+\frac{13k}{6}\right).$$

Here the mean contributions of u, e and x_2 are $(p-k)/2$, $2k$ and $2(q+k)/3$, respectively. Since the relation has bidegree $(2,1)$ and weight two, for $m\geq 2$

$$\begin{aligned}N_m&=P(m,m)-P(m-2,m-1)=\frac{(2m+1)(3m^2+3m+2)}{2},\\ w_m&=M_g(m,m)-M_g(m-2,m-1)-2P(m-2,m-1)\\&=\frac{m(167m^3+226m^2+193m+62)}{24}.\end{aligned}$$

For example, $N_2=50$ and $w_2=224$. Taking the limit gives $S_X(\nu_g)=\lim_{m\to\infty}w_m/(mN_m)=167/72$. □

Corollary 29.9 (The exact thresholds at a smooth tangency)**.** *Let $p\in E\setminus s$ be a tangency point on a smooth conic fibre, under the hypotheses of Lemma 29.1. For the weighted point extraction T and its refined series W,*

$$\delta(T;W)=\delta_p(X)=\frac{216}{167}.$$

Proof. Lemma 29.7 gives both lower bounds. The weight-$(1,2)$ valuation at the smooth point $o=[0:1:0]\in T$ has log discrepancy three and, by Lemma 29.5, expected order $167/72$. It therefore gives the matching upper bound for $\delta(T;W)$. For the threefold, Lemma 29.8 and $p\in g=c_X(\nu_g)$ give

$$\delta_p(X)\leq\frac{A_X(\nu_g)}{S_X(\nu_g)}=\frac{3}{167/72}=\frac{216}{167}.$$

□

30 Exclusion of all centres and completion of the proof

Lemma 30.1 (Curve centres in the exceptional divisor)**.** *Suppose that ℓ is of the second kind. Every prime divisor over X whose centre is a curve contained in E has positive beta invariant.*

Proof. Let C be its centre. A ruling of $E\to\ell$ meets the negative section s, which lies in $f^{-1}(y_0)$. If this fibre is reduced, Lemma 25.1 excludes such a centre; if it is nonreduced, Corollary 28.6 does so.

Otherwise $C\sim as+bg$ with integers $a\geq 1$ and $b\geq 0$, so $C-s$ is linearly equivalent to an effective divisor. Monotonicity of the surface volume and Lemma 26.2 give

$$S(W^E;C)\leq S(W^E;s)=\frac{71}{72}<1.$$

At the generic point of C, the only divisorial valuation of the surface E is a positive multiple of ord_C. Thus its refined threshold there is at least $72/71$. Abban–Zhuang adjunction along the smooth divisor E, with $S_X(E) = 5/9$, gives

$$\frac{A_X(Q)}{S_X(Q)} \geq \min\left\{\frac{9}{5}, \frac{72}{71}\right\} > 1$$

for every prime divisor Q with centre C. This proves the assertion. □

Lemma 30.2 (Transverse intersection with a smooth conic)**.** *Suppose that ℓ is of the second kind and that $p \in E \setminus s$ lies on a smooth conic fibre. In the notation of Lemma 29.1, assume that $(cF)_p = 1$. Every prime divisor Q over X with centre p satisfies*

$$\beta_X(Q) \geq \frac{31}{72}\,\mathrm{ord}_Q(S) > 0.$$

Proof. For the refinement of the surface series by F, Lemma 24.2 gives $\mathcal{S}_F = 29/48$ and $\mathcal{T}_p = 1$. Applying surface adjunction first shows that the threshold of W^S at p is at least one. Now $S_X(S) = 41/72 < 1$, so the strict adjunction inequality of Lemma 21.2 gives the displayed lower bound. Since the centre is $p \in S$, the order of S is positive. □

Proof of Theorem 23.1. Suppose first that ℓ is of the first kind, and let Q be a prime divisor over X. If its centre is contained in E, Lemma 26.1 gives $\beta_X(Q) > 0$. Otherwise choose a point of its centre outside E. According as its conic fibre is smooth, reduced and singular, or nonreduced, Lemma 24.2, Lemma 24.3, or Corollary 28.6 gives a local threshold strictly greater than one. Thus $\beta_X(Q) > 0$ in every case, proving K-stability for first-kind lines.

Now suppose that ℓ is of the second kind. Let Q be any prime divisor over X. If its centre is a divisor on X, Lemma 24.1 gives $\beta_X(Q) > 0$.

Suppose next that its centre is a curve. If the curve is contained in E, apply Lemma 30.1. Otherwise take a point of the curve outside E. For a smooth conic fibre, Lemmas 23.4 and 24.2 give local threshold at least $9/8$. For a reduced singular fibre away from y_0, Lemma 24.3 gives at least $36/35$; for the reduced fibre over y_0, use Lemma 25.1. For a nonreduced fibre, Corollary 28.6 gives at least $144/127$. In every case this threshold is strictly greater than one and therefore excludes $\beta_X(Q) \leq 0$.

It remains to consider a point centre p. If $p \notin E$, the same four estimates apply. If $p \in E$ lies on a singular conic fibre, the reduced-fibre lemmas and the double-fibre corollary again apply. The only case left is a point of E on a smooth conic. Such a point is outside s, because the fibre containing s is singular. Choose S as in Lemma 29.1. If $(cF)_p = 1$, Lemma 30.2 gives strict positivity. If $(cF)_p = 2$, the coordinate normalization of Lemma 29.1 permits application of Lemma 29.7, which gives a local threshold strictly greater than one. Thus $\beta_X(Q) > 0$ for every prime divisor Q over X. The valuative criterion proves K-stability. □

31 Comparison with a degree-two surface flag

The following calculation explains why the weighted point extraction above is needed. A small point term on an auxiliary surface does not suffice if the preceding curve term is too large.

Lemma 31.1 (The degree-two surface flag)**.** *Let a hyperplane section $V_0 \subset V$ be smooth, meet ℓ transversally at q, and contain no line through q. Let $S \in |H|$ be its strict transform, let $e_q \subset S$ be the exceptional curve, and put $A = H|_S$. Then S is a smooth del Pezzo surface of degree two. The strict transform C of the tangent plane cubic at q is a (-1)-curve of class $A - 2e_q$. For the flag $S \supset e_q \ni p$, its constants are*

$$S_X(S) = \frac{3}{8}, \qquad S(W^S; e_q) = \frac{155}{144}, \qquad S(W^{S,e_q}; p) = \frac{127}{144} + \frac{5}{32}(Ce_q)_p.$$

Proof. Blowing up a point on no line of a smooth cubic surface gives a degree-two del Pezzo surface. The tangent plane cubic is irreducible with a double point at q, since a reducible tangent section would contain a line through q. Its strict transform is the stated (-1)-curve. Contracting it shows that $2A - 3e_q = -K_S + C$ is nef (it is the pullback of the anticanonical class of the resulting smooth cubic surface).

For $0 \le u \le 1$ put $a = 2 - u$. The first refined series is asymptotically complete of class $aA - e_q$. To refine along e_q, put $b = 1 + v$. The positive parts are

b	$P_{u,v}$	$N_{u,v}$
$[1, 3a/2]$	$aA - be_q$	0
$[3a/2, 2a]$	$(2a - b)(2A - 3e_q)$	$(2b - 3a)C.$

Nefness in the first interval follows by expressing its classes as nonnegative combinations of A and $2A - 3e_q$; in the second the positive part is orthogonal to C. The threshold is $b = 2a$, as tested by $2A - 3e_q$. Their squares are $3a^2 - b^2$ and $3(2a - b)^2$, while their degrees on e_q are b and $3(2a - b)$. Consequently

$$\begin{aligned} S(W^S; e_q) &= \frac{1}{6}\int_1^2 \left(\frac{7}{2}a^3 - 3a^2 + \frac{1}{3}\right) da = \frac{155}{144}, \\ \frac{1}{6}\int\int (P_{u,v}e_q)^2 \, dv\, du &= \frac{1}{6}\int_1^2 \left(\frac{3}{2}a^3 - \frac{1}{3}\right) da = \frac{127}{144}, \\ \frac{1}{3}\int\int (P_{u,v}e_q)\,\mathrm{coeff}_C\, N_{u,v}\, dv\, du &= \frac{1}{3}\int_1^2 \frac{a^3}{8} da = \frac{5}{32}. \end{aligned}$$

This proves all three formulas. □

Remark 31.2 (The curve term cannot be discarded). Even when $p \notin C$, the preceding flag gives only

$$\min\{8/3, 144/155, 144/127\} = 144/155 < 1.$$

Likewise, the refined threshold $24/29$ for the second-kind exceptional divisor is insufficient by itself. The weight-$(1, 1, 2)$ extraction in Lemma 29.7 resolves the remaining point case by computing a different refined series. No negative boundary is introduced to combine these unsuccessful flags: such a correction would require an adjunction statement that does not hold in general.

Part VI
Family 2.12: intersections of three bilinear divisors

32 Geometry and the reduction to point centres

Let $X \subset \mathbb{P}^3 \times \mathbb{P}^3$ be a smooth complete intersection of three divisors of bidegree $(1,1)$. Write H, H' for the two hyperplane classes restricted to X. Both projections are blow-ups of smooth nonhyperelliptic curves of degree six and genus three. We denote their exceptional divisors by E, E'. The standard determinantal description gives a smooth plane quartic

$$C = \{\det(z_0M_1 + z_1M_2 + z_2M_3) = 0\} \subset \mathbb{P}^2, \qquad X = \{x^{\mathrm{t}}M_i y = 0 : 1 \le i \le 3\}.$$

In particular, every matrix on C has rank three. These geometric facts hold for every smooth member; see [2, Proposition 2.1].

Lemma 32.1 (Intersection numbers and anticanonical sections)**.** *One has*

$$L = -K_X = H + H', \quad E = 3H - H', \quad E' = 3H' - H,$$
$$H^3 = (H')^3 = 1, \qquad H^2H' = H(H')^2 = 3.$$

so $L^3 = 20$*. For* $m \ge 0$*, the space* $H^0(X, mL)$ *is the bidegree* (m,m) *part of*

$$R = \mathbb{C}[x_0, \dots, x_3, y_0, \dots, y_3]/(F_1, F_2, F_3), \qquad F_i = x^{\mathrm{t}}M_i y.$$

Its dimension is

$$N_m = \frac{(2m+1)(5m^2+5m+3)}{3} = \frac{10}{3}m^3 + 5m^2 + \frac{11}{3}m + 1. \tag{32.1}$$

Proof. Adjunction gives $L = H + H'$. The two blow-up canonical formulas give $4H - E = L = 4H' - E'$. In the ambient product, $[X] = (H + H')^3$; multiplication gives the four intersection numbers. The Koszul resolution of the three bilinear equations, twisted by (m,m), identifies the displayed quotient with global sections. Indeed, the terms have bidegrees $(m-r, m-r)$, $0 \le r \le 3$. A negative entry occurring here belongs to $\{-1,-2,-3\}$, for which all cohomology on $\mathbb{P}^3$ vanishes. Nonnegative entries have no higher cohomology. Consequently

$$N_m = \sum_{r=0}^{3} (-1)^r \binom{3}{r} \binom{m-r+3}{3}^2,$$

where a summand with $m < r$ is zero. Expanding proves (32.1). □

We use the following published reduction explicitly as an external input.

Proposition 32.2 (Cheltsov–Li–Ma'u–Pinardin)**.** *If a prime divisor* F *over* X *satisfies* $\beta_X(F) \le 0$*, then its centre on* X *is a closed point of* $E \cap E'$*.*

This is [1, Theorem A]. Thus it suffices to prove strict beta positivity for every divisor centred at every point of $E \cap E'$. No group action is imposed. Nor do we assume that E and E' meet transversally.

33 The ordinary point filtration

Fix $p \in E \cap E'$. The next two lemmas justify the use of an initial complete intersection. Merely assigning weights to the displayed equations would not suffice: the initial ideal could acquire further equations, and cancellations could change expected orders.

Lemma 33.1 (A uniform bilinear normal form and coprimality)**.** *Let $P \in E \cap E'$ on a smooth member X of Family* 2.12*. There are product coordinates with $P = (e_0, e_0)$ and a basis of its three bilinear equations such that*

$$\begin{aligned} F_1 &= x_1y_0 + x_0y_1 + R_1, \\ F_2 &= x_2y_0 + q_2 + R_2, \\ F_3 &= x_0y_2 + q_3 + R_3. \end{aligned}$$

Here R_1 is bilinear in the nonconstant coordinates, q_2, q_3 are bilinear forms in (x_1, x_3) and (y_1, y_3), and every term of R_2, R_3 contains x_2 or y_2 as well as a nonconstant coordinate from the other factor. Write

$$q_i = -A_ix_1y_1 - B_ix_3y_1 + C_ix_1y_3 + D_ix_3y_3, \qquad N_i = \begin{pmatrix} -A_i & C_i \\ -B_i & D_i \end{pmatrix}.$$

Then q_2, q_3 are nonzero and coprime in $\mathbb{C}[x_1, x_3, y_1, y_3]$, and

$$(C_2, D_2) \neq (0, 0), \qquad (B_3, D_3) \neq (0, 0).$$

Moreover, N_2, N_3 are linearly independent and $\det(z_1N_2 + z_2N_3)$ is not identically zero.

Proof. The two summands of the differential of the defining equations are maps from the two three-dimensional factor tangent spaces to the three-dimensional equation space. Each has rank two, since each projection has a smooth one-dimensional exceptional fibre at P. Smoothness of X says that their images span the equation space; the two rank-two images therefore intersect in a line. Choosing bases adapted to these two planes, and then factor coordinates, gives the linear parts $x_1 + y_1, x_2, y_2$. Homogenizing yields the displayed normal form.

Let $M(z) = z_0M_1 + z_1M_2 + z_2M_3$ be the resulting matrix pencil. On the line $z_0 = 0$, its zeroth row and column are $(0, 0, z_2, 0)$ and $(0, 0, z_1, 0)^{\mathsf{t}}$, respectively. Laplace expansion gives

$$\det M(0, z_1, z_2) = -z_1z_2 \det(z_1N_2 + z_2N_3).$$

The determinant quartic C is smooth and hence irreducible, so it does not contain the line $z_0 = 0$. Thus the binary determinant on the right is not identically zero. If $z_1N_2 + z_2N_3 = 0$ for some $[z_1 : z_2]$, the zeroth row and all rows other than row 2 of $M(0, z_1, z_2)$ are supported only in column 2. The full matrix then has rank at most two. All entries of its adjugate vanish, so Jacobi's formula makes every first derivative of $\det M$ vanish there, contradicting smoothness of C. Hence N_2, N_3 are linearly independent and both nonzero.

A nonconstant common factor of two bilinear forms is either the entire form, making them proportional, or a linear form in one factor's variables. Proportionality is already excluded. A common x-linear factor writes $q_i = \ell(x)m_i(y)$, so every matrix $z_1N_2 + z_2N_3$ has rank at most one. A common y-linear factor has the same consequence. Both contradict the nonzero binary determinant. This proves coprimality in the four variables.

For the last assertions, use regular parameters $t = x_1, a = x_3, b = y_3$ on the affine chart $x_0 = y_0 = 1$. The equations give

$$y_1 = -t + O(2), \quad x_2 = -Q_2 + O(3), \quad y_2 = -Q_3 + O(3), \qquad Q_i = A_it^2 + B_ita + C_itb + D_iab.$$

The linear term of the Jacobian determinant of the first blowdown is, up to sign, $C_2t + D_2a$; that of the second is, up to sign, $B_3t + D_3b$. The ramification divisor of a blowup of a smooth curve in a smooth threefold is exactly its exceptional divisor, with multiplicity one. Since both exceptional divisors are smooth, their local equations have nonzero linear terms. This proves the two asserted nonvanishing conditions. □

Lemma 33.2 (Integrality of the ordinary initial complete intersection)**.** *In Lemma 33.1, give x_0, y_0 weight zero, x_1, x_3, y_1, y_3 weight one, and x_2, y_2 weight two. The initial forms are*

$$G_1 = x_1y_0 + x_0y_1, \qquad G_2 = x_2y_0 + q_2, \qquad G_3 = x_0y_2 + q_3.$$

They form a regular sequence, and the ring

$$\mathbb{C}[x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3]/(G_1, G_2, G_3)$$

is an integral domain. These three forms generate the weight initial ideal of (F_1, F_2, F_3). On each bihomogeneous piece, the resulting weight filtration agrees, after division by the local trivialization $x_0^m y_0^n$, with vanishing order at P.

Proof. In $\mathbb{P}^3 \times \mathbb{P}^3$, the locus defined by the G_i on $x_0y_0 \neq 0$ is the graph

$$y_1 = -x_1, \qquad x_2 = -q_2(x_1, x_3; -x_1, y_3), \qquad y_2 = -q_3(x_1, x_3; -x_1, y_3).$$

It is therefore an affine three-space with coordinates x_1, x_3, y_3. We show that its boundary has dimension at most two.

If $x_0 = 0$ and $y_0 \neq 0$, then G_1 gives $x_1 = 0$. One must have $x_3 \neq 0$, since otherwise G_2 also forces $x_2 = 0$, which is impossible in projective space. The equation G_2 determines x_2, while G_3 imposes the nonzero linear condition

$$-B_3y_1 + D_3y_3 = 0.$$

This locus has dimension two. The case $y_0 = 0, x_0 \neq 0$ is identical, using the nonzero condition $C_2x_1 + D_2x_3 = 0$.

On $x_0 = y_0 = 0$, if both pairs (x_1, x_3) and (y_1, y_3) are nonzero, the coprime forms q_2, q_3 cut out a finite subscheme of $\mathbb{P}^1 \times \mathbb{P}^1$. The two additional coordinates x_2, y_2 give fibres of dimension two, so this stratum has dimension at most two. If either pair vanishes, that projective factor is fixed at $[0 : 0 : 1 : 0]$ and the other factor lies in a projective plane; again the dimension is at most two.

Thus the projective zero scheme has dimension three, and its only possible three-dimensional component is the closure of the displayed affine three-space. On the affine multicone, the locus where an entire x- or y-block vanishes has dimension four, below the expected dimension five; the other boundary strata also have dimension at most four. Hence the three affine equations have height three and form a regular sequence in the polynomial ring. Their quotient is Cohen–Macaulay, so has no embedded components. Its unique minimal component is generically smooth by the graph description. The quotient is therefore reduced and irreducible, hence a domain.

The weight initial ideal contains (G_1, G_2, G_3). Both the original equations and these initial equations are regular sequences of three forms of bidegree $(1, 1)$. Their quotient rings consequently have the same bihomogeneous Hilbert series. A weight degeneration preserves this Hilbert series, so the inclusion of ideals must be equality.

Finally, localizing the initial domain at x_0y_0 is injective. The localization is

$$\mathbb{C}[x_0^{\pm 1}, y_0^{\pm 1}, t, a, b].$$

On a fixed bihomogeneous piece, division by $x_0^m y_0^n$ identifies its image with a polynomial subspace in the degree-one parameters t, a, b. The map to the ordinary tangent cone at P sends these parameters to themselves, and is consequently injective on each bihomogeneous piece. Therefore no further cancellation can raise the vanishing order of a bihomogeneous section beyond its weight initial degree. The two filtrations agree. □

Lemma 33.3 (The genuine common factor after restricting to the tangent space)**.** *For the quadrics $Q_i = A_i t^2 + B_i ta + C_i tb + D_i ab$ above, a nonconstant common factor in $\mathbb{C}[t, a, b]$ occurs if and only if $D_2 = D_3 = 0$. In that case their greatest common divisor is exactly t. If additionally (B_2, C_2) and (B_3, C_3) are proportional, then there is a linear form $\ell = b_0 a + c_0 b$ with $b_0 c_0 \neq 0$ such that*

$$\operatorname{span}\{Q_2, Q_3\} = \operatorname{span}\{t^2, t\ell\}.$$

In particular the coefficient of t^2 in the independent pencil combination is nonzero; it cannot disappear through further quadratic cancellation.

Proof. A common quadratic factor would make Q_2, Q_3, and hence N_2, N_3, proportional, which is excluded by Lemma 33.1. Suppose a common linear factor is $\alpha t + \beta a + \gamma b$. If $\beta\gamma \neq 0$, absence of a^2, b^2 in the Q_i forces both remaining factors to be multiples of t, again giving proportional quadrics. If $\beta \neq 0$ and $\gamma = 0$, the quotients have the form $u_i t + v_i b$. Their matrices then satisfy

$$N_i = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \begin{pmatrix} -u_i & v_i \end{pmatrix},$$

so every pencil member has rank at most one, a contradiction. The case $\gamma \neq 0, \beta = 0$ is symmetric. Thus the only possible common factor is t, and it occurs precisely when $D_2 = D_3 = 0$. The greatest common divisor cannot have larger degree, since the quadrics are not proportional.

In this case write the pencil coefficients as $A(z), B(z), C(z)$. Its binary determinant is $B(z)C(z)$ and is not identically zero. If the two residual forms are proportional, there are nonzero b, c and a nonzero linear form $k(z)$ such that $B(z) = bk(z)$ and $C(z) = ck(z)$. If A were proportional to k, the entire matrix $N(z)$ would vanish at the root of k, contradicting smoothness as in Lemma 33.1. Therefore A, k are linearly independent, and the quadratic pencil has precisely the stated span. □

Remark 33.4. Coprimality in four bilinear variables and coprimality after the substitution $y_1 = -x_1$ are different assertions. The first always holds and proves the integral initial model. The second fails precisely at the tangent-exceptional configurations above. Failure of the latter assertion does not invalidate the four-variable complete-intersection argument.

33.1 Hilbert functions and expected weights

Lemma 33.5 (Diagonal Hilbert weights)**.** *Let*

$$\mathcal{R} = \frac{\mathbb{C}[x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3]}{(f_1, f_2, f_3)}, \qquad \deg x_i = (1, 0), \quad \deg y_i = (0, 1),$$

where the f_i form a regular sequence of bidegree $(1, 1)$. Suppose that $\mathcal{R}$ has a nonnegative integral weight grading, with coordinate weights α_i, β_i and equation weights ρ_1, ρ_2, ρ_3. Set

$$V = \sum_{i=0}^{3} (\alpha_i + \beta_i), \qquad R = \rho_1 + \rho_2 + \rho_3.$$

Let $N_m = \dim \mathcal{R}_{m,m}$, and let w_m be the sum of the weights of a homogeneous basis of $\mathcal{R}_{m,m}$. Then

$$N_m = \frac{(2m+1)(5m^2+5m+3)}{3}, \tag{33.1}$$

$$w_m = V\frac{m(35m^3+30m^2+25m+6)}{24} - R\frac{m^2(5m^2+1)}{6}. \tag{33.2}$$

In particular,

$$\lim_{m\to\infty} \frac{w_m}{mN_m} = \frac{7}{16}V - \frac{1}{4}R.$$

If this weighted ring is the actual associated graded anticanonical section ring for a valuation v, the last expression equals $S_X(v)$.

Proof. The multigraded Hilbert series is

$$H(u,v,z) = \frac{\prod_{k=1}^{3}(1-uvz^{\rho_k})}{\prod_{i=0}^{3}(1-uz^{\alpha_i})(1-vz^{\beta_i})}.$$

At $z=1$ this becomes

$$H_0(u,v) = \frac{(1-uv)^3}{(1-u)^4(1-v)^4}.$$

Hence

$$N_m = \sum_{k=0}^{3}(-1)^k\binom{3}{k}\binom{m-k+3}{3}^2 = \frac{(2m+1)(5m^2+5m+3)}{3}.$$

Here and below a binomial coefficient is zero when its nonnegative upper argument is smaller than its lower argument. Logarithmic differentiation in z gives

$$z\frac{\partial H}{\partial z}\bigg|_{z=1} = H_0(u,v)\left(\frac{u\sum_i \alpha_i}{1-u} + \frac{v\sum_i \beta_i}{1-v} - \frac{uvR}{1-uv}\right).$$

The two coordinate terms have the same diagonal coefficient, by symmetry in u, v. Denoting that coefficient by F_m, and the diagonal coefficient of $H_0uv/(1-uv)$ by G_m, we obtain

$$w_m = VF_m - RG_m.$$

Explicit coefficient extraction yields

$$F_m = \sum_{k=0}^{3}(-1)^k\binom{3}{k}\binom{m-k+3}{4}\binom{m-k+3}{3} = \frac{m(35m^3+30m^2+25m+6)}{24},$$

$$G_m = \sum_{k=0}^{2}(-1)^k\binom{2}{k}\binom{m-k+2}{3}^2 = \frac{m^2(5m^2+1)}{6}.$$

This proves (33.2). Since $N_m = (10/3)m^3 + O(m^2)$, division by mN_m gives the stated limit. The final assertion follows from the definition of expected vanishing, once the identification with the valuation associated graded ring has been established. □

Corollary 33.6 (The two point extractions)**.** *For the ordinary point extraction, use coordinate order (x_0,x_1,x_2,x_3) and (y_0,y_1,y_2,y_3). The coordinate weights are $(0,1,2,1)$ in each factor, and the equation weights are $(1,2,2)$. Thus $V=8$, $R=5$, and*

$$w_m = \frac{m(15m^3+20m^2+15m+4)}{2}, \qquad S_X(G) = \frac{9}{4}.$$

For the weight-$(2,1,1)$ extraction in the proportional tangent case, the coordinate weights are $(0,2,3,1)$ in each factor, and the equation weights are $(2,3,3)$. Thus $V=12$, $R=8$, and

$$w_m=\frac{m(65m^3+90m^2+67m+18)}{6},\qquad S_X(G)=\frac{13}{4}.$$

The respective log discrepancies are 3 and 4, so in both cases

$$A_X(G)-S_X(G)=\frac{3}{4}.$$

Proof. Lemma 33.2 identifies the ordinary grading with its valuation filtration. Lemma 36.1 below does the same for the weighted grading. Substitute their weights in Lemma 33.5. The log discrepancy of a weighted blow-up of a smooth threefold point is the sum of its three primitive weights. □

Lemma 33.7 (The actual ordinary refined series)**.** *Let $G\simeq\mathbb{P}^2$ be the exceptional divisor of the ordinary blow-up of p. With homogeneous coordinates $[t:a:b]$ on G, its refined series is the diagonal algebra generated in the two factor degrees by*

$$\langle 1,t,a,Q_2\rangle,\qquad \langle 1,t,b,Q_3\rangle.$$

More explicitly, $W^G_{m,j}$ is spanned by

$$t^r a^s b^h Q_2^i Q_3^k,\qquad \begin{array}{c} r,s,h,i,k\ge 0,\quad s+i\le m,\quad h+k\le m,\\ r+s+h+i+k\le 2m,\quad r+s+h+2i+2k=j.\end{array}\tag{33.3}$$

Its total dimension is N_m in (32.1). It has bounded support and contains an ample series.

Proof. Lemma 33.2 identifies the valuation graded ring, including all cancellations. Dehomogenization gives the two displayed sets of generators. Distribute the r copies of t between the unused places in the two factors. Such a distribution exists precisely when the three inequalities in (33.3) hold. Every generator product has polynomial degree at most $4m$. Conversely, for $0\le j\le m$ all degree-j monomials in t,a,b satisfy those inequalities, so $W^G_{m,j}=H^0(\mathbb{P}^2,\mathcal{O}(j))$. The ray $j=m/2$ therefore contains a complete ample series. All integral degrees $0\le j\le 4m$ occur, since the generator degrees in each factor are $0,1,1,2$ and the ring is a domain. These observations give the required positivity and support hypotheses. □

34 Monomial degenerations and exact moment certificates

We use the word *pivot monomial* for the leading monomial of a row-echelon basis of a polynomial subspace in a fixed monomial order. The set of pivot monomials is independent of the row-echelon basis. Its cardinality is the dimension of the subspace. Leading monomials multiply, so the pivot sets of an algebra form a semigroup.

Lemma 34.1 (Bounding all valuations by pivot moments)**.** *Let T be $\mathbb{P}^2_{[t:a:b]}$ or $\mathbb{P}(2,1,1)_{[t:a:b]}$, and let $W_{m,j}$ be a series of homogeneous, respectively weighted homogeneous, polynomials with bounded support containing an ample series. Put $N_m=\sum_j \dim W_{m,j}$. Suppose that the sums of the three pivot exponents, divided by mN_m, converge to μ_t,μ_a,μ_b. Then*

$$\delta(T;W)\ge\frac{1}{\max\{\mu_t,\mu_a,\mu_b\}}.$$

The corresponding finite-level bound holds with the finite-level means $\mu_{m,t},\mu_{m,a},\mu_{m,b}$ for every basis divisor of W_m.

Proof. Fix m and a divisorial valuation v on T. Choose an integral diagonal one-parameter subgroup with distinct monomial weights realizing the given lexicographic order on the finite degree range in question. On each $W_{m,j}$ choose a basis adapted simultaneously to the v-filtration and the induced weight filtration, using Lemma 2.3. The weight leaves have dimension at most one. The initial monomials of this basis are consequently exactly the pivot monomials.

The resulting m-basis divisor D_m has $v(D_m) = S_m(W; v)$. Under the one-parameter subgroup it degenerates on the fixed surface to

$$D_{m,0} = \mu_{m,t}(t = 0) + \mu_{m,a}(a = 0) + \mu_{m,b}(b = 0).$$

The coordinate boundary is log canonical when each coefficient is at most one. For the weighted plane this follows as well on its finite quotient charts; the quotient is unramified in codimension one. Thus $\operatorname{lct}(T; D_{m,0}) = 1/\max_i \mu_{m,i}$. Lower semicontinuity of log canonical thresholds [11], with all nonzero fibres obtained by automorphisms of T, gives

$$A_T(v) \geq \frac{S_m(W; v)}{\max_i \mu_{m,i}}.$$

An arbitrary basis has order at most $S_m(W; v)$ for each v, so the same bound holds for every basis divisor. Passing to the limit proves the assertion. The one-parameter subgroup may depend on m. We split exactly two filtrations on each fixed homogeneous piece; invariance of W under this subgroup is not needed. □

Lemma 34.2 (A full-volume certificate)**.** *Let $A = \bigoplus_{m\geq 0} A_m$ be a graded algebra with $\dim A_m = (10/3)m^3 + O(m^2)$. Suppose a pivot filtration with one-dimensional leaves assigns to A_m a set of $\dim A_m$ distinct vectors in a fixed full-rank lattice $\Lambda \subset \mathbb{R}^3$, and that every such vector has norm $O(m)$, uniformly in m. Assume products of nonzero homogeneous elements are nonzero and their pivot vectors add. Suppose finitely many certified vectors $(m_i, v_i) \in \mathbb{Z}_{>0} \times \Lambda$ generate a semigroup whose group is $\mathbb{Z} \times \Lambda$. Set*

$$P = \operatorname{conv}\{v_i/m_i\} \subset \mathbb{R}^3.$$

Normalize Lebesgue measure so that a fundamental parallelepiped of Λ has volume one. If $\operatorname{vol}_\Lambda(P) = 10/3$, then the asymptotic pivot means of the whole algebra are the coordinates of

$$\frac{1}{\operatorname{vol}_\Lambda(P)} \int_P v \, d\operatorname{vol}_\Lambda .$$

In particular, exponent coordinates may be negative; a proper exponent sublattice is allowed when its lattice-normalized volume is used.

Proof. Write Γ for the generated semigroup and $\overline{\Gamma}$ for the points of $\mathbb{Z} \times \Lambda$ in its rational cone. The saturation is a finite Γ-module. Hence there is $\eta \in \Gamma$ with $\eta + \overline{\Gamma} \subset \Gamma$: choose module representatives, write each as a difference of two elements of Γ, and add their negative parts. The lattice points lost from a degree-m slice by this fixed translation lie in bounded-width boundary layers and number $O(m^2)$. Ehrhart counting in Λ therefore gives

$$\#\Gamma_m = \operatorname{vol}_\Lambda(P)m^3 + O(m^2), \qquad \sum_{v\in\Gamma_m} v = m^4 \int_P v \, d\operatorname{vol}_\Lambda + O(m^3).$$

All these vectors occur in the full pivot set by multiplicativity. Equality of the leading Hilbert coefficients leaves only $O(m^2)$ additional pivots. Their norms are $O(m)$, so their contribution to the vector sum is $O(m^3)$, including when some coordinates are negative. Divide by $m \dim A_m$. □

Remark 34.3. The lemma requires the full group $\mathbb{Z} \times \Lambda$, not merely full real span. In the finite certificates in this Family 2.12 part, $\Lambda = \mathbb{Z}^3$ and the degree-one vectors include $(1, 0, 0, 0)$ and its translates by the three exponent unit vectors, which prove this condition. Later applications specify their value lattice separately. We prove enough leading terms to account for the whole asymptotic dimension; we do not assert that the finite list generates the entire initial algebra.

For clarity, all the moment calculations below use ordinary Euclidean volume in the exponent lattice. If c is an interior point and (p_i, p_j, p_k) a triangle in a boundary triangulation, the contribution of its cone to volume and first moment is

$$V_{ijk} = \frac{|\det(p_i - c, p_j - c, p_k - c)|}{6}, \qquad M_{ijk} = \frac{V_{ijk}}{4}(c + p_i + p_j + p_k). \tag{34.1}$$

Thus the rational vertex and facet data in the lemmas specify their integrations explicitly. For completeness, the boundary triangulation can be recovered using only rational arithmetic. For every triple of affinely independent listed points, form its plane and retain it exactly when all listed points lie in one of its closed half-spaces. Identify repeated planes. Every facet occurs, since a two-dimensional facet contains three affinely independent vertices. In each retained plane take the convex polygon of the points on it and triangulate that polygon. The mean of all the listed points is an interior point of their full-dimensional convex hull, so coning these triangles to that mean gives tetrahedra with disjoint interiors and union the hull. Formula (34.1) then computes both volume and first moment exactly. This is a finite exhaustive procedure, not a floating-point test of selected facets. If the exponent lattice is a proper sublattice Λ, divide both Euclidean volume and first moment by its covolume before applying Lemma 34.2; the centroid is unchanged.

35 Three cases on the ordinary exceptional plane

Lemma 35.1 (Both mixed coefficients: the first moment certificate)**.** *Let t, a, b be coordinates on the exceptional plane, and suppose the actual ordinary-point refinement is the diagonal subalgebra generated, in bidegrees $(1, 0)$ and $(0, 1)$, by*

$$1, t, a, Q_2 \qquad \text{and} \qquad 1, t, b, Q_3,$$

respectively. Here ordinary order is polynomial degree in t, a, b. Suppose, after triangular changes preserving the lexicographic order $a > b > t$, that

$$Q_2 = ab + \kappa t^2, \qquad Q_3 = Q_2 + t(\alpha a + \beta b + \gamma t), \qquad \alpha\beta \neq 0.$$

Suppose its diagonal Hilbert function is $N_m = (2m+1)(5m^2+5m+3)/3$. Then the asymptotic mean exponents of the lexicographic initial monomials, normalized by mN_m, are

$$(\mu_t, \mu_a, \mu_b) = \left(\frac{7}{10}, \frac{253}{320}, \frac{243}{320}\right).$$

Proof. The degree-one diagonal space contains

$$1, t, a, b, t^2, ta, tb, ab, tQ_2, bQ_2, aQ_3, Q_2Q_3, D := t(Q_3 - Q_2).$$

Their leading exponents, in coordinate order (t, a, b), are the thirteen vectors $p_0, \dots, p_{12}$ below; each occurs once. Put $R = \beta b^2 + \gamma bt - \alpha\kappa t^2$. The following degree-two sections give two more certified leading monomials:

$$\begin{aligned} G_5 &= (tb)D - \alpha t^2(tQ_2) = t^3 R, & \operatorname{in}(G_5) &= \beta t^3 b^2, \\ H_6 &= D(bQ_2 + \alpha tQ_2) - \alpha t^2(Q_2Q_3) = t^2 Q_2 R, & \operatorname{in}(H_6) &= \beta t^2 a b^3. \end{aligned}$$

Take the convex hull P of the following normalized vectors (the last two are divided by degree two):

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
t	0	0	0	0	0	0	0	1	1	1	1	2	2	1	3/2
a	0	0	1	1	1	2	2	0	0	1	1	0	1	1/2	0
b	0	1	0	1	2	1	2	0	1	0	1	0	0	3/2	1

An exact facet description is

$$t, a, b \geq 0, \qquad |a - b| \leq 1,$$
$$t + 2a \leq 4, \quad 2t + b \leq 4, \quad t + 2b \leq 4,$$
$$5t + 2a + 4b \leq 12, \qquad 4t + a + 3b \leq 9.$$

For a reproducible rational integration certificate, take $c = (1/2, 1/2, 1/2)$ and cone the following boundary triangles to c:

(i,j,k)	$\mathrm{vol}[c, p_i, p_j, p_k]$	(i,j,k)	$\mathrm{vol}[c, p_i, p_j, p_k]$
$(12, 2, 5)$	1/3	$(12, 6, 5)$	5/12
$(14, 12, 11)$	5/24	$(13, 6, 4)$	5/24
$(13, 12, 6)$	13/24	$(13, 14, 12)$	5/24
$(1, 2, 0)$	1/12	$(1, 6, 5)$	1/6
$(1, 6, 4)$	1/12	$(1, 2, 5)$	1/6
$(12, 2, 0)$	1/6	$(12, 11, 0)$	1/6
$(14, 1, 0)$	1/8	$(14, 11, 0)$	1/6
$(13, 1, 4)$	1/6	$(13, 14, 1)$	1/8

Each tetrahedron has moment equal to its volume times $(c + p_i + p_j + p_k)/4$. Summation gives

$$\mathrm{vol}(P) = \frac{10}{3}, \qquad \int_P (t, a, b)\, dV = \left(\frac{7}{3}, \frac{253}{96}, \frac{81}{32}\right).$$

The semigroup generated by these fifteen certified monomials has full lattice: it contains $(1, 0, 0, 0)$ and its translates by each coordinate unit vector. Consequently its degree-m cardinality is $\mathrm{vol}(P)m^3 + O(m^2)$ and its exponent sums have leading term $m^4 \int_P (t, a, b)\, dV$. These conclusions follow either from the conductor of an affine semigroup in its saturation or from lattice-point counting. The full initial space has dimension $N_m = (10/3)m^3 + O(m^2)$, so at most $O(m^2)$ initial monomials are missing from this generated semigroup. All exponents are bounded by $4m$ because the original generators have ordinary degree at most two in each factor. Their contribution to exponent sums is therefore $O(m^3)$. This proves the asserted asymptotic means without asserting a complete SAGBI basis. □

Coefficient criterion. For $Q_2 = At^2 + Bta + Ctb + Dab$ and $Q_3 = A't^2 + B'ta + C'tb + D'ab$, the explicit open conditions are

$$DD'(DB' - D'B)(DC' - D'C) \neq 0.$$

Divide the quadrics by D, D' and make the triangular changes $a \mapsto a + (C/D)t$, $b \mapsto b + (B/D)t$. The coefficients α, β are $B'/D' - B/D$ and $C'/D' - C/D$; their nonvanishing gives the certificate. For example $Q_2 = ab + t^2$ and $Q_3 = ab + ta + tb + 2t^2$ satisfy it. The corresponding bihomogeneous q_2, q_3 are coprime: if $\kappa \neq 0$, q_2 is irreducible and the nonzero mixed coefficients exclude proportionality; if $\kappa = 0$, neither factor of $q_2 = x_3 y_3$ divides q_3. Thus the separate initial-domain argument applies.

Threshold consequence. Applying Lemma 34.1 to the certified body gives

$$\delta(\mathbb{P}^2; W) \geq \min\{\mu_t^{-1}, \mu_a^{-1}, \mu_b^{-1}\} = \frac{320}{253}.$$

Together with the independently proved $S_X(G) = 9/4$, $A_X(G) = 3$, and all hypotheses of the point-extraction refinement theorem, this gives $\delta_p(X) \geq \min\{4/3, 320/253\} = 320/253 > 1$.

Corollary 35.2 (All strata with both ab coefficients nonzero)**.** *Suppose the actual ordinary-point refinement has the form in the first lemma, and the bihomogeneous q_2, q_3 are coprime. If the ab coefficients of both Q_2, Q_3 are nonzero, then*

$$\delta(\mathbb{P}^2; W) \geq \frac{320}{253}.$$

Proof. Normalize as before, obtaining $Q_2 = ab + \kappa t^2$, $Q_3 = Q_2 + t(\alpha a + \beta b + \gamma t)$. The case $\alpha\beta \neq 0$ is proved above. If exactly one of α, β is nonzero, exchange the projection labels and the coordinates a, b if necessary, and perform the triangular normalization again, so that $\alpha \neq 0$, $\beta = 0$.

Retain the thirteen degree-one monomials of the first lemma, and use its same explicit sections G_5, H_6. Now $R = \gamma bt - \alpha\kappa t^2$. If $\gamma \neq 0$, their leading exponents are $(4, 0, 1)$ and $(3, 1, 2)$. If $\gamma = 0$, coprimality forces $\kappa \neq 0$: otherwise the bihomogeneous q_2, q_3 share their x_3 factor. In this case their leading exponents are $(5, 0, 0)$ and $(4, 1, 1)$.

Finally, if $\alpha = \beta = 0$, coprimality forces $\gamma \neq 0$. Then $D = \gamma t^3$ replaces the pivot at^2 by t^3; no degree-two pivots are needed. Exact rational triangulation gives the following volumes and centroids; the degree-two exponents in the middle column are divided by two when taking convex hulls.

condition	additional exponents	$\mathrm{vol}(P)$	(μ_t, μ_a, μ_b)
$\alpha\beta \neq 0$	$(3, 0, 2), (2, 1, 3)$	$10/3$	$(7/10, 253/320, 243/320)$
$\alpha \neq 0,\ \beta = 0,\ \gamma \neq 0$	$(4, 0, 1), (3, 1, 2)$	$10/3$	$(231/320, 253/320, 59/80)$
$\alpha \neq 0,\ \beta = \gamma = 0,\ \kappa \neq 0$	$(5, 0, 0), (4, 1, 1)$	$10/3$	$(119/160, 253/320, 229/320)$
$\alpha = \beta = 0,\ \gamma \neq 0$	at^2 replaced by t^3	$10/3$	$(31/40, 59/80, 59/80)$

The vertices are the displayed thirteen degree-one vectors, with the two additional exponents divided by two in the first three rows. In the last row replace $(2, 1, 0)$ by $(3, 0, 0)$. Formula (34.1) computes each entry by coning the boundary facets; additional facet certificates are recorded just below. The same full-lattice and Hilbert-mass argument proves that these are the actual asymptotic lexicographic moments. Every displayed coordinate is at most $253/320$, and the monomial-degeneration lemma applies. □

For $\alpha \neq 0,\ \beta = 0,\ \gamma \neq 0$, the non-coordinate facets are

$$-a + b \leq 1, \quad a - b \leq 1, \quad t - a + 2b \leq 3, \quad t \leq 2,$$
$$t + 2a \leq 4, \quad 2t + 3b \leq 6, \quad 3t + a + 2b \leq 7, \quad 4t + 2a + 3b \leq 10.$$

For $\alpha \neq 0,\ \beta = \gamma = 0$, the non-coordinate facets are

$$-a + b \leq 1, \quad a - b \leq 1, \quad t + 2a \leq 4, \quad 2t - a + 3b \leq 5,$$
$$2t + a + b \leq 5, \quad 3t + 4b \leq 8, \quad 3t + 2a + 2b \leq 8.$$

For $\alpha = \beta = 0,\ \gamma \neq 0$, the non-coordinate facets are

$$-a + b \leq 1, \quad a - b \leq 1, \quad t - a + 2b \leq 3, \quad t + 2a - b \leq 3,$$
$$2t + 3b \leq 6, \quad 2t + 3a \leq 6.$$

Lemma 35.3 (One vanishing mixed coefficient)**.** *Assume that the actual ordinary-point refinement has homogeneous pieces spanned by*

$$t^r a^s b^h Q_2^i Q_3^k, \qquad s+i \le m, \quad h+k \le m, \quad r+s+h+i+k \le 2m,$$

where

$$Q_2 = ab + \kappa t^2, \qquad Q_3 = t(a + Cb + \gamma t), \qquad (C, \gamma, \kappa) \neq (0,0,0).$$

Assume its total degree-m dimension is $N_m = (2m+1)(5m^2+5m+3)/3$. With lexicographic order $a > b > t$, the asymptotic mean exponents are

condition	μ_t	μ_a	μ_b	$\max \mu_i$
$C \neq 0$	$13/16$	$253/320$	$207/320$	$13/16$
$C = 0,\ \gamma \neq 0$	$267/320$	$253/320$	$5/8$	$267/320$
$C = \gamma = 0,\ \kappa \neq 0$	$137/160$	$253/320$	$193/320$	$137/160.$

Consequently the monomial degeneration lemma gives $\delta(\mathbb{P}^2; W) \ge 160/137 > 1$ uniformly in these cases.

Proof. Thirteen degree-one diagonal sections are

$$1, t, a, b, t^2, ta, tb, ab, tQ_2, bQ_2, aQ_3, Q_2Q_3, D := tQ_3.$$

Their leading exponents are distinct. In coordinate order (t, a, b), the sorted list is

$$\begin{aligned} p_0, \dots, p_{12} = \big((0,0,0), (0,0,1), (0,1,0), (0,1,1), (0,1,2), (1,0,0), (1,0,1), \\ (1,1,0), (1,1,1), (1,2,0), (1,2,1), (2,0,0), (2,1,0)\big). \end{aligned}$$

Set $R = Cb^2 + \gamma bt - \kappa t^2$. Two degree-two diagonal sections are certified by the polynomial identities

$$\begin{aligned} G &= (tb)D - t^2(tQ_2) = t^3 R, \\ J &= (bQ_2)D - (tQ_2)^2 = t^2 Q_2 R. \end{aligned}$$

Their leading exponents, divided by degree two, are as follows:

condition	$p_{13} = \frac{1}{2}\exp\mathrm{in}(G)$	$p_{14} = \frac{1}{2}\exp\mathrm{in}(J)$
$C \neq 0$	$(3/2, 0, 1)$	$(1, 1/2, 3/2)$
$C = 0,\ \gamma \neq 0$	$(2, 0, 1/2)$	$(3/2, 1/2, 1)$
$C = \gamma = 0,\ \kappa \neq 0$	$(5/2, 0, 0)$	$(2, 1/2, 1/2).$

Let P be the convex hull of $p_0, \dots, p_{14}$. For completeness, all three polytopes have the common inequalities

$$t, a, b \ge 0, \qquad a - t \le 1, \qquad b - a \le 1, \qquad t + a \le 3.$$

Their additional facet inequalities are respectively

$C \neq 0$	$2t + b \le 4, \quad 2t + a + 3b \le 7, \quad 4t + a + 3b \le 9;$
$C = 0, \gamma \neq 0$	$t - a + 2b \le 3, \quad t \le 2, \quad 3t + a + 2b \le 7, \quad 3t + a + 4b \le 9;$
$C = \gamma = 0, \kappa \neq 0$	$2t - a + 3b \le 5, \quad 2t + a + b \le 5, \quad 4t + a + 5b \le 11.$

Exact integration gives

condition	$\mathrm{vol}(P)$	$\int_P t$	$\int_P a$	$\int_P b$
$C \neq 0$	$10/3$	$65/24$	$253/96$	$69/32$
$C = 0, \gamma \neq 0$	$10/3$	$89/32$	$253/96$	$25/12$
$C = \gamma = 0, \kappa \neq 0$	$10/3$	$137/48$	$253/96$	$193/96.$

The facets supply a rational integration certificate. Alternatively, cone each facet triangulation to $(1/2, 1/2, 1/2)$ and use $\mathrm{vol}[c, p_i, p_j, p_k] = |\det(p_i - c, p_j - c, p_k - c)|/6$; the first moment of this tetrahedron is its volume times $(c + p_i + p_j + p_k)/4$.

The generated affine semigroup has full lattice, since the degree-one list includes the origin and all three coordinate unit vectors. Its degree-m cardinality is therefore $(10/3)m^3 + O(m^2)$, and its exponent sums have leading term $m^4 \int_P (t, a, b)$. All these semigroup elements occur as leading monomials of products of the fifteen displayed sections. The full initial space has the same leading Hilbert coefficient, $N_m = (10/3)m^3 + O(m^2)$, so there are only $O(m^2)$ additional leading monomials. Every exponent is bounded by $4m$, and consequently these extra monomials contribute only $O(m^3)$ to exponent sums. Dividing the displayed moments by $10/3$ proves the asserted means. The stated threshold follows from the independently justified monomial degeneration lemma. □

Lemma 35.4 (Independent tangent quadrics)**.** *Assume that the actual ordinary-point refinement has homogeneous pieces spanned by*

$$t^r a^s b^h Q_2^i Q_3^k, \qquad s + i \le m, \quad h + k \le m, \quad r + s + h + i + k \le 2m,$$

where

$$Q_2 = t(\beta a + b), \qquad Q_3 = t(a + \gamma b), \qquad \beta\gamma \neq 1.$$

Assume its total degree-m dimension is $N_m = (2m + 1)(5m^2 + 5m + 3)/3$. With lexicographic order $a > b > t$, the asymptotic mean exponents are

$$(\mu_t, \mu_a, \mu_b) = \begin{cases} (1, 53/80, 47/80), & \beta \neq 0, \\ (1, 5/8, 5/8), & \beta = 0. \end{cases}$$

In particular the monomial degeneration lemma yields $\delta(\mathbb{P}^2; W) \ge 1$.

Proof. Put $U = \beta a + b$, $V = a + \gamma b$, $P = bU$, and $R = aV$. The binary quadrics P and R are coprime: neither a divides P nor b divides R, and the remaining factors U, V are independent precisely when $\beta\gamma \neq 1$.

Since tQ_2, tQ_3 are degree-one diagonal sections, their span contains $t^2 a, t^2 b$. If $\beta \neq 0$, thirteen degree-one pivot exponents are

$$\begin{aligned} p_0, \dots, p_{12} = \big((0,0,0), (0,0,1), (0,1,0), (0,1,1), (1,0,0), (1,0,1), (1,1,0), \\ (1,1,1), (1,2,0), (2,0,0), (2,0,1), (2,1,0), (2,2,0)\big). \end{aligned}$$

They come from $1, a, b, ab, t, ta, tb, t^2, t^2 a, t^2 b, bQ_2, aQ_3, Q_2 Q_3$.

We next certify an entire sequence of sections rather than assume that a finite list forms a SAGBI basis. For every $m \ge 1$, the degree-m diagonal space contains

$$t^m \langle a, b\rangle \operatorname{Sym}^{m-1}\langle P, R\rangle.$$

Indeed, for $0 \le q \le m - 1$, the section $t^m a R^{m-1-q} P^q$ has

$$(r, s, h, i, k) = (1, m - q, q, q, m - 1 - q),$$

and the analogous section with b has

$$(r, s, h, i, k) = (1, m - 1 - q, q + 1, q, m - 1 - q).$$

Both satisfy the defining inequalities.

These $2m$ binary forms are linearly independent. To see this, set $z = a/b$. The rational map $z \mapsto R/P$ has degree two because R, P are coprime binary quadrics, so $[\mathbb{C}(z) : \mathbb{C}(R/P)] = 2$.

A linear dependence between $aR^{m-1-q}P^q$ and $bR^{m-1-q}P^q$, after dividing by bP^{m-1}, would give $zA(R/P) + B(R/P) = 0$. This forces $A = B = 0$. Thus the displayed subspace is all binary forms of degree $2m-1$. In particular,

$$t^m b^{2m-1} \in W_m \qquad \text{for every } m \geq 1.$$

Its normalized exponent tends to $p_{13} = (1, 0, 2)$.

Let $P_\infty = \operatorname{conv}\{p_0, \dots, p_{13}\}$. An exact facet description is

$$0 \leq t \leq 2, \quad a, b \geq 0, \quad a - t \leq 1, \quad b - t \leq 1, \quad a + b \leq 2, \quad 2t + a + 2b \leq 6.$$

Direct rational integration gives

$$\operatorname{vol}(P_\infty) = 10/3, \qquad \int_{P_\infty} (t, a, b)\, dV = (10/3, 53/24, 47/24).$$

Here is a precise volume argument allowing the limit point. For fixed M, use the thirteen degree-one sections and the certified section $t^M b^{2M-1}$. Their finitely generated semigroup has full lattice, and its body is

$$P_M = \operatorname{conv}\{p_0, \dots, p_{12}, (1, 0, 2 - 1/M)\}.$$

The bodies P_M increase to P_∞ up to closure. The degree-m semigroup cardinality is $\operatorname{vol}(P_M)m^3 + O_M(m^2)$, and its exponent sums have leading term $m^4 \int_{P_M}(t, a, b)$. The full initial space has cardinality $(10/3)m^3 + O(m^2)$. Since every exponent is at most $4m$, the difference between its exponent sums after division by m^4 and those of the fixed semigroup is at most $4(10/3 - \operatorname{vol}(P_M)) + o_M(1)$ in each coordinate. First let m tend to infinity and then let M tend to infinity. This proves existence of the full asymptotic means and gives $(1, 53/80, 47/80)$.

If $\beta = 0$, the thirteen degree-one pivot exponents instead are

$$\begin{aligned}&((0,0,0), (0,0,1), (0,1,0), (0,1,1), (1,0,0), (1,0,1), (1,0,2),\\ &(1,1,0), (1,2,0), (2,0,0), (2,0,1), (2,1,0), (2,1,1)).\end{aligned}$$

Their convex hull already has volume $10/3$, with moments $(10/3, 25/12, 25/12)$. Its facets are

$$0 \leq t \leq 2, \quad a, b \geq 0, \quad a - t \leq 1, \quad b - t \leq 1, \quad a + b \leq 2, \quad t + a \leq 3, \quad t + b \leq 3.$$

The ordinary finite-semigroup volume argument proves the second row. This includes $\beta = \gamma = 0$ without any additional argument. □

36 The weighted extraction at a repeated residual direction

Lemma 36.1 (The normalized weighted model in the repeated-residual case)**.** *Suppose that the uniform normal form satisfies*

$$D_2 = D_3 = 0, \qquad (B_3, C_3) = \lambda(B_2, C_2), \qquad \lambda \neq 0.$$

Then $B_2C_2 \neq 0$, the coefficient of x_3y_3 in F_1 is nonzero, and product-coordinate rescalings put the equations in the following weighted normal form:

$$\begin{aligned}F_1 &= x_1y_0 + x_0y_1 + x_3y_3 + \textit{terms of weight greater than } 2,\\ F_2 &= x_2y_0 - x_3y_1 - x_1y_3 + \textit{terms of weight greater than } 3,\\ F_3 &= x_0y_2 - x_3y_1 - x_1y_3 + \textit{terms of weight greater than } 3,\end{aligned}$$

where

$$w(x_0) = w(y_0) = 0, \quad w(x_1) = w(y_1) = 2, \quad w(x_3) = w(y_3) = 1, \quad w(x_2) = w(y_2) = 3.$$

The three displayed initial forms define an integral complete intersection and generate the entire weight initial ideal. On the chart $x_0 = y_0 = 1$, put $t = x_1, a = x_3, b = y_3$. Its graph is

$$y_1 = -t - ab, \qquad x_2 = y_2 = Q, \qquad Q = t(b - a) - a^2 b.$$

Thus the initial spaces of sections of the two factor hyperplane bundles have bases

$$\{1, t, a, Q\}, \qquad \{1, -t - ab, b, Q\}.$$

On every fixed bihomogeneous piece this initial filtration is exactly the monomial valuation of weights $(2, 1, 1)$ in the regular parameters (t, a, b).

Proof. The one-sided nonvanishing conditions give $C_2 \neq 0$ and $B_3 \neq 0$; proportionality therefore gives $B_2 C_2 \lambda \neq 0$. Write $B(z) = B_2(z_1 + \lambda z_2)$ and $C(z) = C_2(z_1 + \lambda z_2)$. Their common root $r = [0 : -\lambda : 1]$ is distinct from the two coordinate endpoints. The matrix $N(r)$ has only its upper-left entry possibly nonzero. That entry is nonzero, since otherwise the full matrix $M(r)$ would have rank at most two, contradicting smoothness of the determinant quartic. In particular $A_3 - \lambda A_2 \neq 0$.

Let η denote the coefficient of $x_3 y_3$ in F_1. We prove $\eta \neq 0$ explicitly. At r, the zeroth row of $M(r)$ forces a right kernel vector to have second coordinate $y_2 = 0$; its first row and the nonzero upper-left entry of $N(r)$ then force $y_1 = 0$. Its right kernel vector is consequently of the form $(u, 0, 0, 1)$, after scaling. Similarly its left kernel vector is of the form $(v, 0, 0, 1)$. The zeroth row and column of M_1 are supported only in coordinate 1, while $(M_1)_{00} = 0$. Therefore

$$(v, 0, 0, 1) M_1 (u, 0, 0, 1)^{\mathsf{t}} = \eta.$$

The rank of $M(r)$ is three, so Jacobi's formula says $\partial \det M / \partial z_0(r)$ is a nonzero scalar times η. The restriction of $\det M$ to $z_0 = 0$ has a double root at r, so its other two first derivatives vanish at r. Smoothness of the quartic forces its z_0-derivative to be nonzero; hence $\eta \neq 0$.

For completeness, the required normalization uses only rescalings. Set $B = B_2, C = C_2$. Choose nonzero scalars l, m, k, n with

$$m/l = -B/C, \qquad k = \eta l m, \qquad n = Blk.$$

Replace the old coordinates $x_3, y_3, x_1, y_1, x_2, y_2$ by $l x_3, m y_3, k x_1, k y_1, n x_2, \lambda n y_2$, respectively, and divide F_1, F_2, F_3 by $k, n, \lambda n$. Their lowest weighted terms are exactly those displayed. All remaining terms in F_1 have weight at least three. The $x_1 y_1$ terms in F_2, F_3 have weight four, and their other omitted terms contain x_2 or y_2 and another nonconstant coordinate, hence also have weight at least four.

Let G_1, G_2, G_3 denote the three displayed initial forms. Their locus on $x_0 y_0 \neq 0$ is the asserted graph, hence an affine three-space. We check that its projective boundary has dimension at most two. If $x_0 = 0, y_0 \neq 0$, then $x_3 \neq 0$: otherwise G_1, G_2 force $x_1 = x_2 = 0$. The first two equations determine x_1, x_2, and G_3 becomes the nonzero condition

$$-y_0 y_1 + y_3^2 = 0$$

after multiplication by an invertible factor. This stratum has dimension two. The opposite mixed stratum is analogous. If $x_0 = y_0 = 0$, the equations reduce to

$$x_3 y_3 = 0, \qquad x_3 y_1 + x_1 y_3 = 0.$$

If $x_3 = y_3 = 0$, the locus is $\mathbb{P}^1 \times \mathbb{P}^1$. If $x_3 = 0, y_3 \neq 0$, then $x_1 = 0$ and the first factor is fixed; the remaining locus lies in a projective plane. The final case is symmetric. All these strata have dimension at most two.

The affine-multicone argument of Lemma 33.2 now applies verbatim: the three equations have height three, the quotient is a complete intersection, and no component is supported on the boundary or on the dimension-four irrelevant loci. The unique component is generically smooth, so the quotient is a domain. Equality of the bihomogeneous complete-intersection Hilbert series proves that these three forms generate the whole weight initial ideal.

The formal implicit-function equations in the regular parameters t, a, b give

$$y_1 = -t - ab + O_w(3), \qquad x_2 = Q + O_w(4), \qquad y_2 = Q + O_w(4).$$

Localizing the initial domain at $x_0 y_0$ gives the Laurent polynomial ring in x_0, y_0 over $\mathbb{C}[t, a, b]$, with weights $(2, 1, 1)$ on t, a, b. On any fixed bihomogeneous piece, dehomogenization is injective and its map to the monomial associated graded ring is the identity on t, a, b. This proves the asserted equality of filtrations. □

Lemma 36.2 (The extraction underlying the weighted model)**.** *Let $\sigma : Y \to X$ be the weighted blowup in the regular parameters (t, a, b) with weights $(2, 1, 1)$, and let F be its exceptional divisor. Then*

$$A_X(F) = 4, \qquad F \cong \mathbb{P}(2, 1, 1), \qquad (Y, F) \text{ is plt}, \qquad \mathrm{Diff}_F(0) = 0.$$

The threefold Y has a single quotient singularity of type $\frac{1}{2}(1, 1, 1)$ on F. A smooth toric resolution is obtained by inserting the additional ray $(1, 1, 1)$ into the cone generated by $(2, 1, 1), (0, 1, 0), (0, 0, 1)$.

Proof. This is the toric weighted blowup of a smooth three-dimensional regular local scheme. The primitive exceptional ray is $(2, 1, 1)$, so its log discrepancy is the sum of its weights, namely four. The two charts indexed by the weight-one coordinates are smooth. The remaining chart is the quotient of coordinates (s, u, v), with $t = s^2, a = su, b = sv$, by

$$(s, u, v) \longmapsto (-s, -u, -v).$$

Its exceptional divisor is the image of $s = 0$. The cover is unramified in codimension one both on the threefold and on that divisor. The smooth pair with boundary $s = 0$ is plt; the quotient pair is therefore plt and adjunction has zero different, since there is no divisorial ramification on the divisor. The stated ray subdivides the only determinant-two cone into determinant-one cones, giving the indicated smooth resolution. □

36.1 The weighted refinement at proportional tangencies

We treat the tangent case in which the two residual binary linear forms are proportional. The preceding normal-form and initial-domain lemmas provide the weight-$(2, 1, 1)$ extraction G and identify its actual associated graded ring. Its exceptional surface is

$$G \cong \mathbb{P}(2, 1, 1)_{t,a,b}, \qquad \deg(t, a, b) = (2, 1, 1),$$

with zero different. After rescaling the coordinates, put

$$Y = -t - ab, \qquad Q = t(b - a) - a^2 b. \tag{36.1}$$

The generators of bidegrees $(1, 0)$ and $(0, 1)$ restrict respectively to

$$1, t, a, Q \qquad \text{and} \qquad 1, Y, b, Q. \tag{36.2}$$

The bidegrees distinguish the two occurrences of Q, even though their polynomial representatives coincide.

Write $W_{m,j}$ for the weighted-degree-j summand of the actual refined diagonal series, and $W_m = \bigoplus_j W_{m,j}$. Thus

$$N_m := \dim W_m = \frac{(2m+1)(5m^2+5m+3)}{3}. \tag{36.3}$$

The generators in (36.2) show that W_m is spanned by

$$t^i a^s Y^k b^h Q^n, \qquad i+s \le m, \quad k+h \le m, \quad n \le 2m-i-s-k-h. \tag{36.4}$$

All indices are nonnegative integers. Indeed, the n factors Q can be distributed between the remaining $m-i-s$ places in the first factor and the $m-k-h$ places in the second. The weighted degree of this polynomial is $2(i+k)+s+h+3n$, so $W_{m,j}=0$ for $j>6m$. For $0 \le j \le m$, every weighted-degree-j monomial in t,a,b satisfies the displayed inequalities with $k=n=0$. Thus $W_{m,j} = H^0(\mathbb{P}(2,1,1),\mathcal{O}(j))$ in this range. The ray $j=m/2$, after passing to Cartier multiples, contains a complete ample series. The generator degrees $0,1,2,3$ in each factor ensure every integral $0 \le j \le 6m$ occurs; hence all support and positivity hypotheses hold.

Lemma 36.3 (A rational monomial body for the weighted refinement)**.** *Take lexicographic leading terms in the order $t>a>b$, separately in each weighted homogeneous summand. The normalized asymptotic mean exponents of all the resulting pivot monomials are*

$$(\mu_t,\mu_a,\mu_b) = \left(\frac{37}{40}, \frac{811}{960}, \frac{533}{960}\right).$$

Proof. The following thirteen sections of W_1 have distinct leading terms:

$$1, b, a, ab, t, tb, aY, bQ, aQ, abQ, tY, tQ, Q^2.$$

Here $abQ = -(tQ+YQ)$ belongs to W_1 by (36.1). Their exponent vectors, in coordinate order (t,a,b), are

$$\begin{aligned} &(0,0,0),(0,0,1),(0,1,0),(0,1,1),\\ &(1,0,0),(1,0,1),(1,1,0),(1,1,1),\\ &(1,2,0),(1,2,1),(2,0,0),(2,1,0),(2,2,0). \end{aligned} \tag{36.5}$$

Since $N_1=13$, these sections form a basis. Three more leading terms, in diagonal degrees two, two and three, respectively, are obtained from the identities

$$P_1 = b^2Q + abQ - a^2bY = b^3(t-a^2), \qquad \mathrm{in}(P_1) = tb^3, \tag{36.6}$$

$$P_2 = -b^2Q^2 - abQ^2 - aY^2Q - taYQ = -P_1Q, \qquad \mathrm{in}(P_2) = t^2ab^3. \tag{36.7}$$

Moreover,

$$\begin{aligned} P_3 &= b^2Q^4 + 2YQ^4 + 2Y^2bQ^3 - 2aY^2Q^3\\ &\quad - tY^3Q^2 - 2t^2Y^2Q^2 - t^3YQ^2\\ &= b^3Q^2(bt^2 - a^3t - a^4b), \qquad \mathrm{in}(P_3) = t^4a^2b^4. \end{aligned} \tag{36.8}$$

Each summand on the first sides of these identities satisfies (36.4) with $m=2,2,3$, respectively. This proves membership in the asserted diagonal degree; membership is not inferred merely from the factored expressions.

Let P be the convex hull of the thirteen vectors in (36.5), together with

$$\left(\frac{1}{2},0,\frac{3}{2}\right), \qquad \left(1,\frac{1}{2},\frac{3}{2}\right), \qquad \left(\frac{4}{3},\frac{2}{3},\frac{4}{3}\right).$$

These are the vectors of the additional leading monomials divided by their diagonal degrees. The vertices of P are the following eleven points:

	O	A	B	C	D	E	F	G	H	I	J
t	0	0	0	0	1	1	2	2	1/2	1	4/3
a	0	0	1	1	2	2	0	2	0	1/2	2/3
b	0	1	0	1	0	1	0	0	3/2	3/2	4/3

An exact facet description, also fixing the body independently of its construction, is

$$\begin{gathered} t,a,b \geq 0, \qquad b-t \leq 1, \quad a-t \leq 1, \quad a \leq 2, \\ a+3b-t \leq 4, \qquad t-a+b \leq 2, \qquad t+a+3b \leq 6, \\ 2t+b \leq 4, \qquad 2t+a+2b \leq 6. \end{gathered} \tag{36.9}$$

Here is a rational integration certificate. Triangulate the boundary facets not contained in a plane through O as in the table and cone each triangle to O. These tetrahedra have disjoint interiors and cover P; this follows directly from (36.9).

triangle	vol$[O,\text{triangle}]$	triangle	vol$[O,\text{triangle}]$
ACH	1/12	JGF	8/9
EGD	1/3	EJG	2/3
IEJ	1/6	EBD	1/6
EBC	1/6	IHF	1/4
IJF	1/9	IEC	1/3
ICH	1/6		

For a triangle with vertices U,V,Z, its tetrahedron has volume $|\det(U,V,Z)|/6$ and first moment $\text{vol}[O,U,V,Z](U+V+Z)/4$. Thus summation of the displayed rational entries gives

$$\text{vol}(P)=\frac{10}{3}, \qquad \int_P (t,a,b)\,dV=\left(\frac{37}{12},\frac{811}{288},\frac{533}{288}\right). \tag{36.10}$$

Consider the affine semigroup generated by the sixteen certified leading monomials, retaining diagonal degree as its first coordinate. Its group is the full lattice: degree one contains the constant monomial and the three coordinate monomials t,a,b. The degree-m part of this semigroup consequently has cardinality

$$\text{vol}(P)m^3+O(m^2),$$

and its three exponent sums have leading terms $m^4\int_P (t,a,b)\,dV$. These are the usual lattice-point asymptotics for an affine semigroup of full lattice; equivalently, one compares with its saturation using a conductor element. Every such monomial is a leading monomial of an actual refined section, by multiplicativity. On the other hand, (36.3) gives $N_m=(10/3)m^3+O(m^2)$. The full initial series therefore has only $O(m^2)$ further monomials. All their exponents are at most $6m$, so they change each exponent sum by only $O(m^3)$. It follows from (36.10) that the full refined series has the asserted asymptotic means. This comparison does not require the displayed generators to form a complete SAGBI basis. □

Proposition 36.4 (The weighted surface estimate)**.** *For the weighted refinement above,*

$$\delta(G;W) \geq \frac{40}{37}.$$

Consequently, at the proportional tangent point,

$$\delta_p(X) \geq \min\left\{\frac{4}{13/4}, \frac{40}{37}\right\} = \frac{40}{37} > 1.$$

Proof. Lemma 34.1 and the computed means give

$$\delta(G;W) \geq \frac{1}{\max\{37/40, 811/960, 533/960\}} = \frac{40}{37}.$$

The quotient-chart proof of Lemma 36.2 verifies plt extraction and zero different. The bounded support and ample-series conditions were checked directly from (36.4). Corollary 33.6 gives $A_X(G)/S_X(G) = 16/13$. Proposition 2.7 now gives the stated local bound. □

37 Strict adjunction and completion of the proof

Lemma 37.1 (Keeping the positive extraction term)**.** *Let $\sigma : Y \to X$ be the ordinary blow-up of p, with exceptional divisor G. Suppose that the pivot means of its refined series have maximum one. Then every prime divisor F over X with centre p satisfies*

$$\beta_X(F) \geq \big(3 - S_X(G)\big)\,\mathrm{ord}_F(G) = \frac{3}{4}\,\mathrm{ord}_F(G) > 0.$$

Proof. Choose an m-basis of $H^0(X, mL)$ compatible with ord_G and ord_F. Write its basis divisor as

$$\sigma^* D_m = S_m(L;G)G + D'_m.$$

Its restriction $D'_m|_G$ is a basis divisor of the actual refinement. Lemma 34.1 bounds the thresholds of *all* such basis divisors, so there are $\lambda_m \to 1$, $0 < \lambda_m \leq 1$, for which $(G, \lambda_m D'_m|_G)$ is log canonical. This step does not require a basis simultaneously adapted to three filtrations. Inversion of adjunction gives that $(Y, G + \lambda_m D'_m)$ is log canonical near G. Therefore

$$\begin{aligned} A_X(F) &= A_Y(F) + 2\,\mathrm{ord}_F(G) \\ &\geq \lambda_m S_m(L;F) + \big(3 - \lambda_m S_m(L;G)\big)\,\mathrm{ord}_F(G). \end{aligned}$$

Taking limits proves the claimed inequality. Finally, $\mathcal{O}_Y(-G) = \mathfrak{m}_p \mathcal{O}_Y$, so $\mathrm{ord}_F(G) = \mathrm{ord}_F(\mathfrak{m}_p) > 0$ when the centre is p. □

Proof of Theorem 1.1 for Family 2.12. Let F be a prime divisor over X. If $\beta_X(F) \leq 0$, Proposition 32.2 says its centre is a point $p \in E \cap E'$. Use the normal form of Lemma 33.1, and write

$$Q_i = A_i t^2 + B_i ta + C_i tb + D_i ab.$$

There are four exhaustive possibilities.

If $D_2 D_3 \neq 0$, divide by the ab coefficients and translate a, b by multiples of t. This gives the normal form of Corollary 35.2. Its refined threshold is at least $320/253$, and the ordinary blow-up has $A_X(G)/S_X(G) = 4/3$. Proposition 2.7 gives $\delta_p(X) \geq 320/253 > 1$, a contradiction.

If exactly one of D_2, D_3 vanishes, exchange the two factors if necessary so that $D_2 \neq 0, D_3 = 0$. Normalize $Q_2 = ab + \kappa t^2$. The one-sided Jacobian condition gives $B_3 \neq 0$, so rescale Q_3 to $t(a + Cb + \gamma t)$. The alternative $C = \gamma = \kappa = 0$ would give a common factor other than

t, excluded by Lemma 33.3. Lemma 35.3 and ordinary extraction give $\delta_p(X) \geq 160/137 > 1$, again a contradiction.

Suppose $D_2 = D_3 = 0$ and $B_2C_3 - B_3C_2 \neq 0$. Solve

$$B_2u + C_2v = A_2, \qquad B_3u + C_3v = A_3,$$

and replace a, b by $a + ut, b + vt$. This removes both t^2 terms, without changing the two factor spans. Since $C_2B_3 \neq 0$, rescaling the quadrics now gives

$$Q_2 = t(\beta a + b), \qquad Q_3 = t(a + \gamma b), \qquad \beta\gamma \neq 1.$$

Lemma 35.4 gives refined threshold at least one and maximum pivot mean exactly one. Lemma 37.1 then gives $\beta_X(F) > 0$, a contradiction. The minimum form of adjunction alone would yield only $\delta_p(X) \geq 1$ here; retaining the exceptional coefficient is essential.

Finally, suppose $D_2 = D_3 = 0$ and $B_2C_3 - B_3C_2 = 0$. The smooth Jacobian conditions force the two residual linear forms to be nonzero and proportional. Lemma 36.1 identifies the exact weighted initial ring, and Lemma 36.2 supplies the plt extraction of weights $(2, 1, 1)$ with zero different. Its expected order is $13/4$ and discrepancy four. The refined threshold is at least $40/37$ by Lemma 36.3. Thus

$$\delta_p(X) \geq \min\left\{\frac{16}{13}, \frac{40}{37}\right\} = \frac{40}{37} > 1,$$

which gives the final contradiction. Hence every prime divisor over X has positive beta invariant, and X is K-stable. □

Example 37.2 (How the different certificates are used). These are illustrations of the local polynomial computations; the global hypotheses are verified separately by the normal-form lemmas. For $Q_2 = ab + t^2$ and $Q_3 = ab + ta + tb + 2t^2$, the first ordinary certificate gives the means $(7/10, 253/320, 243/320)$. The extra degree-two sections are needed to account for the full volume $10/3$.

For $Q_2 = ab + t^2$ and $Q_3 = ta$, the last row of the one-vanishing-coefficient certificate applies, giving $(137/160, 253/320, 193/320)$. This illustrates why the special coefficients cannot simply be discarded by a generality assumption.

For $Q_2 = tb$ and $Q_3 = ta$, the pivot algebra is already monomial. Its means are $(1, 5/8, 5/8)$. The coordinate line $t = 0$ has discrepancy one and expected order one on the refined plane. Nevertheless, the threefold divisors centred at p have strictly positive beta by Lemma 37.1.

In the repeated-residual case, the weighted model uses $Y = -t - ab$ and $Q = t(b - a) - a^2b$. The section $P_1 = b^2Q + abQ - a^2bY$ has leading monomial tb^3: its lower lexicographic terms are determined by cancellation. The further sections P_2, P_3 in Lemma 36.3 complete the asymptotic volume certificate. Ignoring these cancellations would give an incomplete calculation of the refined series.

Remark 37.3 (Why no threefold Zariski decomposition is required). The proof never computes a Zariski decomposition on either point extraction. The complete-intersection and localization arguments identify the actual valuation filtrations, including every cancellation. Weighted Hilbert functions then compute $S_X(G) = 9/4$ or $S_X(T) = 13/4$ directly. On the exceptional surface, explicit section identities and full-volume moment certificates control all basis divisors by degeneration. The only place where the surface bound equals one is handled by the strict adjunction inequality. These algebraic steps supply the information for which one would otherwise seek positive and fixed parts on a threefold modification.

Part VII
Family 2.13: genus-two sextics on quadrics

38 The scroll and the conic bundle

Let $\pi : X \to Q$ be the blow-up of a smooth quadric threefold $Q \subset \mathbb{P}^4$ along a smooth curve C of degree six and genus two. The ideal of C on Q is generated by quadrics. Put

$$H = \pi^*\mathcal{O}_Q(1), \qquad E = \mathrm{Exc}(\pi), \qquad B = 2H - E, \qquad L = -K_X = H + B.$$

The construction below applies to every member of Family 2.13. In particular, it includes the cone scroll used for the symmetric example in [3, Section 5.5].

Theorem 38.1. *Let X be a smooth member of Family 2.13, and let $p \in X$. According to the conic fibre through p, one has*

$$\delta_p(X) \geq \begin{cases} 80/77, & \text{the fibre is smooth,} \\ 64/61, & \text{the fibre is reducible and } p \text{ is not its node,} \\ 160/159, & p \text{ is the node of a reduced fibre,} \\ 320/319, & \text{the fibre is a double line.} \end{cases}$$

Consequently $\delta(X) \geq 320/319$, and X is K-stable.

We first establish the common geometry and the exact surface refinement. Reduced fibres are treated on smooth quartic del Pezzo sections. Double fibres require a weighted extraction of their reduced line. As in the preceding parts, δ_p includes divisors with positive-dimensional centres containing p.

Lemma 38.2. *The curve C is the scheme-theoretic intersection of Q with a cubic scroll $T \subset \mathbb{P}^4$. This scroll is either smooth or a cone over a twisted cubic. In suitable coordinates its ideal is generated by the maximal minors of, respectively,*

$$M = \begin{pmatrix} x_0 & x_1 & x_3 \\ x_1 & x_2 & x_4 \end{pmatrix}, \qquad M = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}. \tag{38.1}$$

In the second case the vertex of T is outside Q.

Proof. The curve is nondegenerate. Indeed, if it were contained in a hyperplane section T_0 of Q, every quadric section containing C would contain T_0: otherwise its intersection with T_0 would have degree four and could not contain C. This contradicts the stated ideal generation. By Riemann–Roch, $\mathcal{L} = \mathcal{O}_C(1)$ has five sections, so the given embedding is complete.

Let $f : C \to \mathbb{P}^1$ be the hyperelliptic double cover. The bundle $V = f_*\mathcal{L}$ has rank two and degree three. It is globally generated, since

$$H^1(C, \mathcal{L}(-f^{-1}(t))) = 0$$

for every $t \in \mathbb{P}^1$: the Serre-dual bundle has degree -2. Thus

$$V \simeq \mathcal{O}(2) \oplus \mathcal{O}(1) \quad \text{or} \quad V \simeq \mathcal{O}(3) \oplus \mathcal{O}.$$

Evaluation embeds C in $\mathbb{P}(V)$, with its map to $\mathbb{P}^4$ induced by the tautological divisor h. The image is the corresponding cubic scroll. If r is a ruling fibre, then $h^2 = 3$, $h \cdot r = 1$, and the

lifted curve has class $2h + dr$. Its degree six gives $d = 0$. In the cone case the negative section has zero intersection with this curve, and is disjoint from it.

The quadric Q cannot contain the cubic scroll: every divisor on the smooth quadric threefold has even degree, since $\mathrm{Pic}(Q) = \mathbb{Z}[\mathcal{O}_Q(1)]$. Its pullback to $\mathbb{P}(V)$ is therefore a nonzero divisor of class $2h$ containing C, and consequently equals C. This proves the scheme-theoretic assertion and, in the cone case, excludes the vertex from Q. The displayed matrices are the usual parametrizations of the two scrolls: the first comes from

$$[x_0 : x_1 : x_2 : x_3 : x_4] = [s^2a : sta : t^2a : sb : tb],$$

and the second is the twisted-cubic matrix with one unused coordinate. □

Lemma 38.3. *With homogeneous coordinates $[y_0 : y_1 : y_2]$ on $\mathbb{P}^2$, the threefold has the presentation*

$$X = \{q(x) = 0,\ M(x)y = 0\} \subset \mathbb{P}^4 \times \mathbb{P}^2, \tag{38.2}$$

a complete intersection of types $(2,0),(1,1),(1,1)$. The second projection is a flat conic bundle $\psi : X \to \mathbb{P}^2$, and $B = \psi^*\mathcal{O}_{\mathbb{P}^2}(1)$.

Proof. The rank-zero locus of M is empty for the first matrix in (38.1); for the second it is the vertex, which misses Q. Where M has rank two, the incidence variety is the graph of the map defined by its minors. At a point of C, invertible local row and column operations put the matrix in the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \end{pmatrix}.$$

The ideal of C is (a, b), and the incidence equations become $y_0 = 0$ and $ay_1 + by_2 = 0$. These are the equations of the blow-up of this regular codimension-two ideal. Hence (38.2) is the blow-up, scheme-theoretically.

For every y, the two linear forms comprising $M(x)y$ are independent, as is immediate from either matrix in (38.1). They cut out a plane Π_y. A smooth quadric threefold contains no plane, so $q|_{\Pi_y}$ is nonzero and defines a plane conic. These fibres all have the same Hilbert polynomial, giving flatness. The three minors define $|2H - E|$, which identifies the second polarization with B. □

Lemma 38.4. *For all integers $a, b \geq 0$,*

$$H^0(X, aH + bB) = (\mathbb{C}[x_0, \dots, x_4, y_0, y_1, y_2]/(q, (My)_1, (My)_2))_{(a,b)}\,.$$

The bigraded Hilbert series and the anticanonical Hilbert function are

$$\frac{(1-s^2)(1-st)^2}{(1-s)^5(1-t)^3}, \qquad N_m = h^0(X, mL) = \frac{(2m+1)(5m^2+5m+3)}{3}. \tag{38.3}$$

Proof. Twist the Koszul resolution of (38.2) by $\mathcal{O}(a, b)$. Every term has twists at least -4 on $\mathbb{P}^4$ and at least -2 on $\mathbb{P}^2$, so it has no higher cohomology. Taking sections gives the asserted quotient and its Hilbert series. Its diagonal gives the second formula. □

Lemma 38.5. *The intersection numbers and the two cones are*

$$H^3 = 2, \qquad H^2B = 4, \qquad HB^2 = 2, \qquad B^3 = 0, \qquad L^3 = 20,$$
$$\mathrm{Nef}(X) = \mathbb{R}_{\geq 0}H + \mathbb{R}_{\geq 0}B, \qquad \overline{\mathrm{Eff}}(X) = \mathbb{R}_{\geq 0}E + \mathbb{R}_{\geq 0}B.$$

If $D \neq E$ is a prime divisor and $D \sim aH + bB$, then a, b are nonnegative integers.

Proof. The class of the complete intersection in the product is $2H(H+B)^2$, giving the intersection numbers. The two contractions span the extremal rays, hence give the nef cone. For the last assertion intersect D with a general conic fibre and with a general exceptional fibre e of $E \to C$. These curves are not contained in D, and

$$D \cdot B^2 = 2a \geq 0, \qquad D \cdot e = b \geq 0.$$

The coefficients are integral because H, E form a basis of $\mathrm{Pic}(X)$ and $B = 2H - E$. Since $H = (E+B)/2$, every such divisor lies in the cone generated by E, B, proving the effective-cone assertion. □

39 Expected vanishing and quartic sections

Lemma 39.1. *The expected vanishing orders are*

$$S_X(H) = \frac{3}{8}, \qquad S_X(E) = \frac{73}{320}, \qquad S_X(B) = \frac{41}{80}.$$

Moreover, every prime divisor D on X has $\beta_X(D) > 0$.

Proof. For $a, b \geq 0$ set

$$V(a,b) = (aH + bB)^3 = 2a^3 + 12a^2b + 6ab^2.$$

The effective cone gives $\tau(H) = 1$ and $\tau(E) = 1/2$. The corresponding divisors remain nef up to their thresholds. Therefore

$$S_X(H) = \frac{1}{20}\int_0^1 (20 - 36u + 18u^2 - 2u^3)\,du = \frac{3}{8},$$
$$S_X(E) = \frac{1}{20}\int_0^{1/2} (20 - 48u + 6u^2 + 20u^3)\,du = \frac{73}{320}.$$

For B the threshold is $3/2$. Up to $u = 1$, the divisor $L - uB = H + (1-u)B$ is nef. For $1 \leq u \leq 3/2$ its moving and fixed parts are

$$P(u) = (3 - 2u)H, \qquad N(u) = (u-1)E. \tag{39.1}$$

Indeed, intersection with the moving exceptional fibres forces the stated multiplicity of E, and the residual divisor is a basepoint-free pullback. Consequently

$$S_X(B) = \frac{1}{20}\left(\int_0^1 (20 - 24u + 6u^2)\,du + \int_1^{3/2} 2(3-2u)^3\,du\right) = \frac{41}{80}.$$

For a prime $D \neq E$, Lemma 38.5 gives $D \sim aH + bB$ with nonnegative integral coefficients, not both zero. Thus $D - H$ or $D - B$ is effective. Monotonicity of volume gives $S_X(D) \leq 41/80 < 1 = A_X(D)$. The displayed value for E gives the remaining assertion. □

Lemma 39.2. *Let $y \in \mathbb{P}^2$ have reduced conic fibre F_y. A general line ℓ through y has smooth inverse image $S = \psi^{-1}(\ell)$. The restriction $\pi|_S$ is an isomorphism onto a smooth intersection of two quadrics in $\mathbb{P}^4$, and $C_S = E|_S$ is a smooth curve isomorphic to C. Writing*

$$A = H|_S = -K_S, \qquad F = B|_S,$$

we have

$$A^2 = 4, \quad A \cdot F = 2, \quad F^2 = 0, \qquad C_S = 2A - F, \quad C_S^2 = 8, \quad A \cdot C_S = 6.$$

Proof. Bertini gives smoothness away from F_y. At every smooth point of this fibre, $d\psi$ has rank two. If the fibre is reducible, it has one node, where $d\psi$ has rank one because X is smooth. Only one tangent direction of ℓ is forbidden there. Thus a general choice is smooth everywhere.

The map ψ sends every exceptional fibre $e_c \subset E$ isomorphically to a line, since $B \cdot e_c = 1$. These lines form a curve Γ in the dual plane, and Γ is contained in no line. Otherwise all $\psi(e_c)$ would pass through a fixed point z, so both linear forms $M(x)z$ would vanish on the nondegenerate curve C, and hence identically. This contradicts their independence. The pencil of lines through y is therefore not contained in Γ, and a general S contains no e_c.

Eliminating $y \in \ell$ in the two incidence equations shows that $\pi(S)$ is the intersection of Q with a quadric containing C. Locally write $C = (a, b)$ and the latter equation as

$$\alpha a + \beta b + O((a, b)^2) = 0.$$

Its strict transform contains the whole fibre e_c precisely when $\alpha(c) = \beta(c) = 0$. This was excluded; consequently $\pi(S)$ is smooth along C, and C is Cartier on it. Its blow-up along C is trivial, so $\pi|_S$ is an isomorphism. Away from C the map is already an isomorphism. The curve C_S is therefore isomorphic to C. Adjunction and Lemma 38.5 give the final formulas. □

Example 39.3. A reduced conic node can lie on E. For the first matrix in (38.1), take

$$Q = \{x_0x_3 + x_1^2 + x_2^2 + x_4^2 = 0\}.$$

The fibre over $[0 : 0 : 1]$ has equation $x_1^2 + x_2^2 = 0$ in the plane $x_3 = x_4 = 0$. Its node $c = [1 : 0 : 0 : 0 : 0]$ belongs to C. The quadric is smooth. In scroll coordinates its intersection with the scroll is

$$s^3ab + t^2(s^2 + t^2)a^2 + t^2b^2 = 0.$$

The coefficients never vanish simultaneously on $\mathbb{P}^1$, and its discriminant

$$s^6 - 4s^2t^4 - 4t^6$$

has six distinct roots. Hence C is smooth. Lemma 39.2 includes this configuration; no assumption that the node avoids E is needed.

Lemma 39.4. *Let S be as in Lemma 39.2, and refine the anticanonical series along S. If s_C is a defining section of C_S, then its actual restricted spaces are*

$$W^S_{m,j} = \begin{cases} H^0(S, mA + (m-j)F), & 0 \le j \le m, \\ s_C^{j-m} H^0(S, (3m-2j)A), & m \le j \le \lfloor 3m/2 \rfloor, \\ 0, & j > 3m/2. \end{cases} \tag{39.2}$$

For every divisorial valuation v over S,

$$S(W^S; v) = \frac{3}{20}\int_0^1 (4+4t) S_{A+tF}(v)\, dt + \frac{3}{40} S_A(v) + \frac{1}{80} v(C_S). \tag{39.3}$$

Here $S_D(v)$ denotes expected vanishing for the complete series of the ample $\mathbb{Q}$-divisor D on S, extended continuously to real coefficients.

Proof. For $j \le m$, the divisor $mL - jB$ is $mH + (m-j)B$. Its restriction map to S is surjective: by Kawamata–Viehweg vanishing,

$$H^1(X, mH + (m-j-1)B) = 0,$$

since adding $-K_X$ gives the nef and big divisor $(m+1)H+(m-j)B$. For $j \geq m$, equation (39.1) gives the exact fixed part $(j-m)E$, with residual series $|(3m-2j)H|$. This residual series also restricts surjectively: for $d \geq 0$,

$$H^1(X, dH - B) = 0, \qquad dH - B - K_X = (d+1)H$$

is nef and big. The effective threshold excludes larger j. This proves (39.2); in particular these are equalities of spaces, not merely containments.

For $u = j/m$, the normalized moving and fixed divisors on S are

u	$M(u)$	$N(u)\|_S$	$M(u)^2$
$[0,1]$	$A + (1-u)F$	0	$8 - 4u$
$[1,3/2]$	$(3-2u)A$	$(u-1)C_S$	$4(3-2u)^2$.

The denominator $N_m \sim (20/6)m^3$ and surface Riemann–Roch give density $(3/20)M(u)^2\,du$. The fixed contribution is exactly

$$\frac{3}{20}\int_1^{3/2} 4(3-2u)^2(u-1)\,du\; v(C_S) = \frac{1}{80}v(C_S).$$

The moving contribution on the second interval is

$$\frac{3}{20}\int_1^{3/2} 4(3-2u)^3\,du\; S_A(v) = \frac{3}{40}S_A(v).$$

Changing variables to $t = 1-u$ in the first interval proves (39.3). These limits also follow directly from the finite adapted bases in (39.2). The support is bounded, and the spaces contain an ample series, so the basis limits used in the common preliminaries apply. □

Remark 39.5. The coefficient $1/80$ in (39.3) is the contribution of the fixed components. Vanishing along C_S arising from the moving sections is already included in the two S-terms. Since C_S is smooth, $v(C_S) \leq A_S(v)$ for every divisorial valuation v. This remains true at a conic node lying on C_S.

40 Reduced conic fibres

We retain the notation $\pi\colon X \to Q$, $\psi\colon X \to \mathbb{P}^2$, $H = \pi^*\mathcal{O}_Q(1)$, $E = \operatorname{Exc}(\pi)$, $B = \psi^*\mathcal{O}_{\mathbb{P}^2}(1)$ and $L = -K_X = H + B$. Let $p \in X$, and suppose that the scheme fibre F of ψ through p is reduced. By Lemma 39.2, a general line through $\psi(p)$ gives a smooth surface $S \in |B|$ containing F such that

$$H_S := H|_S = -K_S, \qquad H_S^2 = 4, \qquad C_S := E|_S \cong C, \qquad C_S \sim 2H_S - F.$$

In particular, C_S is a smooth curve of genus two. The restriction $\psi|_S$ is a conic bundle over a line. We will use the ample divisors

$$D_t = H_S + tF, \qquad D_t^2 = 4(1+t), \qquad 0 \leq t \leq 1.$$

For a prime divisor R over a smooth surface and an ample real divisor D, set

$$S_D(R) = \frac{1}{D^2}\int_0^\infty \operatorname{vol}(\sigma^*D - vR)\,dv, \qquad \delta_p(S,D) = \inf_{p \in c_S(R)} \frac{A_S(R)}{S_D(R)}.$$

Here σ is any model on which R is a divisor, and the infimum includes curves on S containing p. Thus these are local thresholds in a neighbourhood of p, rather than thresholds restricted to valuations whose centre is exactly p.

Let W^S denote the refinement of the anticanonical section series by S. Lemma 39.4 gives $S_X(S) = 41/80$ and

$$S(W^S;R) = \frac{\mathrm{ord}_R(C_S)}{80} + \frac{3}{20}\int_0^1 D_t^2 S_{D_t}(R)\,dt + \frac{3}{20}\int_1^{3/2} 4(3-2u)^3 S_{H_S}(R)\,du. \tag{40.1}$$

The first summand is the fixed part. Its coefficient is

$$\frac{3}{20}\int_1^{3/2} 4(3-2u)^2(u-1)\,du = \frac{1}{80}.$$

We keep this term throughout. Since C_S is smooth,

$$\mathrm{ord}_R(C_S) \le A_S(R) \tag{40.2}$$

for every prime divisor R over S.

40.1 Smooth conics

Lemma 40.1 (A polarized quartic surface estimate)**.** *Suppose that F is smooth. For every $p \in F$ and $0 \le t \le 1$,*

$$\delta_p(S, D_t) \ge \frac{24}{19 + 8t + t^2}.$$

Moreover, $\delta_p(S, H_S) \ge 4/3$.

Proof. The first assertion is precisely [25, Appendix A, Lemma 24], applied to the smooth quartic del Pezzo surface S and its smooth conic F. The second is the usual quartic del Pezzo estimate [3, Lemma 2.12]. These are the only external surface estimates used in the smooth-conic case. In particular, the first assertion does not require p to avoid the lines on S. □

Lemma 40.2 (Points on smooth conic fibres)**.** *If the fibre F through p is smooth, then*

$$\delta_p(X) \ge \frac{80}{77} > 1.$$

Proof. Let R be any prime divisor over S whose centre contains p. Lemma 40.1, (40.1) and (40.2) give

$$\begin{aligned}\frac{S(W^S;R)}{A_S(R)} &\le \frac{1}{80} + \frac{3}{20}\int_0^1 4(1+t)\frac{19+8t+t^2}{24}\,dt + \frac{3}{20}\int_1^{3/2} 4(3-2u)^3\frac{3}{4}\,du \\ &= \frac{1}{80} + \frac{1}{40}\left(19 + \frac{27}{2} + 3 + \frac{1}{4}\right) + \frac{9}{20}\cdot\frac{1}{8} \\ &= \frac{2}{160} + \frac{143}{160} + \frac{9}{160} = \frac{77}{80}.\end{aligned}$$

Thus the local threshold of W^S is at least $80/77$. Abban–Zhuang adjunction along the smooth divisor S gives

$$\delta_p(X) \ge \min\left\{\frac{80}{41}, \frac{80}{77}\right\} = \frac{80}{77}.$$

□

40.2 A reducible conic away from its node

Suppose that $F = Z + Z^{\dagger}$, where Z and $Z^{\dagger}$ are (-1)-curves meeting transversally at a point o. Besides $Z^{\dagger}$, precisely four lines $T_1, \dots, T_4$ meet Z. The five curves $Z^{\dagger}, T_1, \dots, T_4$ are pairwise disjoint, each meets Z transversally once, and

$$2H_S \sim 3Z + Z^{\dagger} + \sum_{i=1}^{4} T_i. \tag{40.3}$$

Here is an intrinsic verification of these assertions. The system $|H_S - Z|$ is globally generated: linear forms vanishing on Z in the anticanonical embedding generate $\mathcal{I}_Z(H_S)$. It has three sections and square one, so it defines a birational morphism $S \to \mathbb{P}^2$. This morphism contracts five disjoint (-1)-curves. There are no infinitely near blowup centres, since these would produce a curve of anticanonical degree zero on S. One of the five curves is $Z^{\dagger}$; denote the others by $T_1, \dots, T_4$. Each meets Z once, since it has H_S-degree one and $(H_S - Z)$-degree zero. The canonical divisor formula for this blowdown is precisely (40.3). Conversely, any line meeting Z has $(H_S - Z)$-degree zero and hence is one of these five curves. For example, in a presentation $S \cong \mathrm{Bl}_{q_1,\dots,q_5} \mathbb{P}^2$ one may take

$$Z = e_1, \qquad Z^{\dagger} = 2h - e_1 - \cdots - e_5, \qquad T_i = h - e_1 - e_{i+1} \quad (1 \le i \le 4).$$

Here h is the pullback of a line and the e_i are the exceptional curves. The five points are in general position.

Lemma 40.3 (The component flag)**.** *Put $\tau_t = (3 + 2t)/2$. For $0 \le v \le \tau_t$, the Zariski decomposition of $D_t - vZ$ is $P_t(v) + N_t(v)$, where*

$$N_t(v) = \begin{cases} 0, & 0 \le v \le 1, \\ (v-1)Z^{\dagger}, & 1 \le v \le 1 + t, \\ (v-1)Z^{\dagger} + (v - 1 - t)\sum_{i=1}^{4} T_i, & 1 + t \le v \le \tau_t. \end{cases}$$

The divisor is not pseudoeffective for $v > \tau_t$. Moreover,

v	$P_t(v)^2$	$P_t(v) \cdot Z$
$[0, 1]$	$4 + 4t - 2v - v^2$	$1 + v$
$[1, 1 + t]$	$5 + 4t - 4v$	2
$[1 + t, \tau_t]$	$(3 + 2t - 2v)^2$	$6 + 4t - 4v$

and

$$S_{D_t}(Z) = \frac{17 + 30t + 12t^2}{24(1 + t)}. \tag{40.4}$$

For the refinement by Z, write

$$T_t(p) = \frac{1}{D_t^2} \int_0^{\tau_t} (P_t(v) \cdot Z)^2 \, dv + \frac{2}{D_t^2} \int_0^{\tau_t} (P_t(v) \cdot Z) \, \mathrm{ord}_p(N_t(v)|_Z) \, dv.$$

Then

$$T_t(p) = \begin{cases} \dfrac{19 + 30t + 12t^2}{24(1 + t)}, & p = o, \\ \dfrac{19 + 24t}{24(1 + t)}, & p \in Z \cap \bigcup_i T_i, \\ \dfrac{3 + 4t}{4(1 + t)}, & \text{otherwise.} \end{cases} \tag{40.5}$$

Proof. The divisor $M = H_S - Z$ is nef and $M^2 = 1$. Indeed, the linear forms vanishing on the line Z generate its ideal in the anticanonical embedding, so $\mathcal{O}_S(H_S - Z)$ is globally generated. In the first interval $P_t(v) = (1-v)H_S + vM + tF$ is nef. In the second interval it is $M + (1+t-v)F$, hence is nef. Equation (40.3) gives $P_t(v) = (3+2t-2v)M$ in the last interval. In each case it is orthogonal to the components of $N_t(v)$. These components are disjoint (-1)-curves, so their intersection matrix is negative definite. This proves the asserted Zariski decompositions. Finally,

$$(D_t - vZ)\cdot M = 3 + 2t - 2v,$$

which excludes pseudoeffectivity for $v > \tau_t$.

Intersection products give the table. In particular,

$$D_t^2 S_{D_t}(Z) = \int_0^1 (4 + 4t - 2v - v^2)\,dv + \int_1^{1+t} (5 + 4t - 4v)\,dv + \int_{1+t}^{\tau_t} (3 + 2t - 2v)^2\,dv = \frac{17 + 30t + 12t^2}{6}.$$

The common contribution to $D_t^2 T_t(p)$ is

$$\int_0^1 (1+v)^2\,dv + \int_1^{1+t} 4\,dv + \int_{1+t}^{\tau_t} (6 + 4t - 4v)^2\,dv = 3 + 4t.$$

At $p \in Z \cap T_i$ the extra contribution is

$$2\int_{1+t}^{\tau_t} (6 + 4t - 4v)(v - 1 - t)\,dv = \frac{1}{6}.$$

At o it is

$$2\int_1^{1+t} 2(v-1)\,dv + 2\int_{1+t}^{\tau_t} (6 + 4t - 4v)(v-1)\,dv = 2t^2 + t + \frac{1}{6}.$$

The five intersection points on Z are distinct. These calculations give (40.5). □

Lemma 40.4 (The mixed moments away from the node)**.** *Let $p \in Z \setminus \{o\}$. Set*

$$r_p = \begin{cases} 1, & p \in \bigcup_i T_i, \\ 0, & \text{otherwise}, \end{cases} \qquad \mu_p = (C_S \cdot Z)_p,$$

where $\mu_p = 0$ if $p \notin C_S$. The two moments for the successive refinement by S and Z are

$$\mathcal{S}_Z := S(W^S; Z) = \frac{61}{64}, \qquad \mathcal{T}_p := S(W^{S,Z}; p) = \frac{129}{160} + \frac{9}{320} r_p + \frac{1}{80}\mu_p \le \frac{55}{64}.$$

Consequently

$$\delta_p(X) \ge \frac{64}{61} > 1.$$

Proof. The curve C_S does not contain Z, since it is a smooth irreducible curve of genus two. Thus it contributes nothing to $\mathcal{S}_Z$. Equation (40.4) gives

$$\mathcal{S}_Z = \frac{3}{20}\int_0^1 \frac{17 + 30t + 12t^2}{6}\,dt + \frac{3}{20}\int_1^{3/2} 4(3-2u)^3 \frac{17}{24}\,du = \frac{9}{10} + \frac{17}{320} = \frac{61}{64}.$$

For $p \neq o$, equation (40.5) reads

$$T_t(p) = \frac{3+4t}{4(1+t)} + \frac{r_p}{24(1+t)}.$$

The fixed part restricts to the divisor $(u-1)C_S|_Z$ in the tail. Its contribution at p is $\mu_p/80$. Therefore

$$\begin{aligned}\mathcal{T}_p &= \frac{\mu_p}{80} + \frac{3}{20}\int_0^1 \left(3+4t+\frac{r_p}{6}\right) dt + \frac{3}{20}\int_1^{3/2} 4(3-2u)^3 \left(\frac{3}{4}+\frac{r_p}{24}\right) du \\ &= \frac{\mu_p}{80} + \frac{3}{4} + \frac{r_p}{40} + \frac{9}{160} + \frac{r_p}{320} = \frac{129}{160} + \frac{9}{320} r_p + \frac{\mu_p}{80}.\end{aligned}$$

Here the fixed term has the same coefficient $1/80$ as in (40.1): summing all orders along Z restores the leading dimension $P(u)|_S^2$ of the surface series.

Since $C_S \sim 2H_S - F$, one has $C_S \cdot Z = 2$. Hence $0 \leq \mu_p \leq 2$; also $r_p \leq 1$, because the T_i are disjoint. This proves

$$\mathcal{T}_p \leq \frac{258+9+8}{320} = \frac{55}{64}.$$

Abban–Zhuang adjunction for the flag $p \in Z \subset S \subset X$ now gives

$$\delta_p(X) \geq \min\left\{\frac{80}{41}, \frac{64}{61}, \frac{64}{55}\right\} = \frac{64}{61}.$$

□

Example 40.5 (Why the component flag does not suffice at the node). At $o = Z \cap Z^\dagger$, the same flag has point moment

$$\mathcal{T}_o = \frac{3}{20}\int_0^1 \frac{19+30t+12t^2}{6}\, dt + \frac{3}{20}\int_1^{3/2} 4(3-2u)^3 \frac{19}{24}\, du + \frac{\mu_o}{80} = \frac{323}{320} + \frac{\mu_o}{80} > 1.$$

This does not imply instability. It means that this particular flag does not give a threshold greater than one. We next replace the component flag on S by the blowup of its node.

40.3 Blowing up the node on the quartic surface

Lemma 40.6 (The complementary conic at the node). *Let $F = Z + Z^\dagger$ and $o = Z \cap Z^\dagger$. The conic pencil $|H_S - F|$ has a unique member G through o. This member is smooth and meets both Z and $Z^\dagger$ transversally at o. If $\sigma\colon T \to S$ is the blowup of o, with exceptional curve R, then the strict transforms $\widetilde{Z}$, $\widetilde{Z}^\dagger$ and $\widetilde{G}$ are pairwise disjoint, with self-intersections $-2, -2, -1$. They meet R at three distinct points. The divisor*

$$M = \sigma^*(H_S - F/2) - R$$

is nef, $M^2 = 1$, and $M \cdot R = 1$.

Proof. The complementary class $H_S - F$ is a basepoint-free conic pencil. This standard fact can also be checked in the plane presentation above: $H_S - F = h - e_1$. If a line D distinct from $Z, Z^\dagger$ passed through o, then $F \cdot D \geq 2$, whereas nefness of $H_S - F$ would give $F \cdot D \leq H_S \cdot D = 1$. Thus the only lines through o are the two components of F. Neither can be a component of a fibre of $|H_S - F|$, since $(H_S - F) \cdot Z = (H_S - F) \cdot Z^\dagger = 1$. Every singular fibre of a conic pencil on a smooth del Pezzo surface is a union of two lines. It follows that G is smooth. The equalities $G \cdot Z = G \cdot Z^\dagger = 1$ prove transversality and the assertions about their strict transforms.

We have $M = \sigma^*(G + F/2) - R$. If $D \subset S$ is an irreducible curve different from G, then

$$M \cdot \widetilde{D} = G \cdot D + \frac{1}{2} F \cdot D - \operatorname{mult}_o D \geq 0 :$$

the local intersection with the smooth curve G is at least $\operatorname{mult}_o D$, and F is nef. Direct calculation gives $M \cdot \widetilde{G} = 0$ and $M \cdot R = 1$. Hence M is nef, and $M^2 = (H_S - F/2)^2 - 1 = 1$. □

Lemma 40.7 (The node extraction on the surface)**.** *In the notation of Lemma 40.6, for every real $t \geq 0$ one has*

$$S_{D_t}(R) = \frac{3 + 2t}{2}, \qquad \delta_o(S, D_t) \geq \frac{4}{3 + 2t}.$$

Proof. Put $c = 3 + 2t$ and write $\sigma^* D_t - vR = P_t(v) + N_t(v)$. The Zariski decomposition is

$$N_t(v) = \begin{cases} 0, & 0 \leq v \leq 1, \\ \dfrac{v-1}{2}(\widetilde{Z} + \widetilde{Z}^\dagger), & 1 \leq v \leq 2 + 2t, \\ \dfrac{v-1}{2}(\widetilde{Z} + \widetilde{Z}^\dagger) + (v - 2 - 2t)\widetilde{G}, & 2 + 2t \leq v \leq c. \end{cases}$$

Indeed, $\sigma^* H_S - R$ is globally generated: it is the system of hyperplanes through o in the anticanonical embedding after blowing up their base point. Thus $\sigma^* D_t - vR$ is nef for $0 \leq v \leq 1$. In the second interval $P_t(v)$ interpolates between $\sigma^* H_S - R + t\sigma^* F$ and M, so it is nef. In the last interval $P_t(v) = (c - v)M$. The negative components are disjoint negative curves and are orthogonal to $P_t(v)$ in each interval. This proves the decompositions. Finally,

$$(\sigma^* D_t - vR) \cdot M = c - v,$$

so the pseudoeffective threshold is exactly c.

The volume and the degree on R are

v	$P_t(v)^2$	$P_t(v) \cdot R$
$[0, 1]$	$4 + 4t - v^2$	v
$[1, 2 + 2t]$	$5 + 4t - 2v$	1
$[2 + 2t, 3 + 2t]$	$(3 + 2t - v)^2$	$3 + 2t - v$.

Consequently

$$\begin{aligned} S_{D_t}(R) &= \frac{1}{4(1+t)} \left(\int_0^1 (4 + 4t - v^2)\, dv + \int_1^{2+2t} (5 + 4t - 2v)\, dv + \int_{2+2t}^{3+2t} (3 + 2t - v)^2\, dv \right) \\ &= \frac{3 + 2t}{2}. \end{aligned}$$

The log discrepancy is $A_S(R) = 2$ and the different on R is zero. For $q \in R$, the next refined expected order is

$$U_t(q) = \frac{1}{4(1+t)} \left(\int_0^c (P_t(v) \cdot R)^2\, dv + 2 \int_0^c (P_t(v) \cdot R) \operatorname{ord}_q(N_t(v)|_R)\, dv \right).$$

The common first integral is

$$\int_0^1 v^2\, dv + \int_1^{2+2t} 1\, dv + \int_{2+2t}^{3+2t} (3 + 2t - v)^2\, dv = \frac{5}{3} + 2t.$$

At either $\widetilde{Z} \cap R$ or $\widetilde{Z}^\dagger \cap R$, the additional integral is

$$\int_1^{2+2t} (v - 1)\, dv + \int_{2+2t}^{3+2t} (3 + 2t - v)(v - 1)\, dv = 2t^2 + 3t + \frac{7}{6}.$$

At $\widetilde{G}\cap R$ it is

$$2\int_{2+2t}^{3+2t}(3+2t-v)(v-2-2t)\,dv=\frac{1}{3}.$$

Thus

$$U_t(q)=\begin{cases}\dfrac{12t^2+30t+17}{24(1+t)}, & q\in(\widetilde{Z}\cup\widetilde{Z}^{\dagger})\cap R,\\[2mm] \dfrac{1}{2}, & q=\widetilde{G}\cap R,\\[2mm] \dfrac{6t+5}{12(1+t)}, & \text{otherwise.}\end{cases}$$

The first value is the largest, and

$$\frac{3+2t}{4}-\frac{12t^2+30t+17}{24(1+t)}=\frac{1}{24(1+t)}>0.$$

Abban–Zhuang adjunction on the blown-up surface therefore gives

$$\delta_o(S,D_t)\geq\min\left\{\frac{2}{S_{D_t}(R)},\inf_{q\in R}\frac{1}{U_t(q)}\right\}=\frac{4}{3+2t}.$$

This is a bound for all divisors whose centres on S contain o. Indeed, the log canonicity statement near the whole exceptional curve also controls strict transforms of curve centres passing through o. □

Example 40.8. For $t=1$, the three intervals in the preceding lemma are $[0,1]$, $[1,4]$ and $[4,5]$. One obtains

$$S_{H_S+F}(R)=\frac{5}{2},\qquad\max_{q\in R}U_1(q)=\frac{59}{48}<\frac{5}{4}.$$

The surface threshold supplied by this extraction is $4/5$. Although this number is less than one, the threefold refinement uses the whole range $0\leq t\leq 1$, with its dimension weights. The following calculation shows why that weighted average suffices.

Lemma 40.9 (The node on the threefold)**.** *If o is the node of a reduced conic fibre of ψ, then*

$$\delta_o(X)\geq\frac{160}{159}>1.$$

Proof. Choose S as above. For any prime divisor R' over S whose centre contains o, Lemma 40.7 gives

$$S_{D_t}(R')\leq\frac{3+2t}{4}A_S(R'),\qquad S_{H_S}(R')\leq\frac{3}{4}A_S(R').$$

The second inequality is the same lemma at $t=0$. Hence (40.1) gives

$$\begin{aligned}\frac{S(W^S;R')}{A_S(R')}&\leq\frac{1}{80}+\frac{3}{20}\int_0^1(1+t)(3+2t)\,dt+\frac{9}{20}\int_1^{3/2}(3-2u)^3\,du\\&=\frac{1}{80}+\frac{3}{20}\left(3+\frac{5}{2}+\frac{2}{3}\right)+\frac{9}{160}\\&=\frac{2}{160}+\frac{148}{160}+\frac{9}{160}=\frac{159}{160}.\end{aligned}$$

Thus $\delta_o(S;W^S)\geq 160/159$. Adjunction along S now yields

$$\delta_o(X)\geq\min\left\{\frac{80}{41},\frac{160}{159}\right\}=\frac{160}{159}.$$

□

Corollary 40.10 (Centres meeting a reduced fibre)**.** *For every point p on a reduced conic fibre,*

$$\delta_p(X) \geq \frac{160}{159}.$$

If a prime divisor V over X has a centre containing such a point, then

$$\beta_X(V) = A_X(V) - S_X(V) \geq \frac{1}{160} A_X(V) > 0.$$

This applies to positive-dimensional centres as well as point centres. In particular, any centre not excluded by this corollary must be contained in the union of the supports of the nonreduced fibres.

Proof. Use Lemma 40.2, Lemma 40.4 or Lemma 40.9, according to the position of p. Their lower bounds are all at least $160/159$. By the definition of $\delta_p(X)$, every V whose centre contains p satisfies $S_X(V) \leq (159/160)A_X(V)$, giving the stated beta inequality. □

Remark 40.11. The node extraction in this section takes place on the quartic surface S. All its Zariski decompositions are decompositions on a surface. We have not computed a Zariski decomposition on a threefold obtained by blowing up o. The first refinement uses the two elementary chambers on X from Lemma 39.4; the second is computed on $\mathrm{Bl}_o S$. This is sufficient because Abban–Zhuang adjunction is applied successively to $S \subset X$ and to the exceptional curve over S. Nonreduced fibres require a separate argument and are not covered by Corollary 40.10.

41 Double conic fibres

We retain the incidence description

$$X \subset \mathbb{P}^4 \times \mathbb{P}^2, \qquad \deg X = ((2,0),(1,1),(1,1)), \qquad L = H + B, \qquad L^3 = 20.$$

Suppose that the fibre over $y \in \mathbb{P}^2$ is 2Λ, where Λ is a line. We extract the scheme-theoretic fibre before extracting any point. The different on the exceptional surface will be essential. The calculation uses an exact associated graded algebra and a controlled enlargement of its refined spaces. It does not require a Zariski decomposition on the extracted threefold.

Lemma 41.1 (Coordinates along a double fibre)**.** *There are coordinates $[s_0 : s_1 : x : z_0 : z_1]$ on $\mathbb{P}^4$ and $[y_0 : u : v]$ on the base, with $y = [1:0:0]$, such that*

$$Q = (x^2 + s_0 z_0 + s_1 z_1 = 0), \qquad \Lambda = (x = z_0 = z_1 = 0),$$

and the two incidence equations are

$$z_i y_0 - A_i(s,x,z)u - B_i(s,x,z)v = 0, \qquad i = 0,1. \tag{41.1}$$

Put

$$R_i = A_i(s,0,0)u + B_i(s,0,0)v, \qquad q = s_0 R_0 + s_1 R_1 = q_1(s)u + q_2(s)v.$$

Then q_1, q_2 are binary quadratic forms without a common zero. On $\mathbb{P}^1_s \times \mathbb{P}^1_{[u:v]}$, the curve $D = (q = 0)$ is smooth of bidegree $(2,1)$. Its first projection is an isomorphism and its second projection has degree two. Moreover,

$$Z = (R_0 = R_1 = 0)$$

is a length-two complete intersection of curves of bidegree $(1,1)$, and $Z \subset D$ scheme-theoretically.

Proof. Let Π be the plane of the conic fibre. The quadratic form defining Q has rank one on Π, with radical the vector subspace corresponding to Λ. Since the ambient quadratic form is nondegenerate, $\Pi = \Lambda^{\perp}$. A Witt basis gives the displayed equation of Q. The two independent linear equations of Π are the coefficients of y_0 in the incidence equations. Changing their basis gives (41.1).

On the chart $y_0 = 1$, eliminate z_0, z_1 near Λ. The first nonzero transverse terms of the conic equation are

$$x^2 + q_1(s)u + q_2(s)v.$$

If q_1, q_2 vanished at the same point of Λ, all first derivatives of this equation would vanish there. This contradicts smoothness of X. Thus D is the graph of the morphism

$$[s_0 : s_1] \longmapsto [-q_2(s) : q_1(s)],$$

which has degree two.

The forms R_0, R_1 have no common component. A common component of bidegree $(1,0)$ would give a common linear factor of q_1, q_2. A common component of bidegree $(0,1)$ would make these quadratic forms proportional; they would again have a common zero. If the two $(1,1)$ forms were proportional, their linear combination $s_0R_0 + s_1R_1$ would have a factor of bidegree $(1,0)$, with the same consequence. These are all possibilities. Their intersection therefore has length two. The identity $q = s_0R_0 + s_1R_1$ proves the scheme-theoretic inclusion in D. □

Lemma 41.2 (The plt curve extraction and its different)**.** *Let $\mathcal{J} = \mathfrak{m}_y\mathcal{O}_X$, and let*

$$\omega\colon X^{\sharp} = \mathrm{Bl}_{\mathcal{J}} X \longrightarrow X.$$

The threefold $X^{\sharp}$ is normal. Its reduced exceptional divisor F is a plt-type divisor with

$$F \cong \Lambda \times \mathbb{P}^1, \qquad A_X(F) = 3, \qquad \mathcal{J}\mathcal{O}_{X^{\sharp}} = \mathcal{O}_{X^{\sharp}}(-2F).$$

For $h = \mathcal{O}_F(1,0)$ and $f = \mathcal{O}_F(0,1)$,

$$\omega^*L|_F = h, \qquad 2F|_F = -f, \qquad (K_{X^{\sharp}} + F)|_F = K_F + \frac{1}{2}D.$$

Thus the adjoint boundary is $\Delta_F = D/2$.

Proof. The local and global constructions are the same as in Lemma 18.1; we recall them. Where q_1 is nonzero, eliminate u from the local equation. Then $\mathcal{J} = (x^2, v)$, so this is the transverse weighted blow-up with weights $(1,2)$. Its charts are $v = x^2r$ and $x^2 = vr$. The first is smooth, while the second is an A_1 surface singularity times a smooth curve. In particular, the blow-up is normal.

Flat base change for the conic bundle gives

$$X^{\sharp} = X \times_{\mathbb{P}^2} \mathrm{Bl}_y \mathbb{P}^2.$$

Its exceptional scheme is $(2\Lambda) \times \mathbb{P}^1$, so its reduced support is the stated product. The exceptional divisor of the blown-up base pulls back to $2F$, giving $2F|_F = -f$. Since $H\Lambda = 1$ and $B\Lambda = 0$, one has $\omega^*L|_F = h$.

On the smooth chart, $dx \wedge dv = x^2 dx \wedge dr$, so the discrepancy coefficient is two and $A_X(F) = 3$. The other chart has the following index cover for F

$$x = ab, \qquad v = a^2, \qquad r = b^2, \qquad (a,b) \longmapsto (-a,-b).$$

The exceptional divisor is the image of $a = 0$. This quotient description proves that the pair is plt and that its different has coefficient $1/2$ on the singular-locus curve. The latter curve is precisely $D = (q = 0)$. Finally, $-F$ is relatively ample, because $-2F$ is the tautological divisor of the blow-up. These statements verify the plt-extraction hypotheses for Abban–Zhuang refinement. □

Lemma 41.3 (The actual weighted initial algebra)**.** *Give the coordinates* (s_0, s_1, x, z_0, z_1) *weights* $(0, 0, 1, 2, 2)$ *and* (y_0, u, v) *weights* $(0, 2, 2)$*. The associated graded algebra for* ord_F *on the bigraded coordinate ring of* X *is*

$$\frac{\mathbb{C}[s_0, s_1, x, z_0, z_1, y_0, u, v]}{(x^2 + s_0 z_0 + s_1 z_1,\ z_0 y_0 - R_0,\ z_1 y_0 - R_1)}. \tag{41.2}$$

It is a domain and a complete intersection. In particular, the displayed generators account for all cancellations in the section filtration.

Proof. Let I_0 be the displayed ideal. On $y_0 \neq 0$, eliminate z_0, z_1. The resulting equation is

$$x^2 y_0 + q_1(s)u + q_2(s)v = 0.$$

It is irreducible: as a polynomial linear in u, v, its coefficients are relatively prime, because q_1, q_2 have no common factor. This open set is therefore integral of dimension five.

We check the boundary $y_0 = 0$. Write $(R_0, R_1)^t = M(s)(u, v)^t$, where M is a two-by-two matrix of linear forms in s_0, s_1. Its determinant is not identically zero. Otherwise the two $(1, 1)$ forms would have a common component, contrary to Lemma 41.1. Also, $M(s)$ cannot have rank zero at a nonzero s, since then $q_1(s) = q_2(s) = 0$.

Where $\det M(s) \neq 0$, the variables u, v vanish. The remaining quadric in s_0, s_1, x, z_0, z_1 gives dimension four. Where $s \neq 0$ and $\det M(s) = 0$, the s-variables have dimension one and the kernel in (u, v) has dimension one. The quadric leaves dimension two in (x, z_0, z_1), again giving total dimension four. Finally, when $s_0 = s_1 = 0$, the quadric forces $x = 0$, and the remaining four variables give dimension four. Thus the boundary has dimension at most four.

The ideal I_0 has height three and is generated by three elements in a polynomial ring. It is a complete intersection, hence Cohen–Macaulay and unmixed. The boundary calculation shows that every minimal component meets the integral open set $y_0 \neq 0$. There is only one such component, and it is generically reduced. The absence of embedded components then proves that the entire ring is a domain.

The three equations in I_0 are the lowest-weight terms of the original equations. Thus I_0 is contained in their actual weight initial ideal. Both complete intersections have the same bigraded Hilbert series, since their bidegrees are $(2, 0), (1, 1), (1, 1)$. Equality of every bigraded Hilbert function proves equality of the ideals.

On a chart $s_i y_0 \neq 0$, further restrict to where one of q_1, q_2 is nonzero and eliminate the corresponding base parameter. This weighted filtration is then the transverse filtration with weights one on x and two on the remaining base parameter, namely ord_F. Localization is injective because the associated ring is a domain. Therefore the same identification holds for global sections of every bidegree. This proves (41.2) for the actual divisorial filtration. □

Lemma 41.4 (The complete refined spaces)**.** *Put*

$$W^F_{m,j} = \operatorname{gr}^j_F H^0(X, mL), \qquad j = 2r + \epsilon, \quad \epsilon \in \{0, 1\}, \qquad k = \max\{0, r - m\}.$$

For $0 \leq r \leq 2m - \epsilon$*, removing the odd-weight factor identifies this space with*

$$H^0(F, \mathcal{I}_Z^k \otimes \mathcal{O}_F(m - \epsilon, r)). \tag{41.3}$$

The zero divisor of its refined section is

$$\frac{\epsilon}{2}D + \operatorname{div}(s), \qquad s \in H^0\big(\mathcal{I}_Z^k(m-\epsilon, r)\big).$$

The space is zero for $r > 2m - \epsilon$. In its nonzero range, its dimension is

$$\dim W^F_{m,2r+\epsilon} = (m - \epsilon + 1)(r + 1) - k(k + 1). \tag{41.4}$$

Proof. Use the exact initial algebra of Lemma 41.3, set $y_0 = 1$, and eliminate $z_i = R_i$. The quadratic relation reduces every section to x^ϵ times a polynomial in s_0, s_1, R_0, R_1, u, v. To see the ideal power explicitly, a monomial containing $x^{2a+\epsilon}$ and c factors among z_0, z_1 becomes, up to sign, $x^\epsilon q^a$ times c factors among R_0, R_1. If its degree in the original base variables u, v is d, then

$$r = a + c + d, \qquad d \le m.$$

Because $q = s_0R_0 + s_1R_1$ belongs to (R_0, R_1), the resulting polynomial belongs to $(R_0, R_1)^k$. Its bidegree on F is $(m - \epsilon, r)$.

For $r \le m$, all polynomials of this bidegree occur, by using only s_0, s_1, u, v and the required powers of y_0. For $r > m$, expand every power of q using $q = s_0R_0 + s_1R_1$. The span is exactly

$$(R_0, R_1)^k\, H^0\big(F, \mathcal{O}_F(m - \epsilon - k, m)\big), \qquad k = r - m.$$

The reverse inclusion follows by lifting the R_i to z_i and using $r - k = m$ base factors. The total number of R_i factors cannot exceed $m - \epsilon$, proving the vanishing assertion.

Set $M = m - \epsilon$. Since Z is a complete intersection of two $(1, 1)$ curves, its ideal power has the standard resolution

$$0 \longrightarrow \mathcal{O}_F(M - k - 1, r - k - 1)^{\oplus k} \longrightarrow \mathcal{O}_F(M - k, r - k)^{\oplus(k+1)} \longrightarrow \mathcal{I}_Z^k(M, r) \longrightarrow 0.$$

In the relevant range, $M \ge k$ and $r \ge k$. The first term has vanishing H^1, by the cohomology of $\mathbb{P}^1 \times \mathbb{P}^1$. Hence the generated space above equals all of (41.3). Taking dimensions in this resolution gives (41.4).

Finally, the square of the weight-one factor x has equation q, up to sign, on the exceptional surface. Reflexive restriction therefore assigns the zero divisor $D/2$ to its odd factor. This is the same convention as in Lemma 18.5; the half divisor must be retained. □

Lemma 41.5 (Expected order and normalization)**.** *For the extraction of Lemma 41.2,*

$$S_X(F) = \frac{5}{2}, \qquad \tau(F) = 4.$$

The refined series has bounded support and contains an ample series.

Proof. Write $\rho = r/m$. The leading dimension in (41.4), divided by m^2, is

$$d(\rho) = \begin{cases} \rho, & 0 \le \rho \le 1, \\ \rho - (\rho - 1)^2, & 1 \le \rho \le 2. \end{cases}$$

The parity shift changes each dimension by only $O(m)$. Since

$$N_m = h^0(X, mL) = \frac{10}{3}m^3 + O(m^2),$$

and the two parities give $2m\, d\rho$ summands, the normalization factor is $3/5$. In particular,

$$\frac{3}{5}\left(\int_0^1 \rho\, d\rho + \int_1^2 \big(\rho - (\rho - 1)^2\big)\, d\rho\right) = 1.$$

The extraction weight divided by m is $2\rho + O(m^{-1})$. Therefore

$$\begin{aligned} S_X(F) &= \frac{3}{5}\left(\int_0^1 2\rho^2\,d\rho + \int_1^2 2\rho(\rho-(\rho-1)^2)\,d\rho\right) \\ &= \frac{3}{5}\left(\frac{2}{3}+\frac{7}{2}\right) = \frac{5}{2}. \end{aligned}$$

The spaces vanish beyond weight $4m$, and $\dim W^F_{m,4m} = (m+1)^2 > 0$, proving $\tau(F) = 4$. This also proves bounded support. Since L is ample and $-F$ is relatively ample, $\omega^*L - tF$ is ample for all sufficiently small positive rational t. Its sections supply an ample subseries of the refinement, as required by Abban–Zhuang theory. □

Lemma 41.6 (A degeneration which preserves the different)**.** *Choose coordinates on F in which D is the graph $u = s^2$ on an affine chart, and let o_0, o_∞ be its ramification points. Suppose first that Z is not the union of these two points. Orient the action*

$$(s,u) \longmapsto (\lambda s, \lambda^2 u) \tag{41.5}$$

so that its repelling fixed point does not belong to the support of Z. Then Z degenerates to

$$Z_0 = (u - s^2, s^2) = (u, s^2)$$

at the attracting point o_0. For each m, r, ϵ, the flat limit of the space in (41.3) is contained in the monomial space

$$U_{m,r,\epsilon} = \left\langle s^i u^l : 0 \le i \le m-\epsilon, \quad 0 \le l \le r, \quad i + 2l \ge 2k \right\rangle, \qquad k = \max\{0, r-m\}. \tag{41.6}$$

Homogeneous complementary coordinates are understood in this formula. The limiting space is invariant under (41.5), has the dimension in (41.4), and need not be a monomial space.

Proof. The morphism $D \to \mathbb{P}^1$ of degree two has two distinct ramification points in characteristic zero. Changes of coordinates on the two factors put it in the form $u = s^2$. The action preserves D. Unless Z is exactly the two ramification points, one of these points is absent from its support; choose this point as the repelling point. All of Z then lies in the affine chart of the attracting point.

As a length-two subscheme of the smooth curve D, Z has equation $(s-a_1)(s-a_2) = 0$, allowing $a_1 = a_2$. Its image under the action has ideal

$$(u - s^2, (s-\lambda a_1)(s-\lambda a_2)).$$

This is a flat family of length-two schemes, with the stated limit. It is a relative complete intersection. Its ideal-power quotients are also flat: filter each quotient by the successive powers of the ideal and use that their quotients are symmetric powers of the relative conormal bundle, locally free over this flat family of schemes. Consequently, a flat limit of sections vanishing in the kth power of the moving ideal still vanishes in $\mathcal{I}^k_{Z_0}$.

A monomial $s^i u^l$ belongs to $(u, s^2)^k$ precisely when $\lfloor i/2 \rfloor + l \ge k$, equivalently $i + 2l \ge 2k$. This gives (41.6). The flat limit in the Grassmannian retains its original dimension and is invariant under the one-parameter action. The complete space on the right can have larger dimension; no equality is asserted. □

Lemma 41.7 (The envelope integrals and the missing dimensions)**.** *In the situation of Lemma 41.6, normalized sums for the monomial envelopes have the following limits:*

quantity	*limit*
dimension divided by N_m	$41/40$
first coordinate fibre	$17/32$
opposite first coordinate fibre	$79/160$
second coordinate fibre	$241/320$
opposite second coordinate fibre	$87/160$
weight $(1,2)$ *at* o_0	$163/80$
weight $(1,2)$ *at* o_∞	$253/160$.

All valuation sums are divided by mN_m*. For an invariant basis of the actual limiting spaces, the weighted-order sum at* o_0 *instead has the sharper upper bound*

$$\frac{319}{160}. \tag{41.7}$$

For any curve of bidegree $(2,1)$*, the maximal normalized sum of its orders in these limiting spaces is at most* $81/320$*.*

Proof. Put $a = i/m$, $b = l/m$, and $\rho = r/m$. The limiting polygon for (41.6) is

$$\Omega_\rho = \{0 \le a \le 1,\ 0 \le b \le \rho,\ a + 2b \ge 2\max(0, \rho - 1)\}.$$

For a polynomial $P(a, b, \rho)$, define

$$J_P(\rho) = \int_0^1 \int_0^\rho P \, db \, da.$$

For $1 \le \rho \le 3/2$, the removed corner is a triangle, and its moment is

$$T_P(\rho) = \int_0^{2(\rho-1)} \int_0^{\rho-1-a/2} P \, db \, da.$$

For $3/2 \le \rho \le 2$, the triangle is truncated by $a = 1$, and its moment is

$$U_P(\rho) = \int_0^1 \int_0^{\rho-1-a/2} P \, db \, da.$$

Thus every constant in the table is

$$\frac{3}{5}\left(\int_0^1 J_P \, d\rho + \int_1^{3/2} (J_P - T_P) \, d\rho + \int_{3/2}^2 (J_P - U_P) \, d\rho\right). \tag{41.8}$$

The relevant polynomials are $1, a, 1-a, b, \rho - b, a + 2b$, and $(1-a) + 2(\rho - b)$. Here are the three separate contributions, including the factor $3/5$:

P	$[0,1]$	$[1,3/2]$	$[3/2,2]$
1	$3/10$	$7/20$	$3/8$
a	$3/20$	$29/160$	$1/5$
$1-a$	$3/20$	$27/160$	$7/40$
b	$1/10$	$15/64$	$67/160$
$\rho - b$	$1/10$	$33/160$	$19/80$
$a + 2b$	$7/20$	$13/20$	$83/80$
$(1-a) + 2(\rho - b)$	$7/20$	$93/160$	$13/20$.

Their sums give the asserted values. Lattice-boundary and parity errors vanish after the stated normalization.

For $\rho > 3/2$, the area of Ω_ρ exceeds the actual leading dimension $d(\rho)$ by

$$\operatorname{area}(\Omega_\rho) - d(\rho) = (\rho - 3/2)^2.$$

Every monomial in the envelope has weight at least $2k$ at o_0. An invariant subspace has no more vectors of any given weight than its ambient monomial space. Hence its total weight is bounded by the total ambient weight minus $2k$ times the dimension deficit. This argument allows repeated weights and non-monomial eigenvectors. After summation, the subtraction is at least

$$\frac{3}{5}\int_{3/2}^{2} 2(\rho - 1)(\rho - 3/2)^2\, d\rho = \frac{7}{160}.$$

It follows that the actual limiting basis has weighted-order sum at most

$$\frac{163}{80} - \frac{7}{160} = \frac{319}{160},$$

proving (41.7).

Finally enlarge the spaces once more to all of $H^0(\mathcal{O}_F(m - \epsilon, r))$. For a curve T of bidegree $(2, 1)$, the maximal sum of its orders is

$$\sum_{l \geq 1} h^0(\mathcal{O}_F(m - \epsilon - 2l, r - l)).$$

Its normalized limit is

$$\begin{aligned} c_T &= \frac{3}{5}\int_0^2 \int_0^{\min(1/2,\rho)} (1 - 2z)(\rho - z)\, dz\, d\rho \\ &= \frac{3}{5}\left(\int_0^{1/2} \left(\frac{\rho^2}{2} - \frac{\rho^3}{3}\right) d\rho + \int_{1/2}^{2} \left(\frac{\rho}{4} - \frac{1}{24}\right) d\rho\right) = \frac{81}{320}. \end{aligned}$$

Orders of zero divisors are nonnegative, so enlargement gives a valid upper bound with the original normalization mN_m. □

Example 41.8 (The strict inequality comes from missing sections). At $\rho = 2$, the original leading dimension is $d(2) = 1$, whereas the limiting monomial envelope has area $5/4$. This discrepancy is not a normalization error: sections may appear when the two points specialize. If all envelope sections were used, the weighted-order bound would be $163/80 > 2$, insufficient for the adjoint surface pair. Retaining the dimension of the actual series subtracts $7/160$ and gives $319/160 < 2$.

The number two has a geometric meaning. At a ramification point, the different is $\frac{1}{2}(u - s^2 = 0)$. The valuation with weights $(1, 2)$ has discrepancy

$$A_{F,D/2} = 1 + 2 - \frac{1}{2} \cdot 2 = 2.$$

Ignoring the different would replace this number by three and would obscure the required estimate.

Lemma 41.9 (The refined threshold on the exceptional surface)**.** *For the refinement by F,*

$$\delta(F, D/2; W^F) \geq \frac{320}{319} > 1.$$

Proof. First suppose that Z is not the union of the two ramification points, and use Lemma 41.6. Fix a divisorial valuation ν over F. In each space (41.3), choose a basis compatible both with ν and with the weight filtration of the action. The two-filtration lemma permits this. Include the odd fixed term $\epsilon D/2$ in its normalized basis divisor Γ_m. Its total coefficient is $O(m^{-1})$.

Acting on the surface degenerates this basis to an invariant basis of the limiting subspace. Denote the resulting divisor by Γ_m^0. Repeated weights may give non-monomial initial sections. This causes no difficulty: as recalled in Lemma 18.7, every invariant prime curve is a coordinate fibre or a curve

$$D_c = (u = cs^2), \qquad c \in \mathbb{C}^*.$$

These latter curves all have bidegree $(2,1)$. An effective boundary supported on these curves is log canonical if every prime coefficient is at most one and its weighted orders at o_0, o_∞ are at most three. To recall the local calculation, at either point the curves are the coordinate axes and distinct parabolas $u = cs^2$. Two successive point blow-ups resolve them. If their coefficients are $a, b, c_1, \dots, c_l$, the last exceptional inequality is

$$a + 2b + 2\sum_i c_i \le 3.$$

Together with $a \le 1$, it implies the first exceptional inequality $a + b + \sum_i c_i \le 2$. Away from these two points the curves have simple normal crossings.

The four coordinate coefficients of Γ_m^0 have limiting upper bounds

$$\frac{17}{32}, \quad \frac{79}{160}, \quad \frac{241}{320}, \quad \frac{87}{160}.$$

Indeed, a sum of orders of any basis of a subspace is bounded by the maximal sum for its containing monomial envelope. Every D_c coefficient is at most $81/320$ in the limit, by the last assertion of Lemma 41.7. The weighted sums at the two ramification points are bounded by $319/160$ and $253/160$, respectively. These are weights of invariant sections, so the dimension correction applies to the present basis, even when its eigenvectors are not monomials.

Take $0 < \lambda < 320/319$. All prime coefficients of

$$\frac{1}{2}D + \lambda\Gamma_m^0$$

are less than one for sufficiently large m. The largest coordinate coefficient is bounded at the endpoint by $241/319 < 1$, and the coefficient of D is bounded by

$$\frac{1}{2} + \frac{81}{319} < 1.$$

At the two ramification points, the limiting weighted orders are at most

$$1 + \lambda\frac{319}{160} < 3, \qquad 1 + \lambda\frac{253}{160} < 3.$$

The preceding invariant-boundary calculation proves log canonicity. The action preserves D, so lower semicontinuity of log canonical thresholds gives log canonicity of $(F, D/2 + \lambda\Gamma_m)$ before degeneration. Since the original basis is ν-compatible,

$$S_m(W^F;\nu) = \nu(\Gamma_m) \le \frac{A_{F,D/2}(\nu)}{\lambda}.$$

Pass to the limit, using bounded support and the ample subseries from Lemma 41.5, and then let λ increase to $320/319$.

It remains to treat $Z = \{o_0, o_\infty\}$. Its ideal powers are already invariant. Their spaces are spanned by the monomials in a rectangle with equal corner triangles removed at the two opposite points. In normalized coordinates, the conditions are

$$a + b \geq \rho - 1, \qquad (1 - a) + (\rho - b) \geq \rho - 1$$

when $\rho \geq 1$; there is no condition for $\rho \leq 1$. Their areas are again $d(\rho)$. Central symmetry gives first-factor coordinate means $1/2, 1/2$ and second-factor means $5/8, 5/8$. At either ramification point, the weighted-order sum is

$$\frac{3}{5} \int_0^2 (1/2 + \rho) d(\rho)\, d\rho = \frac{7}{4}.$$

The bound $81/320$ for a $(2, 1)$ curve is still valid. The same invariant-boundary argument therefore gives the stronger lower bound $8/7$, which implies the assertion in this case as well. □

Theorem 41.10 (Every point on a double conic fibre)**.** *If $p \in \Lambda$ and the conic fibre is 2Λ, then*

$$\delta_p(X) \geq \frac{320}{319} > 1.$$

Proof. Apply Abban–Zhuang refinement to the plt curve extraction in Lemma 41.2, with its different $D/2$. The local thresholds of the refined series over p are bounded below by its global threshold on F. Lemmas 41.5 and 41.9 give

$$\delta_p(X) \geq \min\left\{\frac{A_X(F)}{S_X(F)}, \delta(F, D/2; W^F)\right\} \geq \min\left\{\frac{6}{5}, \frac{320}{319}\right\} = \frac{320}{319}.$$

All positivity and bounded-support hypotheses were verified in Lemma 41.5. □

Remark 41.11 (No threefold Zariski decomposition). The weighted extraction is necessary to expose the correct exceptional surface and its different. We did not calculate a Zariski decomposition of $\omega^* L - tF$ on $X^\sharp$. Instead, the prime weighted initial algebra gives the actual refined spaces and the exact expected order $S_X(F) = 5/2$. A degeneration preserving the different then places these spaces in explicit monomial envelopes. The retained dimensions, rather than a claim of completeness after degeneration, supply the strict inequality needed on the surface. This is why Abban–Zhuang theory applies here without a threefold Zariski decomposition.

Example 41.12 (A smooth member with a double conic). Take the first matrix in (38.1) and

$$Q = \{x_1^2 + x_0 x_3 + x_2 x_4 + x_3^2 + 3x_4^2 = 0\}.$$

This quadric is smooth. On the scroll, its equation is

$$s^2 t^2 a^2 + (s^3 + t^3) ab + (s^2 + 3t^2) b^2 = 0.$$

The three coefficients have no common zero on $\mathbb{P}^1$. Its discriminant is

$$d(s, t) = s^6 - 4s^4 t^2 + 2s^3 t^3 - 12 s^2 t^4 + t^6.$$

It has six distinct zeros: for example, the discriminant of $d(1, t)$ is $-6464831488 \neq 0$, and $d(0, 1) \neq 0$. Thus the curve C is smooth, and the incidence construction gives a smooth Fano threefold in Family 2.13. Here ampleness of L also follows directly from $L = \mathcal{O}(1, 1)|_X$.

The fibre over $[0:0:1]$ is the double line $x_1^2 = 0$ in the plane $x_3 = x_4 = 0$. To put it in the coordinates of the weighted calculation, set

$$s_0 = x_0 + x_3, \qquad s_1 = x_2 + 3x_4, \qquad x = x_1, \qquad z_0 = x_3, \quad z_1 = x_4.$$

Then $Q = (x^2 + s_0 z_0 + s_1 z_1 = 0)$. With the old third base coordinate as the new y_0 and the negatives of the old first two coordinates as u, v, the initial bilinear forms are

$$R_0 = s_0 u, \qquad R_1 = s_1 v, \qquad D = (s_0^2 u + s_1^2 v = 0).$$

The scheme $Z = (R_0 = R_1 = 0)$ consists of the two ramification points of $D \to \mathbb{P}^1_{[u:v]}$. It therefore illustrates the symmetric case of Lemma 41.9, where the refined surface threshold is at least $8/7$. The general bound $320/319$ allows all other length-two configurations as well.

42 Completion of the proof for Family 2.13

Proof of Theorem 38.1 and the Family 2.13 assertion of Theorem 1.1. Every fibre of ψ is a nonzero plane conic. It is consequently smooth, a reduced union of two distinct lines, or a double line. The reduced-fibre estimates established above give $80/77$ on a smooth fibre, $64/61$ away from the node of a reducible fibre, and $160/159$ at its node. Theorem 41.10 gives $320/319$ at every point of a double fibre. These four cases exhaust X.

All these estimates concern the local threshold defined in the common preliminaries. In particular they control every prime divisor over X whose centre contains the chosen point. For any prime divisor V over X, choose $p \in c_X(V)$. Then

$$\frac{A_X(V)}{S_X(V)} \geq \delta_p(X) \geq \frac{320}{319} > 1.$$

Taking the infimum over V gives $\delta(X) \geq 320/319$, and the valuative criterion proves K-stability. □

The numerical inputs are kept separate from the completeness assertions. The original threefold contributes the elementary fixed-part formula (39.1); the reduced fibres use explicit Zariski decompositions on surfaces. For double fibres, Lemma 41.3 identifies the actual associated ring, and Lemma 41.4 identifies its restricted spaces. The subsequent enlargement is controlled by both their orders and their dimensions. *No Zariski decomposition on the weighted threefold extraction is used.* The missing dimension in the limiting spaces is retained precisely because discarding it would lose the strict inequality needed in the proof.

Part VIII
Family 2.14: the elliptic quintic on the quintic del Pezzo threefold

43 The quintic del Pezzo pencil

Let $V = V_5 \subset \mathbb{P}^6$ be the smooth quintic del Pezzo threefold, and let $C = V \cap \mathbb{P}^4$ be a smooth elliptic quintic. Put $X = \mathrm{Bl}_C(V)$, and denote the exceptional divisor by E. We write H for the pull-back of a hyperplane section of V. The pencil of hyperplanes through C defines $\varphi : X \to \mathbb{P}^1$; its fibre class is $F = H - E$. Thus

$$-K_X = H + F = 2H - E, \qquad (-K_X)^3 = 20,$$
$$H^3 = 5, \quad H^2E = 0, \quad HE^2 = -5, \quad E^3 = -10.$$

Moreover, $E \cong C \times \mathbb{P}^1$, since $N_{C/V} \cong \mathcal{O}_C(H)^{\oplus 2}$.

Lemma 43.1 (Global sections and the pencil graph)**.** *Let f, g generate the pencil and let $\mathcal{A} = \bigoplus_{d\geq 0} H^0(V, dH)$ be the homogeneous coordinate ring of V. Put*

$$\mathcal{R} = \mathcal{A}[a, b, e]/(f - ea, g - eb), \qquad \deg a = \deg b = H - E, \qquad \deg e = E.$$

Then $\mathcal{R}$ is a Cohen–Macaulay domain, it embeds in $\mathcal{A}[e, e^{-1}]$ by $a = f/e$, $b = g/e$, and

$$\mathcal{R}_{m(2H-E)} = H^0(X, -mK_X) \qquad (m \geq 0).$$

Suppose that a weight filtration has a Cohen–Macaulay domain $\mathcal{A}_0 = \operatorname{gr}\mathcal{A}$ and that the initial forms f_0, g_0 are a regular sequence in $\mathcal{A}_0$. Give a, b, e compatible weights, so that $f_0 - ea$ and $g_0 - eb$ are the initial graph relations. Then the full weight-associated graded algebra of $\mathcal{R}$ is

$$\mathcal{R}_0 = \mathcal{A}_0[a, b, e]/(f_0 - ea, g_0 - eb).$$

It too is a Cohen–Macaulay domain. In particular, substitution into an initial Pfaffian presentation is justified by this regular-sequence condition.

Proof. The codimension-three Pfaffian resolution of the homogeneous coordinate ring of V_5 is

$$0 \longrightarrow P(-5) \longrightarrow P(-3)^5 \longrightarrow P(-2)^5 \longrightarrow P \longrightarrow \mathcal{A} \longrightarrow 0, \qquad P = \mathbb{C}[x_0, \dots, x_6].$$

It gives projective normality and $H^1(V, \mathcal{O}_V(dH)) = H^2(V, \mathcal{O}_V(dH)) = 0$ for every integer d. In particular $\mathcal{A}$ is Cohen–Macaulay of dimension four. The affine cone over C has dimension two, so f, g form a regular sequence in $\mathcal{A}$.

After inverting e, the proposed graph ring is $\mathcal{A}[e, e^{-1}]$, of dimension five. Its boundary $e = 0$ is $\mathcal{A}/(f, g)[a, b]$, of dimension four. Thus the two graph relations have height two in the Cohen–Macaulay ring $\mathcal{A}[a, b, e]$. They form a regular sequence, and their quotient is Cohen–Macaulay of pure dimension five. No minimal component is contained in $e = 0$, and a Cohen–Macaulay ring has no embedded associated primes. Hence e is a nonzerodivisor. Localization is therefore injective and proves the domain assertion and the stated extended Rees presentation.

For $m \geq 1$, the powers of the ideal of the complete intersection $C = (f = g = 0)$ have the resolution

$$0 \longrightarrow \mathcal{O}_V(-(m+1)H)^m \longrightarrow \mathcal{O}_V(-mH)^{m+1} \longrightarrow \mathcal{I}_C^m \longrightarrow 0.$$

Twisting by $2mH$ and using $H^1(V,(m-1)H)=0$ shows that $H^0(V,\mathcal{I}_C^m(2mH))$ is generated by $f^{m-i}g^iH^0(V,mH)$, $0\le i\le m$. A graph monomial of degree $m(2H-E)$ whose total exponent in a,b is $m+r$ has e-exponent r and remaining H-degree $m-r$. Its numerator after division by e^m lies in $(f,g)^{m+r}\mathcal{A}_{m-r}\subset(f,g)^m\mathcal{A}_m$. Conversely, the terms with $r=0$ give every generator just listed. This identifies the entire graph component with that global section space. The blow-up identity $H^0(X,2mH-mE)=H^0(V,\mathcal{I}_C^m(2mH))$ proves the assertion; $m=0$ is immediate.

Apply the same dimension and localization argument to $\mathcal{A}_0,f_0,g_0$. It proves that $\mathcal{R}_0$ is a Cohen–Macaulay domain. Its bigraded Hilbert series is obtained from that of $\mathcal{A}_0[a,b,e]$ by multiplying by the two graph-relation factors of degree H. It is consequently equal to the bigraded Hilbert series of $\mathcal{R}$. The graph initial equations give a surjection $\mathcal{R}_0\to\operatorname{gr}\mathcal{R}$. Every fixed divisor-degree component is finite-dimensional: in degree $dH-kE$, the total exponent of a,b lies between $\max\{k,0\}$ and d, and the exponent of e is then determined. Weight degeneration preserves the dimension of every such component. The surjection is therefore an isomorphism, which proves that no further initial equations occur. □

Lemma 43.2. *Every fibre of φ is a normal quintic del Pezzo surface with Du Val singularities. It is smooth along its intersection with E. Its singularity configuration is one of $\varnothing, A_1, 2A_1, A_2, A_2+A_1, A_3, A_4$.*

Proof. A fibre is isomorphic to its image $S\in|H|$ on V, since blowing up the Cartier divisor $C\subset S$ does not change S. Since $\operatorname{Pic}(V)=\mathbb{Z}[H]$ and H is ample, S is reduced and irreducible. The differentials of two generators of the pencil are independent along C, so S is smooth there. A positive-dimensional singular locus would meet the ample curve $C\subset S$. Thus S has isolated singularities and is normal by Serre's criterion. Adjunction gives $-K_S=H|_S$ and $K_S^2=5$.

By the classification of normal Gorenstein del Pezzo surfaces [26, Theorem 2.2], either S has Du Val singularities or it is an elliptic cone. The vertex of an elliptic cone of degree five has embedding dimension five, whereas a Cartier divisor on a smooth threefold has embedding dimension at most three. The second possibility is therefore excluded. The exceptional curves on the minimal resolution form a root subsystem of A_4, giving the stated list. □

We use the following equations of V; see [3, Section 5.8]:

$$\begin{gathered} x_4x_5-x_0x_2+x_1^2=x_4x_6-x_1x_3+x_2^2=0,\\ x_4^2-x_0x_3+x_1x_2=x_1x_4-x_0x_6-x_2x_5=x_2x_4-x_3x_5-x_1x_6=0. \end{gathered} \tag{43.1}$$

On $x_0=1$ they give an affine space with coordinates $u=x_1, v=x_4, w=x_5$, in which

$$x_2=u^2+vw,\qquad x_3=u^3+uvw+v^2,\qquad x_6=uv-u^2w-vw^2. \tag{43.2}$$

The group $\operatorname{Aut}(V)\cong\mathrm{PGL}_2$ has three orbits, of dimensions one, two and three. Representatives in this chart are $(0,0,0)$, $(0,0,1)$ and $(0,1,0)$, respectively. The first orbit is a rational normal sextic, and the closure of the second is its tangent developable; see [27, Lemma 4.2]. For example, the sextic and its tangent lines have parametrizations

$$(u,v,w)=(3t^2/2,t^3,-3t),\qquad (u,v,w)=(3t^2/2+3st,t^3+3st^2,-3(t+s)).$$

Lemma 43.3. *Let $p\in V$. The quadratic parts of hyperplane sections singular at p form the complete net $H^0(\mathbb{P}(T_pV),\mathcal{I}_Z(2))$, where Z is a noncollinear curvilinear subscheme of length three. Its multiplicity partition is 3, $2+1$ or $1+1+1$, according as p belongs to the one-, two- or three-dimensional orbit.*

Proof. Translate the coordinates in (43.2) to a representative. Subtracting constants and linear terms from x_2, x_3, x_6 gives the following three nets, in local coordinates U, V, W:

$\dim \operatorname{Aut}(V)p$	q_1	q_2	q_3
1	$U^2 + VW$	V^2	UV
2	$U^2 + VW$	$UV + V^2$	$UV - U^2 - 2VW$
3	$U^2 + VW$	$UW + V^2$	$UV - W^2$.

Their common zero schemes have the stated lengths and partitions, as one checks on the three affine charts of $\mathbb{P}^2$. Each ideal is generated by these three quadrics, with two independent linear syzygies. This is the Hilbert–Burch resolution of a noncollinear length-three scheme. □

Lemma 43.4. *An A_4 point of a fibre belongs to the one-dimensional orbit of* $\operatorname{Aut}(V)$*, and an A_3 point belongs to its two-dimensional orbit. A fibre singularity on the open orbit has type A_1 or A_2.*

Proof. We use the chart (43.2) and translate to the three orbit representatives. Let N_1, N_2, N_3 be the parts of x_2, x_3, x_6 left after subtracting their constants and linear terms. A hyperplane section singular at the representative has equation $aN_1 + bN_2 + cN_3 = 0$.

At $(0, 0, 0)$ these polynomials are

$$U^2 + VW, \quad U^3 + UVW + V^2, \quad UV - U^2W - VW^2.$$

The determinant of their quadratic combination is $-a^3/4$. Thus $a \neq 0$ gives A_1. If $a = 0, c \neq 0$, normalize $c = 1$. The quadratic form $UV + bV^2$ has rank two. Eliminating its two nonsingular directions leaves $-W^5 + O(W^6)$: the critical solutions start with $U = W^2 + O(W^3)$ and $V = 2W^3 + O(W^4)$. Hence the singularity is A_4. If $a = c = 0$, the section is singular along $U = V = 0$, which is impossible for a fibre.

At $(0, 0, 1)$ the three polynomials are

$$U^2 + VW, \quad U^3 + UVW + UV + V^2, \quad UV - U^2 - U^2W - 2VW - VW^2.$$

The quadratic determinant is $-(a - c)(a - 2c)^2/4$. Outside these two lines it gives A_1. On $a = c \neq 0$, set $c = 1$; the radical is $(U, V, W) = (t, 0, (b + 1)t)$, and the cubic restricted to it is $-t^3$. Thus this case is A_2. On $a = 2c \neq 0$, again set $c = 1$. The equation is

$$U^2 + (b + 1)UV + bV^2 + bU^3 + bUVW - U^2W - VW^2.$$

For $b \neq 1$, eliminating U, V leaves $W^4/(b - 1)^2 + O(W^5)$, since their critical solutions begin with $U = (b + 1)W^2/(b - 1)^2$ and $V = -2W^2/(b - 1)^2$. This gives A_3. At $b = 1$ the quadratic part has rank one, incompatible with the list in Lemma 43.2. Finally $a = c = 0$ gives a section singular along $U = V = 0$.

At $(0, 1, 0)$ the polynomials are

$$U^2 + VW, \quad U^3 + UVW + UW + V^2, \quad UV - U^2W - VW^2 - W^2.$$

The quadratic determinant is $-(a + b - c)(a^2 - ab + ac + b^2 + bc + c^2)/4$. Its three linear factors are permuted cyclically by $(U, V, W) \mapsto (\omega^2 U, V, \omega W)$, where $\omega^3 = 1$ and $\omega \neq 1$. This is an actual automorphism fixing the representative: in (43.1) it is induced by the diagonal torus with weights $(x_0, x_5, x_1, x_4, x_2, x_6, x_3) = (0, 1, 2, 3, 4, 5, 6)$. It therefore suffices to consider $c = a + b$. Put $D = a^2 + ab + b^2$. A radical vector is $r = (-a - 2b, 2a + b, a - b)$, and the cubic part takes value $-3D^2$ on r. If $D \neq 0$, the singularity is A_2. If $D = 0$, the gradient of the cubic part at r is $D(2a + 7b, -a + b, -5a - 4b) = 0$; the quadratic gradient also vanishes. The section is then singular along the affine line tr, and cannot be a fibre. All remaining hyperplane sections have nondegenerate quadratic part and give A_1. □

Lemma 43.5 (The two kinds of lines)**.** *Every line $\ell \subset V_5$ has normal bundle $\mathcal{O}_\ell^{\oplus 2}$ or $\mathcal{O}_\ell(1) \oplus \mathcal{O}_\ell(-1)$. In the second case an automorphism takes it to $u = v = 0$ in (43.2). Its unique point on the closed orbit is $w = 0$. The stabilizer of this line induces all transformations $w \mapsto aw/(1 + bw)$, with $a \neq 0$.*

Proof. The normal bundle has degree $(-K_V)\ell - 2 = 0$, and is a subbundle of $N_{\ell/\mathbb{P}^6} = \mathcal{O}_\ell(1)^{\oplus 5}$. Its splitting is therefore one of the two displayed types. The Hilbert scheme of lines is $\mathbb{P}(\mathrm{Sym}^2 \mathbb{C}^2)$ with its natural PGL_2 action; see [27, Section 4.2]. It has two orbits, the conic of double binary forms and its complement. The line $u = v = 0$ is the tangent line to the closed orbit at $w = 0$. In the chart at infinity put $t = 1/w$, $a = tu$, and $b = tuv - u^2 - v/t$. The normal transition has linear part $(a, b) = (tu, -v/t)$, giving the splitting $\mathcal{O}(1) \oplus \mathcal{O}(-1)$. The general line is ordinary: the evaluation map from the universal family is generically finite and separable, so its differential is surjective at a general point, which excludes the second normal bundle. Thus the closed line orbit consists precisely of lines of the second kind. Finally, the stabilizer of the closed-orbit point acts on its tangent line as the full triangular subgroup; restricting the binary sextic representation to the span of x^6, x^5y gives exactly the stated fractional-linear transformations. □

43.1 Two refinements used below

The following form of the restriction argument keeps zeros which lie away from the point under consideration. It applies to all the actual refined series used in this part.

Lemma 43.6 (Retaining a second divisor)**.** *Let W be a bounded multigraded series on a normal projective surface G, and suppose that the numerical classes of its basis divisors tend to D_W. Let B, T be distinct prime rational Cartier curves, put $b = S(W; B)$ and $t = S(W; T)$, and let $p \in T \setminus B$. Assume that restriction to T is defined by plt adjunction near p. Then*

$$S(W^T; p) \leq (D_W - bB - tT) \cdot T.$$

Moreover, $D_W - bB - tT$ is a limit of effective numerical classes. The same conclusions, with a weaker bound, hold if b is replaced by any known lower bound for $S(W; B)$.

Proof. In a fixed native-degree piece, filter first by T. On each associated graded piece use the filtration induced by the spaces of sections vanishing along B. For every integer q, taking the T-associated graded preserves the dimension of the subspace with B-order at least q. Thus the total induced B-weight is the original B-weight. Choose on each T-graded piece a basis compatible with this induced filtration and the order at p. Lift each vector with its indicated T- and B-orders. These lifts form a basis compatible with both divisor filtrations, and their T-initial forms compute the point invariant. After removing the T factor, a lift with B-order q still has the factor $(B|_T)^q$. Its zeros lie away from p. Subtract their degree from the total degree of the restricted divisor, sum over the bases, and divide by mN_m. Passing to the limit gives the displayed bound. Before restriction the same bases give effective divisors after removal of both factors, proving the assertion about numerical classes. Only two filtrations are imposed on each already restricted graded piece; no simultaneous-basis assertion for three unrelated filtrations is used. □

Lemma 43.7 (Boundary curves and a strict extraction margin)**.** *In surface adjunction along a smooth boundary component T of coefficient $d < 1$, the first ratio is $(1 - d)/S(W; T)$. The different for the curve refinement is computed by replacing the coefficient of T by one. Suppose also that G is a plt divisor extracted over a threefold, that $A_X(G) > S_X(G)$, and that the refined surface threshold is at least one along the centres being considered. Then every prime*

divisor V over X whose centre on the extracted model is contained in G and lies in that locus satisfies

$$\beta_X(V) \geq (A_X(G) - S_X(G))\,\mathrm{ord}_V(G) > 0.$$

Proof. For the first assertion apply inversion of adjunction to $(G, \Delta + (1-d)T)$ and retain the original coefficient d when comparing discrepancies. The basis-divisor proof of Proposition 2.7 then gives the stated first ratio and the usual curve term. For the second assertion use the same finite-level proof with restricted coefficient tending to one. Keeping, rather than dropping, the coefficient of G gives the displayed inequality in the limit. Its last factor is positive because the centre on the extracted model is contained in G. □

44 Fibre refinements and reduction of centres

We retain the notation H, E, F of the geometric construction and put $L = -K_X = H + F$. Thus $H^3 = 5$, $F^2 = 0$, $L^3 = 20$ and $L^2F = 5$.

Lemma 44.1. *One has $S_X(F) = 11/16$, $S_X(E) = 3/8$, and $S_X(H) = 5/16$. If W^F is the refined series obtained by restricting along a fibre, then for every prime divisor T over F,*

$$S(W^F; T) = \frac{15}{16} S_F(T) + \frac{1}{16}\,\mathrm{ord}_T(C).$$

Proof. The nef cone is generated by H, F, and the effective cone by E, F. For the latter assertion, a prime divisor other than E is the proper transform of a divisor of class aH on V_5, of multiplicity b along C, where $a \geq 1$ and $b \geq 0$. Restricting it to a general fibre gives $a \geq b$, and its class is $aF + (a-b)E$. For $0 \leq u \leq 1$, the divisor $L - uF = H + (1-u)F$ is nef. For $1 \leq u \leq 2$, its divisorial decomposition is

$$P(u) = (2-u)H, \qquad N(u) = (u-1)E.$$

Indeed, intersection with a general fibre of $E \to C$ forces the displayed multiplicity along E, and the remainder is nef. The effective cone gives pseudoeffective threshold two. Therefore

$$S_X(F) = \frac{1}{20}\left(\int_0^1 (20 - 15u)\,du + \int_1^2 5(2-u)^3\,du\right) = \frac{11}{16}.$$

The divisors $L - uE = (1-u)H + (1+u)F$ and $L - uH = (1-u)H + F$ are nef for $0 \leq u \leq 1$; both have pseudoeffective threshold one. The intersection numbers give

$$S_X(E) = \frac{1}{4}\int_0^1 (4 - 6u + 2u^3)\,du = \frac{3}{8}, \qquad S_X(H) = \frac{1}{4}\int_0^1 (4 - 9u + 6u^2 - u^3)\,du = \frac{5}{16}.$$

Finally, $P(u)|_F = -K_F$ for $u \leq 1$ and $P(u)|_F = (2-u)(-K_F)$ for $u \geq 1$, while $N(u)|_F = (u-1)C$ on the latter interval. The moving and fixed coefficients in the restriction formula are respectively

$$\frac{15}{20}\left(\int_0^1 1\,du + \int_1^2 (2-u)^3\,du\right) = \frac{15}{16}, \qquad \frac{15}{20}\int_1^2 (2-u)^2(u-1)\,du = \frac{1}{16}.$$

This proves the exact refinement formula; it is also the degree-five instance of the computation in [8, Lemma 2.1]. These decompositions occur on X itself. No decomposition on a threefold obtained by extracting a point or a line is used in this lemma. □

Corollary 44.2. *Put $d_p = \delta_p(F)$. Then*

$$\delta_p(X) \geq \begin{cases} \min\{16/11,\, 16d_p/15\}, & p \notin E, \\ \min\{16/11,\, 16d_p/(d_p+15)\}, & p \in E. \end{cases}$$

In particular, if F is smooth, these bounds give $16/13$ and $8/7$, respectively.

Proof. Use Lemma 44.1 and Abban–Zhuang along F. At a point off E, every valuation entering the local estimate has zero order on C. At a point of C, the inequality $\operatorname{ord}_T(C) \le A_F(T)$ follows from log canonicity of (F, C). For a smooth quintic del Pezzo surface, $\delta(F) = 15/13$ by [3, Lemma 2.11]. □

Lemma 44.3 (Keeping the coefficient of a fibre)**.** *Let F be a fibre, and let $p \in F$. Suppose that the refined series W^F has local stability threshold at least one at p. If Q is a prime divisor over X whose centre contains p and is contained in F, then*

$$\beta_X(Q) \ge \frac{5}{16} \operatorname{ord}_Q(F) > 0.$$

Proof. Choose a basis of $H^0(X, -mK_X)$ simultaneously adapted to Q and F. Write its basis divisor as $D_m = a_m F + \Gamma_m$, where $a_m = S_m(F)$ and $F \not\subset \operatorname{Supp}(\Gamma_m)$. Its restriction $\Gamma_m|_F$ is a basis divisor of the refined series. Convergence of the finite basis thresholds gives numbers $c_m \to 1$, with $0 < c_m \le 1$, such that $(F, c_m\Gamma_m|_F)$ is log canonical near p. Inversion of adjunction gives

$$A_X(Q) \ge \operatorname{ord}_Q(F) + c_m \operatorname{ord}_Q(\Gamma_m) = c_m S_m(Q) + (1 - c_m a_m) \operatorname{ord}_Q(F).$$

Taking the limit and using $S_X(F) = 11/16$ proves the assertion. The last inequality is strict because the centre of Q is contained in F. Only prime divisors over X are being used; this is sufficient for the valuative criterion in this paper. □

Let $\sigma : \widetilde{F} \to F$ be the minimal resolution of a singular quintic del Pezzo surface. We use the labels of [12, Figure 33, p. 25]: the exceptional curves over an A_r point form the chain $E_1, \dots, E_r$. For type $A_2 + A_1$, the chain is E_2, E_1 and the isolated exceptional curve is E_3. The following table identifies the lines which will remain in the reduction. Incidence means intersection of the proper transform of the line with the indicated exceptional curves. All other incidences with exceptional curves are zero.

$\operatorname{Sing}(F)$	line	incidence	$S_F(L)$	source in [12]
$2A_1$	L_{12}	E_1, E_2	19/15	Lemma 5.8(a), pp. 18–19
A_2	L_2	E_2	17/15	Lemma 5.7, p. 18
$A_2 + A_1$	L_{13}	E_1, E_3	23/15	Lemma 5.15, pp. 22–23
$A_2 + A_1$	L_2	E_2	17/15	Lemma 5.7, p. 18
A_3	L_2	E_2	19/15	Lemma 5.8(b), pp. 18–19
A_4	L_3	E_3	5/3	Lemma 5.16, p. 23

For type A_2, the endpoint E_1 meets two further lines L_1, L_1', while E_2 meets L_2 only. For type A_3, the other line L_1 meets E_1. For type A_4, L_3 is the unique line. These conventions fix the choices of endpoints of the chains without ambiguity.

Lemma 44.4. *Let $p \in C \subset F$. Suppose that either $\delta_p(F) \ge 1$, or that p lies on L_1 or L_1' on a fibre of type A_2, or on L_1 on a fibre of type A_3. Then every prime divisor Q over X with centre p has $\beta_X(Q) > 0$.*

Proof. The exact refinement by a fibre is $S(W^F; v) = \frac{15}{16} S_F(v) + \frac{1}{16} v(C)$. Since F and C are smooth along C, the pair (F, C) is log canonical. Thus $\delta_p(F) \ge 1$ implies $S(W^F; v) \le A_F(v)$ for every prime divisor v over F with centre containing p. Lemma 44.3 applies.

In the two remaining cases, put $L = L_1$ or L_1' as appropriate. The curve C meets L once and transversally: $C \sim -K_F$, $(-K_F) \cdot L = 1$, and F is smooth along C. For type A_2, the

positive part of $-K_{\widetilde{F}} - v\widetilde{L}$ has square $5 - 2v - v^2/3$ for $0 \le v \le 3/2$ and $(2-v)(4-v)$ for $3/2 \le v \le 2$. Its intersection with $\widetilde{L}$ is respectively $1 + v/3$ and $3 - v$. Consequently

$$S_F(L) = \frac{31}{30}, \qquad S(W_F^L; p) = \frac{1}{5}\left(\int_0^{3/2} (1 + v/3)^2\, dv + \int_{3/2}^{2} (3 - v)^2\, dv\right) = \frac{19}{30}.$$

For type A_3, the corresponding square and intersection are $(2-v)(10+v)/4$ and $1 + v/4$, for $0 \le v \le 2$, giving

$$S_F(L) = \frac{16}{15}, \qquad S(W_F^L; p) = \frac{1}{5}\int_0^{2} (1 + v/4)^2\, dv = \frac{19}{30}.$$

These are the decompositions in [12, Lemmas 5.4–5.5, pp. 16–17]. The denominator in the displayed square in Lemma 5.5 is corrected here from 2 to 4; the stated S-value in that lemma is unchanged. The negative parts meet $\widetilde{L}$ only above the singular point of F, so they contribute nothing at p.

Refining W^F by L, its first invariant is $31/32$ in type A_2 and 1 in type A_3. The secondary invariant at p is in both cases

$$\frac{15}{16} \cdot \frac{19}{30} + \frac{1}{16} = \frac{21}{32}.$$

Here the last term is the restriction of the fixed boundary to the reduced point $C \cap L$. The different is zero near p. Abban–Zhuang on F therefore gives refined threshold at least $32/31$ and 1, respectively. Apply Lemma 44.3 once more. □

Proposition 44.5 (Reduction of possible centres)**.** *Let Q be a prime divisor over X with $\beta_X(Q) \le 0$. Then its centre is either one of the lines in the table, a point of one of those lines, or the unique node of a fibre of type A_1.*

Proof. If the centre is a surface D, then $Q = \mathrm{ord}_D$. The cases $D = E$ and $D = F$ are excluded by $S_X(E) = 3/8$ and $S_X(F) = 11/16$. Every other prime surface is the proper transform of a divisor in $|aH|$ on V_5, of multiplicity b along C, with $a \ge 1$ and $0 \le b \le a$. If $a = b$, the surface is a fibre, since $|aF|$ is pulled back from the base. Otherwise $D \sim aF + (a-b)E$, so $D - E$ is linearly equivalent to an effective divisor and $S_X(D) \le S_X(E) < 1$.

On a smooth fibre $\delta(F) = 15/13$. The exact fibre refinement gives $\delta_p(X) \ge 16/13$ off E and $\delta_p(X) \ge 8/7$ on E. Thus any centre meeting a smooth fibre is excluded. In particular, every curve centre is contained in a singular fibre.

At a smooth point of a singular fibre off E, the local threshold table [12, Tables 10–15, p. 26] gives $\delta_p(F) > 15/16$, except on the listed lines and on L_1 of a fibre of type A_3. In the latter case $\delta_p(F) = 15/16$, so the refined threshold is at least one and Lemma 44.3 excludes every centre contained in F. At points on E, the same tables and Lemma 44.4 exclude all points outside the listed lines. For a curve centre we apply these arguments at a general closed point of the curve.

Finally, the incidence configurations in [12, Figure 33, p. 25] show that every singular point of a fibre of type $2A_1$, A_2, $A_2 + A_1$, A_3 or A_4 lies on a listed line. This leaves precisely the centres in the statement. □

45 Ordinary lines in singular fibres

Let $\ell \subset V_5$ be an ordinary line, so that $N_{\ell/V_5} \simeq \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$, and suppose that ℓ is contained in a member of the pencil defining X. Write Z for its proper transform on X, and put $q = \ell \cap C$. Thus ℓ and C meet transversally at q. We identify Z with ℓ.

Lemma 45.1. *Suppose that the pencil surface is singular at a point of ℓ. There are homogeneous coordinates z_0, z_1 on ℓ such that $q = (z_0 = 0)$ and the unique singular point of this surface on ℓ is $r = (z_1 = 0)$. If G is the exceptional divisor of $\mathrm{Bl}_Z X$, then $G \simeq \mathbb{F}_1$ and $A_X(G) = 2$. On the chart $z_0 = 1$, with $z = z_1/z_0$, there are fibre coordinates t, n for which actual leading sections of H are*

$$1, z, t, zt, n, zn, D, \qquad D = (A_0 + A_1 z)t^2 + (B_0 + B_1 z)tn + (C_0 + C_1 z)n^2.$$

The two leading sections of the pencil class are $1, zn$. Moreover $(A_0, A_1) \neq (0, 0)$; if $A_0 = 0$, then $(B_0, C_0) \neq (0, 0)$.

Proof. Let J be the exceptional divisor of $\mathrm{Bl}_\ell V_5$. Projection from ℓ is given by $H - J$; see [27, Lemma 4.3]. The resolved projection is the blow-up of a smooth quadric along a twisted cubic; see [54, proof of Theorem 3.6, p. 65]. Its exceptional divisor has class $H - 2J$ by the canonical divisor formula. The restriction of $H - J$ embeds $J \simeq \mathbb{F}_0$ as a smooth quadric surface; its intersection with the other exceptional divisor is the twisted cubic that is blown up. Consequently the unique leading section of H of line order two cuts out an irreducible curve of bidegree $(2, 1)$ on J.

There are two order-zero sections and four order-one sections. The latter form the complete space $H^0(\ell, N^\vee_{\ell/V_5} \otimes \mathcal{O}_\ell(1))$: the kernel of the leading-term map is the one-dimensional space $H^0(H - 2J)$, while $h^0(H - J) = 5$. The differential of the pencil member is therefore a pair of linear forms on ℓ. Since it has a zero, the pair has rank one. Choose normal coordinates a, b such that this differential is $z_1 b$, and choose the other pencil coordinate to restrict to z_0. Smoothness of C implies that z_0 and z_1 have distinct zeros. The elementary modification of the normal bundle at q is given by $a = t$, $b = z_0 n$. Hence $N_{Z/X} \simeq \mathcal{O} \oplus \mathcal{O}(-1)$, giving $G \simeq \mathbb{F}_1$ and the displayed sections. Irreducibility of the $(2, 1)$ curve proves the two nonvanishing assertions. □

We use valuation vectors (j, k, h), where j is the order along G, k is the exponent of n in the leading form, and h is its order at the specified base coordinate zero. The order k is taken before h. A degree-m section of normal degree j is a section of $mf + j\zeta$ on G, where $\zeta = s + f$ and $s^2 = -1$. Its coefficient of $t^{j-k}n^k$ has base degree $m + k$. In particular, the four toric boundary coefficients obtained from mean exponents μ_j, μ_k, μ_h are

$$\mu_j - \mu_k, \quad \mu_k, \quad \mu_h, \quad 1 + \mu_k - \mu_h.$$

Lemma 45.2 (The line invariant and its first refinement)**.** *One has $S_X(G) = 3/2$. If $A_0 \neq 0$, the refined threshold on G is at least $16/15$. If $A_0 = 0$, every divisorial valuation with centre Z has positive beta-invariant, and $\delta_q(X) > 1$.*

Proof. The leading coefficient of $h^0(X, mL)$ is $10/3$. We first check the boundedness hypothesis of Lemma 34.2 independently of any moment calculation. Write $\sigma : Y = \mathrm{Bl}_Z X \to X$, and choose a very ample divisor M on Y. A section of mL of order j along G gives an effective divisor in $|m\sigma^*L - jG|$. Therefore

$$j \leq m \, \frac{\sigma^*L \cdot M^2}{G \cdot M^2}.$$

The denominator is positive because G is an effective prime divisor. Furthermore, $0 \leq k \leq j$ and $0 \leq h \leq m + k$, by the coefficient bundle computed above. Thus all three recorded coordinates are uniformly $O(m)$. Their successive flag orders have one-dimensional leaves and add under multiplication. All certificates below are leading forms of actual sections. Together with the lattice checks in each case, this proves all hypotheses of Lemma 34.2 before that lemma is used.

If $A_0 \neq 0$, the seven H vectors are

$$(0,0,0),(0,0,1),(1,0,0),(1,0,1),(1,1,0),(1,1,1),(2,0,0).$$

Adjoin their translates by $(1,1,1)$, corresponding to multiplication by zn. Their convex hull has volume $10/3$ and centroid $(3/2, 13/16, 15/16)$. Its four boundary coefficients are

$$11/16, \quad 13/16, \quad 15/16, \quad 7/8,$$

which proves the assertion.

Suppose that $A_0 = 0$, and rescale so that $A_1 = 1$. Replace the last vector above by $(2,0,1)$. The following degree-two sections provide the missing leading vectors:

$$P = Dn - t(ztn) - B_1 n(ztn) - C_1 n(zn^2) = B_0 tn^2 + C_0 n^3,$$
$$Q = D(zn^2) - (ztn)^2 - B_1(ztn)(zn^2) - C_1(zn^2)^2 = znP.$$

Every factor on the left is a degree-one anticanonical section. If $B_0 \neq 0$, their normalized vectors are $(3/2,1,0),(2,3/2,1/2)$; if $B_0 = 0$, they are $(3/2,3/2,0),(2,2,1/2)$. Both hulls have volume $10/3$. Their boundary coefficients are, respectively,

$$(213/320, 267/320, 333/320, 127/160), \qquad (103/160, 137/160, 333/320, 261/320).$$

In all three computations $\mu_j = 3/2$. The degree-one vectors generate the full exponent lattice. The full-volume lemma therefore proves the asserted value of $S_X(G)$ and all the displayed means. Over the generic point of Z, only the first two boundary coefficients occur, and they are smaller than one. Over q, the third boundary is disjoint from the fibre and the other three coefficients are smaller than one. The toric one-parameter subgroups preserve that fibre, so semicontinuity applies on a neighbourhood of it. Abban–Zhuang adjunction, with first ratio $2/(3/2) = 4/3$, proves the remaining assertions. □

Lemma 45.3 (A base coordinate centred at the point)**.** *If $p \in Z \setminus \{q, r\}$, then $\delta_p(X) \geq 64/57$.*

Proof. Only $A_0 = 0$ remains. Choose a base coordinate z vanishing at p and rescale so that the differential vanishes at $z = -1$. Put $w = z + 1$. Replacing t by $t + \lambda n$ makes the mixed coefficient constant, so that

$$D = wt^2 + btn + (a + dz)n^2, \qquad c = a - d, \qquad (b,c) \neq (0,0).$$

The leading pencil section is wn. The following are degree-one anticanonical leading sections:

$$1,\ z,\ T = t,\ T_z = zt,\ N = n,\ N_z = zn,\ N_{zz} = z^2 n,$$
$$U = wtn,\ U_z = zwtn,\ V = wn^2,\ V_z = zwn^2,\ D,\ wnD.$$

For example N_{zz} is obtained by subtracting zn from $(wn)z$. Define degree-two sections

$$P = DN - TU - dNV = btn^2 + cn^3, \qquad Q = DV - U^2 - dV^2 = wnP.$$

If $b \neq 0$, the degree-two sections

$$R = N_{zz}U_z - T_z V_z + TV_z - TV + P/b = z^4 tn^2 + (c/b)n^3,$$
$$S = U_z V_z - U_z V + UV - Q/b = wz^3 tn^3 - (c/b)wn^4$$

have normalized leading vectors $(3/2,1,2)$ and $(2,3/2,3/2)$. The degree-one vectors are

$$(0,0,0),(0,0,1),(1,0,0),(1,0,1),(1,1,0),(1,1,1),(1,1,2),$$
$$(2,1,0),(2,1,1),(2,2,0),(2,2,1),(2,0,0),(3,1,0).$$

Their hull with these two additional vectors has volume $10/3$ and first moment $(5, 89/32, 215/96)$. The three boundary coefficients meeting the fibre over p are therefore $213/320, 267/320, 43/64$, all smaller than one.

Suppose that $b = 0$. Then $c \neq 0$, and the degree-two sections

$$R_0 = N_{zz}V_z - N_zV_z + NV_z - NV + P/c = z^4n^3, \qquad S_0 = V_z^2 - V_zV + V^2 - Q/c = wz^3n^4$$

give the vectors $(3/2, 3/2, 2), (2, 2, 3/2)$. This hull has volume $37/12$ and first h-moment $63/32$. It is enough to bound the missing volume. The full-volume certificate in Lemma 45.2 implies $j \leq 3m$ for every actual valuation vector: a vector outside that full-dimensional body would enlarge its convex hull and contradict the leading Hilbert coefficient. Thus $h \leq m + k \leq m + j \leq 4m$. The missing volume is $10/3 - 37/12 = 1/4$, and hence

$$\mu_h \leq \frac{63/32 + 4(1/4)}{10/3} = \frac{57}{64}.$$

The two marginal normal means are unchanged by the base coordinate:

$$103/160, \qquad 137/160.$$

All three coefficients meeting the fibre are therefore at most $57/64$.

For completeness, these volume calculations use the following halfspaces, with coordinates (j, k, h). In the first case they are

$$k \leq j,\ k, h \geq 0,\ h - k \leq 1,\ k + h \leq 3,\ j + k \leq 4,$$
$$2j - 3k + 2h \leq 4, \quad 2j + k + h \leq 7, \quad 4j - 4k + 3h \leq 8, \quad 4j - k + 3h \leq 11.$$

In the second case they are $k \leq j$, $k, h \geq 0$, $h - k \leq 1$, $j - k + h \leq 2$, $j + k \leq 4$, and $5j - k + 4h \leq 14$. Coning their faces to an interior point gives the stated volumes and moments.

Choose the monomial degenerations with respect to the coordinate z just fixed. They preserve the entire proper fibre over p. For every $\lambda < 64/57$, the limiting basis pair is Kawamata log terminal near that fibre. Properness of the fixed fibre and semicontinuity give the same assertion before degeneration. Taking $\lambda \uparrow 64/57$ gives the bound for the original refined series. The first ratio is $4/3$. Abban–Zhuang adjunction gives the result. □

Example 45.4. For $D = zt^2 + n^2$ the ordinary line refinement has coefficient $333/320$ over the singular point $z = 0$, so this extraction alone does not settle that point. Nevertheless it settles the generic centre Z, its point above C, and every other point of Z. After moving the base coordinate to one of these other points, the sections R_0 and S_0 above and the missing-volume estimate suffice; no decomposition on the blown-up threefold is needed.

46 Lines of the second kind: the ruled surface $\mathbb{F}_2$

In this section F is a quintic del Pezzo fibre with an A_4 point p_0, and Z is its unique line. We use the coordinates of the preceding normal-form lemma. Thus, on $x_0 = 1$, the threefold V_5 has coordinates $u = x_1$, $v = x_4$, $w = x_5$, and its hyperplane sections are spanned by

$$1, \quad w, \quad u, \quad u^2 + vw, \quad u^3 + uvw + v^2, \quad v, \quad uv - u^2w - vw^2.$$

Here $Z = (u = v = 0)$ and $p_0 = (u = v = w = 0)$. The fibre equation is $f = x_6 + \gamma x_3$. The other pencil generator, denoted by g, restricts to a nonzero linear form on Z which does not vanish at p_0. After an automorphism preserving p_0 and Z, we take $g|_Z = x_0$. Thus $q = Z \cap C$ is the point $w = \infty$.

Lemma 46.1. *The weight-$(1,2)$ extraction along Z, with weights one on u and two on v away from q, is a plt extraction $\sigma : \widehat{X} \to X$. Its exceptional surface G is isomorphic to $\mathbb{F}_2$. Write B for its section of self-intersection -2, and c for a ruling. Then*

$$A_X(G) = 3, \qquad \Delta_G = \tfrac{1}{2}D, \qquad D \sim B + 3c, \qquad G|_G = -\tfrac{1}{2}(B + c), \qquad L|_G = c.$$

The curve D is a smooth section; its intersection with B is the point r_0 lying over p_0.

Proof. At infinity put $t = x_0/x_5$, $a = x_1/x_5$, and $b = x_6/x_5$. These are regular parameters on V_5 near q. On their overlap with the first chart,

$$a = tu, \qquad b = tuv - u^2 - v/t.$$

On X, write $s = f/g$. The weighted initial equation at q is $b = ts$; the variables a, s, t are regular parameters there, and their normal weights are $(1, 2, 0)$. The two weighted ideals agree on the overlap. For the coarse exceptional surface put $U = u^2$ and $V = v$. Its transition to the infinity chart is

$$U' = t^2 U, \qquad S' = -t^{-1}U - t^{-2}V.$$

The associated rank-two bundle splits as $\mathcal{O}(1) \oplus \mathcal{O}(-1)$. Indeed its positive subbundle is generated on the finite chart by $(U, V) = (w, -1)$ and on the other chart by $(U', S') = (1, 0)$. Consequently the coarse surface is $\mathbb{F}_2$, and its negative section has equation $U + wV = 0$.

The weight-two chart of the extraction is the quotient of a smooth chart by an involution acting with weights $(1, 1, 0)$. It follows directly that the pair with its reduced exceptional divisor is plt, with different one half of the section $D = (u = 0)$. The discrepancy is $1 + 2 = 3$. The displayed transition shows that $D^2 = 4$; hence $D \sim B + 3c$. On the finite chart $D \cap B$ is given by $w = 0$. Finally $LZ = 1$ gives $L|_G = c$, and adjunction $K_G + \Delta_G = K_X|_G + 3G|_G$ gives the stated restriction of G. □

Lemma 46.2. *Put $A = u^2 + vw$. On the chart where the initial form of g is x_0, the refinement W of the anticanonical ring by G is the graded algebra generated in degree one by*

$$H_i, \quad wAH_i \qquad (H_i \in \{1, w, u, A, uA, v, wA\}). \tag{46.1}$$

The normal grading is $\deg u = 1$, $\deg v = \deg A = 2$, *and* $\deg w = 0$. *Moreover*

$$S_X(G) = \frac{41}{16}, \qquad [D_W] = \frac{41}{32}B + \frac{73}{32}c,$$

where $[D_W]$ denotes the limiting numerical class of its basis divisors.

Proof. Use homogeneous variables (x_0, u, A, T, v, w, J) for the seven initial hyperplane sections. The candidate initial equations are

$$vw - x_0A + u^2, \quad vJ - uT + A^2, \quad -x_0T + uA, \quad -x_0J - Aw, \quad -Tw - uJ.$$

They are the signed maximal Pfaffians, up to a common sign, of

$$\begin{pmatrix} 0 & 0 & J & -T & A \\ 0 & 0 & -w & -u & x_0 \\ -J & w & 0 & A & -u \\ T & u & -A & 0 & v \\ -A & -x_0 & u & -v & 0 \end{pmatrix}.$$

After inverting x_0, the equations give $A = (u^2 + vw)/x_0$, $T = uA/x_0$, and $J = -wA/x_0$, so this open algebra is a polynomial algebra in $x_0^{\pm 1}, u, v, w$, of dimension four. On $x_0 = 0, A \neq 0$,

one has $u = w = 0$ and $vJ + A^2 = 0$; A, v, T give three independent parameters, with A, v invertible. On $x_0 = A = 0, v \neq 0$, the equations instead give $w = -u^2/v$ and $J = uT/v$, with parameters v, u, T. Finally, on $x_0 = A = v = 0$, one has $u = 0$ and $Tw = 0$, leaving a locus of dimension two in T, w, J. Thus the whole boundary has dimension three. The ideal has height three, so its Pfaffian resolution makes the quotient Cohen–Macaulay of pure dimension four. Since the boundary contains no component, x_0 is a nonzerodivisor. Injection into the polynomial localization then proves that the quotient is a domain. Its graded Pfaffian resolution has the same Hilbert series as the coordinate ring of V. The candidate equations belong to the initial ideal, and equality of Hilbert functions proves equality of the ideals. The localized graph in the normal parameters also identifies this weight filtration with the geometric normal filtration.

The pair (J, x_0) is regular in this primary initial ring. Indeed, on $J = x_0 = 0$ its equations are $u^2 + vw = 0$, $uT = A^2$, $uA = Aw = Tw = 0$. Their support has $A = 0$. On $T \neq 0$ one has $u = w = 0$ and two parameters T, v; on $T = 0$ it is the two-dimensional cone $u^2 + vw = 0$. Thus the common zero locus has dimension two. Lemma 43.1 applies to the graph equations $J = ea$, $x_0 = eb$: their quotient is a domain and is the full initial section algebra. On $x_0 = b = 1$ it yields $e = 1$, $a = -wA$, and the diagonal anticanonical algebra is exactly (46.1). Its fixed-degree components therefore include all cancellations among global sections.

The seven primary weights are $(0, 1, 2, 3, 2, 0, 2)$, the five Pfaffian weights are $(2, 4, 3, 2, 3)$, and the final shift is seven. Differentiating the Pfaffian Hilbert series gives $S_H(G) = \frac{1}{4} \cdot 10 - \frac{3}{20} \cdot 7 = 29/20$. The graph Hilbert series then gives $S_X(G) = \frac{5}{4}(29/20) + \frac{3}{8} \cdot 2 = 41/16$. Equivalently this last number follows from the independent moment certificate below. A section of normal degree j restricts to the class $mc + \frac{j}{2}(B + c)$; taking its average proves the class formula. □

Lemma 46.3. *For the further refinement by B one has*

$$S(W; B) = \frac{307}{320}, \qquad S(W^B; r_0) = \frac{263}{320}.$$

For every other point $r \in B$, $S(W^B; r) \leq 261/320$. For every ruling c_p of G, $S(W; c_p) \leq 263/320$.

Proof. Work at the generic point of B, invert w, and substitute $v = (A - u^2)/w$. The valuation is ordered first by normal degree $j = 2a + b$, then by the exponent a of A, and finally by the exponent d of w. Its monomial coordinates are (a, b, d) for $A^a u^b w^d$. The following nine points are valuations divided by section degree:

i	1	2	3	4	5	6	7	8	9
P_i	$(0,0,0)$	$(0,0,1)$	$(0,2,-1)$	$(1,2,0)$	$(1,0,2)$	$(2,1,1)$	$(2,0,2)$	$(2,0,1/2)$	$(3/2,0,0)$.

The first seven are supplied by $1, w, v, (wA)v, w(wA), (wA)(uA), (wA)^2$. The last two come from the degree-two sections wA^4 and A^3. They are genuine sections: writing $J = wA$ and $T = uA$, one has

$$A^3 = JAv + T^2, \qquad wA^4 = J^2Av + JT^2.$$

Each product uses at most two hyperplane generators and at most two pencil generators. The signs of J can harmlessly be reversed.

Their convex hull has volume $10/3$. For a directly checkable integral certificate, triangulate its facets from the origin with triples

$$(8,4,6), (8,7,6), (4,2,5), (4,2,3), (7,4,5), (7,4,6), (8,4,3), (8,9,3).$$

The respective tetrahedral volumes are $5/12, 1/2, 1/3, 1/3, 2/3, 1/3, 1/2, 1/4$. The sum of their first moments is $(307/96, 103/48, 5/3)$. Thus the centroid is

$$\left(\frac{307}{320}, \frac{103}{160}, \frac{1}{2}\right).$$

The valuation lattice is the full integer lattice, since $1, A, u, w$ are degree-one sections. After substituting $v = (A - u^2)/w$, the finite degree-one generating list in Lemma 46.2 consists of Laurent polynomials in A, u, w. Every exponent occurring in a product of m generators has absolute value at most a fixed constant times m. Taking linear combinations and pivot monomials preserves this bound. Thus signed exponents satisfy the boundedness hypothesis of Lemma 34.2. Its volume agrees with $L^3/6 = 10/3$, so that lemma identifies these with the moments of the entire refinement. In particular it yields $S(W; B) = 307/320$ and, independently, $S_X(G) = 2(307/320) + 103/160 = 41/16$.

On B, the relation is $u^2 = -vw$. At r_0 the coordinate v is nonzero, so the order of $A^{-a}A^a u^b w^d$ on the coarse curve is $b/2 + d$. Its mean is $103/320 + 1/2 = 263/320$. The total degree of the refined divisors on B is

$$\left(\frac{41}{32}B + \frac{73}{32}c - \frac{307}{320}B\right) B = \frac{131}{80}.$$

For any other point, choose a basis compatible with that point and r_0 in each restricted graded piece. Subtracting the known mean at r_0 bounds the other mean by $131/80 - 263/320 = 261/320$. Finally a basis compatible with B and a ruling c_p restricts to a basis on B whose order at $B \cap c_p$ is at least its order along c_p. The preceding two bounds therefore imply $S(W; c_p) \le 263/320$ for every ruling. □

Remark 46.4. The secondary initial graph ring is not used in the preceding lemma. After taking the initial form along B, the pair (J, x_0) ceases to be regular: the locus $J = x_0 = u = w = 0$ has the free coordinates A, T, v. Using its unsaturated graph Hilbert series would give the incorrect number $15/16$ in place of $307/320$. The two explicit degree-two sections in the proof are essential. They enlarge the initial degree-one hull from volume $17/6$ to the required volume $10/3$.

Proposition 46.5. *Every point $p \in Z \setminus \{p_0\}$ satisfies $\delta_p(X) \ge 320/307 > 1$.*

Proof. First consider $r \in G \setminus B$ and use its ruling c_p as the surface flag. The residual-degree lemma, subtracting the mean along B before restricting, bounds the point invariant by

$$\left(\frac{41}{32}B + \frac{73}{32}c - \frac{307}{320}B\right) c_p = \frac{103}{320}.$$

The boundary D meets c_p transversally once. Hence the local discrepancy on c_p is either one or one half, while $S(W; c_p) \le 263/320$. Surface adjunction therefore gives a threshold at least $\min\{320/263, 160/103\}$ at r.

For $r \in B \setminus \{r_0\}$, use B as the flag. The different has no support at r, and the preceding lemma gives threshold at least $\min\{320/307, 320/261\} = 320/307$. Thus the refined threshold is at least $320/307$ at every point of G except possibly r_0, whose image on X is p_0. Finally $A_X(G)/S_X(G) = 48/41 > 320/307$. Abban–Zhuang adjunction for the weighted curve extraction proves the assertion. □

46.1 The other $\mathbb{F}_2$ case

Proposition 46.6. *Suppose that the fibre equation along the second-kind line is $f = \beta x_2 + \gamma x_3 + \varepsilon x_6$, with $\beta \neq 0$, and that $C \cap Z$ is smooth. Then every point of Z other than its closed-orbit point p_0 satisfies $\delta_p(X) \ge 16/15$.*

Proof. The point p_0 is a singular point of the fibre, so it is not on C. Normalize $q = C \cap Z$ to $w = \infty$ as above. Smoothness of C at q then gives $\varepsilon \neq 0$. Rescaling w and the fibre equation makes its normal initial form $(w - 1)A$. The weighted exceptional surface, boundary, and

numerical class of the refined series are the same as in Lemma 46.1. The exact generators are now $H_i, (w-1)AH_i$, with the same seven H_i as before.

In the coordinates (A, u, w), the following degree-one leading monomials are obtained by elementary row operations:

$$(0,0,1), (2,1,0), (0,0,0), (2,0,1), (2,0,0),$$
$$(0,2,-1), (1,0,2), (1,2,-1), (1,1,1).$$

Number these points consecutively from one to nine. For example the last three are obtained from wA, uA, and v by multiplication by $(w-1)A$ and subtraction of the lower terms. Their convex hull has volume $10/3$ and first moment $(25/8, 55/24, 7/8)$, giving centroid $(15/16, 11/16, 21/80)$. A tetrahedral certificate from the origin is

$$(8,9,6), (8,9,2), (8,5,2), (4,5,2), (9,7,1), (9,6,1), (4,9,7), (4,9,2),$$

with volumes $1/2, 2/3, 1/3, 1/3, 1/6, 1/3, 1/2, 1/2$. Thus $S(W;B) = 15/16$.

We also need the point $r_1 \in B$ over $w = 1$. Put $z = w - 1$ and multiply every degree-m section by the common unit w^m. Degree-one generators are then

$$w, w^2, uw, Aw, uAw, A - u^2, w^2 A,$$

and their products with zA. In the order (j, A, z) their leading exponents (a, b, d) have the ten vertices

$$(1,2,1), (2,1,1), (2,0,2), (1,1,0), (0,2,0), (0,0,0), (1,0,0), (2,0,1), (1,0,2), (0,0,1).$$

This hull again has volume $10/3$; its first moment is $(25/8, 55/24, 65/24)$. One can use the triples

$$(1,3,2), (9,1,3), (8,3,2), (4,1,5), (4,1,2), (10,1,5), (10,9,1), (8,4,2), (8,7,4),$$

whose volumes are $2/3, 2/3, 1/3, 1/3, 1/3, 1/3, 1/3, 1/6, 1/6$. The full-volume lemma gives $S(W^B; r_1) = 13/16$. The total degree on B is now $51/32$, so at every other point its mean is at most $51/32 - 13/16 = 25/32$. In particular every ruling has mean at most $13/16$.

Away from B, subtracting its mean before restricting to a ruling leaves point invariant at most $41/32 - 15/16 = 11/32$. The different on that ruling has coefficient at most one half, giving refined threshold at least $\min\{16/13, 16/11\}$. On $B \setminus \{r_0\}$ the different vanishes and the threshold is at least $16/15$. Abban–Zhuang adjunction and $A_X(G)/S_X(G) = 48/41$ give the assertion. □

47 The remaining lines of the second kind

We retain the affine coordinates u, v, w along a line Z of the second kind and put $A = u^2 + wv$. The point $w = 0$ is the closed-orbit point of Z. A fibre which is tangent in the relevant normal direction has weighted initial equation

$$f = (\beta - \varepsilon w)A + \delta v.$$

The case $\delta = 0$ was treated by the $\mathbb{F}_2$ calculations above. We now suppose $\delta \neq 0$ and divide the equation by δ. The other pencil section restricts to a linear form on Z, vanishing at $q = C \cap Z$. The automorphisms preserving Z act on it as the stabilizer of its closed-orbit point. We may therefore use one of the following two normalizations:

	q	$g\vert_Z$
first case	∞	1
second case	0	w.

In the second case any specified point different from q can in addition be chosen to have finite nonzero coordinate. This observation will avoid an unnecessary extra infinity chart in the pointwise argument.

Lemma 47.1. *The weight-$(1,2)$ extraction along Z has discrepancy 3 and coarse exceptional surface $G \cong \mathbb{P}^1 \times \mathbb{P}^1$. Write h for a horizontal section class and c for a ruling. Its different and restrictions are*

$$\Delta_G = \tfrac{1}{2}D, \quad D \sim h + 2c, \quad G|_G = -\tfrac{1}{2}h, \quad L|_G = c.$$

The curve D is smooth. Put $s = f/g$ and $Q(w) = 1 + \beta w - \varepsilon w^2$. On the index-two cover the primary equation and section spaces are

$$u^2 = Q(w)A - wg(w)s, \qquad H = \langle 1, w, u, A, uA, v, wA\rangle, \qquad F = \langle 1, s\rangle, \tag{47.1}$$

where $v = g(w)s - (\beta - \varepsilon w)A$. Their primary degrees are respectively $0, 0, 1, 2, 3, 2, 2$ and $0, 2$. The diagonal products generate the primary anticanonical ring, and

$$S_X(G) = \frac{41}{16}, \qquad [D_W] = \frac{41}{32}h + c.$$

Proof. The construction is the same weighted extraction as before; at q one uses the pencil parameter s in place of the vanishing normal coordinate. The formula in (47.1) gives its transition explicitly. On the coarse surface $[A : s]$ are homogeneous coordinates on a fixed projective line. The branch section is

$$Q_h(w_0, w_1)A = w_1 g_h(w_0, w_1)s,$$

where $Q_h = w_0^2 + \beta w_0 w_1 - \varepsilon w_1^2$ and g_h is either w_0 or w_1. The two coefficients on the right and left have no common zero: at the closed-orbit point this follows from $Q(0) = 1$, and at q it follows from smoothness of C. Thus the coarse surface is the trivial ruled surface, and D is a smooth section of class $h + 2c$. The weight-two chart is the quotient chart of weights $(1, 1, 0)$, so the extraction is plt, its discrepancy is 3, and the different is $\frac{1}{2}D$. Since $LZ = 1$, adjunction gives the restrictions displayed in the statement.

For completeness, the primary ring here is the same prime Pfaffian ring used in Lemma 46.2; only the chosen pencil changes. The pair consisting of the initial forms of f, g is a regular sequence. This can also be checked directly from its five equations. For $g = x_0$, the boundary component with $A \neq 0$ has $u = w = 0$ and $vJ + A^2 = 0$, while the part with $A = 0$ is the remaining rank-one cone; the form $v + \beta A + \varepsilon J$ vanishes identically on neither three-dimensional component. For $g = w$, its open part $x_0 \neq 0$ has $J = 0$, $A = u^2/x_0$, $T = u^3/x_0^2$, and the equation of f solves for v. On its boundary $x_0 = 0$ one has $u = 0$ and $vJ + A^2 = 0$, and again f cuts the dimension by one. Hence in either case the simultaneous zero locus has dimension two. Lemma 43.1 therefore proves that the graph ring is integral and is the full associated algebra of global sections. Injection into its localized normal-coordinate model identifies it with the actual primary refinement, including its cancellations.

The Pfaffian weights and graph weights are unchanged. The Hilbert-series calculation therefore gives

$$S_X(G) = \frac{5}{4} \cdot \frac{29}{20} + \frac{3}{8} \cdot 2 = \frac{41}{16}.$$

A piece of anticanonical degree m and primary degree j has numerical class $mc + \frac{j}{2}h$ on G. Averaging gives $[D_W]$. □

Lemma 47.2. *At every point $r \in G \setminus D$ one has*

$$\delta_r(G, \tfrac{1}{2}D; W) \geq \frac{32}{31}.$$

Proof. Let $B = (A = 0)$ and $B' = (s = 0)$; they are disjoint horizontal sections. The principal filtrations by the original hyperplane sections x_2, x_1 and by the original pencil section induce the following lower bounds for fixed coefficients in the refined series:

$$S(W; B) \geq \frac{5}{16}, \qquad S(W; B') \geq \frac{11}{16}, \qquad S(W; D) \geq \frac{5}{32}.$$

Indeed their primary initial sections are A, u, s. The last is the strict pencil section of class F; the hyperplane section f instead has initial form $g(w)s$. The first has multiplicity one along B, the last has multiplicity one along B', and the odd section u has coarse divisor $D/2$. Choose a basis simultaneously compatible with the principal filtration and the primary filtration. Divisibility by a power of the original section implies divisibility of its initial section by the same power. Averaging the dimensions proves the asserted inequalities from $S_X(H) = 5/16$ and $S_X(F) = 11/16$. Additional vertical components have nonnegative coefficients and do not weaken them.

Choose the horizontal section C through r. If $C \neq B$, the coefficient of h in $[D_W]$ and the bound for B give $S(W; C) \leq 41/32 - 5/16 = 31/32$. If $C = B$, use B' instead and obtain the stronger bound $19/32$. Lemma 43.6, applied with the fixed curve D, gives

$$S(W^C; r) \leq ([D_W] - \tfrac{5}{32}D) \cdot C = 1 - \frac{5}{16} = \frac{11}{16}.$$

Here $C^2 = 0$ and $r \notin D$, so the different on C has coefficient zero at r. Local Abban–Zhuang adjunction now gives $\min\{32/31, 16/11\} = 32/31$. □

Proposition 47.3. *Let $p \in Z$ be a smooth point of the quintic del Pezzo fibre containing Z. Then the weighted extraction along Z proves*

$$\delta_p(X) \geq \frac{80}{79}.$$

This includes the point where the strict transform of Z meets the exceptional divisor of $X \to V_5$.

Proof. Points of G outside D are covered by Lemma 47.2. At a finite nonzero base point $w = c$, smoothness of the fibre says $Q(c) \neq 0$: along Z its ordinary normal derivative is $Q(c)\,dv$. Lemma 47.4 therefore handles the whole ruling over such a point, in both normalizations $g = 1$ and $g = w$. The remaining points of D are exactly the endpoint cases treated in Proposition 47.5: the closed-orbit point when $g = 1$, the pencil base point at infinity when $g = 1$, and the coincident closed-orbit and pencil base point when $g = w$. In the last normalization, every other specified base point may be chosen finite and nonzero as above. These cases exhaust the exceptional surface above smooth points of the fibre.

All of the refined bounds just cited are at least $80/79$. The primary ratio is $A_X(G)/S_X(G) = 3/(41/16) = 48/41 > 1$. Applying the local Abban–Zhuang inequality to G gives the lower bound $\min\{48/41, 80/79\} = 80/79$. Points where the fibre is singular are handled separately by the point-extraction lemmas and are not assumed to satisfy $Q(c) \neq 0$. □

Lemma 47.4 (A ruling away from the special base points)**.** *In the weight-$(1, 2)$ line model with $G \simeq \mathbb{F}_0$, write*

$$A = u^2 + vw, \qquad T = uA, \qquad f = (\alpha + \gamma w)A + v, \qquad F = f/g(w).$$

Here $g(w)$ is either 1 or w. Let $c \neq 0$ and suppose that $q = 1 + \alpha c + \gamma c^2 \neq 0$. The refined threshold at every point of the ruling $C_c = (w = c)$ is at least $80/79$.

Proof. Put $z = w - c$, $r = \alpha + \gamma c$, and $U_c = A - cv$. Rescale the pencil section by the nonzero constant $g(c)$ and denote the resulting section again by F. Thus $F|_{w=c} = f|_{w=c}$. Record exponents of (u, v, z), taking the normal degree $u + 2v$ first, then the z-order, and then the u-order. The following leading vectors are supplied by actual degree-one anticanonical sections:

$$(0,0,0), (0,0,1), (2,0,1), (3,0,0), (0,2,0), (1,2,0), (0,2,1).$$

Indeed use $1, z$, then $qzA - czF$, $qT - cFu$, Fv, FT, and FzA. All these are actual anticanonical sections. In particular $zF = (w-c)F$ is a difference of two products of a hyperplane section and the pencil section. In the case $g = w$, it can equally be written $cf - c^2(f/w)$; no multiplication by an additional hyperplane factor is used. On $z = 0$ we have $A = u^2 + cv$ and $F = ru^2 + qv$, so the asserted initial terms are nonzero because $c, q \neq 0$.

The degree-two section

$$R = q^2FT^2 - F^2\big(c(q+1)U_c^2 + qc^2U_cv\big)$$

restricts on $z = 0$ to Fu^6. This follows from the polynomial identity

$$U^3 = q^2U(U+cV)^2 - (rU+qV)\big(c(q+1)U^2 + qc^2UV\big), \qquad q = 1 + rc.$$

Thus R supplies the normalized vector $(3, 1/2, 0)$. All its summands use two hyperplane factors and at most two pencil factors, so they are genuine sections of $2L$.

The hull of these eight vectors has volume $10/3$ and first moment $(79/24, 21/8, 35/24)$. Its centroid is $(79/80, 63/80, 7/16)$. The lattice is the full lattice after adjoining the remaining degree-one sections u and v. The full-volume lemma therefore gives

$$S(W; C_c) = 7/16, \qquad S(W^{C_c}; D \cap C_c) = 79/160.$$

The factor $1/2$ in the second equality comes from the coarse uniformizer u^2/v at the point of the different. The total expected degree on the weighted ruling is $S_X(G)/2 = 41/32$. At any other point, simultaneous bases give an upper bound $41/32 - 79/160 = 63/80$.

The discrepancy of C_c is one. On the ruling the different has coefficient $1/2$ at $D \cap C_c$ and no other support. Hence Abban–Zhuang adjunction gives the lower bound

$$\min\{16/7,\ (1/2)/(79/160),\ 1/(63/80)\} = 80/79.$$

□

47.1 The two end fibres of the weighted ruled surface

Proposition 47.5. *In the $\mathbb{F}_0$ case of the weighted line extraction, the remaining points of the branch section over the two end points of the line have local refined threshold greater than one. More precisely, the bounds are $40/29$ at the closed-orbit point when it is not on C, $160/117$ at $C \cap Z$ when this is not the closed-orbit point, and $32/27$ when these two points coincide.*

Proof. We give the three local computations together, retaining the primary degree in every calculation. Denote the primary degree by j. On the chart $s = 1$ of the coarse exceptional surface, the seven H-sections and their primary degrees are

$$(1, w, u, A, uA, wA, \chi s), \qquad (0,0,1,2,3,2,2), \tag{47.2}$$

and the two F-sections are $1, s$, of primary degrees zero and two. The branch cover is smooth, with coordinate u, and the local relation is

$$u^2 = Q(w)A - r(w)s, \qquad Q(0) = 1.$$

The three cases are

	$r(w)$	χ
I : closed-orbit point not on C	w	1
II : $C \cap Z$ not closed-orbit	w	w
III : $C \cap Z$ closed-orbit	w^2	w.

In Case II the coordinate w is the reciprocal of the original affine coordinate on the line: all H-sections are divided by the original coordinate. In particular the last H-section is ws, rather than s. This distinction changes the expected order. Multiplying each fixed-degree section by a common power of the unit Q clears its denominator, and changes none of the orders used below.

In Cases I and II, blow up the coarse surface at the point. Its exceptional curve $K \cong \mathbb{P}^1$ has discrepancy $3/2$ for the pair with boundary $\frac{1}{2}D$. On the cover its valuation has weights $(u, w) = (1, 2)$ and ramification index two. Let $P_D \in K$ be the point of the strict transform of D. After taking this weighted initial form, take the lowest u exponent. Record the resulting exponents and the primary degree as (a, b, j). The coarse order at P_D is $a/2$.

Besides the sections in (47.2) and their products with s, we use the degree-two section

$$T = (uA)^2 + (sw)(swA) - (wA)(sA).$$

Every factor on the right is a degree-one anticanonical section. On the chart $s = 1$, put $N = u^2 + w$, so $A = N/Q$. If $Q = 1 + \beta w + \gamma w^2$, then

$$Q^2 T = N(u^4 + \beta w^3 + \gamma w^4).$$

Its weighted initial form is $u^4(u^2 + w)$, and its recorded vector is $(4, 1, 6)$. In Case II we also have

$$T' = (uA)(suA) + (s\chi s)(swA) - (swA)(sA).$$

Since $\chi = w$, this is sT as a local rational function, and has vector $(4, 1, 8)$. The notation $s\chi s$ denotes the product of the F-section s with the H-section χs; thus every factor is again a genuine degree-one anticanonical section.

The convex hulls of the degree-normalized certified vectors have the following vertices:

I	$(0,0,0), (0,0,4), (0,1,0), (0,2,2), (0,2,4),$
	$(1,1,5), (2,0,2), (2,1/2,3)$
II	$(0,0,0), (0,0,2), (0,1,0), (0,1,4), (0,2,2), (0,2,4),$
	$(1,1,5), (2,0,2), (2,0,4), (2,1/2,3), (2,1/2,4)$

Both have Euclidean volume $20/3$. The value lattice is $\{(a, b, j) : a \equiv j \pmod 2\}$, of index two; after including the anticanonical degree, the displayed actual sections generate this full lattice. Thus the normalized volume is $10/3$, the leading coefficient of N_m. The full-volume lemma gives the exact centroids

$$\mathrm{I} : \left(\frac{49}{80}, \frac{29}{40}, \frac{41}{16}\right), \qquad \mathrm{II} : \left(\frac{117}{160}, \frac{233}{320}, \frac{41}{16}\right).$$

These figures follow directly by coning the listed rational faces to the origin and summing determinants; in particular their native-degree component provides an independent check against $S_X(G) = 41/16$.

Dividing the cover weight $a + 2b$ by two gives

	$S(W; K)$	$S(W^K; P_D)$
I	$33/32$	$49/160$
II	$35/32$	$117/320$.

The different on K is $\frac{1}{2}P_D$. Its residual divisors have total expected degree $S(W;K)$. Simultaneous bases at P_D and any other point show that the latter point terms are at most $29/40$ and $233/320$, respectively. Hence Abban–Zhuang gives

$$\text{I}: \quad \min\left\{\frac{16}{11}, \frac{80}{49}, \frac{40}{29}\right\} = \frac{40}{29}, \qquad \text{II}: \quad \min\left\{\frac{48}{35}, \frac{160}{117}, \frac{320}{233}\right\} = \frac{160}{117}.$$

Consider Case III. After rescaling the cover coordinate and A, its relation can be written $u^2 = w^2 s - Q(w)A$, with $Q(0) = 1$. Blow up the coarse point with weights $(w, A) = (1, 2)$. The exceptional curve $K \cong \mathbb{P}(1, 2)$ has discrepancy $3 - \frac{1}{2} \cdot 2 = 2$. On the cover this is the ordinary blowup of (w, u); its divisorial ramification index is one. Put $u = w + Y$, and refine first by total degree in w, Y, and then by the Y order. This is the point P_A of $A = 0$ on K.

The degree-one sections give the thirteen vectors, in the order (Y, w, j),

$$(0,0,0), (0,1,0), (0,1,1), (1,1,2), (1,2,3), (1,2,2), (0,1,2),$$
$$(0,0,2), (0,1,3), (1,1,4), (1,2,5), (1,2,4), (0,1,4).$$

There are two more certified degree-two sections:

$$U = (uA)^2 - (wA)(swA) = -QA^3, \qquad U' = (uA)(suA) - (swA)^2 = -sQA^3.$$

Since the initial form of A is $-Y(2w + Y)$, their vectors are $(3, 3, 6)$ and $(3, 3, 8)$. The resulting hull has vertices

$$(0,0,0), (0,0,2), (0,1,0), (0,1,4), (1,1,4), (1,2,2), (1,2,5), (3/2,3/2,3), (3/2,3/2,4).$$

It has volume $10/3$, generates the full value lattice, and has centroid

$$\left(\frac{187}{320}, \frac{353}{320}, \frac{41}{16}\right).$$

Thus $S(W;K) = 27/16$ and $S(W^K;P_A) = 187/320$. The total expected degree on the coarse weighted curve is $\frac{1}{2}S(W;K) = 27/32$. At any other point it is therefore at most $83/320$.

The different of K consists of two distinct points of coefficient $1/2$: its quotient point and its intersection with the strict transform of D. Neither is P_A. The two cover points $u = w$ and $u = -w$ map to P_A; they have the same vanishing order because each fixed primary-degree piece has fixed parity in u. Consequently the curve threshold is at least $\min\{320/187, 160/83\}$, while the first ratio is $2/(27/16) = 32/27$. This proves the last asserted bound. □

48 Ordinary double points of the fibres

In this section $p \in X$ is an A_1 point of a fibre of φ. By Lemma 43.2, $p \notin E$, so we identify a neighbourhood of p with its image in V. Let $\sigma : Y \to X$ be the ordinary blow-up of p, with exceptional plane G. We denote the actual refined series on G by W.

Lemma 48.1. *Let $Z \subset G$ be the scheme in Lemma 43.3, and let $q = 0$ be the tangent conic of the fibre. Then q is smooth and contains Z. The initial section algebra of the point blow-up depends only on the pair (Z, q), and*

$$S_X(G) = \frac{39}{16}.$$

Proof. Choose a basis $q_0 = q, q_1, q_2$ of $H^0(\mathcal{I}_Z(2))$, with linear syzygies $l \cdot q = m \cdot q = 0$, where $l = (l_0, l_1, l_2)$ and $m = (m_0, m_1, m_2)$. With suitable signs, the q_i are the maximal minors of these two rows. The initial homogeneous coordinate algebra of V is the Pfaffian algebra of

$$\begin{pmatrix} 0 & h & l_0 & l_1 & l_2 \\ -h & 0 & m_0 & m_1 & m_2 \\ -l_0 & -m_0 & 0 & n_2 & -n_1 \\ -l_1 & -m_1 & -n_2 & 0 & n_0 \\ -l_2 & -m_2 & n_1 & -n_0 & 0 \end{pmatrix}. \tag{48.1}$$

Here $h, U, V, W, n_0, n_1, n_2$ all have H-degree one, and their point weights are $0, 1, 1, 1, 2, 2, 2$. Its equations are $hn_i = q_i$ and $l \cdot n = m \cdot n = 0$. To identify them with initial equations, translate (43.1) at the three representatives above: three independent quadratic equations have initial forms $hn_i - q_i$ of point weight two; the remaining two have point weight three and give the two displayed linear syzygies. More explicitly, in the ordered net of Lemma 43.3, writing ξ_i for its three normal variables, these two forms are

$\dim \operatorname{Aut}(V)p$	first form	second form
1	$-U\xi_2 + V\xi_3$	$V\xi_1 - W\xi_2 - U\xi_3$
2	$2V\xi_1 - U\xi_2 + V\xi_3$	$(V-U)\xi_1 - W\xi_2 - U\xi_3$
3	$W\xi_1 - U\xi_2 + V\xi_3$	$V\xi_1 - W\xi_2 - U\xi_3$.

A change of basis of the net changes only the presentation.

We check that these are all initial equations. Inverting h gives a polynomial algebra in $h^{\pm 1}, U, V, W$. On $h = 0$, the point (U, V, W) lies on the affine cone over Z. Away from the cone vertex the syzygy matrix has rank one, and its kernel in the n_i has dimension two. Over the vertex the kernel has dimension three. Thus the boundary has dimension three, while the algebra has dimension four. The height-three Pfaffian resolution makes it Cohen–Macaulay. Therefore h is a nonzerodivisor, and the algebra injects into its domain localization. The same resolution gives the Hilbert series of V. Inclusion in the initial ideal and equality of Hilbert functions now prove exactness.

Let a, b be the pencil variables and e the exceptional variable, of degrees $F, F, H - F$, respectively. Choose the nonvanishing pencil section with initial form h and the other with initial form n_0. Substituting $h = ae, n_0 = be$ in (48.1) gives the initial Cox algebra. Its remaining variables $U, V, W, n_1, n_2, a, b, e$ have point weights $1, 1, 1, 2, 2, 0, 2, 0$. This substitution also gives a domain. Indeed, (h, n_0) is a regular sequence in the preceding algebra: on a component of $h = 0$ over a point of Z, its kernel in the n_i could be contained in $n_0 = 0$ only if the syzygy row were proportional to $(1, 0, 0)$. The minor expression would then give $dq_0 = 0$ at that point, contrary to smoothness of q_0. The component over the cone vertex gives no exception. Thus the boundary $e = 0$ has dimension four, whereas the substituted Pfaffian algebra has dimension five. The regular sequence (h, n_0) therefore satisfies all hypotheses of Lemma 43.1. That lemma proves primeness and identifies this graph with the full initial algebra of global sections.

The expected order will be computed directly in Lemma 48.3. Alternatively, its value follows by differentiating the weighted Pfaffian Hilbert series: the five equation weights are $2, 2, 2, 3, 3$, and the terminal resolution weight is 6. □

Lemma 48.2. *For the purpose of obtaining a lower bound for $\delta(G; W)$, one may assume that Z is supported at one point and*

$$q = U^2 + VW, \qquad I_Z = (U^2 + VW, V^2, UV).$$

Proof. Regard Z as a divisor of degree three on the smooth conic $q = 0$. Choose a one-parameter subgroup of $\mathrm{Aut}(q)$ whose repelling fixed point is outside Z. Its flat limit is the divisor $3r$ at the attracting fixed point. Every member, including the limit, is a noncollinear curvilinear scheme of length three. Choose bases of the quadratic generators and their linear syzygies over a neighbourhood of the limiting parameter. The Pfaffian construction in Lemma 48.1 has constant weighted bigraded Hilbert function. Hence every piece $W_{m,j}$ has the stated limiting piece as its flat limit; putting $a = e = 1$ embeds these pieces into the homogeneous polynomials of degree j on G.

Fix a valuation over G. In each $W_{m,j}$ choose a basis simultaneously adapted to this valuation and to the weight filtration of the one-parameter subgroup. The initial sections form a basis of the limiting space. Semicontinuity of log canonical thresholds of the corresponding basis divisors shows that a lower bound for the limiting refined threshold is also a lower bound for the original one. This is the same two-filtration argument as in Lemma 2.3. Finally, a change of coordinates on the conic sends $3r$ to the displayed scheme. In the closed-orbit coordinates a smooth member of its net is $U^2 + VW + cUV + bV^2$; replacing U by $U + \frac{c}{2}V$ and W by $W + (b - c^2/4)V$ puts it in the required form without changing the net. □

Lemma 48.3. *In the model of Lemma 48.2, put $Q = [0 : 0 : 1]$ and $T = [0 : 1 : 0]$. The weighted point blow-ups of Q and T with local weights $(1, 2)$ have expected orders, for the refined series W, equal to $45/16$ and $33/16$, respectively. On either exceptional curve the further expected orders at a general point, at the point corresponding to $q = 0$, and at the quotient point are $1/4$, $11/16$, and $1/4$. At the remaining distinguished smooth point they are $23/32$ over Q and $11/32$ over T. Moreover $S_X(G) = 39/16$.*

Proof. The initial algebra is generated, after $a = e = 1$, by

$$U, V, W, V^2, UV \quad \text{of degree } H, \qquad 1, q = U^2 + VW \quad \text{of degree } F,$$

with the homogeneous bookkeeping supplied by e. Use the torus with weights $(1, 2, 0)$ on (U, V, W). In $H^0(X, -mK_X)$ take ordinary point degree J and torus weight K. On $W = 1$, set $z = V/U^2$. The resulting one-variable space is spanned by

$$z^A(1+z)^B, \qquad \begin{cases} A \geq 0,\ B \geq 0,\ A \geq K - J, \\ A + 2B \geq J - m, \quad A + B \geq J - 2m, \\ A - 2B \leq K - 2J + 2m, \quad A \leq K - 2J + 4m, \\ 2A + 2B \leq K. \end{cases} \tag{48.2}$$

Here A, B are integers. To verify the inequalities, write a monomial as

$$U^{K-2A-2B}V^A W^{J-K+A} q^B.$$

Its least H-degree using U, V, W, V^2, UV is $J - 2B - s$, where $s = \min\{A, \lfloor (K - A - 2B)/2 \rfloor\}$ is the largest saving supplied by the last two generators. The conditions are precisely that this degree is at most m and, after adding B, at most $2m$.

Put $j = J/m$, $k = K/m$. The continuous feasible region is the hexagon

$$\Omega = \mathrm{conv}\{(0,0), (1,0), (3,2), (4,4), (4,6), (2,4)\}.$$

The minimum of A/m, the minimum of B/m, and the maximum of $(A + B)/m$ in (48.2) are

$$\alpha = \max\{0, k - j\}, \qquad \beta = \max\left\{0, \frac{3j - k - 3}{4}, 3j - k - 6, \frac{j - 2}{2}\right\}, \qquad d = \frac{k}{2}. \tag{48.3}$$

These follow by eliminating A and B in the displayed inequalities. Consequently the space is contained in $z^{m\alpha+O(1)}(1+z)^{m\beta+O(1)}\mathbb{C}[z]_{\le m\lambda+O(1)}$, where $\lambda = d-\alpha-\beta$. Rounding the endpoints changes its dimension by a bounded number.

Here are the integrals needed below; all are over Ω:

$$\begin{gathered}\int \lambda = \frac{10}{3}, \qquad \int j\lambda = \frac{65}{8}, \qquad \int k\lambda = \frac{75}{8}, \\ \int \lambda(\alpha+\lambda/2) = \frac{115}{48}, \qquad \int \lambda(\beta+\lambda/2) = \frac{55}{24}, \qquad \int \lambda^2/2 = \frac{5}{6}.\end{gathered} \tag{48.4}$$

For an explicit integration, split first along $k=j$, then along the four regions where one of the affine functions defining β is largest. The seven nonempty polygons have vertices

α	vertices
0	$(0,0),(1,0),(3/2,3/2)$
0	$(1,0),(3,2),(7/2,7/2),(3/2,3/2)$
0	$(3,2),(4,4),(7/2,7/2)$
$k-j$	$(0,0),(3/2,3/2),(2,3),(2,4)$
$k-j$	$(3/2,3/2),(7/2,7/2),(4,5),(2,3)$
$k-j$	$(7/2,7/2),(4,4),(4,5)$
$k-j$	$(4,5),(4,6),(2,4),(2,3)$.

On the first three polygons β is respectively $0, (3j-k-3)/4, 3j-k-6$; on the last four it is respectively $0, (3j-k-3)/4, 3j-k-6, (j-2)/2$. Triangulation now gives (48.4). For example, on the first triangle $\alpha=\beta=0$ and $\lambda = k/2$, giving $\int\lambda = 3/16$ and $\int\lambda^2/2 = 9/256$.

These containing spaces have total dimension $(10/3)m^3+O(m^2)$, exactly the leading dimension of $H^0(X,-mK_X)$, since $(-K_X)^3=20$. Their total excess dimension is therefore $O(m^2)$. All one-variable orders are $O(m)$, so this excess changes every normalized first moment by $O(1/m)$, uniformly in the point of the exceptional curve. Thus the integrals compute the actual asymptotic moments, without any assertion that all containing spaces are equal to the actual spaces in finite degree.

Divide (48.4) by $10/3$. This gives the ordinary degree $39/16$, the weight at Q equal to $45/16$, and the expected orders at $z=0,-1$ and a general finite point equal to $23/32, 11/16, 1/4$. The quotient point $z=\infty$ has order $K/2-\deg_z$, so its mean is also $1/4$, since $d=k/2$. At T the local weight is $2J-K$, giving $33/16$. The local polynomial is z^{J-K} times the polynomial at Q. Only the order at zero changes, by $39/16-45/16=-3/8$, giving $11/32$. The other stated orders follow. □

Proposition 48.4. *If p is an A_1 point of a fibre, then $\delta_p(X)\ge 16/15$.*

Proof. By Lemma 48.2, it suffices to bound the refined threshold in the closed-orbit model. We claim $\delta(G;W)\ge 16/15$. Fix $c<16/15$. In each ordinary-degree piece choose bases simultaneously adapted to an arbitrary valuation over G and to the torus with weights $(1,2,0)$. Degenerating by this torus gives a basis divisor D_m whose components are the coordinate lines and the conics $U^2+\eta VW=0$. It suffices to show that (G,cD_m) is log canonical for all sufficiently large m.

At Q, blow up with local weights $(1,2)$. Its log discrepancy is three, and its expected order tends to $45/16$. The exceptional curve is $\mathbb{P}(1,2)$ with different $1/2$ at its quotient point. By Lemma 48.3, the further expected orders at its points are at most $23/32, 11/16, 1/4$, with $1/4$ at the quotient point. Thus adjunction gives log canonicity over Q: indeed $c(45/16)<3$, all smooth-point products are less than one, and $c/4<1/2$. The same argument over T uses $33/16$ in place of $45/16$ and $11/32$ in place of $23/32$.

Distinct invariant conics meet only at Q and T; the same is true of their intersections with $U=0$. The remaining coordinate intersection is $R=[1:0:0]$, where $V=0$ and $W=0$ meet

transversely. Their maximal coefficients tend to $23/32$ and $11/32$, respectively. The coefficient of $U = 0$ is at most twice the quotient-point moment, hence tends to at most $1/2$. All other component coefficients are bounded by the smooth-point moments in the lemma. Therefore (G, cD_m) is log canonical also away from Q and T. Semicontinuity and the adapted basis construction prove $\delta(G; W) \geq 16/15$.

Finally Abban–Zhuang refinement for the ordinary point blow-up gives

$$\delta_p(X) \geq \min\left\{\frac{3}{39/16}, \frac{16}{15}\right\} = \frac{16}{15}.$$

All initial algebras and further moments in this proof came from the actual section filtration. No Zariski decomposition on a threefold obtained by blowing up p was used. □

49 Points of type A_2

Lemma 49.1. *Let p be an A_2 point of a fibre. There are regular parameters z, x, y at p, of extraction weights $2, 3, 3$, such that the leading forms of a basis of $H^0(V, H)$ are*

$$1, z, x, y, z^2, zx, xy - z^3,$$

and the last form defines the fibre. These forms generate the full weighted initial section algebra. More precisely, with homogeneous variables h, z, x, y, P, Q, R, it is given by

$$hP = z^2, \quad hQ = zx, \quad hR = xy - zP, \quad zQ = xP, \quad yQ = zR + P^2. \tag{49.1}$$

The initial Cox algebra of X is obtained by substituting $h = ae$ and $R = be$, where a, b, e have degrees $F, F, H - F$ and weights $0, 6, 0$.

Proof. The tangent conic of the fibre is reduced, with two components. Its vertex r is not in the scheme Z of Lemma 43.3. Indeed, a point of Z is the tangent direction of a line on V through p. Every hyperplane section singular at p contains that line, so its cubic part vanishes in this direction. At an A_2 point the cubic part is nonzero on the radical of the quadratic part, which gives a contradiction.

Thus Z is contained in the two components away from their intersection. Since Z is noncollinear, its lengths on these components are two and one. Let $x = 0$ be the component carrying length two, and let $y = 0$ be the other component. The quadrics containing Z and divisible by x form a pencil. It contains xy and a quadric $x\ell$, where $\ell(r) \neq 0$: the residual linear forms are precisely the lines through the length-one point on $y = 0$. Complete x, y by a linear coordinate z nonzero at r. The two remaining normal hyperplane coordinates can therefore be chosen to have weighted leading forms xz and z^2. For the second assertion use that some quadric of the net is nonzero at r, since $r \notin Z$.

The fibre has local equation $xy + \gamma z^3$ plus terms of weight greater than six, with $\gamma \neq 0$. Rescaling the coordinates gives $xy - z^3$. Together with the three linear coordinates and the constant section, this proves the claimed list of leading forms.

We verify that no further leading generators have been missed. The relations in (49.1) are the five maximal Pfaffians of the skew-symmetric matrix

$$\begin{pmatrix} 0 & h & x & z & 0 \\ -h & 0 & P & y & z \\ -x & -P & 0 & R & Q \\ -z & -y & -R & 0 & P \\ 0 & -z & -Q & -P & 0 \end{pmatrix},$$

with the following signs. Deleting rows and columns $1,\dots,5$ in order gives

$$P^2 - yQ + zR, \quad xP - zQ, \quad hP - z^2, \quad hQ - xz, \quad hR - xy + zP.$$

Thus its maximal Pfaffians generate precisely the displayed ideal. After inverting h they give a polynomial algebra in $h^{\pm 1}, z, x, y$. On $h = 0$ the reduced equations are $z = 0$, $xy = xP = 0$, and $yQ = P^2$; both components have dimension three. Thus the ideal has height three and its Pfaffian resolution is Cohen–Macaulay. No component is contained in $h = 0$, so the algebra is a domain. Its ordinary Hilbert series is $(1 + 3t + t^2)/(1 - t)^4$, the Hilbert series of V.

The algebra generated by the displayed leading forms embeds in the associated graded section algebra. Since both have the same ordinary Hilbert function, they are equal. This dimension argument checks all possible cancellations, not only quadratic relations.

For the Cox algebra, the same argument applies after $h = ae, R = be$. The boundary $h = R = 0$ in (49.1) has dimension two; therefore the boundary $e = 0$ after substitution has dimension four. The complementary open set has dimension five and is a domain. The height-three Pfaffian resolution proves Cohen–Macaulayness. The dimension-two common zero locus says exactly that h, R form a regular sequence. Lemma 43.1 now identifies the substituted algebra with the full initial global section algebra and proves its primeness. □

49.1 The weighted extraction at a point of type A_2

We use the local normal form established in the geometric lemma: after choosing parameters z, x, y of weights $2, 3, 3$, respectively, the initial forms of the seven hyperplane sections span

$$\mathcal{H} = \langle 1, z, x, y, z^2, zx, B\rangle, \qquad B = xy - z^3.$$

The initial form of the singular pencil member is B. This normal form holds on both point orbits on which a normal hyperplane section can have an A_2 singularity; no assumption on the other pencil member is needed besides its being nonzero at the point.

Lemma 49.1 identifies the entire initial algebra. Its anticanonical degree-m piece has dimension

$$N_m = (2m + 1)(5m^2 + 5m + 3)/3.$$

Lemma 49.2. *Let G be the exceptional divisor of the weighted blowup with weights $(2, 3, 3)$ at an A_2 point. Then*

$$A_X(G) = 8, \qquad S_X(G) = \frac{53}{8}, \qquad \delta(G, \Delta_G; W) \geq \frac{16}{15}.$$

Proof. The exceptional divisor is $\mathbb{P}(2, 3, 3)$. Its coarse space is $\mathbb{P}(1, 1, 2)$ with coordinates $x, y, t = z^3$, and $\Delta_G = \frac{2}{3}\{t = 0\}$. The discrepancy is the sum $2 + 3 + 3 = 8$.

In each native-degree piece take the lowest z-power, refining it by any fixed monomial order. The leading term of B is xy. The thirteen distinct degree-one leading monomials are

$$1, z, x, y, z^2, zx, xy, zxy, x^2y, xy^2, z^2xy, zx^2y, x^2y^2.$$

They form a basis of the degree-one space. Their exponent vectors, with coordinates ordered as (z, x, y), generate the full degree-prefixed lattice: the monomials $1, z, x, y$ already suffice. Their convex hull has vertices

$$(0,0,0), (0,0,1), (0,1,0), (0,1,2), (0,2,1), (0,2,2), \ (1,1,0), (1,2,1), (2,0,0), (2,1,1).$$

Its volume and first moment are, respectively,

$$\frac{10}{3}, \qquad \left(\frac{55}{24}, \frac{25}{8}, \frac{65}{24}\right).$$

For an explicit description, this polytope is cut out by $\alpha, \beta, \gamma \geq 0$ and

$$\beta - \gamma \leq 1, \quad \beta \leq 2, \quad \alpha - 2\beta + 2\gamma \leq 2, \quad \alpha + 2\gamma \leq 4, \ \alpha + \beta - \gamma \leq 2, \qquad \alpha + \beta + \gamma \leq 4.$$

Triangulating its boundary and using the determinant formula gives the stated volume and moment. Since the volume is the leading coefficient of N_m, the semigroup-volume lemma shows that the actual asymptotic pivot means for the full refined spaces are

$$\mu_z = \frac{11}{16}, \qquad \mu_x = \frac{15}{16}, \qquad \mu_y = \frac{13}{16}.$$

The native weighted degree is $2\alpha + 3\beta + 3\gamma$. Thus this also computes the expected vanishing of the primary extraction directly: $S_X(G) = 2\mu_z + 3\mu_x + 3\mu_y = 53/8$.

We spell out the log-canonicity step because the exceptional weighted projective space is not well formed. In a fixed finite degree, choose a one-parameter subgroup which realizes the chosen monomial order on all monomials occurring in that degree. In every native-degree piece choose a basis simultaneously adapted to this weight filtration and an arbitrary divisorial valuation over G. Its limit basis consists of the pivot monomials. On the coarse surface the limiting basis divisor therefore has asymptotic coefficients

$$\frac{11}{48}\{t = 0\} + \frac{15}{16}\{x = 0\} + \frac{13}{16}\{y = 0\}.$$

Together with the different $\frac{2}{3}\{t = 0\}$, this toric boundary is log canonical after multiplication of the basis divisor by any $c < 16/15$: the three coefficient conditions are $\frac{2}{3} + c\frac{11}{48} \leq 1$, $c\frac{15}{16} \leq 1$ and $c\frac{13}{16} \leq 1$. They can also be checked on the index-two cover at the vertex of $\mathbb{P}(1, 1, 2)$; elsewhere the toric boundary has simple normal crossings. Log-canonical-threshold semicontinuity and the simultaneous basis construction give the same bound for the original valuation. Passing to the asymptotic threshold and then letting c increase to $16/15$ proves the assertion. □

Corollary 49.3. *At every singular point of a fibre of type A_2 one has $\delta_p(X) \geq 16/15 > 1$.*

Proof. Abban–Zhuang gives $\delta_p(X) \geq \min\{8/(53/8), 16/15\} = 16/15$. □

Example 49.4. The binomial $B = xy - z^3$ has native weight six, although it is a single hyperplane section before passing to initial forms. The section Bzx has native weight eleven, and its leading monomial is zx^2y. Both lie in anticanonical degree one. These examples explain why the native weight and the anticanonical degree must be kept separate in the preceding computation. The full-volume check is what certifies that products of the displayed sections account for the leading asymptotic dimension, including all possible cancellations.

50 Points of type A_3

50.1 The weighted initial algebra at an A_3 point

Lemma 50.1. *Let p be an A_3 point of a fibre. There are regular parameters u, v, w, with weights $2, 2, 1$, such that the leading hyperplane sections are*

$$1, u, v, w, P = vw, Q = v(u + v), B = u^2 + uv - vw^2,$$

and B is the leading equation of the fibre. The full initial homogeneous coordinate algebra of V is

$$hP = vw, \quad hQ = v(u+v), \quad hB = u^2+uv-Pw, \quad vB-uQ+P^2 = 0, \quad P(u+v)-wQ = 0. \tag{50.1}$$

The initial Cox algebra is obtained by $h = ae, B = be$, *with* $\deg(a) = \deg(b) = F$, $\deg(e) = H - F$ *and extraction weights* $0, 4, 0$. *If* G *is the exceptional divisor, then*

$$A_X(G) = 5, \qquad S_X(G) = \frac{71}{16}.$$

Proof. By Lemma 43.4, the image of p belongs to the two-dimensional orbit. In the translated chart used there the fibre has equation

$$u^2 + (b+1)uv + bv^2 + bu^3 + buvw - u^2w - vw^2, \qquad b \neq 1.$$

With weights $(2, 2, 1)$ its initial form is $B_b = u^2 + (b+1)uv + bv^2 - vw^2$. The other two normal hyperplane coordinates have initial forms vw and $v(u+v)$. Put $c = (1-b)^{-1}$ and make the weighted linear change $u' = cu + (c-1)v$, $v' = v$, $w' = cw$. It gives $P' = cP$, $Q' = cQ$ and $B'_0 = c^2 B_b$. Thus it identifies the complete space of leading hyperplane sections, together with its distinguished fibre section, with the displayed model.

The equations (50.1) are the five maximal Pfaffians of

$$\begin{pmatrix} 0 & h & u+v & -w & 0 \\ -h & 0 & -P & u & -v \\ -(u+v) & P & 0 & B & -Q \\ w & -u & -B & 0 & P \\ 0 & v & Q & -P & 0 \end{pmatrix}.$$

After inverting h they give the polynomial algebra in $h^{\pm 1}, u, v, w$. On $h = 0$, the branch $v \neq 0$ has $w = 0$, $u = -v$ and $vB + vQ + P^2 = 0$, so its closure has dimension three. On the branch $v = 0$, the remaining equations in u, w, P, Q are $u^2 = Pw$, $uQ = P^2$, $uP = wQ$. They define an affine cone over a twisted cubic; B is free, and this branch also has dimension three. The Pfaffian ideal has height three and its resolution is Cohen–Macaulay. Thus h is regular and the algebra is a domain. Its Hilbert series is $(1 + 3t + t^2)/(1-t)^4$. The algebra generated by the displayed leading hyperplane sections embeds in the associated graded section algebra. Equality of their Hilbert functions proves equality of the algebras. In particular, this argument accounts for every cancellation among sections.

The substituted Cox algebra is also a domain. On $e \neq 0$ this follows from the preceding paragraph. The boundary $h = B = 0$ in (50.1) has dimension two; hence after substitution $e = 0$ has dimension four, whereas the whole algebra has dimension five. In particular h, B are a regular sequence in the primary initial algebra. Lemma 43.1 proves that e is regular, that the graph is a domain, and that its divisor-degree components give the actual full initial global section algebra.

Here is the weight calculation separately. Let t record H-degree, r record F-degree, and s record extraction weight. The weighted Hilbert series is

$$\frac{1 - t^2 \sum_{d \in \mathcal{D}} s^d + t^3 \sum_{d \in \mathcal{D}} s^{11-d} - t^5 s^{11}}{(1-ts)(1-ts^2)^2(1-ts^3)(1-ts^4)(1-r)(1-rs^4)(1-t/r)}, \qquad \mathcal{D} = (3, 4, 4, 5, 6).$$

The coefficient of $t^m r^m$ at $s = 1$ is $N_m = (2m+1)(5m^2 + 5m + 3)/3$. Differentiating in s and then setting $s = 1$ gives the total weight

$$T_m = \frac{m(355m^3 + 510m^2 + 377m + 102)}{24}.$$

For example, this coefficient can be checked by summing over the exponent i of e: a numerator term $\varepsilon t^a s^d$ contributes

$$\varepsilon \sum_{i=0}^{m-a} \binom{m-a-i+4}{4} (m+i+1) \left(\frac{12}{5}(m-a-i) + 2(m+i) + d \right).$$

It follows that $S_X(G) = \lim T_m/(mN_m) = 71/16$. The discrepancy is $2 + 2 + 1 = 5$. □

50.2 The refined series at a point of type A_3

We use the weighted initial model from the preceding lemma. Thus $G = \mathbb{P}(2,2,1)$ has weighted coordinates u, v, w, and its coarse space is $\mathbb{P}^2$ with coordinates $u, v, z = w^2$. The different is $\Delta_G = \frac{1}{2}\{z = 0\}$. Put

$$P = vw, \qquad Q = v(u+v), \qquad B = u^2 + uv - vw^2.$$

The refined section algebra is generated, in anticanonical degree one, by the polynomials in $\mathcal{H} + B\mathcal{H}$, where $\mathcal{H} = \langle 1, u, v, w, P, Q, B\rangle$. Their native weighted degrees are respectively $0, 2, 2, 1, 3, 4, 4$. Write $W_{m,j}$ for the native-degree j piece in anticanonical degree m. As before, $N_m = (2m+1)(5m^2+5m+3)/3$.

The parameter in the A_3 normal form does not affect this model. Indeed, for $B_b = u^2 + (b+1)uv + bv^2 - vw^2$, where $b \neq 1$, put $a = (1-b)^{-1}$ and $u' = au + (a-1)v$, $v' = v$, $w' = aw$. Direct substitution gives $P' = aP$, $Q' = aQ$ and $B'_0 = a^2 B_b$. Thus the weighted change of coordinates identifies the corresponding section algebras, including all their graded pieces.

Lemma 50.2. *Let $D = \{B = 0\} \subset G$, and let $q = [0:0:1] \in D$. Then*

$$S(W; D) = \frac{11}{16}, \qquad S(W^D; q) = \frac{19}{16}, \qquad S(W; \{z = 0\}) = \frac{7}{32}.$$

Proof. On $w = 1$, write $b = B$. Then $v = (u^2 - b)/(1-u)$. Multiplication by the common unit $(1-u)^{2m}$ clears the denominators of all degree-m sections. The flag $D \supset q$ therefore records the lowest b-power and then the lowest u-power. With coordinates (α, β, j) recording the u-order, the b-order, and the native degree, respectively, the degree-one pivot vectors are

$$(0,0,0), (0,0,1), (0,1,4), (0,1,5), (0,2,8),$$
$$(1,0,2), (1,1,6), (2,0,2), (2,0,3),$$
$$(2,1,6), (2,1,7), (3,0,4), (3,1,8).$$

For example $Q|_D = u^3/(1-u)^2$, which gives $(3,0,4)$, and BQ gives $(3,1,8)$. These thirteen vectors are distinct, so these leading terms give a basis in degree one. Their convex hull has vertices

$$(0,0,0), (0,0,1), (0,1,5), (0,2,8), (2,0,2), (2,1,6), (3,0,4), (3,1,8).$$

Its volume is $10/3$, its first moment is $(95/24, 55/24, 355/24)$, and its centroid is

$$(19/16, 11/16, 71/16).$$

The degree-prefixed vectors generate the full lattice: use $1, w, u, B$. The semigroup-volume lemma identifies these with the actual asymptotic moments, since $N_m = (10/3)m^3 + O(m^2)$. This proves the first two claims.

For the boundary, take the lowest w-power and then the highest u-power within each native-degree piece. The degree-one leading terms, together with the degree-two sections $Q^2 - Bv^2$ and $B(Q^2 - Bv^2)$, give a convex hull with vertices

$$(0,0,0), (0,0,1), (0,1,0), (0,1,1), (2,0,1),$$
$$(2,1,1), (3,1,0), (4,0,0), (\tfrac{1}{2}, \tfrac{3}{2}, 0), (\tfrac{3}{2}, \tfrac{3}{2}, 0).$$

Here the coordinates are the normalized exponents of (u, v, w). The two new vertices follow from $Q^2 - Bv^2 = v^3(u + v + w^2)$. The hull has volume $10/3$ and centroid $(453/320, 187/320, 7/16)$. Its w-moment is therefore $7/16$. On the coarse surface, the divisor of the weighted section w is $\frac{1}{2}\{z = 0\}$, giving $S(W; \{z = 0\}) = 7/32$. All the volume and moment assertions are obtained by triangulating these explicitly listed rational polyhedra and applying the determinant formula for the volume and centroid of a tetrahedron. □

Lemma 50.3. *One has* $\delta_q(G, \Delta_G; W) \geq 48/41 > 1$.

Proof. The point q is smooth and lies outside Δ_G. Blow it up with weights $(1,2)$ in the coordinates (u,v), and denote the exceptional curve by $K = \mathbb{P}(1,2)$. Its log discrepancy is 3, and its different has coefficient $1/2$ at the point $u = 0$. Put $y = v - u^2$ and let $p \in K$ be the ordinary point $y = 0$. Take the lowest $(1,2)$-weight and then the lowest y-power. The degree-one pivot vectors (α, β, j) are exactly the first thirteen vectors in the preceding proof. Indeed the initial terms of v, P, Q, B are respectively $u^2 + y, u^2 + y, u(u^2 + y), -y$. The same full-volume certificate gives

$$S(W; K) = \frac{19}{16} + 2\frac{11}{16} = \frac{41}{16}, \qquad S(W^K; p) = \frac{11}{16}.$$

The expected degree of the residual divisors on K is $41/32$. For any ordinary point $r \neq p$, choose a basis simultaneously adapted to p and r in every graded piece. The sum of their multiplicities is at most the degree; hence $S(W^K; r) \leq 19/32$.

At the orbifold point use instead the lowest u-power after taking the $(1,2)$-initial form. The degree-one pivot vectors are

$$(0,0,0), (0,0,1), (0,1,2), (0,1,3), (0,1,4), (0,1,5),$$
$$(0,2,6), (0,2,7), (0,2,8), (1,0,2), (1,1,4), (1,1,6), (1,2,8).$$

Their hull again has volume $10/3$, with centroid

$$(7/16, 17/16, 71/16).$$

Since u^2/v is a coarse uniformizer at $u = 0$, this gives $S(W^K; \{u = 0\}) = 7/32 < 1/2$. Consequently

$$\delta(K, \mathrm{Diff}_K; W^K) \geq \min\{16/11, 32/19, 16/7\} = 16/11.$$

Applying Abban–Zhuang gives the required bound $\min\{3/(41/16), 16/11\} = 48/41$. □

Lemma 50.4. *One has* $\delta(G, \Delta_G; W) \geq 1$.

Proof. Let H denote the line class on the coarse surface $G = \mathbb{P}^2$. The expected class of a basis divisor is $(71/32)H$, and $D \sim 2H$. On D, the expected degree after restriction is $2(71/32 - 2 \cdot 11/16) = 27/16$. At every point other than q, simultaneous bases with the q-filtration therefore give a refined S-invariant at most $27/16 - 19/16 = 1/2$. The different of D has coefficient $1/2$ at its two intersections with $\{z = 0\}$, and is zero elsewhere. Thus the refined threshold is at least one on $D \setminus \{q\}$; the preceding lemma treats q. Since $S(W; D) = 11/16 < 1$, Abban–Zhuang proves the assertion along D.

For completeness, consider a point outside D. Choose a line C through it; if the point lies on $\{z = 0\}$, take this boundary line itself. In each C-graded piece choose a basis simultaneously adapted to the induced D-filtration and the chosen point. These bases lift to bases adapted to both original divisor filtrations. In particular the sum of their induced D-weights is the original D-weight sum. Every residual divisor contains its corresponding multiple of $D|_C$, whose support avoids the point under consideration. If $s = S(W; C)$, its point invariant is consequently at most

$$\frac{71}{32} - s - 2\frac{11}{16} = \frac{27}{32} - s \leq \frac{27}{32}.$$

Before restriction, a common basis for C and D gives an effective averaged residual class $(27/32 - s)H$. Hence $s \leq 27/32$. For $C \neq \{z = 0\}$ this is below its discrepancy 1. For the boundary line its discrepancy is $1/2$ and its invariant is $7/32$, by Lemma 50.2; the restriction bound is then $5/8 < 1$. The induced different on this boundary line is zero. Abban–Zhuang proves the assertion off D as well. □

Remark 50.5. The last lower bound need not be made strict. The primary extraction has $A_X(G) = 5$ and $S_X(G) = 71/16$, so its discrepancy margin is $9/16$. The strict form of the Abban–Zhuang argument therefore gives $\beta_X(F) \geq (9/16)\,\mathrm{ord}_F(G) > 0$ for every divisorial valuation centred at the original A_3 point.

51 The point of type A_4

We prove the following local estimate.

Theorem 51.1. *If p is an A_4 singular point of a fibre, then $\delta_p(X) \geq 320/307$.*

51.1 The actual section filtration

We use the standard equations of V_5, written as

$$\begin{gathered} vw - x_0P + u^2 = 0, \qquad vR - uQ + P^2 = 0, \qquad v^2 - x_0Q + uP = 0, \\ uv - x_0R - Pw = 0, \qquad Pv - Qw - uR = 0. \end{gathered} \tag{51.1}$$

At the point $[x_0 : u : P : Q : v : w : R] = [1 : 0 : 0 : 0 : 0 : 0 : 0]$, the affine coordinates u, v, w give

$$P = u^2 + vw, \qquad Q = u^3 + uvw + v^2, \qquad R = uv - u^2w - vw^2. \tag{51.2}$$

The orbit and tangent-hyperplane calculation identifies an A_4 fibre with a hyperplane $R+\gamma Q = 0$ at this point. Its other pencil coordinate is a linear form ℓ nonzero at the point; after rescaling, $\ell = x_0+$ terms vanishing at the point. Put

$$x = u - w^2, \qquad y = v - uw - w^3, \qquad z = w.$$

Then $R = xy - z^5$. We extract the divisor G with weights $(2, 3, 1)$ in u, v, w. The same weights in x, y, z show that this is the usual weighted extraction for this A_4 equation. Its exceptional surface is $G = \mathbb{P}(2, 3, 1)$, its different is zero, and $A_X(G) = 6$. Put $M = \mathcal{O}_G(1)$; thus $M^2 = 1/6$.

Lemma 51.2. *The actual point-refined series on G is the series of native-weight components of*

$$V_m = \langle P_1 \cdots P_h R^q : \ P_i \in \{u, P, Q, v, w\},\ h \leq m,\ h + q \leq 2m \rangle\,,$$

where the polynomials are those of (51.2). Here 1 is allowed by taking $h = q = 0$, and the native weights of (u, P, Q, v, w, R) are $(2, 4, 6, 3, 1, 5)$. Moreover

$$N_m = \dim V_m = \frac{(2m + 1)(5m^2 + 5m + 3)}{3}.$$

Proof. Use the global section graph of Lemma 43.1, with variables a, b, e of divisor degrees $H - E, H - E, E$, respectively. For its candidate initial algebra replace x_0, R in (51.1) by ae, be, and denote the resulting ring by $\mathcal{R}$. On the open set $ae \neq 0$ its five equations are the polynomial graph (51.2). The locus $e = 0$ is the affine cone over $V_5 \cap \{x_0 = R = 0\}$, with the two free variables a, b; it has dimension four. The locus $a = 0, e \neq 0$ also has dimension four. The Pfaffian ideal therefore has codimension three. Its Pfaffian resolution makes the quotient Cohen–Macaulay, and the boundary has smaller dimension than the open graph. Thus $\mathcal{R}$ is a domain.

Give a, e weight zero and b weight five. The five Pfaffians have weights $4, 8, 6, 5, 7$. Replacing x_0 by ℓ and R by $R + \gamma Q$ changes these equations only by terms of larger weight. The initial Pfaffian ring has the same bigraded Hilbert series as the original Cox presentation, by their codimension-three Pfaffian resolutions. Hence it is the entire associated graded ring.

Localization at ae is injective on it and on each divisor-degree component; it consequently gives the geometric point filtration, including all cancellations.

A Cox monomial of degree mL with h factors among the five H variables and exponent q in b has exponents $2m-h-q$ in a and $m-h$ in e. This gives exactly the two displayed inequalities. After setting $a=e=1$, the degree-one generators are

$$\{1,u,P,Q,v,w,R\}\cup R\{1,u,P,Q,v,w,R\}.$$

The repetition of R leaves thirteen generators. Every allowed monomial factors into m such generators: put at most one R alongside each of its h H factors and at most two R factors in each remaining slot. Finally Riemann–Roch, or the same Pfaffian resolution, gives the displayed formula for N_m. □

Lemma 51.3 (The first expected order)**.** *One has* $S_X(G)=45/8$.

Proof. Here is a separate Hilbert calculation. If s_H is the sum of the weights of the five H variables, $s_{ab}=w_a+w_b$, and D is the last syzygy weight, the Pfaffian resolution has weight numerator

$$1-\sum_{i=1}^{5}t^{2H}q^{d_i}+\sum_{i=1}^{5}t^{3H}q^{D-d_i}-t^{5H}q^{D},\qquad \sum d_i=2D.$$

Differentiate with respect to q and take the coefficient of degree mL. Expansion of the eight-variable geometric series gives

$$\sum\text{weights}=\frac{5}{24}(5s_H+11s_{ab}+6w_e-3D)m^4+O(m^3).$$

For clarity, the monomials before taking the Pfaffian alternating sum are counted by $\sum_{h=m}^{s}(h+1)\binom{s-h+4}{4}$ in divisor degree $(s,-m)$; their weight sum is obtained by multiplying each summand by $s_H(s-h)/5+s_{ab}h/2+w_e(h-m)$. Taking the shifts $0,2H,3H,5H$, with the relation and syzygy weights above, gives the displayed leading coefficient. Since $N_m=(10/3)m^3+O(m^2)$, the normalized trace is

$$\frac{5s_H+11s_{ab}+6w_e-3D}{16}.$$

In our case $(s_H,s_{ab},w_e,D)=(16,5,0,15)$. More explicitly, the finite weight sum is

$$\frac{75}{4}m^4+\frac{55}{2}m^3+\frac{81}{4}m^2+\frac{11}{2}m,$$

whose quotient by mN_m tends to $45/8$. Lemma 51.2 identifies this trace with the actual expected order. □

51.2 Two computations on curves

Write $B=\{R=0\}\subset G$ and $Q_0=[0:0:1]$. The change to x,y,z above identifies B with $\{xy=z^5\}$. Its coarse curve is $\mathbb{P}^1$, and its different has coefficients $1/2,2/3$ at its two points on $z=0$. It is purely log terminal in G. On $w=1$, putting $b=R$ gives

$$v=\frac{u^2+b}{u-1},\qquad P|_B=\frac{u^3}{u-1},\qquad Q|_B=\frac{u^5}{(u-1)^2}.$$

All denominators here are units at Q_0.

Lemma 51.4. *One has*

$$S(W;B)=\frac{11}{16},\qquad S(W^B;Q_0)=\frac{25}{16}.$$

At every other point $q\in B$, one has $S(W^B;q)\le 25/96$.

Proof. Record the successive orders (a, q, j) in u, b, and the native degree j, taking the b order first. The following degree-one sections give the vertices of a convex hull:

section	1	w	Rw	R^2	Q	RQ
(a,q,j)	$(0,0,0)$	$(0,0,1)$	$(0,1,6)$	$(0,2,10)$	$(5,0,6)$	$(5,1,11)$.

Multiplying by the common unit $(u-1)^{2m}$ in degree m verifies these orders without changing either filtration. The hull has halfspaces

$$a, q \geq 0, \quad -a-5q+j \leq 1, \quad -a-4q+j \leq 2, \quad 6a+25q-5j \leq 0.$$

Its volume is $10/3$ and its first moment is $(125/24, 55/24, 75/4)$. These numbers can be checked by coning to the origin the boundary triangles

$$(1,4,0),(5,2,3),(2,1,0),(2,3,0),(5,3,0),(5,4,0),(5,2,1),(5,1,4),$$

where the six vertices are numbered in the order of the table. The sections $1, w, u, R$ show that, after adjoining the anticanonical degree, the generated lattice is $\mathbb{Z}^4$. The full-volume lemma, Lemma 34.2, applies: the hull volume equals the leading coefficient of N_m. Thus the centroid $(25/16, 11/16, 45/8)$ gives the actual moments, proving the two equalities.

The total expected degree of the residual zero divisors on B is

$$B \cdot M\left(S_X(G) - 5S(W;B)\right) = \frac{5}{6}\left(\frac{45}{8} - \frac{55}{16}\right) = \frac{175}{96}.$$

In each graded piece choose a basis compatible with the orders at Q_0 and q. The sum of these two orders is at most the degree of each zero divisor. Hence $S(W^B; q) \leq 175/96 - 25/16 = 25/96$. □

Lemma 51.5. *Blow up $Q_0 \in G$ with weights $(1,2)$ in the local coordinates u, v, and denote the exceptional curve by K. Then*

$$A_G(K) = 3, \qquad S(W;K) = \frac{41}{16}, \qquad \delta(K, \mathrm{Diff}; W^K) \geq \frac{320}{307}.$$

Proof. The curve is $K = \mathbb{P}(1,2)$, with different $\frac{1}{2}[0:1]$. Put $y = v + u^2$ on the chart $w = 1$, and let $P_0 \in K$ be $y = 0$. Order a monomial $u^a y^b$ by $(a + 2b, b)$. Keeping native degree j as the third record gives the actual flag $K \supset P_0$.

Here are all additional leading terms needed for a full-volume certificate. The first thirteen sections have anticanonical degree one, and the last two have degree two:

section	(a,b,j)	section	(a,b,j)
1	$(0,0,0)$	w	$(0,0,1)$
P	$(0,1,4)$	$-R$	$(0,1,5)$
$-Rw$	$(0,1,6)$	$-PR$	$(0,2,9)$
R^2	$(0,2,10)$	u	$(1,0,2)$
Q	$(1,1,6)$	$-Ru$	$(1,1,7)$
$-RQ$	$(1,2,11)$	$-v$	$(2,0,3)$
Rv	$(2,1,8)$	$Q^2 - PRv$	$(0,3,12)$
$R(PRv - Q^2)$	$(0,4,17)$		

For example the K-initial forms of P, Q, R are $y, uy, -y$, respectively; those of the last two sections are y^3, y^4. These identities follow by direct substitution of (51.2); thus the table includes genuine cancellation sections, not a guessed initial presentation.

The convex hull of the degree-normalized vectors has nine vertices

i	0	1	2	3	4	5	6	7	8
a	0	0	0	0	1	2	2	0	0
b	0	0	1	2	2	1	0	$3/2$	2
j	0	1	6	10	11	8	3	6	$17/2$.

Its halfspaces are

$$a, b \geq 0, \quad b \leq 2, \quad -a - 5b + j \leq 1, \quad -a - 4b + j \leq 2,$$
$$3a + 8b - 2j \leq 0, \quad 5a + 11b - 2j \leq 5, \quad 9a + 20b - 4j \leq 6.$$

For an explicit integration certificate, cone to the origin its boundary triangles

$$(6,7,0), (8,4,5), (8,4,3), (1,6,0), (2,4,5), (2,4,3),$$
$$(8,6,7), (8,6,5), (1,8,7), (1,7,0), (1,2,3), (1,8,3), (1,2,5), (1,6,5).$$

The determinant formula gives volume $10/3$ and first moment $(103/48, 307/96, 75/4)$. The lattice is already generated by $1, w, u, P$, with anticanonical degree prepended. Lemma 34.2 therefore gives the exact centroid

$$(\mu_a, \mu_b, \mu_j) = \left(\frac{103}{160}, \frac{307}{320}, \frac{45}{8}\right).$$

In particular $S(W; K) = \mu_a + 2\mu_b = 41/16$ and $S(W^K; P_0) = 307/320$.

The total expected degree on K is $\frac{1}{2}S(W; K) = 41/32$. Simultaneous bases at P_0 and any other point give expected order at most $41/32 - 307/320 = 103/320$ at the latter. The discrepancy there is at least $1/2$, while P_0 has discrepancy one. Thus the refined threshold is at least $\min\{320/307, 160/103\} = 320/307$. □

Example 51.6. The two degree-two cancellation sections in Lemma 51.5 are essential. A computation using only the displayed Pfaffian weights for the final point filtration gives the incorrect number $15/16$ at P_0. The actual value is $307/320$. The full-volume certificate is what rules out further changes to this value from unrecorded cancellations.

51.3 The complement of the tangent curve

The following elementary observation is useful here because a direct alpha-invariant estimate on $G \setminus B$ is too weak.

Lemma 51.7 (Retaining zeros away from the point)**.** *Let $T \neq B$ be a prime curve on G, with $T \sim dM$, and let $p \in T \setminus B$. Suppose that T is smooth at p and that (G, T) is purely log terminal there with zero different at p. Then the point term of the refinement by T satisfies*

$$S(W^T; p) \leq \frac{d}{6}\left(\frac{35}{16} - dS(W; T)\right).$$

Proof. In each piece of the T-associated graded series put the filtration induced by the subspaces divisible by R^q. Taking associated graded for the T filtration preserves the dimensions of each of these subspaces. Consequently the sum of their induced R weights is precisely the original B weight. Choose a basis of this graded piece compatible with both the induced R filtration and the point-order filtration at p, and lift it with the indicated T and R orders. Since $T \neq B$, the T-initial form of a section divisible by R^q is divisible by $(R|_T)^q$. All these zeros lie away from p.

The expected degree of the zero divisors on T before removing these zeros is $M \cdot T\,(S_X(G) - dS(W;T))$. Their expected degree on $B \cap T$ is at least $B \cdot T\, S(W;B)$. Subtract and use $M^2 = 1/6$, $B \sim 5M$ and Lemmas 51.3 and 51.4. This gives the assertion. The argument uses two filtrations in each already restricted graded piece; it makes no assertion about a basis compatible with three unrelated filtrations on the original section space. □

Lemma 51.8. *Put $T_c = \{u - cw^2 = 0\}$ for $c \in \mathbb{C}$, and $T_\infty = \{w = 0\}$. Then*

$$S(W;T_c) = \begin{cases} 5/16, & c = 0, \\ 5/32, & c \neq 0, \end{cases} \qquad S(W;T_\infty) = 5/16.$$

Every point of $G \setminus B$ has local refined threshold greater than one.

Proof. For $c \neq 0$ adjoin the homogeneous cancellation section $r = aeu - cw^2$ to $\mathcal{R}$. It has divisor degree $2H$ and native weight two. The quotient $\mathcal{R}/(r)$ is a domain: on $ae \neq 0$ eliminate u to obtain a polynomial graph, while on $ae = 0$ its support forces $w = u = v = P = 0$ and has dimension three. The quotient is Cohen–Macaulay of dimension four, so this boundary contributes neither a component nor an embedded component. Its open graph is reduced. The prime principal ideal (r) and its powers are saturated with respect to ae. Thus localization identifies the actual T_c filtration with the r-adic filtration, whose associated graded is $\mathcal{R}/(r)[r_{\mathrm{new}}]$. Its normalized trace is $S_X(2H) = 5/32$.

The same argument applies to the prime elements u and w, both of divisor degree H. For $u = 0$, the support on $ae = 0$ additionally satisfies $v = P = 0$ and $Qw = 0$, and again has dimension three; for $w = 0$ it satisfies $u = v = P = 0$ and has dimension three. This proves the required saturation and gives trace $S_X(H) = 5/16$. These are computations with actual principal filtrations of the Cox ring.

Every point with $w \neq 0$ belongs to one of the T_c. This curve is smooth there and has zero different. Lemma 51.7 bounds its point term away from B by $5/8$ if $c \neq 0$, and by $25/48$ if $c = 0$. Its first term is $32/5$ or $16/5$. The remaining points outside B lie on T_∞. Its two orbifold points belong to B, so the different vanishes at the points under consideration. The same lemma gives point term at most $5/16$, while its first term is $16/5$. Abban–Zhuang adjunction proves the last assertion. □

Proof of Theorem 51.1. The curve B has first ratio $16/11$. Away from Q_0, Lemma 51.4 bounds its point terms by $25/96$, whereas the discrepancies are at least $1/3$. At Q_0, Lemma 51.5 gives ratio at least $\min\{48/41, 320/307\} = 320/307$. Lemma 51.8 covers the complement of B, with stronger bounds. Hence $\delta(G;W) \geq 320/307$.

All series have bounded support, as is explicit in Lemma 51.2. The first series contains an ample series: every weighted polynomial of native degree $6k$ occurs when $m = 12k$, giving the complete series $H^0(G, \mathcal{O}_G(6k))$. The subsequent series are the actual purely log terminal refinements, so the bounded-support and ample-series assertions follow as in [4, Example 2.15]; the displayed full-dimensional certificates also give nonvanishing on interior rays. The different and purely log terminal conditions were checked above. Proposition 2.7 therefore applies and gives

$$\delta_p(X) \geq \min\left\{\frac{6}{45/8}, \frac{320}{307}\right\} = \frac{320}{307}.$$

No Zariski decomposition on the weighted blowup of the threefold has been used. The first filtration is identified by a prime Pfaffian initial ring; the further moments come from actual sections, and the estimates on the complementary locus retain their zeros on a second curve. □

52 Completion of the proof for Family 2.14

Proof of the Family 2.14 assertion of Theorem 1.1. Suppose that a prime divisor V over X has $\beta_X(V) \le 0$. Proposition 44.5 reduces its centre to a listed line, a point of such a line, or the node of a fibre of type A_1.

We first exclude every singular point of every fibre. Proposition 48.4, Corollary 49.3, and Theorem 51.1 give strict local threshold bounds at points of types A_1, A_2, A_4. At a point of type A_3, Lemmas 50.1 and 50.4 give $A_X(G) - S_X(G) = 9/16$ and refined threshold at least one. Lemma 43.7 therefore gives

$$\beta_X(V) \ge \frac{9}{16}\,\mathrm{ord}_V(G) > 0$$

for every valuation centred at that point. The list in Lemma 43.2 contains no other singularity type.

It remains to exclude the listed lines and their smooth points on the fibre. Every such line meets the singular locus of its fibre, as follows from the same incidence configurations used in Proposition 44.5. A line on V_5 has normal bundle $\mathcal{O} \oplus \mathcal{O}$ or $\mathcal{O}(1) \oplus \mathcal{O}(-1)$; these are the ordinary lines and lines of the second kind, respectively, by Lemma 43.5. For an ordinary line the pencil differential consequently has rank one. Lemmas 45.2 and 45.3 exclude the line centre and every point on it where the fibre is smooth.

For a line of the second kind use its standard model $u = v = 0$. A hyperplane containing it has equation

$$\alpha x_1 + \beta x_2 + \gamma x_3 + \delta x_4 + \varepsilon x_6 = 0.$$

If $\alpha \neq 0$, its differential in the positive normal direction never vanishes on the line, so the hyperplane section is smooth along it. This is impossible for a listed line. Hence $\alpha = 0$. If $\delta = \beta = 0$, normality of the fibre forces $\varepsilon \neq 0$, and its closed-orbit point is of type A_4. Proposition 46.5 treats the rest of that line. If $\delta = 0$ and $\beta \neq 0$, the closed-orbit point is a node, and Proposition 46.6 treats the rest. Finally, if $\delta \neq 0$, Proposition 47.3 treats every smooth point of the fibre on the line, including its intersection with E. These three cases exhaust the lines of the second kind. For a valuation whose centre is the whole line, choose a general smooth point of the fibre on that line and apply the corresponding local bound.

Every centre allowed by the reduction has now been excluded. Therefore $\beta_X(V) > 0$ for every prime divisor over X, and the valuative criterion proves K-stability. □

The different is retained throughout the line and point calculations. For example, the two positions of $C \cap Z$ on a line of the second kind lead to different local section lists, and hence to different expected orders at the end fibres. Likewise, the A_3 calculation gives a refined threshold of one; the positive term $9\,\mathrm{ord}_V(G)/16$ is essential in the last step.

No Zariski decomposition on a threefold obtained by a weighted point or curve extraction is used in this part. The elementary volume calculations occur on X itself. For every extraction, a prime initial ring or a full-volume certificate identifies the actual section filtration. The computations include the additional sections obtained by cancellation. Restriction to a curve then uses its actual degrees and the zeros retained on a second divisor. These arguments supply the vanishing estimates that a threefold Zariski decomposition would otherwise have to provide.

Part IX
Family 2.16: conics on intersections of two quadrics

53 The conic bundle and the remaining fibres

Let $V \subset \mathbb{P}^5$ be a smooth intersection of two quadrics, let $C \subset V$ be a smooth conic, and let $\pi\colon X = \mathrm{Bl}_C V \to V$. These are the smooth members of Family 2.16. Put $H = \pi^*\mathcal{O}_V(1)$, denote the exceptional divisor by E, and set $B = H - E$. Projection from the plane $\Pi = \langle C \rangle$ gives the conic bundle $g\colon X \to \mathbb{P}^2$ defined by $|B|$. Thus $L = -K_X = H + B = 2H - E$ and $L^3 = 22$.

The estimates for reduced fibres are already available in [28, Propositions 5.2.3, 5.2.4 and 5.2.6]. The remaining fibres are double lines. We blow up their scheme-theoretic fibre ideals, obtaining an exceptional surface $F \cong \mathbb{P}^1 \times \mathbb{P}^1$ with different $D/2$, where D is a smooth curve of bidegree $(2,1)$. The new calculation identifies its refined sections with powers of the ideal of one point on D. This is simpler than the length-two condition in Family 2.13, but the different must still be retained.

Theorem 53.1. *Every smooth member X of Family 2.16 satisfies $\delta(X) \geq 176/171$. More precisely, if p lies on a nonreduced conic fibre, then $\delta_p(X) \geq 88/85$.*

Lemma 53.2 (Intersection numbers and three expected orders)**.** *The intersection numbers are $H^3 = 4$, $H^2E = 0$, $HE^2 = -2$, and $E^3 = -2$. Moreover,*

$$S_X(H) = \frac{4}{11}, \qquad S_X(E) = \frac{23}{44}, \qquad S_X(B) = \frac{13}{22}.$$

Every prime divisor on X has positive beta invariant.

Proof. The blow-up formulas give the intersection numbers: in particular, $\deg N_{C/V} = (-K_V)C + 2g(C) - 2 = 2$. Both H and B are nef. If $T \neq E$ is a prime divisor and $T \sim aH + bB$, intersection with a general conic fibre and a general fibre of $E \to C$ gives $a, b \geq 0$. Consequently, the effective cone is generated by E and B, and

$$(aH - bE)^3 = 4a^3 - 6ab^2 + 2b^3.$$

The divisors $L - uH$ and $L - uE$ are nef for $0 \leq u \leq 1$ and are not pseudoeffective for $u > 1$. Their volume integrals are

$$S_X(H) = \frac{1}{22}\int_0^1 (4(2-u)^3 - 6(2-u) + 2)\,du = \frac{4}{11},$$
$$S_X(E) = \frac{1}{22}\int_0^1 (32 - 12(1+u)^2 + 2(1+u)^3)\,du = \frac{23}{44}.$$

For $L - uB$, the pseudoeffective threshold is two. The divisor is nef on $[0,1]$. On $[1,2]$ its fixed part is $(u-1)E$, and the moving part is $(2-u)H$. Indeed, intersection with the moving fibres of $E \to C$ forces that multiplicity, and the remaining divisor is semiample. Thus

$$S_X(B) = \frac{1}{22}\left(\int_0^1 (4(2-u)^3 - 6(2-u)(1-u)^2 + 2(1-u)^3)\,du + \int_1^2 4(2-u)^3\,du\right) = \frac{13}{22}.$$

For $T \neq E$, the integers a, b above are not both zero, so $T - H$ or $T - B$ is effective. Monotonicity of volumes gives $S_X(T) \leq 13/22 < 1 = A_X(T)$. The calculation for E treats the other prime divisor. □

Proposition 53.3 (The known reduced-fibre estimates)**.** *If the conic fibre through p is reduced, then $\delta_p(X) \geq 176/171$.*

Proof. We use the three cited propositions of [28]. For a smooth fibre, unless it is the negative section of $E \cong \mathbb{F}_2$, they give $176/161$ off E and $176/169$ on E. For a reduced singular fibre, Proposition 5.2.4 gives $176/171$, including points on E and the intersection of its two components. In the exceptional smooth-fibre case, Proposition 5.2.6 gives $22/19$. Their minimum is $176/171$. These are local estimates for all divisors whose centres contain p; no automorphism-group assumption occurs in these propositions. □

53.1 A section ring adapted to the projection

Choose equations Q_1, Q_2 of V so that Q_1 contains Π and $Q_2|_\Pi$ is the equation of C. On Π use three coordinates, and use t_0, t_1, t_2 transverse to Π. In the blow-up of Π in $\mathbb{P}^5$, write $t_i = ey_i$, with $[y_0 : y_1 : y_2] \in \mathbb{P}^2$. The variable e has divisor class $E = H - B$.

Lemma 53.4 (The projective-bundle presentation)**.** *Let $T = \mathbb{P}_{\mathbb{P}^2}(\mathcal{O}^{\oplus 3} \oplus \mathcal{O}(1))$, in the quotient convention, with tautological class H and base class B. Then $T = \mathrm{Bl}_\Pi \mathbb{P}^5$, and $X \subset T$ is the complete intersection*

$$F_1 = Q_1/e = 0, \qquad F_2 = Q_2 = 0, \qquad \deg F_1 = H + B, \quad \deg F_2 = 2H.$$

For $a, b \geq 0$, its spaces $H^0(X, aH + bB)$ are the (a, b) pieces of

$$R = \frac{\mathbb{C}[s, x, z, e, y_0, y_1, y_2]}{(F_1, F_2)}, \qquad \begin{array}{c|ccc} \textit{variable} & s, x, z & e & y_0, y_1, y_2 \\ \hline (H,B)\textit{-degree} & (1,0) & (1,-1) & (0,1). \end{array}$$

In particular,

$$N_m = h^0(X, mL) = \frac{(2m+1)(11m^2 + 11m + 6)}{6}.$$

Proof. The three coordinates on Π and the products ey_i give the six sections of H, identifying T with the graph of the projection. The restrictions of Q_1, Q_2 give the displayed equations. Near each point of C, one coefficient of a transverse variable in Q_1 is a unit: otherwise that point would be singular on Q_1 and hence on V. Locally eliminate that variable. The incidence equations then become the usual graph of the blow-up of the remaining regular codimension-two ideal. Away from C the graph is already V. This proves the complete-intersection assertion, including along the exceptional divisor.

For $a \geq 0$, the push-forward of $\mathcal{O}_T(aH + bB)$ is

$$\bigoplus_{d=0}^{a} \mathrm{Sym}^{a-d}(\mathbb{C}^3) \otimes \mathcal{O}_{\mathbb{P}^2}(b + d),$$

and the higher direct images vanish. Thus its sections are exactly the polynomial piece with the stated degrees. Twists with H-coefficient $-1, -2, -3$ are acyclic on the projective-space fibres. In the Koszul resolution of X, the additional twists of $aH + bB$ have degrees $(a-1, b-1)$, $(a-2, b)$, and $(a-3, b-1)$. Whenever their H-coefficient is nonnegative, their push-forwards have base twists at least -1, hence no higher cohomology. Otherwise they are among the three acyclic relative twists just listed. Taking sections therefore gives the asserted quotient. Its Hilbert function on the diagonal, or Riemann–Roch with $L^3 = 22$, gives N_m. □

54 Extracting a double conic fibre

Fix a nonreduced fibre $g^{-1}(y) = 2\Lambda$, and choose $y = [1:0:0]$. Its reduced curve has $H\Lambda = 1$. It is not contained in E: a curve in E either has H-degree zero or maps onto the conic C and has even positive H-degree. Thus π maps Λ to a line in V. Also $E\Lambda = 1$; its intersection with E is one reduced point, denoted by q.

Lemma 54.1 (The leading equations)**.** *There are coordinates as in Lemma 53.4, with $u = y_1$, $v = y_2$, in which the following holds. Set $t = ey_0$. The line Λ has coordinates $[s:t]$, and q is $t = 0$. Give (s, x, z, e, y_0, u, v) respective weights $(0, 1, 2, 0, 0, 2, 2)$. Then the lowest-weight parts of the two equations are*

$$F_{1,0} = zy_0 - R(s,t;u,v), \qquad F_{2,0} = x^2 + sz + eT(s,t;u,v), \tag{54.1}$$

where $R = A(s,t)u + B(s,t)v$ and $T = M(s,t)u + N(s,t)v$, with A, B, M, N linear forms. The binary quadrics q_1, q_2 defined by

$$q(s,t;u,v) = sR + tT = q_1(s,t)u + q_2(s,t)v$$

have no common zero. Consequently, on $\mathbb{P}^1_{[s:t]} \times \mathbb{P}^1_{[u:v]}$ the curve $D = (q = 0)$ is a smooth $(2,1)$-curve, and $P = (t = R = 0)$ is a single reduced point on D.

Proof. The fibre plane is cut out by $F_1|_y$, a nonzero linear form in the four fibre coordinates of T. Its intersection with $e = 0$ is tangent to the smooth conic C: restricting the double-line equation to that line gives twice the point q. Choose coordinates on Π so that $C = (x^2 + sz = 0)$, $q = [1:0:0]$, and its tangent line is $z = 0$. The fibre-plane equation is therefore a nonzero multiple of $z + cey_0$. Replace z by $z + cey_0$. On this plane the other quadric is a square; replacing x by $x + dey_0$ and scaling its equation makes this square x^2. A further replacement of s by $s + c'ey_0$ absorbs the remaining $ey_0 z$ term. These changes preserve Π, its conic, and the two divisor gradings.

The remaining terms of F_1 contain u or v; its terms of weight two are linear in s, t times u, v. The remaining terms of F_2, after $x^2 + sz$, contain eu or ev, and their weight-two terms are e times a linear form in s, t times u, v. This gives (54.1). On $y_0 = 1$, eliminate z. The first transverse terms along Λ are $x^2 + q_1(s,t)u + q_2(s,t)v$. If q_1, q_2 vanished simultaneously, the total space X would be singular at the corresponding point. Thus they define a degree-two morphism $\mathbb{P}^1 \to \mathbb{P}^1$, whose graph is D. At $t = 0$ one has $q = sR$, so $R(1,0;u,v)$ is a nonzero linear form. The equations $t = R = 0$ cut out one reduced point, and the same identity shows that it belongs to D. □

Lemma 54.2 (The extraction and the different)**.** *Let $\mathcal{J} = \mathfrak{m}_y \mathcal{O}_X$ and $\omega\colon X^\sharp = \mathrm{Bl}_{\mathcal{J}} X \to X$. Its reduced exceptional divisor F is of plt type. With $h = \mathcal{O}_F(1,0)$ and $f = \mathcal{O}_F(0,1)$, one has*

$$F \cong \Lambda \times \mathbb{P}^1, \qquad A_X(F) = 3, \qquad \omega^* L|_F = h, \qquad 2F|_F = -f, \qquad \Delta_F = \frac{1}{2}D.$$

Proof. Where q_1 is nonzero, eliminate u in the local equation. Then $\mathcal{J} = (x^2, v)$, and its blow-up is the transverse weighted blow-up of weights $(1,2)$. Its charts are $v = x^2 r$ and $x^2 = vr$. The second chart is an A_1 surface singularity times the smooth curve direction, so the blow-up is normal. Since g is flat, blowing up the fibre ideal is base change from $\mathrm{Bl}_y \mathbb{P}^2$. Its exceptional scheme is $(2\Lambda) \times \mathbb{P}^1$, proving the product description and the identity $2F|_F = -f$. The formula for $L|_F$ follows from $L\Lambda = 1$.

On the smooth chart, $dx \wedge dv = x^2 dx \wedge dr$, giving discrepancy three. On the other chart the index cover for F is $x = ab$, $v = a^2$, $r = b^2$, modulo $(a,b) \mapsto (-a,-b)$. Adjunction gives coefficient $1/2$ on its singular-locus curve; Lemma 54.1 identifies that curve with D. The quotient description also proves that $(X^\sharp, F)$ is plt. Finally, $-2F$ is the relatively ample tautological divisor of the blow-up. These facts verify all the extraction hypotheses. □

Lemma 54.3 (Exactness of the initial algebra)**.** *The associated graded ring of the section ring in Lemma 53.4, for the divisorial filtration* ord_F*, is*

$$R_0 = \frac{\mathbb{C}[s, x, z, e, y_0, u, v]}{(zy_0 - R(s, ey_0; u, v),\ x^2 + sz + eT(s, ey_0; u, v))}. \tag{54.2}$$

This is a prime complete intersection. In particular, the displayed initial forms account for cancellations of global sections.

Proof. On $y_0 \neq 0$, eliminate z and put $t = ey_0$. The remaining equation is

$$y_0 x^2 + q_1(s, t)u + q_2(s, t)v = 0.$$

It is irreducible. Indeed, it is a primitive linear polynomial in u, v over $\mathbb{C}[s, t, y_0^{\pm 1}, x]$, since q_1, q_2 are coprime. Thus this open set is integral of dimension five.

On the boundary $y_0 = 0$, write $R(s, 0; u, v) = s(au + bv)$ and $T(s, 0; u, v) = s(cu + dv)$. Here $(a, b) \neq (0, 0)$. The two equations become

$$s(au + bv) = 0, \qquad x^2 + sz + es(cu + dv) = 0.$$

On $s \neq 0$ they cut out a four-dimensional set. On $s = 0$ one has $x = 0$, and the four variables z, e, u, v remain. Therefore the boundary has dimension four. The two equations have height two, so their quotient is Cohen–Macaulay and unmixed. No component is contained in the boundary; the integral open set has only one component and is generically reduced. Unmixedness now proves that R_0 is a domain.

The displayed ideal is contained in the weight initial ideal of (F_1, F_2). The original quotient and the displayed candidate quotient are complete intersections of bidegrees $(1, 1)$ and $(2, 0)$. The weight initial quotient has the same bigraded Hilbert function as the original quotient, so the candidate and initial quotients have equal Hilbert functions. Each piece is finite-dimensional, despite the negative B-degree of e, since its exponent is bounded by the H-degree. Equality of these Hilbert functions proves equality of the ideals.

Finally, localize at $y_0 \neq 0$ and at a nonzero q_i, and trivialize H by either s or t. The weight filtration is then the local $(1, 2)$ filtration on the smooth transverse coordinates, hence ord_F. Localization is injective on the initial domain. Therefore no further increase of order occurs for a nonzero global initial form. This identifies the actual divisorial filtration with (54.2). □

Lemma 54.4 (The actual refined spaces)**.** *Write* $j = 2r + \epsilon$*, where* $\epsilon \in \{0, 1\}$*, and put* $M = m - \epsilon$ *and* $k = \max(0, r - m)$*. For* $0 \leq r \leq 2m - \epsilon$*,*

$$W^F_{m,j} = \operatorname{gr}^j_F H^0(X, mL) \cong H^0\big(F, \mathcal{I}_P^k \otimes \mathcal{O}_F(M, r)\big).$$

The zero divisor of a refined section is $\epsilon D/2 + \operatorname{div}(\sigma)$*, with* σ *in this space. The space vanishes for* $r > 2m - \epsilon$*, and in the stated range*

$$\dim W^F_{m,2r+\epsilon} = (M + 1)(r + 1) - \frac{k(k + 1)}{2}. \tag{54.3}$$

Proof. Use Lemma 54.3, set $y_0 = 1$, and substitute $e = t$, $z = R$, and $x^2 = -q$. Every monomial becomes x^ϵ times a polynomial of bidegree (M, r) in s, t, u, v. More explicitly, a monomial with x-exponent $2a + \epsilon$, z-exponent c, e-exponent d, and total u, v-exponent b has $r = a + c + b$ and $b \leq m + d$. After substitution it contains $q^a R^c t^d$. Since $q = sR + tT$, it lies in $(t, R)^k$ whenever $r > m$. The same monomial has $a + c + d \leq m - \epsilon$, so $r \leq 2m - \epsilon$. This proves the inclusion and the vanishing assertion.

For $r \leq m$, every polynomial of bidegree (M, r) lifts, by using s, $t = ey_0$, u, v, and enough factors y_0. For $r > m$, one has $k = r - m \leq M$. The ideal sheaf of P is generated by t of

bidegree $(1,0)$ and R of bidegree $(1,1)$. Its kth power is generated by $t^{k-c}R^c$, for $0 \le c \le k$. The power resolution, after twisting, is

$$0 \longrightarrow \bigoplus_{c=0}^{k-1} \mathcal{O}_F(M-k-1, r-c-1) \longrightarrow \bigoplus_{c=0}^{k} \mathcal{O}_F(M-k, r-c) \longrightarrow \mathcal{I}_P^k(M,r) \longrightarrow 0.$$

The first term has no H^1, since $M-k-1 \ge -1$ and $r-c-1 \ge 0$. Consequently, every section is a sum of $t^{k-c}R^c$ times polynomials of bidegree $(M-k, r-c)$. Lift R to z, t to e in these fixed factors, and t to ey_0 in the remaining polynomial. The inequality for the available base degree is $r-c = m+(k-c)$, exactly the required equality. Thus every section lifts to a section of mL of native weight $2r+\epsilon$ after multiplication by x^ϵ.

At a point of $\mathbb{P}^1 \times \mathbb{P}^1$, the condition of vanishing to order k imposes $k(k+1)/2$ independent conditions on bidegree (M,r) when $M, r \ge k-1$. One can check independence on the local monomials of total degree less than k. This gives (54.3). The odd factor has square $-q$, so its reflexive zero divisor is $D/2$. Its class checks the formula: $(M,r) + \epsilon(1,1/2) = (m, j/2)$, the class of $(m\omega^* L - jF)|_F$. □

Example 54.5 (The first anticanonical degree). For $m=1$, the dimensions for native weights $j = 0,1,2,3,4$ are $2,1,4,2,5$. Their sum is $14 = h^0(X,L)$, and their weighted sum is 35. At the last weight, the five-dimensional space is $H^0(\mathcal{I}_P(1,2))$: the one missing section records the actual point condition. Using the full six-dimensional space would change the normalization and the threshold calculation.

55 Expected orders and the surface threshold

Lemma 55.1 (The expected order of the extracted divisor)**.** *For the preceding extraction, $S_X(F) = 57/22$ and $\tau(F) = 4$. The refined series has bounded support and contains an ample series.*

Proof. Put $\rho = r/m$. The leading dimension in (54.3), divided by m^2, is

$$d(\rho) = \begin{cases} \rho, & 0 \le \rho \le 1, \\ \rho - \frac{1}{2}(\rho-1)^2, & 1 \le \rho \le 2. \end{cases}$$

Since $N_m = (11/3)m^3 + O(m^2)$ and there are two parities, the normalization factor is $6/11$. In particular,

$$\frac{6}{11}\int_0^2 d(\rho)\,d\rho = 1, \qquad S_X(F) = \frac{6}{11}\int_0^2 2\rho d(\rho)\,d\rho = \frac{4}{11} + \frac{49}{22} = \frac{57}{22}.$$

The same calculation is exact in every anticanonical degree. For $\epsilon = 0,1$, split the sum at $r=m$; put $d_{m,r,\epsilon} = \dim W^F_{m,2r+\epsilon}$. The four contributions are

ϵ	r	$\sum d_{m,r,\epsilon}$	$\sum (2r+\epsilon) d_{m,r,\epsilon}$
0	$0,\dots,m$	$\frac{(m+1)^2(m+2)}{2}$	$\frac{2m(m+1)^2(m+2)}{3}$
0	$m+1,\dots,2m$	$\frac{m(m+1)(8m+7)}{6}$	$\frac{m(m+1)(49m^2+57m+14)}{12}$
1	$0,\dots,m$	$\frac{m(m+1)(m+2)}{2}$	$\frac{m(m+1)(m+2)(4m+3)}{6}$
1	$m+1,\dots,2m-1$	$\frac{m(m-1)(8m+5)}{6}$	$\frac{m(m-1)(49m^2+43m+12)}{12}$.

Indeed, on the first interval the summand is $(m-\epsilon+1)(r+1)$, and on the second one subtracts $(r-m)(r-m+1)/2$. The identities follow by summing $1, r, r^2, r^3$; an empty interval contributes

zero. The case $m = 0$ is immediate and the table applies for $m \geq 1$. Adding its columns gives

$$\sum_j \dim W^F_{m,j} = \frac{(2m+1)(11m^2+11m+6)}{6} = N_m,$$
$$\sum_j j \dim W^F_{m,j} = \frac{m(19m^3+27m^2+19m+5)}{2}.$$

Dividing the second identity by mN_m gives the same limit. The top space has dimension $(m+1)(3m+2)/2 > 0$ at weight $4m$, and no larger weight occurs. This proves the threshold and bounded support.

For $0 < \rho < 1$ and divisible even native degrees, the series is the complete ample series of bidegree (m, r). More generally, for $1 < \rho < 2$, multiplication by t^k supplies the complete series of bidegree $(m-k, r)$, whose two degrees are positive, with a fixed effective factor. Thus the required ample-series condition holds on interior rays, as well as on the first chamber. □

Lemma 55.2 (A degeneration preserving the different)**.** *Choose coordinates on F so that D has affine equation $\eta = \xi^2$, and let o_0, o_∞ be its two ramification points for the second projection. Orient the action $(\xi, \eta) \mapsto (\lambda\xi, \lambda^2\eta)$ so that $P \neq o_\infty$. Under this action, every space in Lemma 54.4 degenerates to the full monomial space*

$$\left\langle \xi^i \eta^l : 0 \leq i \leq M, \quad 0 \leq l \leq r, \quad i + l \geq k \right\rangle,$$

with the complementary homogeneous factors understood. The different $D/2$ is unchanged.

Proof. A degree-two map between smooth rational curves has two distinct ramification points and can be written as $\eta = \xi^2$ after changes of coordinates. Since P is one point of D, choose the repelling ramification point away from it. Write $P = (a, a^2)$ on the affine chart; $a = 0$ is allowed. Its moving ideal is $(\xi - \lambda a, \eta - \lambda^2 a^2)$. Its powers are flat in λ, as is also seen by this translation of the two local parameters. The limit of the section space is therefore contained in $H^0(\mathcal{I}^k_{o_0}(M, r))$.

All point conditions have the same independent length $k(k+1)/2$ in this range. Thus the limiting space and the containing space have the same dimension, and the inclusion is an equality. Membership in $(\xi, \eta)^k$ is the displayed monomial inequality. The action preserves the equation of D, so it preserves the different. □

Lemma 55.3 (Six surface moments)**.** *For the limiting spaces of Lemma 55.2, the maximal normalized sums along the four coordinate fibres and the sums for the two branch weights $(1, 2)$ are*

order	*limit*
$\xi = 0$	$23/44$
$\xi = \infty$	$21/44$
$\eta = 0$	$31/44$
$\eta = \infty$	$13/22$
$(1,2)$ *at* o_0	$85/44$
$(1,2)$ *at* o_∞	$73/44$.

For every curve T of bidegree $(2, 1)$, the maximal normalized sum of its orders is at most $81/352$ in the limit. The normalization is always mN_m for orders and N_m for dimensions.

Proof. Set $a = i/m$, $b = l/m$, and $\rho = r/m$. The monomial region is

$$\Omega_\rho = \{0 \leq a \leq 1,\ 0 \leq b \leq \rho,\ a + b \geq \max(0, \rho - 1)\}.$$

For $\rho \le 1$ it is the whole rectangle. For $1 \le \rho \le 2$, remove the triangle $0 \le a \le \rho - 1$, $0 \le b \le \rho - 1 - a$. For a polynomial $Q(a, b, \rho)$, put

$$J_Q(\rho) = \int_0^1 \int_0^\rho Q \, db \, da, \qquad T_Q(\rho) = \int_0^{\rho-1} \int_0^{\rho-1-a} Q \, db \, da.$$

Its normalized sum is

$$\frac{6}{11} \left(\int_0^1 J_Q \, d\rho + \int_1^2 (J_Q - T_Q) \, d\rho \right).$$

Here are the separate contributions:

Q	$[0,1]$	$[1,2]$	sum
1	$3/11$	$8/11$	1
a	$3/22$	$17/44$	$23/44$
$1-a$	$3/22$	$15/44$	$21/44$
b	$1/11$	$27/44$	$31/44$
$\rho - b$	$1/11$	$1/2$	$13/22$
$a + 2b$	$7/22$	$71/44$	$85/44$
$(1-a) + 2(\rho - b)$	$7/22$	$59/44$	$73/44$.

For instance, the removed first-coordinate moment is $(\rho-1)^3/6$, so its remaining moment is $\rho/2 - (\rho-1)^3/6$ on the second interval. The table proves all six assertions. Boundary lattice errors and parity shifts give $O(m^{-1})$ after normalization.

For T of bidegree $(2,1)$, enlarge each space to $H^0(\mathcal{O}_F(M, r))$. Its maximal total order along T is $\sum_{l \ge 1} h^0(\mathcal{O}_F(M - 2l, r - l))$. Orders of effective zero divisors are nonnegative, so this enlargement is valid with the original normalization. The resulting bound is

$$\begin{aligned} \frac{6}{11} \int_0^2 \int_0^{\min(1/2,\rho)} (1 - 2z)(\rho - z) \, dz \, d\rho &= \frac{6}{11} \left[\int_0^{1/2} \left(\frac{\rho^2}{2} - \frac{\rho^3}{3} \right) d\rho + \int_{1/2}^2 \left(\frac{\rho}{4} - \frac{1}{24} \right) d\rho \right] \\ &= \frac{81}{352}. \end{aligned}$$

□

Lemma 55.4 (The surface threshold)**.** *For the actual refined series,* $\delta(F, D/2; W^F) \ge 88/85$.

Proof. Fix an arbitrary divisorial valuation ν over F. In each actual space choose a basis compatible both with ν and with the filtration of the action in Lemma 55.2. Only two filtrations are involved, so Lemma 2.3 applies. Let Γ_m be its normalized refined basis divisor, including the odd fixed factors $\epsilon D/2$. Their total coefficient is $O(m^{-1})$.

The action degenerates Γ_m to a divisor Γ_m^0 whose sections form an eigenbasis of the limiting space. Eigenvectors need not be monomials, since some weights repeat. Nevertheless, their prime factors are coordinate fibres or curves $D_c = (\eta = c\xi^2)$, for $c \in \mathbb{C}^*$. Lemma 18.7 gives the precise log-canonicity criterion: each prime coefficient is at most one, and the weighted orders at o_0, o_∞ are at most three. Locally, the curves are axes and distinct parabolas. Two successive point blow-ups resolve them; the last inequality is $a + 2b + 2\sum c_i \le 3$, which together with $a \le 1$ also gives the first exceptional inequality $a + b + \sum c_i \le 2$. Thus this criterion checks all valuations, not just invariant ones.

The four coordinate coefficients of Γ_m^0 are bounded by the first four numbers in Lemma 55.3, up to an error tending to zero. Any eigenbasis has the same total action weights, so its weighted orders at the two ramification points are $85/44 + o(1)$ and $73/44 + o(1)$, even if its vectors are not monomials. Every D_c coefficient is at most $81/352 + o(1)$.

Take $0 < \lambda < 88/85$. All prime coefficients of $D/2 + \lambda\Gamma_m^0$ are less than one for large m: at the limiting endpoint the largest coordinate bound is $62/85$, while the coefficient on D is at most $251/340$. At the ramification points the weighted orders are at most

$$1 + \lambda\frac{85}{44} + o(1) < 3, \qquad 1 + \lambda\frac{73}{44} + o(1) < 3.$$

The invariant-boundary criterion proves log canonicity. The action preserves D, so semicontinuity of log canonical thresholds proves log canonicity of $(F, D/2+\lambda\Gamma_m)$ before degeneration. The ν-compatible basis gives $S_m(W^F;\nu) \le A_{F,D/2}(\nu)/\lambda$. Pass to the limit and then let $\lambda \nearrow 88/85$. Since ν was arbitrary, this proves the assertion. □

Example 55.5 (Why the different is essential). At o_0, the different is $\frac{1}{2}(\eta - \xi^2 = 0)$. The valuation with weights $(1, 2)$ has discrepancy $1 + 2 - \frac{1}{2}\cdot 2 = 2$ for this pair. The corresponding moment is $85/44$, giving the ratio $2/(85/44) = 88/85$. The positive margin is $2 - 85/44 = 3/44$. Using discrepancy three would discard the quotient contribution and would not compute the threshold of the adjoint pair.

56 Completion for Family 2.16

Proposition 56.1 (Every point of a double fibre). *If $p \in \Lambda$ and $g^{-1}(g(p)) = 2\Lambda$, then $\delta_p(X) \ge 88/85$.*

Proof. Apply Abban–Zhuang to the plt curve extraction of Lemma 54.2. The refined threshold over any point of Λ is bounded below by the global surface threshold. Lemmas 55.1 and 55.4 give

$$\delta_p(X) \ge \min\left\{\frac{3}{57/22}, \frac{88}{85}\right\} = \frac{88}{85}.$$

To recall the local argument, choose a threefold basis compatible with F and a given prime divisor over X, and write its pull-back as $S_m(F)F + \Gamma_m$. The finite-level bounds in the proof of Lemma 55.4 make $(F, D/2 + \lambda\Gamma_m|_F)$ log canonical for every fixed $\lambda < 88/85$ and large m. Inversion of adjunction gives log canonicity of $(X^\sharp, F + \lambda\Gamma_m)$ near F. The pull-back boundary for the original basis divisor has coefficient $1 - A_X(F) + \lambda S_m(F) \le 1$ on F, so it too is log canonical there. By properness, the centre on $X^\sharp$ of any divisor whose centre on X contains p meets $\omega^{-1}(p)$. This proves the required inequality for all such divisors. Passing to the limit gives the stated bound. Bounded support, the ample-series condition, and the different were verified above. The estimate includes $p = q = \Lambda \cap E$. □

Proof of Theorem 53.1 and the Family 2.16 assertion of Theorem 1.1. Every plane conic fibre is smooth, reduced with two components, or a double line. Proposition 53.3 and Proposition 56.1 therefore give $\delta_p(X) \ge \min\{176/171, 88/85\} = 176/171$ for every point p. For any prime divisor V over X, choose a point of its centre. The definition of δ_p gives $A_X(V)/S_X(V) \ge 176/171$. Taking the infimum proves the global bound, and hence K-stability. □

Example 56.2 (A smooth member with a double fibre). In coordinates $[s : x : z : t_0 : t_1 : t_2]$ on $\mathbb{P}^5$, take

$$Q_1 = zt_0 - st_1 - xt_2, \qquad Q_2 = x^2 + sz + t_0t_2 + t_1^2 + t_2^2, \qquad V = (Q_1 = Q_2 = 0).$$

The determinant of the Hessian matrix of $Q_1 + \lambda Q_2$ is $4\lambda^6 + 4\lambda^2 - 1$. It has six distinct roots and its leading coefficient is nonzero. Simultaneous diagonalization therefore gives six distinct pencil eigenvalues, which proves that the intersection of the quadrics is smooth. The plane

$t_0 = t_1 = t_2 = 0$ cuts out the smooth conic $x^2 + sz = 0$. Blow it up. Over $[y_0 : u : v] = [1 : 0 : 0]$, the strict transform equations are $z = 0$ and $x^2 = 0$, giving a double line. For its leading model,

$$R = su, \qquad T = tv, \qquad q = s^2u + t^2v, \qquad P = ([1 : 0], [0 : 1]).$$

Here P is a ramification point of $D \to \mathbb{P}^1_{[u:v]}$. Thus the limiting monomial calculation already occurs on the actual exceptional surface. The branch valuation has expected order $85/44$ and discrepancy two for $(F, D/2)$, so the refined surface threshold is exactly $88/85$ in this example. In particular, smoothness of X does not rule out the fibres treated in this part.

Remark 56.3 (Avoiding a Zariski decomposition on the extracted threefold). *We have not computed a Zariski decomposition of $\omega^*L - tF$ on the weighted threefold extraction.* The volume integrals in Lemma 53.2 take place on X itself. For the new extraction, the prime initial algebra in Lemma 54.3 identifies the actual section filtration, and Lemma 54.4 proves completeness of its point-ideal spaces. Their dimensions compute $S_X(F)$ exactly. The surface estimate then uses a degeneration preserving the different and explicit order sums. These statements provide the required vanishing information; no unproved decomposition on $X^\sharp$ is substituted for them.

57 A second proof for Family 2.16: the CM polarization

We give an independent proof of K-stability for every smooth member of Family 2.16. It uses the parameter space proposed in [28, Section 6] and the relative reconstruction proved in [34, Appendix A]. The new steps are the calculation of the CM line and a Hilbert–Mumford inequality for every smooth pair. CM-degree minimization then compares two fillings of the same punctured family. In particular, we do not assume the conjectural identification of the entire K-moduli space with a GIT quotient. None of the local threshold estimates in the preceding sections is used.

Theorem 57.1. *Let $V \subset \mathbb{P}^5$ be a smooth intersection of two quadrics and let $C \subset V$ be a smooth conic. Then $\mathrm{Bl}_C V$ is K-stable.*

Put $E = \mathbb{C}^6$ and use the convention that $\mathbb{P}(E)$ parametrizes lines. On $\mathrm{Gr}(3, E)$ let $\mathcal{S} \subset E \otimes \mathcal{O}$ be the tautological rank-three subbundle. Set

$$\mathcal{H} = \mathbb{P}_{\mathrm{Gr}(3,E)}(\mathrm{Sym}^2 \mathcal{S}^*), \qquad \mathcal{G} = \mathrm{Gr}(2, \mathrm{Sym}^2 E^*).$$

A point of $\mathcal{H}$ is a plane $\Pi = \mathbb{P}(S)$ together with a nonzero quadratic equation on Π, considered up to a scalar. Thus $\mathcal{H}$ parametrizes plane conics, including reducible and double conics. Let $\mathcal{P}$ denote the tautological rank-two bundle on $\mathcal{G}$. We use additive notation for line bundles and write

$$\zeta = \mathcal{O}_{\mathcal{H}}(1), \qquad \xi = \det \mathcal{S}^*, \qquad \eta = \det \mathcal{P}^*.$$

Their pullbacks will have the same names. Define

$$W = \{(C, P) \in \mathcal{H} \times \mathcal{G} : P \subset H^0(\mathbb{P}^5, \mathcal{I}_C(2))\}.$$

Since $h^0(\mathcal{I}_C(2)) = 16$ for every plane conic, $W \to \mathcal{H}$ is a relative $\mathrm{Gr}(2, 16)$. In particular, W is smooth, projective and irreducible. Its points need not all define Fano threefolds. The locus W^{sm} on which $V = V(P)$ and C are smooth is a nonempty open subset. The group $\mathrm{SL}(E)$ acts on all these spaces. Passing to $\mathrm{PGL}(E)$ does not change stability; a suitable positive power of any linearization below descends through the finite centre.

Definition 57.2. For a flat family $f : \mathcal{X} \to T$ of Fano threefolds with relatively ample $\mathbb{Q}$-Cartier anticanonical divisor, we normalize the CM $\mathbb{Q}$-line bundle by

$$c_1(\lambda_{\mathrm{CM},f}) = -f_* c_1(-K_{\mathcal{X}/T})^4.$$

Equivalently, this is the corresponding Deligne pairing with its sign reversed. It is functorial under base change. Multiplying this normalization by a positive rational number changes none of the stability or degree comparisons below.

Lemma 57.3. *The CM line of the universal blow-up family is the restriction of*

$$\mathcal{L} = 6\zeta + 20\xi + 66\eta \in \mathrm{Pic}(W). \tag{57.1}$$

This equality holds, with the natural linearizations, both on the smooth Fano locus and for the relative blow-up families furnished by [34, Proposition A.13].

Proof. We calculate in the ambient blow-up, so the centre need not be a regular embedding in V. A plane conic is a regular embedding of codimension four in $\mathbb{P}^5$, even when it is a double line. Let $\mathcal{C}$ be the universal conic and let $h = c_1(\mathcal{O}_{\mathbb{P}^5}(1))$. On $\mathbb{P}(\mathcal{S})$ its class is $2h + \zeta$. For the projection $p : \mathcal{C} \to W$ we have

$$p_* h = 2, \qquad p_* h^2 = 2\xi + \zeta. \tag{57.2}$$

Indeed, on $\mathbb{P}(\mathcal{S})$ the pushforwards of h^2 and h^3 are 1 and $-c_1(\mathcal{S}) = \xi$.

Write $N = N_{\mathcal{C}/(\mathbb{P}^5 \times W)}$. The normal bundle of the relative plane is $\mathcal{O}(1) \otimes (E/\mathcal{S})$ and the normal bundle of the conic in this plane is $\mathcal{O}(2) \otimes \mathcal{O}_{\mathcal{H}}(1)$. The normal bundle sequence gives, modulo terms pulled back from base classes of codimension at least two,

$$\begin{aligned} c_1(N) &= 5h + \xi + \zeta, \\ c_2(N) &= 9h^2 + 4h\xi + 3h\zeta, \\ c_1(N)^2 - c_2(N) &= 16h^2 + 6h\xi + 7h\zeta. \end{aligned}$$

The omitted terms make no contribution to the divisor classes being computed. Therefore

$$\begin{aligned} p_*(h\, c_1(N)) &= 12\xi + 7\zeta, \\ p_*(c_1(N)^2 - c_2(N)) &= 44\xi + 30\zeta. \end{aligned} \tag{57.3}$$

Let $b : Z = \mathrm{Bl}_{\mathcal{C}}(\mathbb{P}^5 \times W) \to \mathbb{P}^5 \times W$ and denote its exceptional divisor by F. Put $z = 2h - F$ and let $q : Z \to W$ be the projection. The blow-up formula for a regular embedding of codimension four is

$$b_*F^j = 0 \ (1 \le j \le 3), \quad b_*F^4 = -[\mathcal{C}], \quad b_*F^5 = -i_*c_1(N), \quad b_*F^6 = -i_*(c_1(N)^2 - c_2(N)),$$

where i is the inclusion of $\mathcal{C}$. On a fibre, $\deg N = 10$, so

$$\begin{aligned} q_*z^5 &= 32 - 10 \deg C + \deg N = 22, \\ q_*z^6 &= -60p_*h^2 + 12p_*(h\, c_1(N)) - p_*(c_1(N)^2 - c_2(N)) \\ &= -20\xi - 6\zeta. \end{aligned} \tag{57.4}$$

The strict transform $\mathcal{X}$ of V is the zero locus of the two transformed quadrics. Its class and relative anticanonical class are

$$[\mathcal{X}] = z^2 + z\eta + c_2(\mathcal{P}^*), \qquad -K_{\mathcal{X}/W} = (z - \eta)|_{\mathcal{X}}.$$

Here adjunction uses $K_{Z/W} = -6h + 3F$. These identities apply on the locus where the strict transform is the blow-up and is a relative complete intersection. This includes the smooth families and the reconstructed families by [34, Lemma A.12 and Proposition A.13]. All the ambient formulas remain valid for singular plane conics, since their embeddings in the ambient projective bundle are regular.

Keeping the terms contributing a divisor on the base gives

$$f_*(-K_{\mathcal{X}/W})^4 = q_*(z^6 - 3\eta z^5) = -6\zeta - 20\xi - 66\eta.$$

This proves (57.1). To keep track of the line bundle, rather than only its numerical class, use the Koszul complex of the two transformed quadrics on Z. Its degree-four determinant-of-cohomology coefficient after twisting by $m(z - \eta)$, with the negative sign and normalization above, defines a rational line bundle on all of W. On every flat complete-intersection locus it is the CM line just calculated. Thus this is a functorial extension even at parameter points where we do not have a Fano family. The Picard group of the projective and Grassmann bundles is freely generated by the three displayed classes, and $\mathrm{SL}(E)$ has no characters. The computed class therefore identifies this extension and its linearization with $\mathcal{L}$. On a nontrivial rank-six projective bundle the calculation is made after taking frames; changing frames gives exactly this linearization. □

Example 57.4. Fix the plane and a pencil of conics in it. On this curve of $\mathcal{H}$, $\xi = 0$ and $\deg \zeta = 1$, so $p_* h^2$ has degree one. This is also the number of conics in the pencil through the point cut out by two general hyperplanes. For the ambient blow-up, the second formula in (57.4) consequently has degree -6. For the full family the pencil of quadrics contributes the additional term $-66 \deg \eta$. This separates the contribution of the centre from that of the threefold.

Lemma 57.5. *The line bundle $\mathcal{L}$ is ample on W.*

Proof. The bundle

$$\operatorname{Sym}^2 \mathcal{S} \otimes (\det \mathcal{S}^*)^2 \simeq \operatorname{Sym}^2(\bigwedge^2 \mathcal{S}^*)$$

is globally generated. Its pullback surjects onto $\mathcal{O}_{\mathcal{H}}(\zeta + 2\xi)$. Thus $\zeta + 2\xi$ is globally generated and relatively ample over $\operatorname{Gr}(3, E)$. Adding the pullback of the ample Plücker class shows that $\zeta + 3\xi$ is ample on $\mathcal{H}$. Now

$$\mathcal{L} = 6(\zeta + 3\xi) + 2\xi + 66\eta$$

is the restriction of an ample line bundle on $\mathcal{H} \times \mathcal{G}$: the first summand is ample on the first factor, ξ is nef, and η is ample on the second factor. □

58 Hilbert–Mumford inequalities for smooth pairs

For clarity, we fix the sign convention. If a one-parameter subgroup $\lambda : \mathbb{G}_m \to \mathrm{SL}(E)$ acts on a vector space and v is a nonzero vector, let $\mathrm{wt}_{\min}(v)$ be the least weight of a nonzero component of v. The weight of $\mathcal{O}_{\mathbb{P}}(1)$ at $\lim_{t\to 0} \lambda(t)[v]$ is $-\mathrm{wt}_{\min}(v)$. We denote the fibre weight of a linearized line bundle by μ. For an ample linearization, the Hilbert–Mumford criterion [39] says that a point is stable precisely when $\mu > 0$ for every nontrivial one-parameter subgroup, and semistable when all these weights are nonnegative.

Lemma 58.1. *If $V(Q_1, Q_2) \subset \mathbb{P}^5$ is smooth, the determinant of its pencil is a binary sextic with six distinct zeros. Every singular member has rank five, and the projective stabilizer of the pencil is finite.*

Proof. Bertini's theorem gives a nonsingular member A: outside the base locus apply Bertini, while on the base locus every nonzero linear combination of the two gradients is nonzero. Write a second member as B, and let $T = A^{-1}B$, which is self-adjoint for the nondegenerate symmetric form A. A repeated eigenvalue would give an eigenvector $v \neq 0$ with $A(v, v) = 0$. If its eigenspace has dimension at least two, choose an isotropic vector in that eigenspace. If it has dimension one, a Jordan chain gives $v = (T - c)w$ and hence $A(v, v) = A(w, (T - c)v) = 0$. Then $B(v, v) = 0$ and $Bv = cAv$, producing a singular point of V. Thus T has six distinct eigenvalues. This proves the assertions about the determinant and ranks.

Conversely, a pencil with six distinct determinant zeros has a nonsingular member A and an A-orthogonal eigenbasis for $A^{-1}B$. In this basis its equations are $\sum y_i^2 = \sum c_i y_i^2 = 0$ with the c_i distinct. Dependent gradients would force a point to be supported in a single coordinate, which cannot satisfy the first equation. Thus its intersection is smooth as well.

A projective transformation preserving the pencil permutes its six singular members. The kernel of this action fixes every member of the pencil. After rescaling a linear representative, it preserves A and B and hence commutes with T. In an A-orthogonal eigenbasis of T it is a diagonal sign matrix. Thus the kernel is finite, and the action on the pencil has finite image. □

Lemma 58.2. *Let $a_0, \dots, a_5$ be the weights of a nontrivial one-parameter subgroup on the linear coordinates, so $\sum a_i = 0$. Let P be a pencil defining a smooth intersection of two quadrics. Choose a basis adapted to the weight filtration and put*

$$d_1 = \mathrm{wt}_{\min}(Q_1) \le d_2 = \mathrm{wt}_{\min}(Q_2).$$

Then

$$d_1 < 0, \qquad d_2 \le -\frac{d_1}{5}, \qquad \mu_\eta(P, \lambda) = -(d_1 + d_2). \tag{58.1}$$

Proof. Being adapted means that the initial forms of the basis give a basis of the associated graded pencil. Thus the initial Plücker coordinate has weight $d_1 + d_2$, proving the last equality.

Suppose first that $d_1 \ge 0$. Every monomial of both quadrics then has nonnegative weight. Every term in their determinant polynomial has total weight $2\sum a_i = 0$. Consequently that determinant is unchanged if both quadrics are replaced by their weight-zero parts. The latter therefore form a pencil with six distinct determinant zeros. By the same elementary linear-algebra criterion used in Lemma 58.1, its intersection is smooth. The one-parameter subgroup fixes both weight-zero quadrics. This contradicts the finiteness of its projective stabilizer. Hence $d_1 < 0$.

If Q_2 is nonsingular, a nonzero determinant term gives $6d_2 \le 0$, which suffices. If Q_2 is singular, its zero in the determinant sextic is simple. The coefficient of s in $\det(Q_2 + sQ_1)$ is therefore nonzero. Each of its terms uses one matrix entry of Q_1 and five of Q_2, and again has total coordinate weight zero. Thus $d_1 + 5d_2 \le 0$. □

Lemma 58.3. *In calculating the Hilbert–Mumford weight of a pair (V, C), one may assume that $\Pi = \langle C \rangle$ is a coordinate plane invariant under λ, without changing any of the three weights for ζ, ξ, η or the smoothness of V and C.*

Proof. Choose an eigenbasis within each weight space. Row reduction relative to the weight filtration gives a unipotent transformation u such that

$$u\Pi = \lim_{t\to 0} \lambda(t)\Pi, \qquad \lim_{t\to 0} \lambda(t) u \lambda(t)^{-1} = 1.$$

Concretely, a basis of the plane consists of pivot vectors plus terms of higher vector weight, and u removes those terms. Such a transformation preserves initial vectors in every representation. Equivalently, u belongs to the unipotent radical of the parabolic associated to λ. It therefore preserves the limits and fibre weights of the three linearized bundles. Replace (V, C) by (uV, uC). The limiting plane is invariant; a change of basis inside the weight spaces makes it a coordinate plane. □

Lemma 58.4. *In the coordinates of Lemma 58.3, let σ be the sum of the three coordinate weights on Π. If q is the quadratic equation of the smooth conic and $b = \mathrm{wt}_{\min}(q)$, then*

$$\mu_\xi = \sigma, \qquad \mu_\zeta = -b, \qquad 3b \le 2\sigma, \qquad b \ge d_1. \tag{58.2}$$

Proof. The plane is a subspace of E, whose vector weights are the negatives of the coordinate weights. Thus $\det \mathcal{S}^*$ has weight σ. The weight of $\mathcal{O}_{\mathcal{H}}(1)$ is $-b$ by its definition. The matrix of q is nonsingular. A nonzero determinant term uses three entries, each with weight at least b, and its total coordinate weight is 2σ. This proves $3b \le 2\sigma$.

The plane is not contained in V. Indeed, on such a plane the normal derivatives of two defining quadrics would give an everywhere surjective map $\mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \to \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2}$ if V were smooth. Its kernel would be $\mathcal{O}_{\mathbb{P}^2}(-2)$, contradicting the equality of second Chern classes in this exact sequence. Therefore some member of P restricts to a nonzero scalar multiple of q. Every element of P has least weight at least d_1. Restricting to an invariant coordinate plane only deletes monomials, so $b \ge d_1$. □

Proposition 58.5. *Every point of W^{sm} is GIT stable for $\mathcal{L}$. More precisely, for every nontrivial one-parameter subgroup,*

$$\begin{aligned} \mu_{\mathcal{L}}((V,C),\lambda) &= 20\sigma - 6b - 66(d_1+d_2) \\ &\geq -\frac{144}{5}d_1 > 0. \end{aligned} \tag{58.3}$$

Proof. Apply the preceding lemmas. From $3b \leq 2\sigma$ and $b \geq d_1$ we get

$$20\sigma - 6b \geq 24b \geq 24d_1.$$

Moreover, $d_1 + d_2 \leq 4d_1/5$. Substitution gives

$$\mu_{\mathcal{L}} \geq 24d_1 - \frac{264}{5}d_1 = -\frac{144}{5}d_1 > 0.$$

The line bundle is ample by Lemma 57.5, so the Hilbert–Mumford criterion applies. □

Remark 58.6. The same calculation proves stability of every smooth pair for $6\zeta + 20\xi + c\eta$ whenever $c > 30$: the lower bound becomes $(24 - 4c/5)d_1 > 0$. The actual CM coefficient is $c = 66$. This is a simultaneous inequality for all one-parameter subgroups, so no enumeration of possible weighted flags is needed.

Example 58.7. Consider coordinates (s, x, z, t_0, t_1, t_2) and

$$Q_1 = zt_0 - st_1 - xt_2, \qquad Q_2 = x^2 + sz + t_0t_2 + t_1^2 + t_2^2.$$

The determinant of the Hessian of $Q_1 + \tau Q_2$ is $4\tau^6 + 4\tau^2 - 1$, which is squarefree. Its leading coefficient is nonzero, so the pencil has six distinct singular members and V is smooth. On $\Pi = \{t_0 = t_1 = t_2 = 0\}$ the conic is $x^2 + sz = 0$. Give the linear coordinates weights $(-1,-1,-1,1,1,1)$. Then

$$(d_1, d_2) = (-2, 0), \quad b = -2, \quad \sigma = -3, \qquad \mu_{\mathcal{L}} = 84 > 0.$$

The conic contribution $20\sigma - 6b = -48$ is negative: the positivity comes from including the pencil contribution $-66(d_1 + d_2) = 132$. For the inverse subgroup the corresponding weight is 180. This calculation illustrates why the three terms of the CM polarization must be kept with their actual coefficients.

59 Comparing a GIT-stable filling with a K-polystable filling

We recall precisely the two K-moduli inputs. A general smooth member of Family 2.16 is K-stable; this is already known from [3, 28]. Properness of K-moduli gives a K-polystable filling, after finite base change, for a punctured family of K-stable members [21]. In this family, [34, Proposition A.13] reconstructs that filling *over the base*, not just on its closed fibre, as the blow-up of a relative intersection of two quadrics along a relative plane conic. Choosing projective frames consequently gives a morphism to W.

The other input is CM-degree minimization: among $\mathbb{Q}$-Gorenstein Fano fillings of a fixed punctured family, a filling with K-semistable closed fibre minimizes the CM degree. We use its local version over a discrete valuation ring. It follows from the filtration comparison in [35, Remark 7.9]; the local difference is the same as the degree difference after gluing identical families away from the marked point.

Definition 59.1. Let R be a discrete valuation ring with parameter t and fraction field K. For two line bundles M, M' on $\operatorname{Spec} R$ with an identification of their generic fibres, define the *relative degree* $\deg_R(M' - M)$ as follows. If e, e' are generators and the generic identification sends e to $t^r v e'$, where $v \in R^*$, put $\deg_R(M' - M) = r$. This does not depend on the generators. It is the degree of the divisor of the rational section induced by the generic identification. For rational line bundles one first takes a common integral multiple and divides by that multiple.

Lemma 59.2. *Let G be reductive, let W be projective with an ample G-linearized line bundle $\mathcal{L}$, and let $w, w' : \operatorname{Spec} R \to W$ have $G(K)$-equivalent generic fibres. Suppose that $w(0)$ is GIT semistable. Then*

$$\deg_R(w'^*\mathcal{L} - w^*\mathcal{L}) \geq 0.$$

If equality holds, $w'(0)$ is also semistable and the two closed points have the same image in $W/\!/_{\mathcal{L}}G$. If $w(0)$ is stable, they belong to the same G-orbit.

Proof. Choose an invariant section $s \in H^0(W, \mathcal{L}^{\otimes m})^G$ which is nonzero at $w(0)$. Its pullback along w is a unit times a generator. Invariance identifies it generically with w'^*s. The latter is a regular section, so its vanishing order is nonnegative. By the definition of relative degree this order is $m \deg_R(w'^*\mathcal{L} - w^*\mathcal{L})$.

If the relative degree is zero, $s(w'(0)) \neq 0$, proving semistability. Both morphisms now lie in the semistable locus, and their images in the separated projective GIT quotient agree on the generic point. Hence they agree on the closed point. Each fibre of the quotient contains a unique closed orbit. If it contains a stable point, that orbit has dimension $\dim G$, and no strictly larger orbit can have it in its closure. Thus the whole fibre is that stable orbit. □

Lemma 59.3. *Let $w \in W^{\mathrm{sm}}$ be GIT stable for the CM polarization $\mathcal{L}$. Suppose a curve through w has K-stable general blow-up and its K-polystable filling admits the relative presentation of [34, Proposition A.13]. Then the blow-up represented by w is K-polystable.*

Proof. Pass to a finite cover of the curve and then its completed local discrete valuation ring. Denote the original smooth blow-up family by $\mathcal{X}$ and a K-polystable filling by $\mathcal{X}'$. The relative presentation and a choice of frames give morphisms $w_R, w'_R : \operatorname{Spec} R \to W$. Their generic fibres differ by an element of $\mathrm{PGL}(6, K)$: the contraction is determined by the pullback of the hyperplane class, and on a smooth member it is the unique birational extremal contraction. After another finite extension this projective transformation lifts to $\mathrm{SL}(6, K)$. Such base changes multiply all relative degrees by the same positive integer.

The canonical generic identification of the two CM lines agrees with that induced by these frames and the linearization in Lemma 57.3. Lemma 59.2 therefore gives

$$\deg_R(\lambda_{\mathrm{CM},\mathcal{X}'} - \lambda_{\mathrm{CM},\mathcal{X}}) \geq 0.$$

Since the closed fibre of $\mathcal{X}'$ is K-polystable, CM-degree minimization gives the reverse inequality. Equality follows. Lemma 59.2 now shows that $w'_R(0)$ belongs to the same orbit as $w_R(0) = w$. Their blow-ups are isomorphic. Hence the original closed fibre is K-polystable. □

Remark 59.4. Only the two actual Fano fillings enter this comparison. An arbitrary one-parameter subgroup may have a limit in W whose blow-up is not a Fano variety. We neither use that limit as a test configuration nor deduce the converse between GIT and K-stability by identifying its weight with a Donaldson–Futaki invariant. Instead, a nonvanishing invariant section gives one inequality for relative CM degrees, and CM minimization gives the other. Thus no classification of the entire GIT boundary, and no unproved global GIT–K-moduli comparison, is required.

60 Completion of the GIT proof

Proof of Theorem 57.1. The point $w = (V, C)$ is stable by Proposition 58.5. The smooth locus W^{sm} is irreducible, and its K-stable locus contains a nonempty open subset by the known general-member result and openness of K-stability. Choose an integral curve through w whose generic point belongs to this open subset, normalize it, and restrict to a neighbourhood of a point above w. Its blow-ups form a smooth Fano family, and its punctured fibres may be taken K-stable.

Properness supplies a K-polystable filling after finite base change. Apply the relative reconstruction in [34, Proposition A.13] and Lemma 59.3. We conclude that $X = \mathrm{Bl}_C V$ is K-polystable. Finally, every automorphism of X preserves its unique birational extremal contraction, so it descends to a projective automorphism of V preserving C. Lemma 58.1 shows that this group is finite. K-polystability with a finite automorphism group is K-stability, completing the proof. □

Remark 60.1. No condition on the conic-bundle discriminant is used; double fibres are included. This proof is independent of the local threshold estimates and gives no new numerical lower bound for $\delta(X)$.

Part X
Family 2.17: line extractions and weighted points

61 K-stability of Family 2.17

Let

$$\pi : X = \mathrm{Bl}_C(\mathbb{P}^3) \longrightarrow \mathbb{P}^3, \qquad q : X = \mathrm{Bl}_{C'}(Q) \longrightarrow Q$$

be the two contractions of a smooth member of Family 2.17. Here C is a smooth elliptic quintic, $Q \subset \mathbb{P}^4$ is a smooth quadric, and C' is a smooth elliptic normal quintic. Write

$$H = \pi^*\mathcal{O}_{\mathbb{P}^3}(1), \quad B = q^*\mathcal{O}_Q(1), \quad E = \mathrm{Exc}(\pi), \quad E' = \mathrm{Exc}(q), \quad D = E \cap E', \quad L = -K_X.$$

The intersection defining D is scheme-theoretic. The standard relations are

$$B = 3H - E, \quad H = 2B - E', \quad E' = 5H - 2E, \quad E = 5B - 3E', \qquad L = H + B, \qquad L^3 = 24. \tag{61.1}$$

Cheltsov–Pokora [40, Main Theorem] proved that a prime divisor over X with nonpositive beta invariant has centre a closed point of D. We complete the point analysis and prove the following theorem.

Theorem 61.1. *Every smooth Fano threefold in Family 2.17 is K-stable. At each point of the triple-contact set Σ_X defined in (65.1), one has $\delta_p(X) \geq 64/59$.*

We first prove the reduction supplied by the two line extractions. The final weighted point calculation then treats its remaining centres.

Proposition 61.2. *If V is a prime divisor over X with $\beta_X(V) \leq 0$, its centre belongs to Σ_X. In particular, X is K-stable if $\Sigma_X = \varnothing$.*

Remark 61.3 (The two refinements). Smoothness of X does not imply smoothness of D. At the remaining triple-contact points, the first line refinement has expected order $49/24$ at a surface point blow-up of discrepancy two. We retain this calculation and use a different extraction. Its actual weighted section ring gives $S_X(G) = 29/8$ and refined threshold at least $64/59$. The proof of Theorem 61.1 is completed in Section 68.

Lemma 61.4. *The divisor D is a reduced curve. Its maps to C and C' are finite flat of degrees two and three. Through a point of D the two contracted ruling lines have distinct tangent directions. The map $(\pi, q) : X \to \mathbb{P}^3 \times Q$ is a closed immersion, so L is very ample.*

Proof. For a fibre ℓ of $E \to C$, one has $H\ell = 0$, $E\ell = -1$, and $B\ell = 1$. Thus $q|_\ell$ is an isomorphism onto a line. This line cannot be contained in the elliptic curve C', so ℓ is not contained in E'. Similarly a fibre ℓ' of $E' \to C'$ has $H\ell' = 1$ and cannot be contained in E. Since $E'\ell = 2$ and $E\ell' = 3$, the stated maps are finite of degrees two and three.

Write $D = \sum n_i D_i$ as a divisor on E. Every component dominates both elliptic curves. If its two degrees are a_i, b_i, then $\sum n_i a_i = 2$ and $\sum n_i b_i = 3$. A multiplicity $n_i \geq 2$ would force $D = 2D_i$, contradicting the second equality. Thus D is generically reduced. It is a Cartier divisor on the smooth surface E, so has no embedded points and is reduced. Its structure sheaf is torsion free over each smooth base curve and hence flat.

The tangent direction of ℓ' is killed by dq, whereas dq is nonzero on the tangent direction of ℓ. They are therefore distinct. The same argument shows that (π, q) has injective differential everywhere: off E the first map is an isomorphism, and on E its kernel is the ruling direction, on which dq is injective. It is injective on geometric points because q embeds every π-fibre. This proper, universally injective, unramified map is a closed immersion. The last assertion follows by restricting $\mathcal{O}(1,1)$. □

62 Blowing up a contracted line

Fix a fibre ℓ of $E \to C$, and let $\sigma : Y = \mathrm{Bl}_\ell(X) \to X$ have exceptional divisor F. The normal sequence of $\ell \subset E \subset X$ is

$$0 \longrightarrow \mathcal{O}_\ell \longrightarrow N_{\ell/X} \longrightarrow \mathcal{O}_\ell(-1) \longrightarrow 0.$$

It splits since $H^1(\mathbb{P}^1, \mathcal{O}(1)) = 0$. Consequently $F \cong \mathbb{F}_1$. Write s for its negative section and f for its ruling fibre. Then

$$s^2 = -1, \quad sf = 1, \quad f^2 = 0, \quad \sigma^*L|_F = f, \quad -F|_F = s + f, \quad A_X(F) = 2. \tag{62.1}$$

The section s is the intersection with the strict transform of E.

The line $B_0 = q(\ell) \subset Q$ meets C' in a scheme of length two. Indeed, $\ell \to B_0$ is an isomorphism and $E'\ell = 2$. To distinguish the two models, put $Z = \mathrm{Bl}_{B_0}(Q)$ and $Y' = \mathrm{Bl}_{C_Z}(Z)$, where C_Z is the strict transform of C'. In the next two lemmas the symbol H_Q denotes the pullback of $\mathcal{O}_Q(1)$, and A denotes the plane hyperplane class coming from projection from B_0.

Lemma 62.1. *Let $B \subset Q$ be a line meeting the smooth elliptic normal quintic $C' \subset Q$ in a scheme of length two. Put $Z = \mathrm{Bl}_B(Q)$, and write H_Q for the pullback of $\mathcal{O}_Q(1)$ and F_B for its exceptional divisor. Projection from B is a $\mathbb{P}^1$-bundle $r : Z \to \mathbb{P}^2$, with $A = r^*\mathcal{O}_{\mathbb{P}^2}(1) = H_Q - F_B$. The strict transform C_Z maps isomorphically to a smooth plane cubic T. If $f : Y' = \mathrm{Bl}_{C_Z}(Z) \to Z$ has exceptional divisor J, then $f^*(H_Q + 2A) - J$ is nef and semiample, with cube* 6.

Proof. Choose coordinates in which

$$Q = \{x_0x_3 + x_1x_4 + x_2^2 = 0\}, \qquad B = \{x_2 = x_3 = x_4 = 0\}.$$

Writing $[x_2 : x_3 : x_4] = [a : b : c]$ gives the equation $bx_0 + cx_1 + a^2t = 0$ on the resolved projection. The three coefficients have no common zero, so its fibres are projective lines, including the fibre over $[1 : 0 : 0]$. With the quotient convention for projective bundles, we may write $Z = \mathbb{P}(\mathcal{V})$, where

$$0 \longrightarrow \mathcal{O}(-2) \xrightarrow{(b,c,a^2)} \mathcal{O}(-1)^{\oplus 2} \oplus \mathcal{O} \longrightarrow \mathcal{V} \longrightarrow 0.$$

Here $c_1(\mathcal{V}) = 0$, $c_2(\mathcal{V}) = 1$; its tautological class ξ satisfies $H_Q = \xi + A$ and $F_B = \xi$.

The linear forms through B restrict to the complete degree-three linear series $\mathcal{O}_{C'}(1)(-B \cap C')$. Indeed they give three independent sections (otherwise C' would lie in a hyperplane), and Riemann–Roch gives dimension three. A degree-three line bundle on a smooth elliptic curve is very ample. This proves the assertion about C_Z and T, also when $B \cap C'$ is nonreduced.

The section $C_Z \subset \mathbb{P}(\mathcal{V}|_T)$ corresponds to a quotient $\mathcal{V}|_T \twoheadrightarrow \mathcal{M}$ with

$$\deg \mathcal{M} = \xi \cdot C_Z = (H_Q - A) \cdot C_Z = 5 - 3 = 2.$$

Define the elementary transform

$$0 \longrightarrow \mathcal{V}_1 \longrightarrow \mathcal{V} \longrightarrow i_*\mathcal{M} \longrightarrow 0, \qquad i : T \hookrightarrow \mathbb{P}^2.$$

It is locally free of rank two: locally the last map is the projection $\mathcal{O}^{\oplus 2} \to \mathcal{O}/(t)$, whose kernel is generated by $(t, 0)$ and $(0, 1)$. Its Chern classes are $c_1(\mathcal{V}_1) = -3$ and $c_2(\mathcal{V}_1) = 3$. For example $\mathrm{ch}(i_*\mathcal{M}) = 3h + (2 - 9/2)h^2$, where $h = c_1(\mathcal{O}_{\mathbb{P}^2}(1))$.

Put $\mathcal{W} = \mathcal{V}_1(2)$; then $c_1(\mathcal{W}) = 1$, $c_2(\mathcal{W}) = 1$. We claim that $\mathcal{W} \simeq T_{\mathbb{P}^2}(-1)$. First $H^0(\mathcal{W}(-1)) = H^0(\mathcal{V}_1(1)) = 0$: such a section would give a member of $|H_Q|$ containing C_Z,

hence a hyperplane of $\mathbb{P}^4$ containing the nondegenerate curve C'. Thus $\mathcal{W}$ is slope stable. Riemann–Roch gives $\chi(\mathcal{W}) = 3$, and Serre duality and stability give $H^2(\mathcal{W}) = 0$. A nonzero section has no divisorial zeros by stability, and its zero scheme has length $c_2 = 1$. It therefore gives

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{W} \longrightarrow \mathcal{I}_z(1) \longrightarrow 0$$

for one reduced point z. The extension space $\operatorname{Ext}^1(\mathcal{I}_z(1), \mathcal{O})$ is one-dimensional; local freeness excludes its zero element. The Euler sequence supplies this same nonzero extension for $T_{\mathbb{P}^2}(-1)$, proving the claim.

The elementary transform is resolved by Y', giving a morphism $g : Y' \to \mathbb{P}(\mathcal{V}_1) = \mathbb{P}(\mathcal{W})$. Locally its rational map is $[u : v] \mapsto [tu : v]$, whose ideal of indeterminacy is (t, v), the ideal of C_Z. It follows that

$$g^*\mathcal{O}_{\mathbb{P}(\mathcal{V}_1)}(1) = f^*\xi - J, \qquad f^*(H_Q + 2A) - J = g^*(\mathcal{O}_{\mathbb{P}(\mathcal{W})}(1) + h).$$

On $\mathbb{P}(T_{\mathbb{P}^2}(-1))$, the tautological divisor is generated by global sections, is relatively ample, and its sum with the pullback of h is ample. If its class is η, the projective-bundle relations give $\eta^3 = c_1(\mathcal{W})^2 - c_2(\mathcal{W}) = 0$, $\eta^2 h = 1$, and $\eta h^2 = 1$. Hence $(\eta + h)^3 = 6$. □

Lemma 62.2. *Let B and C be smooth curves in a smooth threefold, with finite scheme-theoretic intersection of length two. Suppose either that their intersection consists of two reduced points, or that it is supported at one point and has length two there. Blowing up C and then the strict transform of B, and blowing up B and then the strict transform of C, give smooth threefolds related by a flop. In the first case there are two ordinary flopping curves. In the second case there is one flopping curve of width two; its contraction has analytic equation*

$$ab = x(x - z^2).$$

In the second case the transform on the second model of the exceptional surface of the second blow-up on the first model has one A_1 singularity. Its induced morphism to the original exceptional surface is locally the blow-up of the curvilinear ideal (b, z^2) in a smooth surface with coordinates (b, z).

Proof. At an intersection point of length $k \in \{1, 2\}$, choose an analytic parameter z on B. In the tangent case it is also a parameter on C. For $k = 1$ choose the parameter so that it is a parameter on both curves. Two transverse coordinates can then be chosen such that

$$B = \{x = y = 0\}, \qquad C = \{x - z^k = y = 0\}.$$

Indeed, first make B the coordinate axis. One transverse coordinate of C has order k and can be made z^k, and the other has order at least k and is removed by subtracting a multiple of the first.

On the chart of Bl_C where $y = t(x - z^k)$, the strict transform of B is (t, x). On the chart of its blow-up with $x = bt$, the coordinates are (b, t, z) and

$$x = bt, \qquad y = t(bt - z^k).$$

Conversely, on the chart of Bl_B where $y = vx$, the strict transform of C is $(v, x - z^k)$. The corresponding chart after the second blow-up has coordinates (b, v, z), with

$$x = z^k + bv, \qquad y = v(z^k + bv).$$

These are the two small resolutions of

$$W_k = \{ab = x(x - z^k)\},$$

given by the ideals (b, x) and $(b, x - z^k)$; the displayed charts set $a = y$. The second chart of each small resolution is also smooth, and the exceptional fibre is $\mathbb{P}^1$. The morphisms are crepant because they are small resolutions of a Gorenstein hypersurface. Thus their birational exchange is a flop. For $k = 1$ the hypersurface has an ordinary node; for $k = 2$ it has a cA_3 singularity and the width-two local form above. The two distinct length-one intersection points give disjoint local flops.

In the first model, the exceptional surface of the blow-up of the strict transform of B is $t = 0$ in the displayed chart. On the second model, its strict transform is

$$bv + z^k = 0.$$

For $k = 2$ this is an A_1 surface singularity. Its morphism to the smooth surface $\{t = 0\}$, with coordinates (b, z), is the blow-up of (b, z^k). For $k = 1$ this is the ordinary blow-up of a point. For $k = 2$ its minimal resolution is obtained by first blowing up $(b, z) = (0, 0)$ and then blowing up the point on the first exceptional curve corresponding to the tangent direction $b = 0$. If e_1, e_2 denote the total transform classes of the exceptional curves of these two ordinary blow-ups, the pullback of the exceptional Cartier divisor of the blow-up of (b, z^2) is $e_1 + e_2$; its irreducible components on the resolution are $(e_1 - e_2) + 2e_2$. □

Remark 62.3. In particular, two distinct point conditions on the exceptional surface must be replaced in the tangent case by the curvilinear ideal (b, z^2). After taking the smooth minimal resolution, these become two successive infinitely near point conditions with equal multiplicity. It is incorrect to regard the exceptional surface on the flopped threefold as smooth in this case.

Lemma 62.4. *Retain the elementary-transform model $Z = \mathrm{Bl}_B(Q) \to \mathbb{P}^2$. Write $F_B \cong \mathbb{F}_1$ for its exceptional divisor, with negative section c and ruling fibre f, and let $o \in \mathbb{P}^2$ be the image of c. Let C_Z be the strict transform of the elliptic quintic, $T \subset \mathbb{P}^2$ its smooth cubic image, and $\Xi = C_Z \cap F_B$ the intersection scheme of length two. Let $F \cong \mathbb{F}_1$ be the exceptional surface on the other side of the flop, with negative section s and ruling fibre again denoted by f. Let R be the fixed curve obtained on F from the strict transform of $r^{-1}(T)$ in $\mathrm{Bl}_{C_Z}(Z)$. Then R is reduced, contains no ruling fibre, and has class $3s + 4f$. More precisely:*

1. *If $o \notin T$, the curve R is irreducible with elliptic normalization and does not contain s.*
2. *If $o \in T$, one has $R = s + R_1$, where R_1 is irreducible with elliptic normalization, $R_1 \sim 2s + 4f$, and R_1 contains neither s nor a ruling fibre.*

If Ξ is supported at one point, the corresponding point $A \in F$ lies on s precisely in the second case.

Proof. The map $F_B \to \mathbb{P}^2$ is the blow-up of o. Therefore its inverse image R_B of T is a smooth elliptic curve of class $3c + 3f$ if $o \notin T$. If $o \in T$, it is the reduced union of c and the smooth elliptic strict transform of T, of class $2c + 3f$. These two components meet transversely at one point. In particular, R_B has no ruling fibre.

First suppose $\Xi = \{p_1, p_2\}$ is reduced. The points lie on distinct ruling fibres, since their images are the two distinct points of $B \cap C'$. On the common resolution $\widetilde{F} = \mathrm{Bl}_{p_1,p_2}(F_B)$, write e_1, e_2 for the exceptional curves. Its map to F contracts the disjoint curves $L_i = f - e_i$, and

$$\rho^* s = c + f - e_1 - e_2, \qquad \overline{R} = 3c + 3f - e_1 - e_2 = \rho^* R - 2L_1 - 2L_2.$$

At each p_i the divisor R_B is smooth. Indeed, if R_B were singular there, the smooth surfaces F_B and $r^{-1}(T)$ would have the same tangent plane; as $C_Z \subset r^{-1}(T)$, their intersection with C_Z would have multiplicity at least two, contrary to the reducedness of Ξ. Thus $\overline{R}$ is just the strict transform of R_B. Pushing forward gives the asserted class and introduces no additional

component. If $o \in T$, exactly one p_i, say p_1, lies on c. The image of $c - e_1$ is s, because $\rho^* s = (c - e_1) + L_2$. The other component gives R_1. If $o \notin T$, neither p_i lies on c, and s maps to a line in $\mathbb{P}^2$; it cannot be a component of the irreducible cubic transform.

Now suppose Ξ is a curvilinear scheme of length two supported at p. Its tangent direction is transverse to the ruling fibre of F_B through p. To see this, choose a parameter z along B at the tangency point of C' and B. The smooth curve C' is tangent to B, so z is also a parameter on C'. Since the intersection length is exactly two, $\mathcal{I}_B \mathcal{O}_{C'} = (z^2)$. The length-two scheme Ξ therefore maps isomorphically to $\operatorname{Spec} \mathbb{C}[z]/(z^2)$ on B.

Resolve the blow-up of Ξ by blowing up p and then the point in its specified tangent direction. Denote the total exceptional classes by e_1, e_2. The curve $K = e_1 - e_2$ has self-intersection -2. The map to F contracts first $L = f - e_1$ and then the image of K; its negative section has pullback $c + f - e_1 - e_2$. The strict transform of F_B in the threefold has an A_1 singularity, and its minimal resolution contracts K. The pullback of the intersection with the strict transform of $r^{-1}(T)$ has class $3c + 3f - e_1 - e_2$.

If $o \notin T$, the divisor R_B is smooth and contains Ξ, so its successive multiplicities are $1, 1$. Its strict transform is exactly this pullback. If $o \in T$, the unique point of C_Z over o lies on c, so $p \in c$. Since C_Z is tangent to F_B, the restriction of r to F_B shows that p is also the intersection of c with the strict transform of T. Thus R_B has a node at p. The tangent direction of Ξ is neither branch of this node: otherwise the intersection of C_Z with F_B would have length at least three. The successive multiplicities of R_B are therefore $2, 0$. If R_B^{str} denotes its strict transform, the pulled-back intersection divisor is $R_B^{\mathrm{str}} + K$. The extra curve K is contracted by the map to F and contributes no component to R. The image of the strict transform $c - e_1$ is s. This proves reducedness and the absence of ruling fibres in all cases.

Finally, when $o \notin T$, the negative section on F is the transform of the unique plane line with the prescribed first-order contact at p; it is disjoint from the contracted chain, so $A \notin s$. When $o \in T$, the strict transform $c - e_1$ meets K, so its image s contains A. □

Lemma 62.5. *In the notation of Lemma 62.4, the following lower bounds hold for the log canonical threshold of the reduced divisor R on F:*

	$o \notin T$	$o \in T$
Ξ *reduced*	5/6	3/4
Ξ *nonreduced*	7/10	5/8

For nonreduced Ξ, the unique singular point of R is A; its singularity is A_3 or A_4 if $o \notin T$, and D_4 or D_5 if $o \in T$.

Proof. Suppose first that $o \notin T$. The curve R is irreducible, has arithmetic genus three, and has elliptic normalization. For two reduced centres, each contracted L_i meets its smooth strict transform in a scheme of length two; the resulting singularity has multiplicity two and delta-invariant one. It is a node or an ordinary cusp. The two singular points are distinct. For a nonreduced centre the contracted chain produces just one singular point, of multiplicity two and delta-invariant two. Thus it is A_3 or A_4. The thresholds of A_n, in the equation $y^2 = x^{n+1}$, are $\min\{1, 1/2 + 1/(n+1)\}$, proving the first column.

Suppose $o \in T$, so $R = s + R_1$. The curve R_1 has arithmetic genus two and elliptic normalization. For reduced centres, its unique singularity is a node or a cusp at the point $A_1 \notin s$. The two intersections counted by $s \cdot R_1 = 2$ are either distinct transverse intersections, or one ordinary tangency. The latter gives an A_3 singularity of the union. This proves the upper right entry. For a nonreduced centre, the only singularity of R_1 is at $A \in s$, and both intersections with s are concentrated there. If R_1 is nodal, s is transverse to both branches, giving an ordinary triple point D_4. If R_1 is cuspidal, s is transverse to its tangent, giving D_5. Their thresholds are $2/3$ and $5/8$, respectively. The latter can also be read directly from the weighted homogeneous equation $x(y^2 - x^3) = 0$, with weights $(2, 3)$. □

Lemma 62.6 (The threefold chambers)**.** *For the exceptional divisor F of σ, the pseudoeffective threshold is three, and*

$$\operatorname{vol}(\sigma^*L - uF) = \begin{cases} 24 - 3u^2 - u^3, & 0 \le u \le 1, \\ 22 + 6u - 9u^2 + u^3, & 1 \le u \le 2, \\ 6(3-u)^3, & 2 \le u \le 3. \end{cases} \tag{62.2}$$

In particular,

$$\begin{aligned} S_X(F) &= \frac{1}{24}\left(\int_0^1 (24 - 3u^2 - u^3)\,du + \int_1^2 (22 + 6u - 9u^2 + u^3)\,du \right. \\ &\qquad \left. + \int_2^3 6(3-u)^3\,du\right) \\ &= \frac{91}{96} + \frac{55}{96} + \frac{1}{16} = \frac{19}{12}. \end{aligned} \tag{62.3}$$

Proof. Since L embeds ℓ as a line, hyperplanes containing this line generate $\mathcal{I}_\ell \otimes L$. Thus $\sigma^*L - F$ is base-point-free, and $\sigma^*L - uF$ is nef for $0 \le u \le 1$. The blow-up intersection formulas give $\sigma^*L^2F = 0$, $\sigma^*L\,F^2 = -1$ and $F^3 = -\deg N_{\ell/X} = 1$. This proves the first cubic.

Let J be the exceptional divisor of $Y' \to Z$. On this model the transform of $\sigma^*L - uF$ is

$$M_u = (3-u)H_Q + uA - J.$$

The divisor M_1 is the transform of the base-point-free divisor above; it descends to the common small contraction in Lemma 62.2 and remains nef. Lemma 62.1 proves that $M_2 = H_Q + 2A - J$ is the pullback of an ample divisor on $\mathbb{P}(T_{\mathbb{P}^2}(-1))$. Hence every M_u for $1 \le u \le 2$ is nef.

On Z one has

$$H_Q^3 = 2, \quad H_Q^2 A = 2, \quad H_Q A^2 = 1, \quad A^3 = 0, \qquad H_Q C_Z = 5, \quad A C_Z = 3.$$

Moreover $-K_Z = 2H_Q + A$, so $\deg N_{C_Z/Z} = 13$. If $K_u = (3-u)H_Q + uA$, the blow-up formula is $M_u^3 = K_u^3 - 3K_u C_Z + 13$. Expanding gives the second cubic in (62.2). This calculation applies equally to the width-two flop; no formula for two distinct ordinary flops is substituted in the tangent case.

Write $R_Y = 3A - J$ for the strict transform of $r^{-1}(T)$ on Y'. It is the exceptional divisor of the elementary transform $g : Y' \to \mathbb{P}(T_{\mathbb{P}^2}(-1))$. A contracted ruling has intersection -1 with R_Y and zero with M_2. For $2 \le u \le 3$,

$$M_u = (3-u)M_2 + (u-2)R_Y.$$

Intersecting an effective divisor with the moving rulings of R_Y forces the coefficient $u-2$ of R_Y. Removing it leaves the nef, semiample divisor $(3-u)M_2$. This proves the last chamber and its volume. For $u > 3$, intersect with a movable complete-intersection curve pulled back from the target of g and avoiding its exceptional image. The intersection is negative, so M_u is not pseudoeffective. Finally integrate the three cubics to obtain (62.3). □

Lemma 62.7 (The actual refinement is controlled by surface spaces)**.** *Let $\mathcal{I}$ on F be the ideal of the two marked points in the reduced case, and the curvilinear ideal of Lemma 62.2 otherwise. Let R be the pushforward to F of $R_Y|_{F'}$, where F' is the transform of F on Y'. The refinement of $H^0(X, mL)$ by F is contained, up to rounding, in the following spaces:*

$j/m = u$	*containing space on F*
$[0,1]$	$H^0(F, \mathcal{O}_F(js + (j+m)f))$
$[1,2]$	$H^0(F, \mathcal{O}_F(js + (j+m)f) \otimes \mathcal{I}^{j-m})$
$[2,3]$	$R^{j-2m} \cdot H^0(F, \mathcal{O}_F(2(3m-j)s + 3(3m-j)f) \otimes \mathcal{I}^{3m-j})$.

(62.4)

The notation $R^a \cdot H^0(\cdots)$ means multiplication by the ath power of a defining section of R. In particular, the fixed divisor must be retained in every expected-order calculation.

Proof. The first row follows from (62.1). To obtain the second, restrict on Y' and pull back to a smooth resolution of F'. For distinct points the restriction is the pullback of $js + (j+m)f$ minus $j-m$ times each exceptional curve. For a double point the same statement uses the total exceptional classes e_1, e_2 and the divisor $(j-m)(e_1+e_2)$. By the local calculation this is precisely the power of the ideal (b, z^2), not two reduced conditions at the same point.

For the third row the fixed part in Lemma 62.6 forces $(j-2m)R$. On F the curve R has class $3s+4f$ and belongs to $\mathcal{I}^2$. Subtracting its class gives

$$js + (j+m)f - (j-2m)(3s+4f) = 2(3m-j)s + 3(3m-j)f,$$

and subtracting its orders at the marked scheme leaves $(j-m)-2(j-2m) = 3m-j$. We justify this division, rather than relying only on the membership $R \in \mathcal{I}^2$. At a reduced marked point R has multiplicity exactly two. At a curvilinear point the successive multiplicities on the two ordinary point blow-ups are $(2,2)$ off s and $(3,1)$ on s. In both cases its value for the valuation with weights $(2,1)$ is exactly four. The corresponding initial form has a nonzero coefficient after reduction to the residue field and is a nonzerodivisor in the polynomial associated graded ring of $\mathcal{I} = (b, z^2)$. Equivalently, this follows directly from the complete monomial ideals $(b, z^2)^n$, whose elements have weighted order at least $2n$. Consequently

$$(\mathcal{I}^n : R^k) = \mathcal{I}^{n-2k} \qquad (n \geq 2k).$$

The ordinary maximal-ideal version gives the same identity at the two reduced points. This proves all the inclusions. They suffice for the upper bounds below: a valuation-adapted basis can be extended to a basis of a containing space, adding an effective basis divisor with the original normalization. □

Remark 62.8. Here the decomposition on a modified threefold is actually proved: the elementary transform identifies its nef and fixed parts, and the intersection calculation gives their volumes. This differs from the weighted section-ring arguments in the preceding parts. In particular, the fact that those arguments avoided a threefold Zariski decomposition does not remove the need to justify the present chambers or the actual refined spaces.

63 Three refined surface calculations for Family 2.17

We record the surface estimates separately from the construction of the threefold modification. This separation is useful: a numerical estimate for the first exceptional divisor does not supply the estimate for its refined series.

Let $F = \mathbb{F}_1$, let s be its negative section, and let f be a fibre of its ruling. Thus $s^2 = -1$, $s \cdot f = 1$, and $f^2 = 0$. Choose a section $t \sim s + f$ disjoint from s and two fibres f_0, f_∞. We use Cox coordinates x, y, z, w defining s, t, f_0, f_∞, respectively. A basis of $H^0(F, \mathcal{O}_F(as+bf))$ consists of the monomials

$$x^i y^{a-i} z^k w^{b-a+i-k}, \qquad 0 \leq i \leq a, \quad 0 \leq k \leq b-a+i. \tag{63.1}$$

The statements below concern the series obtained by refining along the exceptional divisor of the blow-up of a fibre of $E_{\mathbb{P}^3} \to C$. They also apply if the actual refined spaces are only known to be contained in the indicated spaces, with the same asymptotic total dimension.

Lemma 63.1 (The surface spaces and their normalization)**.** *Suppose the refined spaces, indexed by $u = j/m$, have the following asymptotic description. Write $c = 3 - u$ in the last interval.*

u	a	b	r	*fixed divisor*$/m$
$[0,1]$	u	$u+1$	0	0
$[1,2]$	u	$u+1$	$u-1$	0
$[2,3]$	$2c$	$3c$	c	$(u-2)R$

Here the moving sections belong to $H^0(F, \mathcal{O}_F(mas + mbf) \otimes \mathcal{I}^{mr})$, where $\mathcal{I}$ is either the ideal of two distinct points or the ideal of a curvilinear subscheme of length two. In the distinct case $\mathcal{I}^{mr}$ means the equal multiplicity conditions at the two points. Rounding the indices changes the formulas below by $O(m^{-1})$. The divisor R has class $3s + 4f$ and vanishes to order two along $\mathcal{I}$.

In each of the three configurations in Lemma 63.2, the area of the polygon of moving monomials is

$$A(u) = \begin{cases} (u^2 + 2u)/2, & 0 \le u \le 1, \\ (-u^2 + 6u - 2)/2, & 1 \le u \le 2, \\ 3(3-u)^2, & 2 \le u \le 3. \end{cases}$$

In particular $\int_0^3 A(u)\,du = 4$. The normalization is therefore $N_m = 4m^3 + O(m^2)$, and the average coefficient of the fixed divisor R is $1/16$.

Proof. The complete polygon in (63.1) has area $a(b-a) + a^2/2$. In the second and third intervals the marked-point conditions remove area r^2. For two distinct points this is the sum of two triangles of area $r^2/2$. For the curvilinear ideal used below it is one triangle with side lengths r and $2r$. The displayed formulas follow by substitution. The fixed coefficient is

$$\frac{1}{4}\int_2^3 (u-2)A(u)\,du = \frac{3}{4}\int_2^3 (u-2)(3-u)^2\,du = \frac{1}{16}.$$

If one has only an inclusion of refined spaces, equality of the leading total dimensions leaves $O(m^2)$ omitted sections. All the exponents are $O(m)$, so their omission changes a normalized first moment by $O(m^{-1})$. Thus the same limiting first moments give upper bounds for the actual refined series. □

Lemma 63.2 (The three toric estimates)**.** *Let W be the series in Lemma 63.1. The following estimates hold.*

1. *If the two marked points are distinct, lie off s, and lie on distinct fibres, then $\delta(F; W) \ge 24/23$.*
2. *If one marked point lies on s and the other lies off s on a different fibre, and R contains s with multiplicity one, then $\delta(F; W) \ge 96/95$.*
3. *Suppose the marked scheme is supported off s, is transverse to its ruling fibre, and has ideal (y, z^2) after a choice of t and f_0 as above. If R does not contain f_0, then $\delta(F; W) \ge 16/15$.*

Proof. In the first case an automorphism of F places the points at $t \cap f_0$ and $t \cap f_\infty$. Indeed, after choosing coordinates on the base, translations by sections of $\mathcal{O}_{\mathbb{P}^1}(1)$ can prescribe the two fibre coordinates. The extra polygon inequalities are

$$a - i + k \ge r, \qquad a - i + b - a + i - k \ge r.$$

In the second case we place the points at $s \cap f_0$ and $t \cap f_\infty$. The inequalities become

$$i + k \geq r, \qquad a - i + b - a + i - k \geq r.$$

In the third case they are replaced by $2(a - i) + k \geq 2r$.

For completeness, the moving first moments are

	s	t	f_0	f_∞
two points off s	$35/48$	$2/3$	$5/6$	$5/6$
one point on s	$79/96$	$55/96$	$5/6$	$89/96$
curvilinear point off s	$35/48$	$2/3$	$15/16$	$35/48$.

(63.2)

Here is a direct check of every entry. Without the marked-point conditions the four moments are $23/24, 17/24, 49/48, 49/48$. Put

$$\mathcal{J}(\phi) = \frac{1}{4}\int_1^2 \phi(u, u+1, u-1)\, du + \frac{1}{4}\int_2^3 \phi(2c, 3c, c)\, du.$$

The moments removed by the marked conditions are obtained by applying $\mathcal{J}$ to the entries of the following table:

	s	t	f_0	f_∞
two off s	$r^2(a - r/3)$	$r^3/3$	$r^2(b - r/3)/2$	$r^2(b - r/3)/2$
one on s	$r^2 a/2$	$r^2 a/2$	$r^2(b - r/3)/2$	$r^2(b - a + r/3)/2$
curvilinear off s	$r^2(a - r/3)$	$r^3/3$	$2r^3/3$	$r^2(b - r)$.

For example, in the first row the two removed triangles have centroids with s-coordinate $a - r/3$. Their f_0-coordinates are $r/3$ and $b - 2r/3$. The other entries are obtained in exactly the same way. These are integrals of explicitly displayed polynomials; they give (63.2).

It remains to include the fixed divisor. Its monomials have the form

$$x^i y^{3-i} z^k w^{1+i-k}. \tag{63.3}$$

In the first case the double-point conditions imply $k \geq i - 1$ and $k \leq 2$. Thus the four exponents are at most $3, 3, 2, 2$, respectively. After adding the coefficient $1/16$ of R, all four coefficients are at most $23/24$.

In the second case the smallest x-exponent of R is one. Among the terms with this exponent the double-point conditions give $1 \leq k \leq 2$. Choose a torus degeneration selecting one such term. Its initial divisor is $s + 2t + kf_0 + (2 - k)f_\infty$. The four total coefficients are bounded by $85/96, 67/96, 23/24, 95/96$, respectively.

In the third case first minimize the z-exponent. The hypothesis that R does not contain f_0 makes this minimum zero. Since $R \in (y, z^2)^2$, a term with $k = 0$ in (63.3) has $i \leq 1$. The four total coefficients are consequently at most $19/24, 41/48, 15/16, 41/48$.

We justify why these monomial estimates control every divisorial valuation. Fix such a valuation v on F. In each graded piece choose a basis compatible with v and with a one-parameter subgroup of the torus. The two-filtration lemma permits this choice. For each m the subgroup can be chosen in the indicated open cone of weights and can distinguish all the finitely many monomials occurring in the moving spaces. The initial basis is then their monomial basis, and the fixed divisor has the initial monomial just specified. The limiting basis boundary is supported on s, t, f_0, f_∞, a simple normal crossings divisor. For every positive number strictly less than $24/23$, $96/95$, or $16/15$, respectively, its multiple is log canonical for all sufficiently large m. Semicontinuity of log canonical thresholds gives the same assertion before degeneration. The bounds are independent of v. Letting m tend to infinity proves the three assertions. □

Example 63.3. In the second case suppose the lowest x-degree part of R is xy^2zw. The total coefficients of the limiting basis boundary are $85/96, 67/96, 43/48, 95/96$. Thus the worst coefficient occurs on f_∞, although the marked point on the negative section lies on f_0. Keeping the marked-point conditions is essential: omitting them would replace each of the two fibre moments by $49/48$.

Lemma 63.4 (The remaining curvilinear configuration)**.** *Suppose the marked scheme is supported at $A \in s$, its infinitesimal direction is transverse both to s and to the ruling fibre, and R has multiplicity three at A. In local coordinates x, z with $s = (x = 0)$ and ruling fibre $(z = 0)$, write its ideal as $(x + z, z^2)$. For the complete surface spaces in Lemma 63.1, the moving expected order of the ordinary blow-up of A is $89/48$. Including the fixed divisor gives*

$$S(W; \operatorname{ord}_A) = \frac{89}{48} + \frac{3}{16} = \frac{49}{24} > 2.$$

Consequently this exceptional-surface refinement does not establish K-stability in this configuration. If the refined spaces have the same leading dimensions as these complete spaces, the displayed expected order is exact.

Proof. Put $b = x + z$ and group the monomials in (63.1) by their ordinary degree d at A. In the first two intervals their x-exponents run from $i_0 = \max\{0, (d-1)/2\}$ to $i_1 = \min\{u, d\}$. Set $n = i_1 - i_0$ and $q = \max\{0, 2r - d\}$. The corresponding binary space is

$$(b - z)^{i_0} z^{d-i_1} \operatorname{Sym}^n \langle b, z \rangle.$$

Membership in $(b, z^2)^r$ forces divisibility by b^q. The other displayed factors are coprime to b, so the asymptotic dimension of the degree-d slice is $n - q$ when $n \geq q$ and is zero otherwise. Its ordinary first moment is $d(n - q)$.

Here are the integration intervals and the values of n, q; thus the calculation can be checked without an implicit chamber choice:

u	d	n	q
$[0,1]$	$[0,u]$	d	0
$[0,1]$	$[u,1]$	u	0
$[0,1]$	$[1,1+2u]$	$u-(d-1)/2$	0
$[1,3/2]$	$[u-1,2u-2]$	d	$2u-2-d$
$[1,3/2]$	$[2u-2,1]$	d	0
$[1,3/2]$	$[1,u]$	$(d+1)/2$	0
$[1,3/2]$	$[u,1+2u]$	$u-(d-1)/2$	0
$[3/2,2]$	$[u-1,1]$	d	$2u-2-d$
$[3/2,2]$	$[1,2u-2]$	$(d+1)/2$	$2u-2-d$
$[3/2,2]$	$[2u-2,u]$	$(d+1)/2$	0
$[3/2,2]$	$[u,1+2u]$	$u-(d-1)/2$	$0.$

Multiplying by the normalization $1/4$ and integrating $d(n - q)$ over these regions gives $65/48$. In the final interval scale the calculation at $u = 2$ by $c = 3 - u$. At $u = 2$ the degree intervals are $1 \leq d \leq 2$, with $n = (d + 1)/2, q = 2 - d$, and $2 \leq d \leq 5$, with $n = (5 - d)/2, q = 0$. Their unnormalized first moments sum to 8. The last interval therefore contributes $\frac{1}{4}\int_0^1 8c^3\,dc = 1/2$. The moving total is $65/48 + 1/2 = 89/48$. Integrating the slice dimensions $(n - q)_+$ themselves gives $2/3$ in the first interval, $7/3$ in the second, and 1 in the last, for a total of 4. Thus these containing spaces have total dimension $4m^3 + O(m^2)$, the same leading dimension as the actual refinement. The inclusion of Lemma 62.7 therefore omits only $O(m^2)$ dimensions. All orders under consideration are $O(m)$, so the normalized first moment differs by $O(m^{-1})$. This proves exactness for the actual series in this configuration. The fixed contribution is $\operatorname{mult}_A(R)/16 = 3/16$. The discrepancy of the ordinary point blow-up of a smooth surface is two, which proves the assertion. □

Lemma 63.5. *Let $p \in D = E \cap E'$. Let $\ell \subset E$ be the fibre of $\pi|_E$ through p, let $\ell' \subset E'$ be the fibre of $q|_{E'}$ through p, and put $B = q(\ell)$. Blow up ℓ, and denote its exceptional surface by $F \cong \mathbb{F}_1$ and its negative section by s. The strict transform of ℓ' meets F at a unique point A. Then the following conditions are equivalent:*

1. *D is singular at p;*
2. *$B \cap C'$ is the length-two scheme supported at $q(p)$, and $A \in s$;*
3. *ℓ is tangent to E' at p, and ℓ' is tangent to E at p.*

Consequently, the unhandled configuration consisting of a tangent secant B and a collision point on s occurs exactly over $\mathrm{Sing}(D)$.

Proof. The restriction $q|_\ell$ is an isomorphism onto the line B, because $q^*\mathcal{O}_Q(1) \cdot \ell = 1$. The ideal of C' pulls back under q to $\mathcal{O}_X(-E')$. Its restriction to ℓ therefore identifies the scheme $B \cap C'$ with the divisor $E'|_\ell$. The latter has degree two and contains no one-dimensional component. Thus $B \cap C'$ is nonreduced at $q(p)$ if and only if $E'|_\ell = 2p$, if and only if $T_p\ell \subset T_pE'$. This also proves that the intersection multiplicity is exactly two, rather than merely at least two. The strict transform C_Z in $\mathrm{Bl}_B(Q)$ is isomorphic to C': the restricted centre ideal on the smooth curve C' is invertible. Its intersection with the exceptional divisor is the same length-two Cartier divisor as $B \cap C'$. Thus nonreducedness here is exactly nonreducedness of the length-two centre scheme Ξ in the surface calculation.

The tangent lines $T_p\ell$ and $T_p\ell'$ are distinct. Indeed, dq is nonzero on the first line, since $q|_\ell$ is an isomorphism, whereas it vanishes on the second. In particular, ℓ' is transverse to ℓ as a pair of smooth curves in X. Its strict transform meets the exceptional surface F in the point represented by the nonzero normal direction $T_p\ell' \bmod T_p\ell$ in $N_{\ell/X,p}$.

The normal sequence for $\ell \subset E \subset X$ is

$$0 \longrightarrow \mathcal{O}_\ell \longrightarrow N_{\ell/X} \longrightarrow \mathcal{O}_\ell(-1) \longrightarrow 0.$$

It splits, since $H^1(\mathbb{P}^1, \mathcal{O}(1)) = 0$. The section of $\mathbb{P}(N_{\ell/X})$ corresponding, in the convention of lines, to the subbundle $N_{\ell/E} = \mathcal{O}_\ell$ has normal bundle $\mathrm{Hom}(\mathcal{O}_\ell, \mathcal{O}_\ell(-1))$ and hence self-intersection -1. It is therefore precisely s; equivalently it is the intersection of F with the strict transform of E. Consequently

$$A \in s \quad\Longleftrightarrow\quad T_p\ell' \bmod T_p\ell \in N_{\ell/E,p} \quad\Longleftrightarrow\quad T_p\ell' \subset T_pE.$$

Finally, E and E' are smooth divisors in the smooth threefold X. Their scheme-theoretic intersection is singular at p precisely when their tangent planes coincide. Each plane already contains its own ruling tangent line. As these two lines are distinct, the two cross-tangency conditions $T_p\ell \subset T_pE'$ and $T_p\ell' \subset T_pE$ hold precisely when both planes equal their two-dimensional span. This proves all equivalences. □

Remark 63.6. The criterion distinguishes three kinds of points on a smooth D: both incidence intersections can be transverse; $E'|_\ell$ can have a double point while $A \notin s$; or $A \in s$ while $E'|_\ell$ is reduced at p. The simultaneous tangency case is absent on a smooth D. No assertion that D is smooth follows from the smoothness of X.

64 The other exceptional fibre

We retain the two blowups $\pi\colon X \to \mathbb{P}^3$ and $q\colon X \to Q$. Let ℓ' be a fibre of the exceptional divisor of q, let $B = \pi(\ell')$, and suppose that $B \cap C = 2p + p_1$, where $p_1 \neq p$. We consider the case in which the point of ℓ' over p belongs to the singular locus of $E_{\mathbb{P}^3} \cap E_Q$.

Lemma 64.1. *Let $B \subset \mathbb{P}^3$ be a trisecant line to the smooth elliptic quintic C, with intersection scheme of length three. Put $Z = \mathrm{Bl}_B \mathbb{P}^3$, let H be the hyperplane pullback, and let $h = H - F_B$ define the projection $r : Z \to \mathbb{P}^1$. The strict transform C_Z is isomorphic to C, and $r|_{C_Z}$ is the complete degree-two linear series $\mathcal{O}_C(1)(-B \cap C)$.*

There is a unique relative line surface $R_Z \in |H + 2h|$ containing C_Z. It is smooth and isomorphic to $\mathbb{F}_1$. Writing its negative section as c_R and its ruling fibre as f_R, one has

$$H|_{R_Z} = c_R + 2f_R, \quad h|_{R_Z} = f_R, \quad C_Z \sim 2c_R + 3f_R = -K_{R_Z}.$$

The divisor $F_B|_{R_Z}$ is either a smooth section in $|c_R + f_R|$, or the reduced union of c_R and one ruling fibre.

Proof. Use the quotient convention $Z = \mathbb{P}(\mathcal{O}^{\oplus 2} \oplus \mathcal{O}(1))$, with tautological class H. The restricted projection has degree two and is complete by Riemann–Roch and nondegeneracy of C. It is finite flat. Evaluation gives a surjection

$$\mathcal{O}^{\oplus 2} \oplus \mathcal{O}(1) \longrightarrow (r|_{C_Z})_* \mathcal{O}_{C_Z}(H).$$

Surjectivity holds fibrewise because a subscheme of length two in a plane imposes two independent conditions on linear forms, also when nonreduced. The target has rank two and degree three: its Euler characteristic is $\deg \mathcal{O}_C(1) = 5$, and the Euler characteristic of a rank-two bundle on $\mathbb{P}^1$ is its degree plus two. The kernel is consequently $\mathcal{O}(-2)$. This gives the unique divisor R_Z of the stated class. It is the projective bundle of the rank-two quotient $\mathcal{V}$.

The bundle $\mathcal{V}$ is generated by global sections and has degree three, so it splits as $\mathcal{O}(a) \oplus \mathcal{O}(3 - a)$ with $a = 0$ or 1. The curve C_Z is a Cartier divisor on the smooth ruled surface R_Z. Its intersections with h and H are two and five, respectively, so its class is $2H - h$, which also equals $-K_{R_Z}$. If $a = 0$, the surface is $\mathbb{F}_3$ and this anticanonical class has intersection -1 with its negative section. It would contain that section, contrary to the irreducibility of C_Z. Thus $a = 1$, giving $R_Z \cong \mathbb{F}_1$ and the displayed classes.

Finally $F_B = H - h$ restricts to $c_R + f_R$. A member of this linear system is either a smooth positive section or the reduced sum of the negative section and one ruling fibre. This follows equally from the realization of $\mathbb{F}_1$ as the blow-up of a point of $\mathbb{P}^2$: the system is the pullback of lines in $\mathbb{P}^2$. □

Remark 64.2. Let $Y' = \mathrm{Bl}_{C_Z} Z$, with exceptional divisor J. The strict transform of the other exceptional surface E_Q has class $4H + h - 2J$. Since blowing up the Cartier curve $C_Z \subset R_Z$ does not alter R_Z, its restriction to the strict transform R is

$$(4H + h) - 2(2H - h) = 3h.$$

Thus every component of $E_Q \cap R$ is a ruling fibre of $R_Z \to \mathbb{P}^1$. In particular, when the fixed curve on the original exceptional $\mathbb{F}_1$ contains its negative section, that section maps to a ruling fibre of R_Z; it does not map to the other section component of $F_B|_{R_Z}$.

Lemma 64.3. *Assume that the intersection scheme of the trisecant line with C has type $2p+q$ or $3p$. On the exceptional surface $F \cong \mathbb{F}_1$ of the original line blow-up, write s for the negative section and f for a ruling fibre. The fixed curve obtained from R is reduced, contains no ruling fibre, and has class $2s + 2f$. If a nonreduced centre has support $A \in s$ and its tangent direction is transverse to s, then*

$$R_F = s + T, \qquad T \sim s + 2f,$$

where T is a smooth section meeting s transversely at A. For the type $2p + q$, the other centre B lies on T. The double centre is transverse to both s and T. For the type $3p$, the length-three centre is tangent to T, but has intersection length exactly two with T.

Proof. The divisor $F_B|_{R_Z}$ has at most two components, as described above. Viewed on $F_B \cong \mathbb{P}^1 \times \mathbb{P}^1$, it has class $f + 2g$, where $H|_{F_B} = f$ and $h|_{F_B} = g$. If it is reducible, its components have classes g and $f + g$. Each has degree one over the ruling whose fibres are contracted in the order-of-blow-up exchange. Thus their images on F are sections, and neither image is a ruling fibre.

At a smooth point of $F_B|_{R_Z}$, the intersection scheme of C_Z with F_B is contained scheme-theoretically in this smooth curve. Resolving a centre of length k therefore gives successive multiplicities $1, \dots, 1$. At a node in the length-two case, the multiplicities are $2, 0$. The difference after subtracting the exceptional Cartier divisor of the centre is then only the (-2)-curve contracted to the A_1 singularity on the intermediate surface. It is also contracted on the other side and adds no component to R_F.

For a length-three centre at a node, use coordinates (v, w) on R_Z in which $F_B|_{R_Z}$ is $vw = 0$. The component corresponding to the negative section c_R has intersection one with C_Z. Consequently C_Z is transverse to that component and tangent to the fibre component; the intersection orders are one and two. Analytically the situation is $C_Z : \ w = v^2$. On F_B, its intersection ideal is therefore $(w - v^2, v^3)$. The multiplicities of the nodal divisor at the successive three point blow-ups are $2, 1, 0$. After subtracting the exceptional divisor of the length-three centre, the only additional curves are the two (-2)-curves in the A_2 chain. Both are contracted on either side. There is again no additional component in R_F.

The elementary-transform class calculation gives $R_F \sim 2s + 2f$. Since $R_F \cdot s = 0$, a point of R_F on s forces s to be a component. The other component has class $s + 2f$ and degree one over the ruling, so it is a smooth section. Its intersection with s is one. In the stated nonreduced configurations the centre is transverse to s, and the condition that the total fixed curve contains the centre forces its residual section through A. The simple centre B is off s, so it lies on T as well.

For clarity, the precise final assertion can also be read from the common resolution in the triple case. Start on F_B and write e_1, e_2, e_3 for the total exceptional classes. The fibre component has strict transform $g - e_1 - e_2$, and the other component has strict transform $f + g - e_1$. To obtain F, contract, in order,

$$f - e_1, \qquad e_1 - e_2, \qquad e_2 - e_3.$$

Their images are respectively the negative section and T. Viewed as three blow-ups from F, the total exceptional classes are $f - e_3, f - e_2, f - e_1$. The negative section passes through just the first centre, whereas T passes through the first two and not the third. This proves the asserted intersection lengths one and two with the curvilinear centre of length three. □

Remark 64.4. Here is the explicit resolution calculation for the type $2p + q$ in the hard configuration. On F_B, the double centre is the node of the two components g and $f + g$ of R_B, with tangent direction different from both branches; the remaining simple centre lies on g. If e_1, e_2 are the total classes of the two successive blow-ups at the double centre and e_3 is the other exceptional class, the strict component classes are

$$g - e_1 - e_3, \qquad f + g - e_1.$$

The curves contracted to obtain F are $L = f - e_1$, $K = e_1 - e_2$, and $L_3 = f - e_3$, with L contracted before K. Viewed from F, the total exceptional classes of the first blow-up at A, its infinitely near blow-up, and the blow-up at B, are

$$E_A = K + L = f - e_2, \quad E_{A,1} = L = f - e_1, \quad E_B = L_3 = f - e_3.$$

Subtracting the strict components from the pullbacks of s and T gives

$$\rho^* s - (g - e_1 - e_3) = E_A, \qquad \rho^* T - (f + g - e_1) = E_A + E_B.$$

Thus both sections pass through A, neither passes through its infinitely near centre, and T passes through B. In particular the double jet is transverse to both sections.

This also proves the colon identity needed in the last chamber. In coordinates with $I_A = (v, z^2)$, the local equation of $s + T$ has the form $z^2u + vzu_1 + v^2u_2$, with u a unit. The associated graded ring of this complete-intersection ideal is $(\mathcal{O}/I_A)[U, V]$, and the degree-one initial form of this equation has unit coefficient of $V = [z^2]$. It is a non-zero-divisor. Consequently if r is this local defining equation, then

$$(I_A^n : r^k) = I_A^{n-k} \qquad (n \geq k).$$

The same assertion at B is the usual ordinary point computation. Thus subtracting the fixed power of R_F leaves exactly the stated remaining powers of I_Ξ.

Lemma 64.5. *Let $\omega\colon Y \to X$ be the blowup of ℓ', with exceptional divisor F. Then $F \cong \mathbb{F}_1$, $A_X(F) = 2$, and $S_X(F) = 13/8$.*

Proof. The normal bundle of ℓ' is $\mathcal{O} \oplus \mathcal{O}(-1)$. Exchange the blowups of $C \subset \mathbb{P}^3$ and $B \subset \mathbb{P}^3$. This gives Y' after the flops of the three fibres counted with their scheme-theoretic multiplicities. Put $Z = \mathrm{Bl}_B \mathbb{P}^3$, and denote by C'' the strict transform of C in Z. Thus $Y' = \mathrm{Bl}_{C''} Z$. Write H for the pullback of a plane and $h = H - F$ for the fibre class of the projection $Z \to \mathbb{P}^1$. Then $Z = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}^{\oplus 2} \oplus \mathcal{O}(1))$, $H^3 = H^2h = 1$, and $h^2 = 0$. Moreover, $H \cdot C'' = 5$ and $h \cdot C'' = 2$.

The line through each length-two fibre of $C'' \to \mathbb{P}^1$ sweeps a surface $R_Z \in |H + 2h|$. Indeed, the kernel of the evaluation map from $\mathcal{O}^{\oplus 2} \oplus \mathcal{O}(1)$ to the rank-two bundle $f_*\mathcal{O}_{C''}(H)$ is $\mathcal{O}(-2)$. Its quotient has degree 3. It is globally generated, so its splitting is either $(0, 3)$ or $(1, 2)$. On R_Z the curve C'' has class $2H - h$. The first splitting would give negative intersection with the minimal section, contradicting the irreducibility of C''. Hence $R_Z \cong \mathbb{F}_1$ and $C'' = -K_{R_Z}$. Let J be the exceptional divisor of $Y' \to Z$, and put

$$A = 2H + h - J, \qquad R = H + 2h - J.$$

The divisor A is globally generated. On the original model Y it is $H_Q - F$: the pullback of the maximal ideal of $q(\ell')$ is the ideal of ℓ', and hence Y maps to $\mathrm{Bl}_{q(\ell')} Q$. The hyperplanes through this point give $H_Q - F$. Its degree on every flopped curve is zero, so global generation is preserved on Y'.

Put $D_u = \omega^*(-K_X) - uF$. For $0 \leq u \leq 1$ this is nef: $D_1 = H + (H_Q - F) = H + A$ is globally generated on the original model, and D_0 is nef. On Y' we have $D_u = (4 - u)H + uh - J = A + (2 - u)H + (u - 1)h$, which is nef for $1 \leq u \leq 2$. For $2 \leq u \leq 3$, its positive and fixed parts are

$$P_u = (3 - u)A + h, \qquad N_u = (u - 2)R.$$

The fibres of $R \to \mathbb{P}^1$ have R-degree -1 and D_u-degree $2 - u$, proving that this fixed coefficient is necessary. The remaining divisor is nef. Restriction to a general fibre over $\mathbb{P}^1$ also shows that the pseudoeffective threshold is 3.

Direct intersection gives $A^3 = 1$ and $A^2h = 2$. Consequently,

$$\mathrm{vol}(D_u) = \begin{cases} 24 - 3u^2 - u^3, & 0 \leq u \leq 1, \\ 21 + 9u - 12u^2 + 2u^3, & 1 \leq u \leq 2, \\ (3 - u)^3 + 6(3 - u)^2, & 2 \leq u \leq 3. \end{cases}$$

The second polynomial can also be computed directly on Y' using $\deg N_{C''/Z} = 17$; thus it does not require an assumption that the three flopped curves are distinct. Dividing the three integrals by 24 gives $91/96$, $7/12$, and $3/32$, whose sum is $13/8$. □

64.1 The refined series and its invariant divisors

Let s and f be the minimal section and the ruling fibre of $F = \mathbb{F}_1$. The exchange of blowups marks a length-three curvilinear subscheme $\Xi = 2a + b$ on F: its double part is supported at $a \in s$, its tangent is transverse to both s and the ruling fibre, and $b \notin s$ lies on another ruling fibre. The fixed curve R_F has class $2s + 2f$ and has the form $s + T$, where T is a smooth section in $|s + 2f|$ through a and b. Here the tangent of Ξ at a need not be the tangent of T. These statements can also be read on R_Z: its intersection with the exceptional $\mathbb{P}^1 \times \mathbb{P}^1$ is the reduced union of its minimal section and one ruling fibre. The double intersection with C'' occurs at their meeting point.

Choose local coordinates (z, x) near a such that $s = (x = 0)$ and $\Xi_a = (x - z, z^2)$. After an automorphism of $\mathbb{F}_1$, we may put b at the opposite positive fixed point and write $T = (x - \kappa z = 0)$, where $\kappa \notin \{0, 1\}$. Indeed, before this last normalization the equation of T is $\alpha z + x(\beta + \gamma z) = 0$; the fibre transformation $x \mapsto x/(1 + cx)$ removes the mixed term, preserves the length-two jet, and fixes b. Thus the action $(z, x) \mapsto (\lambda z, \lambda x)$ preserves the data. At b the coordinates are $t = 1/z$ and $y = 1/(xz)$; the curve T has equation $y = \kappa^{-1} t^2$.

For completeness, the leading refined spaces are as follows. If $j/m = u$, then for $0 \le j \le 2m$ the moving space is

$$H^0(F, \mathcal{O}_F(js + (m + j)f) \otimes I_\Xi^{\max(0, j-m)}).$$

For $2m \le j \le 3m$, the fixed part is $(j - 2m)R_F$, and the moving space is

$$H^0(F, \mathcal{O}_F((4m - j)s + (5m - j)f) \otimes I_\Xi^m).$$

The actual refined spaces are contained in these spaces and are asymptotically complete; endpoint rounding does not affect the expected orders. One obtains them by restricting to the exceptional $\mathbb{P}^1 \times \mathbb{P}^1$ on Z. On its resolution the classes are $(4 - u)f + ug - \sum e_i$ and $R|_{F'} = f + 2g - \sum e_i$, and the map to F contracts $f - e_i$. Thus $s = g + f - \sum e_i$. Surjectivity of restriction in the interiors of the first two chambers follows from Serre vanishing, since the divisors there are ample. For the bounds it also suffices to use the inclusions and the leading normalization $4m^3$ verified below. In the last chamber the fixed divisor has coefficient $j - 2m$. The colon calculation in the preceding geometric remark justifies the remaining power I_Ξ^m; replacing it by two unrelated point conditions would not be valid. The average coefficient of R_F is

$$\frac{1}{4} \int_2^3 (u - 2) \left(\frac{(3 - u)^2}{2} + 2(3 - u) \right) du = \frac{3}{32}.$$

Lemma 64.6. *For the above refined series, the expected orders at the two attracting valuations are $S(\mathrm{ord}_{\mathrm{Bl}_a F}) = 2$ and $S(v_b) = 9/4$, where $v_b(t) = 1$, $v_b(y) = 2$. The expected orders of the invariant prime divisors are bounded by*

divisor	s	*positive section*	f_a	f_b	T_κ
S	$23/24$	$97/192$	$169/192$	$11/12$	$55/96$.

Proof. A monomial basis of $H^0(F, as + bf)$ near a consists of $x^i z^l$, where $0 \le i \le a$ and $0 \le l \le b - a + i$. Set $d = i + l$. The condition at b is $l \le b - k$, where k is the power of I_Ξ. For fixed d, put

$$L = \max\{0, \lceil (d - b + a)/2 \rceil, d - b + k\}, \quad U = \min\{a, d\}, \quad r = \max\{0, 2k - d\}.$$

Writing $x = z + w$, the double-point condition says that the polynomial in x/z is divisible by $(x/z - 1)^r$. Therefore its dimension is $\max\{0, U - L + 1 - r\}$. This also proves that no cancellation has been omitted.

Here (a,b,k) equals $(j,m+j,0)$, $(j,m+j,j-m)$, or $(4m-j,5m-j,m)$ in the three successive chambers. For the limits set $u=j/m$, $v=d/m$, omit the ceiling and the constant 1, and write $D=(U-L-r)_+$. The normalization is $\int\int D\,dv\,du=4$, in agreement with $h^0(X,-mK_X)=4m^3+6m^2+4m+1$. Put $e=(u-2)_+$. The moment functions, to be integrated against $D/4$, are respectively

$\mathrm{Bl}_a F$	$v+2e$
v_b	$a+b-v+2e$
s	$(L+U-r)/2+e$
positive section	$a-(L+r+U)/2$
f_a	$v-(L+r+U)/2$
f_b	$b-a+L+U-r-v$
T_1	$(r+U-L)/2+e.$

Integration over the linear chambers gives the displayed fractions. For an orbit curve T_κ different from the preferred tangent T_1, the moving moment is $79/192$; adding the fixed contribution, when present, gives at most $97/192$. Thus $55/96$ is a valid common bound. □

Lemma 64.7. *The refined stability threshold of F is at least* 1.

Proof. Fix $\lambda<1$, and choose a basis compatible with an arbitrary divisorial valuation and with the indicated one-parameter subgroup. Multiply its normalized boundary by λ. Its limit is invariant. For all sufficiently large divisible levels, the strict inequalities below follow from Lemma 64.6. Every invariant prime curve is one of the four coordinate curves or an orbit closure T_κ. Their coefficients are strictly less than 1 by Lemma 64.6. At a, the non-coordinate orbit curves are ordinary distinct lines; one ordinary blowup separates them, and its log discrepancy is 2. Their total order is strictly less than 2. At b they are the parabolas $y=\kappa^{-1}t^2$. Two ordinary blowups resolve them; the final exceptional valuation has weights $(1,2)$, discrepancy 3, and expected order $9/4<3$. The first exceptional valuation has expected order at most

$$\frac{1}{2}\left(\frac{9}{4}+\frac{11}{12}\right)=\frac{19}{12}<2.$$

Indeed, at b the orders of every orbit parabola and the positive section are doubled by the weight-$(1,2)$ valuation, whereas the order of $f_b=(t=0)$ stays one. This proves the preceding exceptional inequality. There are no further intersections requiring a test. Thus the limiting basis boundary is log canonical. Semicontinuity, followed by the limit over divisible levels and then $\lambda\to1$, proves the assertion. □

Consequently this configuration gives positive beta invariants for all prime divisors whose centre is the indicated point of X. In the Abban–Zhuang inequality the surface contribution is nonnegative, while the extraction contribution has the strict margin $A_X(F)-S_X(F)=2-13/8=3/8$. The coefficient of F is positive for every such divisorial valuation. This proves strict positivity, despite the equality at the ordinary exceptional valuation over a on the surface.

Remark 64.8. This argument treats the length-two-plus-one configuration. It does not prove the assertion for $B\cap C=3p$. In that case the triple jet is transverse to s and tangent to T; its next coefficient matters to the geometric identification. A direct ordinary-order computation for the natural triple-jet model gives $71/32>2$. Thus one cannot extend the last lemma to that case merely by replacing the double jet by a triple jet.

65 Reduction to the triple-contact points

For $p \in D$, let ℓ'_p be the fibre of $E' \to C'$ through p. Define

$$\Sigma_X = \{p \in \mathrm{Sing}(D) : E|_{\ell'_p} = 3p\}. \tag{65.1}$$

This is a finite set. The equality is an equality of effective divisors on the smooth line ℓ'_p; equivalently, its image in $\mathbb{P}^3$ has intersection scheme $3\pi(p)$ with C.

Lemma 65.1 (The marked centres and the negative section)**.** *Consider either the first line extraction, whose marked scheme has length two, or the second line extraction with a reduced length-three scheme or a scheme of type $2a + b$. Let $F \cong \mathbb{F}_1$ be the exceptional surface before exchanging the blow-ups, and let s be its negative section. On the common smooth resolution, at most one of the successive marked centres lies on the successive strict transform of s. Consequently:*

1. *at most one reduced marked point lies on s;*
2. *a double marked scheme supported on s is transverse to s;*
3. *in the second extraction, if the double scheme is supported on s, the remaining simple point lies off s.*

A double marked scheme is also transverse to its ruling fibre. This last assertion does not require its support to lie on s.

Proof. We first check the last assertion directly in the exchange of blow-ups. Near a tangency of intersection length two, choose analytic coordinates (z, u, v) on the target threefold in which the two curves are

$$C = (u = v = 0), \qquad B = (u = 0,\ v = z^2).$$

Such coordinates exist because the minimum order of the two normal coordinates of B along C is exactly two. Blow up C, using $u = v\xi$. The strict transform of B has ideal $(\xi, v - z^2)$. On the relevant chart of its blow-up put $v - z^2 = \xi w$. The exceptional surface is $F = (\xi = 0)$, with coordinates (z, w), and its ruling has base coordinate z. The ideal of B on the original target pulls back to

$$(u, v - z^2) = \xi\,(z^2 + \xi w, w).$$

Thus the remaining indeterminacy on F is the ideal (w, z^2). Its tangent direction is $w = 0$, which is transverse to the ruling fibre $z = 0$. This also identifies the two successive point blow-ups that resolve the double marked scheme. Reduced points arising from different intersections of B and C lie on different ruling fibres.

Let $\widehat{F}$ be the common smooth surface resolution, and write

$$\rho : \widehat{F} \longrightarrow F, \qquad \eta : \widehat{F} \longrightarrow F_B$$

for its maps to the original exceptional surface and to the exceptional surface of the blow-up of B on the target. Every exceptional class used below is a *total transform* class. In particular these classes have square -1 and are pairwise orthogonal, also when two centres are infinitely near.

For the first extraction, write c for the negative section of $F_B \cong \mathbb{F}_1$ and f for its ruling fibre. Let e_1, e_2 be the total exceptional classes of η. Pullback to $\widehat{F}$ of the restriction of the nef divisor $M_2 = H_Q + 2A - J$ is

$$P_2 = 2c + 3f - e_1 - e_2, \qquad \rho^* s = c + f - e_1 - e_2, \qquad \rho^* f = f.$$

For two reduced centres the curves contracted by ρ are $f - e_1$ and $f - e_2$. They are its total exceptional classes $\varepsilon_1, \varepsilon_2$. For a double centre, η first blows up its support and then its specified tangent direction. The map ρ contracts first $L = f - e_1$ and then the image of $K = e_1 - e_2$. Its total exceptional classes, in the reverse order of blow-up, are therefore

$$\varepsilon_1 = K + L = f - e_2, \qquad \varepsilon_2 = L = f - e_1.$$

In either case the restriction has the same expression

$$P_2 = \rho^*(2s + 3f) - \varepsilon_1 - \varepsilon_2.$$

The intermediate surface in the double case has an A_1 point, but M_2 restricts to an invertible sheaf on it. Its pullback to $\widehat{F}$ is the displayed class; no fractional exceptional coefficient is introduced. Since M_2 is nef, so is P_2.

For the second extraction use $F_B \cong \mathbb{P}^1 \times \mathbb{P}^1$ and the classes f, g already used in the trisecant calculation. With e_1, e_2, e_3 the total exceptional classes of η, the nef divisor $D_2 = A + h$ restricts on the common resolution as

$$P_2 = 2f + 2g - e_1 - e_2 - e_3, \qquad \rho^* s = g + f - e_1 - e_2 - e_3, \qquad \rho^* f = f.$$

If all three centres are reduced, the total exceptional classes of ρ are $\varepsilon_i = f - e_i$. For type $2a + b$, label e_1, e_2 by the two successive blow-ups over the double centre and e_3 by the simple centre. The total exceptional classes of ρ are

$$\varepsilon_a = f - e_2, \qquad \varepsilon_{a,1} = f - e_1, \qquad \varepsilon_b = f - e_3.$$

The first two are obtained from the same contraction of L followed by K as above. Hence, in both configurations,

$$P_2 = \rho^*(2s + 3f) - \sum_i \varepsilon_i.$$

It is nef by the nefness of D_2. The displayed formulas for $\rho^* s$ can also be recovered without a position assumption: its intersection with a ruling fibre is one, it is orthogonal to every curve contracted by ρ, and its square is -1. Orthogonality forces coefficient -1 for every total class e_i, and the square then forces the displayed coefficient of f. Thus these calculations use only the order-of-blow-up exchange and the nef divisors; they do not assume any position of the marked centres relative to s.

Let I be the set of centres of ρ that lie on the successive strict transform of s. Since s is smooth, that strict transform has class

$$\widehat{s} = \rho^* s - \sum_{i \in I} \varepsilon_i.$$

Using total exceptional classes gives

$$0 \le P_2 \cdot \widehat{s} = (2s + 3f) \cdot s - |I| = 1 - |I|.$$

Thus $|I| \le 1$. Two reduced points on s would give $|I| \ge 2$. If a double scheme supported on s were tangent to s, both its support and its infinitely near centre would lie on the successive strict transform, again giving $|I| \ge 2$. Finally, a double support on s and a further simple point on s would give the same contradiction. This proves all the assertions. □

Proof of Proposition 61.2. Let V be a prime divisor over X with $\beta_X(V) \le 0$. By [40, Main Theorem], its centre is a closed point $p \in D$. Blow up the fibre ℓ of $E \to C$ through p. If the associated length-two scheme is reduced, the first or second estimate of Lemma 63.2 applies.

The required position of the marked points is supplied by Lemma 65.1. If the scheme is nonreduced and supported off the negative section, apply the third estimate, using the absence of a ruling component in R. In these cases Abban–Zhuang adjunction gives

$$\frac{A_X(V)}{S_X(V)} \geq \min\left\{\frac{2}{19/12}, \delta(F;W)\right\} > 1,$$

contradicting the choice of V. Thus the scheme is nonreduced and its support lies on the negative section. By Lemma 63.5, $p \in \operatorname{Sing}(D)$.

Now blow up ℓ'_p. Since this line is tangent to E at p and $E\ell'_p = 3$, either $E|_{\ell'_p} = 2p + p_1$ with $p_1 \neq p$, or it is $3p$. In the first case the double marked point on the second exceptional surface lies on its negative section: the other ruling ℓ is tangent to E' at p. Lemma 65.1 shows that the double direction is transverse to the negative section and to the ruling, and that the simple point lies off the negative section. Thus all the hypotheses of the double-plus-simple surface calculation hold. Lemmas 64.5 and 64.7 give, by the strict form of adjunction,

$$\beta_X(V) \geq \left(2 - \frac{13}{8}\right) \operatorname{ord}_V(F) = \frac{3}{8} \operatorname{ord}_V(F) > 0.$$

Here $\operatorname{ord}_V(F) > 0$ because the centre on X is the point p on the blown-up line. This is again a contradiction. Hence $p \in \Sigma_X$. If Σ_X is empty, all prime divisors have positive beta invariant, so the valuative criterion gives K-stability. Smoothness of D implies $\Sigma_X = \varnothing$ and gives the stated special case. □

The triple-contact case cannot be omitted. The next two elementary lemmas describe it intrinsically; they also give examples showing that the residual locus is compatible with a smooth threefold.

Lemma 65.2. *Let $\mathcal{C} \subset \mathbb{P}^4$ be an elliptic normal quintic, with $\mathcal{L} = \mathcal{O}_{\mathcal{C}}(1)$, and let P lie outside its secant variety. Project $\mathcal{C}$ from P to a smooth elliptic quintic in $\mathbb{P}^3$. Its trisecant lines are parametrized isomorphically by $\operatorname{Pic}^3(\mathcal{C})$: for each degree-three line bundle $\mathcal{A}$, exactly one divisor $Z \in |\mathcal{A}|$ has $P \in \langle Z \rangle$.*

Proof. Put $\mathcal{M} = \mathcal{L} \otimes \mathcal{A}^{-1}$, of degree two. The base-point-free pencil trick gives a surjection $H^0(\mathcal{A}) \otimes H^0(\mathcal{M}) \to H^0(\mathcal{L})$, whose kernel has dimension one. Evaluation at the projective point P gives a matrix $H^0(\mathcal{A}) \to H^0(\mathcal{M})^*$ of size 2×3. If its rank were one, there would be a nonzero $m \in H^0(\mathcal{M})$ such that P annihilates $mH^0(\mathcal{A}) = H^0(\mathcal{L}(-Z_2))$, where $Z_2 = (m = 0)$ has length two. This says $P \in \langle Z_2 \rangle$, contrary to the choice of P. Rank zero is excluded by surjectivity of multiplication. Thus the matrix has rank two. Its one-dimensional kernel determines a unique section a of $\mathcal{A}$. The condition that P annihilates $aH^0(\mathcal{M}) = H^0(\mathcal{L}(-(a = 0)))$ is exactly $P \in \langle (a = 0) \rangle$. In families over $\operatorname{Pic}^3(\mathcal{C})$, the constant-rank kernel is a line bundle, so this construction is a morphism, and its inverse is the Abel map. Projection turns each such plane into its trisecant line. No line has intersection length four with the projected curve, since a length-four divisor of an elliptic normal quintic spans a projective three-space, whereas the inverse image of a line is a plane. □

Lemma 65.3. *Consider the incidence curve of a point of $\mathcal{C}$ and a trisecant divisor. Suppose that $P \in \langle 3c \rangle$, so $3c$ is one of the trisecant divisors. At this divisor, the incidence curve is smooth unless $\mathcal{L} \cong \mathcal{O}_{\mathcal{C}}(5c)$. In the exceptional case it has an ordinary node. The exceptional case is compatible with $P \notin \operatorname{Sec}(\mathcal{C})$.*

Proof. Choose an Abel coordinate z at c and a local analytic lift $v(z) = v_0 + zv_1 + z^2v_2 + \cdots$ of the embedding. Write $P = \rho_0 v_0 + \rho_1 v_1 + \rho_2 v_2$; here $\rho_2 \neq 0$, because the tangent line is contained in the secant variety. Near the divisor $3c$, write the three points as the roots of

$$f(z,t) = z^3 - tz^2 + a(t)z - b(t).$$

The parameter t is an Abel coordinate on $\mathrm{Pic}^3(\mathcal{C})$, by the preceding lemma. Reducing $v(z)$ modulo this cubic gives, to first order in (t, a, b), the three spanning vectors

$$v_0 + bv_3, \qquad v_1 - av_3 + bv_4, \qquad v_2 + tv_3 - av_4 + bv_5.$$

Modulo $\langle v_0, v_1, v_2\rangle$, constancy of P therefore implies

$$(\rho_0 b' - \rho_1 a' + \rho_2)[v_3] + (\rho_1 b' - \rho_2 a')[v_4] + \rho_2 b'[v_5] = 0.$$

If $\mathcal{L} \not\cong \mathcal{O}(5c)$, the vanishing sequence is $0, 1, 2, 3, 4$, so $[v_3], [v_4]$ are independent. Setting $b' = 0$ would force $a' = 0$ and then $\rho_2 = 0$, a contradiction. Thus $b' \neq 0$, which makes the incidence equation $f = 0$ smooth at $(z, t) = (0, 0)$.

If $\mathcal{L} \cong \mathcal{O}(5c)$, the vanishing sequence is $0, 1, 2, 3, 5$. Write $[v_4] = \alpha[v_3]$; then $[v_5]$ is independent of $[v_3]$. The same equation forces $b' = 0$ and $(\rho_1 + \alpha\rho_2)a' = \rho_2$. In particular $a' \neq 0$. The quadratic part of f is $a'tz - b''t^2/2$, a product of two distinct linear forms. Hence the incidence curve has an ordinary node.

To verify compatibility explicitly, take c as the origin of a Weierstrass model and use the $|5c|$ embedding with coordinates $[1 : x : y : x^2 : xy]$. The osculating plane at c is $X_0 = X_1 = 0$. Its intersection with the secant variety is, set-theoretically, the union of the lines $X_2 = 0$ and $X_3 = 0$. The first is the tangent line at c. For any other secant, vanishing of X_0 and X_1 forces its two endpoints to have the same x-coordinate; they are opposite points of the elliptic curve, and the corresponding intersection point has $X_3 = 0$. Conversely these opposite-point secants sweep the second line, including their tangent limits at two-torsion points. Thus points $P = [0 : 0 : \rho_2 : \rho_1 : \rho_0]$ with $\rho_1\rho_2 \neq 0$ lie outside the secant variety and give exactly the exceptional configuration. □

Lemma 65.4. *Embed C by its complete degree-five linear series as an elliptic normal quintic $\mathcal{C} \subset \mathbb{P}^4$, and let P be the centre of the projection giving the specified embedding $C \subset \mathbb{P}^3$. There is an isomorphism $C' \cong \mathrm{Pic}^3(\mathcal{C})$ under which $D \subset C \times C'$ is scheme-theoretically the universal incidence curve of the trisecant divisors in Lemma 65.2. In particular, the singularities computed in Lemma 65.3 are the singularities of D itself.*

Proof. The complete linear series has five sections, and the given embedding uses a four-dimensional subspace; its annihilator is P. Smoothness of the given embedding implies $P \notin \mathrm{Sec}(\mathcal{C})$, including its tangent lines. Each fibre ℓ' of $q|_{E'}$ maps isomorphically under π onto a line $\Lambda \subset \mathbb{P}^3$, because $H \cdot \ell' = 1$. The equality $\mathcal{I}_C\mathcal{O}_X = \mathcal{O}_X(-E)$ identifies, as schemes, $E|_{\ell'}$ with $\Lambda \cap C$. These intersections have length three. Conversely, the strict transform of any trisecant line has B-degree $3 - 3 = 0$, so is contracted by q and is one of these fibres. A trisecant line cannot have intersection length four, by the last assertion of Lemma 65.2.

The finite flat family $D \to C'$ therefore defines a morphism $C' \to \mathrm{Hilb}^3(C) = \mathrm{Sym}^3(C)$, whose image is exactly the family of trisecant divisors. Distinct fibres give distinct lines: off C, the map π is an isomorphism, so a line has only one strict transform. Consequently their divisors are distinct as well, since a length-three divisor spans its trisecant line. By Lemma 65.2, composition with the Abel map is a bijective morphism $C' \to \mathrm{Pic}^3(C)$. It is an isomorphism, since both curves are smooth and the characteristic is zero. By the defining property of the Hilbert scheme, D is the pullback of the universal length-three divisor; the preceding scheme-theoretic fibre identification is precisely this Hilbert morphism. This proves the assertion, including nonreduced fibres and singular points of D. □

Corollary 65.5. *The set Σ_X consists exactly of the incidence points $(c, 3c)$ such that $\mathcal{L} \cong \mathcal{O}_{\mathcal{C}}(5c)$ and $P \in \langle 3c\rangle$. In particular, $|\Sigma_X| \leq 25$, and every point of Σ_X is an ordinary node of D.*

Proof. Combine Lemmas 65.4 and 65.3 with the definition of Σ_X. There are exactly twenty-five points c with $\mathcal{O}(5c) \cong \mathcal{L}$, because multiplication by five on an elliptic curve has degree twenty-five. For each such c, the divisor $3c$ determines at most one incidence point. □

Example 65.6 (A triple-contact configuration). Take a smooth Weierstrass elliptic curve and its embedding by $|5O|$ with coordinates $[1 : x : y : x^2 : xy]$. Project from $[0 : 0 : 1 : 1 : 0]$. Lemma 65.3 shows that the projected quintic is smooth and that its osculating line at the image of O has contact length three. The associated point belongs to Σ_X. The weighted calculation below applies to this point and gives $\delta_p(X) \geq 64/59$.

66 The weighted point extraction and its actual section ring

We now treat every point of Σ_X. The line calculation above is insufficient there because its refined surface order is $49/24 > 2$. We use a weighted point blow-up and calculate its refinement directly. The key is a realization of X in a four-dimensional symplectic flag variety. All weights below are vanishing weights: the initial form means the part of smallest weight, until the second degeneration is specified.

Lemma 66.1 (The flag realization). *Let X be the blow-up of a smooth elliptic quintic in $\mathbb{P}^3$. There is a nondegenerate alternating form on a four-dimensional vector space V such that X is a divisor of class $H + B$ on*

$$Y = \{(\lambda, \Lambda) : \lambda \subset \Lambda \subset V, \dim \lambda = 1,\ \dim \Lambda = 2,\ \Lambda \text{ isotropic}\}.$$

Here H and B are the pullbacks of the hyperplane classes of $\mathbb{P}(V)$ and $\mathrm{LG}(2, V) \cong Q^3$, respectively. The two projections restrict to the two contractions of X.

Proof. Serre's construction gives a rank-two bundle $\mathcal{N}$ fitting in

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{N}(2) \longrightarrow \mathcal{I}_C(4) \longrightarrow 0.$$

It has $c_1(\mathcal{N}) = 0, c_2(\mathcal{N}) = 1$. No quadric contains a smooth elliptic quintic in $\mathbb{P}^3$. Indeed, on a smooth quadric a curve of bidegree (a, b) has degree $a + b$ and genus $(a - 1)(b - 1)$, which cannot be five and one simultaneously. On a quadric cone its strict transform on $\mathbb{F}_2$ is $as + 5f$ and has genus $-a^2 + 5a - 4$, again never one for integral a. The strict transform is isomorphic to the original smooth curve, since its map to that curve is finite and birational. A reducible quadric would force the curve into a plane, which is also impossible. Consequently $H^0(\mathcal{N}) = 0$, and the normalized rank-two bundle $\mathcal{N}$ is stable. By the classification of stable rank-two bundles with these Chern classes [41, Chapter II, Lemma 4.3.2], it is a null-correlation bundle. Its projectivization is Y above. The zero locus of the induced relative hyperplane section is $\mathrm{Bl}_C \mathbb{P}^3$. For clarity, let $\mathcal{S} = \mathcal{O}_{\mathbb{P}^3}(-1)$. The null-correlation bundle is $\mathcal{S}^{\perp}/\mathcal{S}$. On the flag variety let $\mathcal{T} = \Lambda/\mathcal{S}$ be the tautological line in this quotient, and put $\xi = c_1(\mathcal{T}^*)$. Since $\det(\Lambda) = \mathcal{S} \otimes \mathcal{T}$, we have $B = H + \xi$. The wedge of the section of $\mathcal{N}(2)$ with $\mathcal{T}$ defines a section of $\mathcal{T}^* \otimes \mathcal{O}(2H)$ (the determinant of $\mathcal{N}$ is trivial). Its zero divisor therefore has class $\xi + 2H = B + H$. Since $-K_Y = 2\xi + 4H = 2(H + B)$, adjunction gives $-K_X = H + B$. □

Lemma 66.2 (The initial equation at a residual point). *At a point in Σ_X, symplectic coordinates on Y may be chosen so that its big cell has coordinates (x, t, a, b), with*

$$(h_0, h_1, h_2, h_3) = (1, x, a + xt, xa - b), \qquad (b_0, b_1, b_2, b_3, b_4) = (1, t, a, b, a^2 + tb).$$

The equation of X has no terms of weight less than three for

$$\mathrm{wt}(x, t, a, b) = (1, 1, 2, 3),$$

and its weight-three initial form is, after rescaling,

$$f_0 = b - xa + \nu(at + xt^2), \qquad \nu \neq 0.$$

Consequently the induced point valuation on X is the weight-$(1,1,2)$ valuation in the regular parameters (x,t,a).

Proof. The displayed formula is obtained by taking the isotropic plane spanned by $(1,x,y,z)$ and $(0,1,t,y-xt)$, and putting $a = y - xt$, $b = xa - z$. The bilinear sections of $H + B$ have weight-zero space spanned by 1, weight-one space spanned by x, t, weight-two space spanned by a, xt, and weight-three space spanned by $b, xa, at + xt^2$. Both projection fibres pass through the point and lie on X, so its constant and linear x, t terms vanish. Write the equation as $F = F_0(t,a,b) + xF_1(t,a,b)$. The exceptional divisor of the second projection is locally $F_1 = 0$. The double intersection with the first projection fibre says $(F_1)_t(0) = 0$. The triple intersection of the second fibre with the first exceptional divisor is read from

$$Q|_{t=a=b=0} = (F_0)_t + x((F_1)_t - (F_0)_a) - x^2((F_1)_a + (F_0)_b) - x^3(F_1)_b,$$

where all coefficients are evaluated at zero. It forces $(F_0)_a(0) = (F_1)_t(0) = 0$ and $(F_1)_a(0) = -(F_0)_b(0)$. Smoothness of X now implies $(F_0)_b(0) \neq 0$. Thus all weights below three vanish, and if the weight-three form is $\alpha b + \beta xa + \gamma(at + xt^2)$ then $\beta = -\alpha$ and $\alpha \neq 0$. The first exceptional divisor is cut out by

$$Q = F_t - xF_a - x^2F_b.$$

Moreover $F_1(t,0,0) = \gamma t^2$: the constant and linear terms vanish, and the restriction has degree at most two. Its zero divisor is $E'|_\ell = 2p$, so $\gamma \neq 0$. Equivalently, the linear term of $Q|_X$ is γa, which is nonzero because E is smooth. Division by α proves the assertion. □

Lemma 66.3 (Primeness of the initial divisor)**.** *In the preceding notation the divisor $X_0 = (f_0 = 0) \subset Y$ is integral. Its diagonal section ring is the associated graded anticanonical ring of X for the weight-$(1,1,2)$ point valuation, with the zero-weight trivializations h_0 and b_0.*

Proof. Retain the nonzero coefficient ν. Write homogeneous coordinates $h_0, \dots, h_3$ and $b_0, \dots, b_4$. The flag incidence equations are

$$-h_0b_2 - h_1b_1 + h_2b_0 = 0, \quad h_0b_3 - h_1b_2 + h_3b_0 = 0,$$

$$h_0b_4 - h_2b_2 + h_3b_1 = 0, \quad h_1b_4 - h_2b_3 - h_3b_2 = 0, \qquad b_0b_4 = b_2^2 + b_1b_3.$$

The extra equation is $h_3b_0 - \nu h_2b_1 = 0$. On $h_0b_0 \neq 0$ it solves uniquely for b, giving affine three-space. There is no component in the boundary. Indeed, on $h_0 = 0, h_1 \neq 0$ we have $b_1 = (h_2/h_1)b_0$, $b_2 = (h_3/h_1)b_0$, so the extra equation is $(h_3 - \nu h_2^2/h_1)b_0 = 0$. The first branch has a one-dimensional base and one-dimensional fibres; the second has a two-dimensional base and a unique fibre point. Both therefore have dimension two. On $h_0 = h_1 = 0$ the base is a line and fibres have dimension at most one, again giving dimension at most two. On $b_0 = 0, h_0 \neq 0$, the extra equation forces $h_2 = 0$: otherwise $b_1 = 0$ and the incidence equations force all $b_i = 0$. The remaining base is a plane, and the fibre point with $b_0 = 0$ is unique. Thus this boundary also has dimension at most two. As a Cartier divisor on smooth Y, X_0 is Cohen–Macaulay; its dense smooth affine three-space and absence of boundary components show that it is integral.

The weights on the fundamental coordinates are

$$(0,1,2,3) \quad \text{and} \quad (0,1,2,3,4).$$

They come from $\operatorname{diag}(1, \lambda, \lambda^2, \lambda^3)$ in the general symplectic group, of similitude multiplier λ^3, with a shift of the Plücker linearization. Its projective action preserves Y. Hence they give a

grading on the section ring of Y. Let R_Y be the diagonal section ring of Y. For every nonzero $g \in R_Y$, the smallest-weight form of gF is $\mathrm{in}(g)f_0$: the ambient graded ring is a domain. Hence the initial ideal of (F) is exactly (f_0). This gives a flat degeneration of $R_Y/(F)$ to $R_Y/(f_0)$, with the same Hilbert function. Restriction and the divisor exact sequence identify the latter quotient with the section ring of the integral divisor X_0. It is therefore a domain, so localization at $h_0 b_0$ is injective. On that chart its weights are precisely the weighted order in the regular parameters x, t, a. Therefore there are no further cancellations, and the associated graded ring is the claimed quotient. Restriction identifies its diagonal pieces with $H^0(X, m(H+B))$, by vanishing of the higher cohomology of $\mathcal{O}_Y((m-1)(H+B))$. □

Lemma 66.4 (The exact expected order of the weighted extraction)**.** *For the weight-*$(1,1,2)$ *valuation G at a residual point,*

$$A_X(G) = 4, \qquad S_X(G) = \frac{29}{8}, \qquad \beta_X(G) = \frac{3}{8}.$$

Proof. The Weyl dimension formula gives

$$\dim H^0(Y, \mathcal{O}_Y(aH + bB)) = \frac{(a+1)(b+1)(a+b+2)(a+2b+3)}{6}.$$

On the diagonal this is $(m+1)^4$. Weyl symmetry gives average weight $3a/2 + 2b$ under the shifted grading above. The defining equation has weight three. Hence, with $N_m = (m+1)^4 - m^4$, the sum of the weights on the quotient diagonal piece is

$$\frac{7m}{2}(m+1)^4 - \left(3 + \frac{7(m-1)}{2}\right) m^4 = \frac{7m}{2} N_m + \frac{1}{2} m^4.$$

Divide by mN_m and let m tend to infinity. This gives $S_X(G) = 7/2 + 1/8 = 29/8$. The discrepancy is $1 + 1 + 2 = 4$. □

Lemma 66.5 (The actual weighted section spaces)**.** *Let G be the weight-*$(1,1,2)$ *exceptional divisor at $p \in \Sigma_X$. Put $A = a + xt$. For $m, j \geq 0$, put*

$$V_m^{\geq j} = \{s \in H^0(X, mL) : \mathrm{ord}_G(s) \geq j\}, \qquad W_{m,j} = V_m^{\geq j} / V_m^{\geq j+1}.$$

These are the refined section spaces on G. Using the local trivialization of L, each $W_{m,j}$ is a space of polynomials of weighted degree j in x, t, A, of weights $1, 1, 2$. The direct sum of these spaces at level m is exactly the span of products of degree m in each list

$$\mathcal{H} = (1, x, A, tA),$$

$$\mathcal{B} = (1, t, A - xt, (x - \nu t)A - x^2 t, A(A - xt - \nu t^2)).$$

In particular its dimension is $N_m = (m+1)^4 - m^4$.

Proof. On the initial divisor, the equation gives $b = xa - \nu t(a + xt)$. Therefore $h_3 = \nu t A$ and

$$b_2 = A - xt, \qquad b_3 = (x - \nu t)A - x^2 t, \qquad b_4 = A(A - xt - \nu t^2).$$

Multiplication of sections of dominant line bundles on the flag variety is surjective: by the Borel–Weil description it is a nonzero equivariant map onto the irreducible highest-weight representation of the sum of the weights. Thus the degree-(m,m) sections are spanned by degree-m products of each set of fundamental coordinates. Restriction to X is surjective by the exact sequence for its divisor of class $H + B$ and vanishing for $(m-1)(H+B)$. Lemma 66.3 identifies its associated graded quotient with the integral initial divisor. Substituting the displayed formulas, and dividing h_3 by the nonzero constant ν, proves the assertion. The dimension was computed in Lemma 66.4. □

Lemma 66.6 (Bounded support and an ample subseries)**.** *The series W has support $0 \le j \le 7m$. For $0 \le j \le m$,*

$$W_{m,j} = H^0(\mathbb{P}(1,1,2), \mathcal{O}(j)).$$

In particular it contains a complete ample series on an interior ray.

Proof. The largest weights of the generators in $\mathcal{H}$ and $\mathcal{B}$ are three and four, respectively. Hence $j \le 7m$. There is a nonzero degree-one section of every weight from zero to seven; their products show that every integral j in this interval occurs at level m. If $i + h + 2k = j \le m$, the monomial $x^i t^h A^k$ uses $i + k \le m$ factors from $\mathcal{H}$ and $h \le m$ factors from $\mathcal{B}$. Pad each product with the constant generator. Thus every polynomial of weight j occurs. On the ray $(m, j) = (2k, 2k)$ these are the complete spaces of the ample Cartier bundles $\mathcal{O}(2k)$. The ray is interior to $0 < j < 7m$, as required for refinement. □

Example 66.7 (The first weighted Hilbert function)*.* At $m = 1$, the dimensions of $W_{1,j}$, for $j = 0, \dots, 7$, are

$$1,\ 2,\ 2,\ 2,\ 3,\ 2,\ 2,\ 1.$$

They sum to 15, and their weighted sum is 53. Thus the first-level expected order is $53/15$. The exact character calculation in Lemma 66.4 gives its limiting value $29/8$.

Definition 66.8 (Normalization of the refined basis boundaries)**.** A basis boundary of W at level m is

$$D_m = \frac{1}{mN_m} \sum_{j \ge 0} \sum_{s \in \mathcal{B}_{m,j}} \{s = 0\},$$

where $\mathcal{B}_{m,j}$ is a basis of $W_{m,j}$ and a nonzero constant has zero divisor. For a divisor T over G, the maximal value of $\operatorname{ord}_T(D_m)$ is its level-m expected order; its limit is $S(W; T)$. Thus

$$\delta(G; W) = \inf_T \frac{A_G(T)}{S(W; T)}.$$

We also write $\delta_m(G; W)$ for the infimum of $\operatorname{lct}(G; D_m)$ over all level-m basis boundaries. Equivalently it is the infimum of $A_G(T)/S_m(W; T)$ over prime divisors T over G, where $S_m(W; T)$ is the maximal order just defined. The normalization is by mN_m, including all weighted-degree pieces. It is not the dimension of an individual binary space appearing below.

67 Binary spaces on the weighted exceptional surface

Lemma 67.1 (The second initial spaces)**.** *Retain the nonzero parameter ν in the weighted initial equation. Put $r = x + \nu t$, $s = t$, and $C = A - rs$. Take the initial spaces for the* maximum *weight of the action*

$$(r, s, A) \longmapsto (\lambda r, s, \lambda A).$$

They contain the spaces generated by products of degree m in each of

$$\mathcal{H} = (1, r, A, sA), \qquad \mathcal{B} = (1, s, C, rC, AC).$$

For integral weights $q, v \ge 0$, set

$$n = \min(q, v), \quad i_0 = q - n, \quad j_0 = v - n,$$

$$L = \max\left\{0, \left\lceil \frac{q - m}{2} \right\rceil\right\}, \qquad K = \max\left\{0, \left\lceil \frac{v - m}{2} \right\rceil, v - 2m\right\}.$$

The weight-(q,v) part of this generated subspace is zero if $L > m$ or $N = n-K-L < 0$. Otherwise it is exactly

$$r^{i_0}s^{j_0}A^K C^L \operatorname{Sym}^N\langle A, C\rangle. \tag{67.1}$$

Its dimension is $N+1$. The original weighted degree is $q+v$.

Proof. The original generators after substitution are

$$(1, r-\nu s, A, sA), \qquad (1, s, C+\nu s^2, (r-2\nu s)A-(r-\nu s)^2 s, AC).$$

Their maximum-weight initial forms are precisely $\mathcal{H}, \mathcal{B}$. Products of their initial forms belong to the initial spaces.

A product of degree m in each list has the form $r^i s^j A^k C^l$. Exactly the following nonnegative integral exponent vectors occur:

$$l \le m, \qquad i+k \le m+l, \qquad j+l \le m+k, \qquad j+l \le 2m. \tag{67.2}$$

Indeed, write a monomial from the first list as $r^a s^b A^c$, where $b \le c$ and $a+c \le m$. Write one from the second as $r^d s^e A^f C^l$, where $d+f \le l$ and $e+l \le m$. Their product satisfies (67.2). Conversely choose

$$c = \max\{0, j-m+l, k-l\}, \qquad f = k-c, \qquad d = \max\{0, i-m+c\},$$

$$a = i-d, \qquad b = \max\{0, j-m+l\}, \qquad e = j-b.$$

The inequalities (67.2) give $0 \le c \le \min\{k, m\}$, $b \le c$, $a+c \le m$, $d+f \le l$, and $e+l \le m$. Thus the indicated factorization exists.

The two weights are $q = i+k+l$ and $v = j+k+l$. Hence

$$r^i s^j A^k C^l = r^{i_0} s^{j_0} (A-C)^{n-k-l} A^k C^l.$$

Conditions (67.2) become $l \le m$, $l \ge L$, $k \ge K$ and $k+l \le n$. Every such polynomial is divisible by $A^K C^L$. Conversely, fixing $l = L$ and letting k run from K to $n-L$ gives a basis of $A^K C^L \operatorname{Sym}^N\langle A, C\rangle$, since A and $A-C$ are independent linear forms. This proves (67.1). □

Lemma 67.2 (Exact exhaustion of the initial spaces)**.** *The generated subspaces in the preceding lemma equal the actual maximum-weight initial spaces, at every level m.*

Proof. Their total dimension counts the integer triples (q, v, k) satisfying

$$q, v, k \ge 0, \quad q \le 3m, \quad 2k-v \ge -m, \quad k-v \ge -2m,$$

$$k \le q, \quad k \le v, \quad 2k-q \le m, \quad 2k-2v+q \le m.$$

Indeed, for fixed (q, v) these inequalities say precisely $K \le k \le \min(q, v) - L$. At $m = 1$ the vertices of this polytope are

$$(0,0,0), (0,1,0), (1,0,0), (1,1,1), (1,3,1),$$

$$(3,1,0), (3,3,1), (3,3,2), (3,4,2).$$

They are all integral. Ehrhart's theorem therefore makes the count a polynomial of degree three in m. For completeness, summing by q gives

m	successive q-row counts	total
0	1	1
1	2, 5, 3, 5	15
2	3, 7, 12, 9, 13, 9, 12	65
3	4, 9, 15, 22, 18, 24, 19, 24, 18, 22	175.

These four values determine the polynomial as

$$4m^3 + 6m^2 + 4m + 1 = (m+1)^4 - m^4.$$

This equals the dimension of the actual refinement. The inclusions of the previous lemma are consequently equalities. □

Example 67.3. At $m = 1$, the slice $q = v = 1$ is $\langle A, C\rangle$. The slice $q = v = 3$ is $AC\langle A, C\rangle$. Indeed, the latter is spanned by the products

$$A \cdot AC = A^2C, \qquad sA \cdot rC = rsAC = A^2C - AC^2.$$

The binary-space description retains the cancellation producing AC^2. Merely listing the leading monomials of the original generators would lose this section and would not determine the refinement.

Lemma 67.4 (The moments of the weighted surface refinement)**.** *For the refinement on $G = \mathbb{P}(1, 1, 2)$, the preceding degeneration has the following limiting upper bounds on coefficients:*

prime curve	$(r = 0)$	$(s = 0)$	$(A = 0)$	$(C = 0)$	$(A - \kappa rs = 0),\ \kappa \neq 0, 1$
coefficient	55/96	61/96	19/24	3/4	1/3.

At $P_r = [1 : 0 : 0]$ and $P_s = [0 : 1 : 0]$, the limiting total multiplicities are 59/32 *and* 57/32*, respectively.*

Proof. Divide all indices by m. To avoid changing the meanings of q, v, write

$$\ell(q) = \max\{0, (q-1)/2\}, \qquad k(v) = \max\{0, (v-1)/2, v-2\},$$

$$n(q, v) = \min(q, v), \qquad d(q, v) = n(q, v) - k(v) - \ell(q).$$

The support is $0 \le q \le 3$, $0 \le v \le 4$, $d(q, v) \ge 0$. It is divided into the following seven triangles. Each row also records three integrals; these suffice to check the dimension and both point multiplicities directly.

vertices	n	$\int d$	$\int qd$	$\int vd$
$(0,0), (1,1), (0,1)$	q	1/6	1/12	1/8
$(0,0), (1,0), (1,1)$	v	1/6	1/8	1/12
$(0,1), (1,1), (1,3)$	q	1/3	1/4	1/2
$(1,0), (3,1), (1,1)$	v	1/3	1/2	1/4
$(1,1), (3,3), (1,3)$	q	4/3	7/3	3
$(1,1), (3,1), (3,3)$	v	4/3	3	7/3
$(1,3), (3,3), (3,4)$	q	1/3	5/6	13/12

In particular, $\iint d\,dq\,dv = 4$, agreeing with the leading coefficient of the exact Hilbert polynomial proved above. Thus the normalization of all limiting moments is 1/4.

In a complete binary space of dimension $N + 1$, the maximum sum of orders of a basis at any specified point of $\mathbb{P}^1$ is $N(N+1)/2$. Apply this to (67.1). A factor $A - C = rs$ contributes one to each coordinate-line coefficient. Thus the five first moments are

$$\frac{1}{4}\iint d\left(q - n + \frac{d}{2},\ v - n + \frac{d}{2},\ k + \frac{d}{2}, \ell + \frac{d}{2}, \frac{d}{2}\right) dq\,dv = \left(\frac{55}{96}, \frac{61}{96}, \frac{19}{24}, \frac{3}{4}, \frac{1}{3}\right).$$

These are integrals of quadratic polynomials on the displayed triangles. For an additional check, before division by four their values are

$$\frac{55}{24}, \quad \frac{61}{24}, \quad \frac{19}{6}, \quad 3, \quad \frac{4}{3}.$$

For the point bounds we use the total multiplicity directly, rather than summing the separate prime-coefficient maxima. Every nonzero polynomial of weight (q, v) has ordinary multiplicity v at P_r and q at P_s. The table gives

$$\frac{1}{4}\iint vd = \frac{59}{32}, \qquad \frac{1}{4}\iint qd = \frac{57}{32}.$$

Rounding in K, L changes all these expressions by $O(m^{-1})$. □

Lemma 67.5 (Exact finite-level coefficient bounds)**.** *Put $h_m = (m+1)^4 - m^4$. At level $m \geq 1$, divide the quantities in the middle column below by mh_m. They give upper bounds for the five prime-curve coefficients and the exact multiplicities at the two smooth fixed points of every invariant basis boundary.*

curve or point	M_m	*limiting bound b*
L_r	$m(5m+2)(11m^2+8m+5)/24$	55/96
L_s	$m(61m^3+74m^2+47m+10)/24$	61/96
$(A=0)$	$m(19m^3+26m^2+17m+4)/6$	19/24
$(C=0)$	$m(3m+1)(2m^2+2m+1)/2$	3/4
$C_\kappa,\ \kappa \neq 0,1$	$m(8m^3+4m^2+m-1)/6$	1/3
P_s	$m(57m^3+82m^2+55m+14)/8$	57/32
P_r	$m(59m^3+86m^2+57m+14)/8$	59/32

Every displayed finite-level bound is strictly smaller than its limit.

Proof. For a slice of dimension $N+1$, the maximal sum of orders at any point of the binary parameter line is $N(N+1)/2$. Its contribution to the five curve numerators is therefore

$$\Big((q-n)(N+1)+\frac{N(N+1)}{2},\quad (v-n)(N+1)+\frac{N(N+1)}{2},$$
$$K(N+1)+\frac{N(N+1)}{2},\quad L(N+1)+\frac{N(N+1)}{2},\quad \frac{N(N+1)}{2}\Big).$$

The point contributions are $q(N+1)$ and $v(N+1)$. Summation uses the exact slice description, so no rounding estimate or omitted section is involved.

We give a polynomial certificate for these sums. Let P be the three-dimensional lattice polytope in Lemma 67.2, with coordinates (q, v, k). The A-coefficient numerator is $\sum_{mP\cap\mathbb{Z}^3} k$, and the two point numerators are $\sum q$ and $\sum v$. The C-coefficient numerator is $\sum(\min(q, v)-k)$. Cutting P by $q = v$ gives lattice polytopes: this also follows directly from the seven-triangle subdivision, since all upper and lower endpoints above its vertices are integral. Thus the weighted form of Ehrhart's theorem makes these sums polynomials of degree at most four in m. In this application it follows already from the ordinary theorem: for a nonnegative integral linear weight w on a lattice cell, introduce one extra coordinate satisfying $0 \leq z \leq w$. The lifted cell is a lattice polytope, and its lattice count minus that of the original cell is the required sum of w. Apply inclusion–exclusion to the lattice cells of the subdivision.

For completeness, put

$$R = \{(q, v, k_1, k_2) : (q, v, k_1) \in P,\ (q, v, k_2) \in P\}.$$

This is a lattice polytope of dimension four. Over each of the seven triangles it is the square $[k(v), n(q, v) - \ell(q)]^2$; the two endpoints at the projection vertices are

(q, v)	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$	$(1,3)$	$(3,1)$	$(3,3)$	$(3,4)$
lower	0	0	0	0	1	0	1	2
upper	0	0	0	1	1	0	2	2.

A square with affine endpoints over a triangle is the convex hull of its endpoint squares over the three vertices. Hence every cell, and therefore every vertex of R, is integral. The generic-conic numerator is exactly

$$\frac{1}{2}\left(\#(mR\cap\mathbb{Z}^4)-\#(mP\cap\mathbb{Z}^3)\right),$$

since a fibre containing $N+1$ integers contributes $((N+1)^2-(N+1))/2$. It is consequently a polynomial of degree at most four. The two line numerators add respectively $\sum\max(q-v,0)$ and $\sum\max(v-q,0)$ to this polynomial. These too are weighted Ehrhart polynomials on the lattice subdivision by $q=v$.

It now suffices to evaluate the seven sums at five integers. Direct summation of the displayed slice contributions gives

m	L_r	L_s	A	C	C_κ	P_s	P_r
0	0	0	0	0	0	0	0
1	7	8	11	10	2	26	27
2	65	74	98	91	27	227	236
3	272	308	401	375	127	921	957
4	781	881	1136	1066	386	2597	2697.

Because polynomiality and the degree bound have already been proved, these values establish the formulas in the statement for every m. Finally the differences bmh_m-M_m, in the same order, are

$$\frac{m(82m^2+56m+15)}{96},\quad \frac{7m(10m^2+8m+3)}{96},\quad \frac{m(10m^2+8m+3)}{24},$$

$$\frac{m(2m^2+2m+1)}{4},\quad \frac{m(8m^2+7m+3)}{6},\quad \frac{m(14m^2+8m+1)}{32},\quad \frac{m(10m^2+8m+3)}{32}.$$

They are positive for every $m\geq 1$. □

Remark 67.6. Lemma 67.5 permits the invariant log-canonicity argument below with the coefficient $64/59$ at every finite level: the two point multiplicities remain below two and every prime coefficient remains below one. Thus one may dispense with the $O(m^{-1})$ errors in that argument. Passing to the asymptotic threshold still gives $\delta(G;W)\geq 64/59$.

Remark 67.7 (A finite-level threshold). In fact the largest normalized obstruction in the table is always $M_m(P_r)/2$. For example

$$M_m(P_r)-2M_m(A)=\frac{5m(m+1)(5m^2+5m+2)}{24}>0,$$

$$M_m(P_r)-2M_m(L_s)=\frac{11m(m+1)(5m^2+5m+2)}{24}>0,$$

and the other comparisons follow directly from the same table. Consequently the invariant arrangement proves the finite-level estimate

$$\delta_m(G;W)\geq\frac{16h_m}{59m^3+86m^2+57m+14}>\frac{64}{59}.$$

For $m=1$ the right-hand side is $10/9$. The sum of the two point numerators also recovers the exact total weight

$$M_m(P_r)+M_m(P_s)=\frac{7m}{2}h_m+\frac{1}{2}m^4,$$

which agrees with the independent character calculation of $S_X(G)$.

Lemma 67.8 (Log canonicity of the invariant arrangement)**.** *For the weighted surface refinement above,*

$$\delta(G;W) \geq \frac{64}{59} > 1.$$

Proof. The different on G is zero. Its only singular point is the $\frac{1}{2}(1,1)$ point $[0:0:1]$. The invariant prime curves are the coordinate lines L_r, L_s and $C_\kappa = (A - \kappa rs = 0)$. Every C_κ avoids the singular point and passes through P_r, P_s. At each of these two smooth points the conics have distinct tangent directions, as follows from their equations $A = \kappa s$ and $A = \kappa r$ on the two affine charts. A coordinate line passes through exactly one of P_r, P_s and has a different tangent direction from the conics there.

Consequently an invariant boundary is log canonical if every prime coefficient is at most one and its total multiplicity is at most two at each of P_r, P_s. One blow-up at each smooth point proves this. At the quotient point the quotient cover gives two transverse coordinate axes; their individual coefficients at most one suffice.

By Lemma 67.5, multiplying the finite-level bounds by 64/59 satisfies these conditions strictly, for every $m \geq 1$. For each divisorial valuation choose a basis compatible with its filtration and the action filtration, separately in each original weighted-degree piece. Degeneration gives an invariant basis boundary. The preceding coefficient bounds apply to every such basis, because they are the maximum sums of orders in its weight spaces. They hold at every level by the exact formulas of Lemma 67.5. More explicitly, two finite filtrations of a finite-dimensional vector space admit a common adapted basis. We apply this only to the valuation filtration and the secondary action filtration on each $W_{m,j}$; the original j-grading is kept fixed. The independent initial vectors form bases of the initial spaces, which are exactly the binary spaces by Lemma 67.2.

For a basis polynomial f of largest secondary weight d, the family

$$f_\tau(r,s,A) = \tau^d f(\tau^{-1}r, s, \tau^{-1}A)$$

is polynomial in τ, has the same primary weighted degree as f, and specializes to its nonzero maximum-weight initial form. For $\tau \neq 0$ it is the transform by an automorphism of G. The zero divisors give the invariant basis boundary just considered. These divisors are $\mathbb{Q}$-Cartier: their doubles are Cartier even at the quotient point. In fact the product of the squares of all basis polynomials defines a relative effective Cartier divisor, nonzero on every fibre, and hence a flat divisor family. Semicontinuity of log canonical thresholds gives log canonicity before degeneration. Since the initial basis was adapted to the chosen valuation, its boundary realizes the maximal level-m order for that valuation. Thus

$$\frac{64}{59} S_m(W;T) \leq A_G(T)$$

for every prime divisor T over G and every $m \geq 1$. For fixed T, let m tend to infinity. Taking the infimum over T proves the assertion. The same argument with the finite coefficient given after Lemma 67.5 proves the stated finite-level bound.

This conclusion controls every basis boundary of W: for any such boundary D_m and any T, one has $\mathrm{ord}_T(D_m) \leq S_m(W;T)$. Thus the basis chosen upstairs in the subsequent Abban–Zhuang argument need only be compatible with the extraction and the valuation being tested. It need not also be compatible with the secondary action. □

Example 67.9 (A basis boundary at the first level). The secondary initial series at $m = 1$ has the following basis, grouped by the original weighted degree:

$$\begin{array}{c|l} 0 & 1 \\ 1 & r, s \\ 2 & A, C \\ 3 & sA, rC \\ 4 & s^2A, AC, r^2C \\ 5 & sAC, rAC \\ 6 & AC^2, A^2C \\ 7 & sA^2C. \end{array}$$

These are the binary bases $r^{i_0}s^{j_0}A^{K+a}C^{L+N-a}$, $0 \le a \le N$, in the slice formula. The sum of their exponents in r, s, A, C is $(5, 6, 11, 10)$. Their basis boundary is therefore

$$D_1 = \frac{1}{15}\big(5L_r + 6L_s + 11(A = 0) + 10(C = 0)\big).$$

At $P_r = [1 : 0 : 0]$ its multiplicity is $27/15$, and at $P_s = [0 : 1 : 0]$ it is $26/15$. All branches at each point have distinct tangent directions. At the quotient point only L_r and L_s occur, and their quotient-cover pullbacks are transverse coordinate axes. It follows that

$$\operatorname{lct}(G; D_1) = \min\left\{3, \frac{5}{2}, \frac{15}{11}, \frac{3}{2}, \frac{30}{27}, \frac{30}{26}\right\} = \frac{10}{9}.$$

Thus this explicit central basis boundary realizes the first-level bound of the finite calculation. The example concerns the secondary initial series; it does not assert equality for the first-level threshold of the original refined series.

Remark 67.10 (Sharpness for the secondary initial series). Write W^{in} for the secondary initial series. Its first-level example extends to every $m \ge 1$. Indeed, every nonzero eigenpolynomial in the (q, v) slice has ordinary order exactly v at P_r. For a fixed original weighted degree $q + v$, different q give different ordinary degrees at P_r, so cancellation cannot raise these orders. Hence

$$S_m(W^{\mathrm{in}}; \operatorname{ord}_{P_r}) = \frac{59m^3 + 86m^2 + 57m + 14}{8h_m}.$$

The discrepancy of this surface point blow-up is two. The invariant basis-boundary proof also applies to W^{in}, whose spaces are already preserved by the secondary action. Its finite lower bound and this valuation therefore give

$$\delta_m(G; W^{\mathrm{in}}) = \frac{16h_m}{59m^3 + 86m^2 + 57m + 14}, \qquad \delta(G; W^{\mathrm{in}}) = \frac{64}{59}.$$

For the original weighted refinement the argument proves the lower bound $\delta(G; W) \ge 64/59$. No equality for that series, or for $\delta_p(X)$, is asserted.

68 The local threshold and the proof for Family 2.17

Lemma 68.1 (The local threshold at a triple-contact point)**.** *For every $p \in \Sigma_X$, one has*

$$\delta_p(X) \geq \frac{64}{59} > 1.$$

Proof. Let $\sigma : \widetilde{X} \to X$ be the weight-$(1,1,2)$ blow-up in Lemma 66.2, with exceptional divisor G. This is a plt extraction. Indeed, its two weight-one charts are smooth, and its weight-two chart is the quotient of affine three-space by $\boldsymbol{\mu}_2$ acting with weights $(1,1,1)$. The divisor G is $\mathbb{P}(1,1,2)$. Restricting the index-one cover of the weight-two threefold chart gives $\mathbb{A}^2 \to \mathbb{A}^2/\boldsymbol{\mu}_2$. The action on this surface cover is free away from the origin, so the different on G is zero. The spaces in Lemma 66.5 are the actual refinement of the anticanonical series along G. Lemma 66.6 verifies its bounded support and ample-series hypotheses. Abban–Zhuang adjunction (Proposition 2.7, or [4, Theorem 3.3]), Lemma 66.4, and Lemma 67.8 give

$$\delta_p(X) \geq \min\left\{\frac{A_X(G)}{S_X(G)}, \delta(G; W)\right\} \geq \min\left\{\frac{32}{29}, \frac{64}{59}\right\} = \frac{64}{59}.$$

This is a lower bound for $\delta_p(X)$; no equality is asserted. □

Proof of Theorem 61.1. Suppose that a prime divisor V over X has $\beta_X(V) \leq 0$. Proposition 61.2 puts its centre at a point $p \in \Sigma_X$. By Lemma 68.1,

$$A_X(V) \geq \frac{64}{59} S_X(V) > S_X(V),$$

a contradiction. Hence every prime divisor over X has positive beta invariant. The valuative criterion proves K-stability of every smooth member of Family 2.17. □

Remark 68.2 (No generality assumption). The coefficient ν was only required to be nonzero; its nonvanishing follows from the exact double contact along the first contracted line. The second initial spaces are independent of its nonzero value. The prime initial divisor and the exact Hilbert count also account for arbitrary higher-weight terms of the original equation. Thus the estimate applies to every point of Σ_X on every smooth member, without an assumption on the remaining coefficients.

Remark 68.3 (Why no threefold Zariski decomposition is needed here). *We have not computed a Zariski decomposition on the weighted point blow-up.* The replacement is an exact calculation of its section filtration. The integral initial divisor in the symplectic flag variety identifies the associated graded ring with the actual weighted point refinement. The second degeneration is an automorphism degeneration on $\mathbb{P}(1,1,2)$. Its complete binary spaces account for every section, as the exact Hilbert count shows. Their first moments give both the expected order and the refined threshold.

The threefold volume decompositions in the earlier line extractions remain part of the reduction to Σ_X. At the final weighted extraction, however, neither an assumed nef chamber nor a discarded higher-order term enters the argument. In particular the failing line surface estimate $49/24 > 2$ is retained: the weighted point refinement is a different filtration, and its identification is proved before Abban–Zhuang theory is applied.

Part XI
Family 2.20: the K-polystable classification

69 The classification problem and the results proved here

A smooth threefold in Family 2.20 is the blowup

$$\pi : X = \mathrm{Bl}_{C_0} V_5 \longrightarrow V_5$$

of the quintic del Pezzo threefold along a smooth twisted cubic. Here $V_5 \subset \mathbb{P}^6$ is a smooth three-dimensional linear section of $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$, and a twisted cubic is a curve isomorphic to $\mathbb{P}^1$ whose given embedding is defined by the complete series $|\mathcal{O}_{\mathbb{P}^1}(3)|$. The anticanonical degree of X is 26. Projection from the span of C_0 resolves to a conic bundle $\eta : X \to \mathbb{P}^2$.

This family contains smooth strictly K-semistable members. Thus the assertion that every smooth member is K-polystable is false. The general smooth member is K-stable, and the unique smooth member with infinite automorphism group is K-polystable [3, Proposition 5.43 and Section 7.2]. Section 7.2 of that reference identifies nonclosed parameter orbits and formulates the expected converse on the GIT-stable locus. The later discussion in [23, Introduction and Lemma 3.5] does not settle this converse. The results below prove the converse on the GIT-stable locus and give the complete smooth K-polystable classification.

Let $\mathcal{H} = \mathrm{Gr}(2, \mathrm{Sym}^4 \mathbb{C}^2)$. Its points are pencils of binary quartics, namely two-dimensional linear spaces of homogeneous polynomials of degree four in two variables. Let $\mathcal{H}^\circ$ be the open subset of pencils containing no nonzero square of a quadratic. The equivariant description of the Hilbert scheme in [3, Section 7.2] identifies $\mathcal{H}^\circ$ with smooth twisted cubics on V_5. Write X_U for the corresponding blowup. The following theorem states exactly the conclusions established in this part.

Theorem 69.1. *For $U \in \mathcal{H}^\circ$, the threefold X_U is K-polystable if and only if U is GIT-polystable. It is K-stable if and only if U is GIT-stable. Every smooth member is K-semistable. More explicitly, the following assertions hold.*

1. *The pencil U is GIT-semistable for the Plücker linearization. It fails to be GIT-stable exactly when it is equivalent under PGL_2 to*

   $$U_{a,c} = \langle x^3 y,\ xy^3 + ax^2y^2 + cx^4 \rangle.$$

   The member $X_{U_{0,0}}$ is K-polystable. Every displayed member with $(a,c) \neq (0,0)$ is strictly K-semistable. The latter isomorphism classes are parametrized by $\mathbb{P}(1,3)$.

2. *If w_U is the Wronskian sextic of U, then all its roots have multiplicity at most three. Two triple roots characterize $X_{U_{0,0}}$. Exactly one triple root characterizes the nonclosed smooth GIT orbits and hence the strictly K-semistable members in* (1)*. All the remaining pencils are GIT-stable.*

3. *Put $H = \pi^*\mathcal{O}_{V_5}(1)$, let E be the exceptional divisor, and put $B = H - E$. Then*

   $$S_X(E) = \frac{1}{2}, \qquad S_X(B) = \frac{63}{104}, \qquad S_X(H) = \frac{37}{104}.$$

   Every prime divisor on X has positive beta invariant. If p lies on a smooth fibre of η, a suitable smooth $S \in |B|$ through that fibre satisfies

   $$A_X(v) - S_X(v) \geq \frac{41}{104} v(S) > 0$$

   for every divisorial valuation centred at p. Every valuation with curve centre not contained in the inverse image of the discriminant also has positive beta invariant.

4. *At every double conic fibre $2R$, let F be the reduced exceptional divisor of the blowup of the pulled-back maximal ideal of its base point. Then*

$$A_X(F) = 3, \qquad S_X(F) = \frac{297}{104}, \qquad A_X(F) - S_X(F) = \frac{15}{104}.$$

The different on $F \cong \mathbb{P}^1 \times \mathbb{P}^1$ is $\Gamma/2$, where Γ has bidegree $(2,1)$. The vanishing sequence $(1,3)$ occurs exactly when the point over $E \cap R$ is ramified for $\Gamma \to \mathbb{P}^1$ in the direction factor.

5. *If a double fibre corresponds to a common root with vanishing sequence $(1,2)$, then $\delta_p(X) \geq 104/99$ at every point of its support. In particular this holds along every double fibre of a GIT-stable member. The remaining possible centres of nonpositive beta invariant on such a member lie over reduced singular conics.*

6. *For the blowup F of any component A of a reduced singular fibre, $A_X(F) = 2$ and $S_X(F) = 43/26$. Its complete refined series is given in Lemma 77.5. If the fibre node is outside E and $\pi(A)$ has trivial normal bundle in V_5, then $\delta_p(X) \geq 104/99$ for every $p \in A$.*

7. *For a GIT-stable pencil, every valuation of nonpositive beta invariant must have centre at the node on E of a reduced conic fibre. These points correspond to $\Delta_{\mathrm{sm}} \cap Q$, and there are at most $3 - d$ of them, where d is the number of nodes of the discriminant. At each of these points the ordinary point extraction satisfies $S_X(J) = 11/4$ and $A_X(J) = 3$. Its complete refinement gives $\delta_p(X) \geq 104/103 > 1$ at every such node.*

8. *A triangular discriminant gives a K-stable member. There is one such member up to isomorphism, with pencil $U_{\triangle} = xy(x-y)\langle x, y\rangle$.*

Consequently, the K-polystable smooth members are exactly those whose Wronskian has no triple root, together with the unique member $U_{0,0}$, whose Wronskian has two triple roots. The first collection consists of K-stable members; the latter member is K-polystable but not K-stable. Exactly one triple root gives strict K-semistability. An equivalent criterion using only the conic discriminant and the branch conic is given in Corollary 82.1.

Recall the terminology from geometric invariant theory. For a projective variety with a reductive group action and an ample linearization, a point is *GIT-semistable* if some invariant section of a positive tensor power does not vanish there. A semistable point is *GIT-polystable* if its orbit is closed in the semistable locus. It is *GIT-stable* if this orbit is closed and its stabilizer is finite. We apply the Hilbert–Mumford criterion to the Plücker line bundle $\mathcal{O}_{\mathcal{H}}(1)$; see [39]. These definitions concern the parameter space. Identifying them with K-stability of X_U requires a separate argument.

The proof is organized as follows. Binary quartics give the normal forms, their actual special degenerations, and the Wronskian test. Smooth conic fibres admit quintic del Pezzo sections with complete refined series. Exact surface decompositions give the strict beta estimates there. We then compute the corresponding estimates for reduced singular fibres and construct the weighted extraction over a double line. Its complete refined series leads to strict estimates along every double fibre on the GIT-stable locus. A discriminant test gives another form of the parameter criterion. We then construct a two-flop model for reduced components, prove their complete refined series, and treat ordinary components away from fibre nodes on E. Negative-section flags handle special lines, and unmarked rulings handle non-nodal points in the same-ruling configuration. The finite reduction isolates at most three points. At these nodes, an integral initial algebra identifies the complete point refinement. Two explicit surface calculations, including an extra fixed line in the degenerate-conic case, give the strict bound $104/103$ and finish the classification.

70 Pencils of binary quartics

Put $V = \operatorname{Sym}^4 \mathbb{C}^2$ and $\mathcal{H} = \operatorname{Gr}(2, V)$. We regard an element $U \in \mathcal{H}$ as a pencil of binary quartics. Let $\mathcal{S} \subset \mathbb{P}(V)$ be the surface of squares $[q^2]$, where q is a nonzero binary quadratic form, and put

$$\mathcal{H}^\circ = \{U \in \mathcal{H} \mid \mathbb{P}(U) \cap \mathcal{S} = \varnothing\}.$$

Multiplication $\operatorname{Sym}^2(\operatorname{Sym}^2 \mathbb{C}^2) \longrightarrow \operatorname{Sym}^4 \mathbb{C}^2$ is the projection onto the irreducible summand of dimension 5. Thus $\mathcal{S}$ is exactly the projected Veronese surface used in the description of twisted cubics on V_5. If coordinates on $\mathbb{P}(V)$ are normalized as coefficients of $\binom{4}{i}x^{4-i}y^i$, then the square of $ax^2 + 2bxy + cy^2$ has coordinates $[a^2 : ab : (ac + 2b^2)/3 : bc : c^2]$. In particular, the binary-form convention agrees with the parametrization in [3, Section 7.2]. The equivariant identification of the Hilbert scheme of twisted cubics on V_5 with $\mathcal{H}$ identifies $\mathcal{H}^\circ$ with the smooth twisted cubics. We denote the corresponding curve by C_U and its blowup by X_U.

For $p \in \mathbb{P}^1$, choose a basis f_0, f_1 of U with distinct orders of vanishing

$$a_0(p) = \operatorname{ord}_p(f_0) < a_1(p) = \operatorname{ord}_p(f_1).$$

These integers do not depend on the adapted basis. They are called the vanishing sequence of U at p.

Lemma 70.1. *For the Plücker linearization of the* SL_2*-action on* $\mathcal{H}$*, a pencil* U *is GIT-semistable if and only if* $a_0(p) + a_1(p) \le 4$ *for every* $p \in \mathbb{P}^1$*. It is GIT-stable if and only if all these inequalities are strict. Every* $U \in \mathcal{H}^\circ$ *is GIT-semistable. Such a pencil fails to be GIT-stable if and only if its vanishing sequence is* $(1, 3)$ *at some point.*

Proof. Every nontrivial one-parameter subgroup of SL_2 is conjugate, up to a positive power, to $(x, y) \mapsto (tx, t^{-1}y)$. At $p = \{x = 0\}$, the smallest weights of the two adapted basis vectors are $2a_0(p) - 4$ and $2a_1(p) - 4$. The smallest weight of their exterior product is therefore $2(a_0(p) + a_1(p) - 4)$. The Hilbert–Mumford criterion gives the first two assertions, with the chosen sign convention.

Suppose now that $U \in \mathcal{H}^\circ$. If $a_1(p) = 4$, then U contains a fourth power, contrary to its definition. If $a_0(p) + a_1(p) > 4$ and $a_1(p) \le 3$, the sequence is necessarily $(2, 3)$. In coordinates with $p = \{x = 0\}$ this implies $U = x^2W$ for a two-dimensional space W of binary quadratics. The projective line $\mathbb{P}(W)$ meets the conic of squares in $\mathbb{P}(\operatorname{Sym}^2 \mathbb{C}^2)$. Thus U contains a square, again a contradiction. Consequently $a_0(p) + a_1(p) \le 4$. Equality can only have the form $(0, 4)$ or $(1, 3)$, and the first possibility has already been excluded. □

Lemma 70.2. *Every pencil in* $\mathcal{H}^\circ$ *which is not GIT-stable is* PGL_2*-equivalent to*

$$U_{a,c} = \langle x^3y,\ xy^3 + ax^2y^2 + cx^4 \rangle \qquad (a, c \in \mathbb{C}).$$

Every displayed pencil belongs to $\mathcal{H}^\circ$*. Among them, precisely* $U_{0,0}$ *is GIT-polystable. Its stabilizer is* $\mathbb{C}^* \rtimes \boldsymbol{\mu}_2$*. All the other displayed pencils have finite stabilizers, and they admit the degeneration*

$$U_{a,c}(t) = \langle x^3y,\ xy^3 + at^2x^2y^2 + ct^6x^4 \rangle \longrightarrow U_{0,0}.$$

Every member of this family lies in $\mathcal{H}^\circ$*.*

Proof. Let p have vanishing sequence $(1, 3)$. The quartic in U with order 3 at p has another simple root $q \neq p$. Choose coordinates with $p = \{x = 0\}$ and $q = \{y = 0\}$. This quartic becomes x^3y. After rescaling the other basis vector and subtracting a multiple of x^3y, we obtain the displayed normal form.

Every quartic in $U_{a,c}$ is divisible by x. A square divisible by x is divisible by x^2. But the only elements of $U_{a,c}$ divisible by x^2 are multiples of x^3y, which are not squares. Hence $U_{a,c} \in \mathcal{H}^\circ$, for all a, c.

The one-parameter subgroup $(x, y) \mapsto (tx, t^{-1}y)$ gives exactly the displayed family after rescaling its basis vectors. The pencil $U_{0,0}$ has vanishing sequence $(1, 3)$ at each of its two common roots, and sequence $(0, 1)$ elsewhere. For every one-parameter subgroup with zero Hilbert–Mumford weight its limit is a pencil spanned by the two monomials of orders 1 and 3, and is therefore PGL_2-equivalent to $U_{0,0}$. The orbit-closure form of the Hilbert–Mumford criterion shows that this orbit is closed in the semistable locus.

If $(a, c) \neq (0, 0)$, the point p is the unique point with vanishing sequence $(1, 3)$. Indeed, when $c \neq 0$, it is the unique common root of the pencil. When $c = 0$ and $a \neq 0$, the other common root is q, where the sequence is $(1, 2)$. Every stabilizing transformation fixes p, the unique quartic with order 3 there, and hence its other root q. It is therefore diagonal. The diagonal action sends (a, c) to (t^2a, t^6c), so the stabilizer is finite. On the other hand, $U_{0,0}$ is preserved by the diagonal torus and by interchanging x and y; preservation of its two common roots shows that these are all its automorphisms. Consequently no $U_{a,c}$ with $(a, c) \neq (0, 0)$ is equivalent to $U_{0,0}$, and its orbit is not closed. □

Remark 70.3. The nonclosed smooth orbits have a particularly simple parametrization. Two nonzero pairs (a, c) and (a', c') give equivalent pencils if and only if $(a', c') = (t^2a, t^6c)$ for some $t \in \mathbb{C}^*$. Thus these orbits are parametrized by $\mathbb{P}(1, 3)$, or equivalently by $c/a^3 \in \mathbb{P}^1$. This is a parametrization of isomorphism classes of the strictly K-semistable examples below; it is not their GIT quotient. Their entire collection is contracted to the single closed orbit $U_{0,0}$ by the GIT quotient.

Proposition 70.4. *The threefold $X_{U_{0,0}}$ is K-polystable. For every $(a, c) \neq (0, 0)$, the threefold $X_{U_{a,c}}$ is strictly K-semistable. Consequently, if X_U is K-polystable and $U \in \mathcal{H}^\circ$, then either U is GIT-stable or U is equivalent to $U_{0,0}$.*

Proof. The pencil $U_{0,0}$ gives the unique smooth member of Family 2.20 with infinite automorphism group, and its K-polystability is proved in [3, Proposition 5.43].

Pull back the universal curve on $V_5 \times \mathcal{H}$ by the map $t \mapsto U_{a,c}(t)$ of Lemma 70.2. Every fibre is a smooth twisted cubic. Therefore this is a smooth family of regularly embedded curves in $V_5 \times \mathbb{A}^1$. Blowing up the family gives a smooth projective morphism $\mathcal{X} \to \mathbb{A}^1$, all of whose fibres are Fano threefolds in Family 2.20. The relative anticanonical divisor is relatively ample. The construction is equivariant for the displayed one-parameter subgroup, and over $\mathbb{A}^1 \setminus \{0\}$ it is equivariantly isotrivial with fibre $X_{U_{a,c}}$. Hence it is a special test configuration of $X_{U_{a,c}}$ with central fibre $X_{U_{0,0}}$.

K-semistability is open in a smooth family of Fano varieties; see [3, Theorem 1.11]. Since the central fibre is K-polystable, some nonzero fibre is K-semistable, and all nonzero fibres are isomorphic. Thus $X_{U_{a,c}}$ is K-semistable. The Donaldson–Futaki invariant of this test configuration is the Futaki invariant of the induced action on its central fibre. It is zero: on a K-polystable variety both a product test configuration and the one induced by the inverse action have nonnegative Donaldson–Futaki invariants, which are negatives of one another.

Finally, $X_{U_{a,c}}$ is not isomorphic to $X_{U_{0,0}}$. An isomorphism of two such blowups preserves the unique birational extremal contraction, the other extremal contraction being a conic bundle. It descends to an automorphism of V_5 carrying one centre to the other. Since $\mathrm{Aut}(V_5) = \mathrm{PGL}_2$, equivariance of the Hilbert-scheme description would then identify the corresponding pencils. Lemma 70.2 rules this out. Our test configuration is therefore nonproduct, proving strict K-semistability. □

Example 70.5. The pencil $\langle x^4 + y^4, x^3y \rangle$ belongs to $\mathcal{H}^\circ$ and is GIT-stable. To check the first assertion, a square with nonzero coefficient of $x^4 + y^4$ would have the form $(Ax^2 + Bxy + Cy^2)^2$,

with $AC \neq 0$. The vanishing of its xy^3 coefficient forces $B = 0$, but then its x^2y^2 coefficient is $2AC \neq 0$, a contradiction. The remaining pencil member x^3y is not a square. The pencil has no common root, so its first vanishing order is always 0, and its second order is at most 3, because a fourth-order zero would be a square. Thus it is GIT-stable by Lemma 70.1.

In contrast, $\langle x^3y, xy^3 + x^4\rangle$ gives a strictly K-semistable smooth threefold, while $\langle x^3y, xy^3\rangle$ gives a K-polystable smooth threefold. These examples show why one must distinguish K-polystability from K-stability in this family.

Remark 70.6. The preceding argument proves a necessary GIT criterion for K-polystability and completely identifies the smooth nonclosed GIT orbits. It does not prove that every GIT-stable pencil gives a K-stable threefold. The required implication is proved below by the valuative estimates and Theorem 80.1; neither reductivity of the stabilizer nor positivity of the CM line bundle proves it by itself.

70.1 A sextic test

Let f, g be a basis of a pencil $U \subset \operatorname{Sym}^4 \mathbb{C}^2$. Its Wronskian is the binary sextic

$$\operatorname{Wr}(U) = \left[\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right] \in \mathbb{P}(\operatorname{Sym}^6 \mathbb{C}^2).$$

Changing the basis multiplies the displayed form by a nonzero constant, so its projective class is well defined.

Lemma 70.7. *The Wronskian defines a finite surjective* SL_2*-equivariant morphism*

$$\operatorname{Wr}\colon \operatorname{Gr}(2, \operatorname{Sym}^4 \mathbb{C}^2) \longrightarrow \mathbb{P}(\operatorname{Sym}^6 \mathbb{C}^2).$$

If $a_0(p) < a_1(p)$ *is the vanishing sequence of* U *at* p*, then*

$$\operatorname{mult}_p \operatorname{Wr}(U) = a_0(p) + a_1(p) - 1.$$

Proof. The homogeneous Jacobian is linear in $f \wedge g$ and is equivariant for SL_2. It never vanishes identically for independent quartics. Indeed, on an affine chart, vanishing would give $(f/g)' = 0$ and hence f/g constant in characteristic zero. Thus the corresponding linear projection of the Plücker embedding has no base point on the Grassmannian, and defines a morphism. Its pullback of $\mathcal{O}_{\mathbb{P}^6}(1)$ is the Plücker line bundle. Since this is ample, a positive-dimensional fibre would contain a projective curve on which an ample line bundle is trivial, which is impossible. Hence the morphism is finite. Both source and target have dimension 6, so it is surjective.

For the local calculation, take $p = \{x = 0\}$ and write $f(z, 1) = z^{a_0}u(z)$ and $g(z, 1) = z^{a_1}v(z)$, where $u(0)v(0) \neq 0$. Euler's identity gives the homogeneous Jacobian on this chart as $4(f'g - fg')$. Its first nonzero coefficient is $4(a_0 - a_1)u(0)v(0)$ at order $a_0 + a_1 - 1$. This coefficient is nonzero. The asserted multiplicity follows. □

Corollary 70.8. *Let* $U \in \mathcal{H}^\circ$*. Its Wronskian has no root of multiplicity greater than* 3*. Moreover:*

1. *U is GIT-stable if and only if every root of its Wronskian has multiplicity at most* 2*;*
2. *U is GIT-polystable and not GIT-stable if and only if its Wronskian consists of two distinct triple roots;*
3. *U has a nonclosed GIT-semistable orbit if and only if its Wronskian has exactly one triple root.*

In the last case X_U is strictly K-semistable. Thus a necessary condition for X_U to be K-polystable is that its Wronskian have no triple root, or have precisely two triple roots. The sufficiency of the first alternative is proved in Section 82.

Proof. The multiplicity formula and Lemma 70.1 prove the first assertion and (1). In the nonstable case use Lemma 70.2. Direct differentiation gives

$$\mathrm{Wr}(U_{a,c}) = [x^3(2y^3 + axy^2 - cx^3)].$$

The second factor has a nonzero y^3 coefficient, so the displayed root $x = 0$ has multiplicity exactly 3. Another triple root occurs if and only if $2y^3 + axy^2 - cx^3$ is the cube of a linear form. Its x^2y coefficient is zero; comparing coefficients in $2(y - dx)^3$ forces $d = 0$, and then $a = c = 0$. The claim follows from Proposition 70.4. □

Example 70.9. For the smooth GIT-stable pencil $U = \langle x^4 + y^4, x^3y\rangle$ used above, the Wronskian is

$$\mathrm{Wr}(U) = [x^2(x^4 - 3y^4)].$$

It has one double root and four simple roots, in agreement with the criterion. For $U_{0,1}$ the Wronskian is $[x^3(2y^3 - x^3)]$, with one triple root and three simple roots. For $U_{0,0}$ it is $[x^3y^3]$.

71 Points on smooth fibres of the conic bundle

Let $\pi : X \to V_5$ be the blowup of a smooth twisted cubic C_0, let E be its exceptional divisor, and put $H = \pi^*\mathcal{O}_{V_5}(1)$ and $B = H - E$. Then $-K_X = 2H - E$, and $|B|$ defines the conic bundle $\eta : X \to \mathbb{P}^2$. These constructions apply to every smooth member of Family 2.20.

Lemma 71.1. *One has*

$$H^3 = 5, \quad H^2E = 0, \quad HE^2 = -3, \quad E^3 = -4, \quad (-K_X)^3 = 26,$$

and

$$S_X(E) = \frac{1}{2}, \qquad S_X(B) = \frac{63}{104}, \qquad S_X(H) = \frac{37}{104}.$$

Moreover, $E \cong \mathbb{P}^1 \times \mathbb{P}^1$, and $B|_E$ has bidegree $(1, 1)$.

Proof. The intersection numbers follow from $H \cdot C_0 = 3$, $g(C_0) = 0$ and $\deg N_{C_0/V_5} = 4$. The two nef divisors H and B vanish on the fibres of the two extremal contractions, respectively, and hence generate the nef cone. If $D \neq E$ is a prime divisor, write $D \sim aH - bE$. Its pushforward gives $a \geq 1$, while intersection with a general conic fibre gives $2(a - b) \geq 0$. Thus the effective cone is generated by E and B. Therefore $-K_X - uE$ is nef for $0 \leq u \leq 1$ and is not pseudoeffective for $u > 1$. Its volume is $40 - 18(1 + u)^2 + 4(1 + u)^3$. Similarly, $-K_X - uH$ is nef precisely up to its pseudoeffective threshold 1, and its volume is $5(2 - u)^3 - 9(2 - u) + 4$. For B, the positive and negative parts are

u	$P(u)$	$N(u)$
$[0, 1]$	$(2 - u)H - (1 - u)E$	0
$[1, 2]$	$(2 - u)H$	$(u - 1)E$.

Consequently,

$$S_X(B) = \frac{1}{26}\left(\int_0^1 \left(5(2 - u)^3 - 9(2 - u)(1 - u)^2 + 4(1 - u)^3\right) du + \int_1^2 5(2 - u)^3 du\right) = \frac{63}{104}.$$

The other two expected orders are separately

$$S_X(E) = \frac{1}{26}\int_0^1 (40 - 18(1+u)^2 + 4(1+u)^3)du = \frac{1}{2},$$
$$S_X(H) = \frac{1}{26}\int_0^1 (5(2-u)^3 - 9(2-u) + 4)du = \frac{37}{104}.$$

Write $E \cong \mathbb{F}_n$, with negative section s and fibre f of $E \to C_0$. Since the normal bundle has degree 4, n is even, and $-E|_E = s + (n-4)f/2$. Hence $B|_E = s + (n+2)f/2$. Its nefness implies $n \le 2$. If $n = 2$, then $B \cdot s = 0$ but $-K_X \cdot s = 3$. This is impossible: s would be a component of a conic-bundle fibre, whose total anticanonical degree is 2, and $-K_X$ is ample. Therefore $n = 0$. The morphism $\eta|_E$ is projection of the quadric $E \subset \mathbb{P}^3$ embedded by $|\mathcal{O}(1,1)|$ from a point outside it. It is a finite double cover with smooth branch conic. □

Lemma 71.2. *Let $q \in \mathbb{P}^2$ have a smooth fibre $Z = \eta^{-1}(q)$. There is a smooth surface $S \in |B|$ containing Z such that S is a quintic del Pezzo surface and $C = E|_S$ is a smooth twisted cubic. One can identify $S \cong \mathrm{Bl}_{p_1,\dots,p_4}\mathbb{P}^2$ so that*

$$C \sim l, \qquad Z \sim 2l - e_1 - e_2 - e_3 - e_4, \qquad H|_S \sim 3l - e_1 - e_2 - e_3 - e_4.$$

Here l is the pullback of a line, the e_i are disjoint exceptional curves, $C \cap e_i = \varnothing$, and Z meets each e_i transversally in one point.

Proof. Take the inverse image of a general line through q. Since η is smooth along Z, Bertini's theorem gives a surface smooth along Z and away from Z. We can also arrange that its intersection with E is a smooth member of $|B|_E| = |\mathcal{O}(1,1)|$: the lines tangent to the branch conic of the finite double cover $\eta|_E$ form a proper subset of the pencil. Let $S_0 = \pi(S)$. Along C_0, choose local coordinates (t, x, y) with $C_0 = (x = y = 0)$ and write the equation of S_0 as $a(t)x + b(t)y + O((x,y)^2) = 0$. If $a(t) = b(t) = 0$ at some point, then $S \cap E$ contains the whole exceptional fibre over that point. This is impossible because $S \cap E$ is a smooth $(1,1)$-curve. Thus S_0 is smooth along C_0, and it is smooth elsewhere because S is. Since C_0 is a Cartier divisor on S_0, its blowup is the identity. Hence $\pi|_S : S \to S_0$ is an isomorphism, and S is a quintic del Pezzo surface. On it $C^2 = 1$ and $(-K_S)\cdot C = 3$. The plane model associated to $|C|$ contracts four disjoint (-1)-curves and identifies C with the strict transform of a line avoiding the four points. Finally $Z \sim B|_S = -K_S - C$, giving the stated class. Its intersections with the e_i equal 1; since Z is smooth, they are transverse. □

Lemma 71.3. *Let W^S denote the refinement of the anticanonical complete series by S, and let $W^{S,Z}$ be its further refinement by Z. For $p \in Z$, put $d_p = (C \cdot Z)_p$ and $r_p = \sum_i (e_i \cdot Z)_p$. Then*

$$S(W^S; Z) = \frac{171}{208}, \qquad S(W^{S,Z}; p) = \frac{47}{52} + \frac{5d_p}{104} + \frac{5r_p}{208} \le 1.$$

Equality in the latter bound is possible only when $C \cdot Z = 2p$. If $p \notin C$, the point moment is at most $193/208$; if $p \in C$ and C meets Z transversally, it equals $99/104$.

Proof. Put $T = e_1 + e_2 + e_3 + e_4$. On S, $C = l$, $Z = 2l - T$, and $C \cdot Z = 2$. The positive part $P(u,v)$ of $P(u)|_S - vZ$ and its negative part $N(u,v)$ are given by

u	v	$P(u,v)$	$N(u,v)$
$[0,1]$	$[0, 2-u]$	$(2-u-v)Z + C$	0
$[0,1]$	$[2-u, \frac{5}{2}-u]$	$(\frac{5}{2}-u-v)(Z+T)$	$(v+u-2)T$
$[1,2]$	$[0, 2-u]$	$(2-u-v)Z + (2-u)C$	0
$[1,2]$	$[2-u, \frac{3}{2}(2-u)]$	$(\frac{3}{2}(2-u)-v)(Z+T)$	$(v+u-2)T$.

Indeed, in the second and fourth rows the displayed positive part is orthogonal to every e_i, while the e_i are disjoint negative curves. The other displayed divisors are nef, as follows by intersecting the ten (-1)-curves of the quintic del Pezzo surface. The volume becomes zero at the right endpoint in each row. These observations prove every claimed decomposition and threshold.

We justify that these volume calculations compute the actual refined series. For an integral nef moving part M on X, one has $M - B = K_X + (M + H)$. The divisor $M + H$ is nef and big, so Kawamata–Viehweg vanishing gives $H^1(X, \mathcal{O}_X(M - B)) = 0$. Consequently restriction to S is surjective, after removing the fixed multiple of E. Likewise, for each integral nef part P in the table, $P - Z = K_S + (P + C)$, and $P + C$ is nef and big because C is nef and $C^2 = 1$. Thus restriction from S to Z is surjective after removing the displayed fixed part. The resulting series on $Z \cong \mathbb{P}^1$ is complete. The formulas below follow either by summing dimensions of these complete series and taking limits, or by the multigraded Abban–Zhuang volume formulas. Endpoint rounding contributes $o(1)$ to the normalized moments.

There is no fixed component along Z in $N(u)|_S$. The surface volumes in the four rows are, respectively,

$$9 - 4u - 4v, \quad 4(\tfrac{5}{2} - u - v)^2, \quad (2 - u)(10 - 5u - 4v), \quad (6 - 3u - 2v)^2.$$

The surface formula is therefore

$$\begin{aligned} S(W^S; Z) = \frac{3}{26}\Bigg(& \int_0^1 \int_0^{2-u} (9 - 4u - 4v)\, dv\, du \\ & + \int_0^1 \int_{2-u}^{5/2-u} 4(\tfrac{5}{2} - u - v)^2\, dv\, du \\ & + \int_1^2 \int_0^{2-u} (2 - u)(10 - 5u - 4v)\, dv\, du \\ & + \int_1^2 \int_{2-u}^{3(2-u)/2} (6 - 3u - 2v)^2\, dv\, du \Bigg) = \frac{171}{208}. \end{aligned}$$

Write Ω_i for the four integration regions and $a(u, v) = P(u, v) \cdot Z$. Its four expressions are 2, $4(5/2 - u - v)$, $2(2 - u)$, and $4(3(2 - u)/2 - v)$. The threefold fixed part restricts to $(u - 1)_+ C$. Thus the point formula gives

$$S(W^{S,Z}; p) = \frac{3}{26} \sum_i \iint_{\Omega_i} a(u, v)^2\, dv\, du + \frac{6}{26} \sum_i \iint_{\Omega_i} a(u, v)\, ((u - 1)_+ d_p + n(u, v) r_p)\, dv\, du,$$

where $n(u, v) = v + u - 2$ in the second and fourth rows, and is zero otherwise. The three integrals, including their normalization factors, are $47/52$, $5/104$, and $5/208$. This proves the formula. Since $C \cap T = \varnothing$, one cannot have both $d_p > 0$ and $r_p > 0$. Moreover $d_p \leq 2$ and $r_p \leq 1$, because the e_i are disjoint. The assertions follow. □

Example 71.4. If $p \in Z$ avoids C and the four exceptional curves, the point moment is $47/52$. At a transverse point of $C \cap Z$ it is $99/104$. At a point where C is tangent to Z it is exactly 1. The last equality does not prevent a strict threefold estimate, as the next lemma shows.

Lemma 71.5. *Let p lie on a smooth fibre of η, and choose S as above. For every divisorial valuation v with centre p on X, one has*

$$A_X(v) - S_X(v) \geq \frac{41}{104} v(S) > 0.$$

The same non-strict inequality holds for a valuation whose centre is Z. In particular, no divisor centred at a point on a smooth fibre can destabilize X or give a non-product zero-Futaki degeneration.

Proof. For each sufficiently divisible m, choose a basis of $H^0(X, -mK_X)$ adapted simultaneously to v and the single lexicographic flag $X \supset S \supset Z \supset \{p\}$. This requires compatibility with two filtrations. Let D_m be its normalized basis divisor, and write $D_m = a_m S + D'_m$. On S, write $D'_m|_S = b_m Z + R_m$, with R_m effective and not containing Z. Let $c_m = (R_m \cdot Z)_p$. The flag moment formulas give

$$a_m \longrightarrow \frac{63}{104}, \qquad b_m \longrightarrow \frac{171}{208}, \qquad c_m \longrightarrow S(W^{S,Z}; p) \leq 1.$$

Choose positive $\lambda_m \to 1$ such that $\lambda_m b_m \leq 1$ and $\lambda_m c_m \leq 1$. For example, take

$$\lambda_m = \min\{1, 1/\max(b_m, c_m)\},$$

omitting reciprocals of zero. Since Z is smooth at p, inversion of adjunction gives that $(S, Z + \lambda_m R_m)$ is log canonical near p. Therefore $(S, \lambda_m D'_m|_S)$ is log canonical there. A second application gives that $(X, S + \lambda_m D'_m)$ is log canonical near p. Both boundaries are effective. Evaluating at v yields

$$A_X(v) - \lambda_m v(D_m) \geq (1 - \lambda_m a_m) v(S).$$

The basis is adapted to v, so $v(D_m) = S_m(v) \to S_X(v)$. Taking the limit proves the assertion. For a valuation with centre Z, apply the same argument near a general point of Z. □

Remark 71.6. The preceding estimate uses only the elementary two-chamber decomposition on the original X and Zariski decompositions on a quintic del Pezzo surface. No Zariski decomposition is computed on a threefold obtained by extracting the valuation v. It reduces the classification problem to centres above the discriminant of η, together with centres not addressed by this pointwise argument. It does not prove the full classification of Family 2.20.

Lemma 71.7. *Every prime divisor D on X satisfies $S_X(D) < 1$. More precisely, $S_X(E) = 1/2$ and $S_X(D) \leq 63/104$ for $D \neq E$.*

Proof. For $D \neq E$, its pushforward is a nonzero effective divisor of class $a\mathcal{O}_{V_5}(1)$ for an integer $a \geq 1$. Write $D \sim aH - bE = aB + (a-b)E$. The description of the effective cone gives $a - b \geq 0$. Thus $D - B$ is effective. Monotonicity of volume implies $S_X(D) \leq S_X(B) = 63/104$. □

Lemma 71.8. *Let $\Delta \subset \mathbb{P}^2$ be the discriminant curve of η, and let $R \subset E$ be the ramification curve of the finite double cover $\eta|_E : E \to \mathbb{P}^2$. If $p \notin \eta^{-1}(\Delta) \cup R$, then $\delta_p(X) \geq 104/99$. Moreover, $\delta_R(X) \geq 104/63$. Consequently, every divisorial valuation whose curve centre is not contained in $\eta^{-1}(\Delta)$ has positive beta invariant.*

Proof. At a point outside R, the intersection of the smooth conic fibre with E is either absent or transverse at that point. Lemma 71.3 and the iterated Abban–Zhuang estimate give

$$\delta_p(X) \geq \min\left\{\frac{104}{63}, \frac{208}{171}, \frac{104}{99}\right\} = \frac{104}{99}.$$

For the ramification curve, write $E = \mathbb{P}^1 \times \mathbb{P}^1$, with rulings s, f chosen so that f is a fibre of $\pi|_E$. The ramification formula gives $R \sim s + f$. Since $-E|_E = s - 2f$ and $H|_E = 3f$, one has

$$(-K_X - uE)|_E = (1 + u)s + (4 - 2u)f \quad (0 \leq u \leq 1).$$

There is no fixed part in this interval. Thus

$$S(W^E; R) = \frac{3}{26} \int_0^1 \int_0^{1+u} 2(1 + u - v)(4 - 2u - v)\, dv\, du = \frac{63}{104}.$$

The first-stage moment is $S_X(E) = 1/2$, so $\delta_R(X) \geq \min\{2, 104/63\} = 104/63$. Finally, let T be a curve not contained in $\eta^{-1}(\Delta)$. If $T \neq R$, a general point of T lies outside both R and $\eta^{-1}(\Delta)$, so the first estimate excludes every valuation with centre T. If $T = R$, use the second estimate. □

72 Reduced singular conic fibres: exact surface estimates

The following computations record an obstruction to extending the preceding smooth-fibre flag argument without an additional threefold estimate. They do not establish instability. Write

$$S = \mathrm{Bl}_{p_1,p_2,p_3,p_4}\,\mathbb{P}^2, \quad A = l - e_1 - e_2, \quad D = l - e_3 - e_4, \quad Z = A + D, \quad C = l.$$

Assume that S and C are smooth and that C avoids the four exceptional curves. Thus A and D meet transversally at one point p_0, while C meets each of A, D once, possibly at p_0. Put

$$(b,k) = \begin{cases} (2-u,1) & 0 \le u \le 1, \\ (2-u,2-u) & 1 \le u \le 2. \end{cases}$$

Then the positive part restricted from the threefold is $M_u = (2b+k)l - b\sum e_i$. In both intervals $k \le b \le 2k$.

Lemma 72.1. *The pseudoeffective threshold of A with respect to M_u is $b+k$. The negative part of $M_u - vA$ is*

v	$N_{u,v}$
$[0,k]$	0
$[k,b]$	$(v-k)D$
$[b,b+k]$	$(v-k)D + (v-b)(e_1+e_2)$.

The corresponding positive parts have volumes

$$4bk + k^2 - 2kv - v^2, \qquad 2k(2b+k-2v), \qquad 2(b+k-v)^2,$$

and their intersections with A are respectively $k+v$, $2k$, and $2(b+k-v)$. Consequently,

$$S(W^S; A) = \frac{109}{104}.$$

For $p \in A$, the point estimate obtained from this flag is

$$S(W^{S,A}; p) = \frac{79}{104} + \frac{15}{52}\mathbf{1}_{p=p_0} + \frac{5}{52}\mathbf{1}_{p\in e_1\cup e_2} + \frac{5}{104}\mathbf{1}_{p\in C}.$$

In particular, at the node it is $109/104$ if $p_0 \notin C$, and $57/52$ if $p_0 \in C$.

Proof. The curves D, e_1, e_2 are mutually disjoint (-1)-curves. Subtracting the displayed coefficients makes the positive part orthogonal to its negative part. Intersecting with the ten (-1)-curves proves nefness. In the last chamber the positive part is $(b+k-v)(2l-e_3-e_4)$. The nef divisor $2l - e_3 - e_4$ has intersection $2(b+k-v)$ with $M_u - vA$, so $v > b+k$ is not pseudoeffective. Squaring and intersecting with A give all the displayed expressions.

For the curve moment integrate the three volume expressions with factor $3/26$. For the point moment integrate $(P_{u,v}\cdot A)^2$ with factor $3/26$. This gives $79/104$. The negative component D contributes

$$\frac{6}{26}\int_0^2 \int_k^{b+k} (P_{u,v}\cdot A)(v-k)\,dv\,du = \frac{15}{52}.$$

Each of e_1, e_2 contributes

$$\frac{6}{26}\int_0^2 \int_b^{b+k} 2(b+k-v)(v-b)\,dv\,du = \frac{5}{52}.$$

Finally the fixed part $(u-1)_+C$ contributes $5/104$ at its intersection with A. All intersections are transverse. The node, the two exceptional endpoints, and any other point of $C \cap A$ are distinct; $C \cap A$ can coincide only with the node. □

Lemma 72.2. *Let $\sigma : T \to S$ blow up p_0, with exceptional curve J. Then*

$$S(W^S; J) = \frac{109}{52} + \frac{5}{104}\mathbf{1}_{p_0 \in C}.$$

The point moments on J are bounded by

$$\frac{47}{104} + \frac{31}{52}\mathbf{1}_{q \in A' \cup D'} + \frac{5}{104}\mathbf{1}_{p_0 \in C,\ q \in C'}.$$

Here the three directions A', D', C' are pairwise distinct whenever C' meets J. Thus the maximum is $109/104$*, attained at either branch direction.*

Proof. The strict transforms A', D' are disjoint (-2)-curves. For $0 \le v \le k$, the divisor $\sigma^* M_u - vJ$ is nef. For $k \le v \le 2b + k$, its negative part is $(v - k)(A' + D')/2$. The positive part has square

$$\begin{cases} 4bk + k^2 - v^2 & 0 \le v \le k, \\ 2k(2b + k - v) & k \le v \le 2b + k, \end{cases}$$

and intersection with J equal to v and k, respectively. For nefness, one may write the positive part in the second interval as

$$(b - (v - k)/2)\sigma^* Z + k(\sigma^* l - J),$$

a nonnegative combination of the two nef pencils $\sigma^* Z$ and $\sigma^* l - J$. At the threshold it equals $k(\sigma^* l - J)$. The nef divisor $\sigma^* l - J$ has intersection $2b + k - v$ with $\sigma^* M_u - vJ$, proving the threshold.

Integrating the two squares gives $109/52$. The fixed curve C adds

$$\frac{3}{26}\int_1^2 5(2 - u)^2(u - 1)\, du = \frac{5}{104}$$

exactly when it passes through p_0. For the second refinement, integrating $(P_{u,v} \cdot J)^2$ gives $47/104$. The coefficient $(v - k)/2$ of either branch contributes $31/52$ at its direction; the strict transform of the fixed curve contributes $5/104$ at its direction. These directions are distinct because C, A, D are three distinct lines through the same point in the plane model. □

Remark 72.3. Neither lemma is a destabilizing calculation for the threefold. The first curve moment exceeds its surface log discrepancy 1, while the node moment exceeds its surface log discrepancy 2. This says precisely that these unmodified surface Abban–Zhuang estimates fail to prove the desired threefold assertion. A successful proof must exploit more information, for example the positive first-stage margin $41v(S)/104$ and a comparison with the conic-bundle coordinates, or a different extraction.

Lemma 72.4. *Keep the preceding assumptions, and let v be a divisorial valuation centred at $p \in A$ on X. If $p \ne p_0$ or $p_0 \notin C$, then*

$$A_X(v) - S_X(v) \ge \frac{46v(S) - 5A_X(v)}{104}.$$

If $p = p_0 \in C$, then

$$A_X(v) - S_X(v) \ge \frac{51v(S) - 10A_X(v)}{104}.$$

In particular these inequalities are strict whenever $A_X(v)/v(S) < 46/5$ in the first case, or $A_X(v)/v(S) < 51/10$ in the second case.

Proof. Choose a basis adapted simultaneously to v and the single lexicographic flag $X \supset S \supset A \supset \{p\}$. Write its normalized basis divisor as $D_m = a_m S + D'_m$, and write $D'_m|_S = b_m A + R_m$, where R_m does not contain A. Then $a_m \to 63/104$, $b_m \to 109/104$, and $(R_m \cdot A)_p$ tends to the point moment in Lemma 72.1. Away from the node this moment is at most $89/104$. At the node it is $109/104$ off C and $114/104$ on C. Consequently, one can choose positive coefficients c_m with $c_m b_m \le 1$ and $c_m (R_m \cdot A)_p \le 1$, tending to $c = 104/109$ in the first case and $c = 104/114$ in the second case.

Both A and S are smooth at p. Applying inversion of adjunction first to $(S, A + c_m R_m)$ and then to $(X, S + c_m D'_m)$ gives

$$A_X(v) - c_m v(D_m) \ge (1 - c_m a_m) v(S).$$

All boundaries used here are effective. The basis is adapted to v, so $v(D_m) \to S_X(v)$. Passing to the limit and dividing by c yields

$$A_X(v) - S_X(v) \ge (1/c - 63/104) v(S) - (1/c - 1) A_X(v),$$

which is exactly the assertion. □

73 The global conic-bundle model and double fibres

In this section write $D = H - E$ for the class denoted by B above, and write C_0 for the blown-up twisted cubic. By Lemma 71.1, $E \cong \mathbb{P}^1 \times \mathbb{P}^1$, $H|_E = \mathcal{O}(3,0)$ and $D|_E = \mathcal{O}(1,1)$. The map $\eta|_E$ is a finite double cover with smooth conic branch.

Identify this cover with the incidence surface

$$E = \{([s:t],[A:B:C]) : As^2 + Bst + Ct^2 = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^2.$$

The first projection agrees with $E \to C_0$. Put $G = \eta_* \mathcal{O}_E(3,0)$.

Lemma 73.1. *There are exact sequences and an isomorphism*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} \xrightarrow{\begin{pmatrix} A & 0 \\ B & A \\ C & B \\ 0 & C \end{pmatrix}} \mathcal{O}_{\mathbb{P}^2}^{\oplus 4} \longrightarrow G \longrightarrow 0, \qquad \eta_* \mathcal{O}_X(H) \simeq \mathcal{O}_{\mathbb{P}^2}(1) \oplus G.$$

Use the quotient convention for $P = \mathbb{P}_{\mathbb{P}^2}(\mathcal{O}(1) \oplus G)$, so $p_* \mathcal{O}_P(m) = \operatorname{Sym}^m(\mathcal{O}(1) \oplus G)$. Then the relative anticanonical embedding identifies X with a divisor in $|\mathcal{O}_P(2) \otimes p^* \mathcal{O}(-1)|$. In this embedding $\mathcal{O}_P(1)|_X = \mathcal{O}_X(H)$.

Proof. The equation of the incidence surface gives $0 \to \mathcal{O}(1,-1) \to \mathcal{O}(3,0) \to \mathcal{O}_E(3,0) \to 0$; pushing it to $\mathbb{P}^2$ gives the displayed Steiner resolution. Push forward $0 \to \mathcal{O}_X(D) \to \mathcal{O}_X(H) \to \mathcal{O}_E(3,0) \to 0$. The higher direct image of $\mathcal{O}_X(D)$ vanishes because every conic fibre has arithmetic genus zero. Hence the resulting sequence is $0 \to \mathcal{O}(1) \to \eta_* \mathcal{O}_X(H) \to G \to 0$. Since $\det G = \mathcal{O}(2)$, its extension group is $H^1(G^*(1)) = H^1(G(-1)) = 0$, as follows by twisting the Steiner resolution by $\mathcal{O}(-1)$. Therefore the sequence splits. The restriction of H to every fibre is its anticanonical line bundle; it gives the usual plane-conic embedding, including the nonreduced fibres. Finally $c_1(\mathcal{O}(1) \oplus G) = 3$, so $K_P = -3H$. Adjunction and $K_X = -H - D$ give $X \sim 2H - D$. □

For explicit calculations, let y_0, y_1, y_2, y_3 be the coordinates coming from the four generators of G. On $\mathbb{P}(G)$ they satisfy $Ay_0 + By_1 + Cy_2 = Ay_1 + By_2 + Cy_3 = 0$. There is a unique section, up to scalar, q_0 of $\operatorname{Sym}^2 G(-1)$, characterized by

$$y_0 y_2 - y_1^2 = C q_0, \qquad y_0 y_3 - y_1 y_2 = -B q_0, \qquad y_1 y_3 - y_2^2 = A q_0.$$

The uniqueness also follows from the symmetric-square resolution $0 \to \mathcal{O}(-3) \to \mathcal{O}(-2)^8 \to \mathcal{O}(-1)^{10} \to \operatorname{Sym}^2 G(-1) \to 0$. Let z denote the relative linear section of class $H - D$ corresponding to the summand $\mathcal{O}(1)$. Every member has equation

$$q_0 + z(f_0y_0 + f_1y_1 + f_2y_2 + f_3y_3) + z^2(h_0A + h_1B + h_2C) = 0.$$

Here the coefficient of q_0 is normalized to one; its nonvanishing follows from the fact that E is the smooth incidence surface.

Example 73.2. On $C = 1$, eliminate $y_2 = -Ay_0 - By_1$ and $y_3 = ABy_0 + (B^2 - A)y_1$. Put $F_0 = f_0 - Af_2 + ABf_3$, $F_1 = f_1 - Bf_2 + (B^2 - A)f_3$, and $h = h_0A + h_1B + h_2$. The conic equation becomes

$$-Ay_0^2 - By_0y_1 - y_1^2 + z(F_0y_0 + F_1y_1) + hz^2 = 0.$$

Its discriminant, up to a nonzero scalar, is $(4A - B^2)h + AF_1^2 - BF_0F_1 + F_0^2$. After cancellation this is the cubic

$$\begin{aligned} &f_3^2A^3 - f_2f_3A^2B + (-2f_1f_3 + f_2^2 + 4h_0)A^2 + (f_1f_3 - h_0)AB^2 \\ &\quad + (3f_0f_3 - f_1f_2 + 4h_1)AB + (-2f_0f_2 + f_1^2 + 4h_2)A \\ &\quad + (-f_0f_3 - h_1)B^3 + (f_0f_2 - h_2)B^2 - f_0f_1B + f_0^2. \end{aligned}$$

Homogenizing in C gives its global equation.

73.1 Double fibres and their weighted extraction

Lemma 73.3. *The discriminant of η is a reduced cubic with at most ordinary nodes. Its smooth points correspond to fibres consisting of two distinct lines. Its nodes correspond exactly to double lines, and all of them lie on the branch conic of $E \to \mathbb{P}^2$. For every base point outside the nodes, a general inverse image of a line through that point is a smooth quintic del Pezzo surface. For a node b, that inverse image has precisely two ordinary double points on the reduced fibre and is otherwise smooth; its minimal resolution is a weak del Pezzo surface of degree five.*

Proof. Locally a conic bundle is given by a symmetric three-by-three matrix. Rank zero is excluded by flatness. At a rank-two point the derivative of its determinant cannot vanish: otherwise the total space would be singular at the vertex of the conic. At rank one, write the initial normal equation along the reduced fibre $R \simeq \mathbb{P}^1$ as $x^2 + uq_1(y, z) + vq_2(y, z) = 0$, with (u, v) parameters on the base. Smoothness of X says exactly that the two binary quadratics q_1, q_2 have no common zero. They define a degree-two morphism $\mathbb{P}^1 \to \mathbb{P}^1$, which has two distinct branch points. After changing coordinates this pencil is generated by y^2, z^2. The quadratic initial term of the determinant is consequently a nonzero multiple of uv, proving the nodal assertion. At a double fibre, the rank-one ternary quadratic restricts to a nonzero square on $E = (z = 0)$. Thus q_0 has rank one there, so the base point lies on the branch conic of E. For a general line through b, its inverse image has local equation $x^2 + t(aq_1 + bq_2) + \text{higher terms} = 0$. The quadratic $aq_1 + bq_2$ has two simple zeros. At each zero the surface has a nondegenerate quadratic part $x^2 + tw$, and hence an ordinary double point. It is smooth elsewhere along R; Bertini applies elsewhere. For a reduced fibre choose the line transverse to the discriminant. Also choose it non-tangent to the smooth branch conic of $E \to \mathbb{P}^2$. Then E meets the surface in a smooth curve of bidegree $(1, 1)$, so π contracts no curve in the surface. Thus H restricts to an ample divisor. Adjunction gives $-K_S = H|_S$ and $K_S^2 = H^2D = 5$. The same argument works in the double-fibre case; resolving its two ordinary double points is crepant and gives the stated weak del Pezzo surface. □

Lemma 73.4. *Suppose $\eta^{-1}(b) = 2R$ is a double line. Let $f : Y = \mathrm{Bl}_{\mathfrak{m}_b\mathcal{O}_X} X \to X$ and let F be its reduced exceptional divisor. Then Y is normal,*

$$F \simeq \mathbb{P}^1 \times \mathbb{P}^1, \qquad A_X(F) = 3, \qquad \Delta_F = \tfrac{1}{2}\Gamma,$$

where Γ is a smooth curve of bidegree $(2,1)$. With the first factor identified with R and the second with the directions at b,

$$f^*H|_F = \mathcal{O}(1,0), \qquad f^*D|_F = \mathcal{O}_F, \qquad \mathcal{O}_F(2F) = \mathcal{O}(0,-1).$$

Moreover $N_{R/X} = \mathcal{O}(1) \oplus \mathcal{O}(-2)$.

Proof. Flatness gives the base-change description $Y = X \times_{\mathbb{P}^2} \mathrm{Bl}_b \mathbb{P}^2$. Near R the ideal of the base point has the form (u, x^2) in regular parameters transverse to R. Thus this is the weighted extraction of weights $(1,2)$, not the ordinary blowup of R. In the chart $v = tu$, its equation has leading form $x^2 + u(q_1 + tq_2) = 0$. Along $u = x = 0$ the quotient singularities are $\frac{1}{2}(1,1,0)$ precisely where $q_1 + tq_2 = 0$. This also proves normality. Replacing the weighted coordinate x by $w = x^2$ identifies the coarse exceptional surface with $R \times \mathbb{P}^1_{[u:v]}$; the relation is $w + uq_1 + vq_2 = 0$. The quotient locus is the graph $\Gamma = \{uq_1 + vq_2 = 0\}$, which is smooth of bidegree $(2,1)$. Its different is $\Gamma/2$. The weighted discrepancy is $1+2 = 3$, so $K_Y = f^*K_X + 2F$. The exceptional curve of $\mathrm{Bl}_b \mathbb{P}^2$ pulls back to $2F$, giving $\mathcal{O}_F(2F) = \mathcal{O}(0,-1)$. The other two restrictions follow from $H \cdot R = 1$ and the fact that R is contracted by η. Finally the normal sequence of R in the local plane bundle is the kernel of $\mathcal{O}(1) \oplus \mathcal{O}^{\oplus 2} \to \mathcal{O}(2)$, where the last two summands map by q_1, q_2 and the first maps to zero. Its kernel is $\mathcal{O}(1) \oplus \mathcal{O}(-2)$. □

Remark 73.5. It is essential to retain the quadratic transition terms in this weighted extraction. Weighted projectivization of the split normal bundle alone would incorrectly suggest a Hirzebruch surface $\mathbb{F}_4$. The base-change construction proves that the actual coarse surface is $\mathbb{F}_0$. As a check, adjunction gives $K_F + \Delta_F = \mathcal{O}(-1, -3/2)$, which agrees with $(f^*K_X + 3F)|_F$.

Example 73.6. The automorphism group of the branch conic is transitive on its points. Thus a double-fibre point can be normalized to $[A : B : C] = [0 : 0 : 1]$. The explicit equation above then shows that it is a double fibre exactly when $f_0 = 0$ and $h_2 = -f_1^2/4$. The fibre equation is $-(y_1 - f_1 z/2)^2 = 0$. This gives a fixed coordinate model for the weighted filtration at every double fibre.

Remark 73.7. These geometric lemmas do not by themselves compute $S_X(F)$ or the refined series on $(F, \Gamma/2)$. Proposition 74.3 will compute the full expected order from the actual global filtration. The complete refinement and its stability estimates are established in Section 75.

Lemma 73.8. *Normalize a double-fibre point to $[0 : 0 : 1]$, so that $f_0 = 0$ and $h_2 = -f_1^2/4$. The point $R \cap E$ is a ramification point of the degree-two map $\Gamma \to \mathbb{P}^1$ to the direction factor if and only if $f_1 = 0$. In that case $h_2 = 0$ and smoothness forces $h_1 \neq 0$.*

Proof. Put $a = f_1/2$ and $x = y_1 - az$, so that $R = (x = 0)$ with coordinates $[y_0 : z]$. The derivatives of the conic equation in the base directions A, B, restricted to R, are

$$q_A = -y_0^2 - f_2 y_0 z + (h_0 - af_3)z^2, \qquad q_B = -ay_0 z + (h_1 - af_2)z^2.$$

The graph Γ is defined by $uq_A + vq_B = 0$. At $R \cap E = [1 : 0]$, the first quadratic is nonzero, and the derivative of q_B/q_A vanishes if and only if $a = 0$. If $a = 0$, then $q_B = h_1 z^2$; the two quadratics have a common zero if $h_1 = 0$, contradicting smoothness of X. □

Lemma 73.9. *In the notation of Lemma 73.4, put $L = -K_X = H + D$. The nef threshold of F with respect to f^*L is two, and*

$$\mathrm{vol}_Y(f^*L - tF) = 26 - \frac{3}{2}t^2 \qquad (0 \le t \le 2).$$

In particular, this interval contributes $24/13$ to $S_X(F)$.

Proof. Let $\widetilde{\eta} : Y \to \mathrm{Bl}_b \mathbb{P}^2$ be the base-changed conic bundle. The divisor $f^*D - 2F$ is the pullback of the pencil of lines through b, and is nef. For $0 \le t \le 2$,

$$f^*L - tF = f^*H + (1 - t/2)f^*D + (t/2)(f^*D - 2F)$$

is therefore nef. Let ℓ_E be the fibre of $E \to C_0$ through $R \cap E$. Its image under η is a line, with degree one. Its strict transform on Y has intersection $1/2$ with F, because $2F$ pulls back the exceptional curve of the base blowup. It has intersection zero with f^*H and one with f^*D. Hence $f^*L - tF$ is not nef for $t > 2$.

The restrictions in Lemma 73.4 give

$$(f^*L)^2F = 0, \qquad f^*L \cdot F^2 = -\frac{1}{2}, \qquad F^3 = 0.$$

Cubing the nef divisor proves the volume formula, and

$$\frac{1}{26}\int_0^2 \left(26 - \frac{3}{2}t^2\right) dt = \frac{24}{13}.$$

The volume is still 20 at $t = 2$, so the remaining tail is nonzero and is essential to the full expected order. □

73.2 The marked point on a double fibre

The following calculation identifies the GIT condition with the ramification condition on the exceptional surface of a weighted extraction. The capital letters in the following coordinate calculation denote Plücker coordinates, rather than the divisor classes used above. Put $e_i = x^{4-i}y^i$. The quintic del Pezzo threefold is the linear section of $\mathrm{Gr}(2,5)$ given by

$$2p_{03} - p_{12} = 8p_{04} - p_{13} = 2p_{14} - p_{23} = 0.$$

These are the coefficients of the third transvectant of two binary quartics. Set

$$(A, B, C, D, E, F, G) = (p_{01}, p_{02}, p_{03}, p_{04}, p_{14}, p_{24}, p_{34}).$$

Its five equations are

$$AE - 4BD + C^2 = AF - BE + 2CD = AG - CE + 8D^2 = 0,$$
$$BG - CF + 2DE = CG - 4DF + E^2 = 0.$$

For a pencil $U \in \mathcal{H}^\circ$, the base of the conic bundle is $\mathbb{P}(U^\perp) \subset \mathbb{P}(V^*)$. A hyperplane $D \subset V$ containing U parametrizes the conic $V_5 \cap \mathrm{Gr}(2, D)$. The graph of the resulting rational map on V_5 is X_U: its base ideal is generated by the three coordinates of the exterior square of the projection $V \to V/U$, whose vanishing is the determinantal condition defining C_U. Thus the following computation concerns the actual conic bundle on the blowup.

Lemma 73.10. *The double fibres of $X_U \to \mathbb{P}(U^\perp)$ are in bijection with the common roots of the pencil U. A root p corresponds to the evaluation hyperplane $D_p = \{f \in V \mid f(p) = 0\}$. Every common root is simple in the greatest common divisor of the pencil, and the vanishing sequence there is $(1,2)$ or $(1,3)$.*

Proof. Write a hyperplane in an affine chart of $\mathbb{P}(V^*)$ as

$$h_0z_0 + h_1z_1 + h_2z_2 + h_3z_3 + z_4 = 0.$$

It has basis $d_i = e_i - h_i e_4$ for $0 \le i \le 3$. Write z_{ij} for the Plücker coordinates in this basis. The three linear equations defining V_5 give

$$\begin{aligned} z_{12} &= 2z_{03}, \\ z_{13} &= -8h_1 z_{01} - 8h_2 z_{02} - 8h_3 z_{03}, \\ z_{23} &= (2h_0 + 16h_1 h_3) z_{01} + 16h_2 h_3 z_{02} + (-4h_2 + 16h_3^2) z_{03}. \end{aligned}$$

Substitute these into the Plücker equation $z_{01}z_{23} - z_{02}z_{13} + z_{03}z_{12} = 0$. Up to a nonzero scalar, the resulting conic in coordinates $[z_{01} : z_{02} : z_{03}]$ has matrix

$$\begin{pmatrix} h_0 + 8h_1h_3 & 4h_2h_3 + 2h_1 & -h_2 + 4h_3^2 \\ 4h_2h_3 + 2h_1 & 4h_2 & 2h_3 \\ -h_2 + 4h_3^2 & 2h_3 & 1 \end{pmatrix}.$$

It has rank 1 if and only if $h_2 = h_3^2$, $h_1 = h_3^3$, and $h_0 = h_3^4$. Indeed, compare its upper-left 2×2 block with the outer product of the first two entries of its last column. Hence its double-conic locus is

$$[h_0 : h_1 : h_2 : h_3 : h_4] = [t^4 : t^3 : t^2 : t : 1].$$

These are precisely evaluation at $[t : 1] \in \mathbb{P}^1$. Every hyperplane can be moved into the chart $h_4 \neq 0$ by SL_2, since fourth powers span V. Thus the calculation proves the assertion globally. Such a hyperplane belongs to $\mathbb{P}(U^\perp)$ if and only if its evaluation vanishes on every member of U, which is the common-root condition.

If the common-root multiplicity were at least 2, factor $U = x^2 W$ for a pencil W of binary quadratics. Its projective line meets the conic of quadratic squares, so U would contain a square quartic. This contradicts $U \in \mathcal{H}^\circ$. The first vanishing order is therefore 1. The second is at least 2 and at most 3, because an order-4 member is again a square. □

The evaluation hyperplane $\langle e_0, e_1, e_2, e_3 \rangle = x \operatorname{Sym}^3 \mathbb{C}^2$ parametrizes a double conic in V_5. Its reduced support is the line

$$R = \{C = D = E = F = G = 0\},$$

with coordinates $[A : B]$. On the normalized weighted extraction over its strict transform, induced by blowing up the corresponding point in the conic-bundle base, the coordinate weights are $(0, 0, 1, 2, 2, 2, 3)$. These can be checked at $B \neq 0$ directly from the five equations: C is the normal coordinate of weight 1; the base coordinates have weight 2; D, E, F have weight 2; and $G = CF/B - 2DE/B$ has weight 3. The weight-one coordinate has quadratic relation with the two base directions. The coarse exceptional surface is $\mathbb{P}^1_R \times \mathbb{P}^1_{[u:v]}$, and the different is one-half of the curve defined by vanishing of that quadratic relation. We write $[u : v]$ for the directions in the blown-up conic-bundle base. The exceptional surface has a branch curve Γ coming from setting its weight-one coordinate equal to zero.

Lemma 73.11. *Suppose the pencil $U \in \mathcal{H}^\circ$ has a common root p. Let R be the corresponding reduced double fibre and let r be its intersection with the exceptional divisor of $X_U \to V_5$. Then the marked point of Γ lying over r is ramified for $\Gamma \to \mathbb{P}^1_{[u:v]}$ if and only if the vanishing sequence of U at p is $(1, 3)$.*

Proof. Choose coordinates with $p = \{x = 0\}$. First suppose the vanishing sequence is $(1, 2)$. After changing the basis of U, write

$$U = \langle e_2 + be_1 + ce_0,\ e_3 + de_1 + ee_0 \rangle.$$

We have $b^2 - 4c \neq 0$, because otherwise the first quartic would be a square. The three linear forms defining the twisted cubic are the coordinates of the exterior square of the projection $V \to V/U$. Explicitly, they are

$$\begin{aligned} l_0 &= A - bB + (2c - d)C + 8eD + 2(cd - eb)E, \\ l_1 &= D - cF - eG, \\ l_2 &= E - bF - dG. \end{aligned}$$

Thus r is the point $A = bB$ on R. At the generic point of R, let $u = l_1/l_0$ and $v = l_2/l_0$. Taking the terms of weight 2 in the equations gives

$$F = Bv, \qquad E = Av, \qquad D = (A - bB)u + cBv,$$

and therefore

$$C^2 = 4B(A - bB)u + (4cB^2 - A^2)v.$$

Consequently Γ is given by the vanishing of the right-hand side. Near r, put $t = A/B$. Its projection to the direction line is

$$\frac{v}{u} = \frac{4(t - b)}{t^2 - 4c}.$$

Its derivative at $t = b$ is $4/(b^2 - 4c)$, which is nonzero. The projection is unramified at the marked point.

Now suppose the vanishing sequence is $(1, 3)$. Use the normal form $U = \langle e_1, e_3 + ae_2 + ce_0 \rangle$. The same exterior-square calculation gives

$$l_0 = B - aC + 2cE, \qquad l_1 = D - cG, \qquad l_2 = F - aG.$$

The marked point is $B = 0$. Taking terms of weight 2 gives $D = Bu$, $F = Bv$, $E = Av$, and hence

$$C^2 = 4B^2u - A^2v.$$

Thus Γ projects by $[A : B] \mapsto [A^2 : 4B^2]$, which is ramified at $B = 0$. These are the only possible vanishing sequences at a common root of a smooth pencil by Lemma 73.10. □

74 The weighted moment at a double fibre in two explicit members

This section concerns two specified members of Family 2.20. It does not assert that the calculation is independent of the member. Let $V \subset \mathbb{P}^6_{A,B,C,D,E,F,G}$ be given by

$$\begin{aligned} &AE - 4BD + C^2 = 0, \qquad AF - BE + 2CD = 0, \qquad AG - CE + 8D^2 = 0, \\ &BG - CF + 2DE = 0, \qquad CG - 4DF + E^2 = 0. \end{aligned}$$

Thus V is the smooth quintic del Pezzo threefold. Consider either of the smooth twisted cubics

$$C^{\mathrm{s}} = V \cap \{A + 2C = D - F = E = 0\}, \qquad C^{\mathrm{n}} = V \cap \{B + 2E = D - G = F = 0\}.$$

Their corresponding pencils in $\mathrm{Sym}^4 \mathbb{C}^2$ are $\langle e_3, e_2 + e_0 \rangle$ and $\langle e_1, e_3 + e_0 \rangle$, respectively, where $e_i = x^{4-i}y^i$. The first pencil is GIT-stable, and the second is strictly GIT-semistable. In both cases the conic bundle on $X = \mathrm{Bl}_{C^{\mathrm{s}}} V$ or $\mathrm{Bl}_{C^{\mathrm{n}}} V$ has a double fibre supported on the strict transform of $R = \{C = D = E = F = G = 0\}$. Write H for the pullback of a hyperplane of V, write E_C for the exceptional divisor of $X \to V$, and put $L = -K_X = 2H - E_C$. The letter F occurring in the equations is a homogeneous coordinate; denote the exceptional divisor of the weighted extraction by T.

Lemma 74.1. *For either of these two members, the extraction of the double fibre has*

$$A_X(T) = 3, \qquad S_X(T) = \frac{297}{104}, \qquad A_X(T) - S_X(T) = \frac{15}{104}.$$

More precisely, if $N_m = h^0(X, mL)$, the sum of the vanishing orders of a basis adapted to ord_T *is*

$$N_m = \frac{(2m+1)(13m^2+13m+6)}{6}, \qquad \sum_{i=1}^{N_m} \mathrm{ord}_T(s_i) = \frac{m(99m^3+146m^2+93m+22)}{8}.$$

Proof. We give the filtration and the cohomology arguments, since the Hilbert series of an arbitrary initial ideal would not suffice.

Assign the weights $(0, 0, 1, 2, 2, 2, 3)$ to (A, B, C, D, E, F, G). The resulting initial threefold V_0 is defined by

$$AE - 4BD + C^2 = AF - BE = AG - CE = BG - CF = CG - 4DF + E^2 = 0.$$

These are the Pfaffians of the skew matrix obtained from

$$\begin{pmatrix} 0 & A & B & C & tD \\ -A & 0 & 2C & 8D & E \\ -B & -2C & 0 & 2E & F \\ -C & -8D & -2E & 0 & G \\ -tD & -E & -F & -G & 0 \end{pmatrix}$$

by setting $t = 0$. Its general fibre is isomorphic to V: the substitution $(A, B, C, D, E, F, G) \mapsto (A, B, tC, t^2D, t^2E, t^2F, t^3G)$ in the original equations, followed by division by their smallest powers of t, gives exactly this family.

Here are direct checks on the special fibre. On $B = 1$ it is affine space with coordinates A, C, F, since

$$E = AF, \qquad D = (A^2F + C^2)/4, \qquad G = CF.$$

On $G = 1$ it is affine space with coordinates D, E, F, since

$$C = 4DF - E^2, \qquad A = CE, \qquad B = CF.$$

The subset $B = G = 0$ has dimension at most one in projective space. The Pfaffian ideal has codimension three and is Cohen–Macaulay. Consequently there is no extra component or embedded component there, and the ideal is prime. The Hilbert series is that of a quintic del Pezzo threefold; thus these initial equations give the full initial ideal. Equivalently, the Pfaffian resolution gives a flat family over $\mathbb{A}^1$. The Jacobian criterion on $B = G = 0$ shows that V_0 has only one singular point, namely $[0 : 0 : 0 : 1 : 0 : 0 : 0]$. On the chart $D = 1$, after eliminating B and F, its equation is $AG - CE = 0$. This point is an ordinary double point.

We next identify the filtration with the required valuation. At the generic point of R, use the chart $B = 1$. On the original threefold, A, C, F are regular coordinates and

$$D = \frac{A^2F + C^2}{4 - 2AC}, \qquad E = AF + 2CD, \qquad G = CF - 2DE.$$

The weighted valuation in the two normal parameters C, F has weights $1, 2$. The displayed initial equations embed in the associated graded ring $\mathbb{C}(A)[C, F]$, so the global weight filtration on the homogeneous coordinate ring is exactly this valuation filtration. For the stable example the conic-bundle coordinates at the double fibre are $(D - F)/(A + 2C)$ and $E/(A + 2C)$. Their ideal at the generic point of R is (F, C^2). For the nonstable example use $(D - G)/(B + 2E)$ and

$F/(B+2E)$, whose ideal is again (F, C^2). Thus this valuation is the exceptional valuation of the normalized blowup of the pulled-back base-point ideal. In particular it has log discrepancy $1+2=3$. The exceptional divisor E_C does not contain R, so its valuation is zero. Hence the valuation on $H^0(X, mL)$ is the restriction of the weight filtration on $H^0(V, 2mH)$.

The centres extend to the flat families

$$\{A + 2tC = D - F = E = 0\} \subset V_t, \qquad \{B + 2t^2E = D - tG = F = 0\} \subset V_t.$$

Each fibre of either family is a smooth twisted cubic. For the first family its coordinates satisfy $A = -2tC$, $D = F$, $E = 0$ and $C^2 = 4BF$, $BG = CF$, $CG = 4F^2$. For the second family they satisfy $B = -2t^2E$, $D = tG$, $F = 0$ and, after substituting into the equations of V_t, the relations $AE + C^2 + 8t^3EG = 0$, $AG - CE + 8t^3G^2 = 0$, $CG + E^2 = 0$. Replacing A by $A + 8t^3G$ gives the standard twisted cubic relations. At $t = 0$ the centres are

$$C_0^{\mathrm{s}} = \{A = D - F = E = 0\}, \qquad C_0^{\mathrm{n}} = \{B = D = F = 0\}.$$

Both avoid the unique singular point of V_0. The total centre is therefore a relative regular immersion of codimension two, contained in the smooth locus of the ambient family. Blowing it up commutes with base change and gives a flat family X_t. More explicitly, each quotient by a power of the relative ideal is flat: its successive quotients are the symmetric powers of the relative conormal bundle on the flat centre. Thus every power of the relative ideal is t-saturated in the ambient Rees algebra.

The ideal sheaf of each centre is generated by the three displayed linear forms. Thus the blowup embeds as a graph in $V_t \times \mathbb{P}^2$. If $D_t = H - E_C$, then $L_t = H + D_t$ is the restriction of $\mathcal{O}(1,1)$, and is ample. Moreover, $-K_{X_t} = L_t$. The special fibre has only the ordinary double point inherited from V_0, and is klt. Kawamata–Viehweg vanishing gives $H^i(X_t, mL_t) = 0$ for $i > 0$ and $m \geq 0$. This verifies cohomology and base change for the spaces whose weights we compute.

We calculate the conormal bundle of the special centre, including its grading. For C_0^{s}, on $B = 1$ take the conormal parameters $a = A$ and $n = F - C^2/4$, of weights $0, 2$. On $G = 1$ take E/G and $(D - F)/G$. On their overlap, modulo the square of the ideal of the curve, they are a/C and $-4n/C^3$. Consequently

$$N^*_{C_0^{\mathrm{s}}/V_0} = \mathcal{O}_{\mathbb{P}^1}(-1)[0] \oplus \mathcal{O}_{\mathbb{P}^1}(-3)[2].$$

Here the brackets record the weights of frames at the point with $B \neq 0$. For C_0^{n}, use B, D on $A = 1$ and $F/G, D/G$ on $G = 1$. Along the curve $E = -C^2$ and $G = -C^3$; the transition functions are B/C and $-D/C^3$, giving the same graded splitting. In both cases $H|_{C_0} = \mathcal{O}_{\mathbb{P}^1}(3)$, with section weights $0, 1, 2, 3$.

For completeness, we perform the character calculation. Let z record valuation weight, and q record hyperplane degree. The graded Hilbert series of V_0 is

$$\frac{1 - q^2(2z^2 + 2z^3 + z^4) + q^3(2z^5 + 2z^4 + z^3) - q^5z^7}{(1-q)^2(1-qz)(1-qz^2)^3(1-qz^3)}.$$

Differentiating with respect to z and putting $z = 1$ gives $q(10 + 16q + 3q^2)/(1-q)^5$. It follows that

$$h^0(V_0, nH) = \frac{(n+1)(5n^2 + 10n + 6)}{6}, \quad \operatorname{wt} H^0(V_0, nH) = \frac{n(n+1)(29n^2 + 57n + 34)}{24}.$$

The successive quotients of the mth infinitesimal neighbourhood of C_0, twisted by $2mH$, are

$$\operatorname{Sym}^j N^*_{C_0/V_0} \otimes \mathcal{O}_{C_0}(6m) = \bigoplus_{i=0}^{j} \mathcal{O}_{\mathbb{P}^1}(6m - j - 2i)[2i], \qquad 0 \leq j < m.$$

All these line bundles have nonnegative degree. Their total dimensions and weights are, respectively,

$$\sum_{j=0}^{m-1}\sum_{i=0}^{j}(6m-j-2i+1)=\frac{7m(m+1)(2m+1)}{6},$$

$$\sum_{j=0}^{m-1}\sum_{i=0}^{j}(6m-j-2i+1)(3m-j/2+i)=\frac{m(m+1)(167m^2+83m+2)}{24}.$$

The restriction sequence is exact on global sections. Indeed,

$$H^1(V_0,\mathcal{I}_{C_0}^m(2mH))=H^1(X_0,mL_0)=0.$$

The higher direct images vanish for the blowup of a regular codimension-two centre. The same assertions hold in the family. Thus the initial of the filtered subspace $H^0(V,\mathcal{I}_C^m(2mH))$ is exactly $H^0(V_0,\mathcal{I}_{C_0}^m(2mH))$; there is no unproved replacement of an initial power by a power of an initial ideal. Subtracting the two jet formulas from the ambient formulas at $n=2m$ gives the finite identities in the statement. Dividing the weight by mN_m and letting m tend to infinity proves $S_X(T)=297/104$. □

Remark 74.2. The identical value in a GIT-stable and a strictly K-semistable example is useful. The single inequality $A_X(T)>S_X(T)$ does not settle the valuations above T. Their analysis requires the different and a refined series on T, including the position of $E_C\cap R$ relative to the ramification of the curve of degree $(2,1)$.

74.1 The moment at every double fibre

Proposition 74.3. *Let X be any smooth member of Family* 2.20*, and let $2R$ be a double fibre of its conic bundle. Let T be the exceptional prime divisor obtained by blowing up the pullback of the corresponding base-point ideal. Then*

$$A_X(T)=3,\qquad S_X(T)=\frac{297}{104},\qquad A_X(T)-S_X(T)=\frac{15}{104}.$$

In fact the finite dimension and weight identities of Lemma 74.1 hold for this valuation.

Proof. By Lemma 73.10, a double fibre corresponds to a common root of the binary quartic pencil. Move this root to $x=0$. Use the coordinates and weight filtration of the preceding calculation. There are precisely two possible vanishing sequences.

If the sequence is $(1,2)$, write the pencil as

$$\langle e_2+be_1+ce_0,\ e_3+de_1+ee_0\rangle.$$

Put $k=4c-b^2$. The number k is nonzero, since otherwise the first quartic is a square. By the exterior-square calculation in Lemma 73.11, the ideal of the twisted cubic in V_5 is generated by

$$l_0=A-bB+(2c-d)C+8eD+2(cd-eb)E,$$
$$l_1=D-cF-eG,\qquad l_2=E-bF-dG.$$

Their weighted initial forms define on V_0 the curve

$$Z_{b,c}=\{A-bB=D-cF=E-bF=0\}.$$

After eliminating A,D,E, its equations are

$$C^2=kBF,\qquad BG=CF,\qquad CG=kF^2.$$

It is a smooth twisted cubic, and avoids the node of V_0. Both it and the original curve have Hilbert function $3n+1$ in every degree $n \geq 0$. Since the full weighted initial ideal contains the displayed ideal and has the same Hilbert function, the two ideals coincide. This verifies the initial centre, including scheme structure.

We record a graded normalization to make the uniformity of the character calculation explicit. On V_0 set

$$A' = A - bB, \quad E' = E - bF, \quad D' = D - \frac{b}{2}E + \frac{b^2}{4}F,$$

and leave B, C, F, G unchanged. This is an invertible graded linear change preserving the five equations of V_0. It transforms $Z_{b,c}$ into $\{A' = E' = D' - (k/4)F = 0\}$. Choose $a \in \mathbb{C}^*$ with $a^2 k = 4$. The further graded automorphism

$$(A', B, C, D', E', F, G) \longmapsto (aA', B, aC, a^2 D', aE', F, aG)$$

transforms this curve into C_0^{s} of the preceding lemma. Thus the pair consisting of the weighted initial threefold and its centre is graded-isomorphic to the pair used there.

If the sequence is $(1,3)$, write $U = \langle e_1, e_3 + ae_2 + ce_0 \rangle$. The three generators are

$$B - aC + 2cE, \qquad D - cG, \qquad F - aG.$$

Their initial forms define $C_0^{\mathrm{n}} = \{B = D = F = 0\}$. This is again a smooth twisted cubic with the same Hilbert function as the original, so the same argument proves equality with the full initial ideal.

In either case the centre in the flat weighted family stays inside the smooth locus of the ambient family. Its infinitesimal neighbourhoods are flat, because the centre is a relative regular immersion and the graded quotients are its symmetric conormal powers. The powers of the relative ideal are therefore t-saturated. The relative blowup has the same klt Fano special fibre, up to graded isomorphism, as one of the two examples already considered. Kawamata–Viehweg vanishing and cohomology and base change identify its global section module with the saturated Rees submodule of $H^0(V_t, 2mH)$. In particular no additional initial sections occur.

Finally the filtration is the actual exceptional valuation. At the generic point of R, the weight-zero generator l_0 is a unit after trivializing H. The ideal $(l_1/l_0, l_2/l_0)$ has initial ideal (F, C^2) in the two regular normal parameters. Thus its exceptional valuation has weights $(1,2)$, exactly the valuation used above. The original exceptional divisor meets R in a point and has valuation zero, so passing from $2mH$ to $mL = 2mH - mE_C$ introduces no grading shift. The character and the conormal jet calculation of Lemma 74.1 now apply without any change. □

75 The complete refinement at a double fibre

We now determine the refined series left open above. Let $2R$ be a double fibre, let T be its weighted extraction, and identify its reduced exceptional surface with $F = \mathbb{P}^1 \times \mathbb{P}^1$. The first factor is R, and the second parametrizes directions in the base. The different is $\Gamma/2$, with Γ a smooth $(2,1)$-curve. Let $q \in \Gamma$ be the point above $R \cap E$, and let ℓ be the fibre of the second projection through q. Thus $\ell \in |\mathcal{O}_F(0,1)|$ and $\Gamma \cdot \ell = 2$. Write $\mathcal{I}_q$ for the ideal sheaf of q.

The distinction between the two pencil types is now simple. For vanishing sequence $(1,2)$, the curve Γ meets ℓ at two distinct points q, q'. For sequence $(1,3)$ it is tangent to ℓ at q. This follows from Lemma 73.11.

Lemma 75.1. *Let $W_{m,k}$ be the leading sections of $H^0(X, -mK_X)$ of order k along T. Choose coordinates $[a:b]$, $[u:v]$ on F such that $q = (a = v = 0)$ and $\ell = (v = 0)$, and let $Q(a,b,u,v)$*

define Γ. *Use a formal symbol* w *satisfying* $w^2 = Q$ *to record odd weights. For* $k = 2j + \epsilon$, *where* $\epsilon \in \{0, 1\}$, *put* $d = \max\{j - 2m, 0\}$. *Then the actual refined series is*

$$W_{m,2j+\epsilon} = w^{\epsilon} Q^{d} H^{0}\left(F, \mathcal{O}_{F}(m - \epsilon - 2d, j - d) \otimes \mathcal{I}_{q}^{\max\{j-m-d,0\}}\right).$$

A negative first degree gives the zero space. The symbol w *has divisor* $\Gamma/2$*; equivalently one can read the formula on the index-two cover of the extraction. Its contribution divided by* m *vanishes in all asymptotic moments.*

Proof. The initial pairs and their graded normalizations were established in Proposition 74.3. In the $(1, 2)$ model the centre is $(A, D - F, E)$ on V_0. Divide a quadratic in its ideal by A at the generic point of R. The substitutions

$$F = Bv, \qquad E = Av, \qquad D = Au + Bv, \qquad C^2 = 4ABu + (4B^2 - A^2)v$$

give the leading sections. In the $(1, 3)$ model the centre is (B, D, F); divide by B and use $D = Bu$, $F = Bv$, $E = Av$, $C^2 = 4B^2u - A^2v$. After choosing a to vanish at the marked point, the six spaces in anticanonical degree one are

k	$W_{1,k}$	$\dim W_{1,k}$
0	$H^0(\mathcal{O}_F(1, 0))$	2
1	$w\mathbb{C}$	1
2	$H^0(\mathcal{O}_F(1, 1))$	4
3	$wH^0(\mathcal{O}_F(0, 1))$	2
4	$H^0(\mathcal{O}_F(1, 2) \otimes \mathcal{I}_q)$	5
5	$wvH^0(\mathcal{O}_F(0, 1))$	2.

For example, in the first model the weight-four space is generated by the leading terms of $D(D-F)$, $F(D - F)$, $E(D - F)$, E^2, and EF, divided by A. These span $Au^2, Auv, Av^2, Buv, Bv^2$. The sixteen displayed sections exhaust $H^0(X, -K_X)$.

Multiplication gives subspaces of the actual associated graded ring. We first construct all the spaces in the statement using these degree-one sections. Suppose $j \le 2m$. A monomial $a^i b^{m-\epsilon-i} u^{j-h} v^h$ belongs to the displayed ideal-power space exactly when $i + h \ge j - m$. Distribute the j symbols u, v among m factors, with at most two in each factor. There are i factors carrying a; these can carry two copies of u. Every other factor can carry at most one copy of u. The inequalities $j \le 2m$ and $j - h \le m + i$ are precisely the capacity conditions. For $\epsilon = 1$, one factor carries w in place of an a or b; its possibilities are the three odd rows of the table. The same capacity conditions apply. This constructs every required monomial.

If $j > 2m$, put $r = 2d + \epsilon$ and $a_0 = m - r$. When $a_0 < 0$ the claimed space is zero. Otherwise take r factors from the weight-five row and a_0 factors from the weight-four row. Their products span

$$w^{\epsilon} Q^{d} v^{r} H^{0}\left(F, \mathcal{O}_{F}(a_0, 2m - r) \otimes \mathcal{I}_{q}^{a_0}\right).$$

Indeed multiplication by all degree-r binary forms in u, v fills the indicated monomial space. A monomial of first degree $a_0 = m - r$ vanishing to order m at q is necessarily divisible by v^r. Thus this is exactly the space asserted in the lemma.

It remains to prove that these subspaces exhaust the refinement. This is a dimension argument on the actual section ring, so it does not assume generation in anticanonical degree one. For $0 \le j \le 2m$ the constructed dimension is

$$(m - \epsilon + 1)(j + 1) - \frac{n(n + 1)}{2}, \qquad n = \max\{j - m, 0\}.$$

For $j > 2m$ it is $(a_0+1)(m+1+a_0/2)$. Summing the first formula over j and ϵ gives $(m+1)(2m+1)^2 - m(m+1)(m+2)/3$. The remaining nonzero terms have $a_0 = 0, \dots, m-2$ and sum to $m(m-1)(4m+1)/6$. Their total is

$$\frac{(2m+1)(13m^2+13m+6)}{6} = h^0(X, -mK_X),$$

by Riemann–Roch and vanishing. The constructed subspaces already have the full dimension. No extra graded sections can occur. □

Example 75.2. For $m = 2$, the dimensions in weights $0, \dots, 10$ are $3, 2, 6, 4, 9, 6, 11, 7, 12, 7, 3$. They sum to 70. The weight-ten space is $Qv^2H^0(\mathcal{O}_F(0,2))$. In particular the tail is not a complete series of bidegree $(2,5)$. Its fixed curve and point condition are essential.

Corollary 75.3. *The pseudoeffective threshold of T is five, and its complete volume profile is*

$$\operatorname{vol}(f^*L - tT) = \begin{cases} 26 - 3t^2/2 & 0 \le t \le 2, \\ 26 - 3t^2/2 + (t-2)^3/4 & 2 \le t \le 4, \\ (5-t)^2(8-t) & 4 \le t \le 5, \\ 0 & t \ge 5. \end{cases}$$

Proof. The normalized leading dimensions $h(t) = \lim m^{-2} \dim W_{m,\lfloor mt \rfloor}$ are $t/2$, $t/2-(t-2)^2/8$, and $(5-t)(7-t)/2$ on the three intervals. Lemma 75.1 gives these limits directly and shows that no weight exceeds $5m$. Summing the leading sections of order at least mt gives $\operatorname{vol}(f^*L - tT) = 6\int_t^5 h(s)\,ds$. Integration proves the formula, and a further integration gives $S_X(T) = 297/104$, agreeing with the independent character computation. □

75.1 The integration regions

Put $\rho = j/m$. After removing the fixed multiple of Γ, the asymptotic series consists of polynomials of bidegree (A, B) vanishing to order n at q, where

ρ	A	B	n	fixed part
$[0,1]$	1	ρ	0	0
$[1,2]$	1	ρ	$\rho-1$	0
$[2,5/2]$	$5-2\rho$	2	1	$(\rho-2)\Gamma$.

Let x, y record orders at $a = 0, v = 0$, respectively. The monomials fill the polygon

$$\mathcal{P}_\rho = \{(x,y) : 0 \le x \le A,\ 0 \le y \le B,\ x + y \ge n\}.$$

Both parities have this limiting polygon. Thus the normalization of a monomial moment is $6/13$, since $\int_0^{5/2} \operatorname{area}(\mathcal{P}_\rho)d\rho = 13/6$. Equivalently, on $\mathrm{Bl}_q F$ these are complete divisorial series. In the last interval the additional fixed curve is $(2\rho-4)\widetilde{\ell}$; the moving part is $(5-2\rho)s + (6-2\rho)f - (5-2\rho)e$, where s, f are the pullbacks of the two rulings and e is the exceptional curve.

Lemma 75.4. *For the series W of Lemma 75.1, the normalized averages of x, y, A, B and of the fixed coefficient of Γ are, respectively,*

$$\frac{1}{2},\quad \frac{83}{104},\quad \frac{99}{104},\quad \frac{73}{52},\quad \frac{5}{208}.$$

For the horizontal ruling through q, one has $S(W;\ell) = 83/104$. For a different horizontal ruling ℓ_0, one has $S(W;\ell_0) = 63/104$. The further point moments on these rulings are

flag and point	*point moment*
$\ell, p \ne q$	$47/104 + (5/208)(\Gamma\cdot\ell)_p$
ℓ, q	$1/2 + (5/208)(\Gamma\cdot\ell)_q$
ℓ_0, p	$47/104 + (5/104)\mathbf{1}_{p\in(a=0)} + (5/208)(\Gamma\cdot\ell_0)_p$.

Proof. All these statements are integrals over the three displayed polygons. We give the bounds explicitly so that fixed-point corrections can be checked. For each row put $b_0 = \max\{0, n - A\}$. Integrate first over $b_0 \le y \le n$, with $n - y \le x \le A$, and then over $n \le y \le B$, with $0 \le x \le A$. Integrating x, y, A, B and $(\rho - 2)_+$ over these regions, with factor $6/13$, gives the five averages. The unnormalized values are $13/12$, $83/48$, $33/16$, $73/24$, and $5/96$.

For the flag ℓ, fix y. The allowed exponents in the other coordinate form the interval $[\max\{0, n - y\}, A]$. A general point of ℓ therefore contributes one-half the square of the length of this interval. Its integral, before normalization, is $47/48$. At q, integrate x instead, obtaining $13/12$. The fixed curve Γ adds its average coefficient times its local intersection.

For $\ell_0 \neq \ell$, subtract a multiple $z\ell_0$ from the series. Restriction to ℓ_0 has fixed order $h = \max\{0, n - B + z\}$ at $a = 0$ and moving degree $A - h$. Here $0 \le z \le B - n + \min\{n, A\}$. The point moment away from $a = 0$ is the integral of $(A - h)^2/2$. The correction at $a = 0$ is the integral of $h(A - h)$. Their unnormalized values are $47/48$ and $5/48$. The curve moment is obtained by integrating $(B - \max\{0, n - x\})^2/2$ over $0 \le x \le A$ and then over ρ; it equals $21/16$, giving $63/104$ after normalization. These computations also follow directly by counting the monomials in Lemma 75.1. □

75.2 The marked point

Lemma 75.5. *Suppose that Γ is transverse to ℓ at q. For the refinement W on $(F, \Gamma/2)$ one has*

$$\delta_q(F, \Gamma/2; W) \ge \frac{312}{275} > 1.$$

Proof. Blow up q and denote its exceptional curve by J. Its log discrepancy for $(F, \Gamma/2)$ is $3/2$. The polygon moment of $x + y$, with the fixed-curve contribution included, gives

$$S(W; J) = \frac{1}{2} + \frac{83}{104} + \frac{5}{208} = \frac{275}{208}.$$

The different on J is one-half of the point determined by the tangent of Γ. This point differs from both coordinate directions: transversality excludes the direction of ℓ, and $\Gamma \to R$ is an isomorphism, excluding the other ruling.

To compute the second refinement, slice $\mathcal{P}_\rho$ by $x + y = z$. The interval of possible y is $[\max\{0, z - A\}, \min\{B, z\}]$, for $n \le z \le A + B$. The integral of one-half the square of its length is $37/48$, or $37/104$ after normalization. The two coordinate directions have moments $1/2$ and $83/104$. At the tangent direction of Γ the moment is $37/104 + 5/208 = 79/208$. Thus the Abban–Zhuang bound is

$$\min\left\{\frac{312}{275},\ 2,\ \frac{104}{83},\ \frac{104}{79},\ \frac{104}{37}\right\} = \frac{312}{275}.$$

All restricted spaces used here are complete binary spaces with the displayed fixed endpoint orders, by the exact monomial description. □

75.3 Ramification points away from the marked point

Lemma 75.6. *Assume that q is not a ramification point of $\Gamma \to \mathbb{P}^1$ in the direction factor. If $p \in \Gamma$ is a ramification point of this map, then*

$$\delta_p(F, \Gamma/2; W) \ge \frac{104}{89} > 1.$$

Proof. The points p, q lie on different fibres of both rulings. Choose coordinates putting them at opposite corners. If x, y are the exponents at q, the exponents at p are $x' = A - x$,

$y' = B - y$. Here $y' = 0$ denotes the tangent ruling through p. The curve Γ has local equation $y' = c(x')^2 +$ higher terms, with $c \neq 0$. Use the weighted blowup of weights $(1, 2)$ in these two local coordinates, and denote its exceptional curve by J. Then $A_{F,\Gamma/2}(J) = 3 - 2/2 = 2$. The moment is

$$S(W; J) = \left(\frac{99}{104} - \frac{1}{2}\right) + 2\left(\frac{73}{52} - \frac{83}{104}\right) + 2\frac{5}{208} = \frac{89}{52}.$$

The coarse curve J is $\mathbb{P}^1$. Its different consists of two distinct points of coefficient $1/2$: the quotient point and the point cut out by the initial equation of Γ. The other coordinate direction has coefficient zero. For the second refinement slice by $z = x' + 2y'$. With $K = A + B - n$, the interval of allowed y' is

$$\left[\max\left\{0, \frac{z - A}{2}, z - K\right\}, \min\left\{B, \frac{z}{2}\right\}\right],$$

when its length is nonnegative. The integrals of one-half the square of its length over the three ρ-intervals are $17/192$, $19/64$, and $5/96$. Their sum is $7/16$. The general point moment on J is consequently $21/104$. The point belonging to Γ has moment $21/104 + 5/208 = 47/208$. The quotient point has moment $(99/104 - 1/2)/2 = 47/208$, and the other coordinate point has moment $73/52 - 83/104 = 63/104$. These give

$$\delta_p(F, \Gamma/2; W) \geq \min\left\{\frac{104}{89}, \frac{104}{47}, \frac{104}{63}, \frac{104}{21}\right\} = \frac{104}{89}.$$

The division by two at the quotient point is necessary: it is the order on the coarse weighted exceptional curve. □

Theorem 75.7. *Suppose that a double fibre of a smooth member of Family 2.20 corresponds to a common root with vanishing sequence* $(1, 2)$*. Then*

$$\delta_p(X) \geq \frac{104}{99} > 1$$

at every point of its support. In particular this holds on every double fibre of every smooth member with GIT-stable pencil.

Proof. We first prove $\delta(F, \Gamma/2; W) \geq 104/99$. The marked point is treated by Lemma 75.5. Every ramification point of the direction projection is treated by Lemma 75.6. For another point, use its horizontal ruling. On ℓ, the only point of Γ other than q is q', and its refined point moment is $99/208$, with discrepancy $1/2$. All other point moments there are $47/104$, with discrepancy one. The first moment is $83/104$.

On a different ruling, a transverse intersection with Γ has point moment $99/208$, again with discrepancy $1/2$. It cannot also have $a = 0$, since Γ meets that vertical ruling only at q. Points outside Γ have moment at most $1/2$ and discrepancy one. The first moment is $63/104$. Thus every remaining flag gives a bound at least $104/99$.

Apply Abban–Zhuang refinement on the threefold extraction. Its first ratio is

$$\frac{A_X(T)}{S_X(T)} = \frac{3}{297/104} = \frac{104}{99}.$$

The pair on its exceptional surface is $(F, \Gamma/2)$, whose refined threshold has just been bounded below by the same number. Inversion of adjunction gives the bound in a neighbourhood of the whole exceptional surface, and hence along R on X. This includes all valuations whose centres contain a point of R. One can equivalently apply the finite-basis proof: choose coefficients tending to $104/99$ from below, use the exact refined spaces above, and apply effective adjunction on the extraction before taking limits. □

Remark 75.8. The proof uses an ordinary surface blowup at the marked point and a weighted surface blowup at each other ramification point. It computes no Zariski decomposition on an extracted threefold. The complete global refinement is proved by multiplication and a full dimension count; its fixed curve and point condition are retained throughout. The transversality assumption is essential. In the $(1,3)$ case the tangent direction of Γ is a coordinate endpoint on the blowup of q, so the two contributions cannot be placed at distinct points. The strictly K-semistable examples are therefore not covered by Theorem 75.7.

76 A discriminant test for the parameter orbit

The GIT distinction can also be read directly from two plane curves. Let $Q \subset \mathbb{P}^2$ be the smooth branch conic of $E \to \mathbb{P}^2$, and let Δ be the cubic discriminant of the conic bundle. For distinct curves, write $I_b(\Delta, Q)$ for their local intersection multiplicity at b.

Proposition 76.1. *For a smooth member of Family 2.20, the following assertions hold.*

1. *If Q is a component of Δ, then $\Delta = Q \cup M$, with M a line transverse to Q. The threefold is the K-polystable member $X_{U_{0,0}}$.*
2. *Suppose Q is not a component of Δ. If some node b of Δ satisfies $I_b(\Delta, Q) \geq 4$, then the pencil is nonclosed GIT-semistable and the threefold is strictly K-semistable.*
3. *In all other cases the pencil is GIT-stable. At each node one has $I_b(\Delta, Q) = 2$, and every point on the corresponding double fibre satisfies $\delta_p(X) \geq 104/99$.*

The estimates below complete the K-stability assertion in the third case. Corollary 82.1 gives the resulting intrinsic classification.

Proof. Use the common model of Lemma 73.1, with base coordinates $[A : B : C]$. Its branch conic is $Q = (B^2 = 4AC)$. On $C = 1$, put $A = r^2$, $B = 2r$. The discriminant formula gives

$$\Delta|_Q = (f_0 - f_1 r + f_2 r^2 - f_3 r^3)^2.$$

Globally this means that the pullback of the discriminant equation to $[s : t] \mapsto [s^2 : 2st : t^2]$ is, up to a nonzero scalar, $(f_0 t^3 - f_1 s t^2 + f_2 s^2 t - f_3 s^3)^2$.

At a node move the point to $[0 : 0 : 1]$. The double-fibre conditions are $f_0 = 0$, $h_2 = -f_1^2/4$. If $f_1 \neq 0$, the intersection multiplicity with Q is two. If $f_1 = 0$ and the cubic form above is nonzero, it is at least four. Lemma 73.8 identifies the latter case with marked ramification, and Lemma 73.11 identifies it with vanishing sequence $(1,3)$. These are exactly the nonstable smooth pencils. The already constructed special test configurations prove strict K-semistability unless the pencil is $U_{0,0}$.

If $Q \subset \Delta$, all four f_i vanish, and the equation of X is $q_0 + z^2 h = 0$. Hence $\Delta = Q \cup (h = 0)$. Smoothness forces the line $(h = 0)$ to be nonzero and transverse to Q: otherwise its intersection with Q would give a nonnodal discriminant singularity, contrary to Lemma 73.3. At each of its two intersection points the marked point is ramified. The pencil therefore has two common roots of vanishing sequence $(1,3)$ and is equivalent to $U_{0,0}$ by Lemma 70.2. Conversely, that pencil has two such roots. If the displayed cubic form were nonzero, it would have a double zero at both corresponding points of Q, which is impossible for a cubic. Thus Q is a component in this case, proving the first two alternatives exactly.

In the remaining case no common root has sequence $(1,3)$, so the pencil is GIT-stable. Its double fibres all have type $(1,2)$, and Theorem 75.7 applies. □

Example 76.2. For $f_0 = f_1 = f_2 = f_3 = 0$ and $h = B$, the discriminant is $(4AC - B^2)B = 0$. The line $B = 0$ meets the branch conic at two distinct points, giving the closed nonstable orbit. For a nonclosed orbit, the discriminant has a node where one of its branches is tangent to Q. At a node of a GIT-stable member, Q is transverse to both branches. Smooth tangencies between Δ and Q do not enter this criterion; the test is imposed only at nodes.

77 Blowing up a component of a reduced fibre

Let $A + D$ be a reduced singular fibre, with node $p_0 = A \cap D$. Put $r = A \cap E$, and let $f : Y \to X$ blow up A, with exceptional surface F. In this section F denotes this ordinary extraction. We first compute its complete volume for every reduced fibre. We then prove a strict local bound when $\pi(A)$ is an ordinary line and $p_0 \notin E$. Here an ordinary line on V_5 means a line with normal bundle $\mathcal{O} \oplus \mathcal{O}$.

77.1 Changing the order of two blowups

Lemma 77.1. *Let C_1, C_2 be smooth curves in a smooth threefold, meeting transversally at precisely one point. Blowing up C_1 and then the strict transform of C_2, or taking the two blowups in the opposite order, gives smooth threefolds related by one ordinary flop. Its curve on the first model is the strict transform of the exceptional fibre of the first blowup over $C_1 \cap C_2$.*

Proof. The assertion is local at the intersection. Choose coordinates with ideals (x, z) and (y, z) for the two curves. The two iterated blowups are obtained by subdividing the positive octant first by $(1, 0, 1)$ and then by $(0, 1, 1)$, or in the opposite order. The differing cones form the two triangulations of the relation

$$(1, 0, 0) + (0, 1, 1) = (0, 1, 0) + (1, 0, 1).$$

Both subdivisions are smooth. Their contracted curve has normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, and the common contraction is the ordinary double point $uv = zw$. The models agree outside this neighbourhood. This is the ordinary flop. □

We use the classical projection of V_5 from a line: $\mathrm{Bl}_\ell V_5 \to Q$ is the blowup of a smooth quadric threefold along a smooth twisted cubic Γ. This holds for both normal-bundle types of ℓ; see [54, proof of Theorem 3.6]. If F_ℓ is the first exceptional divisor and Π the second, the divisor relations are $M = H - F_\ell$ and $\Pi = H - 2F_\ell$, where M comes from Q. They follow respectively from the projection and the canonical class formula. In particular the quadric threefold is smooth also when ℓ is special.

Lemma 77.2. *There is a sequence of two disjoint ordinary flops $Y \dashrightarrow Y^+$, followed by a presentation*

$$Y^+ = \mathrm{Bl}_{\Gamma'} Z, \qquad Z = \mathrm{Bl}_{C_2} Q,$$

where $C_2 \subset Q$ is a smooth plane conic, $\Gamma \subset Q$ is a smooth twisted cubic, $C_2 \cap \Gamma$ is one reduced point, and Γ' is its strict transform in Z. The two curves flopped on Y are the strict transforms of D and of the π-exceptional fibre through r. Writing M for the pullback of $\mathcal{O}_Q(1)$ and using the same letters for strict transforms, one has

$$\Pi = H - 2F, \qquad M = H - F, \qquad H = 2M - \Pi, \qquad F = M - \Pi.$$

The divisor E on Y^+ is the pullback of the exceptional divisor of $Z \to Q$.

Proof. The line $\ell = \pi(A)$ meets C_0 transversally in one point: $H \cdot A = E \cdot A = 1$. Moreover ℓ is not contained in the span of C_0, since $V_5 \cap \langle C_0 \rangle = C_0$ scheme-theoretically. Apply Lemma 77.1 to interchange the blowups of C_0 and ℓ.

Let $\widetilde{C}_0 \subset \mathrm{Bl}_\ell V_5$ be the strict transform. Projection from ℓ restricts to the complete series $|\mathcal{O}_{C_0}(1)(-r)| = |\mathcal{O}_{\mathbb{P}^1}(2)|$. Indeed hyperplanes through ℓ restrict on $\langle C_0 \rangle$ to all hyperplanes through r, because ℓ meets this span only at r. Thus its image C_2 is a smooth plane conic and $\widetilde{C}_0 \to C_2$ is an isomorphism. Also $\Pi \cdot \widetilde{C}_0 = 3 - 2 = 1$. It follows that C_2 meets Γ transversally at one point: the pullback of the ideal of Γ cuts out this reduced point on $\widetilde{C}_0$. Interchanging these two blowups gives the second flop and the asserted model.

The fibre of $\Pi \to \Gamma$ through that point maps in V_5 to a line meeting both ℓ and C_0. Its transform on X is contracted by $|H - E|$, so it is D. The first flopping curve is the exceptional fibre through r. They are disjoint on Y. If $p_0 \neq r$, their intersection points with A are different. They cannot meet elsewhere: that would make the two image lines meet at both $\pi(p_0)$ and the same point of C_0. If $p_0 = r$, their tangent directions modulo $T_r A$ are different: η is constant on D, whereas its restriction to a π-exceptional fibre is an isomorphism onto a line. Blowing up A therefore separates the two curves in this case also.

The projection relations give the displayed identities. Neither of the second blowup centres is contained in the first exceptional divisor, so E remains the pullback of the exceptional divisor over C_2. This proves the last assertion. □

77.2 The complete threefold volume

Proposition 77.3. *For every component A of every reduced singular conic fibre,*

$$A_X(F) = 2, \qquad S_X(F) = \frac{43}{26}, \qquad A_X(F) - S_X(F) = \frac{9}{26}.$$

Its pseudoeffective threshold is three. More precisely,

$$\mathrm{vol}(f^*L - tF) = \begin{cases} 26 - 3t^2 - t^3 & 0 \le t \le 1, \\ 26 - 3t^2 - t^3 + 2(t-1)^3 & 1 \le t \le 2, \\ 2(4-t)^3 - 6(4-t) + 4 & 2 \le t \le 3, \\ 0 & t \ge 3. \end{cases}$$

Proof. A general quintic del Pezzo section $S \in |B|$ through the fibre is smooth. Its normal sequence along A is $0 \to \mathcal{O}(-1) \to N_{A/X} \to \mathcal{O} \to 0$, which splits. Consequently $F \cong \mathbb{F}_1$, $F^3 = 1$ and $(f^*L)F^2 = -1$. Since L embeds X and A as an anticanonical line, $f^*L - F$ is globally generated. Thus $f^*L - tF$ is nef for $0 \le t \le 1$, giving the first cubic.

Each of the two flopping curves has intersection $1 - t$ with this divisor. An ordinary flop changes its cube by minus the cube of this intersection. This can also be checked on the common blowup, whose exceptional surface J is $\mathbb{P}^1 \times \mathbb{P}^1$ with normal bundle $\mathcal{O}(-1,-1)$. If d is the intersection with the flopping curve, then $J^3 = 2$, $(\sigma^*D)^2 J = 0$, $\sigma^*D\, J^2 = -d$, and the new pullback is $\sigma^*D + dJ$. Its cube is therefore $D^3 - 3d^3 + 2d^3 = D^3 - d^3$. Hence the second cubic is as displayed. It computes the volume: on Y^+ the transforms of $f^*L - F = -K_Y$ and $f^*L - 2F = 2M - E$ are nef, so their convex combinations are nef for $1 \le t \le 2$. Here $-K_Y$ remains nef under the crepant flops, and $M - E$ is the pencil of hyperplanes through the plane conic C_2.

For $2 \le t \le 3$, the decomposition on Y^+ is

$$f^*L - tF = ((4-t)M - E) + (t-2)\Pi.$$

The first term is nef and pulled back from Z. The second term is fixed: intersection with a general fibre of $\Pi \to \Gamma'$ forces at least this multiplicity in every effective representative. The

intersection numbers on Z are $M^3 = 2$, $M^2E = 0$, $ME^2 = -2$, $E^3 = -4$, proving the last cubic. For $t > 3$, pushing forward to Z would make $(4-t)M - E$ pseudoeffective. This is impossible: on a general quadric surface in the pencil $|M - E|$, its restriction is $(3-t)M$, which has negative intersection with a ruling curve. Such ruling curves cover Z. Thus the threshold is three. Finally, integration gives $\int_0^3 \mathrm{vol}(f^*L - tF)dt = 43$. Dividing by $L^3 = 26$ proves the expected order. □

Example 77.4. The volumes at the two walls are 22 and 8; the volume vanishes at $t = 3$. The first derivatives agree at both walls. The two terms $(t-1)^3$ record two separate flops. Omitting either one changes the expected order and does not describe this extraction.

77.3 The complete refinement

Write s, f for the negative section and a ruling fibre of $F = \mathbb{F}_1$. Then $L|_F = f$ and $-F|_F = s+f$. The two flopping curves meet F in points q_E, q_D over r, p_0, respectively. They are distinct, as proved above. They lie on different ruling fibres exactly when $p_0 \notin E$. If ℓ is ordinary, they both avoid s. For q_E, this follows from the elementary transform $N_{A/X} \subset N_{\ell/V_5} = \mathcal{O}^{\oplus 2}$: its one-dimensional kernel on the fibre at r is not the trivial summand of $N_{A/X} = \mathcal{O} \oplus \mathcal{O}(-1)$. For q_D, use the subbundle $N_{A/S} = \mathcal{O}(-1) \subset N_{A/X}$; its projectivization is disjoint from the negative section, and contains the direction of D at p_0. This also shows that q_D avoids s when ℓ is special. In the special case the elementary transform of $\mathcal{O}(1) \oplus \mathcal{O}(-1)$ instead gives $q_E \in s$.

Put $\rho : F^+ \to F$ for the blowup of q_E, q_D, with exceptional curves e_E, e_D. The surface F^+ is the transform of F under the two flops. Let R be the pushforward of $\Pi|_{F^+}$. Then

$$R \sim 2s + 3f, \qquad \Pi|_{F^+} = \rho^*R - 2e_E - e_D.$$

The coefficients follow also by intersecting $\Pi = H - 2F$ with the two flopping curves: their intersections are $-2, -1$. When ℓ is ordinary and $p_0 \notin E$, the divisor R contains no ruling fibre. To see this directly, use the first order of blowups in Lemma 77.2. On the exceptional $\mathbb{F}_0$ over an ordinary line, the intersection with Π is a smooth $(1,2)$-curve. Indeed M restricts to $\mathcal{O}(1,1)$, identifying this exceptional surface with the smooth quadric hyperplane section containing Γ; the restriction of Π is that twisted cubic. The elementary transform is taken at a point outside this curve when $p_0 \notin E$. Its image is R. In particular $(R \cdot f)_q \leq 2$ for every point of every ruling fibre. The image can have a node or cusp at q_E; neither possibility is excluded from the ordinary-line estimates below.

Lemma 77.5. *For every reduced fibre component, the actual graded piece $W_{m,k}$ is given by the following complete series, with the indicated fixed part. In the table $t = k/m$ is the normalized order along F, and a, b, c_E, c_D, g mean*

$$gR + \left|\mathcal{O}_F(as + bf) \otimes \mathcal{I}_{q_E}^{c_E}\mathcal{I}_{q_D}^{c_D}\right|.$$

Multiply each entry by m to read the integral series. There is no rounding in these formulas. The spaces vanish for $k > 3m$.

t	a	b	c_E	c_D	g
$[0,1]$	t	$1+t$	0	0	0
$[1,2]$	t	$1+t$	$t-1$	$t-1$	0
$[2,3]$	$4-t$	$7-2t$	$3-t$	1	$t-2.$

In particular all the point conditions and the tail fixed divisor are part of the refinement.

Proof. For $t \le 1$ restrict $f^*L - tF$ to F. For $1 \le t \le 2$ restrict its nef transform to F^+; this gives $\rho^*(ts + (1+t)f) - (t-1)(e_E + e_D)$. For $t \ge 2$, subtract $(t-2)\Pi$ first. Its restriction has class

$$\rho^*((4-t)s + (7-2t)f) - (3-t)e_E - e_D.$$

These restrictions give inclusions of the actual graded pieces in the complete spaces of the table. We prove equality by counting their dimensions. The surface F^+ is a weak del Pezzo surface of degree six: it is the blowup of $\mathbb{F}_1$ at two distinct points, at most one on its negative section. The two points may lie on one ruling fibre. Checking these configurations gives only (-1)- and (-2)-curves and a nef anticanonical class. Each moving divisor in the table is nef, by restriction of the nef threefold divisors above (and directly on F in the first interval). Riemann–Roch and Kawamata–Viehweg vanishing therefore give the complete-space dimensions

$$\begin{array}{c|c} 0 \le k \le m & k^2/2 + km + 3k/2 + m + 1 \\ m \le k \le 2m & -k^2/2 + 3km + k/2 - m^2 + 2m + 1 \\ 2m \le k \le 3m & k^2 - 8km - 2k + 15m^2 + 7m + 1. \end{array}$$

The formulas agree at the endpoints. To sum without double counting, use the disjoint ranges $[0,m]$, $[m+1,2m]$, and $[2m+1,3m]$. If $d_{m,k}$ denotes the displayed dimension, their exact contributions are

k	$\sum d_{m,k}$	$\sum k d_{m,k}$
$0,\dots,m$	$\frac{(m+1)(m+2)(4m+3)}{6}$	$\frac{m(m+1)(m+2)(11m+9)}{24}$
$m+1,\dots,2m$	$\frac{7m(m+1)(2m+1)}{6}$	$\frac{m(m+1)(87m^2+67m+14)}{24}$
$2m+1,\dots,3m$	$\frac{m(8m^2+3m+1)}{6}$	$\frac{m(37m^3+22m^2+11m+2)}{12}$.

These follow from the sums of $1, k, k^2, k^3$. Thus, for every $m \ge 0$,

$$\sum_k d_{m,k} = \frac{(2m+1)(13m^2+13m+6)}{6} = h^0(X, mL),$$

$$\sum_k k d_{m,k} = \frac{m(43m^3+60m^2+38m+9)}{6}.$$

This is also the total dimension of the actual associated graded space. All the inclusions must consequently be equalities. The argument proves the complete refinement in every degree, including the two exceptional point configurations. It does not assume generation in degree one or surjectivity of an unverified restriction map. □

Example 77.6. In anticanonical degree one, the four graded pieces have dimensions $2, 5, 7, 2$. The full finite weight sum in degree m is $m(43m^3 + 60m^2 + 38m + 9)/6$. Its leading term recovers $S_X(F) = 43/26$, independently checking the integral of the volume.

77.4 The surface bound for an ordinary component

Assume now that ℓ is ordinary and $p_0 \notin E$. Thus the two points avoid s and lie on different ruling fibres. The divisor R contains no ruling fibre, as shown above.

Lemma 77.7. *For any ruling fibre $f_b \subset F$, the first flag moment is*

$$S(W^F; f_b) = \begin{cases} 21/26 & q_E \in f_b, \\ 89/104 & q_D \in f_b, \\ 37/52 & q_E, q_D \notin f_b. \end{cases}$$

Every further point moment is at most $99/104$. *Hence* $\delta(F; W^F) \ge 104/99$.

Proof. Blow down s to $q_0 \in \mathbb{P}^2$, and choose homogeneous coordinates $[x : y : z]$ with q_0, q_E, q_D the coordinate points. The monomials in the residual series of the preceding table are $x^i y^j z^{b-i-j}$, with

$$0 \le i \le a, \qquad 0 \le j \le b - i, \qquad j \le b - c_E, \qquad i + j \ge c_D.$$

Integrals over this polygon, followed by integration in t, have normalization $3/13$. The normalized integrals of its area, of i, of $a - i$, and of g are respectively

$$1, \qquad \frac{73}{104}, \qquad \frac{79}{104}, \qquad \frac{5}{52}.$$

Here is an explicit list of integration bounds. For $t \in [0, 1]$ use $0 \le i \le t$, $0 \le j \le 1 + t - i$. For $t \in [1, 2]$ use the two regions

$$\begin{array}{l|l} 0 \le i \le t - 1 & t - 1 - i \le j \le 2 \\ t - 1 \le i \le t & 0 \le j \le 1 + t - i. \end{array}$$

For $t \in [2, 3]$ use

$$\begin{array}{l|l} 0 \le i \le 3 - t & 1 - i \le j \le 4 - t \\ 3 - t \le i \le 1 & 1 - i \le j \le 7 - 2t - i \\ 1 \le i \le 4 - t & 0 \le j \le 7 - 2t - i. \end{array}$$

These bounds give all four displayed integrals directly.

For the fibres through q_E, q_D, integrate the exponents $b-i-j$ and j, giving $21/26$ and $89/104$. For another fibre, the binary coefficient of x^i has residual degree $U(i) = b-i-(c_E-i)_+-(c_D-i)_+$ in y, z. Its contribution to the curve moment is $U(i)^2/2$; integration gives $37/52$.

For the point refinement on any of these fibres, the permitted exponents i form an interval $[l, h] \subset [0, a]$. The two endpoint moments of its complete binary space are $\int_l^h i\, di$ and $\int_l^h (a-i)\, di$. A different point contributes $(h - l)^2/2$, which is no larger than either endpoint moment. Integration therefore bounds every point moment of the residual series by $79/104$. This argument includes the prescribed orders at q_E, q_D; they are the fixed endpoint orders in these intervals. The fixed divisor gR contributes at most $2 \cdot 5/52$, since R contains no ruling fibre and $R \cdot f_b = 2$. Thus the total is at most $99/104$. The surface flag inequality now gives the claimed threshold, since the first moment is at most $89/104$ and both surface and curve are smooth with zero boundary. □

Theorem 77.8. *Let $A + D$ be a reduced singular fibre of a smooth member of Family 2.20. Suppose $p_0 = A \cap D \notin E$ and $N_{\pi(A)/V_5} \cong \mathcal{O}^{\oplus 2}$. Then*

$$\delta_p(X) \ge \frac{104}{99} > 1 \qquad \textit{for every } p \in A.$$

In particular, if both component lines are ordinary, this bound holds along the whole reduced fibre.

Proof. Apply Abban–Zhuang refinement to the ordinary extraction F. Its discrepancy ratio is $2/(43/26) = 52/43$. The refined surface threshold is at least $104/99$ by the preceding lemma. Their minimum is $104/99$. Since the estimate holds on all of F, inversion of adjunction gives the asserted bound along all of A, including $A \cap E$ and the node p_0. □

Lemma 77.9. *Suppose ℓ is ordinary but $p_0 \in E$, and let f_0 be the ruling fibre of F containing q_E, q_D. Then*

$$S(W^F; f_0) = \frac{57}{52} > 1.$$

Thus this particular refined surface threshold is less than one. This is not a destabilizing valuation of X.

Proof. In the elementary transform used above, the centre now belongs to the smooth $(1,2)$-curve on $\mathbb{F}_0$. Its total transform contains the exceptional curve once. On F this gives f_0 as a component of R with coefficient one. The remaining component has class $2s+2f$ and contains no ruling fibre.

Blow down s. The images of q_0, q_E, q_D are three distinct collinear points. For the residual degree-b plane system in Lemma 77.5, put $c_0 = b-a$. After subtracting order v along their line, the dimension density of the restricted binary space is

$$d(v) = (b - v - (c_0 - v)_+ - (c_E - v)_+ - (c_D - v)_+)_+ .$$

Use the entries of that lemma. In the three t-intervals, the successive expressions for $d(v)$ and their v-ranges are

t	first range	second range	third range
$[0,1]$	$t\ (0 \le v \le 1)$	$1+t-v\ (1 \le v \le 1+t)$	
$[1,2]$	$2-t+2v\ (0 \le v \le t-1)$	$t\ (t-1 \le v \le 1)$	$1+t-v\ (1 \le v \le 1+t)$
$[2,3]$	$2v\ (0 \le v \le 3-t)$	$6-2t\ (3-t \le v \le 1)$	$7-2t-v\ (1 \le v \le 7-2t)$.

Direct integration gives $(3/13)\int\int v d(v)\,dv\,dt = 1$. The fixed component gf_0 contributes $5/52$, giving $57/52$. The first extraction still has $A_X(F) - S_X(F) = 9/26$; the failed surface inequality does not determine the sign of a threefold beta invariant. □

Remark 77.10. The threefold calculation in this section differs from the weighted calculation at a double fibre. Here we explicitly pass through two flops and compute the nef and fixed parts on the resulting threefold. For double fibres the preceding proof instead obtained the complete volume and refinement from section counts. Thus the manuscript uses both methods, and does not claim that the ordinary line calculation avoids threefold birational decompositions.

The positions of q_E, q_D matter. If $p_0 \in E$, they lie on the same ruling fibre. If $\pi(A)$ is special, then $q_E \in s$. Lemma 77.7 is not applied in either configuration. The volume in Proposition 77.3 remains valid, but its positivity alone does not settle these cases.

78 Special lines and points away from the fibre node

We retain the notation of Section 77. In particular $F = \mathbb{F}_1$ is the exceptional surface of the blowup of a component A of a reduced fibre $A + D$, and $W = W^F$ is its actual refined series. The divisor R and the numbers a, b, c_E, c_D, g are those of Lemma 77.5. A special line on V_5 has normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(-1)$.

78.1 The fixed curve for a special line

Lemma 78.1. *Suppose $\pi(A)$ is special. If $p_0 \notin E$, then*

$$R = s + T, \qquad T \in |s + 3f|,$$

where T is a smooth section of the ruling, passes through q_E, q_D, and has intersection number two with s. The point $s \cap f_D$ does not belong to T. If $p_0 \in E$, let f_0 be the ruling through q_E, q_D. Then

$$R = s + f_0 + T, \qquad T \in |s + 2f|,$$

where T is a smooth section, avoids q_E, and meets s once, away from f_0. If $\pi(A)$ is ordinary and $p_0 \in E$, then $R = f_0 + T$, where $T \sim 2s + 2f$ contains no ruling and is disjoint from s.

Proof. For a special line, the exceptional surface over the line on $\mathrm{Bl}_{\pi(A)} V_5$ is $\mathbb{F}_2$. Write s_2, f for its negative section and ruling. The projection divisor M restricts to $s_2 + 2f$. It contracts s_2 to the vertex of a quadric cone. The smooth twisted cubic Γ in the projection link is contained in this cone, and the restriction of Π has class $2s_2 + 3f$. It is

$$s_2 + T_2, \qquad T_2 \sim s_2 + 3f.$$

Indeed its nonexceptional image is the irreducible curve Γ; the remaining component is s_2, with the displayed multiplicity forced by the class. The curve T_2 is a section, hence smooth, and $T_2 \cdot s_2 = 1$.

The elementary transformation taking this surface to F is centred at a point $u \notin s_2$. Otherwise the resulting ruled surface would be $\mathbb{F}_3$, contrary to $F = \mathbb{F}_1$. Blow up u and contract the strict transform of its ruling. The contracted ruling maps to q_E, and the new ruling through q_E is the image of the exceptional curve over u. The condition $p_0 \in E$ says precisely that $u \in T_2$: the second curve in the projection link then meets the original exceptional fibre at the intersection with A.

If $u \notin T_2$, both sections meet the contracted ruling once. Their images are s and a section $T \sim s+3f$, both through q_E. The second marked point belongs to R and avoids s, so it belongs to T. As a section, T meets f_D only at q_D, proving the assertion about $s \cap f_D$. Tangency of T and s at q_E is allowed.

If $u \in T_2$, the strict transform of T_2 is disjoint from the contracted ruling. Its image therefore avoids q_E. The exceptional curve over u supplies f_0 with coefficient one. Subtracting $s + f_0$ from $2s + 3f$ gives $T \sim s + 2f$. Its intersection with s is one and is away from q_E.

For an ordinary line, perform the same transformation on $\mathbb{F}_0$ and its irreducible $(1,2)$-curve, as in Lemma 77.9. The residual image T is irreducible, has class $2s + 2f$, and has intersection zero with s. It is distinct from s, so it is disjoint from s. □

78.2 A polygon calculation for the negative section

Lemma 78.2. *Suppose $\pi(A)$ is special and $p_0 \notin E$. For the residual series, before multiplication by the fixed equation of gR, the normalized moments are*

quantity	i	$a-i$	f_E	f_D	*other ruling*	g
moment	63/104	89/104	21/26	99/104	79/104	5/52.

Here the moment of g means its average against the dimensions of the residual spaces. On every ruling, every residual point moment is at most 89/104. *On s, the residual point moments at q_E and $s \cap f_D$ are* 84/104 *and* 99/104, *respectively. At every other point of s they are at most* 84/104.

Proof. Contract s to $q_0 = [1:0:0] \in \mathbb{P}^2$, with homogeneous coordinates $[X:Y:Z]$. The point q_E is infinitely near q_0 in direction $Z = 0$, and take $q_D = [0:0:1]$. This normalization is possible because the two ruling fibres are different. For a normalized degree-b monomial $X^i Y^{b-i-k} Z^k$, the conditions of Lemma 77.5 are

$$0 \le i \le a, \qquad \max(0, c_E - a + i) \le k \le \min(b - i, b - c_D).$$

For the infinitely near condition, use $Y = w$, $Z = wz$ near q_0, and divide by w^{b-a}. The monomial becomes $w^{a-i} z^k$, whose order at q_E is $a - i + k$. At q_D its plane order is $b - k$.

The following bounds give the entire polygon explicitly:

t	i	k
$[0,1]$	$[0,t]$	$[0,1+t-i]$
$[1,2]$	$[0,t-1]$	$[0,2]$
	$[t-1,1]$	$[0,1+t-i]$
	$[1,t]$	$[i-1,1+t-i]$
$[2,3]$	$[0,1]$	$[0,6-2t]$
	$[1,4-t]$	$[i-1,7-2t-i]$.

Integrate over these polygons and then over t, with factor $3/13$. The integrands $1, i, a-i, k, b-i-k, g$ give respectively

$$1, \quad 63/104, \quad 89/104, \quad 84/104, \quad 99/104, \quad 10/104.$$

For another ruling put

$$U(i) = \min(b-i, b-c_D) - \max(0, c_E - a + i).$$

After removing the two prescribed endpoint factors, the coefficient space is the complete binary series of degree $U(i)$. Its order sum at another base point has leading term $U(i)^2/2$. Integrating this expression gives $79/104$.

After restriction to a ruling and removal of its fixed order, the allowed exponents i form an interval inside $[0,a]$. The endpoint point moments are the integrals of i and $a-i$. Any other point has moment at most either endpoint moment, because for $l \le i \le h$ its order sum is $(h-l)^2/2$. This proves the ruling bound, also for f_E, f_D.

For the flag with first curve s, fix i instead. The remaining binary exponents k form the displayed interval. The two endpoint moments are the integrals of k and $b-i-k$, and other points have no larger moment than either endpoint. This gives exactly the asserted point estimates on s. □

Proposition 78.3. *Suppose $\pi(A)$ is special and $p_0 \notin E$. Then*

$$\delta(F; W) \ge 1.$$

For $p \in A \setminus \{r\}$ one has $\delta_p(X) \ge 104/99$. For every divisorial valuation whose centre on X is contained in A, one has

$$A_X(v) - S_X(v) \ge \frac{9}{26} v(F) > 0.$$

If T meets s transversally, the stronger bound $\delta_p(X) \ge 104/99$ holds at r also.

Proof. At a point of $F \setminus s$, use its ruling. The first moment is at most $99/104$. The fixed component s contributes nothing at the chosen point, while the section T contributes at most $5/52$. The further point moment is at most $(89+10)/104 = 99/104$.

At a point of s, use s as first curve. Its full moment is $(89+10)/104 = 99/104$. At $s \cap f_D$, the point moment is $99/104$, since T avoids this point. Elsewhere it is at most

$$\frac{84}{104} + \frac{10}{104}(T \cdot s)_q \le 1.$$

Except possibly at q_E, it is at most $99/104$: the section T already meets s at q_E, and has total intersection two. If the intersections are transverse, the same strict bound holds everywhere. These are surface flag estimates with all fixed components retained. They prove the surface assertions.

The first extraction has ratio $52/43 > 104/99$. The only possible surface point at which the estimate equals one is q_E, which maps to r. This proves the asserted strict local bounds away from r.

For completeness, the equality case on the surface still gives a strict threefold beta invariant. Choose a basis compatible with v and the refinement, and write its pullback basis divisor as $a_m F + D'_m$, where $a_m \to 43/26$ and D'_m does not contain F. For coefficients $c_m \to 1$ from below, the preceding estimates give log canonicity of $(F, c_m D'_m|_F)$. Inversion of adjunction gives, since $K_Y = f^*K_X + F$,

$$A_X(v) - c_m v(D_m) \geq (2 - c_m a_m) v(F).$$

Taking the limit proves the displayed inequality. Its right-hand side is positive for centres contained in A. □

Example 78.4. When $(T \cdot s)_{q_E} = 2$, the surface point bound is exactly $(84 + 20)/104 = 1$. Discarding the first-stage discrepancy would lose the conclusion. Keeping it gives $9v(F)/26 > 0$. No inference from GIT-stability is used to exclude this tangency.

78.3 Both marked points on one ruling

Here $p_0 \in E$, so f_0 lies over the fibre node. We estimate all points of $F \setminus f_0$. The failure on f_0 itself will remain explicit.

Lemma 78.5. *Suppose $p_0 \in E$, and let $f_b \neq f_0$ be a ruling. For an ordinary image line, its full first moment is $37/52$, and every point moment on f_b is at most $11/13$. For a special image line, points of $F \setminus (f_0 \cup s)$ are controlled by the flag f_b with first moment $79/104$ and point moment at most $73/104$. Points of $s \setminus f_0$ are controlled by s, with first moment $99/104$ and point moment at most $89/104$.*

Proof. We give the surface decompositions used in this computation. On F^+ use the total-transform plane basis h, e_0, e_E, e_D, with intersection matrix $\mathrm{diag}(1, -1, -1, -1)$. The moving divisors are

$$D_t = \begin{cases} (1+t)h - e_0 & 0 \leq t \leq 1, \\ (1+t)h - e_0 - (t-1)(e_E + e_D) & 1 \leq t \leq 2, \\ (7-2t)h - (3-t)(e_0 + e_E) - e_D & 2 \leq t \leq 3. \end{cases}$$

The class of the unmarked ruling is $C = h - e_0$. In the ordinary case put $S = e_0$ and $Z = h - e_0 - e_E - e_D$; in the special case put $S = e_0 - e_E$ and use the same class Z. Thus Z is the strict transform of f_0.

Here is a full chamber list. A domain specified by vertices means their convex hull in the (t, v)-plane. On each domain $P = D_t - vC - N$ is the positive part. For the last table take $C = S$. The coefficients not displayed in N are zero.

ordinary line, $C = h - e_0$

t	vertices	N
$[0,1]$	$(0,0), (1,0), (1,1), (0,1)$	0
	$(0,1), (1,1), (1,2)$	$(v-1)S$
$[1,2]$	$(1,0), (2,0), (2,1), (1,1)$	0
	$(1,1), (2,1), (1,2)$	$(v-1)S$
	$(2,1), (2,2), (1,2)$	$(t+2v-4)S + (t+v-3)Z$
$[2,3]$	$(2,0), (3,0), (2,1)$	0
	$(3,0), (2,2), (2,1)$	$(2t+2v-6)S + (t+v-3)Z.$

special line, $C = h - e_0$

t	vertices	N
$[0,1]$	$(0,0),(1,0),(1,1),(0,1)$	0
	$(0,1),(1,1),(1,2)$	$(v-1)(S+e_E)$
$[1,2]$	$(1,0),(2,0),(1,1)$	0
	$(2,0),(2,2),(1,1)$	$(t+v-2)S/2$
	$(1,1),(2,2),(1,2)$	$(v-1)S+(v-t)e_E$
$[2,3]$	$(2,0),(3,0),(2,2)$	$vS/2$.

special line, $C = S = e_0 - e_E$

t	vertices	N
$[0,1]$	$(0,0),(1,0),(1,1)$	ve_E
$[1,2]$	$(1,0),(2,0),(2,1)$	0
	$(1,0),(2,1),(1,1)$	$(v-t+1)e_E$
	$(1,1),(2,1),(2,2)$	$(2v-t)e_E+(v-1)Z$
$[2,3]$	$(2,0),(3,0),(2,1)$	0
	$(3,0),(3,1),(2,2),(2,1)$	$(2t+2v-6)e_E+(t+v-3)Z$.

To verify the decompositions, intersect P with S, e_E, e_D, Z. These generate the negative curves in the respective weak del Pezzo configurations; P has nonnegative intersection with each and is orthogonal to every component of N. The upper edges give the pseudoeffective thresholds.

The curve moment is $(3/26)\iint P^2\,dv\,dt$; the free point moment is $(3/26)\iint (P\cdot C)^2\,dv\,dt$. A negative component with coefficient n contributes $(3/13)\iint (P\cdot C)n\,dv\,dt$ at its intersection with C. The displayed domains give

flag	curve	free point	negative-part corrections relevant away from f_0
ordinary, f_b	$37/52$	$17/26$	$2/13$ at $s\cap f_b$
special, f_b	$79/104$	$63/104$	$1/4$ at $s\cap f_b$
special, s	$89/104$	$79/104$	0 away from q_E.

The exceptional curves over the marked points and the curve Z are disjoint from an unmarked ruling. In the last row their intersections with s lie over q_E, which is excluded here. Thus no marked-point correction is omitted in this table.

In the ordinary case, Lemma 78.1 gives $R = f_0 + T$, with $T\cdot f_b = 2$ and $T\cap s = \varnothing$. At $s\cap f_b$ the total moment is $(68+16)/104 = 84/104$. Elsewhere it is at most $(68+20)/104 = 88/104 = 11/13$.

In the special case, $R = s + f_0 + T$, with T a section. At points off s the moment is at most $(63+10)/104$. For the flag s, its fixed component adds $10/104$ to the first moment. The residual section T meets s transversally once, away from f_0, so the further point moment is at most $(79+10)/104$. This proves all assertions. □

Corollary 78.6. *For any reduced singular fibre $A+D$ whose node p_0 lies on E,*

$$\delta_p(X) \geq 104/99 \qquad (p \in (A\cup D)\setminus\{p_0\}).$$

For an ordinary component the preceding calculation gives the stronger bound $13/11$ *away from* p_0.

Proof. The inverse image of such a point on the component extraction is an unmarked ruling. The preceding flags cover this entire ruling. Apply Abban–Zhuang refinement, including the first ratio $52/43$. For an ordinary component the minimum is $13/11$; for a special component the stated common bound is $104/99$. □

Example 78.7. In the special same-ruling configuration, let $q = T \cap s$. The ruling flag at q would give the larger point bound $(63+26+10+10)/104 = 109/104$. Changing the first curve to s gives first moment $99/104$ and point moment $89/104$. This explains why a single choice of ruling flag is insufficient.

79 Reduction to finitely many points and a triangle example

79.1 Curve centres in the original exceptional divisor

Lemma 79.1. *Every divisorial valuation whose centre is a curve contained in E has positive beta invariant.*

Proof. Write $E = \mathbb{P}^1 \times \mathbb{P}^1$, with f a fibre of $E \to C_0$ and s the other ruling class. The complete containing series for the E-refinement has class $(1+u)s + (4-2u)f$, for $0 \le u \le 1$, and no fixed part. Its moment along s is

$$\frac{3}{26}\int_0^1 \int_0^{1+u} 2(1+u-v)(4-2u)\,dv\,du = \frac{3}{4}.$$

For an irreducible curve $T \sim as + bf$ other than a fibre of $E \to C_0$, one has $a \ge 1$, $b \ge 0$. Monotonicity therefore gives $S(W^E;T) \le 3/4$. Together with $S_X(E) = 1/2$, adjunction at the generic point of T gives $\delta_T(X) \ge 4/3$.

A fibre f of $E \to C_0$ is not contained in $\eta^{-1}(\Delta)$. Indeed $\eta|_f$ maps isomorphically onto a line. If this line were contained in Δ, its general reduced conic fibre would have a component whose image is a line in V_5 through the fixed point $\pi(f)$. This would give infinitely many distinct lines through that point, which is impossible.

Here is an elementary justification of the last fact. Since V_5 is cut out by quadrics, the directions of lines through a point are cut out by quadrics in $\mathbb{P}(T_pV_5) = \mathbb{P}^2$. At least one is nonzero, for otherwise V_5 would contain its projective tangent three-space. A positive-dimensional component would thus have degree one or two. The union of the corresponding lines would be a surface of degree one or two in V_5. But $\mathrm{Pic}(V_5) = \mathbb{Z}[H]$ makes the degree of every effective surface a positive multiple of $H^3 = 5$. This contradiction proves finiteness. Now apply Lemma 71.8 to f. □

79.2 An upper bound for the ordinary point extraction

Lemma 79.2. *Let A be a smooth curve in a smooth threefold X, let $p \in A$, and let F be the exceptional divisor of its blowup. Write f_p for the ruling over p, and J for the divisor of the ordinary blowup of p. For a big line bundle L and its actual refinement W^F, one has*

$$S_L(J) \le S_L(F) + S(W^F; f_p).$$

This is an inequality; identifying the right-hand side with $S_L(J)$ requires additional information about the point filtration.

Proof. Choose regular parameters (x,y,z) at p with $A = (y = z = 0)$. If a local section has order k along A, its first nonzero normal term is

$$\sum_{i+j=k} a_{ij}(x) y^i z^j.$$

Let $q = \min \operatorname{ord}_x a_{ij}$. Then q is exactly the order of the leading section along f_p, whereas the order of the original section at p is at most $k+q$. Terms of higher normal order may lower that total order; they cannot invalidate this upper bound.

In each $H^0(X, mL)$ choose a basis compatible with the point-order filtration and the single lexicographic refinement by (k, q). Two filtrations of a finite-dimensional vector space admit a common adapted basis. The sum of the point orders is therefore at most the sum of the k-orders plus the sum of the q-orders. Divide by $mh^0(X, mL)$ and pass to the limit. These three sums give respectively the three moments in the assertion. □

Example 79.3. For $A = (y = z = 0)$, the local function $x^2y + z^2$ has normal order $k = 1$ and leading-coefficient order $q = 2$, but its order at the origin is two, strictly smaller than $k + q = 3$. Thus one cannot replace the initial-term upper bound by an equality without checking the actual point filtration.

Proposition 79.4. *At the node $p_0 \in E$ of any reduced fibre, the ordinary point extraction J satisfies*

$$A_X(J) = 3, \qquad S_X(J) \le \frac{11}{4}, \qquad A_X(J) - S_X(J) \ge \frac{1}{4}.$$

Proof. For either component extraction, $S_X(F) = 43/26$. The marked ruling has full moment $57/52$ for both normal-bundle types. The ordinary case was proved in Lemma 77.9. For the special case, use $q_0 = [1 : 0 : 0]$, $q_D = [0 : 1 : 0]$, and the infinitely near point q_E in direction $Z = 0$. The residual monomials $X^iY^{b-i-k}Z^k$ satisfy

$$0 \le i \le a, \qquad \max(0, c_E - a + i, c_D - i) \le k \le b - i.$$

Integrating k over this polygon with factor $3/13$ gives $11/104$, $59/104$, and $34/104$ on the intervals $[0, 1]$, $[1, 2]$, and $[2, 3]$, respectively. Their sum is one. The fixed ruling in $R = s + f_0 + T$ adds $5/52$. Thus its full moment is $57/52$ also. The preceding lemma now gives

$$S_X(J) \le \frac{43}{26} + \frac{57}{52} = \frac{11}{4}.$$

The discrepancy of the ordinary point blowup is three. □

79.3 The residual point set

Theorem 79.5. *Let $U \in \mathcal{H}^\circ$ be GIT-stable, and put*

$$\mathcal{N}_E = \{p \in E : \ p \text{ is the node of a reduced conic fibre}\}.$$

Every divisorial valuation of nonpositive beta invariant must have centre a point of $\mathcal{N}_E$. The map η identifies this finite set with $\Delta_{\mathrm{sm}} \cap Q$. If d is the number of nodes of Δ, then

$$\#\mathcal{N}_E \le 3 - d.$$

At every point of this residual set the ordinary point extraction has beta invariant at least $1/4$.

Proof. The earlier results treat divisor centres, smooth conic fibres, and every double fibre on the GIT-stable locus. Theorem 77.8, Proposition 78.3, and Corollary 78.6 treat closed point centres on every reduced fibre outside the set displayed above. In the possible equality case on a special-line surface, the retained term $9v(F)/26$ gives strict positivity.

Curve centres contained in E are excluded by Lemma 79.1. For a curve not contained in E, choose a general point outside E. Over a reduced singular conic the new estimates there have the strict bound $104/99$; over a double fibre use Theorem 75.7; otherwise use Lemma 71.8. Thus all curve centres are excluded as well.

To identify the residual set, move a point of Q to $[0 : 0 : 1]$ in the model of Lemma 73.1. The fibre equation is

$$-y_1^2 + z(f_0y_0 + f_1y_1) + h_2z^2 = 0.$$

Its discriminant on Q is f_0^2. At an intersection with Δ, therefore $f_0 = 0$. If the discriminant is smooth there, this is a union of two distinct lines with vertex $[y_0 : y_1 : z] = [1 : 0 : 0]$, which lies on $E = (z = 0)$. Conversely, a reduced conic whose node lies on E has a double intersection with E, so its base point lies on Q. This proves the claimed bijection.

For a GIT-stable pencil Q is not a component of Δ, and every node of Δ has intersection multiplicity two with Q, by Proposition 76.1. The restriction $\Delta|_Q$ is the square of a nonzero binary cubic. Each remaining point of intersection also has multiplicity at least two. Since $\Delta \cdot Q = 6$, there are at most $3 - d$ of them. Finally apply Proposition 79.4. □

Corollary 79.6. *Every smooth member whose conic discriminant is a triangle is K-stable. There is one such member up to isomorphism. Its pencil can be written*

$$U_\triangle = xy(x - y)\langle x, y\rangle.$$

Proof. A triangle has three nodes. All lie on Q. Bezout's theorem forces intersection multiplicity two at each and no further intersection. The pencil is GIT-stable by Proposition 76.1. Theorem 79.5 now leaves no possible centre of nonpositive beta invariant. The valuative criterion gives K-stability.

By Lemma 73.10, the three double fibres correspond to three distinct common roots of the pencil. Hence $U = g\langle x, y\rangle$ for a squarefree binary cubic g. Conversely such a pencil contains no square: multiplying g by a linear form leaves at least two of its distinct roots with odd multiplicity. It therefore belongs to $\mathcal{H}^\circ$ and has three double fibres. A reduced nodal plane cubic with three nodes is a triangle. Finally PGL_2 is transitive on unordered triples of distinct points, giving the displayed normal form and uniqueness. □

Remark 79.7. The finite reduction alone does not finish the classification. There can be three, two, or one points left when the discriminant has zero, one, or two nodes. The bound for the ordinary point extraction controls only that divisor. Sections 80 and 81 identify its entire refined series and control every valuation above these points. Thus the conclusion does not rely on the positivity of a single beta invariant.

80 The full refinement at a reduced-fibre node on E

We finish the finite reduction of Theorem 79.5. The point extraction alone did not suffice: the inequality $S_X(J) < A_X(J)$ controls only one valuation. We now identify its entire refinement and estimate all further valuations.

Theorem 80.1. *Let $p \in E$ be the node of a reduced singular fibre of η. Let $J \cong \mathbb{P}^2$ be the exceptional divisor of the ordinary blowup of p. Then*

$$S_X(J) = \frac{11}{4}, \qquad A_X(J) = 3, \qquad \delta_p(X) \geq \frac{104}{103} > 1.$$

80.1 The initial algebra in every degree

Use Lemma 73.1. Move the image of p to $[A : B : C] = [0 : 0 : 1]$. On $C = y_0 = 1$ put $b = B$, $y = y_1$. The point p is $b = y = z = 0$, and the condition that it is a fibre node gives $f_0 = 0$. Put $f = f_1$, $h = h_2$. The fibre is reduced if and only if $\kappa = f^2 + 4h \neq 0$. The equation solves uniquely for A as a convergent, or formal, power series in the regular parameters b, y, z, with quadratic initial term

$$q = y^2 - fyz - hz^2, \qquad Q = -by - q, \qquad g = yq + b(fyz + hz^2), \qquad \mathrm{in}_p(A) = Q.$$

We use ordinary total degree in these three parameters.

Lemma 80.2. *Let $R_m = H^0(X, mL)$ and let $W_{m,j}$ be its leading forms of order j at p, regarded as a subspace of $H^0(J, \mathcal{O}_J(j))$. The bigraded algebra $\bigoplus_{m,j} W_{m,j}$ is generated in anticanonical degree one by the following homogeneous polynomials:*

j	$W_{1,j}$ *(a basis)*
0	1
1	$b,\ y,\ z$
2	$q,\ by,\ bz$
3	$g,\ zQ,\ bq,\ zb^2$
4	$Qq,\ bg,\ bzQ$
5	$Qg,\ zQ^2.$

This is an equality of the actual section algebra, in every degree, for all f, h with $\kappa \neq 0$.

Proof. The seven H-sections on this chart are

$$1,\ y,\ -A - by,\ Ab + (b^2 - A)y,\ zA,\ zb,\ z,$$

and the three B-sections are $1, b, A$. Among their products take the sixteen expressions obtained from the displayed table by replacing q, Q, g by $-A - by, A, Ab + (b^2 - A)y$, respectively. Their initial forms are exactly the table. It remains to prove generation; generation of the original ring alone would not imply generation of its initial algebra.

Inside the same projective bundle P consider instead

$$X_0 = (q_0 + fzy_1 + hCz^2 = 0).$$

On $C = y_0 = 1$ its equation is exactly $A = Q$, so this chart is $\mathbb{A}^3_{b,y,z}$. Its generic conic is nonsingular: its discriminant is $C(\kappa AC - hB^2)$, a nonzero polynomial. The quadratic q_0 is nonzero on every base fibre, including the branch conic, where it has rank one. Thus this hypersurface has no vertical component and is integral. We do not assume that X_0 is smooth or K-semistable.

Here is the needed section calculation. For $r \geq 0$, the Steiner resolution of G gives the symmetric Koszul resolution

$$0 \longrightarrow \mathcal{O}(-2) \otimes \operatorname{Sym}^{r-2} \mathbb{C}^4 \longrightarrow \mathcal{O}(-1)^{\oplus 2} \otimes \operatorname{Sym}^{r-1} \mathbb{C}^4 \longrightarrow \mathcal{O} \otimes \operatorname{Sym}^r \mathbb{C}^4 \longrightarrow \operatorname{Sym}^r G \longrightarrow 0,$$

where terms with negative symmetric exponent are omitted. After twisting by any $d \geq 0$, this proves vanishing of higher cohomology and surjectivity on sections from the third term. Consequently

$$H^0(P, mH + mB) = \bigoplus_{a=0}^{m} z^a H^0(\mathbb{P}^2, \operatorname{Sym}^{m-a} G(m + a))$$

is generated by degree-one sections: a monomial of base degree $m + a$ factors into a quadratic and $m - a$ linear factors. The ideal of X_0 in this degree is $(m - 2)H + (m + 1)B$. For $m \geq 2$ its pushforward has summands $\operatorname{Sym}^{m-2-a} G(m + 1 + a)$ and has no higher cohomology. For $m = 0, 1$ all fibre cohomology vanishes because the relative degrees are $-2, -1$. Restriction is therefore surjective in every degree. In particular the section ring of (X_0, L) is generated in degree one and has the same Hilbert function as (X, L):

$$N_m = \frac{(2m + 1)(13m^2 + 13m + 6)}{6}.$$

Restriction to the displayed dense affine chart embeds this section ring in $\mathbb{C}[b, y, z, u]$, where u records the anticanonical degree. Its sixteen degree-one generators are precisely u times the table. Their independence also follows directly, degree by degree, from the polynomials in the table. The algebra they generate is contained in the actual initial algebra of X. In degree m the former already has dimension N_m, while the latter has dimension $\dim R_m = N_m$. The inclusion is consequently an equality in every degree. □

80.2 Six point conditions and the extra fixed line

Factor $q = (y - \lambda z)(y - \mu z)$, where $\lambda \neq \mu$, $\lambda + \mu = f$, $\lambda\mu = -h$. The degree-two space has three base points

$$p_a = [0 : \lambda : 1], \qquad p_d = [0 : \mu : 1], \qquad p_c = [1 : 0 : 0].$$

At these points mark the respective tangent directions

$$t_a : \lambda b + (\lambda - \mu)(y - \lambda z) = 0, \quad t_d : \mu b + (\mu - \lambda)(y - \mu z) = 0, \quad t_c : z = 0.$$

Let $\sigma : T \to J$ blow up the three points and then the three indicated infinitely near points. Write ℓ for a line class, e_i for the total transform of the first exceptional curve, and f_i for the second exceptional curve, for $i = a, d, c$. Their intersection matrix is $\operatorname{diag}(1, -1, -1, -1, -1, -1, -1)$ in the order $\ell, e_a, f_a, e_d, f_d, e_c, f_c$. Put $r_i = e_i - f_i$.

Lemma 80.3. *For $0 \leq j \leq 5m$, put*

$$r = \max\{j - m, 0\}, \quad a = \lceil r/2 \rceil, \quad b_0 = \lfloor r/2 \rfloor, \quad a_c = \max\{a, j - 2m\}, \quad b_c = r - a_c,$$

and set

$$D_{m,j} = j\ell - a(e_a + e_d) - b_0(f_a + f_d) - a_c e_c - b_c f_c.$$

The following descriptions give the complete spaces $W_{m,j}$.

If $h \neq 0$, let Q_c be the strict transform of the smooth conic $Q = 0$. Its class is

$$Q_c = 2\ell - e_a - f_a - e_d - f_d - e_c.$$

The series on T, after the six indicated point conditions, is

$$\max\{j - 4m, 0\}Q_c + |D_{m,j} - \max\{j - 4m, 0\}Q_c|.$$

If $h = 0$, normalize $f = 1$, $\lambda = 0$, $\mu = 1$. Put

$$L_1 = (y = 0)^{\sim} = \ell - e_a - f_a - e_c, \qquad L_2 = (b + y - z = 0)^{\sim} = \ell - e_d - f_d.$$

The series on T is

$$n_1 L_1 + n_2 L_2 + |D_{m,j} - n_1 L_1 - n_2 L_2|, \qquad n_1 = \max\{j - 3m, 0\}, \quad n_2 = \max\{j - 4m, 0\}.$$

In both cases the divisor in the complete moving series is nef. The spaces with $j > 5m$ vanish.

Proof. At each marked point, use regular coordinates whose second coordinate vanishes on the specified tangent. Direct substitution in Lemma 80.2 shows that the weight-$(1, 2)$ order of every degree-one generator is at least $j - 1$. Its ordinary order at p_c is at least $j - 2$. These inequalities multiply, so all products satisfy the conditions defining $D_{m,j}$. The two successive blowups express the first condition as ordinary order at least $\lceil r/2 \rceil$ and weighted order at least r.

When $h \neq 0$, every generator of degree five is divisible by Q, and every generator has degree at most five. Thus products of anticanonical degree m and point degree j are divisible by $Q^{\max\{j-4m,0\}}$. When $h = 0$, the table gives the stronger divisibilities $y^{\max\{j-3m,0\}}$ and $(b + y - z)^{\max\{j-4m,0\}}$. This proves inclusion in the asserted spaces. These are algebraic fixed-component conditions; in particular the first coefficient in the second case must not be replaced by the smaller coefficient in the Zariski decomposition of $D_{m,j}$ alone.

We verify equality by dimensions. For reference the negative curves on T in the first case are

$$r_i = e_i - f_i, \quad f_i, \quad t_i = \ell - e_i - f_i, \quad l_{ik} = \ell - e_i - e_k \quad (i \neq k),$$
$$Q_i = 2\ell - \sum_{k=a,d,c} (e_k + f_k) + f_i.$$

There are fifteen, with $l_{ik} = l_{ki}$. In the second case they are the twelve curves

$$r_i,\ f_i\ (i = a, d, c), \qquad L_1, L_2, t_c, l_{ad}, l_{dc}, Q_a.$$

Here Q_a is the smooth conic $y^2 - yz + bz = 0$. These are weak del Pezzo surfaces of degree three. Here is a direct check of nefness of the anticanonical class. In the first case, for each i, the effective divisor $Q_i + (\ell - e_i) + r_i$ is anticanonical, where the middle term is a general line through p_i. These divisors have no common curve component. In the second case use the same divisors for $i = a, d$, interpreting $Q_d = L_1 + l_{dc} + r_c$. Again they have no common component. Thus $-K_T$ is nef, and its square is three. To check the negative-curve lists, the three points are noncollinear. In the first case none of the three specified tangents joins two marked points, and the three five-point conics are smooth. There is no conic through all six points. In the second case the only additional root is L_1, meeting r_c; the other roots are r_a, r_d. A line or conic gives all remaining negative curves listed above. Equivalently one may enumerate the plane classes with square $-1, -2$ and anticanonical degree $1, 0$; the proximity inequalities give exactly these lists. There are no higher-degree candidates: Cauchy's inequality applied to $\sum m_i = 3d - 1$, $\sum m_i^2 = d^2 + 1$ in the (-1) case, and to $\sum m_i = 3d$, $\sum m_i^2 = d^2 + 2$ in the (-2) case, forces $d \leq 2$ in both cases. Adjunction and nefness of $-K_T$ exclude more negative irreducible curves.

We now prove nefness in every degree using six fixed classes. Put $P = -K_T$, $U = 2\ell - e_a - e_d - e_c$, and put $L = 0$ in the first case and $L = L_1$ in the second case. Define

$$A_2 = 8\ell - 3(e_a + f_a + e_d + f_d) - 4e_c - 2f_c - 2L,$$
$$A_1 = 4\ell - 2(e_a + e_d) - f_a - f_d - 2e_c - f_c - L,$$
$$A_0 = 3\ell - e_a - f_a - e_d - f_d - 2e_c - f_c - L.$$

The classes ℓ, P, U are nef: the first is a pullback of a line, the second was treated above, and the third is the pullback of the base-point-free conic system through the three proper points. For A_2, A_1, A_0, intersection with each curve in the relevant negative-curve list is nonnegative. Their squares are $8, 1, 0$, respectively. Their line degrees are $8, 4, 3$ in the first case and $6, 3, 2$ in the second. They therefore lie in the closure of the positive cone, in its component containing ℓ. Every irreducible curve of nonnegative square lies in the same closed component; the Hodge index theorem gives nonnegative intersection with these curves as well. Thus all six classes are nef.

Write $M_{m,j}$ for the proposed moving divisor. The following identities express it as a nonnegative integral combination of these classes and also account for every parity:

range	indices	$M_{m,j}$
$0 \leq j \leq m$		$j\ell$
$m \leq j \leq 3m$	$j - m = 2k + \epsilon$	$(m - k - \epsilon)\ell + kP + \epsilon U$
$3m \leq j \leq 4m$	$j - 3m = q = 2k + \epsilon$	$(m - q)P + kA_2 + \epsilon A_1$
$4m \leq j \leq 5m$	$q = j - 4m, \quad m - q = 2k + \epsilon$	$qA_0 + kA_2 + \epsilon A_1,$

where $\epsilon \in \{0, 1\}$. In the second row $2k + \epsilon \leq 2m$ implies $m - k - \epsilon \geq 0$; all other coefficients are visibly nonnegative. The identities follow by substituting the coefficients of $D_{m,j}$ and its stated fixed part. In particular, in the last two rows the subtraction of L is retained when $h = 0$. This proves nefness for all m, j, without extrapolating from finitely many degrees. For

a nef integral divisor M on either surface, Kawamata–Viehweg vanishing applied to $M - K_T$ gives

$$h^0(T, M) = 1 + \frac{M^2 - M \cdot K_T}{2}.$$

Let $M_{m,j}$ denote the moving divisor in either case. Its Riemann–Roch value is the same in the two cases. Summing the displayed quadratic polynomial gives the identities

$$\sum_{j=0}^{5m} h^0(T, M_{m,j}) = \frac{(2m+1)(13m^2+13m+6)}{6} = N_m,$$
$$\sum_{j=0}^{5m} j h^0(T, M_{m,j}) = \frac{143m^4+208m^3+133m^2+32m}{12}.$$

For an explicit way to perform the sum, take $j \le m$ first, where the summand is $\binom{j+2}{2}$. In the remaining terms write $j = m + 2k + \epsilon$, $\epsilon = 0, 1$, and split at $j = 3m, 4m$. Substitute $a = k + \epsilon$, $b_0 = k$ in $D_{m,j}$ and use $K_T = -3\ell + \sum(e_i + f_i)$. The sums of $1, k, k^2, k^3$ give the two identities, for both parities of m. This is an all-degree polynomial identity.

The inclusions already proved have total dimension at most the left side of the first identity. Their actual total dimension is N_m by Lemma 80.2. Every inclusion is therefore an equality. Generation in anticanonical degree one also proves vanishing above $5m$. □

Example 80.4. For $m = 1$ the dimensions, in order $j = 0, \dots, 5$, are $1, 3, 3, 4, 3, 2$. If $h = 0$ and $j = 4$, all three polynomials Qq, bg, bzQ contain the line $y = 0$. This line is forced by the section algebra, even though its coefficient in the Zariski negative part of $D_{1,4}$ is only $1/2$. Using that negative part as if it specified the complete series would give an incorrect refinement.

80.3 The point moment

Lemma 80.5. *The volume function of the point extraction is*

$$\operatorname{vol}(L - tJ) = \begin{cases} 26 - t^3 & 0 \le t \le 1, \\ 26 - t^3 + \frac{3}{2}(t-1)^3 & 1 \le t \le 3, \\ t^3 - 9t^2 + 18t + 11 & 3 \le t \le 4, \\ 3(5-t)^2 & 4 \le t \le 5, \\ 0 & t \ge 5. \end{cases}$$

Consequently $S_X(J) = 11/4$.

Proof. Put $t = j/m$. The asymptotic moving divisors of Lemma 80.3, in the case $h \neq 0$, are

$$\begin{array}{l|l} 0 \le t \le 1 & P_t = t\ell \\ 1 \le t \le 3 & P_t = t\ell - \frac{t-1}{2}\sum_i (e_i + f_i) \\ 3 \le t \le 4 & P_t = t\ell - \frac{t-1}{2}(e_a + f_a + e_d + f_d) - (t-2)e_c - f_c \\ 4 \le t \le 5 & P_t = (8-t)\ell - \frac{7-t}{2}(e_a + f_a + e_d + f_d) - 2e_c - f_c. \end{array}$$

If $h = 0$, replace the third row by $P_t - (t-3)L_1$ and the fourth by $P_t - L_1$, leaving the first two rows unchanged. In either case the squares are, respectively,

$$t^2, \qquad -\tfrac{1}{2}t^2 + 3t - \tfrac{3}{2}, \qquad -t^2 + 6t - 6, \qquad 10 - 2t.$$

The normalization is $\frac{3}{26}\int_0^5 P_t^2\,dt = 1$. The exact section spaces give $\mathrm{vol}(L - tJ) = 3\int_t^5 P_u^2\,du$, hence the displayed function. Finally

$$S_X(J) = \frac{1}{26}\int_0^5 \mathrm{vol}(L - tJ)\,dt = \frac{3}{26}\int_0^5 tP_t^2\,dt = \frac{11}{4}.$$

The weighted Hilbert identity in the preceding lemma is an independent finite-degree computation of the same limit. □

Remark 80.6. No Zariski decomposition on the threefold $\mathrm{Bl}_p X$ has been used. The integral initial algebra and its Hilbert function identify the actual refinement in every degree. The volume follows by summing those dimensions. All Zariski decompositions below take place on the explicit rational surface T. The auxiliary threefold X_0 is used only to prove equality of section algebras; no assertion about its K-stability or singularities is needed.

81 Surface flags at the remaining nodes

We give the surface computations separately. They keep the exceptional fixed coefficients, the extra line when $h = 0$, and the different at a weighted extraction.

81.1 Moment conventions and computations

For a curve $C \subset T$, decompose

$$P_t - vC = P_{t,v} + \sum_R n_R(t,v)R$$

on its pseudoeffective range, with $P_{t,v}$ nef and the sum negative definite. Set $d(t,v) = P_{t,v}\cdot C$ and define

$$u_C = \frac{3}{26}\iint P_{t,v}^2\,dv\,dt, \qquad b_C = \frac{3}{26}\iint d(t,v)^2\,dv\,dt,$$
$$z_{C,R} = \frac{3}{13}\iint d(t,v)n_R(t,v)\,dv\,dt.$$

Here u_C is the moving first moment, b_C is the free point moment, and $z_{C,R}(R\cdot C)_x$ is its negative-part correction at $x \in C$. Fixed components of the original series must be added separately. Their average coefficient is computed with probability density $3P_t^2/26$.

The normalized asymptotic fixed part of the pullback series is

$$a_a r_a + a_d r_d + a_c r_c + c(f_a + f_d + f_c) + N_t,$$

where

$$c = (t-1)_+, \quad a_a = a_d = c/2, \quad a_c = \begin{cases} c/2 & t \le 3, \\ t-2 & t \ge 3, \end{cases}$$

and $N_t = (t-4)_+ Q_c$ if $h \neq 0$, whereas $N_t = (t-3)_+ L_1 + (t-4)_+ L_2$ if $h = 0$. The integral rounding errors are bounded and disappear in these moments. For a function a write $\overline{a} = \frac{3}{26}\int P_t^2 a(t)\,dt$. Direct integration gives

$$\overline{c} = \frac{183}{104}, \quad \overline{a_c} = \frac{107}{104}, \quad \overline{(t-4)_+} = \frac{1}{26}, \quad \overline{(t-3)_+} = \frac{31}{104}.$$

Lemma 81.1. *In the case $h \neq 0$, the flag moments are as follows. Entries not displayed in the last column are zero.*

C	u_C	b_C	$R : z_{C,R}$
f_a	17/26	103/208	$r_a : 17/52,\ t_a : 3/52,\ r_d : 1/52,$ $r_c : 1/208,\ Q_d : 1/26,\ Q_c : 17/208$
r_c	35/52	59/104	$r_a, r_d : 5/104,\ f_c : 19/104,$ $l_{ac}, l_{dc}, Q_c : 5/52$
$\ell - e_a$	19/52	323/416	$r_a : 89/416,\ f_a : 3/104,$ $Q_a : 7/208,\ Q_c : 1/32$
l_{ad}	63/104	47/104	$r_a, r_d : 67/208,\ f_a, f_d : 1/26,\ t_c : 5/104$
l_{ac}	69/104	43/104	$r_a : 75/208,\ f_a, f_c : 1/26,\ t_d : 3/52,$ $r_c : 27/104,\ Q_a : 1/52,\ Q_c : 1/208$
t_a	49/104	33/52	$r_a : 17/208,\ f_a : 7/52,\ r_d : 3/208,$ $l_{dc}, Q_a : 3/104.$

In the case $h = 0$, the table is

C	u_C	b_C	$R : z_{C,R}$
f_a	11/13	71/208	$r_a : 45/104,\ r_c : 17/104,\ f_c : 3/208,$ $L_1 : 5/16,\ L_2 : 3/104,\ Q_a : 1/52$
f_d	17/26	103/208	$r_a : 1/52,\ r_d : 17/52,\ r_c : 1/208,$ $L_1 : 1/104,\ L_2 : 29/208,\ Q_a : 1/26$
r_c	43/52	35/78	$r_d : 97/624,\ f_c : 25/104,$ $L_1 : 8/39,\ L_2 : 1/208,\ l_{dc} : 97/312$
$\ell - e_a$	19/52	323/416	$r_a : 89/416,\ f_a : 3/104,$ $L_2 : 1/32,\ Q_a : 7/208$
l_{ad}	63/104	47/104	$r_a, r_d : 67/208,\ f_a, f_d : 1/26,\ t_c : 5/104$
L_1	17/26	25/52	$r_a : 19/104,\ f_a : 19/52,\ r_c : 9/26,$ $f_c : 1/26,\ L_2 : 2/13.$

Proof. All entries can be calculated by linear algebra and integration of quadratic polynomials. We specify the computation to make the negative parts and the normalization reproducible. Use the negative-curve lists in Lemma 80.3. The auxiliary class

$$A = 10\ell - 2(e_a + e_d + e_c) - f_a - f_d - f_c$$

is ample: it has square 85 and positive intersection with every listed negative curve, while the Hodge index theorem treats the curves of nonnegative square. On each of the four t-intervals, $A \cdot P_t \leq 26$, as is checked at its endpoints, and $A \cdot C \geq 1$ for every flag in the tables. Consequently, pseudoeffectivity of $P_t - vC$ implies $0 \leq v \leq 26$. Thus the whole calculation takes place in bounded rational rectangles. On each t-interval, enumerate all negative definite subsets $R_1, \ldots, R_s$ of the list and form

$$M = (R_i \cdot R_j)_{i,j}, \quad (n_i)_i = M^{-1}((P_t - vC) \cdot R_i)_i, \quad P_{t,v} = P_t - vC - \sum_i n_i R_i.$$

Keep the polygon where $v \geq 0$, all $n_i \geq 0$, and $P_{t,v} \cdot R \geq 0$ for every listed negative curve. Also require $P_{t,v} \cdot \ell \geq 0$. These are linear inequalities in t, v. Check also that the positive part has nonnegative square at each vertex; the convexity of the positive cone gives this throughout each polygon. The negative-curve list and the Hodge index theorem then verify nefness. These rational polygons give the Zariski chambers, with disjoint interiors. Substitution in the three preceding integrals gives the tables.

For completeness an elementary integration rule avoids any numerical approximation. Triangulate each polygon, with a triangle parametrized by $p + x(a-p) + y(b-p)$, $x, y \geq 0$, $x + y \leq 1$. Multiply by its oriented Jacobian. For every resulting monomial use

$$\int_{x,y\geq 0,\ x+y\leq 1} x^i y^j \, dx\, dy = \frac{i!j!}{(i+j+2)!}.$$

The numbers of nonempty chambers in the successive rows of the first table are $11, 11, 10, 8, 12, 7$, and in the second are $13, 11, 15, 10, 8, 11$. The chamber count here ignores boundaries. In each row the additional check

$$\frac{3}{13} \iint (P_{t,v} \cdot C)\, dv\, dt = 1$$

holds. Indeed $-\partial_v(P_{t,v}^2) = 2P_{t,v} \cdot C$, and the volume vanishes at the final endpoint. This identity checks that the chambers account for the full slice mass. The script `check_f220_chambers_exact.py`, included in the audit certificates, carries out this finite enumeration entirely in rational arithmetic, including the negative-definiteness and polygon-feasibility tests. Its files `chambers_f220_False_*.json` and `chambers_f220_True_*.json` record the polygons, negative supports, coefficients n_i, and positive parts; the file `chambers_f220_results.json` records their integrals. The script also checks orthogonality, nefness, and the displayed mass identity. The linear systems, inequalities, and integration rule needed to reproduce the computation have all been given above. □

Example 81.2. The fixed conic contribution is

$$\overline{(t-4)_+} = \frac{3}{26} \int_4^5 (10-2t)(t-4)\, dt = \frac{1}{26}.$$

In the second case the extra fixed line contributes

$$\frac{3}{26} \left(\int_3^4 (-t^2+6t-6)(t-3)\, dt + \int_4^5 (10-2t)(t-3)\, dt \right) = \frac{31}{104}.$$

For the first row of either table and $0 \leq t \leq 1$, the initial chamber is $0 \leq v \leq t$, with

$$N_{t,v} = \frac{v}{2} r_a, \qquad P_{t,v} = t\ell - v f_a - \frac{v}{2} r_a, \qquad P_{t,v} \cdot f_a = \frac{v}{2}.$$

For example this chamber contributes $\frac{3}{26} \int_0^1 \int_0^t (v/2)^2\, dv\, dt = 1/416$ to b_{f_a}. The other chambers are computed by the same rule.

81.2 The three marked points

Lemma 81.3. *For the complete series W on J, one has*

$$\delta_{p_a}(J; W), \quad \delta_{p_d}(J; W), \quad \delta_{p_c}(J; W) \ \geq \frac{104}{103}.$$

Proof. At p_a and p_d use the weight-$(1,2)$ extraction in the specified tangent direction. Its discrepancy over J is three. Its exceptional curve has different $1/2$ at the orbifold point and zero elsewhere. Its strict transform on T is f_i, and the curve contracted over the orbifold point is r_i. The full first moment is $u_{f_i} + \overline{c}$. The point moment at the orbifold point is

$$b_{f_i} + z_{f_i, r_i} - \frac{1}{2} u_{f_i}.$$

To verify the subtraction, the pullback of the weighted exceptional curve is $f_i + r_i/2$. The fixed coefficient along r_i before subtracting this curve is $a_i = c/2$, so the initial fixed contribution

cancels. The further negative coefficient must be reduced by $v/2$. Since $\frac{3}{13}\iint v(P_{t,v}\cdot f_i) = u_{f_i}$, the correction is exactly $u_{f_i}/2$.

If $h \neq 0$, the first moment is $251/104$, and the orbifold point moment is $103/208$. Away from that point only t_i, Q_k, Q_c can contribute from the first table. They have intersection one with f_i. Even adding all their corrections and the fixed Q_c coefficient gives at most

$$\frac{103+12+8+17+8}{208} = \frac{148}{208} < 1.$$

The curves over the other two marked points are disjoint from f_i. Consequently the smallest possible ratio is $\frac{1/2}{103/208} = 104/103$.

If $h = 0$, at p_a the first moment is $271/104$, and the orbifold moment is $73/208 < 1/2$. The only further contributing curve meeting f_a is L_1, away from the orbifold point. At this point the full moment is

$$\frac{71}{208} + \frac{5}{16} + \frac{31}{104} = \frac{99}{104} < 1.$$

At p_d the first moment is $251/104$ and the orbifold moment is again $103/208$. At the ordinary points of f_d, even adding the contributions of L_2, Q_a and the fixed L_2 gives $(103+29+8+8)/208 = 148/208 < 1$. Both weighted flags therefore give the asserted bound.

At p_c use the ordinary blowup, of discrepancy two. Its exceptional curve is represented by r_c on T, and its pullback is $r_c + f_c$. Thus its full moment is $u_{r_c} + \overline{a_c}$, and the point moment in the marked direction t_c is

$$b_{r_c} + z_{r_c,f_c} + \overline{c} - \overline{a_c} - u_{r_c}.$$

If $h \neq 0$, these are $177/104$ and $21/26$, respectively. At every other direction an upper bound is $(59+10+10+10+4)/104 = 93/104$, adding the three possible line/conic corrections and the fixed conic.

If $h = 0$, the first moment is $193/104$ and the marked point moment is $185/312$. The L_1 direction is different from t_c and from the l_{dc} direction. Its point moment is

$$\frac{35}{78} + \frac{8}{39} + \frac{31}{104} = \frac{99}{104}.$$

At the l_{dc} direction the moment is $(140+97)/312 = 79/104$. Elsewhere it is $35/78$. The other negative curves in that row are disjoint from r_c. All these ratios, including $2/(193/104)$, exceed $104/103$. Abban–Zhuang refinement proves the lemma. □

81.3 All other points of the exceptional plane

Lemma 81.4. *For every $x \in J \setminus \{p_a, p_d, p_c\}$,*

$$\delta_x(J; W) \geq \frac{104}{103}.$$

Proof. First suppose that the line through p_a and x contains neither of the other marked points and is not t_a. Its strict transform has class $\ell - e_a$. Its first moment is $19/52$. Corrections supported above p_a do not meet x. If $h \neq 0$, the conics Q_a, Q_c each already meet the line at p_a, so each has local intersection at most one at x. The point moment is at most

$$\frac{323+14+13+16}{416} = \frac{183}{208} < 1.$$

If $h = 0$, the fixed line L_1 meets this line only at p_a. The line L_2 has intersection one, and Q_a has remaining intersection at most one. The same bound follows from the second table.

It remains to use the excluded lines themselves. At points of l_{ad} away from its endpoints, the first moment is $63/104$ and the point moment is at most $(47+5)/104 = 1/2$, in either

case. All fixed components meet this line at its marked endpoints; the only further correction is from t_c.

If $h \neq 0$, use l_{ac} or t_a at the remaining points. Their first moments are $69/104$ and $49/104$. Discarding the corrections over marked endpoints, and using local intersection at most one for lines and at most two for conics, bounds their point moments by, respectively,

$$\frac{43+6+4+1+8}{104} = \frac{62}{104}, \qquad \frac{66+3+6+8}{104} = \frac{83}{104}.$$

These bounds include the fixed conic coefficient.

If $h = 0$, the line t_a is the line joining p_a to p_c, with strict transform L_1. Its first moment includes its own fixed coefficient and is

$$\frac{17}{26} + \frac{31}{104} = \frac{99}{104}.$$

Away from its marked endpoints the only correction is from L_2, including its fixed coefficient, so the point moment is at most

$$\frac{25}{52} + \frac{2}{13} + \frac{1}{26} = \frac{70}{104}.$$

Every first and point moment just listed is at most $103/104$. The ordinary surface-flag inequality proves the assertion. The upper bounds allow contributing curves to meet at the same point; no general-position assumption on their other intersections has been used. □

Proof of Theorem 80.1. Lemmas 81.3 and 81.4 cover every point of J, and give $\delta(J;W) \geq 104/103$. The ordinary point blowup is a plt extraction with $A_X(J) = 3$ and trivial different on J. Lemma 80.5 and the Abban–Zhuang inequality [4, Theorem 3.3] now give

$$\delta_p(X) \geq \min\left\{\frac{3}{11/4}, \frac{104}{103}\right\} = \frac{104}{103} > 1.$$

This proves a bound for every valuation involved in the local threshold, rather than only for the first exceptional divisor. □

Example 81.5. The choices $(f,h) = (0,1)$ and $(1,0)$ illustrate the two surface calculations. In the first, $q = y^2 - z^2$ and Q_c is smooth. In the second, $q = y(y-z)$ and $Q = -y(b+y-z)$, so its two line factors have different fixed coefficients: $(t-3)_+$ and $(t-4)_+$. Nevertheless both point extractions have expected order $11/4$, and both full refinements have threshold at least $104/103$. Equality of the first moments alone would not have justified transferring the surface bound between them.

82 The complete classification

Proof of Theorem 69.1. The first assertion follows from Lemmas 70.1 and 70.2 and Proposition 70.4. Corollary 70.8 proves the second assertion. The moments and the valuative reductions are Lemmas 71.1, 71.5, 71.7 and 71.8. Finally, Lemmas 73.4 and 73.11, together with Proposition 74.3, prove the fourth assertion. The fifth assertion is Theorem 75.7, together with the preceding reductions. The sixth follows from Proposition 77.3 and Theorem 77.8. The seventh combines Theorem 79.5 with Theorem 80.1, and the eighth is Corollary 79.6. If U is GIT-stable, the finite-node reduction and the strict point estimate leave no valuation of nonpositive beta invariant. The valuative criterion gives K-stability. The nonstable pencils are exactly the displayed $U_{a,c}$. Their K-polystability and strict K-semistability are Proposition 70.4; $U_{0,0}$ is not K-stable because its automorphism group is positive-dimensional. These facts prove every direction of the classification, and also prove K-semistability of all smooth members. □

The classification can be read directly from the Wronskian sextic:

Wronskian condition	K-stability of X_U
Every root has multiplicity at most two	K-stable
Exactly two triple roots	K-polystable, not K-stable
Exactly one triple root	Strictly K-semistable

There are no other cases on $\mathcal{H}^\circ$. The second row is the single orbit $\langle x^3y, xy^3\rangle$. The third consists of $\langle x^3y, xy^3 + ax^2y^2 + cx^4\rangle$ with $(a, c) \neq (0, 0)$, parametrized up to isomorphism by $\mathbb{P}(1, 3)$. This explicitly describes all exceptions to K-stability, including which exception is K-polystable.

82.1 An intrinsic classification by the conic bundle

The criterion does not require finding a binary quartic pencil. Let $\pi : X \to V_5$ be the birational contraction, with exceptional divisor E, and let $\eta : X \to \mathbb{P}^2$ be the other extremal contraction. Let Δ be the reduced nodal cubic discriminant of η, and let Q be the smooth branch conic of the finite double cover $\eta|_E : E \to \mathbb{P}^2$. If Q is not a component of Δ, put

$$I_b(\Delta, Q) = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^2,b}/(\Delta, Q).$$

Here the equations in the denominator are local equations of the two curves. This finite integer is their local intersection multiplicity.

Corollary 82.1. *Every smooth member X of Family 2.20 is K-semistable. Exactly one of the following alternatives holds.*

1. *If $Q \subset \Delta$, then $\Delta = Q \cup M$, where M is a line transverse to Q. The threefold is K-polystable but not K-stable, and is isomorphic to $X_{\langle x^3y, xy^3\rangle}$.*
2. *If $Q \not\subset \Delta$ and $I_b(\Delta, Q) = 2$ at every node b of Δ, then X is K-stable. In particular every member with smooth discriminant is K-stable.*
3. *If $Q \not\subset \Delta$ and $I_b(\Delta, Q) > 2$ at some node b of Δ, then X is strictly K-semistable. Such a node is unique, and its intersection multiplicity is either four or six.*

Thus X is K-polystable precisely in the first two alternatives.

Proof. Proposition 76.1 identifies the three alternatives with, respectively, the closed nonstable orbit, the GIT-stable locus, and the nonclosed semistable locus. Theorem 69.1 now supplies every claimed K-stability conclusion; the sufficiency in the second case uses Theorem 80.1.

For the additional numerical assertion, when Q is not a component the restriction $\Delta|_Q$ is the square of a nonzero binary cubic. All intersection multiplicities are therefore even and their sum is six. Every discriminant node lies on Q and contributes at least two. An intersection multiplicity greater than two is four or six, and there cannot be two such points. The second alternative includes the case with no discriminant nodes: its condition is then vacuous. □

Remark 82.2. The third alternative tests only the nodes of Δ. A smooth point of Δ with $I_b(\Delta, Q) = 4$ or 6 does not satisfy its hypothesis. Such a point corresponds to a reduced conic fibre, and the point-refinement theorem treats its node on E. Confusing these two types of nodes would give an incorrect classification.

82.2 Explicit examples of all alternatives

Use the projective bundle and coordinates of Lemma 73.1. Thus

$$Ay_0 + By_1 + Cy_2 = Ay_1 + By_2 + Cy_3 = 0,$$

and q_0 is specified by

$$y_0y_2 - y_1^2 = Cq_0, \quad y_0y_3 - y_1y_2 = -Bq_0, \quad y_1y_3 - y_2^2 = Aq_0.$$

Here $E = (z = 0)$ and $Q = (B^2 = 4AC)$.

Equation of X	Equation of Δ	Stability
$q_0 + zy_0 + Az^2 = 0$	$4A^2C - AB^2 + C^3 = 0$	K-stable
$q_0 + Bz^2 = 0$	$B(4AC - B^2) = 0$	K-polystable, not K-stable
$q_0 + zy_2 + Bz^2 = 0$	$A^2C + 4ABC - B^3 = 0$	Strictly K-semistable
$q_0 + zy_3 + Bz^2 = 0$	$A^3 + 4ABC - B^3 = 0$	Strictly K-semistable

Lemma 82.3. *All assertions in the table hold. In its first row, Δ is smooth and has contact order six with Q at $[1 : 0 : 0]$. In its third and fourth rows the only discriminant node is $[0 : 0 : 1]$, with intersection multiplicity four and six, respectively.*

Proof. The determinants of the conic matrices give the displayed cubics. For the first cubic the partial derivatives are $8AC - B^2, -2AB, 4A^2 + 3C^2$, which have no common projective zero. For each of the last two cubics their only common zero is $[0 : 0 : 1]$. On $C = 1$ the quadratic terms there are $A(A+4B)$ and $4AB$, respectively, so these singularities are ordinary nodes. The second cubic is the union of Q and the transverse line $B = 0$.

Parametrize Q by $[A : B : C] = [s^2 : 2st : t^2]$. The restrictions of the four discriminant equations are, in their displayed order,

$$t^6, \qquad 0, \qquad s^4t^2, \qquad s^6.$$

This proves the contact assertions, including the distinction between a smooth discriminant point in the first row and a singular discriminant point in the last row.

We check smoothness of the threefolds as well as of their discriminants. Over a smooth discriminant point the conic matrix has rank two, and its determinant has nonzero differential; this makes the total space smooth at the fibre node. At every discriminant node in the table the conic matrix has rank one. Near such a base point, completing a square writes the equation as

$$u^2 + s\,q_1(v, w) + t\,q_2(v, w) + \text{terms of base order at least two} = 0.$$

The quadratic initial term of the determinant is a nonzero multiple of $\det(sq_1 + tq_2)$. Since the discriminant has an ordinary node, this is a product of distinct linear forms. The quadratics q_1, q_2 have no common zero in $\mathbb{P}^1$: otherwise, choosing that zero as $[1 : 0]$ would make this determinant a square. Thus at every point of the double line at least one base derivative of the equation is nonzero. The total space is smooth. Points on smooth fibres and the other points of reduced fibres are automatically smooth.

Finally consider the flat seven-parameter family $q_0 + z\sum f_iy_i + z^2(h_0A + h_1B + h_2C) = 0$ in the fixed projective bundle. Its smooth parameter locus is a nonempty open subset of $\mathbb{A}^7$, hence connected, and contains the model of a member of Family 2.20. Adjunction gives $-K_X = (H + B)|_X$. The evaluation map of $\mathcal{O}(1) \oplus G$ embeds the projective bundle in $\mathbb{P}^6 \times \mathbb{P}^2$; thus $H + B$ is very ample. All four equations are therefore in the same Fano deformation family. Their stability follows from Corollary 82.1. □

Part XII
Family 3.2: double conics in a projective bundle

83 The statement for Family 3.2

We consider smooth Fano threefolds of Picard rank three and anticanonical degree fourteen. Belousov–Loginov [43] proved their K-stability under a condition on the cubic surface containing each double conic: its singularities along that conic must be ordinary double points. We use the corrected third version of their paper. In particular, its reduced-conic estimate is non-strict; the first adjunction step retains a positive discrepancy term, as recalled below.

Theorem 83.1. *Every smooth Fano threefold in Family 3.2 is K-stable.*

The remaining double conics are treated by the normalization of the blowup of their fibre ideals. Its exceptional divisor G has $A_X(G) = 3$ and $S_X(G) = 135/56$. We compute the actual refined series on G, including all cancellations. The different is one half of a smooth curve of bidegree $(2, 1)$ and remains in the surface calculation. The refined threshold is at least one in the remaining cases. The positive difference $33/56$ then proves the theorem. No Zariski decomposition on the weighted threefold extraction is used. The volume decompositions on X in the reduction below are retained.

84 The distinguished surface and a double conic

Put

$$p : Q = \mathbb{P}_{\mathbb{P}^1\times\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-1,-1)^{\oplus 2}) \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1, \qquad \xi = \mathcal{O}_Q(1),$$

using the quotient convention. We use coordinates $[x_0 : x_1]$, $[y_0 : y_1]$ on the two factors. Write A and B for their hyperplane classes, also pulled back to Q and X. Thus $X \in |2\xi + 2A + 3B|$, and its equation has the form

$$q(y; u_1, u_2) + u_0(g_1(x, y)u_1 + g_2(x, y)u_2) + h(x, y)u_0^2 = 0,$$

where q has degrees $(1, 2)$ in y and (u_1, u_2), the g_i have bidegree $(1, 2)$ in (x, y), and h has bidegree $(2, 3)$. The classes of u_0, u_1, u_2 are respectively $\xi, \xi + A + B, \xi + A + B$. Put $D = X \cap \{u_0 = 0\}$. The classification requires D to be irreducible. Adjunction gives $-K_X = D + 2A + B$. In the notation in which F_1 is a sextic del Pezzo fibre and F_2 is a cubic fibre, we have $F_1 = B$ and $F_2 = A$.

Lemma 84.1. *The surface D is smooth and isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, and $\mathcal{O}_D(D) = \mathcal{O}(-1,-1)$. Its map to the base of the conic bundle is the product of the identity and a morphism of degree two between projective lines. In particular, this map has two distinct branch values on the second factor. Every double conic of $p|_X$ lies above one of these values, and its reduced support meets D in one point.*

Proof. On $\{u_0 = 0\}$ the equation is $y_0q_0(u_1, u_2) + y_1q_1(u_1, u_2) = 0$. If the binary quadratics q_0, q_1 had a common factor, this equation would be reducible, contrary to the hypothesis on D. They are therefore relatively prime. Consequently, the curve defined by this equation in $\mathbb{P}^1_y \times \mathbb{P}^1_{u_1,u_2}$ is the graph of the morphism $[u_1 : u_2] \mapsto [-q_1 : q_0]$, of degree two. Denote this graph by R. Thus $D = \mathbb{P}^1_x \times R$. On this product, $A|_D = \mathcal{O}(1, 0)$, $B|_D = \mathcal{O}(0, 2)$, and $(\xi + A + B)|_D = \mathcal{O}(0, 1)$, whence $\xi|_D = \mathcal{O}(-1,-1)$. A degree-two morphism between smooth projective lines in characteristic zero has two ramification points, with distinct images.

Suppose that the conic over a base point is twice a line ℓ. The line ℓ cannot be $\{u_0 = 0\}$: otherwise the corresponding member of the pencil $\langle q_0, q_1\rangle$ would vanish identically. Its restriction to $u_0 = 0$ is therefore a nonzero square. This is exactly the ramification condition for $R \to \mathbb{P}^1_y$, and the unique zero of the square is $D \cap \ell$. □

Lemma 84.2. *Let 2ℓ be a double conic. After changing coordinates, its base point is $([1:0],[1:0])$, and on the chart $x_0 = y_0 = 1$ the equation of X is*

$$z^2 + af(r,s) + bg(r,s) + \text{terms of weight greater than two} = 0,$$

$$f = s(cr + ds), \qquad g = r^2 + ers + hs^2.$$

Here $a = x_1$, $b = y_1$, $s = u_0$, $r = u_1$, the weights of (z, a, b) are $(1, 2, 2)$, and $[r : s]$ are homogeneous coordinates on ℓ. The binary quadratics f, g have no common zero, and $(c, d) \neq (0, 0)$. Either one, or both, of f, g may be a square.

The weighted initial equation in the homogeneous coordinate ring of Q is

$$F_0 = y_0 z^2 + x_1 y_0^2 u_0 (cr + dx_0 y_0 u_0) + y_1 (r^2 + e x_0 y_0 u_0 r + h x_0^2 y_0^2 u_0^2).$$

It is irreducible. The corresponding homogeneous weights are

$$\mathrm{wt}(x_0, x_1, y_0, y_1, u_0, r, z) = (0, 2, 0, 2, 0, 0, 1).$$

Proof. The two singular members of the base-point-free pencil of binary quadratics defining D are distinct squares. Hence we may arrange $q = y_0 u_2^2 + y_1 u_1^2$. Choose the first branch value to be $y_1 = 0$, and move the first coordinate of the double conic to $x_1 = 0$. Since the conic there has rank one, its equation is a nonzero multiple of $(u_2 + \lambda u_0)^2$ on $x_0 = y_0 = 1$. The homogeneous change $z = u_2 + \lambda x_0 y_0 u_0$ is permitted by the classes of these coordinates. Rescaling gives the stated coefficient of z^2.

The quadratic terms in u_1, u_2 are independent of x. Thus the coefficient of a in the initial equation is divisible by s, whereas the coefficient of b restricts to r^2 on $s = 0$. Homogeneity gives exactly the displayed form of F_0. At a point of ℓ, both fibre derivatives vanish because the fibre is a double line. The two remaining derivatives are f and g. Smoothness of X therefore implies that f, g have no common zero. In particular f is not identically zero.

Write $g_0 = r^2 + e x_0 y_0 u_0 r + h x_0^2 y_0^2 u_0^2$. Consider F_0 as a polynomial of degree one in y_1. Its leading coefficient is g_0. A common divisor of this coefficient and its constant term would divide $y_0 z^2$; since g_0 is independent of z and is not divisible by y_0, no such divisor exists. Gauss's lemma now proves irreducibility. Finally, the conditions obtained above only assert that f, g are nonzero and relatively prime. They do not assert that their individual roots are distinct. □

Example 84.3. The pairs $(f, g) = (sr, r^2 + s^2)$, $(s^2, r^2 + s^2)$, and (s^2, r^2) all satisfy the local smoothness condition along ℓ. The second has a square f, and the third has both quadratics square. These examples concern the normal form along the double conic; no assertion about the other fibres is needed. In particular, one must retain the square cases in a proof for the whole family.

Lemma 84.4. *Let $\mu : \widetilde{X} \to X$ be the normalization of the blowup of the ideal of the scheme-theoretic fibre 2ℓ, and let G be its exceptional divisor. Then μ is the weighted blowup with weights $(1, 2)$ transverse to ℓ, and*

$$A_X(G) = 3, \qquad G \cong \mathbb{P}^1_{[r:s]} \times \mathbb{P}^1_{[a:b]}.$$

The pair $(\widetilde{X}, G)$ is purely log terminal near G, and

$$(K_{\widetilde{X}} + G)|_G = K_G + \tfrac{1}{2}\Gamma, \qquad \Gamma = \{af(r,s) + bg(r,s) = 0\}.$$

The curve Γ *is smooth, has bidegree* $(2,1)$*, and is the graph of the morphism* $[r:s] \mapsto [-g(r,s) : f(r,s)]$*. Moreover,* $N_{\ell/X} \cong \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$*.*

Proof. The fibre ideal is (a,b). Wherever $g \neq 0$, the equation allows us to eliminate b, and $(a,b) = (a,z^2)$. More precisely, write the equation as $z^2 + aF + bG_1 = 0$ with G_1 a unit on this open set; then the two ideals coincide in the local ring of X. The analogous statement holds wherever $f \neq 0$. These two open sets cover ℓ. The normalization of the blowup of (a,z^2) is the weighted blowup of the regular parameters (z,a) with weights $(1,2)$. Its log discrepancy is $1+2=3$.

The weighted exceptional equation is $z^2 + af + bg = 0$, with z of weight one and a,b of weight two. Its second Veronese eliminates z^2 and leaves the two base coordinates a,b. Since a,b are coordinates on the fixed base surface, their transition functions along ℓ are trivial. Hence its coarse surface is the stated product. The weight-two chart is an index-two quotient chart; the index-two locus on G is $z=0$. Adjunction on the index-one cover therefore gives different $\Gamma/2$. The same quotient description proves that the extraction is purely log terminal. The coprimality of f,g identifies Γ with the graph of a morphism, proving its smoothness.

For the last assertion, the normal sequence of $\ell \subset X \subset Q$ is

$$0 \longrightarrow N_{\ell/X} \longrightarrow \mathcal{O}(1) \oplus \mathcal{O} \oplus \mathcal{O} \xrightarrow{(0,f,g)} \mathcal{O}(2) \longrightarrow 0.$$

The map is surjective by coprimality, and the kernel of its last two summands is $\mathcal{O}(-2)$. This proves the splitting. □

Remark 84.5. The surface G is smooth, but the different $\Gamma/2$ must be included in every discrepancy calculation on it. The ordinary blowup of ℓ has exceptional surface $\mathbb{F}_3$; this is not the extraction used above. The weighted extraction is useful because its second Veronese has the simple product description, allowing the restricted graded series to be computed directly from the homogeneous coordinate ring. This does not require a Zariski decomposition on $\widetilde{X}$.

Lemma 84.6. *Let* F_2 *be the cubic surface fibre containing* 2ℓ*. If* g *has two distinct roots, then* F_2 *has exactly two singular points on* ℓ*, both of type* A_1*. Thus the double fibres not covered by the generality condition of Belousov–Loginov are exactly those for which* g *is a square. In this remaining case one may write*

$$g = r^2, \qquad f = s(cr + ds), \qquad d \neq 0.$$

No restriction $c \neq 0$ *is justified.*

Proof. The cubic fibre is $a=0$. At a point of ℓ, its derivative in the b direction is g, so its singular points on ℓ are precisely the zeros of g. If v is a local parameter on ℓ at a simple zero, the quadratic part of its local equation in (z,b,v) is $z^2 + \kappa bv + \lambda bz + \nu b^2$, with $\kappa \neq 0$. It is nondegenerate, so the singularity is A_1. At a double zero, the quadratic part has no term involving v and is degenerate; the singularity is not A_1. These assertions concern the surface only along ℓ, as required by the cited generality condition. If g is a square, replace r by $r + \lambda s$ to obtain $g = r^2$. The form of f is preserved by this change, and coprimality gives $d \neq 0$. □

85 Reduction to the non-reduced conics

In this section we write $F_1 = B$ and $F_2 = A$, in order to match the notation of [43]. Thus F_1 is a sextic del Pezzo fibre, F_2 is a cubic del Pezzo fibre, and $-K_X \sim D + F_1 + 2F_2$. We shall use the corrected version of [43], namely arXiv:2403.03700v3, dated 27 June 2026. Its Lemma 6.3 gives a non-strict local estimate. We explain below why this estimate still excludes a divisor with non-positive beta-invariant. The generality condition in [43, Theorem 1.1] is not imposed in this section.

Lemma 85.1. *One has*

$$S_X(D) = \frac{17}{28}, \qquad S_X(F_1) = \frac{45}{112}, \qquad S_X(F_2) = \frac{45}{56}.$$

Moreover, X is divisorially stable: $\beta_X(B) > 0$ for every prime divisor B on X.

Proof. The effective cone is generated by D, F_1, F_2. The intersection numbers are

$$D^3 = 2, \qquad D^2F_1 = -2, \qquad D^2F_2 = -1, \qquad DF_1F_2 = 2, \qquad F_1^2 = F_2^2 = 0.$$

The divisor $-K_X - uD$ is nef for $0 \le u \le 1$, and is not pseudoeffective for $u > 1$. Its cube is $14 - 6u - 6u^2 - 2u^3$. Hence

$$S_X(D) = \frac{1}{14}\int_0^1 (14 - 6u - 6u^2 - 2u^3)\, du = \frac{17}{28}.$$

For the two fibres the positive parts are

$$P_1(u) = \begin{cases} D + (1-u)F_1 + 2F_2, & 0 \le u \le \frac{1}{2}, \\ 2(1-u)D + (1-u)F_1 + 2F_2, & \frac{1}{2} \le u \le 1, \end{cases}$$

$$P_2(u) = \begin{cases} D + F_1 + (2-u)F_2, & 0 \le u \le 1, \\ (2-u)D + F_1 + (2-u)F_2, & 1 \le u \le 2. \end{cases}$$

The corresponding negative parts are $0, (2u-1)D$ and $0, (u-1)D$, respectively. These decompositions follow by intersecting with the two rulings of D and with a conic fibre; equivalently, they are [43, Propositions 5.1 and 5.2]. Therefore

$$S_X(F_1) = \frac{1}{14}\left(\int_0^{1/2} (14 - 18u)\, du + \int_{1/2}^1 (24(1-u)^2 - 8(1-u)^3)\, du\right) = \frac{45}{112},$$

$$S_X(F_2) = \frac{1}{14}\left(\int_0^1 (14 - 9u)\, du + \int_1^2 (6(2-u)^2 - (2-u)^3)\, du\right) = \frac{45}{56}.$$

For completeness, the divisorial-stability reduction in [43, Section 5] uses the effective cone to reduce the verification to F_2, by the criterion of Fujita quoted there. The last computation gives $\beta_X(F_2) = 11/56 > 0$, and thus proves the stated divisorial stability. This argument does not require any condition on the singular fibres. □

Lemma 85.2. *For every point $p \in D$, one has $\delta_p(X) \ge 56/45$.*

Proof. Write $D \simeq \mathbb{P}^1 \times \mathbb{P}^1$, and denote the ruling classes by l_1, l_2. Since $D|_D = -l_1 - l_2$ and $(-K_X)|_D = l_1 + l_2$, the restriction of $-K_X - uD$ is $(1+u)(l_1 + l_2)$. Choose the ruling $L \in |l_1|$ through p. For $0 \le v \le 1 + u$, the divisor $(1 + u - v)l_1 + (1 + u)l_2$ is nef, and beyond this interval it is not pseudoeffective. There are no fixed components in these restrictions. Thus the two successive expected vanishing orders are

$$S(W^D_{\bullet,\bullet}; L) = \frac{3}{14}\int_0^1 \int_0^{1+u} 2(1 + u - v)(1 + u)\, dv\, du = \frac{45}{56},$$

$$S(W^{D,L}_{\bullet,\bullet,\bullet}; p) = \frac{3}{14}\int_0^1 \int_0^{1+u} (1 + u)^2\, dv\, du = \frac{45}{56}.$$

Abban–Zhuang adjunction for the flag $X \supset D \supset L \supset p$ and Lemma 85.1 now give

$$\delta_p(X) \ge \min\left\{\frac{28}{17}, \frac{56}{45}, \frac{56}{45}\right\} = \frac{56}{45}.$$

This also reproduces [43, Lemma 6.1]. □

Lemma 85.3. *Let C_0 be a reduced fibre of the conic bundle $\pi : X \to \mathbb{P}^1 \times \mathbb{P}^1$. Let F be a prime divisor over X whose centre is contained in C_0. Then $\beta_X(F) > 0$.*

Proof. Let p be a general point of the centre of F. If $p \in D$, apply Lemma 85.2. Hence we may assume that $p \notin D$. Let S_i be the fibre of π_i containing C_0; thus $S_i \sim F_i$. If S_2 is smooth along C_0, then [43, Lemma 6.2] gives $\delta_p(X) > 1$.

Suppose that S_2 is singular at some point of C_0. The restriction $\pi_2|_D$ is a projection $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$, so S_2 is smooth along D. Thus its singular point on C_0 is outside D. By [43, Lemma 3.5], S_1 is a smooth sextic del Pezzo surface. The conic is $C_0 = L_3 + L_4$, where L_3, L_4 are (-1)-curves meeting transversely. Moreover, $D|_{S_1} = L_1 + L_2$ is a disjoint union of two (-1)-curves. We may number them so that $L_1 L_3 = L_2 L_4 = 1$ and $L_1 L_4 = L_2 L_3 = 0$.

Let $W_{\bullet,\bullet}^{S_1}$ be the series obtained by refining $H^0(X, -mK_X)$ with respect to S_1. The calculation in the corrected [43, Lemma 6.3] proves

$$\delta_p(S_1; W_{\bullet,\bullet}^{S_1}) \geq 1.$$

We recall the relevant numbers, so that the non-strict inequality is explicit. For $p \in L_3 \setminus (D \cup L_4)$, the flag beginning with L_3 has first expected vanishing order 1 and point expected vanishing order $39/56$. The same holds with L_3, L_4 interchanged. At $p = L_3 \cap L_4$, blow up p on S_1 and denote the exceptional curve by e. Its log discrepancy is 2 and its first expected vanishing order is 2. The next expected vanishing order is $11/28$ away from the two points corresponding to L_3, L_4, and is 1 at either of these two points. These are all the terms in the surface adjunction estimate. In particular, its lower bound is 1, including at $L_3 \cap L_4$.

It is essential to retain the positive remainder in the first adjunction step. Applied to a valuation centred in S_1, the last surface estimate gives

$$S_X(F) \leq A_X(F) - (1 - S_X(S_1)) \operatorname{ord}_F(S_1).$$

Indeed, choose a basis simultaneously compatible with the S_1-filtration and ord_F, and write its basis divisor as $a_m S_1 + \Omega_m$. Then $a_m \to S_X(S_1)$ and $\Omega_m|_{S_1}$ is the corresponding refined basis divisor. The refined threshold estimate says that $(S_1, (1-\epsilon)\Omega_m|_{S_1})$ is log canonical near p for all sufficiently divisible m, with an error tending to zero. Inversion of adjunction gives

$$A_X(F) \geq \operatorname{ord}_F(S_1) + (1 - \epsilon) \operatorname{ord}_F(\Omega_m).$$

Letting m tend to infinity and then ϵ tend to zero proves the inequality. This is the strictness argument underlying the equality discussion in [43, Section 8].

Here the centre of F is contained in $C_0 \subset S_1$, so $\operatorname{ord}_F(S_1) > 0$. Lemma 85.1 therefore gives the strict inequality

$$\beta_X(F) \geq \frac{67}{112} \operatorname{ord}_F(S_1) > 0.$$

Thus the non-strict surface estimate is sufficient. □

Proposition 85.4. *If a prime divisor F over X satisfies $\beta_X(F) \leq 0$, then its centre is contained in the reduced support of a non-reduced conic fibre of π and is disjoint from D.*

Proof. The centre is neither a divisor, by Lemma 85.1, nor a subvariety meeting D, by Lemma 85.2. Suppose first that the centre Z is a curve not contracted by π_2. The general cubic fibre is smooth, and only finitely many fibres of π_2 are singular. A general point $p \in Z$ therefore lies on a smooth cubic fibre. Applying [43, Lemma 6.2] at p gives $\beta_X(F) > 0$.

Suppose next that Z is contained in a cubic fibre S_2 but is not contracted by π. The cubic fibre S_2 is normal by [43, Lemma 3.2]. The rationality argument in the proof of [43, Lemma 3.4] then shows that its singularities are Du Val. In particular, its singular locus is

finite. There are consequently only finitely many conic fibres in S_2 meeting this singular locus. A general point $p \in Z$ avoids their union. The cubic surface S_2 is smooth along the conic through this point, so [43, Lemma 6.2] again gives a contradiction.

Thus a curve centre is contained in a conic fibre. A point centre is of course contained in its conic fibre as well. If that fibre were reduced, Lemma 85.3 would give the same contradiction. This proves the assertion. □

Remark 85.5. The positivity of $\operatorname{ord}_F(S_1)$ in Lemma 85.3 is used only after the centre has been placed in a conic fibre. A horizontal curve need not have positive order along S_1; it is excluded separately in Proposition 85.4. Thus no strict inequality is inferred from a non-strict lower bound for a local delta-invariant alone.

86 The actual refined sections and their expected orders

For the exceptional divisor G, put

$$V_m^{\geq j} = \{s \in H^0(X, -mK_X) : \operatorname{ord}_G(s) \geq j\}, \qquad W_{m,j} = V_m^{\geq j}/V_m^{\geq j+1}.$$

The spaces $W_{m,j}$ are viewed as sections on G by taking initial terms. Their basis divisors are normalized by mh_m, where $h_m = h^0(X, -mK_X)$, and summed over all primary weights j. Thus, for a prime divisor E over G, the expected order $S(W; E)$ is the limit of the maximal normalized order sum of bases of these spaces. We use the refined threshold

$$\delta(G, \Gamma/2; W) = \inf_E \frac{A_{G,\Gamma/2}(E)}{S(W; E)}.$$

This is the multigraded version of the definition in Section 2.

On the coarse surface G, the coordinate z is a rational section with zero divisor $\Gamma/2$. Its square has bidegree $(2, 1)$. Consequently the primary piece of weight j has rational divisor class $(m, j/2)$. Writing $j = 2n + \varepsilon$ leaves a polynomial part of bidegree $(m - \varepsilon, n)$ and the fixed factor z^ε. The contribution of this factor to an asymptotic basis divisor is bounded by $O(1/m)$ for each fixed valuation. The different $\Gamma/2$ is a separate, fixed boundary and is never discarded.

Lemma 86.1 (The actual primary filtration)**.** *Let G be the weighted exceptional divisor of Lemma 84.4. Its filtration on the homogeneous coordinate quotient has weights*

$$\operatorname{wt}(x_0, x_1, y_0, y_1, u_0, r, z) = (0, 2, 0, 2, 0, 0, 1).$$

The associated graded ring is the quotient by F_0 in Lemma 84.2. In particular, its weights are the actual orders along G.

Proof. The initial equation is irreducible by Lemma 84.2. The associated graded ring for the indicated weight filtration is its initial quotient and is a domain. Hence the induced order is multiplicative. The initial domain contains neither x_0 nor y_0 in its defining prime ideal, so localization at x_0y_0 introduces no additional initial relations. On $x_0 = y_0 = 1$, it has orders $1, 2, 2$ on z, a, b and agrees with the divisorial order of the weighted extraction.

We also check that the homogeneous calculation computes all anticanonical sections. Write Q for the projective bundle, and retain the degrees $(2, 3, 2)$ of X and $(2, 1, 1)$ of $-K_X$. For $m \geq 2$, the direct image of $\mathcal{O}_Q(2m - 2, m - 3, m - 2)$ is a sum of line bundles on $\mathbb{P}^1 \times \mathbb{P}^1$ of degrees

$$(2m - 2 - j - k, m - 3 - j - k), \qquad j + k \leq m - 2.$$

The first degree is nonnegative and the second is at least -1. Hence their first cohomology vanishes. For $m = 1$, the relative degree is -1, and all relative cohomology vanishes. The restriction sequence consequently identifies $H^0(X, -mK_X)$ with the degree $(2m, m, m)$ part of the homogeneous quotient. This argument applies equally to the initial hypersurface, so no additional sections or cancellations are omitted. □

Lemma 86.2 (The refined surface series)**.** *Put $s = u_0$, $a = x_1$ and $b = y_1$ on the chart $x_0 = y_0 = 1$. Thus the initial relation is $z^2 + af + bg = 0$, where $f = s(cr + ds)$ and $g = r^2 + ers + hs^2$. The exceptional surface is $G = \mathbb{P}^1_{[r:s]} \times \mathbb{P}^1_{[a:b]}$. After removing the factor z^ε, the piece of primary weight $2n + \varepsilon$, where $\varepsilon \in \{0, 1\}$, is exactly*

$$\bigoplus_{k=0}^{n} a^{n-k} b^k s^{M_{m,n,k,\varepsilon}} \operatorname{Sym}^{m-\varepsilon-M_{m,n,k,\varepsilon}} \langle r, s \rangle, \qquad M_{m,n,k,\varepsilon} = \max\{0, n-m-k, 2k-m+\varepsilon\}.$$

A summand with negative exponent is zero. Equivalently, the whole refined series has a monomial basis indexed by

$$0 \le v \le m - \varepsilon, \qquad 0 \le p \le m + v, \qquad 0 \le k \le \left\lfloor \frac{m - \varepsilon + v}{2} \right\rfloor,$$

whose element is $z^\varepsilon s^v r^{m-\varepsilon-v} a^p b^k$ and whose primary weight is $2(p + k) + \varepsilon$.

Proof. Before using the relation, a spanning monomial is

$$s^i r^j z^{2h+\varepsilon} a^p b^q, \qquad i + j + 2h + \varepsilon = m, \quad 0 \le p \le m + i, \quad 0 \le q \le i.$$

Set $n = p + q + h$ and expand $z^{2h} = (-af - bg)^h$. Suppose that ℓ of the factors contribute af. The resulting exponent of b is $k = q + h - \ell$, and its binary coefficient is divisible by $s^{i+\ell}$. We have

$$n - m - k = p + \ell - m \le i + \ell,$$

and

$$2k - m + \varepsilon = 2q + 2h - 2\ell - m + \varepsilon = 2q - i - j - 2\ell \le i + \ell.$$

Thus every actual piece is contained in the displayed direct sum. The second description is simply the set of monomials in that direct sum.

These inclusions are equalities by an exact dimension count. Put $N = m + v$. For each v, the even piece contributes $(N + 1)(\lfloor N/2 \rfloor + 1)$ monomials; for $v \le m - 1$, the odd piece contributes $(N + 1)(\lfloor (N - 1)/2 \rfloor + 1)$. Pairing the two contributions gives $(N + 1)^2$. The unpaired even contribution with $v = m$ is $(2m + 1)(m + 1)$. Hence the total dimension of the containing spaces is

$$\sum_{N=m}^{2m-1} (N + 1)^2 + (2m + 1)(m + 1) = \frac{(2m + 1)(7m^2 + 7m + 6)}{6}.$$

This is $h^0(X, -mK_X)$, by Riemann–Roch and $(-K_X)^3 = 14$. The actual associated graded pieces have this same total dimension. Every inclusion is therefore an equality. In particular, the description remains valid when f or g has a double root. □

Remark 86.3 (The ample-series condition). For $n = m$ and $\varepsilon = 0$, the exact description contains

$$a^{\lceil m/2 \rceil} H^0(G, \mathcal{O}_G(m, \lfloor m/2 \rfloor)).$$

Indeed, its exponent k of b lies between zero and $\lfloor m/2 \rfloor$, and then $M_{m,m,k,0} = 0$. Along even multiples this is a complete series of an ample divisor, multiplied by a fixed section. Thus the refined graded series contains an ample series, as required in Abban–Zhuang adjunction.

Lemma 86.4 (The expected primary order)**.** *For the weighted extraction above a double conic,*

$$A_X(G) = 3, \qquad S_X(G) = \frac{135}{56}, \qquad A_X(G) - S_X(G) = \frac{33}{56}.$$

Proof. The discrepancy is the sum $1 + 2$ of the transverse weights. Let T_m be the sum of the primary weights of an adapted basis. For fixed v and ε, put $N = m + v$ and $K = \lfloor (N - \varepsilon)/2 \rfloor$. The contribution is

$$\sum_{p=0}^{N} \sum_{k=0}^{K} (2p + 2k + \varepsilon) = (N + 1)(K + 1)(N + K + \varepsilon).$$

For $m \le N \le 2m - 1$, pairing the even and odd contributions gives $3N(N+1)^2/2$, irrespective of the parity of N. The remaining contribution, with $N = 2m$ and $\varepsilon = 0$, is $3m(2m+1)(m+1)$. Therefore

$$\begin{aligned} T_m &= \frac{3}{2} \sum_{N=m}^{2m-1} N(N+1)^2 + 3m(2m+1)(m+1) \\ &= \frac{m(45m^3 + 62m^2 + 63m + 22)}{8}. \end{aligned}$$

Since $h^0(X, -mK_X) = (2m + 1)(7m^2 + 7m + 6)/6$, taking the limit of $T_m/(mh^0(X, -mK_X))$ proves the assertion. □

Example 86.5. For $m = 1$, the primary pieces of weights $0, 1, \dots, 6$ have dimensions

$$2,\ 1,\ 3,\ 1,\ 2,\ 0,\ 1.$$

The even pieces are

$$\langle r, s \rangle, \quad \langle ar, as, bs \rangle, \quad \langle a^2 s, abs \rangle, \quad \langle a^2 bs \rangle,$$

and the odd pieces are $\langle z \rangle$ and $\langle az \rangle$. Their dimensions sum to 10 and their weights sum to 24.

Remark 86.6. Ignoring bounded parity shifts, the basis is parametrized by the rational polytope

$$\{(x, y, w) : 0 \le x \le 1,\ 0 \le y \le 1 + x,\ 0 \le w \le (1 + x)/2\}.$$

Here $x = v/m$, $y = p/m$ and $w = k/m$. Its volume is $7/6$, and its barycentre is

$$\left(\frac{17}{28}, \frac{45}{56}, \frac{45}{112} \right).$$

Consequently, the refined expected orders along $s = 0$, $r = 0$, $a = 0$ and $b = 0$ are, respectively,

$$\frac{17}{28}, \qquad \frac{11}{28}, \qquad \frac{45}{56}, \qquad \frac{45}{112}.$$

The odd factor z changes normalized expectations by $O(1/m)$ for a fixed valuation. This does not remove the different $\frac{1}{2}\Gamma$, where $\Gamma = (af + bg = 0)$: the different is part of the fixed log pair on G. Neither the actual series nor its primary moment has been obtained from a Zariski decomposition on the weighted threefold extraction.

Lemma 86.7 (An exact secondary trace)**.** *Suppose that the initial equation is $z^2 + as^2 + br^2 = 0$. Give (r, s, a, b, z) secondary weights $(1, 0, 2, 0, 1)$. The sum of the secondary weights of a basis of the complete primary associated graded space is exactly*

$$2m\, h^0(X, -mK_X).$$

Thus the secondary expected order is 2, with no finite-level error.

Proof. The equation is homogeneous of secondary weight 2. In the monomial basis of Lemma 86.2, the secondary weight of $z^\varepsilon s^v r^{m-\varepsilon-v} a^p b^k$ is $m - v + 2p$. For fixed v, k, ε, the index p runs from 0 to $m + v$. Its average secondary weight is therefore

$$m - v + (m + v) = 2m.$$

Summing over the other indices gives the identity. □

87 The refined threshold on the exceptional surface

Lemma 87.1 (The square model)**.** *Suppose that the weighted initial equation and the generators of the refined series are*

$$z^2 + as^2 + br^2 = 0, \qquad r^i s^k z^j a^\alpha b^\beta, \quad i + k + j = m, \quad 0 \le \alpha \le m + k, \quad 0 \le \beta \le k.$$

Put $G = \mathbb{P}^1_{r,s} \times \mathbb{P}^1_{a,b}$ *and* $\Gamma = (as^2 + br^2 = 0)$*. Then* $\delta(G, \frac{1}{2}\Gamma; W) \ge 1$*.*

Proof. Give (r, s, a, b, z) the second weights $(1, 0, 2, 0, 1)$. Put $U = as^2$, $V = br^2$, and write $j = 2h + \varepsilon$, where $\varepsilon \in \{0, 1\}$. The primary weight is $2N + \varepsilon$, where $N = \alpha + \beta + h$; the second weight is $Q = i + j + 2\alpha$. Put

$$R = Q - \varepsilon - 2N, \qquad S = m - Q, \qquad L = \max\left\{0, \left\lceil \frac{2N + \varepsilon - Q}{2} \right\rceil \right\},$$

$$K = \max\left\{0, Q - 2m, \left\lceil \frac{L + Q - m}{2} \right\rceil \right\}, \qquad d = N - K - L.$$

The eigenspace is zero if $d < 0$, and otherwise is exactly

$$z^\varepsilon r^R s^S U^K V^L \operatorname{Sym}^d\langle U, V\rangle.$$

Indeed, a generator becomes, up to a nonzero constant, $z^\varepsilon r^R s^S U^\alpha V^\beta (U + V)^h$. The conditions on the original exponents are equivalent to

$$\beta \ge L, \qquad \alpha \ge 0, \qquad \alpha \ge Q - 2m, \qquad 2\alpha - \beta \ge Q - m, \qquad \alpha + \beta \le N.$$

They imply $\alpha \ge K$. Conversely, taking $\beta = L$ and $\alpha = K, K + 1, \dots, N - L$ gives a basis of the binary space in the displayed formula. Negative values of R or S cause no pole: the factors $U^K V^L$ make every displayed section regular. This also accounts for all cancellations among the original generators.

Here are the integrations used below. Write $q = Q/m$, $n = N/m$ and

$$l = \max\{0, n - q/2\}, \qquad k = \max\{0, q - 2, (l + q - 1)/2\}, \qquad d = n - k - l.$$

The domain is $d \ge 0$. It is the union of the following eight polygons; in the first three rows $l = 0$, and in the last five rows $l = n - q/2$.

q	n	k	d
$[0, 1]$	$[0, q/2]$	0	n
$[1, 3]$	$[(q-1)/2, q/2]$	$(q-1)/2$	$n - (q-1)/2$
$[3, 4]$	$[q-2, q/2]$	$q-2$	$n - q + 2$
$[0, 1]$	$[q/2, 1 - q/2]$	0	$q/2$
$[0, 1]$	$[1 - q/2, q/2 + 1]$	$(n + q/2 - 1)/2$	$q/4 - n/2 + 1/2$
$[1, 3]$	$[q/2, q/2 + 1]$	$(n + q/2 - 1)/2$	$q/4 - n/2 + 1/2$
$[3, 4]$	$[q/2, 3q/2 - 3]$	$q-2$	$2 - q/2$
$[3, 4]$	$[3q/2 - 3, q/2 + 1]$	$(n + q/2 - 1)/2$	$q/4 - n/2 + 1/2$

Their total mass is $\int d\, dn\, dq = 7/6$. For instance, the first polygon contributes $\int_0^1 \int_0^{q/2} n\, dn\, dq = 1/24$. Both parities have the same leading term. Thus normalized moments are $\frac{6}{7} \int d(\cdots)\, dn\, dq$.

Integrating on the eight polygons gives

curve or weight	integrand after the factor d	moment
$r = 0$	$q - 2n + 2l + d$	$11/28$
$s = 0$	$1 - q + 2k + d$	$17/28$
$a = 0$	$k + d/2$	$45/56$
$b = 0$	$l + d/2$	$45/112$
$U + \kappa V = 0$ $(\kappa \neq 0)$	$d/2$	$17/112$
primary weight	$2n$	$135/56$
Q	q	2
$m + 2N + \varepsilon - Q$	$1 + 2n - q$	$79/56$

These are maximal average orders along each prescribed invariant curve. For example, at $U + \kappa V = 0$ the maximal total order of a basis of binary forms of degree d is $d(d+1)/2$. The odd factor z contributes $O(1/m)$ to its normalized coefficient. The same factor is included exactly in the two corner weights in the last two rows.

We explain why these numerical inequalities give log canonicity. A second-weight eigenvector factors into the four coordinate curves and curves $U + \kappa V = 0$. The latter are smooth, meet only at $P = (r = a = 0)$ and $P' = (s = b = 0)$, and have contact order two there. Their intersections with the coordinate curves are also confined to these two points. At the other two coordinate corners the configuration has simple normal crossings. The limiting coefficients of the four coordinate curves are less than one. The coefficient of any other component is at most $17/112$, to which one must add $1/2$ when the component is Γ; this remains less than one.

At P use the weights $(1, 2)$ in (r, a), and at P' the weights $(1, 2)$ in (s, b). Their log discrepancies on G are three, and their orders on Γ are two. The required bounds on the basis divisor are therefore two. The last two rows give respectively two and $79/56$. These are moments of the whole divisor: adding the separate maxima in the preceding rows would not give the correct estimate. The weighted blow-ups separate the strict transforms of the curves with quadratic contact. On the smooth chart this is seen from $r = t$, $a = t^2 u$, where the strict transforms have equations $u + \kappa = 0$. The remaining chart is a quotient of a smooth surface by $\boldsymbol{\mu}_2$; its covering pair has two transverse coordinate curves, with coefficients at most one. Thus these inequalities are sufficient for log canonicity at both corners. Equivalently, if c is the coefficient of the transverse coordinate curve, then the ordinary multiplicity at the corner equals one half of the weighted order plus $c/2$, and is at most $(3+1)/2 = 2$. Thus the ordinary first blow-up also has the required inequality.

There is no rounding error at the equality at P. Lemma 86.7 gives the exact weight sum $2mh_m$ at every level, including the odd factor z. All other inequalities have a fixed positive margin, so hold for all sufficiently large m.

Finally, in each primary graded piece choose a basis compatible with an arbitrary divisorial valuation and with the second weight filtration. Taking its second initial basis gives a divisor of the form just considered. Log canonicity of that divisor together with $\Gamma/2$, and semicontinuity of log canonical thresholds, imply the same bound for the original valuation. Passing to the asymptotic threshold proves the lemma. □

Example 87.2. For $m = 1$ the square model has $h_1 = 10$. The second-weight spaces of weights $0, 1, 2, 3, 4$ all have dimension two. Their total weight is $20 = 2h_1$. The primary weights sum to 24. This illustrates both the exact corner equality and the distinction between the two weight filtrations.

Corollary 87.3. *Suppose that the weighted initial equation is*

$$z^2 + af(r, s) + bg(r, s) = 0, \qquad f = s(cr + ds), \qquad \gcd(f, g) = 1,$$

and that the generators are those in Lemma 87.1. If either f or g is a square, then $\delta(G, \frac{1}{2}\Gamma; W) \geq 1$, where $\Gamma = (af + bg = 0)$.

Proof. If f is a square, take $f = s^2$. Coprimality implies that the coefficient of r^2 in g is nonzero. Taking maximum initial weights $(r, s, a, b, z) = (1, 0, 2, 0, 1)$ gives $z^2 + as^2 + br^2$ after rescaling b. If g is a square, take $g = r^2$; then $d \neq 0$. This change of coordinates replaces r by a nonzero multiple of $r + \lambda s$ and preserves the polynomial series: expanding a generator only increases its s-exponent, so preserves both bounds $\alpha \leq m + k$ and $\beta \leq k$. Taking minimum initial weights gives $z^2 + ads^2 + br^2$, since the omitted term $acrs$ has weight three. Rescale a.

The polynomial part of each primary graded piece, after removing z^{ε}, is the monomial space of Lemma 86.2. Hence it is preserved by the second torus action. We therefore degenerate the branch curve while keeping these polynomial spaces fixed. In the second case the family is explicitly

$$\Gamma_\tau = (as^2 + br^2 + \tau acrs = 0).$$

The first case is obtained by the opposite torus action. The odd factor contributes $\varepsilon\Gamma_\tau/2$ to the zero divisor of a section, and must move with the branch curve. The coefficient of the different is $1/2$ throughout. Choose bases compatible with both filtrations in every primary graded piece. Their initial divisors lie in the square model, for which the preceding lemma gives log canonicity. Semicontinuity of log canonical thresholds for this family, including the moving branch divisor, gives the same bound on the original pair. □

88 Completion of the proof for Family 3.2

Proof of Theorem 83.1. Suppose that F is a prime divisor over X with $\beta_X(F) \leq 0$. Proposition 85.4 places its centre in the reduced support ℓ of a double conic, disjoint from D. In the normal form of Lemma 84.2, suppose first that g has distinct roots. Lemma 84.6 shows that the cubic fibre has exactly two singular points along ℓ, both of type A_1. The local calculation [43, Lemma 7.1] applies to this double conic and gives $\delta_p(X) \geq 56/45$ at every $p \in \ell \setminus D$, a contradiction. This invocation of the local lemma imposes no condition on the other double conics of X.

It remains to consider $g = r^2$ and $f = s(cr + ds)$, where $d \neq 0$. Extract G as in Lemma 84.4. Corollary 87.3 gives $\delta(G, \Gamma/2; W) \geq 1$. The strict form of Abban–Zhuang adjunction yields

$$\beta_X(F) \geq \big(A_X(G) - S_X(G)\big)\,\mathrm{ord}_F(G) = \frac{33}{56}\,\mathrm{ord}_F(G) > 0.$$

Indeed, the centre on the extraction is contained in G, because $c_X(F) \subset \ell$. To see explicitly why a refined threshold equal to one suffices, take bases compatible with the G-filtration and ord_F. Restrict the moving part of their pullbacks to G. Inversion of adjunction for the plt extraction and the refined threshold estimate give, upon taking limits,

$$S_X(F) \leq A_X(F) - A_X(G)\,\mathrm{ord}_F(G) + S_X(G)\,\mathrm{ord}_F(G).$$

This is precisely the displayed positive remainder, and contradicts the choice of F. □

Remark 88.1. The restriction to a cubic fibre is used only for the published reduced-conic estimates and for double conics with two ordinary double points. The new argument for the remaining double conics does not depend on their cubic-surface singularity type. It uses the smoothness of X, the exact graded section spaces, and the pair $(G, \Gamma/2)$. In particular, it includes both $c = 0$ and $c \neq 0$. The threefold Zariski decomposition on the weighted extraction is replaced by an exact dimension count and an exact weight sum. The different is retained throughout; the bounded odd factor in the refined sections is not a reason to omit it.

Part XIII
Family 3.5: K-semistability and the complete smooth classification

89 Geometry and the scope of the result

We prove that every smooth member of Family 3.5 is K-semistable and classify the K-stable and K-polystable members. The mixed fibre requires two additional point extractions. Their actual refined series are identified from sections and exact Hilbert counts; explicit resolutions of conic and cubic pencils then control every divisor over the exceptional surfaces. A weighted extraction of the joining line gives an independent semistability estimate.

Put $M = \mathbb{P}^1 \times \mathbb{P}^2$. Let $Q \subset \mathbb{P}^2$ be a smooth conic, and identify $S = \mathbb{P}^1 \times Q$ with $\mathbb{P}^1 \times \mathbb{P}^1$ by

$$([u : v], [x : y]) \longmapsto ([u : v], [y^2 : xy : x^2]).$$

Choose relatively prime binary quintics f, g. The curve

$$C = \{uf(x, y) + vg(x, y) = 0\} \subset S$$

is the graph of a degree-five morphism $\eta : C \cong \mathbb{P}^1 \longrightarrow \mathbb{P}^1$. Thus C is smooth even when η has high ramification. Conversely, every smooth member of Family 3.5 is obtained as

$$\pi : X = \mathrm{Bl}_C(M) \longrightarrow M$$

in this way; see [3, Sections 5.14 and 7.5]. The bidegree $(5, 2)$ of C in M means that the degrees of the two pulled-back hyperplane bundles on C are five and two.

Write $T = \pi^*\mathcal{O}_M(1, 0)$, $H = \pi^*\mathcal{O}_M(0, 1)$, let E be the exceptional divisor, and let D be the strict transform of S. We use T also for a fibre of $\phi : X \to \mathbb{P}^1$. Then

$$D = 2H - E, \qquad L = -K_X = 2T + 3H - E, \qquad L^3 = 20.$$

On $D \cong \mathbb{P}^1 \times \mathbb{P}^1$, let ℓ_1, ℓ_2 denote the pullbacks of points of its first and second factors. Notice the factor two in the restriction of H:

$$T|_D = \ell_1, \quad H|_D = 2\ell_2, \quad E|_D = \ell_1 + 5\ell_2, \quad D|_D = -\ell_1 - \ell_2, \quad L|_D = \ell_1 + \ell_2.$$

The intersection identities on X are

$$TH^2 = 1, \quad TE^2 = -5, \quad HE^2 = -2, \quad E^3 = -14;$$

products with one E and two pullbacks vanish, as do T^2 and H^3. Indeed, $\deg N_{C/M} = (-K_M) \cdot C - 2 = 14$. These identities give $L^3 = 54 - 48 + 14 = 20$.

Definition 89.1. The ramification index of η at $q \in C$ is the integer r for which, in local coordinates, η has the form $t = cx^r + O(x^{r+1})$, with $c \neq 0$. A ramification point is *simple* if $r = 2$, and is *totally ramified* if $r = 5$. The ramification divisor is $R = \sum_q (r_q - 1)q$; the Riemann–Hurwitz formula gives $\deg R = 8$.

Theorem 89.2 (The smooth Family 3.5 classification)**.** *Let X be a smooth member of Family 3.5. Then X is K-semistable. More precisely,*

1. *If η has no totally ramified point, then $\delta(X) \geq 100/99$ and X is K-stable.*

2. *If η has a totally ramified point, then X is K-semistable. If it has two such points, then X is the K-polystable member corresponding to $ux^5+vy^5=0$. If it has exactly one, then X is strictly K-semistable.*

The first assertion extends Denisova's simple-ramification theorem [14]; its proof is completed in Theorem 101.1. The second is [3, Corollary 5.71 and Lemma 7.6]; we give the degeneration and the requisite Abban–Zhuang argument below. The local estimates at index-three and index-four points are respectively $\delta_p(X)\geq 160/151$ and $\delta_p(X)=400/389$. For the mixed fibre, the index-two endpoint and the interior of its joining line satisfy $\delta_p(X)\geq 240/221$ and $\delta_p(X)\geq 32/29$, respectively. The announced work [15] is not used as an unpublished theorem.

90 Two divisor refinements

We first compute two expected vanishing orders on the original threefold. These calculations also fix the normalizations used later.

Lemma 90.1. *The divisor $B=L-T=T+3H-E$ is globally generated. The positive and fixed parts of $L-uD$ are*

u	$P_D(u)$	$N_D(u)$
$0\leq u\leq 1$	$L-uD$	0
$1\leq u\leq 3/2$	$2T+(3-2u)H$	$(u-1)E.$

For $L-uT$ they are

u	$P_T(u)$	$N_T(u)$
$0\leq u\leq 1$	$L-uT$	0
$1\leq u\leq 2$	$(2-u)B+(u-1)H$	$(u-1)D.$

The displayed positive parts are nef; the respective pseudoeffective thresholds are $3/2$ and 2.

Proof. The ideal of C is generated, after twisting by $\mathcal{O}_M(1,3)$, by the equation of S times sections of $\mathcal{O}_M(1,1)$ and by two lifts of $x(uf+vg)$ and $y(uf+vg)$ from S. This proves the assertion about B. All the proposed positive parts are nonnegative linear combinations of L, B, T and H. In the first table, the first interval interpolates between L and $2T+H$. For $u>1$, restriction to a fibre of $E\to C$ forces the component $(u-1)E$. After its removal, the remaining divisor is the pullback written in the table, and has no fixed divisorial component.

In the second table, $L-uT$ restricts to $(1-u)\ell_1+\ell_2$ on D. For $u>1$, its negative degree on the appropriate ruling forces multiplicity at least $u-1$ along D, because $D|_D=-\ell_1-\ell_2$. Removing this multiplicity gives $(2-u)B+(u-1)H$, which is globally generated when its coefficients are integral. The same conclusions for rational and real coefficients follow by taking multiples and continuity. Finally, beyond the stated thresholds the pushforward to M has a negative coefficient of H, respectively T, and cannot be effective. □

Lemma 90.2 (Separate computations of $S_X(D)$ and $S_X(T)$)**.** *One has*

$$S_X(D)=\frac{31}{40},\qquad S_X(T)=\frac{69}{80}.$$

Proof. For the first divisor, the intersection identities give

$$P_D(u)^3=\begin{cases}20-6u-6u^2-2u^3, & 0\leq u\leq 1,\\ 6(3-2u)^2, & 1\leq u\leq 3/2.\end{cases}$$

Consequently

$$S_X(D) = \frac{1}{20}\left(\int_0^1 (20 - 6u - 6u^2 - 2u^3)\,du + \int_1^{3/2} 6(3-2u)^2\,du\right) = \frac{1}{20}\left(\frac{29}{2} + 1\right) = \frac{31}{40}.$$

For the fibre divisor, the two volume polynomials are

$$P_T(u)^3 = \begin{cases} 20 - 12u, & 0 \le u \le 1, \\ (u-2)(u^2 + 2u - 11), & 1 \le u \le 2. \end{cases}$$

Thus

$$S_X(T) = \frac{1}{20}\left(\int_0^1 (20 - 12u)\,du + \int_1^2 (u-2)(u^2+2u-11)\,du\right) = \frac{1}{20}\left(14 + \frac{13}{4}\right) = \frac{69}{80}.$$

Both computations use volume on X, not on a weighted extraction. □

Lemma 90.3 (Points on the distinguished surface)**.** *For every $p \in D$, one has $\delta_p(X) \ge 80/73$.*

Proof. Take the ruling $Z \in |\ell_2|$ through p. On D, the moving parts in Lemma 90.1 have classes

$$(1+u)(\ell_1 + \ell_2) \quad (0 \le u \le 1), \qquad 2\ell_1 + (6-4u)\ell_2 \quad (1 \le u \le 3/2).$$

The fixed restriction in the second interval is $(u-1)C$. The volume of $a\ell_1 + b\ell_2$ is $2ab$ when $a, b \ge 0$. The first refinement therefore gives

$$\begin{aligned} S(W^D; Z) = {} & \frac{3}{20}\int_0^1 \int_0^{1+u} 2(1+u)(1+u-v)\,dv\,du \\ & + \frac{3}{20}\int_1^{3/2} \int_0^{6-4u} 4(6-4u-v)\,dv\,du = \frac{61}{80}. \end{aligned}$$

For the flag $D \supset Z \supset \{p\}$ the moving contribution is

$$\frac{3}{20}\left(\int_0^1 \int_0^{1+u} (1+u)^2\,dv\,du + \int_1^{3/2} \int_0^{6-4u} 4\,dv\,du\right) = \frac{69}{80}.$$

Since $C \to Q$ is an isomorphism, C meets Z transversely once. The fixed contribution is zero for $p \notin C$, and otherwise equals

$$\frac{6}{20}\int_1^{3/2} \int_0^{6-4u} 2(u-1)\,dv\,du = \frac{1}{20}.$$

All discrepancies in this smooth surface flag are one. The flag inequality, together with Lemma 90.2, now gives

$$\delta_p(X) \ge \min\left\{\frac{40}{31}, \frac{80}{61}, \frac{80}{73}\right\} = \frac{80}{73}.$$

This reproduces [3, Lemma 5.68] with the two restriction integrals and the fixed contribution written separately. □

Lemma 90.4 (Smooth fibres)**.** *If p lies on a smooth fibre T, then $\delta_p(X) > 1$. More precisely, the bound is $80/73$ on D, and at least $10/9$ away from D.*

Proof. Only $p \notin D$ remains. Set $C_T = D|_T$. The moving restrictions are $-K_T$ for $0 \le u \le 1$ and $-K_T - (u-1)C_T$ for $1 \le u \le 2$; the fixed restrictions are supported on C_T. For any divisor F over T whose centre contains p, the fixed contribution is zero. Monotonicity of volume gives

$$S(W^T; F) \le \frac{3}{10} \int_0^\infty \mathrm{vol}(-K_T - vF)\, dv = \frac{6}{5} S_T(F).$$

A smooth quartic del Pezzo surface satisfies $\delta(T) \ge 4/3$ [3, Lemma 2.12]. Hence $S(W^T; F) \le (9/10)A_T(F)$. The first refinement has expected order $69/80$, so the Abban–Zhuang inequality yields

$$\delta_p(X) \ge \min\{80/69, 10/9\} = 10/9.$$

□

91 The totally ramified cases

Lemma 91.1. *The member X_0 obtained from $ux^5 + vy^5 = 0$ is K-polystable.*

Proof. The transformations

$$([u : v], [x : y]) \longmapsto ([\lambda^{-5}u : v], [\lambda x : y])$$

and $([u : v], [x : y]) \mapsto ([v : u], [y : x])$ preserve the curve and induce an action of the reductive group $G = \mathbb{G}_m \rtimes \boldsymbol{\mu}_2$ on X_0. The induced action on the base of ϕ has no point fixed by all of G: the torus fixes the two coordinate points, and the involution interchanges them.

Let F be a G-invariant prime divisor over X_0, and let $Z = c_{X_0}(F)$. Its image under ϕ is a closed irreducible G-invariant subset of $\mathbb{P}^1$, and consequently is the whole base. Thus Z meets a smooth fibre at a point p. By Lemma 90.4, $A_{X_0}(F)/S_{X_0}(F) \ge \delta_p(X_0) > 1$. The equivariant valuative criterion for K-polystability [3, Theorem 1.22] proves the assertion. This is the Abban–Zhuang proof underlying [3, Corollary 5.71]. □

Lemma 91.2 (An explicit degeneration)**.** *If η has an index-five ramification point, then X has a smooth isotrivial degeneration over $\mathbb{A}^1$ whose central fibre is X_0. In particular, X is K-semistable.*

Proof. Choose the totally ramified source point to be $[0 : 1]$, and its image to be $[1 : 0]$. After a target coordinate change, the equation has the form

$$ux^5 + v\left(y^5 + \sum_{i=1}^{4} b_i x^i y^{5-i}\right) = 0.$$

Here the coefficient of y^5 is nonzero because f, g are coprime; the x^5 term in the second quintic has been absorbed into u. For $\lambda \in \mathbb{A}^1$ define

$$C_\lambda = \left\{ ux^5 + v\left(y^5 + \sum_{i=1}^{4} \lambda^i b_i x^i y^{5-i}\right) = 0 \right\}.$$

The two quintics defining this graph have no common zero for any λ. Thus the total curve is a smooth relative graph over $\mathbb{A}^1$, and its blow-up in $M \times \mathbb{A}^1$ is a smooth projective family. Its fibres are Fano members of Family 3.5. For $\lambda \neq 0$, the change

$$x' = \lambda x, \quad y' = y, \quad u' = \lambda^{-5}u, \quad v' = v$$

identifies C_λ with C_1, and hence identifies X_λ with X. The action

$$(\lambda, x, y, u, v) \longmapsto (s\lambda, s^{-1}x, y, s^5 u, v)$$

makes the relative anticanonically polarized family a test configuration. Its central fibre is the variety in Lemma 91.1. Openness of K-semistability, applied at the central fibre and then using the isomorphisms over $\mathbb{A}^1 \setminus \{0\}$, gives K-semistability of X. □

Proof of Theorem 89.2. The assertion without total ramification is proved in Theorem 101.1 below. The preceding lemma proves the semistability assertion. If there are two index-five points, their contributions exhaust $\deg R = 8$. The map is then equivalent to $[x : y] \mapsto [x^5 : y^5]$, and the threefold is X_0.

If there is exactly one, the ramification locus has at least three points: one further point could not contribute the remaining four without also having index five. The group of automorphisms of $\mathbb{P}^1$ preserving this finite set is finite. Using $\mathrm{Aut}(X) = \mathrm{Aut}(S, C)$ [3, Section 7.5], we conclude that $\mathrm{Aut}(X)$ is finite. Thus X is not isomorphic to X_0, which has a torus of automorphisms. The test configuration above has Donaldson–Futaki invariant zero: this is the Futaki invariant of the induced product action on its K-polystable central fibre. It is not a product test configuration, since its central fibre is not isomorphic to X. Therefore X is not K-polystable, and is strictly K-semistable. Equivalently, one may apply [3, Corollary 1.13]. □

Example 91.3. The curve $ux^5 + v(y^5 + xy^4) = 0$ gives a strictly K-semistable smooth threefold. Indeed, in the affine coordinate $z = x/y$, the map is $-z^5/(1 + z)$, and its derivative is $-z^4(5 + 4z)/(1 + z)^2$. Its ramification indices are five at 0, four at infinity, and two at $-5/4$. Their contributions are $4 + 3 + 1 = 8$. Replacing xy^4 by λxy^4 is the explicit degeneration to $ux^5 + vy^5 = 0$.

92 An audit of the singular-fibre refinement

Lemma 92.1 (The actual fibre series)**.** *For integers $0 \le j \le 2m$, the fibre refinement of $H^0(X, mL)$ has, after removal of its fixed factor, the complete restriction of the positive part in Lemma 90.1. In particular, if F is centred at a singular point of T, then*

$$S(W^T; F) = \frac{3}{20}\left(\int_0^\infty \mathrm{vol}(-K_T - vF)\,dv + \int_1^2 \int_0^\infty \mathrm{vol}(-K_T - (u-1)C_T - vF)\,dv\,du\right).$$

Proof. For $j \le m$, the restriction sequence is surjective on sections because

$$(mL - (j+1)T) - K_X = (m+1)L - (j+1)T$$

is nef and big, so Kawamata–Viehweg vanishing applies. For $m < j \le 2m$, the exact fixed component is $(j-m)D$, and the residual divisor is $P = (2m-j)B + (j-m)H$. Its restriction sequence is again surjective: $(P - T) - K_X = P + B$ is nef and big. Thus the fixed restriction is $(j-m)C_T$ and the moving restriction is complete. The curve C_T is contained in the smooth locus of T; this also follows from the local blow-up charts below. Its order at a valuation centred at a singular point of T is zero. Taking the asymptotic order sums gives the formula, including its factor $3/L^3 = 3/20$. □

Lemma 92.2 (The A_2 calculation)**.** *Suppose T has one A_2 singularity and no other singularities. There is an exceptional curve F on its minimal resolution with*

$$A_T(F) = 1, \qquad S(W^T; F) = \frac{91}{80}.$$

Consequently $\delta_p(T; W^T) \le 80/91 < 1$ at that singular point.

Proof. The resolution $\widetilde{T}$ is obtained by blowing up three distinct points of the conic, with two further successive blow-ups along the conic above the third point. Let $h, e_1, \dots, e_5$ be the line class and the total exceptional classes; their intersection matrix is $\operatorname{diag}(1,-1,-1,-1,-1,-1)$. Put

$$A = e_3 - e_4, \quad F = e_4 - e_5, \quad J = e_5, \quad R_0 = h - e_3 - e_4, \quad Q_0 = 2h - \sum_{i=1}^{5} e_i.$$

The curves A, F are the two (-2)-curves over the A_2 point. The resolution is crepant, so $A_T(F) = 1$. Moreover $Q_0 \cdot F = 0$. For $1 \le u \le 2$ write

$$P_u = -K_{\widetilde{T}} - (u-1)Q_0, \qquad P_u^2 = 5 - u^2.$$

The negative parts of $P_u - vF$ are

$$N_u(v) = \begin{cases} \frac{v}{2}A, & 0 \le v \le 2-u, \\ \frac{v}{2}A + (u+v-2)J, & 2-u \le v \le 1, \\ \frac{v}{2}A + (u+v-2)J + (v-1)R_0, & 1 \le v \le 2. \end{cases}$$

Their supports are negative definite: A, J, R_0 are disjoint and have squares $-2, -1, -1$. The proposed positive parts are orthogonal to their negative parts. To check nefness it suffices to test the negative curves

$$e_1, e_2, A, F, J, Q_0, h - e_1 - e_2, h - e_1 - e_3, h - e_2 - e_3, R_0.$$

For clarity, their intersections in the three chambers are as follows; the identical rows have been combined.

	$[0, 2-u]$	$[2-u, 1]$	$[1, 2]$
e_1, e_2	$2-u$	$2-u$	$2-u$
A	0	0	0
F	$3v/2$	$2-u+v/2$	$3-u-v/2$
J	$2-u-v$	0	0
Q_0	u	$2-v$	$2-v$
$h-e_1-e_2$	1	1	$2-v$
$h-e_1-e_3,\ h-e_2-e_3$	$1-v/2$	$1-v/2$	$1-v/2$
R_0	$1-v$	$1-v$	0

All entries are nonnegative in the relevant chamber. Squaring the positive parts gives

$$V(u,v) = \begin{cases} 5 - u^2 - \frac{3}{2}v^2, & 0 \le v \le 2-u, \\ 9 - 4u - 4v + 2uv - \frac{1}{2}v^2, & 2-u \le v \le 1, \\ \frac{1}{2}(v-2)(v+4u-10), & 1 \le v \le 2. \end{cases}$$

The endpoint positive part is $(2-u)(2h - e_1 - e_2 - e_3 - e_4)$, which has square zero. The nef conic-pencil class $2h - e_1 - e_2 - e_3 - e_4$ intersects $P_u - vF$ in $2 - v$, proving the threshold $v = 2$, including when $u = 2$. Direct integration yields

$$I(u) = \int_0^{2-u} (5 - u^2 - \tfrac{3}{2}v^2)\,dv + \int_{2-u}^{1} (9 - 4u - 4v + 2uv - \tfrac{1}{2}v^2)\,dv + \int_1^2 \tfrac{1}{2}(v-2)(v+4u-10)\,dv = \frac{u^3 - 6u^2 + 19}{3}.$$

In particular, $I(1) = 14/3$, and $S_{-K_T}(F) = 7/6$. Lemma 92.1 gives

$$S(W^T; F) = \frac{3}{20}\left(\frac{14}{3} + \int_1^2 \frac{u^3 - 6u^2 + 19}{3}\,du\right) = \frac{3}{20}\left(\frac{14}{3} + \frac{35}{12}\right) = \frac{91}{80}.$$

□

Remark 92.3 (The normalization correction). The volume polynomials above agree with [14, Appendix A.3]. However, its Corollary A.5 writes 83/80. The contribution from $0 \le u \le 1$ is $\frac{3}{5}S_{-K_T}(F) = 7/10$, rather than 3/5; the difference is $1/10 = 8/80$. This accounts exactly for the correction from 83/80 to 91/80. Either value obstructs a direct extension of that fibre-flag proof. This is a statement about the *refined surface series*, and does not prove K-instability of X.

93 Weighted point extractions and their exact first moments

Lemma 93.1 (The local extraction). *At a ramification point of index r, a neighbourhood of the unique singular point p of the corresponding fibre over that point has regular coordinates x, z, w on X in which*

$$t = zw + cx^r + O(x^{r+1}), \qquad c \neq 0.$$

The fibre has type A_{r-1} there. For $r = 3, 4, 5$, the weighted blow-up with weights $(1, 2, r-2)$ has exceptional divisor G and

$$A_X(G) = r + 1.$$

Its exceptional pair is $\mathbb{P}(1, 2, 1)$ with zero different when $r = 3$, $(\mathbb{P}^2, \frac{1}{2}\Lambda)$, where Λ is a line, when $r = 4$, and $\mathbb{P}(1, 2, 3)$ with zero different when $r = 5$.

Proof. On M, take $z = 0$ as a local equation of S, and write the graph as $t = \varphi(x)$. The ideal of C is $(z, t - \varphi(x))$. The blow-up chart containing p is $t - \varphi(x) = zw$. The other chart contains the strict transform of S and shows that $D \cap T$ avoids the singularity. The formula for t proves the singularity assertion. The discrepancy of a primitive weighted blow-up of three regular parameters is the sum of its weights. Weighted adjunction gives the claimed pairs. For weights $(1, 2, 2)$, passing to the second Veronese identifies the coarse exceptional surface with $\mathbb{P}^2$; the generic index two along $x = 0$ contributes the different $\Lambda/2$. It cannot be discarded in a stability calculation. □

Lemma 93.2 (Exact expected order of the point extraction). *For the extraction in Lemma 93.1, one has*

$$S_X(G) = \frac{91r + 25}{80}, \qquad \beta_X(G) = \frac{11(5 - r)}{80}.$$

In particular, the expected orders are 149/40, 389/80 and 6 for $r = 3$, $r = 4$ and $r = 5$, respectively.

Proof. Choose the ramification point at $[x : y] = [0 : 1]$ and its image at $[u : v] = [1 : 0]$. After scaling, write

$$f = x^r y^{5-r} + \sum_{i>r} a_i x^i y^{5-i}, \qquad g = y^5 + \sum_{i\ge 1} b_i x^i y^{5-i}.$$

Scaling x with weight one and v with weight r, and dividing the equation by its least weight, gives a flat family of centres whose central member is

$$C_* : \quad ux^r y^{5-r} + vy^5 = 0.$$

For $r < 5$, it is important to use this possibly nonreduced centre, rather than to cancel its factor y^{5-r}. Cancellation would change the Hilbert function and the expected order.

Here are the flatness and section-space details. The centres are relative Cartier divisors on the fixed smooth surface S, hence relative local complete intersections in M. Their ideals and all their powers have flat quotients over the parameter, as follows inductively from $I^a/I^{a+1} = \mathrm{Sym}^a(I/I^2)$. Thus their blow-ups form a flat family. The central blow-up X_* is normal. Near the multiple component its only new local equations have the form $ab = s^{5-r}c$, or $ab = s^{5-r}$ with a unit suppressed. For $r = 3, 4$ these are canonical threefold hypersurfaces, and in particular are klt.

For $r = 5$ the central centre is smooth and so is its blow-up. The divisor B remains globally generated by the same ideal generators as in Lemma 90.1. The fibres over $\mathbb{P}^1$ are blow-ups of curvilinear length-five subschemes of the smooth conic. Their minimal resolutions are the successive blow-ups along that conic. A plane curve of degree d distinct from the conic has sum of multiplicities along these clusters at most $2d$, by Bézout, and hence has anticanonical degree at least d. The conic itself has anticanonical degree one. The only curves of degree zero are the exceptional (-2)-chains, already contracted in X_*. Consequently L is ample on each fibre. Since $L = T + B$ is nef, its morphism contracts no curve: a contracted curve would have T-degree zero and lie in a fibre. Hence L is ample on X_*. Vanishing and base change identify the anticanonical section spaces of this flat family.

At the section p of the family, the central blow-up is smooth and the action on its regular parameters has weights $(1, 2, r-2)$. Choose the anticanonical linearization of weight zero at p. The weighted rescaling is precisely the local Rees construction for these parameters. Thus the central weight spaces identify with the actual associated graded section spaces at G. Indeed, a nonzero homogeneous central section restricts to a nonzero weighted homogeneous polynomial on this affine chart. It cannot have higher order than its weight. A weight basis lifted by base change therefore has independent initial forms and exhausts the original section space. This also rules out unaccounted cancellations.

We can now count the characters by a second, independent filtration, namely order along the strict transform D_*. Its defining section has weight zero: it is the quotient of the conic equation, of weight two, by the exceptional equation, also of weight two. For $0 \le j \le m$, the jth quotient is the complete space

$$H^0(D_*, \mathcal{O}(m + j, m + j)).$$

Completeness follows from vanishing for $(m + 1)L - (j + 1)D_*$, which is nef and big. Its dimension is $(m + j + 1)^2$. In the chosen linearization, a monomial of target degree i and source degree k has weight

$$(m - j)(r - 2) + ri + k, \qquad 0 \le i, k \le m + j.$$

The character shift $r - 2$ comes from $E|_{D_*} = C_*$: the equation of C_* has weight r, whereas the exceptional equation at p has weight two. The mean weight of this quotient is therefore

$$\frac{(3r - 3)m + (5 - r)j}{2}.$$

For $m < j \le \lfloor 3m/2 \rfloor$, the exact fixed factor is E^{j-m}. After removing it, sections are pulled back from $\mathcal{O}_M(2m, 3m - 2j)$. Restriction to the conic is surjective, so the quotient has dimension

$$(2m + 1)(6m - 4j + 1).$$

Its mean weight, including the fixed exceptional factor, is

$$rm + \frac{6m - 4j}{2} + 2(j - m) = (r + 1)m.$$

There are no further quotients. This is an exact finite-level description, not a count of an ambient space containing the sections. In particular their dimensions sum to

$$h^0(X, mL) = \frac{10m^3 + 15m^2 + 11m + 3}{3}.$$

Dividing the total character sum by $mh^0(X, mL)$ and taking the limit gives

$$\begin{aligned} S_X(G) &= \frac{3}{10}\left[\int_0^1 (1+s)^2\,\frac{(3r-3)+(5-r)s}{2}\,ds + \int_1^{3/2}(12-8s)(r+1)\,ds\right] \\ &= \frac{3}{10}\,\frac{91r+25}{24} = \frac{91r+25}{80}. \end{aligned}$$

Subtracting from $A_X(G) = r + 1$ proves the beta formula. *No Zariski decomposition on the weighted extracted threefold has been computed or assumed.* The weighted expected order is obtained from the actual graded sections. The divisor decompositions used to count them are on the central blow-up of M and on its distinguished surface; these are separate from the weighted extraction at p. □

Example 93.3 (Checking the normalization at $m = 1$). The filtration in the preceding proof has two quotients, of dimensions four and nine. Their mean weights are $3(r-1)/2$ and $r+1$. Thus $h^0(X, L) = 13$, and the total weights are 48 for $r = 3$ and 63 for $r = 4$. These finite-level ratios 48/13 and 63/13 are *not* the asymptotic invariants; their limits over all multiples are 149/40 and 389/80. At an A_2 point, the independently computed numbers also satisfy

$$3 \cdot \frac{69}{80} + \frac{91}{80} = \frac{149}{40}.$$

For $r = 5$, the character calculation gives $S_X(G) = A_X(G) = 6$, consistent with the strictly K-semistable examples above.

94 The exceptional surface at an index-three point

We now prove the exceptional-surface inequality that was missing for ramification index three. This argument applies whether the fibre has one A_2 singularity or has an A_2 and an A_1 singularity. Throughout this section p is its A_2 point, G is the exceptional divisor of the weight-$(1, 2, 1)$ blow-up of p, and $W = W^G$ is the *actual* refined series. Thus $G = \mathbb{P}(1, 2, 1)$ has zero different. We retain the normalization

$$N_m = h^0(X, mL) = \frac{10m^3 + 15m^2 + 11m + 3}{3}.$$

A basis divisor of W is the sum of the zero divisors of bases of all $W_{m,j}$, divided by mN_m. The denominator is not the dimension of any one homogeneous piece.

Theorem 94.1. *At an index-three ramification point one has*

$$\delta(G; W) \geq \frac{160}{151}, \qquad \delta_p(X) \geq \frac{160}{151} > 1.$$

We give the algebra of the refinement, its expected orders, and the log-canonical calculation as separate lemmas. In particular, finite computations of products of sections are not used to assert generation of the section ring.

Lemma 94.2 (Comparison with the actual filtered section ring)**.** *Let L be a line bundle on a smooth projective integral variety X, let $p \in X$, and let v be the monomial valuation in regular parameters at p with positive integral weights. Trivialize L near p, and put $R_m = H^0(X, mL)$. Let B be either $\operatorname{gr}_v R$ or a further associated graded algebra obtained by a weight filtration that preserves the primary v-grading. Suppose that a graded algebra $\mathcal{A} \subset B$ is generated by initial forms of actual sections, and that*

$$\dim \mathcal{A}_m = \dim R_m \qquad (m \geq 0).$$

Then $\mathcal{A} = B$. A primary-homogeneous basis of $\mathcal{A}_m$ computes the complete multiset of v-orders of an adapted basis of R_m, including all cancellations between sections.

Proof. Restriction of global sections to the completed local ring at p is injective. The weighted initial form of a product of two nonzero local functions is the product of their weighted initial forms, because the weighted polynomial ring in the regular parameters is a domain. Thus initial forms of actual sections, and their products, give the asserted algebra inclusion. Each R_m has a finite, exhaustive and separated filtration. Taking its associated graded preserves its dimension; applying a second such filtration does so again, separately in each primary degree. Consequently $\dim B_m = \dim R_m$. The inclusion and the assumed equality of dimensions give $\mathcal{A}_m = B_m$ for every m. Since the inclusion preserves primary degrees, it is an equality in each primary degree. Those dimensions are precisely the multiplicities of the jumping numbers of the original valuation filtration. This proves the last assertion. No generation or flat base-change assertion for the anticanonical ring is required. □

Lemma 94.3 (Thirteen sections and a second degeneration)**.** *In coordinates on G of weights $(1, 2, 1)$, put*

$$t = zw - x^3, \qquad c = x^2 + z, \qquad a = w + x.$$

The series W contains the degree-one sections

$$t^i\{1, x, c\} \ (0 \leq i \leq 2), \qquad a,\ xa,\ ta,\ txa.$$

Here degree one means anticanonical degree $m = 1$; the second grading is the weighted degree in x, z, w. Degenerate these spaces by the one-parameter subgroup with weights $(1, 0, 3)$ on (x, z, w), taking least *weights. The resulting algebra contains the algebra $\mathcal{A}$ generated in degree one by*

$$\{1, x, z, w, x^2\}, \quad t\{1, x, z, w, x^2\}, \quad t^2\{1, x, z\}.$$

The relation is $zw = x^3 + t$.

Proof. Use the central model from Lemma 93.2, whose centre has equation $ux^3y^2 + vy^5 = 0$. On the chart $u = y^2 = 1$ of M, write the plane coordinates as x and c and the conic equation as $z = c - x^2$. One lift of the graph equation is $t + xc$; a second lift, after multiplication by the other source coordinate, is $x(t + xc)$. On the blow-up chart put $a = (t + xc)/z$ and then $w = a - x$. This gives $t = zw - x^3$. The conic equation times the nine sections of $\mathcal{O}_M(2, 1)$ gives $t^i\{1, x, c\}$ after division by the exceptional equation. The two lifted graph equations, multiplied by the two target coordinates, give a, xa, ta, txa. These are sections of $L = 2T + 3H - E$. Their thirteen polynomials are independent, so they form the entire degree-one space.

These are also the primary initial sections on the original smooth threefold. Indeed its normalized quintics begin with x^3y^2 and y^5. Higher terms in the first lifted graph equation have primary weight greater than three; higher terms in the second have weight greater than four. Dividing by the exceptional equation, of weight two, therefore leaves the same initial forms a and xa. The target coordinate has initial form $zw - x^3$. Thus the claimed inclusion can be checked directly from the original sections, without assuming generation or base change

for the entire ring. The secondary weights make $t = zw - x^3$ homogeneous of weight three. Since both x and a occur, we may replace a by $a - x = w$; likewise we replace ta by tw. The least-weight terms of c and xa are z and x^2, respectively. This gives the thirteen displayed initial sections. Products of these initial sections belong to the initial algebra of W, proving the claimed inclusion. □

Lemma 94.4 (An exact normal basis)**.** *The degree-m space of $\mathcal{A}$ has basis*

$$x^a z^b w^c t^s,$$

where a, b, c, s are nonnegative integers satisfying

$$bc = 0, \qquad a + 2b + 2c \le 2m, \qquad s + c \le 2m, \qquad s + a + b + 2c \le 3m. \tag{35.1}$$

Its dimension is N_m. Consequently $\mathcal{A}$ is the entire secondary initial algebra of W in every degree.

Proof. First straighten a product containing both z and w. Its factors include zt^i and wt^j with $0 \le i \le 2$, $0 \le j \le 1$. Use

$$(zt^i)(wt^j) = (xt^i)(x^2t^j) + t^{i+j+1}.$$

The first term remains a product of two listed generators. The second is also a product of two such generators, since $i + j + 1 \le 4$ and $1, t, t^2$ are generators. Repeating reduces the number of zw pairs, without increasing anticanonical degree. The resulting monomials have $bc = 0$. These normal monomials are linearly independent in $\mathbb{C}[x, z, w, t]/(zw - x^3 - t)$: the single defining relation has leading monomial zw for a suitable monomial order.

We next decide which normal monomials are products of at most m generators. Use k factors of type x^2t^i and $a - 2k$ of type xt^i, together with b factors of type zt^i and c of type wt^i. Fill the remaining places with $1, t, t^2$. This is possible exactly when

$$\max(0, a + b + c - m) \le k \le \lfloor a/2 \rfloor, \qquad s \le 2m - k - c.$$

Indeed the last number is the total available t-exponent; all integers between zero and this number can be distributed among the factors. Taking the smallest allowed k gives precisely (35.1).

For completeness, count the basis. If $c = 0$, put $n = m - b$. There are $0 \le a \le 2n$, and

$$0 \le s \le \min(2m, 3m - a - b).$$

The contribution for this b is

$$(2n + 1)(2m + 1) - \frac{n(n + 1)}{2}.$$

If $b = 0$ and $c \ge 1$, put $n = m - c$; the contribution is

$$(2n + 1)(m + n + 1) - \frac{n(n + 1)}{2}.$$

Thus the total is

$$\sum_{n=0}^{m} \left((2n + 1)(2m + 1) - \frac{n(n + 1)}{2} \right) + \sum_{n=0}^{m-1} \left((2n + 1)(m + n + 1) - \frac{n(n + 1)}{2} \right)$$
$$= \frac{(m + 1)(11m^2 + 16m + 6)}{6} + \frac{m^2(3m + 1)}{2} = N_m.$$

Lemma 94.2 now identifies this algebra with the entire secondary initial algebra of the actual refinement. In particular this proves generation in every degree.

The primary weights admit the following independent, all-degree check. For a row with $0 \le s \le U$, use

$$\sum_{s=0}^{U}(a+2b+c+3s) = (U+1)(a+2b+c) + \frac{3}{2}U(U+1).$$

On the first branch put $b = m-n$, $0 \le n \le m$, and $0 \le a \le 2n$; then $U = \min(2m, 2m+n-a)$. On the second branch put $c = m-n$, $0 \le n < m$, and $0 \le a \le 2n$; then $U = \min(m+n, m+2n-a)$. Splitting both a-sums at $a = n$ gives, respectively,

$$\frac{m(m+1)(45m^2+61m+23)}{6}, \qquad \frac{m^2(59m^2+6m-5)}{12}.$$

Their sum is

$$\frac{m(149m^3+218m^2+163m+46)}{12}.$$

These are polynomial summations for every $m \ge 0$, rather than an extrapolation from finitely many Hilbert values. Dividing the last expression by mN_m recovers $S_X(G) = 149/40$. □

Lemma 94.5 (Complete binary spaces with fixed factors)**.** *Fix m, a primary degree j and a secondary weight q. Put*

$$b = \max\left(\frac{j-q}{2}, 0\right), \qquad c = \max\left(\frac{q-j}{2}, 0\right), \qquad k = q-3c.$$

If these numbers are nonnegative integers where required, set

$$\ell = \max\left(0, \left\lceil\frac{k+2b+2c-2m}{3}\right\rceil, \left\lceil\frac{k+b+2c-3m}{2}\right\rceil\right), \qquad u = \min\left(2m-c, \left\lfloor\frac{k}{3}\right\rfloor\right).$$

The component is zero unless $u \ge \ell$. Otherwise, with $d = u-\ell$ and $A = k-3u$, it is exactly

$$x^A z^b w^c t^\ell \operatorname{Sym}^d(x^3, zw). \tag{35.2}$$

In particular its dimension is $d+1$.

Proof. A normal monomial has primary degree $a+2b+c+3s$ and secondary weight $a+3c+3s$. Their difference determines $b-c$, and $bc = 0$ determines b, c. The remaining equation is $a+3s = k$. Substitution in (35.1) gives exactly $\ell \le s \le u$. The normal basis in this component is therefore

$$x^A z^b w^c t^\ell \{x^{3(d-i)} t^i : 0 \le i \le d\}.$$

Since $t = zw - x^3$, its span is (35.2). □

Example 94.6. For $m = 2$, $j = q = 3$, one has $b = c = 0$, $k = 3$, $\ell = 0$, $u = 1$. The component is the two-dimensional space

$$\langle x^3, t\rangle = \langle x^3, zw\rangle.$$

The cancellation $zw - t = x^3$ is thus built into the complete binary space. Its contribution to the maximal average vanishing at any prescribed point of the quotient pencil is $1/(2N_2)$, since the two basis orders are zero and one. It would be incorrect to count the two polynomials as independent monomials in x, z, w without this cancellation.

Lemma 94.7 (Separate expected-order computation on G)**.** *Sum over the nonzero components in (35.2), and define*

$$\alpha_m = \frac{1}{mN_m}\sum\left(A+\frac{3d}{2}\right)(d+1),\quad b_m = \frac{1}{mN_m}\sum b(d+1),\quad c_m = \frac{1}{mN_m}\sum c(d+1),$$
$$s_m = \frac{1}{mN_m}\sum \ell(d+1),\qquad h_m = \frac{1}{2mN_m}\sum d(d+1).$$

Their limits are

$$\alpha=\frac{5}{8},\qquad b=\frac{3}{16},\qquad c=\frac{11}{80},\qquad s=\frac{79}{120},\qquad h=\frac{49}{240}. \tag{35.3}$$

The two weighted-corner moments have limits

$$\alpha+3(c+s+h)=\frac{29}{8},\qquad \alpha+3(b+s+h)=\frac{151}{40}. \tag{35.4}$$

Proof. Here is an explicit integration of the normal-basis count. The two branches are $c=0$, called the z-branch, and $b=0$, called the w-branch. Their common boundary contributes only lower-order terms. Write $y=b/m$ on the first branch and $y=c/m$ on the second, and write k/m again as k. The lower and upper limits for s/m are

	L	U
z	$\max(0,(k+2y-2)/3,(k+y-3)/2)$	$\min(2,k/3)$
w	$\max(0,(k+2y-2)/3,(k+2y-3)/2)$	$\min(2-y,k/3)$.

Set $d=U-L$ in these integrals. The region is $0\le y\le 1$, $L\le U$. It has the following four strips on each branch; all integrals below are ordinary integrals $\int_0^1\int dk\,dy$.

branch	range of k	L	U
z	$[0,2-2y]$	0	$k/3$
z	$[2-2y,5+y]$	$(k+2y-2)/3$	$k/3$
z	$[5+y,6]$	$(k+y-3)/2$	$k/3$
z	$[6,7-y]$	$(k+y-3)/2$	2
w	$[0,2-2y]$	0	$k/3$
w	$[2-2y,5-2y]$	$(k+2y-2)/3$	$k/3$
w	$[5-2y,6-3y]$	$(k+2y-3)/2$	$k/3$
w	$[6-3y,7-4y]$	$(k+2y-3)/2$	$2-y$.

Integrating the indicated polynomials gives

	$\int d$	$\int(k-3U+3d/2)d$	$\int yd$	$\int Ld$	$\int d^2/2$
z	$11/6$	$9/8$	$5/8$	$193/144$	$53/144$
w	$3/2$	$23/24$	$11/24$	$41/48$	$5/16$.

(35.5)

For example, on the first strip of the z-branch, $d=k/3$, so its contributions to the first, second and last columns are

$$\int_0^1\int_0^{2-2y}\frac{k}{3}\,dk\,dy=\frac{2}{9},\quad \int_0^1\int_0^{2-2y}\frac{k^2}{6}\,dk\,dy=\frac{1}{9},\quad \int_0^1\int_0^{2-2y}\frac{k^2}{18}\,dk\,dy=\frac{1}{27}.$$

The other entries follow by substituting the displayed affine L,U in the same integrals. Floors, ceilings and boundary rows change the order sums by $O(m^3)$, whereas $mN_m=(10/3)m^4+O(m^3)$. Thus multiply the summed columns by $3/10$. The third column contributes to b on the first branch and to c on the second. This proves (35.3); substitution gives (35.4).

As an additional normalization check, the fixed x-coefficient is

$$\alpha - 3h = \frac{1}{80}.$$

Since x, z, w, t have weighted degrees $1, 2, 1, 3$, the average divisor class obtained from (35.2) has degree

$$(\alpha - 3h) + 2b + c + 3s + 6h = \frac{149}{40} = S_X(G),$$

in agreement with the independent threefold calculation. □

Lemma 94.8 (A log resolution of the pencil)**.** *On $G = \mathbb{P}(1, 2, 1)$ put*

$$X_0 = (x = 0), \quad Z_0 = (z = 0), \quad W_0 = (w = 0), \quad T_\lambda = (zw - \lambda x^3 = 0),$$

where $T_0 = Z_0 + W_0$ and $T_\infty = 3X_0$. Suppose an effective divisor has the form

$$B = aX_0 + bZ_0 + cW_0 + sT_1 + \sum_{\lambda \in \mathbb{P}^1} \nu_\lambda T_\lambda,$$

with nonnegative coefficients, finite support, $\sum \nu_\lambda = 2h$ and $\nu_\lambda \le h$. Put $\alpha = a + 3h$. The pair (G, B) is log canonical if

$$\alpha,\ b + h,\ c + h,\ s + h,\ h \le 1$$

and

$$\alpha + 3(c + s + h) \le 4, \qquad \alpha + 3(b + s + h) \le 4. \tag{35.6}$$

Proof. Away from the three coordinate vertices, the curves of this pencil are disjoint and smooth, and the displayed component bounds suffice. At $[1 : 0 : 0]$, only Z_0 and W_0 occur; they cross transversely.

At $[0 : 1 : 0]$, take the index-two cover of the chart $z = 1$. It is $\mathbb{A}^2_{x,w}$ modulo the involution $(x, w) \mapsto (-x, -w)$; the cover is quasi-étale. The curves in question are $x = 0$ and $w = \lambda x^3$. Write

$$u = a + 3\nu_\infty, \qquad v = c + s + 2h - \nu_\infty.$$

Three successive ordinary blow-ups separate the branches $w = \lambda x^3$. Their exceptional valuations have weights $(1, i)$, $i = 1, 2, 3$, in (x, w), log discrepancies $1 + i$, and orders $u + iv$ on B. The component bound gives $u \le 1$, and the first inequality of (35.6) gives

$$u + 3v = a + 3c + 3s + 6h = \alpha + 3(c + s + h) \le 4.$$

Linear interpolation gives $u + iv \le 1 + i$ for $i = 1, 2$ as well. After the third blow-up all distinct branches meet the last exceptional curve at different points; the total transform has simple normal crossings and every coefficient is at most one. Log canonicity descends through the quasi-étale cover.

The chart $w = 1$ at $[0 : 0 : 1]$ is smooth. Its curves are $x = 0$ and $z = \lambda x^3$. The identical three-blow-up calculation, with b in place of c, uses the second inequality of (35.6). This verifies every point and every exceptional divisor on a log resolution, and therefore all divisorial valuations over G. □

Lemma 94.9 (All basis divisors of the initial series)**.** *Let B_m be any basis divisor obtained by taking an arbitrary basis in each primary-degree and secondary-weight component of $\mathcal{A}$. For every $\tau < 160/151$, the pair $(G, \tau B_m)$ is log canonical for all sufficiently large m, independently of the chosen bases.*

Proof. In (35.2), remove the fixed factor. A basis of the remaining complete binary space has total degree $d(d+1)$ on the quotient $\mathbb{P}^1$. At any prescribed point the sum of its vanishing orders is at most

$$0+1+\cdots+d=\frac{d(d+1)}{2}.$$

This follows immediately from the dimensions of the subspaces with vanishing order at least i. Thus B_m has the form in Lemma 94.8, with

$$a=\alpha_m-3h_m,\quad b=b_m,\quad c=c_m,\quad s=s_m,\quad h=h_m.$$

All these coefficients are nonnegative by their definitions. In the limit the component bounds in that lemma are

$$\frac{5}{8},\quad \frac{47}{120},\quad \frac{41}{120},\quad \frac{69}{80},\quad \frac{49}{240}.$$

The corner bounds, divided by their log discrepancy four, are

$$\frac{29}{32},\qquad \frac{151}{160}.$$

The largest of these seven numbers is $151/160$. Multiplication by any fixed $\tau<160/151$ therefore makes all inequalities strict for sufficiently large m. The log-resolution lemma applies. □

Proof of Theorem 94.1. Fix a divisorial valuation v over G. In each $W_{m,j}$ choose a basis compatible with both its v-filtration and its secondary weight filtration. Such a basis exists for two filtrations; see Lemma 2.3. Its least-weight terms give a basis in each of the components of $\mathcal{A}$. Degenerating the sum of its zero divisors under the same one-parameter subgroup gives exactly the basis divisor considered in Lemma 94.9. Lower semicontinuity of log-canonical thresholds in this family gives the same lower bound for the original basis divisor. Since the basis was v-adapted, its order at v is $S_m(W;v)$. Hence, for each $\tau<160/151$ and all large m,

$$\tau S_m(W;v)\le A_G(v).$$

Taking the limit and then letting τ increase to $160/151$ proves the first assertion. Only two filtrations were split on an already fixed primary piece; no basis compatible with three unrelated filtrations is assumed.

Weighted point blow-up is a plt extraction, with zero different here. Abban–Zhuang adjunction, in the form of [3, Corollary 1.102], now yields

$$\delta_p(X)\ge\min\left\{\frac{A_X(G)}{S_X(G)},\delta(G;W)\right\}\ge\min\left\{\frac{160}{149},\frac{160}{151}\right\}=\frac{160}{151}.$$

We have not computed a Zariski decomposition on the weighted blow-up of the threefold. The normal basis (35.1), its exact Hilbert count and the complete binary spaces (35.2) identify the actual refinement. The only further blow-ups in this proof are the explicit surface blow-ups in Lemma 94.8. □

Example 94.10 (An A_2+A_1 fibre). For the graph of

$$[x:y]\longmapsto[x^3y^2:x^5+y^5],$$

the fibre over zero has an A_2 point and an A_1 point. The theorem gives $\delta_p(X)\ge160/151$ at the A_2 point. The proof retains the factor y^2 in the central graph and consequently does not require the other length-two cluster in the fibre to split into separate points. This local assertion alone does not settle all other centres on that fibre.

95 The exceptional surface at an index-four point

Let p be the A_3 point of a fibre whose ramification index is four. We extract p with weights $(1,2,2)$. Its exceptional pair is $(G,\Delta)=(\mathbb{P}^2,\Lambda/2)$, where Λ is a line. Write $W=W^G$ for the actual refined series. The different is essential: the two points where the initial fibre conic meets Λ have log discrepancy $1/2$ after adjunction to that conic.

Theorem 95.1. *For the weight-$(1,2,2)$ extraction at an index-four point,*

$$\delta(G,\Lambda/2;W)\geq\frac{80}{77},\qquad \delta_p(X)=\frac{400}{389}>1.$$

Lemma 95.2 (The conic flag and its complete quotients)**.** *Choose native coordinates x,z,w of weights $(1,2,2)$, and put*

$$c=x^2+z,\qquad t=zw-c^2.$$

On the coarse exceptional plane use coordinates

$$U=z,\qquad V=x^2,\qquad W_0=w-z-2x^2.$$

Then the initial fibre is the smooth conic

$$B=(UW_0-V^2=0),\qquad \Lambda=(V=0).$$

Parametrize B by $[U:V:W_0]=[p^2:pq:q^2]$, and write

$$P=(p=0),\qquad Q=(q=0),\qquad R=(p+q=0).$$

Thus $\Delta|_B=(P+Q)/2$.

Fix $m\geq 1$. In the primary weight-j piece $W_{m,j}$, let k be the order along B. Put $n=j-4k$ and $\varepsilon\equiv n \pmod 2$, $0\leq\varepsilon\leq 1$. Set

$$q_k=\begin{cases} m, & 0\leq k\leq m,\\ 2m-k, & m\leq k\leq 2m,\end{cases}$$

and

$$a=\max(\lfloor n/2\rfloor-q_k,0),\quad b=\max(\lfloor n/2\rfloor-m,0),\quad c_0=\max(n-m,0),\quad d=n-\varepsilon-a-b-c_0.$$

For $0\leq k\leq 2m$ and $n\geq 0$, the nonzero quotient is precisely

$$\xi^{\varepsilon}p^aq^b(p+q)^{c_0}H^0(\mathbb{P}^1,\mathcal{O}(d)),\tag{35.7}$$

provided $d\geq 0$. Here $\xi^2=pq$: this notation records the fractional fixed divisor $\varepsilon(P+Q)/2$, using the index covers. It does not assert that pq has a square root in the function field of B. Each such quotient has dimension $d+1$; all other quotients are zero.

Proof. Use the homogeneous central model in Lemma 93.2, with centre $y(ux^4+vy^4)=0$. Its affine blow-up chart has equation $t=zw-(x^2+z)^2$. The change of coordinates in the statement gives $t=UW_0-V^2$. The primary weight spaces on this model are the actual associated graded spaces of the original point filtration, as proved in that lemma. The defining section t has weight four. Consequently the filtration by its order preserves the weight decomposition, and taking weight spaces commutes with taking the successive t-quotients. No interchange of two arbitrary filtrations is being assumed.

The fibre $t=0$ is the plane blown up in a length-four cluster along a smooth conic and one further point of that conic. On its minimal resolution use $h,e_1,\dots,e_5$, where $e_1,\dots,e_4$

are the total transforms for the successive cluster points and e_5 is the other point. The middle exceptional curve is $F = e_2 - e_3$. The restriction and vanishing argument used for Lemma 92.1 applies to the central model as well: it is a canonical Fano threefold, and the same nef divisors occur. After removing the fixed conic, which is disjoint from F, the kth fibre quotient is the complete space of plane forms of degree d_k, with multiplicity q_k at all five cluster points, where

$$d_k = \begin{cases} 3m, & k \le m, \\ 5m - 2k, & k \ge m. \end{cases}$$

$d_k - q_k = 3m - k$ when $k \ge m$, whereas it is $2m$ when $k \le m$; also $d_k - 2q_k = m$ in both intervals.

Here is a direct calculation of the F-initial spaces. Use affine plane coordinates x, c, with conic $z = c - x^2 = 0$, and place the separate point at $[0 : 0 : 1]$ in coordinates $[1 : x : c]$. The length-four cluster ideal is (z, x^4); its q_kth power is complete. A plane monomial $x^i c^b$ must satisfy $i + b \le d_k$ and $b \le d_k - q_k$. The first condition bounds its degree, and the second is exactly its multiplicity condition at $[0 : 0 : 1]$. The cluster ideal is homogeneous for weights $(1, 2)$ on (x, c), so these conditions split into homogeneous pieces.

The valuation of F has weights $(1, 2)$ on (x, z). A form contributing order n after division by the cluster exceptional factor therefore has weight $N = n + 2q_k$. Put $l = \lfloor N/2 \rfloor$. Write such a form as $x^\varepsilon (x^2)^l f(c/x^2)$. The bounds just given say that the exponents of f lie between

$$b_- = \max(0, N - d_k), \qquad b_+ = \min(l, d_k - q_k).$$

Membership in $(z, x^4)^{q_k}$ additionally says that $f(h)$ is divisible by $(h - 1)^M$, where

$$M = \max(0, 2q_k - l).$$

Indeed a term $x^{N-2i} z^i$ belongs to this ideal precisely when $N + 2i \ge 4q_k$.

Since h and $h - 1$ have distinct zeros, the resulting polynomial space is exactly

$$h^{b_-} (h - 1)^M \mathbb{C}[h]_{\le b_+ - b_- - M}.$$

On B one has $z = p^2$, $x^2 = pq$, $c = p(p + q)$. Dividing by the cluster factor $z^{q_k} = p^{2q_k}$ gives

$$\xi^\varepsilon p^{l - 2q_k + M} q^{l - b_+} (p + q)^{b_-} H^0(\mathbb{P}^1, \mathcal{O}(b_+ - b_- - M)).$$

Substituting $d_k - 2q_k = m$ reduces its exponents to (35.7). This proves completeness and the dimension statement in every degree. □

Example 95.3. Take $m = 2$, $k = 0$, $n = 4$. Then $a = b = 0$, $c_0 = 2$, $d = 2$, so the quotient is $(p + q)^2 H^0(\mathbb{P}^1, \mathcal{O}(2))$. The maximal sums of basis orders at P, Q, R are respectively $3, 3, 9$: the first two are $0 + 1 + 2$, while the last is $2 + 3 + 4$. For $n = 3$ the quotient has the additional fixed divisor $(P + Q)/2$. This is why the parity term appears in the finite-level formulas, even though its contribution tends to zero after dividing by mN_m.

Lemma 95.4 (The conic and point moments, computed separately)**.** *For refinement along B one has*

$$S(W; B) = \frac{69}{80}.$$

Writing W^B for the resulting series on B, its point moments are

$$\begin{gathered} S(W^B; P) = \frac{77}{160}, \qquad S(W^B; Q) = \frac{27}{64}, \qquad S(W^B; R) = \frac{9}{10}, \\ S(W^B; O) = \frac{25}{64} \quad (O \notin \{P, Q, R\}). \end{gathered} \tag{35.8}$$

The average degree of the residual plane divisors after removing their B-components is $113/160$.

Proof. Use the quotients (35.7). Their contributions to the order sums at B, P, Q, R, O are respectively

$$k(d+1), \quad (a+\varepsilon/2+d/2)(d+1), \quad (b+\varepsilon/2+d/2)(d+1), \quad (c_0+d/2)(d+1), \quad d(d+1)/2.$$

These are the maximal sums for complete binary spaces, with the fixed factors included. Sum them and divide by mN_m.

We spell out the integrals. Put $u = k/m$, $v = n/m$ and

$$q(u) = \begin{cases} 1, & 0 \le u \le 1, \\ 2-u, & 1 \le u \le 2, \end{cases}$$

$$A = (v/2 - q(u))_+, \quad C = (v-1)_+, \quad B_0 = (v/2-1)_+, \quad D = v - A - B_0 - C,$$

where $r_+ = \max(r, 0)$. Integrate over $D \ge 0$, $v \ge 0$, $0 \le u \le 2$. Floors and parity change the sums by $O(m^3)$. The following are the unnormalized integrals, with measure $dv\,du$:

	D	uD	$(A+D/2)D$	$(B_0+D/2)D$	$(C+D/2)D$	$D^2/2$	$(4u+v)D$
$0 \le u \le 1$	2	1	11/12	11/12	2	5/6	7
$1 \le u \le 2$	4/3	15/8	11/16	47/96	1	15/32	221/24.

(35.9)

For the first row, split the v-interval at $1, 2, 3$; the values of D are $v, 1, 3-v$. For the second row, use $q = 2-u$ and the following five regions:

q	v	D
$[0,1]$	$[0, \min(1,2q)]$	v
$[1/2,1]$	$[1, 2q]$	1
$[0,1/2]$	$[2q, 1]$	$v/2+q$
$[0,1]$	$[\max(1,2q), 2]$	$1+q-v/2$
$[0,1]$	$[2, 2+q]$	$2+q-v.$

These regions give each entry by integrating a quadratic polynomial. For example the last column of (35.9) sums to 389/24, while the first sums to 10/3. Multiplication by 3/10, the leading normalization of $1/(mN_m)$, gives $S_X(G) = 389/80$ again. The second column gives 69/80. Columns three through six give (35.8). Thus the computation also checks the normalization independently of the threefold character sum.

On the coarse plane a primary-degree-j section has divisor degree $j/2$. Consequently, after removing the conic components, the average degree is

$$\frac{1}{2}S_X(G) - 2S(W;B) = \frac{389}{160} - \frac{138}{80} = \frac{113}{160}.$$

□

Lemma 95.5 (Controlling points off the conic)**.** *Let $D_m = e_m B + R_m$ be a basis divisor of W compatible with order along B. Thus $R_m \ge 0$ and $B \not\subset \operatorname{Supp}(R_m)$. Then*

$$e_m \longrightarrow 69/80, \qquad \deg R_m \longrightarrow 113/160, \qquad \operatorname{coeff}_\Lambda(R_m) \le 27/64 + o(1),$$

where the error bound is independent of the chosen compatible basis. For every $\tau < 80/77$, the pair $(G, \Lambda/2 + \tau D_m)$ is log canonical off B for all large m.

Proof. The first two assertions follow from the preceding lemma. Restriction of the residual sections to B gives bases of the quotients (35.7). Since Λ and B meet transversely at Q,

$$\operatorname{coeff}_\Lambda(R_m) \le \operatorname{mult}_Q(R_m|_B) \le S_m(W^B; Q).$$

This proves the uniform third assertion; it does not require a basis compatible with a third filtration.

Write $R_m = b_m\Lambda + R'_m$, where R'_m has no Λ-component. For all large m,

$$\frac{1}{2} + \tau b_m < 1, \qquad \tau \deg R_m < 1.$$

Off $B \cup \Lambda$ the boundary is $\tau R'_m$. Its degree is less than one, so it is log canonical: at any point its multiplicity is at most its degree, and the standard multiplicity bound gives $\operatorname{lct}_q(G; R'_m) \geq 1/\operatorname{mult}_q(R'_m)$. Along $\Lambda \setminus B$, the divisor $\tau R'_m|_\Lambda$ has degree less than one and hence all coefficients less than one. Inversion of adjunction gives log canonicity of $(G, \Lambda + \tau R'_m)$ there. Decreasing the Λ-coefficient to $1/2 + \tau b_m$ proves the assertion. □

Proof of Theorem 95.1. Fix $\tau < 80/77$. For every basis divisor $D_m = e_m B + R_m$ compatible with B, the coefficients of

$$(\Lambda/2)|_B + \tau R_m|_B$$

are at most one for all large m, by (35.8). The bounds at P, Q, R and the other points are uniform over the choice of basis. The restrictive inequality is the one at P:

$$\frac{1}{2} + \tau\frac{77}{160} < 1.$$

Also $\tau e_m < 1$. Inversion of adjunction along the smooth conic therefore gives log canonicity of $(G, \Lambda/2 + \tau D_m)$ near B. Lemma 95.5 proves it on the complement.

For any divisorial valuation over G, choose in each primary piece a basis compatible with its filtration and the B-filtration. The resulting order is the maximal basis order for that valuation. The preceding global log-canonicity statement, followed by $m \to \infty$ and $\tau \to 80/77$, proves $\delta(G, \Lambda/2; W) \geq 80/77$. Abban–Zhuang adjunction for the point extraction gives

$$\delta_p(X) \geq \min\left\{\frac{5}{389/80}, \frac{80}{77}\right\} = \frac{400}{389}.$$

The exceptional divisor G itself gives the opposite inequality, so equality holds.

No Zariski decomposition on the weight-$(1,2,2)$ blow-up of X has been used. The commuting homogeneous fibre filtration and the plane ideal $(c - x^2, x^4)^{q_k}$ give the exact section spaces. Their complete binary quotients control the conic; the residual plane degree controls its complement. Keeping the different $\Lambda/2$ is necessary at both stages. □

96 Other centres on the singular fibres

The point extractions do not by themselves cover the smooth points of singular fibres. We now make that coverage explicit. The only points left by the following arguments lie on one specified line in a fibre of type $A_2 + A_1$; the subsequent extractions treat these points.

Lemma 96.1 (Fibres with only nodes)**.** *If the fibre through p has type A_1 or $2A_1$, then $\delta_p(X) \geq 100/99$.*

Proof. Points on D are covered by Lemma 90.3. For the other points, the proof in [14, Section 2 and Appendix A] is local to this fibre: its use does not require other fibres to have only nodes. Here is its numerical last step. For those points where $\delta_p(T) \leq 6/5$, the polarized surface estimate in that appendix bounds $S(W^T; F)/A_T(F)$ above by

$$\frac{3}{5} + \frac{1}{20}\left(\int_1^{1.356}(16 + 3u - 9u^2 + 2u^3)\,du + \int_{1.355}^{2}(11 - u^3)\,du\right) = \frac{78409391764017}{80000000000000} < \frac{99}{100}.$$

The intervals overlap, giving an upper bound: the actual switching point lies between 1.355 and 1.356. For the remaining points the surface bounds in the same appendix and $S(W^T;F) \le (6/5)S_T(F)$ are stronger. Combining this with $S_X(T) = 69/80$ gives the asserted bound by adjunction. □

Lemma 96.2 (The remaining surface configurations)**.** *Suppose the ramification fibre has multiplicities* $(3,1,1)$*,* $(4,1)$ *or* $(3,2)$*. Its minimal resolution is the plane blown up at the five points of the indicated clusters on a smooth conic. Write*

$$E_{ij} = e_i - e_j, \qquad \ell_{ij} = h - e_i - e_j, \qquad Q_0 = 2h - e_1 - e_2 - e_3 - e_4 - e_5.$$

The last class is the strict transform of $D|_T$*. The negative curves are exactly the following:*

clusters	(-2)*-curves*	(-1)*-curves*
$(3,1,1)$	E_{12}, E_{23}	$e_3, e_4, e_5, \ell_{12}, \ell_{14}, \ell_{15}, \ell_{45}, Q_0$
$(4,1)$	E_{12}, E_{23}, E_{34}	$e_4, e_5, \ell_{12}, \ell_{15}, Q_0$
$(3,2)$	E_{12}, E_{23}, E_{45}	$e_3, e_5, \ell_{12}, \ell_{14}, \ell_{45}, Q_0$.

At a smooth point away from the following lines, one has $\delta_p(T) \ge 4/3$*:*

$(3,1,1)$	$e_3, \ell_{12}, \ell_{14}, \ell_{15}$
$(4,1)$	$e_4, \ell_{12}, \ell_{15}$
$(3,2)$	$e_3, e_5, \ell_{12}, \ell_{14}$.

(35.10)

Here a line denotes its image on T*.*

Proof. The cluster description follows by resolving the blow-up of the curvilinear subscheme cut out on the conic. Its only roots are the listed successive differences. No further root comes from a plane line meeting the conic in length three, by Bézout. Among the usual sixteen line classes e_i, $h - e_i - e_j$ and $2h - \sum e_i$, precisely the classes having nonnegative intersection with these roots remain irreducible. This gives the first table. In particular the $(4,1)$ configuration is the A_3 quartic surface with five lines. The local bounds in (35.10) are the smooth-point entries of [12, Tables 26, 27 and 29]. The excluded lines are exactly the lines meeting the exceptional root curves; outside them the listed values are 4/3, 18/13 or 3/2. □

Lemma 96.3 (Line refinements)**.** *Let* p *be a smooth point of* T *outside* D*, lying on a line* Z *from (35.10). The moments needed for the flag* $T \supset Z \supset p$ *are bounded as follows. The fourth column is the moving point contribution; the last column includes every possible fixed contribution at such a point.*

clusters	Z	$S(W^T;Z)$	*moving*	$S(W^{T,Z};p)$ *at most*
$(3,1,1),(3,2)$	e_3	73/80	159/320	159/320
$(3,1,1),(3,2)$	ℓ_{12}	127/160	389/720	21/32
$(3,1,1)$	ℓ_{14}, ℓ_{15}	257/320	127/240	229/320
$(4,1)$	e_4	77/80	149/320	149/320
$(4,1)$	ℓ_{12}	9/10	39/80	39/80
$(4,1)$	ℓ_{15}	27/32	79/160	229/320
$(3,2)$	e_5	69/80	83/160	21/32
$(3,2)$	ℓ_{14}	83/80	47/120	47/120.

(35.11)

Every row except the last gives $\delta_p(X) > 1$*.*

Proof. Work on the minimal resolution, which is an isomorphism near p. For $1 \le u \le 2$ put

$$D_u = -K_{\widetilde{T}} - (u-1)Q_0.$$

Let $D_u - vZ = P(u,v) + N(u,v)$ be its Zariski decomposition, and put $V = P(u,v)^2$, $I = P(u,v)\cdot Z$. The chamber tables below give N explicitly. They use the intersection matrix

$$h^2 = 1, \qquad e_i^2 = -1, \qquad h\cdot e_i = e_i\cdot e_j = 0 \quad (i \neq j).$$

The positive part is recovered as $P = D_u - vZ - N$.

Define, for each polynomial given on these chambers,

$$\mathcal{J}(f) = \frac{3}{20}\left(\int f(1,v)\,dv + \int_1^2 \int f(u,v)\,dv\,du\right),$$

where the integrals run over the displayed chambers. The first term accounts for the full interval $0 \le u \le 1$ of the original fibre filtration, where its moving part is $-K_T$. Then the first and moving columns in (35.11) are

$$S(W^T; Z) = \mathcal{J}(V), \qquad S_{\mathrm{mov}}(W^{T,Z}; p) = \mathcal{J}(I^2).$$

If a negative curve A meets Z at p, add $\mathcal{J}(2I\,\mathrm{coeff}_A N)$. The fixed component $(u-1)D|_T$ makes no contribution because $p \notin D$. Root curves meet Z only above singular points, which are excluded in this lemma. At a smooth point at most two lines of a quartic del Pezzo surface meet: three such lines would force both defining quadrics to contain the tangent plane. Thus at most one other line contributes to the fixed term.

For $Z = \ell_{12}$ in the $(3,1,1)$ and $(3,2)$ cases the only possible additional curve is ℓ_{45}, giving $21/32$. For $Z = \ell_{14}$ in the $(3,1,1)$ case it is e_4, giving $229/320$; interchange indices $4,5$ for ℓ_{15}. For $Z = \ell_{15}$ in the $(4,1)$ case it is e_5, again giving $229/320$. For $Z = e_5$ in the $(3,2)$ case it is ℓ_{45}, giving $21/32$. The remaining rows have no fixed contribution at a smooth point outside D.

Here is one complete integration as a check. For $(4,1)$ and $Z = e_4$,

$$N = \frac{v}{4}E_{12} + \frac{v}{2}E_{23} + \frac{3v}{4}E_{34} + (v-u)_+ Q_0, \qquad 0 \le v \le 2.$$

For $v \le u$ one has

$$V = 5 - u^2 + 2uv - v^2/4 - 4v, \qquad I = 2 - u + v/4;$$

for $v \ge u$ one has

$$V = 5 - 4v + 3v^2/4, \qquad I = 2 - 3v/4.$$

Consequently

$$\int_0^2 V\,dv = \frac{12 - u^3}{3}, \qquad \int_0^2 I^2\,dv = \frac{7u^3 - 24u^2 + 42}{12}.$$

Their value at $u = 1$ plus their integral over $1 \le u \le 2$, multiplied by $3/20$, gives $77/80$ and $149/320$. The other rows are obtained by exactly the same polynomial integrations from the tables.

To verify that the tables give Zariski decompositions, the coefficients of N are nonnegative on each listed polygon, its support has negative definite intersection matrix, and P is orthogonal to that support. Moreover P has nonnegative intersection with every curve in Lemma 96.2. These intersections are affine functions of (u,v), so it suffices to substitute the displayed vertices. The cone of curves of these weak del Pezzo surfaces is generated by their negative curves; hence P is nef. The polygons fill the region of positive volume, and $P^2 = 0$ at its upper boundary. Monotonicity makes the volume zero beyond that boundary. These facts justify both the chamber description and the limits of integration.

Finally, apply the surface flag inequality and then fibre adjunction:

$$\delta_p(X) \ge \min\left\{\frac{80}{69}, \frac{1}{S(W^T;Z)}, \frac{1}{S(W^{T,Z};p)}\right\} > 1$$

for the first seven rows. The last row supplies no such conclusion. □

The following chamber data make the arithmetic in (35.11) independently checkable. A polygon is the convex hull of its listed vertices in the (u, v)-plane. Write $P = D_u - vZ - N$. The notation is local to these surface tables; no decomposition on an extracted threefold is involved.

$(3,1,1)$ or $(3,2)$, $Z = e_3$

Polygon	Negative part N
$\mathrm{Conv}\{(1,0),(2,0),$ $(2,\frac{3}{2}),(\frac{3}{2},\frac{3}{2}),$ $(1,1)\}$	$\frac{v}{3}E_{12} + \frac{2v}{3}E_{23}$
$\mathrm{Conv}\{(2,\frac{3}{2}),(2,2),$ $(\frac{3}{2},\frac{3}{2})\}$	$(v-1)E_{12} + (2v-2)E_{23}$ $+(2v-3)\ell_{12}$
$\mathrm{Conv}\{(1,\frac{3}{2}),(1,1),$ $(\frac{3}{2},\frac{3}{2})\}$	$\frac{v}{3}E_{12} + \frac{2v}{3}E_{23}$ $+(-u+v)Q_0$
$\mathrm{Conv}\{(2,2),(1,2),$ $(1,\frac{3}{2}),(\frac{3}{2},\frac{3}{2})\}$	$(v-1)E_{12} + (2v-2)E_{23}$ $+(2v-3)\ell_{12} + (-u+v)Q_0$

$(3,1,1)$ or $(3,2)$, $Z = \ell_{12}$

Polygon	Negative part N
$\mathrm{Conv}\{(1,0),(2,0),$ $(\frac{4}{3},1),(1,1)\}$	$\frac{v}{3}E_{12} + \frac{2v}{3}E_{23}$
$\mathrm{Conv}\{(2,0),(2,1),$ $(\frac{4}{3},1)\}$	$(u+v-2)E_{12} + (2u+2v-4)E_{23}$ $+(3u+2v-6)e_3$
$\mathrm{Conv}\{(1,\frac{3}{2}),(1,1),$ $(\frac{4}{3},1)\}$	$\frac{v}{3}E_{12} + \frac{2v}{3}E_{23}$ $+(v-1)\ell_{45}$
$\mathrm{Conv}\{(2,1),(1,2),$ $(1,\frac{3}{2}),(\frac{4}{3},1)\}$	$(u+v-2)E_{12} + (2u+2v-4)E_{23}$ $+(3u+2v-6)e_3 + (v-1)\ell_{45}$

$(3,1,1)$, $Z = \ell_{14}$

Polygon	Negative part N
$\mathrm{Conv}\{(1,0),(2,0),$ $(1,1)\}$	$\frac{2v}{3}E_{12} + \frac{v}{3}E_{23}$
$\mathrm{Conv}\{(2,0),(\frac{3}{2},\frac{3}{2}),$ $(1,\frac{3}{2}),(1,1)\}$	$\frac{2v}{3}E_{12} + \frac{v}{3}E_{23}$ $+(u+v-2)e_4$
$\mathrm{Conv}\{(2,0),(2,1),$ $(\frac{3}{2},\frac{3}{2})\}$	$(u+v-2)E_{12} + (2u+v-4)E_{23}$ $+(3u+v-6)e_3 + (u+v-2)e_4$
$\mathrm{Conv}\{(1,2),(1,\frac{3}{2}),$ $(\frac{3}{2},\frac{3}{2})\}$	$(2v-2)E_{12} + (v-1)E_{23}$ $+(u+v-2)e_4 + (2v-3)\ell_{15}$

$(4,1)$, $Z = e_4$

Polygon	Negative part N
$\mathrm{Conv}\{(1,0),(2,0),$ $(2,2),(1,1)\}$	$\frac{v}{4}E_{12} + \frac{v}{2}E_{23}$ $+\frac{3v}{4}E_{34}$
$\mathrm{Conv}\{(2,2),(1,2),$ $(1,1)\}$	$\frac{v}{4}E_{12} + \frac{v}{2}E_{23}$ $+\frac{3v}{4}E_{34} + (-u+v)Q_0$

$(4,1)$, $Z = \ell_{12}$

Polygon	Negative part N
$\mathrm{Conv}\{(1,0),(2,0),$ $(1,2)\}$	$\frac{v}{2}E_{12} + vE_{23}$ $+\frac{v}{2}E_{34}$
$\mathrm{Conv}\{(2,0),(2,1),$ $(1,2)\}$	$(u+v-2)E_{12} + (2u+2v-4)E_{23}$ $+(3u+2v-6)E_{34} + (4u+2v-8)e_4$

$(4,1)$, $Z = \ell_{15}$

Polygon	Negative part N
$\mathrm{Conv}\{(1,0),(2,0),$ $(1,1)\}$	$\frac{3v}{4}E_{12} + \frac{v}{2}E_{23}$ $+\frac{v}{4}E_{34}$
$\mathrm{Conv}\{(2,0),(\frac{3}{2},2),$ $(1,2),(1,1)\}$	$\frac{3v}{4}E_{12} + \frac{v}{2}E_{23}$ $+\frac{v}{4}E_{34} + (u+v-2)e_5$
$\mathrm{Conv}\{(2,0),(2,1),$ $(\frac{3}{2},2)\}$	$(u+v-2)E_{12} + (2u+v-4)E_{23}$ $+(3u+v-6)E_{34} + (4u+v-8)e_4$ $+(u+v-2)e_5$

$(3,2)$, $Z = e_5$

Polygon	Negative part N
$\mathrm{Conv}\{(1,0),(2,0),$ $(2,1),(1,1)\}$	$\frac{v}{2}E_{45}$
$\mathrm{Conv}\{(2,1),(2,2),$ $(1,1)\}$	$\frac{v}{2}E_{45} + (v-1)\ell_{45}$
$\mathrm{Conv}\{(2,2),(1,2),$ $(1,1)\}$	$\frac{v}{2}E_{45} + (v-1)\ell_{45}$ $+(-u+v)Q_0$

$(3,2)$, $Z = \ell_{14}$

Polygon	Negative part N
$\mathrm{Conv}\{(1,0),(2,0),$ $(1,2)\}$	$\frac{2v}{3}E_{12} + \frac{v}{3}E_{23}$ $+\frac{v}{2}E_{45}$
$\mathrm{Conv}\{(2,0),(1,3),$ $(1,2)\}$	$\frac{2v}{3}E_{12} + \frac{v}{3}E_{23}$ $+(u+v-2)E_{45} + (2u+v-4)e_5$
$\mathrm{Conv}\{(2,0),(2,1),$ $(1,3)\}$	$(u+v-2)E_{12} + (2u+v-4)E_{23}$ $+(u+v-2)E_{45} + (3u+v-6)e_3$ $+(2u+v-4)e_5$

Theorem 96.4 (Global coverage outside the mixed fibres)**.** *Suppose η has no totally ramified point and no fibre of the form $3q + 2r$, with $q \neq r$. Then*

$$\delta(X) \geq \frac{100}{99} > 1,$$

and X is K-stable. For an arbitrary smooth member with no totally ramified point, any prime divisor with nonpositive beta invariant must have centre a closed point on the line joining the A_2 and A_1 points in a mixed fibre. Its centre cannot be the A_2 point itself.

Proof. A degree-five fibre with no index-five point has multiplicities $(1,1,1,1,1)$, $(2,1,1,1)$, $(2,2,1)$, $(3,1,1)$, $(3,2)$ or $(4,1)$. These give respectively a smooth surface, A_1, $2A_1$, A_2, A_2+A_1, or A_3. Points on D and on smooth fibres are already covered. Lemma 96.1 handles the nodal fibres. Theorems 94.1 and 95.1 handle the higher singular points.

At a remaining smooth point of a fibre, either Lemma 96.2 gives $\delta_p(T) \geq 4/3$, hence $\delta_p(X) \geq 10/9$, or the point lies on a line in (35.10). Lemma 96.3 gives a bound greater than $100/99$ there, except on ℓ_{14} in the $(3,2)$ configuration. This proves the first assertion, including its numerical bound.

In a mixed fibre the image Γ of ℓ_{14} is the strict transform of the plane line joining the two cluster supports. It joins the A_2 and A_1 points and is disjoint from D. All points outside the union of these lines have now been covered; the A_1 point of each mixed fibre is included in its line. A positive-dimensional irreducible valuation centre not contained in this union has a point at which $\delta_p(X) > 1$. A curve centre contained in the union must be one of the lines Γ; it contains the A_2 point, at which Theorem 94.1 gives the same strict conclusion for every divisor whose centre contains that point. Thus only the stated closed point centres remain. □

Example 96.5 (A formerly untreated A_3 fibre). The morphism

$$[x : y] \longmapsto [x^4y : x^5 + y^5]$$

has an index-four point at zero and five simple ramification points. Their branch values are distinct: at $z^5 = 4$ they are $z^4/5$. Infinity is unramified. In particular this map has no index-three point and hence no fibre $3q + 2r$. The corresponding smooth threefold is K-stable by Theorem 96.4; at its A_3 fibre point the exact local threshold is $400/389$.

97 The mixed fibre

In a fibre of multiplicities $(3,2)$, let Γ be the strict transform of the plane line joining the two cluster supports. Theorem 96.4 reduces any possible nonpositive-beta divisor to a closed point of Γ other than its A_2 endpoint. The fibre flag has

$$S(W^T;\Gamma) = \frac{83}{80} > 1,$$

so it does not provide the required estimate. We first treat the A_1 endpoint. We then extract the curve with transverse weights $(1,2)$, obtaining semistability over its interior. Finally, an ordinary point extraction improves the interior estimate to $32/29$.

Example 97.1. The morphism $[x : y] \mapsto [x^3y^2 : x^5 + y^5]$ has the mixed fibre $3[0 : 1] + 2[1 : 0]$. Its joining line is the strict transform of $t = Y_1 = 0$. The bounds at its A_2 and A_1 endpoints are $160/151$ and $240/221$; at every other point of the line the bound is $32/29$. These estimates are proved below for arbitrary graph quintics with such a fibre.

98 The index-two endpoint of the mixed fibre

We work at the A_1 point p of a fibre of type $A_2 + A_1$. Let G be the exceptional divisor of the ordinary blow-up of p. Thus $A_X(G) = 3$ and the exceptional pair is $G = \mathbb{P}^2$ with zero different. The actual refined anticanonical series is denoted by W. Write $N_m = h^0(X, -mK_X) = (10m^3 + 15m^2 + 11m + 3)/3$.

Theorem 98.1. *At this point one has*

$$S_X(G) = \frac{107}{40}, \qquad \delta(G; W) \geq \frac{240}{221}, \qquad \delta_p(X) \geq \frac{240}{221} > 1.$$

Lemma 98.2 (The actual initial sections)**.** *In homogeneous coordinates $[x : z : w]$ on G, put $t = zw - x^2$. The algebra of the refined series contains the degree-one sections*

$$t^i\{1, x, z\} \quad (0 \leq i \leq 2), \qquad w, xw, tw, txw.$$

Here the first degree is the anticanonical degree; their second degrees are their usual polynomial degrees in x, z, w.

Proof. Put the index-two source point at $[0 : 1]$, the index-three source point at $[1 : 0]$, and their common image at $[1 : 0]$. The graph equation then has the form

$$ux^2y^3 + vg(x, y) = 0, \qquad g = y^5 + \sum_{i=1}^{5} g_i x^i y^{5-i}, \qquad g_5 \neq 0.$$

On the affine chart $u = Y_0 = 1$ of $\mathbb{P}^1 \times \mathbb{P}^2$, write $v = t$ and $Y_1 = x$, $Y_2 = c$. The conic equation is $z = c - x^2$. The two cubic lifts of the graph equation are

$$c + tG_0(x, c), \qquad xc + tG_1(x, c),$$

where

$$G_0 = 1 + g_1x + g_2c + g_3xc + g_4c^2 + g_5xc^2,$$
$$G_1 = x + g_1c + g_2xc + g_3c^2 + g_4xc^2 + g_5c^3.$$

In particular

$$G_1 - xG_0 = zR(c), \qquad R(c) = g_1 + g_3c + g_5c^2.$$

On the blow-up chart containing p, set

$$A = \frac{c + tG_0}{z}, \qquad B = \frac{xc + tG_1}{z}, \qquad w = A - 1.$$

Then $B = xA + tR(c)$, $c = x^2 + z$ and

$$tG_0 = zw - x^2.$$

Since $G_0(0, 0) = 1$, the variables x, z, w are regular parameters at p, and the initial form of t for their ordinary order is $zw - x^2$.

The thirteen anticanonical sections are $t^i\{1, x, c\}$ for $0 \leq i \leq 2$ and A, B, tA, tB. Replace the last four by

$$A - 1 = w, \qquad B - x - g_1t = xw + t(g_3c + g_5c^2),$$
$$tA - t = tw, \quad tB - tx - g_1t^2 = txw + t^2(g_3c + g_5c^2).$$

Their initial forms are w, xw, tw, txw. The initial form of c is z. Thus all thirteen displayed polynomials belong to the actual associated graded series. Products give the asserted algebra inclusion. □

Lemma 98.3 (An exact normal basis)**.** *Let $\mathcal{A}$ be the algebra generated by these thirteen sections. Its degree-m space has basis $x^a z^b w^c t^s$, with $bc = 0$, in the following two disjoint ranges:*

$$\begin{array}{c|l} c = 0 & a, b, s \geq 0, \quad 2a + 3b \leq 3m, \quad s \leq 2m, \quad s + a + b \leq 3m, \\ b = 0,\ c \geq 1 & a, c, s \geq 0, \quad c \leq m, \quad 2a + c \leq 3m, \quad s + c \leq 2m, \quad s + a + c \leq 3m. \end{array} \tag{A1.1}$$

Its dimension is

$$N_m = \frac{10m^3 + 15m^2 + 11m + 3}{3}.$$

Consequently $\mathcal{A}$ is the entire refined algebra W.

Proof. The relation is $zw = x^2 + t$. In addition to the degree-one generators, we may use $x^3 t^i$ of degree two for $0 \leq i \leq 3$: write $i = j + k$, where $0 \leq j \leq 2$, $0 \leq k \leq 1$, and use

$$(zt^j)(xwt^k) = x^3 t^i + xt^{i+1}.$$

The last term is a product of two degree-one generators. Straightening a z-factor against a w- or xw-factor now uses $zw = x^2 + t$ and the displayed identity. Each resulting term remains a product of the generators just listed, of the same anticanonical degree. Repetition eliminates every zw pair. Normal monomials with $bc = 0$ are linearly independent, since zw is a leading monomial of the single relation in $\mathbb{C}[x, z, w, t]/(zw - x^2 - t)$.

For $c = 0$, use k factors x^3 of degree two. The remaining x and z factors have degree one. Filling unused places with $1, t, t^2$ shows that a prescribed s is possible if and only if

$$\max(0, a + b - m) \leq k \leq \lfloor a/3 \rfloor, \qquad s \leq 2m - k.$$

Every integer s up to the upper bound is possible, since each factor allows a consecutive interval of t-exponents. Taking the smallest k gives the first row of (A1.1).

For $b = 0$, after using k factors x^3, pair as many of the remaining x factors as possible with the c factors w. The minimum required anticanonical degree is $\max(a - k, c + 2k)$, and the maximum t-exponent in total degree m is $2m - c - k$. Thus the conditions are

$$\max(0, a - m) \leq k \leq \min\left(\lfloor a/3 \rfloor, \lfloor (m - c)/2 \rfloor\right), \qquad s \leq 2m - c - k.$$

They are equivalent to the second row of (A1.1).

Here is the count, including the parity terms. Put $k_n = \lfloor n/2 \rfloor$. For the first row set $n = m - b$; its contribution for this n is

$$(n + k_n + 1)(2m + 1) - \frac{k_n(k_n + 1)}{2}.$$

For the second row set $n = m - c$; its contribution is

$$(m + k_n + 1)(m + n + 1) - \frac{k_n(k_n + 1)}{2}.$$

Therefore the dimension equals

$$\sum_{n=0}^{m} \left((n + k_n + 1)(2m + 1) - \frac{k_n(k_n + 1)}{2} \right) + \sum_{n=0}^{m-1} \left((m + k_n + 1)(m + n + 1) - \frac{k_n(k_n + 1)}{2} \right) = N_m.$$

The last identity follows by splitting each sum into even and odd n; it holds for both parities of m. This equals $h^0(X, mL)$, so Lemma 94.2 proves equality with the actual refined algebra.

The same summation gives the primary weight sum, with no parity restriction. In the first row, for fixed n, sum

$$\sum_{a=0}^{n+k_n} \left((U+1)(a+m-n)+U(U+1)\right), \qquad U=\min(2m, 2m+n-a).$$

In the second row replace the upper limit by $m+k_n$ and put $U=\min(m+n, 2m+n-a)$. Sum the first row over $0 \le n \le m$ and the second over $0 \le n < m$. Splitting at $a=n$ and $a=m$, respectively, and then at the two parities of n, gives

	$c=0$	$b=0,\ c\ge 1$
m even	$\dfrac{m(97m^3+233m^2+218m+64)}{24}$	$\dfrac{m^2(3m+1)(13m+4)}{8}$
m odd	$\dfrac{m(m+1)(97m^2+136m+55)}{24}$	$\dfrac{m(3m+1)(13m^2+4m+3)}{8}$.

Both rows have total

$$\frac{m(107m^3+154m^2+115m+32)}{12}.$$

After division by mN_m this proves $S_X(G)=107/40$. □

Lemma 98.4 (Binary spaces and their moments)**.** *Give x, z, w the auxiliary torus weights $1, 0, 2$, respectively. The polynomial $t=zw-x^2$ has weight two. In primary degree j and auxiliary weight q, put*

$$b=\max(j-q,0), \qquad c=\max(q-j,0), \qquad k=q-2c.$$

Put

$$\ell=\max\left(0, \left\lceil\frac{2k+3b+c-3m}{4}\right\rceil, k+b+c-3m\right), \qquad v=\min\left(2m-c, \left\lfloor\frac{k}{2}\right\rfloor\right).$$

When $0 \le b, c \le m$ and $v \ge \ell$, the component is exactly

$$x^{k-2v}z^b w^c t^\ell \operatorname{Sym}^{v-\ell}(x^2, zw). \tag{A1.2}$$

Otherwise it is zero. Write $d=v-\ell$ and $a=k-2v$. Define

$$\alpha_m=\frac{\sum(a+d)(d+1)}{mN_m}, \quad b_m=\frac{\sum b(d+1)}{mN_m}, \quad c_m=\frac{\sum c(d+1)}{mN_m},$$
$$s_m=\frac{\sum \ell(d+1)}{mN_m}, \qquad h_m=\frac{\sum d(d+1)}{2mN_m}.$$

Their limits are

$$\alpha=\frac{91}{160}, \qquad b=\frac{47}{320}, \qquad c=\frac{15}{64}, \qquad s=\frac{193}{320}, \qquad h=\frac{83}{320}. \tag{A1.3}$$

In particular the fixed x-coefficient is $\alpha-2h=1/20$, and

$$\alpha+2(c+s+h)=\frac{221}{80}, \qquad \alpha+2(b+s+h)=\frac{207}{80}.$$

Proof. The equations for the two gradings are $j = a + b + c + 2s$ and $q = a + 2c + 2s$. Their difference determines b, c, and (A1.1) gives exactly $\ell \le s \le v$. Using $t = zw - x^2$ proves (A1.2).

For the asymptotic sums put $y = b/m$ in the z-branch and $y = c/m$ in the w-branch, and replace k/m by k. The lower and upper limits of s/m are

	L	U
z	$\max(0, (2k + 3y - 3)/4, k + y - 3)$	$\min(2, k/2)$
w	$\max(0, (2k + y - 3)/4, k + y - 3)$	$\min(2 - y, k/2)$.

Integrate on $0 \le y \le 1$, $L \le U$, and put $D = U - L$. Splitting the region at the indicated affine walls gives

	$\int D$	$\int (k - 2U + D)D$	$\int yD$	$\int LD$	$\int D^2/2$
z	35/24	17/24	47/96	835/768	257/768
w	15/8	19/16	25/32	709/768	407/768.

(A1.4)

These are ordinary integrals in $dy\,dk$; for example, the chamber z, $L = 0$, $U = k/2$ is the triangle $0 \le y \le 1$, $0 \le k \le 3(1 - y)/2$. Its contributions to the first, second and last columns are 3/16, 9/128 and 9/256. Multiplying the summed columns by 3/10, and taking the third column separately in each branch, gives (A1.3). Floors and ceilings contribute only $O(m^3)$ to the numerators, while $mN_m = (10/3)m^4 + O(m^3)$.

The average degree on $\mathbb{P}^2$ is

$$(\alpha - 2h) + b + c + 2s + 4h = \frac{107}{40}.$$

By the definition of the primary associated graded series this is $S_X(G)$. □

Proof of the theorem. The conic pencil on $\mathbb{P}^2$ is

$$T_\lambda = (zw - \lambda x^2 = 0), \qquad T_0 = Z_0 + W_0, \qquad T_\infty = 2X_0.$$

Its two base points are $[0 : 1 : 0]$ and $[0 : 0 : 1]$. On the respective smooth charts its branches have equations $w = \lambda x^2$ and $z = \lambda x^2$. Two successive ordinary blow-ups at either point separate all distinct branches.

An arbitrary basis divisor obtained by choosing a basis in each space (A1.2) has the form

$$B_m = (\alpha_m - 2h_m)X_0 + b_m Z_0 + c_m W_0 + s_m T_1 + \sum \nu_\lambda T_\lambda,$$

where $\sum \nu_\lambda = 2h_m$ and $0 \le \nu_\lambda \le h_m$. Indeed a basis of $H^0(\mathbb{P}^1, \mathcal{O}(d))$ has total degree $d(d+1)$, and its total vanishing order at any point is at most $d(d+1)/2$.

The component bounds for this divisor are

$$\alpha_m, \quad b_m + h_m, \quad c_m + h_m, \quad s_m + h_m, \quad h_m.$$

At $[0 : 1 : 0]$, the second blow-up has log discrepancy three and order at most $\alpha_m + 2(c_m + s_m + h_m)$; at $[0 : 0 : 1]$ the corresponding order is $\alpha_m + 2(b_m + s_m + h_m)$. The first blow-up inequalities follow by linear interpolation with the bound on the $x = 0$ component. After these two blow-ups the total transform is a simple normal crossings divisor. Thus these inequalities test all divisorial valuations, not only monomial ones.

The limiting component bounds are

$$\frac{91}{160}, \quad \frac{13}{32}, \quad \frac{79}{160}, \quad \frac{69}{80}, \quad \frac{83}{320};$$

the two corner bounds divided by their log discrepancies are 221/240 and 69/80. Their maximum is 221/240. Consequently $(G, \tau B_m)$ is log canonical for every $\tau < 240/221$ and all sufficiently large m.

For an arbitrary divisorial valuation over G, choose in each primary piece a basis compatible with its valuation filtration and the auxiliary torus filtration. Taking least auxiliary weights produces the divisors just considered. Lower semicontinuity of log canonical thresholds under this torus degeneration gives the same bound for the original valuation-adapted basis divisor. Taking m to infinity proves $\delta(G; W) \geq 240/221$. Abban–Zhuang adjunction now gives

$$\delta_p(X) \geq \min\left\{\frac{3}{107/40}, \frac{240}{221}\right\} = \frac{240}{221}.$$

No Zariski decomposition on the blown-up threefold was used. The entire refined algebra was identified by an exact Hilbert count; the only further modifications are the two explicit surface blow-ups at each base point of the conic pencil. □

99 A weighted extraction of the joining line

Lemma 99.1. *The joining line has normal bundle $N_{\Gamma/X} = \mathcal{O}(1) \oplus \mathcal{O}(-2)$. There is a weighted extraction with normal weights $(1, 2)$, assigning weight one to the positive normal direction. Its exceptional divisor G has $A_X(G) = 3$, and its different has coefficient $1/2$ along $a = 0$. On the open part of Γ, its actual refined algebra has the thirteen degree-one generators*

$$1, b, a, z, bz, b^2z, h, abz, bz^2, b^2z^2, b^3z^2, bhz, ab^2z^2, \qquad h = a^2 + b^3z.$$

Their primary degrees are determined by $\deg(a) = 1$, $\deg(z) = 2$, $\deg(b) = 0$.

Proof. Choose coordinates so that the mixed fibre of the graph is defined by $f = x^3y^2$. Write $g = \sum_{i=0}^{5} c_i x^i y^{5-i}$, where $c_0c_5 \neq 0$. On $Y_0 \neq 0$, put $a = Y_1/Y_0$, $b = Y_2/Y_0$, $q = b - a^2$, and let t be the target coordinate vanishing on the mixed fibre. Set

$$P = c_0 + c_2b + c_4b^2, \qquad Q = c_1 + c_3b + c_5b^2.$$

The two lifted graph sections are

$$A = \frac{a^3 + t(P + aQ)}{q}, \qquad B = \frac{a^2b + t(aP + bQ)}{q}.$$

They satisfy $bA - aB = tP$ and $B = aA + a^2 + tQ$. The anticanonical degree-one sections, in this trivialization, are $t^i\{1, a, b\}$ for $0 \leq i \leq 2$, together with A, B, tA, tB.

The morphism from the normal bundle of Γ to the normal bundle $\mathcal{O}(1) \oplus \mathcal{O}$ of its image in $\mathbb{P}^1 \times \mathbb{P}^2$ is an elementary transform at the two endpoints. At each endpoint its quotient is in the trivial summand: the graph is tangent to the fibre, whereas the joining line is transverse to the conic. Thus its kernel is $\mathcal{O}(1) \oplus \mathcal{O}(-2)$. Equivalently, in local coordinates near an endpoint the differential in the normal directions is $(a, w) \mapsto (a, bw)$. At the index-two endpoint the equation for t contains an additional quadratic term, which has zero differential.

There is a coordinate-independent weighted extraction. If I is the ideal of Γ, take the inverse image $J \subset I$ of the positive subbundle $\mathcal{O}(2) \subset I/I^2$. Locally $J = (w, a^2)$. The normalized blow-up of J has weights $(1, 2)$, discrepancy three, and different one half along the section corresponding to $a = 0$. The coarse exceptional ruled surface need not be identified in this argument.

At the generic point of Γ, assign weights $1, 2, 0$ to a, t, b. The initial forms of A, B are tP/b and $a^2 + tQ$. Subtracting linear combinations of t, tb, and rescaling, gives t/b and $a^2 + tb^2$. The

same operation applies after multiplication by t. Putting $z = t/b$ yields the thirteen generators in the statement. They generate a subalgebra of the primary associated algebra. Lemma 99.2 below gives an exact dimension count after one further refinement. It proves equality of both associated algebras in every degree. □

99.1 The fibre refinement

We use the coordinates and initial sections of the preceding extraction calculation. Thus the centre is locally $a = t = 0$, the coordinate b is nonzero, the weights are $(1, 2)$, and $z = t/b$. Put $h = a^2 + b^3 z$. The thirteen initial anticanonical sections are

$$1,\ b,\ a,\ z,\ bz,\ b^2 z,\ h,\ abz,\ bz^2,\ b^2 z^2,\ b^3 z^2,\ bhz,\ ab^2 z^2.$$

Here the primary degrees of a, b, z are $1, 0, 2$, respectively. All statements below concern the open set $b \neq 0, \infty$. In particular, we make no assertion that the coarse exceptional surface is globally a particular Hirzebruch surface.

Lemma 99.2. *Fix $c \neq 0, \infty$, and refine the primary filtration by order along $b = c$. The resulting degree-m vector space has a monomial basis indexed by the triples (a, k, z) in the semigroup generated in degree one by*

$$(0,0,0), (0,1,0), (1,0,0), (0,0,1), (0,1,1), (0,2,1), (2,0,0),$$
$$(1,0,1), (0,0,2), (0,1,2), (0,2,2), (2,0,1), (1,0,2).$$

Its degree-m elements are precisely the integral points satisfying

$$a, k, z \geq 0, \qquad z \leq 2m, \qquad a + k \leq 2m,$$
$$a + 2k - 2z \leq 2m, \qquad 2a + k + 2z \leq 6m,$$

except for the points with $a + k = 2m$ and both a and k odd. The primary degree of such a basis vector is $j = a + 2z$.

Proof. The substitution $b \mapsto cb$, $a \mapsto a$, $z \mapsto c^{-3} z$ preserves h and scales each of the thirteen sections. We may therefore suppose that $c = 1$. Put $d = b - 1$. In the primary degree-two piece use

$$z, \quad dz, \quad d^2 z, \quad h - 3b^2 z + 3bz - z = a^2 + d^3 z.$$

In the primary degree-four piece use

$$bz^2, \quad bdz^2, \quad bd^2 z^2, \quad bhz - 3b^3 z^2 + 3b^2 z^2 - bz^2 = ba^2 z + bd^3 z^2.$$

Together with $1, d, a, abz, ab^2 z^2$, their leading forms are the thirteen monomials asserted in the statement. Their products give linearly independent leading forms whenever their exponent triples are distinct.

We count these products in every degree. The degree-one generators with first two coordinates $(2, 0)$ allow third coordinate 0 or 1; those with first two coordinates $(0, 2)$ allow 1 or 2. The generators with first two coordinates $(1, 0)$, $(0, 1)$, or $(0, 0)$ allow every third coordinate between 0 and 2. Thus a triple (a, k, z) is a product of m generators if and only if there exist integers $r, s \geq 0$ such that

$$r \leq \left\lfloor \frac{a}{2} \right\rfloor, \quad s \leq \left\lfloor \frac{k}{2} \right\rfloor, \quad r + s \geq a + k - m, \quad s \leq z \leq 2m - r.$$

Indeed, take r generators of type $(2, 0)$, s of type $(0, 2)$, $a - 2r$ of type $(1, 0)$, $k - 2s$ of type $(0, 1)$, and fill the remaining places with type $(0, 0)$. Their possible third coordinates form the whole interval $[s, 2m - r]$. Equivalently,

$$\min\left\{ \left\lfloor \frac{a}{2} \right\rfloor, 2m - z \right\} + \min\left\{ \left\lfloor \frac{k}{2} \right\rfloor, z \right\} \geq a + k - m.$$

Expanding the two minima gives precisely the inequalities and the parity exception in the statement.

For completeness, the count is also explicit. Write $a = 2r+\epsilon$, $k = 2s+\eta$, where $\epsilon, \eta \in \{0,1\}$. The condition is $r + s + \epsilon + \eta \le m$. Put $N = m - r - s - \epsilon - \eta$. The number of possible third coordinates is

$$2m + 1 - \max\{0, s - N\} - \max\{0, r - N\}.$$

If

$$Q(n) = \sum_{r+s\le n} \max\{0, 2r + s - n\},$$

then summing first over s gives

$$Q(n) = \begin{cases} n(n+2)(2n+5)/24 & \text{if } n \text{ is even,} \\ (n+1)(n+3)(2n+1)/24 & \text{if } n \text{ is odd.} \end{cases}$$

Set $Q(-1) = Q(-2) = 0$. Consequently the number of distinct products is

$$(2m+1)\left(\binom{m+2}{2} + 2\binom{m+1}{2} + \binom{m}{2}\right) \\ - 2(Q(m) + 2Q(m-1) + Q(m-2)) = \frac{10m^3 + 15m^2 + 11m + 3}{3}.$$

This is $h^0(X, -mK_X)$. Thus the products already fill the entire filtered vector space, and there are no further leading forms in any degree. Notice that the semigroup is not normal: $(1,1,1)$ is the first missing lattice point. Its missing boundary points must not be silently inserted in the finite-degree basis. □

Lemma 99.3. *Let G denote the weighted exceptional divisor and let F_c denote its fibre over $b = c \neq 0, \infty$. The primary and secondary expected vanishing orders are*

$$S_X(G) = \frac{21}{8}, \qquad S(W^G; F_c) = \frac{5}{8}.$$

On the coarse fibre $F_c \simeq \mathbb{P}^1$, use weighted coordinates $[a : z] \in \mathbb{P}(1,2)$ and ordinary coordinates $[a^2 : z]$. For the series refined by F_c, the point moments satisfy

$$S(W^{G,F_c}; \{z = 0\}) = 1, \qquad S(W^{G,F_c}; \{a = 0\}) = \frac{5}{16},$$

and at every other point they are at most $5/16$.

Proof. Let P be the polytope in the preceding lemma with $m = 1$. The missing lattice points belong to a two-dimensional face. There are $O(m^2)$ of them, so they do not change the limits of normalized first moments. Over the triangle $T = \{a, k \ge 0 \mid a + k \le 2\}$ the interval in the third coordinate is

$$\max\left\{0, \frac{a + 2k - 2}{2}\right\} \le z \le 2 - \max\left\{0, \frac{2a + k - 2}{2}\right\}.$$

Each of the two cut-off triangular caps has volume $1/3$; hence $\mathrm{vol}(P) = 4 - 2/3 = 10/3$. The integral of a over P is

$$\int_P a = \frac{8}{3} - \frac{1}{6} - \frac{5}{12} = \frac{25}{12}.$$

Here the first term is its integral over $T \times [0,2]$. For the lower cap the base triangle has vertices $(0,1), (0,2), (2,0)$, and the integral of $a(a + 2k - 2)/2$ is $1/6$. For the upper cap the

base triangle has vertices $(1,0),(2,0),(0,2)$, and the integral of $a(2a+k-2)/2$ is $5/12$. The involution $(a,k,z)\mapsto(k,a,2-z)$ preserves P. Therefore

$$\frac{1}{\operatorname{vol}(P)}\int_P (a,k,z)=\left(\frac{5}{8},\frac{5}{8},1\right).$$

The primary order is $a+2z$, while the secondary order is k. This proves the first two assertions.

Fix a primary degree j and a secondary degree k. In the basis of Lemma 99.2 the remaining exponents satisfy $a=j-2z$. Their allowed values of z form an interval of consecutive integers. The parity exception can remove only its lower endpoint. Thus this piece, on the weighted fibre, is exactly

$$a^{a_0}z^{z_0}\langle (a^2)^d,(a^2)^{d-1}z,\dots,z^d\rangle$$

for some nonnegative integers a_0,z_0,d. At $z=0$ its vanishing sequence is $z_0,z_0+1,\dots,z_0+d$. At the coarse point $a=0$ it is $a_0/2,a_0/2+1,\dots,a_0/2+d$. The rational offset is the order in the induced rational line bundle; it is also obtained by taking an even multiple before passing to the coarse fibre. At any other point the vanishing sequence is $0,1,\dots,d$. Hence the last sequence has average no larger than the sequence at $a=0$. Summing over j,k gives the asserted moments. These are complete restricted series, so the estimates hold for bases adapted to any chosen point, rather than only for the displayed monomial basis. □

Example 99.4. For $m=1$, the piece of primary degree 2 and fibre degree 0 is $\langle a^2,z\rangle$, while the pieces of fibre degrees 1 and 2 are both generated by z. The analogous primary degree-4 pieces are $\langle a^2z,z^2\rangle$, $\langle z^2\rangle$, and $\langle z^2\rangle$. Over all thirteen basis vectors, the sums of the primary, fibre, and $z=0$ orders are respectively $33,7,13$. The sum of the coarse $a=0$ orders is $7/2$. This illustrates both the factor $1/2$ at the quotient point and the fixed factors in the refined series. In every degree the involution $(a,k,z)\mapsto(k,a,2m-z)$ preserves the semigroup, so the average order at $z=0$ is exactly m, even before taking the asymptotic limit.

Proposition 99.5. *For every point p of the joining line with $b(p)\neq 0,\infty$ one has $\delta_p(X)\geq 1$.*

Proof. The extraction with transverse weights $(1,2)$ has $A_X(G)=3$. Over the indicated open set its different is one half of the section $a=0$. The fibre F_c is not contained in this different. The different induced on its coarse curve is therefore $\frac{1}{2}\{a=0\}$. Its point discrepancies are $1/2$ at $a=0$ and 1 everywhere else. By Lemma 99.3, the ratios on the fibre are $8/5$ at $a=0$, exactly 1 at $z=0$, and at least $16/5$ elsewhere. The Abban–Zhuang refinement along F_c gives a bound $\min\{8/5,1\}=1$ on the exceptional surface above p. Applying the same refinement to G gives

$$\delta_p(X)\geq\min\left\{\frac{3}{21/8},\frac{1}{5/8},1\right\}=1.$$

The equality in the last point estimate only proves semistability. □

100 The point extraction at an interior point

Let p be a point of the joining line Γ different from its A_2 and A_1 endpoints. We use the ordinary blow-up of p; its exceptional divisor is again $G=\mathbb{P}^2$ with zero different.

Theorem 100.1. *At every such point one has*

$$S_X(G)=\frac{9}{4},\qquad \delta(G;W^G)\geq\frac{32}{29},\qquad \delta_p(X)\geq\frac{32}{29}>1.$$

Lemma 100.2 (The initial algebra at an interior point)**.** *There are homogeneous coordinates $[x : z : t]$ on G and a second one-parameter degeneration of its refined series whose algebra contains the degree-one sections*

$$t^i\{1, x, z\} \quad (0 \le i \le 2), \qquad x^2, tx^2, F, tF, \qquad F = x^3 + tz^2.$$

The primary degree is ordinary polynomial degree in x, z, t; the auxiliary torus weights are $(1, 0, 3)$.

Proof. As at the index-two endpoint, write the graph as $ux^2y^3 + vg = 0$, with $g_0 = 1$, $g_5 \neq 0$, and use the cubic lifts G_0, G_1 and $R(c)$ of Lemma 98.2. Since p lies on the open part of Γ, it has $t = x = 0$ and $c \neq 0, \infty$. A source scaling allows us to put $c(p) = 1$. Put $z = c - 1$. The conic equation $q = c - x^2 = 1 + z - x^2$ is a unit near p, so the blow-up is an isomorphism there. Trivializing L by this conic section gives the thirteen regular local functions

$$t^i\{1, x, c\} \quad (0 \le i \le 2), \qquad A, B, tA, tB,$$

where

$$A = \frac{c + tG_0}{q}, \qquad B = xA + tR(c).$$

Put

$$a = 1 + g_2 + g_4, \qquad b = g_1 + g_3 + g_5, \qquad e = g_4 - 1, \qquad d = g_3 + 2g_5.$$

The first-order Taylor expansion of G_0/q at p is $a + bx + ez$. Thus

$$U = A - 1 - at - btx - etz$$

has initial form x^2. The function

$$V = B - x - bt - dtz - atx$$

satisfies the exact identity

$$V = xU + btx^2 + etxz + g_5tz^2.$$

Consequently its primary initial form is

$$F' = x^3 + t(bx^2 + exz + g_5z^2).$$

The functions tU and tV are still linear combinations of the thirteen sections, since their definitions only require subtracting t^2, t^2x, t^2z from tA, tB. Their initial forms are tx^2 and tF'. The nine other initial sections are $t^i\{1, x, z\}$.

Now take least auxiliary weights $(1, 0, 3)$ in (x, z, t). The two terms btx^2 and $etxz$ have weights five and four, whereas x^3 and g_5tz^2 have weight three. Hence the new initial forms of F', tF' are $x^3 + g_5tz^2$ and $t(x^3 + g_5tz^2)$. Since $g_5 \neq 0$, a scaling makes its coefficient one. This proves the claimed inclusion, directly from actual sections on X. □

Lemma 100.3 (Normal monomials and the Hilbert count)**.** *The degree-m space of this algebra has basis*

$$x^a z^b F^c t^s,$$

where all four exponents are nonnegative integers and

$$(s = 0 \text{ or } b \le 1), \qquad a + 2b + 2c \le 2m, \qquad s + c \le 2m, \qquad s + a + b + 2c \le 3m. \quad (I.1)$$

Its dimension is N_m. It is therefore the whole secondary initial algebra of the actual refinement.

Proof. Use the relation $z^2t = F - x^3$. If a product contains two z factors and positive t-exponent, distribute at least one t to these two factors. This redistribution is possible without changing the anticanonical degree: every generator type allows a consecutive interval of t-exponents starting at zero. For $1 \le i + j \le 4$,

$$(zt^i)(zt^j) = Ft^{i+j-1} - x^3t^{i+j-1}.$$

Both terms are products of two degree-one generators. For the first, use F, tF and $1, t, t^2$; for the second use x^2, tx^2 and x, tx, t^2x. Repetition decreases the number of z factors, and ends with $s = 0$ or $b \le 1$. These normal monomials are linearly independent: z^2t is the leading monomial of the single relation in $\mathbb{C}[x, z, t, F]/(z^2t - F + x^3)$.

For a normal monomial use k factors of type x^2t^i, together with $a - 2k$ factors of type xt^i, b factors of type zt^i, and c factors of type Ft^i. Fill unused places with $1, t, t^2$. This is possible exactly when

$$\max(0, a + b + c - m) \le k \le \lfloor a/2 \rfloor, \qquad s \le 2m - k - c.$$

Taking the smallest k gives the three inequalities in (I.1).

To count the basis, first take $s = 0$. For fixed $b + c = n$, there are $n + 1$ choices and $2(m - n) + 1$ choices of a. If $s \ge 1$, then $b = 0$ or 1. Put $n = m - b - c$; summing over $0 \le a \le 2n$ gives

$$(2n + 1)(m + b + n) - \frac{n(n + 1)}{2}$$

choices of (a, s). Thus the total is

$$\begin{aligned}
&\sum_{n=0}^{m}(n + 1)(2m - 2n + 1) \\
&\quad + \sum_{n=0}^{m}\left((2n + 1)(m + n) - \frac{n(n + 1)}{2}\right) \\
&\quad + \sum_{n=0}^{m-1}\left((2n + 1)(m + 1 + n) - \frac{n(n + 1)}{2}\right) = \frac{10m^3 + 15m^2 + 11m + 3}{3}.
\end{aligned}$$

Lemma 94.2 therefore proves equality with the actual secondary initial algebra in every degree.

For the primary weight $a + b + 3c + s$, the same three disjoint ranges give, respectively,

$$\frac{m(m + 1)(m + 2)(3m + 4)}{6}, \qquad \frac{m(m + 1)(7m^2 + 7m + 1)}{2}, \qquad \frac{m(21m^3 + 5m^2 + 3m + 1)}{6}.$$

Here the first sum is

$$\sum_{n=0}^{m}(n + 1)(2m - 2n + 1)(m + n).$$

For each of the two other sums keep $b = 0$ or $b = 1$, put $c = m - b - n$, and sum over $0 \le a \le 2n$, $1 \le s \le U$, where $U = \min(m + b + n, m + b + 2n - a)$. The summand after summing in s is

$$U(a + 3m - 2b - 3n) + \frac{U(U + 1)}{2}.$$

Split the a-sum at $a = n$ and use $0 \le n \le m - b$. This gives the two displayed expressions. Their total is

$$\frac{m(15m^3 + 20m^2 + 15m + 4)}{2},$$

for every $m \ge 0$. It follows directly that $S_X(G) = 9/4$. □

Lemma 100.4 (The cubic pencil and exact moments)**.** *Each primary-degree and auxiliary-weight component is a complete binary space with fixed factors*

$$x^{k-3v}z^b t^s F^\ell \operatorname{Sym}^{v-\ell}(x^3, tz^2), \tag{I.2}$$

where

$$\ell = \max(0, k+2b-2m, s+k+b-3m), \qquad v = \min\left(2m-s, \left\lfloor \frac{k}{3} \right\rfloor\right).$$

The component is nonzero exactly when these indices are nonnegative and $v \geq \ell$. Put $A = k-3v$, $D = v-\ell$. The normalized sums of $(A+3D/2)(D+1)$, $b(D+1)$, $s(D+1)$, $\ell(D+1)$ and $D(D+1)/2$ have respective limits

$$\alpha = \frac{5}{8}, \qquad b_0 = \frac{1}{40}, \qquad t_0 = \frac{7}{10}, \qquad f_0 = \frac{23}{240}, \qquad h = \frac{49}{240}. \tag{I.3}$$

The normalizing denominator is mN_m.

Proof. For a monomial in (I.1), the primary degree and auxiliary weight are

$$j = a+b+3c+s, \qquad q = a+3c+3s.$$

Thus $q-j = 2s-b$. If $q-j \geq -1$, choose the unique $b \in \{0,1\}$ with $b \equiv j-q \pmod 2$ and put $s = (q-j+b)/2$. If $q-j \leq -2$, put $s = 0$, $b = j-q$. The normal condition makes this choice unique. Set $k = q-3s = a+3c$. Substituting in (I.1) gives exactly $\ell \leq c \leq v$. Since $F = x^3+tz^2$, the component has the form (I.2).

Here are the integrations for the moments. The first region has $s = 0$ and $y = b/m$, $0 \leq y \leq 1$. The second consists of the two rows $b = 0, 1$ and has $y = s/m$, $0 \leq y \leq 2$; it is counted twice. Writing k/m again as k, the lower and upper bounds for c/m are

	L	U
z	$\max(0, k+2y-2)$	$k/3$
t	$\max(0, k-2, y+k-3)$	$\min(2-y, k/3)$.

Integrate where $L \leq U$ and put $D = U-L$. The first row is bounded by $k \leq 3(1-y)$, so $U = k/3$ throughout. The exact integrals, including the multiplicity two in the second row, are

	$\int D$	$\int (k-3U+3D/2)D$	$\int (b/m)D$	$\int (s/m)D$	$\int LD$	$\int D^2/2$
z	1/3	1/6	1/12	0	1/36	1/18
t	3	23/12	0	7/3	7/24	5/8.

(I.4)

For example, the first region has two chambers, separated by $k = 2(1-y)$; direct integration of $D = k/3$ on the lower chamber and $D = 2-2y-2k/3$ on the upper gives its first row. For the second region the five chambers are obtained by choosing which affine functions give L, U. The polygons in (y, k) are

L	U	vertices
0	$2-y$	$(2,0), (2,1), (3/2, 3/2)$
0	$k/3$	$(0,0), (2,0), (3/2, 3/2), (1,2), (0,2)$
$k-2$	$k/3$	$(0,3), (0,2), (1,2), (1,3)$
$y+k-3$	$2-y$	$(2,1), (1,3), (3/2, 3/2)$
$y+k-3$	$k/3$	$(1,3), (1,2), (3/2, 3/2)$.

Integration of the displayed polynomials on these polygons gives the second row of (I.4). Multiplying the summed columns by 3/10 gives (I.3); lattice boundaries, floors and ceilings give only lower-order terms.

The fixed x-coefficient is $\alpha - 3h = 1/80$. The average ordinary degree is consequently

$$S_X(G) = (\alpha - 3h) + b_0 + t_0 + 3f_0 + 6h = \frac{9}{4}.$$

□

Lemma 100.5 (Resolution of the cubic pencil)**.** *On $\mathbb{P}^2_{x,z,t}$ put*

$$T_\lambda = (tz^2 - \lambda x^3 = 0), \qquad T_0 = T_0' + 2Z_0, \qquad T_\infty = 3X_0,$$

where X_0, Z_0, T_0' are the coordinate lines. An effective divisor

$$B = aX_0 + bZ_0 + cT_0' + fT_{-1} + \sum \nu_\lambda T_\lambda$$

with nonnegative coefficients, $\sum \nu_\lambda = 2h$ and $\nu_\lambda \le h$ is log canonical provided

$$a + 3h, \quad b + 2h, \quad c + h, \quad f + h, \quad h \le 1,$$

and

$$a + 3c + 3f + 6h \le 4, \qquad 2a + 3b + 6f + 12h \le 5.$$

Proof. Away from the coordinate vertices the pencil members are disjoint and smooth, so the component bounds suffice. At $[1:0:0]$, only the two transverse coordinate lines occur.

At $[0:1:0]$, put $z = 1$. The branches have equations $t = \lambda x^3$. Three successive ordinary blow-ups separate them. The final exceptional valuation has weights $(1, 3)$ and log discrepancy four. Its order on B is $a+3c+3f+6h$, independently of ν_0, ν_∞. The preceding two exceptional inequalities follow by interpolation with the coefficient bound on $x = 0$.

At $[0:0:1]$, put $t = 1$. The nonzero finite members are cusps $z^2 = \lambda x^3$. Write

$$u = a + 3\nu_\infty, \qquad v = b + 2\nu_0, \qquad e = f + 2h - \nu_\infty - \nu_0.$$

The regular toric subdivision with exceptional primitive vectors $(1,1), (2,3), (1,2)$ resolves all these cusps simultaneously. Their strict transforms meet the $(2,3)$ exceptional curve transversely at distinct points, away from the coordinate corners. The corresponding orders and log discrepancies are

$(1,1)$	$u + v + 2e$	2
$(2,3)$	$2u + 3v + 6e$	5
$(1,2)$	$u + 2v + 3e$	3.

The middle order is $2a + 3b + 6f + 12h$. The first equals one third of this order plus $u/3$; the last equals half this order plus $v/2$. Since $u, v \le 1$, the asserted bounds imply all three exceptional inequalities. After this subdivision the total transform has simple normal crossings and all coefficients are at most one. This verifies log canonicity at every point and for every divisorial valuation. □

Proof of the theorem. An arbitrary basis of each binary component (I.2) produces a divisor of the form in the preceding lemma, with

$$a = \alpha_m - 3h_m, \quad b = b_m, \quad c = t_m, \quad f = f_m.$$

Indeed the sum of orders of a basis of degree-D binary forms at any point is at most $D(D+1)/2$, while its total degree is $D(D+1)$. The limiting component bounds are

$$\frac{5}{8}, \quad \frac{13}{30}, \quad \frac{217}{240}, \quad \frac{3}{10}, \quad \frac{49}{240}.$$

The smooth tangency corner has order $29/8$ and discrepancy four; the cusp corner has order $25/8$ and discrepancy five. The largest of all these ratios is $29/32$. Thus all these basis divisors have log canonical threshold at least any fixed $\tau < 32/29$ for all sufficiently large m.

For a divisor over G, choose a basis of each primary piece compatible with its valuation filtration and the auxiliary weight filtration. The least-weight degeneration has the binary components (I.2), because the normal-basis count proved equality of algebras. Lower semicontinuity of log canonical thresholds therefore gives the same bound for the original valuation-adapted basis divisor. It follows that $\delta(G; W^G) \geq 32/29$. Finally, Abban–Zhuang adjunction gives

$$\delta_p(X) \geq \min\left\{\frac{3}{9/4}, \frac{32}{29}\right\} = \frac{32}{29}.$$

Again the proof needs no Zariski decomposition on the threefold extraction: the actual refined series is identified by sections and an exact Hilbert count, and its thresholds are controlled by two explicit surface resolutions of a pencil. □

101 Completion of the smooth classification

Theorem 101.1. *If η has no totally ramified point, then*

$$\delta(X) \geq \frac{100}{99} > 1.$$

Every smooth member of Family 3.5 is K-semistable. It is K-stable if and only if η has no totally ramified point. It is K-polystable if and only if η has either no totally ramified point or two such points. In the latter case η is equivalent to $[x : y] \mapsto [x^5 : y^5]$.

Proof. Assume first that η has no totally ramified point. The proof of Theorem 96.4 gives the bound $100/99$ at every point except, possibly, points on the joining lines of mixed fibres. Their A_2 endpoints are covered by Theorem 94.1. Their A_1 endpoints and smooth interior points are covered by Theorems 98.1 and 100.1. All three bounds $160/151$, $240/221$ and $32/29$ exceed $100/99$. Thus the same lower bound holds at every closed point of X. Every divisorial valuation has a centre containing a closed point, so the valuative criterion gives $\delta(X) \geq 100/99$ and K-stability.

The totally ramified cases were proved in Section 91: one such point gives strict K-semistability, whereas two give the monomial K-polystable member. There cannot be more than two, by Riemann–Hurwitz. The monomial member has a nontrivial one-parameter automorphism group, so is not K-stable. This proves every assertion. □

The weighted curve extraction gives another proof of semistability at the interior points. Its last point moment is exactly one; the ordinary point extraction is needed here for the stronger K-stability statement. In both arguments, *no Zariski decomposition on the extracted threefold is assumed or computed.* The complete associated series are identified from actual sections by exact Hilbert counts. The subsequent estimates concern complete binary spaces and explicit surface resolutions.

102 Liu's method: the CM line, GIT, and the reconstruction problem

We now compare the proof with Yuchen Liu's approach. There are two relevant arguments. In [36], one degenerates a log pair to a pair with a torus action, checks the latter, and uses interpolation. The moduli continuity argument of [37] first identifies the CM polarization and

then reconstructs the geometry of K-semistable limits. For Family 3.5 the CM calculation and the GIT problem admit explicit solutions. The smooth classification has already been proved above by point extractions. Reconstruction of arbitrary K-semistable limits is a separate question, needed for a K-moduli compactification.

102.1 The graph parameter space

Put $V = H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,5))$ and $P = \mathbb{P}(V) \cong \mathbb{P}^{11}$. Let $U \subset P$ be the open subset of equations $F = uf + vg$ for which f, g have no common zero. The group $G = \mathrm{SL}_2 \times \mathrm{SL}_2$ acts by changes of target and source coordinates. We use its natural linearization on $\mathcal{O}_P(1)$.

For clarity, our Hilbert–Mumford convention is the following. If a one-parameter subgroup λ has weights w_i on a basis of V, set $\mu(F, \lambda) = -\min\{w_i : F_i \neq 0\}$. A point is GIT-semistable if all these numbers are nonnegative, and GIT-stable if they are positive for every nontrivial one-parameter subgroup. A semistable point is GIT-polystable if its orbit is closed in the semistable locus. See [39].

Theorem 102.1 (GIT classification of smooth graphs)**.** *Every point of U is GIT-semistable. Such a point is GIT-stable if and only if its degree-five map has no totally ramified point. Among the points of U that are not stable, the only polystable orbit is that of $ux^5 + vy^5$. A graph with exactly one totally ramified point is strictly semistable and has this orbit in its orbit closure.*

Proof. Diagonalize a nontrivial one-parameter subgroup. Interchanging the eigenvectors in each factor if necessary, we may take the weights on u, v, x, y to be $-a, a, b, -b$, where $a, b \geq 0$ are integers and $(a, b) \neq (0, 0)$. The weights of the two sorts of monomials are

$$-a + (2i - 5)b, \qquad a + (2i - 5)b.$$

If $f(0,1) \neq 0$, the monomial uy^5 has negative weight. Otherwise $g(0,1) \neq 0$. Write $r = \mathrm{ord}_{(0:1)} f$; then $1 \leq r \leq 5$, since the two quintics are linearly independent. The support contains monomials of weights

$$-a + (2r - 5)b, \qquad a - 5b.$$

Since $r \leq 5$, their minimum is at most $\min\{-a + 5b, a - 5b\} = -|a - 5b| \leq 0$. This proves semistability. If $b = 0$, the minimum is negative. If $b > 0$, it can be zero only when $a = 5b$ and $r = 5$. Conversely, when $f = cx^5$ and $g(0,1) \neq 0$, the subgroup with $a = 5$, $b = 1$ has minimum weight zero. This proves the assertion about stability.

In the equality case its limit is $cux^5 + g(0,1)vy^5$. Thus every zero-weight limit of a smooth graph is in the displayed monomial orbit. For the monomial graph itself, such a limit is again in its orbit; the polystable form of the Hilbert–Mumford criterion proves that this orbit is closed. A graph with one totally ramified point is not equivalent to the monomial graph, which has two. If there are two, their ramification contributions exhaust eight, and the map is equivalent to $[x : y] \mapsto [x^5 : y^5]$. This proves all the remaining assertions. □

Example 102.2. For the mixed-fibre example, write $F = ux^3y^2 + v(x^5 + y^5)$. Its associated map has ramification indices three at zero and two at infinity; its five other ramification points satisfy $z^5 = 3/2$ and are simple. It has no totally ramified point, so it is GIT-stable. In the coordinates of the proof, the three monomial weights are $-a + b$, $a + 5b$, $a - 5b$. Their minimum is negative for every $a, b \geq 0$ with $a + b > 0$. The theorem checks the other choices of eigenvectors as well. Theorem 101.1 proves that its threefold is K-stable, in agreement with this GIT calculation.

102.2 The CM line, with its normalization

Let $\mathcal{C} \subset S \times P$ be the universal divisor, where $S = \mathbb{P}^1 \times Q \subset M = \mathbb{P}^1 \times \mathbb{P}^2$. Write $\rho : \mathcal{X} = \mathrm{Bl}_{\mathcal{C}}(M \times P) \to M \times P$, let $\mathcal{E}$ be its exceptional divisor, and let $p : \mathcal{X} \to P$ be the projection. Set $\mathcal{L} = \rho^*(-K_M) - \mathcal{E} = -K_{\mathcal{X}/P}$. The anticanonical CM normalization used here is

$$c_1(\lambda_{\mathrm{ac}}) = -p_*(\mathcal{L}^4).$$

Equivalently, λ_{ac} is the inverse of the fourfold Deligne pairing of $\mathcal{L}$. It is defined on P even where $\mathcal{L}$ is not relatively ample. On the open locus of Q-Fano fibres it is the anticanonical CM line in the normalization above; compare [37, Section 5.1].

Lemma 102.3 (The universal blow-up)**.** *The total space $\mathcal{X}$ is smooth and p is flat. Formation of its fibres commutes with the blow-up. Over U it is the smooth family of Family 3.5 threefolds.*

Proof. The universal divisor $\mathcal{C} \subset S \times P$ is smooth: evaluation at a point of S is a nonzero linear functional on V, so the projection $\mathcal{C} \to S$ is a projective bundle. It is a codimension-two regular embedding in $M \times P$. Moreover, it is a relative Cartier divisor on $S \times P$, flat over P. For its ideal I, each I^j/I^{j+1} is $\mathrm{Sym}^j(I/I^2)$ and is flat over P. Induction gives flatness of $\mathcal{O}/I^j$, hence of I^j, and base change for these ideals. Taking the relative Proj of the Rees algebra proves flatness and the fibre assertion. The total blow-up is smooth because its centre and ambient space are smooth. Over U every centre is a smooth graph and every fibre is the smooth Fano threefold described above. □

Proposition 102.4 (Exact CM class)**.** *There is a G-equivariant isomorphism*

$$\lambda_{\mathrm{ac}} \cong \mathcal{O}_P(22).$$

Proof. We compute on $S \times P$. Let A, B be the two ruling classes of S and let $h = c_1(\mathcal{O}_P(1))$. Thus $A^2 = B^2 = 0$, $\int_S AB = 1$, and $H|_S = 2B$. The universal centre has class $F = A + 5B + h$. Its normal bundle N in $M \times P$ fits into

$$0 \longrightarrow \mathcal{O}_{\mathcal{C}}(F) \longrightarrow N \longrightarrow \mathcal{O}_{\mathcal{C}}(4B) \longrightarrow 0.$$

In particular, $c_1(N) = A + 9B + h$ and $c_2(N) = 4B(A + 5B + h)$. The class $K = -K_M$ restricts to $2A + 6B$.

For the inclusion $i : \mathcal{C} \hookrightarrow M \times P$, the blow-up pushforward identities are

$$\rho_*\mathcal{E} = 0, \quad \rho_*\mathcal{E}^2 = -i_*1, \quad \rho_*\mathcal{E}^3 = -i_*c_1(N), \quad \rho_*\mathcal{E}^4 = -i_*(c_1(N)^2 - c_2(N)).$$

These follow by restricting to the projective bundle $\mathcal{E} \to \mathcal{C}$; in particular the last two signs are negative. Since $K^4 = 0$ on $M \times P$, expansion of $(\rho^*K - \mathcal{E})^4$ gives

$$-p_*(\mathcal{L}^4) = \mathrm{pr}_{P*}\big((6K^2 - 4Kc_1(N) + c_1(N)^2 - c_2(N))F\big).$$

Using $A^2 = B^2 = 0$, the factor in parentheses is

$$62AB - 6Ah - 10Bh + h^2.$$

Only the coefficient of ABh survives the pushforward. It is $62 - 30 - 10 = 22$. Hence $c_1(\lambda_{\mathrm{ac}}) = 22h$. Since $\mathrm{Pic}(P) = \mathbb{Z}[\mathcal{O}_P(1)]$, the underlying line is $\mathcal{O}_P(22)$. Finally, G has no characters, so two G-linearizations on the same line differ trivially. This proves the equivariant statement. □

Remark 102.5. Computing on the compact parameter space is useful here. The complement of U is the resultant divisor, so the class of a line on U alone would not determine its extension across the boundary. The universal construction fixes that extension and its linearization.

Corollary 102.6 (Test configurations induced by graph coordinates)**.** *Suppose a one-parameter subgroup of G acting on a smooth graph induces, by the universal blow-up, a normal test configuration with relatively ample anticanonical polarization. Its Donaldson–Futaki invariant is nonnegative. It is positive if the graph map has no totally ramified point and the subgroup is nontrivial.*

Proof. The CM weight has the same sign as the Donaldson–Futaki invariant. By Proposition 102.4, it is $22\mu(F,\lambda)$. Apply Theorem 102.1. The ampleness and normality assumptions ensure that this is a test configuration of the required kind; a blow-up of an arbitrary boundary graph is not asserted to have these properties. □

102.3 Why the earlier weighted limits do not finish the argument

Lemma 102.7 (The ramification limits are not semistable)**.** *For $r = 3, 4$, the canonical Fano threefold*

$$X_r = \mathrm{Bl}_{C_r}(\mathbb{P}^1 \times \mathbb{P}^2), \qquad C_r : \ ux^r y^{5-r} + vy^5 = 0 \quad \text{on } S,$$

is not K-semistable.

Proof. These are the central models in Lemma 93.2. The subgroup having weights $-r, r, 1, -1$ on u, v, x, y fixes $[C_r]$; both monomials have weight $r - 5$. Its weight on the CM line is $22(5 - r) \neq 0$. Consequently the product test configurations associated to this subgroup and its inverse have nonzero opposite Donaldson–Futaki invariants. One of them is negative.

There is also a direct check using the earlier first moment. In the primitive local normalization, the two Futaki values have absolute value

$$(r + 1) - S_{X_r}(G) = \frac{11(5 - r)}{80}.$$

The subgroup just used doubles the local weights, giving the same nonvanishing test. The common factor y^{5-r} in the centre must be retained throughout this calculation. □

Thus openness cannot be applied to these limits to establish K-semistability of the original smooth member. Nor does the interpolation step in [36] apply without additional input: there the desired variety is a double cover, corresponding to an interior boundary coefficient. Here the desired boundary coefficient on X is zero. For example, for $B \sim_{\mathbb{Q}} -K_X$ one has

$$\beta_{X,cB}(v) = \beta_X(v) + c(S_X(v) - v(B)).$$

Nonnegativity at a positive value of c does not imply nonnegativity at $c = 0$.

102.4 The volume estimate and the K3 lattice

For a klt three-dimensional germ $z \in Z$, recall that its normalized volume is

$$\widehat{\mathrm{vol}}(z, Z) = \inf_v A_Z(v)^3 \,\mathrm{vol}(v), \qquad \mathrm{vol}(v) = \limsup_{m\to\infty} \frac{6\,\ell(\mathcal{O}_{Z,z}/\mathfrak{a}_m(v))}{m^3},$$

where v is centred at z and $\mathfrak{a}_m(v) = \{f : v(f) \geq m\}$. A smooth threefold germ has normalized volume 27.

Lemma 102.8 (Restrictions on K-semistable limits)**.** *Let Z be a K-semistable $\mathbb{Q}$-Fano threefold admitting a $\mathbb{Q}$-Gorenstein smoothing to smooth members of Family 3.5. Then*

$$\widehat{\mathrm{vol}}(z, Z) \geq \frac{135}{16} > 8$$

for every closed point z. Moreover, Z is Gorenstein canonical, and a general anticanonical member is a K3 surface with du Val singularities and polarization of degree 20*. If a Q-Cartier Weil divisor on Z extends to a Q-Cartier Weil divisor on the total space of a Q-Gorenstein smoothing of Z, then it is Cartier.*

Proof. The volume is constant in the smoothing and equals 20. Liu's local-to-global inequality [38] gives $\widehat{\mathrm{vol}}(z, Z) \geq 20(3/4)^3 = 135/16$. The remaining assertions follow from [37, Theorem 4.2], whose volume assumption is exactly $(-K_Z)^3 \geq 20$. □

Lemma 102.9 (The anticanonical lattice on a smooth member)**.** *Let $S_X \in |-K_X|$ be a smooth anticanonical surface on a smooth member of Family 3.5. The restrictions of T, H, D span a rank-three lattice with Gram matrix*

$$\begin{pmatrix} 0 & 3 & 1 \\ 3 & 2 & 2 \\ 1 & 2 & -2 \end{pmatrix}.$$

Its determinant is 28*, and it is isometric to* $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \langle -28 \rangle$*. The anticanonical polarization is $2T + H + D$ and has square* 20*.*

Proof. Intersect the threefold classes with $L = -K_X$. The identities in Section 89 give

$$T^2L = 0, \quad THL = 3, \quad TDL = 1, \quad H^2L = 2, \quad HDL = 2, \quad D^2L = -2.$$

This proves the matrix and its nonzero determinant. Put $F_1 = T$, $F_2 = T + D$, $R = H - 8T - 3D$. Then $F_1^2 = F_2^2 = 0$, $F_1F_2 = 1$, $RF_1 = RF_2 = 0$, and $R^2 = -28$. The change of basis is integral with determinant one up to sign. Finally, $L = 2T + H + D = 6F_1 + 4F_2 + R$, whose square is $48 - 28 = 20$. No assertion that this lattice is the entire Picard group of every anticanonical surface is needed. □

These results supply the numerical inputs for Liu's moduli continuity method. What remains is to recover the contractions to $\mathbb{P}^1$ and $\mathbb{P}^2$, and the graph centre, on the relevant K-semistable limits. Lemma 102.8 does not say that the closures of T, H, D are Q-Cartier, and Cartierness alone would not prove their nefness or semiampleness. The K3 lattice calculation does not supply those missing assertions either.

In particular, Corollary 102.6 only excludes test configurations obtained from the fixed-conic graph parameter space. An arbitrary special test configuration need not arise that way. The smooth mixed-fibre case was settled instead by the exact local series in Sections 98–101. Together with Theorem 102.1, the smooth classification shows that GIT stability and K-stability agree for smooth graphs, and likewise for semistability and polystability. A reconstruction theorem of the kind proved in [37] is still required to identify the entire compactified K-moduli space with a GIT quotient.

102.5 The boundary of the graph parameter space

The following observations make the parameter space in Liu's method more explicit. They do not identify its GIT quotient with a K-moduli space.

Lemma 102.10 (GIT for pencils with base points)**.** *Let $F = uf + vg$ be a nonzero element of $H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,5))$. If f, g are linearly dependent, then $[F]$ is GIT-unstable. Otherwise, for each $p \in \mathbb{P}^1$, let $s_p < r_p$ be the vanishing sequence of the pencil $\langle f, g \rangle$ at p. Then $[F]$ is GIT-semistable if and only if $s_p + r_p \leq 5$ for every p, and is GIT-stable if and only if all these inequalities are strict. There are exactly three nonstable polystable orbits, represented by*

$$ux^5 + vy^5, \qquad xy(ux^3 + vy^3), \qquad x^2y^2(ux + vy).$$

Proof. In the dependent case, a change of target coordinates gives $F = uf$. A target one-parameter subgroup acts on this vector with strictly positive weight, proving instability.

Suppose that the pencil has rank two. Diagonalize a one-parameter subgroup and use weights $-a, a, b, -b$ on u, v, x, y, where $b \geq 0$. The case $b = 0$ is strictly positive in the Hilbert–Mumford convention used above, since both target coefficients are nonzero. Suppose $b > 0$, and put $p = (0 : 1)$. If the orders of the chosen coefficients are k, l, the minimum weight is

$$\min\{-a + (2k - 5)b,\ a + (2l - 5)b\}.$$

For any basis of the pencil, its two orders are either s_p, r_p or s_p, s_p. In particular, $k+l \leq s_p+r_p$. The displayed minimum is at most $(k+l-5)b$. It is therefore nonpositive, respectively negative, under the stated semistability, respectively stability, assumption.

Conversely, choose a basis with orders r_p, s_p, and take $a = r_p - s_p$, $b = 1$. The minimum weight is $r_p + s_p - 5$. This proves the two criteria. If $r_p + s_p = 5$, its zero-weight limit is, after rescaling,

$$x^{s_p} y^{s_p} (u x^{5-2s_p} + v y^{5-2s_p}).$$

Here $s_p = 0, 1, 2$. In each of these three examples, equality in the criterion occurs only at zero and infinity, and every zero-weight limit is in its own orbit. The polystable Hilbert–Mumford criterion proves that these orbits are closed in the semistable locus. Every nonstable semistable pencil has one of them in its orbit closure, so these are all the nonstable polystable orbits. □

Lemma 102.11 (Canonical Fano blow-ups of rank-two pencils)**.** *Let f, g be linearly independent binary quintics, possibly with a common factor. Set $C = (uf + vg = 0) \subset S \subset M$ and $X_F = \mathrm{Bl}_C M$. Then X_F is a normal Gorenstein canonical Fano threefold with $(-K_{X_F})^3 = 20$. If p is a common root of multiplicity s, the singularities over the corresponding vertical component of C are locally*

$$zw = x^s h.$$

In particular, over the GIT-semistable locus one has $s \leq 2$.

Proof. Write $uf + vg = c(x, y)(uf_0 + vg_0)$, where f_0, g_0 are relatively prime. Near a root of c, choose a local source coordinate x. The residual graph has an equation $h = t - \eta_0(x)$ after choosing an affine target coordinate. Together with a local equation z of S, the ideal of C is $(z, x^s h)$. One chart of its blow-up is smooth, and the other is the displayed hypersurface. The same description with h a unit applies away from the residual graph. The hypersurface is normal: its singular locus has codimension at least two, and it is Cohen–Macaulay. It is Gorenstein and toric. Its toric Gorenstein height function has value one on every primitive ray generator, so every nonzero lattice point in the cone has positive integral height. Thus the singularity is canonical. Where the centre is reduced and smooth the blow-up is smooth. This proves the assertions about singularities.

The ideal of C twisted by $\mathcal{O}_M(1, 3)$ is globally generated. Indeed, use the equation of S multiplied by the sections of $\mathcal{O}_M(1, 1)$ and the two cubic lifts of $x(uf + vg)$ and $y(uf + vg)$ from S. Consequently $B = T + 3H - E$ is globally generated and $-K_{X_F} = T + B$.

The centre is finite flat of degree five over the target $\mathbb{P}^1$: on each target fibre the quintic $uf + vg$ is nonzero, because f, g are independent. Since C is a relative complete intersection, the powers of its ideal commute with this base change. Thus a fibre V of $X_F \to \mathbb{P}^1$ is the blow-up of $\mathbb{P}^2$ in a length-five Cartier subscheme $Z \subset Q$. Its exceptional Cartier divisor is E_V, and $B|_V = 3h - E_V$. This is ample. To see this, its square is four. If a curve is contracted to a point in $\mathbb{P}^2$, positivity follows from the relative ampleness of $-E_V$. The strict transform of Q is isomorphic to Q and has intersection $6 - 5 = 1$ with $3h - E_V$. For the strict transform R of any other irreducible plane curve of degree d, the effective Cartier divisor $Q' = 2h - E_V$

does not contain R. Therefore

$$(3h - E_V) \cdot R = d + Q' \cdot R \geq d > 0.$$

Nakai–Moishezon gives ampleness on V.

The globally generated line bundles T and B define a morphism to a product of projective spaces. It has no contracted curve: such a curve would be in a target fibre and have zero intersection with $B|_V$. The morphism is finite, and $T + B$ is its pullback of an ample line bundle. Thus $-K_{X_F}$ is ample. Its cube is twenty by flatness of the universal blow-up, or by the same intersection calculation as for a smooth graph. Finally, the GIT criterion gives $2s + 1 \leq s + r \leq 5$, hence $s \leq 2$. □

Remark 102.12. The first boundary example $xy(ux^3 + vy^3)$ has two ordinary double points; the second example $x^2y^2(ux + vy)$ has two singular curves with generic transverse type A_1. At the intersections with the residual graph the latter has equation $zw = x^2h$. The absence of a Futaki character for their visible one-dimensional tori does not alone prove K-semistability. For these particular models there is, however, a further equivariant argument, as follows.

Lemma 102.13 (The distinguished surface at a boundary pencil)**.** *Suppose that the rank-two pencil F is GIT-semistable. Then $\delta_p(X_F) \geq 80/73$ for every $p \in D$.*

Proof. The strict transform D is smooth, the threefold is smooth along D, and $D|_D = \mathcal{O}(-1,-1)$. Let Z be the source ruling through p, and let s be its multiplicity as a component of C. The GIT criterion gives $s \leq 2$. The nef and fixed parts of $L - uD$ have the same classes as in Lemma 90.1. In the second interval the fixed restriction is $(u-1)C$. Consequently the first moment along Z is the old moving moment plus the contribution of its fixed multiplicity:

$$S(W^D; Z) = \frac{61}{80} + \frac{3s}{20}\int_1^{3/2} 4(6 - 4u)(u-1)\,du = \frac{61 + 4s}{80}.$$

After removing sZ, the remaining divisor $C - sZ$ meets Z transversely at one point: the other base rulings are disjoint from Z, and the residual graph has degree one over the source. Thus the secondary moving moment is $69/80$ and its fixed contribution is at most $1/20$, exactly as in Lemma 90.3. The flag inequality gives

$$\delta_p(X_F) \geq \min\left\{\frac{40}{31}, \frac{80}{61 + 4s}, \frac{80}{73}\right\} = \frac{80}{73}.$$

□

Proposition 102.14 (The two additional closed orbits)**.** *Let $b = 1, 2$, and let X_b be the canonical Fano threefold obtained by blowing up*

$$C_b: \quad x^b y^b (ux^{5-2b} + vy^{5-2b}) = 0.$$

Then X_b is K-polystable.

Proof. Put $d = 5 - 2b$. The torus

$$([u : v], [x : y]) \longmapsto ([\lambda^{-d}u : v], [\lambda x : y])$$

and the involution $([u : v], [x : y]) \mapsto ([v : u], [y : x])$ preserve the centre, including its multiplicities. They act on X_b and generate a reductive group $G = \mathbb{G}_m \rtimes \boldsymbol{\mu}_2$. There is no point of the target $\mathbb{P}^1$ fixed by all of G.

The surface D is still $\mathbb{P}^1 \times \mathbb{P}^1$, with normal bundle $\mathcal{O}(-1,-1)$. The nef and fixed parts in Lemma 90.1 hold on X_b with the same proof: B is globally generated; restriction to D forces

the stated multiples of D; and the fibres of every component of E force the stated multiples of E. Over a base component of multiplicity s, locally $E = sE_{\mathrm{red}}$ and $E_{\mathrm{red}} \cdot f = -1/s$ on an exceptional fibre; hence the forced fixed part is still $(u-1)E$. Thus the expected orders and the moving series on a target fibre are unchanged. Vanishing for the nef and big positive parts applies because X_b is canonical.

The preceding lemma gives $\delta_p(X_b) \geq 80/73$ at all points of D, including points on the multiple components of C_b.

For $b = 1$, a general target fibre is a smooth quartic del Pezzo surface: the length-five scheme consists of the two simple base points and the three distinct points of a general residual fibre. At its points outside D, the proof of Lemma 90.4 gives $\delta_p(X_b) \geq 10/9$. For $b = 2$, a general fibre has cluster type $(2, 2, 1)$ and two nodes. At its smooth points outside D, the polarized-surface computation in [14, Appendix A], used in Lemma 96.1, gives the bound $100/99$. The restricted classes and the marked conic are exactly the same. The pair (X_b, T) is plt by inversion of adjunction, since T is Cartier and has Du Val singularities; thus the same refinement applies. We need this estimate only at points where the ambient threefold is smooth. The two singular curves of X_2 are disjoint and exchanged by the involution, so no irreducible G-invariant centre lies in their union.

Let F be a G-invariant prime divisor over X_b, and put $Z = c_{X_b}(F)$. Its image in the target is irreducible, closed and G-invariant, so it is the whole target. Choose a general point p of Z on a general target fibre, outside the singular locus when $b = 2$. If Z lies in D, use the bound $80/73$; otherwise choose p outside D and use the smooth or nodal fibre bound. In all cases $A_{X_b}(F) > S_{X_b}(F)$. The equivariant valuative criterion for K-polystability, [3, Theorem 1.22], proves the assertion. □

Corollary 102.15. *Every rank-two pencil which is GIT-semistable but not GIT-stable gives a K-semistable canonical Fano threefold X_F.*

Proof. The zero-weight one-parameter subgroup in the GIT lemma gives an isotrivial degeneration to one of the three displayed closed orbits. Its universal blow-up is flat, and all its fibres are canonical Fano threefolds by the rank-two pencil lemma. Thus it is a Q-Gorenstein Fano family. Its central fibre is K-polystable by Lemma 91.1 or the preceding proposition. Openness of K-semistability, followed by the isomorphisms of the nonzero fibres, proves the assertion. □

Remark 102.16. These boundary results supply two additional K-polystable models for the moduli-continuity argument. The smooth mixed-fibre case is proved by the local extractions above. Its graph is GIT-stable and therefore has neither of these boundary orbits in its semistable orbit closure; the boundary argument is a separate application of Liu's method.

Lemma 102.17 (A reducible-conic boundary)**.** *Let $Q = L_1 \cup L_2 \subset \mathbb{P}^2$ be two distinct lines. Suppose that C is a Cartier divisor on $\mathbb{P}^1 \times Q$ whose restrictions to $S_i = \mathbb{P}^1 \times L_i$ are graphs of degrees a_i, where $a_1 + a_2 = 5$. Assume these graphs have the same restriction at $S_1 \cap S_2$, so that the Cartier divisor is defined there. Let Y be the blow-up of M along C. Then $-K_Y$ is not ample. If $-K_Y$ is nef, then $\{a_1, a_2\} = \{2, 3\}$, and its restriction to the strict transform of the degree-three component is $\mathcal{O}(1, 0)$.*

Proof. The ideal of C restricts to the principal ideal of its graph on each S_i. Consequently its strict transform D_i is isomorphic to S_i, and the exceptional Cartier divisor restricts to $\mathcal{O}(1, a_i)$. Since $-K_M|_{S_i} = \mathcal{O}(2, 3)$, one has

$$-K_Y|_{D_i} = \mathcal{O}(1, 3 - a_i).$$

Ampleness would require $a_1, a_2 \leq 2$, contrary to their sum. Nefness requires $a_1, a_2 \leq 3$, leaving only the stated pair. □

Thus even the simplest conic degeneration needs an anticanonical contraction before it can give a Fano boundary model. The preceding lemma does not exclude that contracted model from K-moduli; its singularities and its K-stability require a separate argument.

Part XIV
Family 3.6: double conics and weighted curve extractions

103 Geometry and the statement for Family 3.6

Let $C \subset \mathbb{P}^3$ be a smooth elliptic quartic, let $\Lambda \subset \mathbb{P}^3$ be a line disjoint from C, and let $\pi : X \to \mathbb{P}^3$ blow up $C \sqcup \Lambda$. These are exactly the smooth members of Family 3.6. Write E, R for the exceptional divisors over C, Λ, respectively, and put

$$H = \pi^*\mathcal{O}_{\mathbb{P}^3}(1), \qquad A = H - R, \qquad B = 2H - E.$$

The two pencils define morphisms $\sigma : X \to \mathbb{P}^1$ and $\phi : X \to \mathbb{P}^1$. Their general fibres are del Pezzo surfaces of degrees five and six, respectively. The product $\eta = (\sigma, \phi)$ is a conic bundle over $\mathbb{P}^1 \times \mathbb{P}^1$. We prove the following assertion.

Theorem 103.1. *Every smooth Fano threefold in Family 3.6 is K-stable.*

Cheltsov proved this when every conic fibre is reduced [42, Theorem 1]. The omitted configurations are quintic fibres with two A_1 points or one A_3 point. We first reduce possible nonpositive-beta centres to their double conics. We then blow up the scheme-theoretic conic fibre. The resulting exceptional surface has a nonzero different, which is retained in every subsequent estimate.

Lemma 103.2. *Write $C = (q_0 = q_1 = 0)$ and $\Lambda = (l_0 = l_1 = 0)$. The variety X is the complete intersection*

$$u_0q_1 - u_1q_0 = 0, \qquad v_0l_1 - v_1l_0 = 0$$

in $\mathbb{P}^3 \times \mathbb{P}^1 \times \mathbb{P}^1$, of classes $2H + B$ and $H + A$. In particular,

$$-K_X = H + A + B = 4H - E - R, \qquad (-K_X)^3 = 22.$$

Every quintic fibre is normal and has only Du Val singularities. Its possible singularity types are $A_1, A_2, 2A_1, A_3$.

Proof. Each equation is the graph of a pencil whose base ideal is a regular sequence. Since the two centres are disjoint, the product of the two graphs is their simultaneous blow-up. Adjunction gives the asserted anticanonical class. The nonzero intersection numbers needed below are

$$H^3 = 1, \quad HE^2 = -4, \quad E^3 = -16, \quad HR^2 = -1, \quad R^3 = -2;$$

also $H^2E = H^2R = ER = 0$. They give the degree 22.

A plane through Λ cuts out on the smooth curve C a curvilinear scheme of length four. On that plane the fibre is the blow-up of the scheme. A length-k part has local ideal (y, x^k); one chart of its blow-up has equation $yt = x^k$ and gives an A_{k-1} singularity. The partitions of four give the stated list. The pencil of restricted quadrics supplies a conic bundle on each such fibre. The partitions $(2, 2)$ and (4) give a double conic, as described explicitly below. □

Lemma 103.3 (The two fibre moments)**.** *For a fibre $S \in |A|$ and a fibre $T \in |B|$,*

$$S_X(S) = \frac{67}{88}, \qquad S_X(T) = \frac{109}{176}.$$

More precisely, the positive and negative parts of $-K_X - uS$ are

u	$P(u)$	$N(u)$	$P(u)^3$
$[0,1]$	$(4-u)H - E + (u-1)R$	0	$22 - 15u$
$[1,2]$	$(4-u)H - E$	$(u-1)R$	$(4-u)^3 - 12(4-u) + 16,$

and those of $-K_X - uT$ are

u	$P(u)$	$N(u)$	$P(u)^3$
$[0,1]$	$(4-2u)H + (u-1)E - R$	0	$22 - 18u$
$[1,3/2]$	$(4-2u)H - R$	$(u-1)E$	$(4-2u)^3 - 3(4-2u) + 2.$

The pseudoeffective thresholds are 2 and $3/2$, respectively.

Proof. The nef classes H, A, B express the positive parts in the displayed intervals. For example, in the second interval of the first table, $P(u) = (2-u)H + B$; in that of the second table, $P(u) = (3-2u)H + A$. Intersection with the fibres of $R \to \Lambda$ or $E \to C$ forces the indicated fixed part when $u > 1$. The effective cone is generated by E, R, A, B [42, Section 2]; it gives the two thresholds. The intersection numbers in Lemma 103.2 give the cubics in the tables. Thus, explicitly,

$$22S_X(S) = \int_0^1 (22 - 15u)\,du + \int_1^2 \left((4-u)^3 - 12(4-u) + 16\right) du = \frac{67}{4},$$
$$22S_X(T) = \int_0^1 (22 - 18u)\,du + \int_1^{3/2} \left((4-2u)^3 - 3(4-2u) + 2\right) du = \frac{109}{8}.$$

□

Lemma 103.4 (Previously established local estimates)**.** *The following estimates hold for the actual fibre refinements.*

1. *If $S \in |A|$ is smooth, then $\delta_p(S; W^S) > 1$ for every $p \in S$.*
2. *If $S \in |A|$ has just one singularity, of type A_1 or A_2, then $\delta_p(S; W^S) > 1$ away from the conic fibre through that singularity.*
3. *If $T \in |B|$ is smooth, then $\delta_p(T; W^T) \geq 1$. The inequality is strict unless $p \in E$ and p lies on a (-1)-curve of T.*

Every prime divisor whose centre on X is a surface has positive beta.

Proof. The three estimates are [42, Corollaries 2–4]; they are local statements about the indicated fibre and do not assume that other fibres of X are reduced. The divisorial assertion is the divisorial-stability calculation recalled in [42, Section 2]. □

104 Reduction to the double conics

104.1 The two remaining quintic surfaces

We retain the notation E, R, H for the two exceptional divisors and the pull-back of a plane. In particular, $E \cap R = \varnothing$. Let $S \in |H - R|$, and put $D_a = ah - e_1 - e_2 - e_3 - e_4$ on its minimal resolution, where h is the pull-back of a line and the e_i are total-transform exceptional classes. The latter convention is important: $e_i^2 = -1$ and $e_i \cdot e_j = 0$ for $i \neq j$.

Lemma 104.1. *Suppose that S has singularities $2A_1$ or A_3. The conic bundle on S has a unique non-reduced fibre 2ℓ. Its minimal resolution is the blow-up of the plane at a curvilinear scheme of type $(2,2)$ or (4), respectively. The pencil of conics through this scheme is base-point-free on the resolution and has class*

$$F = 2h - e_1 - e_2 - e_3 - e_4.$$

In type $2A_1$, the only other singular member is $L_1 + L_2$, where

$$L_1 = h - e_1 - e_2, \qquad L_2 = h - e_3 - e_4.$$

The curves L_1 and L_2 are smooth (-1)-curves, intersect once, and avoid the singularities of S. In type A_3, every fibre other than 2ℓ is smooth.

Proof. Let Π be the plane defining S, and let $Z = C \cap \Pi$. By Lemma 103.2, Z is curvilinear of type $(2,2)$ or (4), respectively. It is the complete intersection of the two restricted quadrics. In type $(2,2)$ let $\lambda = 0$ be the line joining the two support points; its square vanishes on both length-two parts of Z. In type (4) let $\lambda = 0$ be the tangent line to Z. In $\mathcal{O}_Z \cong \mathbb{C}[t]/(t^4)$ its equation belongs to (t^2), so its square again vanishes on Z. The twisted Koszul resolution

$$0 \longrightarrow \mathcal{O}_\Pi(-2) \longrightarrow \mathcal{O}_\Pi^{\oplus 2} \longrightarrow \mathcal{I}_Z(2) \longrightarrow 0$$

shows that $H^0(\Pi, \mathcal{I}_Z(2))$ is exactly the given pencil. Thus λ^2 is one of its members.

Write the pencil as $\langle q, \lambda^2 \rangle$, with $q = 0$ smooth. Such a choice exists: Bertini gives smoothness away from Z, and curvilinearity makes the differential of every member independent of λ^2 nonzero at each support point of Z. In type $(2,2)$, the line meets $q = 0$ in two distinct points. Suitable coordinates give $q = xz - y^2$ and $\lambda = y$. Its only singular members are y^2 and xz, the latter being the two tangent lines to $q = 0$ at these points. In type (4), the line is tangent to $q = 0$, and we may take $q = xz - y^2$ and $\lambda = x$. Every finite member $xz - y^2 + tx^2$ is smooth. The graph equation identifies the fibres on S with these plane conics scheme-theoretically. In particular, there is precisely one non-reduced fibre, namely 2ℓ.

A length-k part of Z has local ideal (v, u^k). Its k successive point blow-ups resolve the pencil; their first $k-1$ exceptional curves form the A_{k-1} chain contracted on S. All base multiplicities are one, giving $F = 2h - e_1 - e_2 - e_3 - e_4$. In type $(2,2)$ the two tangent lines have strict transforms $L_1 = h - e_1 - e_2$ and $L_2 = h - e_3 - e_4$. They are smooth, have self-intersection -1, intersect once away from the base scheme, and are disjoint from the exceptional roots $e_1 - e_2, e_3 - e_4$. In type (4) the exceptional roots are $e_1 - e_2, e_2 - e_3, e_3 - e_4$. This proves all the assertions. □

For the refinement along S, the positive restriction is $D_3 = -K_S$ for $0 \le u \le 1$, and D_{4-u} for $1 \le u \le 2$. The fixed part is $(u-1)R|_S$ in the second interval. For a function g on $[2,3]$, write its integrated value, without introducing an additional invariant, as

$$\frac{3}{22}\left(g(3) + \int_2^3 g(a)\,da\right).$$

Lemma 104.2. *Let F be a smooth fibre different from 2ℓ, and let $p \in F$. The integrated curve moment of the moving restriction is $109/176$. Its integrated point moment is $37/44$ if $p \notin E$, is $79/88$ if $p \in E$ and S has type $2A_1$, and is $21/22$ if $p \in E$ and S has type A_3.*

Proof. The Zariski decomposition of $D_a - vF$ is

v	P	N
$0 \le v \le 1$	$D_a - vF$	0
$1 \le v \le a/2$	$(a-2v)h$	$(v-1)(e_1+e_2+e_3+e_4)$.

The first positive part equals $(a-2)h+(1-v)F$, so it is nef. In the second interval P is nef, N is exceptional for the morphism to the plane, and $P\cdot N=0$. The pseudo-effective threshold is $a/2$. Thus the volume and the degree of the moving restriction to F are

v	P^2	$P\cdot F$
$0\le v\le 1$	$a^2-4-4(a-2)v$	$2(a-2)$
$1\le v\le a/2$	$(a-2v)^2$	$2(a-2v)$.

Their relevant integrals are

$$\int_0^{a/2} P^2\,dv = a(a-2)+\frac{(a-2)^3}{6}, \qquad \int_0^{a/2}(P\cdot F)^2\,dv = 4(a-2)^2+\frac{2}{3}(a-2)^3.$$

The integrated values are 109/176 and 37/44.

The restriction of $e_1+\cdots+e_4$ to F has coefficient $r=2$ at each of its two points in type $2A_1$, and coefficient $r=4$ at its single point in type A_3. Indeed, in the latter case

$$e_1+\cdots+e_4=(e_1-e_2)+2(e_2-e_3)+3(e_3-e_4)+4e_4,$$

and F meets only the last curve. The additional point moment is obtained by integrating

$$2\int_1^{a/2}(P\cdot F)r(v-1)\,dv=\frac{r}{6}(a-2)^3.$$

Its integrated value is $5r/176$. This gives the asserted numbers. □

Lemma 104.3. *Suppose that S has type $2A_1$, and let $p\in L_1$. The integrated curve moment of L_1 is 73/88. The integrated point moment is 59/88 away from $L_2\cup E$, is 73/88 at $L_1\cap L_2$, and is 79/88 at $L_1\cap E$. The same statements hold with the two components interchanged.*

Proof. The Zariski decomposition of D_a-vL_1 is

v	P	N
$0\le v\le a-2$	D_a-vL_1	0
$a-2\le v\le 1$	$D_a-vL_1-(v-a+2)L_2$	$(v-a+2)L_2$
$1\le v\le a-1$	$(a-v-1)(2h-e_3-e_4)$	$(v-a+2)L_2+(v-1)(e_1+e_2)$.

The classes h, F and $2h-e_3-e_4$ are nef. At the first wall the positive part is $(3-a)F+(a-2)(2h-e_3-e_4)$, and at the second wall it is $(a-2)(2h-e_3-e_4)$. This proves nefness throughout the first two intervals by convexity. The negative supports are negative definite and orthogonal to P. The last displayed positive part is nef, and the pseudo-effective threshold is $a-1$.

In these three intervals the volumes are, respectively,

$$(a-v)^2-2(1-v)^2-2, \qquad (a-v)^2-2(1-v)^2-2+(v-a+2)^2, \qquad 2(a-v-1)^2,$$

while the degrees on L_1 are $a-2+v$, $2a-4$ and $2(a-v-1)$. Direct integration gives

$$\int P^2\,dv=\frac{(a-2)(a^2+2a-2)}{3}, \qquad \int(P\cdot L_1)^2\,dv=\frac{(14-a)(a-2)^2}{3}.$$

The extra fixed-part point integral at $L_1\cap L_2$ is

$$\frac{2(a-2)(a^2-7a+13)}{3},$$

and that at $L_1\cap E$ is $4(a-2)^3/3$. For the latter coefficient, use $e_1+e_2=(e_1-e_2)+2e_2$. Applying the preceding integration in a gives 73/88, 59/88, 7/44 and 5/22, respectively. Adding the appropriate fixed-part term gives the claim. □

Proposition 104.4. *If S has type $2A_1$ or A_3, then*

$$\delta_p(S, W^S_{\bullet,\bullet}) > 1$$

at every point $p \in S \setminus \ell$.

Proof. All such points are smooth. Apply the surface flag inequality to the fibre through p, using its component if the fibre is reducible. The preceding lemmas bound the moving contribution, divided by the log discrepancy, by the larger of the integrated curve and point moments. These bounds also apply to a divisorial valuation whose centre is the flag curve itself.

Here the integrations precede the maximum. To see this directly, work at a sufficiently divisible finite level and choose, in each restricted section space, a basis simultaneously adapted to the given valuation and to the flag curve C. Remove the prescribed fixed divisor and the powers of C from its elements. The resulting normalized basis divisor has the form $c_m C + B_m$, where C is not a component of B_m. The calculations above bound c_m by the integrated curve moment plus a term tending to zero, and bound the coefficient of $B_m|_C$ at p by the integrated point moment plus a term tending to zero. Inversion of adjunction gives

$$\operatorname{ord}_T(c_m C + B_m) \le c_m \operatorname{ord}_T(C) + b_m\big(A_S(T) - \operatorname{ord}_T(C)\big),$$

where b_m is the latter bound. Passing to the limit gives the maximum used above. This argument uses the bases of the individual spaces; it does not require the moving parts alone to form a graded algebra.

It remains to include the fixed part $(u-1)R|_S$. The curve $R|_S$ is smooth and avoids the singularities of S, so $\operatorname{ord}_T(R|_S) \le A_S(T)$ for every divisor T over S. Its contribution is therefore at most

$$\frac{3}{22}\int_1^2 (u-1)(2-u)(6-u)\,du\,A_S(T) = \frac{9}{88}A_S(T).$$

This contribution is zero if $p \notin R$. Moreover, $R \cap E$ is empty. Thus a smooth conic flag gives at most $83/88$ at points in R, at most $79/88$ at points in E in type $2A_1$, and at most $21/22$ at points in E in type A_3. At all other points it gives at most $37/44$. A component of a reducible fibre gives at most $82/88$ on R, at most $79/88$ on E, and at most $73/88$ elsewhere. Every displayed number is strictly smaller than one. □

Remark 104.5. The use of the smoothness of $R|_S$ in the last proof is useful when $R|_S$ is tangent to the chosen conic. Restricting the fixed divisor to that conic would count the tangency twice. The inequality $\operatorname{ord}_T(R|_S) \le A_S(T)$ applies before this restriction and avoids this unnecessary loss.

Proposition 104.6. *If a prime divisor F over X satisfies $\beta_X(F) \le 0$, its centre is contained in the support of a double fibre of η.*

Proof. Let $Z = c_X(F)$ and choose a general point $p \in Z$. Lemma 103.4 excludes surface centres and smooth quintic fibres. Since the singular quintic fibres are finite in number, Z lies in one of them, say S. The estimates just proved exclude the complement of its double conic if S has type $2A_1$ or A_3.

It remains to consider a fibre with one A_1 or A_2 point o. By Lemma 103.4, Z is contained in the conic fibre C_o through o. This fibre is a pair of distinct lines meeting at o. Let $T \in |B|$ be the fibre containing it. Its image in $\mathbb{P}^3$ is smooth: a rank-three quadric has only double lines or pairs of generator lines as singular plane sections, and in the latter case their intersection is its vertex, which cannot lie on the smooth complete intersection C. Here $\pi(o) \in C$. The line

Λ meets the two components at distinct smooth points, since it is disjoint from C and their intersection is $\pi(o)$. Thus T is a smooth sextic del Pezzo surface.

The third estimate of Lemma 103.4 confines Z to the finitely many intersections of the smooth elliptic curve $E|_T$ with the six (-1)-curves. Consequently Z is a point. The strict Abban–Zhuang inequality along T, whose refined threshold is at least one, now gives

$$\beta_X(F) \geq \operatorname{ord}_F(T)\left(1 - \frac{109}{176}\right) > 0,$$

a contradiction. This proves the reduction. □

105 The weighted extraction and its actual section ring

Write a double conic as 2ℓ. We use the normalized blow-up of its fibre ideal, rather than the ordinary blow-up of its support. The latter has exceptional surface $\mathbb{F}_3$, since $N_{\ell/X} = \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$. Indeed, the derivative of the conic equation along its double line gives a surjection $\mathcal{O}(1) \oplus \mathcal{O}^{\oplus 2} \to \mathcal{O}(2)$. It vanishes on $\mathcal{O}(1)$, and the other two entries are relatively prime binary quadratics, so the kernel is as stated. In the weighted construction below the exceptional pair is a product of two projective lines with a smooth boundary. Put $E_C = E$ and $E_\Lambda = R$ in this section.

Lemma 105.1. *Let X be the blowup of $\mathbb{P}^3$ along a smooth complete intersection $C = \{q_0 = q_1 = 0\}$ of two quadrics and a line Λ disjoint from C. Put $A = H - E_\Lambda$ and $B = 2H - E_C$. Suppose that a fibre of the conic bundle $\eta = (|A|, |B|) : X \to \mathbb{P}^1 \times \mathbb{P}^1$ is 2ℓ. There are homogeneous coordinates $[r : s : z : w]$ on $\mathbb{P}^3$ such that*

$$\Lambda = \{s = w = 0\}, \qquad \pi(\ell) = \{z = w = 0\}, \qquad q_0 = z^2 + w(c_0 r + c_1 s + c_2 z + c_3 w),$$

where $(c_0, c_1) \neq (0, 0)$. If

$$f(r, s) = s(c_0 r + c_1 s), \qquad g(r, s) = q_1(r, s, 0, 0),$$

then f and g have no common zero on $\mathbb{P}^1$. In particular, $g(1, 0) \neq 0$. On a neighbourhood of the whole curve ℓ, the threefold is given in $\mathbb{P}^2_{[r:s:z]} \times \mathbb{A}^2_{a,b}$ by an equation whose terms of smallest weight, for $\operatorname{wt}(z) = 1$ and $\operatorname{wt}(a) = \operatorname{wt}(b) = 2$, are

$$z^2 + af(r, s) + bg(r, s).$$

Proof. The plane corresponding to the first coordinate of $\eta(\ell)$ contains Λ, and its intersection with the corresponding quadric is the double line $2\pi(\ell)$. The quadric cannot be smooth, since a tangent plane section of a smooth quadric is a pair of distinct lines. Every singular member of the pencil through a smooth elliptic quartic is a rank-three quadric. Choose the plane $w = 0$ and its double line $z = w = 0$; the displayed form follows. The rank-three condition is precisely $(c_0, c_1) \neq (0, 0)$. The zeroes of f are the point $\Lambda \cap \pi(\ell) = [1 : 0 : 0 : 0]$ and the vertex of $q_0 = 0$, counted with multiplicity. The first point does not belong to C. The vertex does not belong to C either, since otherwise the complete intersection $\{q_0 = q_1 = 0\}$ would be singular. Thus f and g are relatively prime.

In the chart of the pencil of planes through Λ with $w = as$, the blowup of Λ is a trivial $\mathbb{P}^2$-bundle with fibre coordinates $[r : s : z]$. The second pencil has local equation $q_0 + bq_1 = 0$. Substitution gives

$$z^2 + as(c_0 r + c_1 s + c_2 z + c_3 as) + bq_1(r, s, z, as) = 0.$$

This equation contains the whole strict transform of $\pi(\ell)$ and has the stated initial form. The use of the blowup chart is essential at $\Lambda \cap \pi(\ell)$: the coordinate a remains regular there. □

Lemma 105.2. *In the notation of Lemma 105.1, let $\sigma : Y \to X$ be the normalized blowup of the ideal (a, b), and let G be its reduced exceptional divisor. Then σ is the weighted blowup of ℓ with transverse weights $(1, 2)$, and*

$$A_X(G) = 3, \qquad G \cong \mathbb{P}^1 \times \mathbb{P}^1, \qquad (K_Y + G)|_G = K_G + \frac{1}{2}D,$$

where

$$D = \{Uf(r, s) + Vg(r, s) = 0\} \subset \mathbb{P}^1_{[r:s]} \times \mathbb{P}^1_{[U:V]}$$

is a smooth curve of bidegree $(2, 1)$. The pair (Y, G) is purely log terminal.

Proof. At every point of ℓ at least one of f and g is nonzero. If $g \neq 0$, the equation of X allows us to eliminate b, and $(a, b) = (a, z^2)$ in the local ring. If $f \neq 0$, the same argument, with a eliminated, gives $(a, b) = (b, z^2)$; this equality follows as well from the implicit function theorem, or from the completed local ring. The normalized blowup of (a, z^2) is the weighted blowup with $\operatorname{ord}_G(z) = 1$ and $\operatorname{ord}_G(a) = 2$. Its log discrepancy is $1 + 2 = 3$.

The exceptional divisor is the relative weighted hypersurface

$$Z^2 + Uf(r, s) + Vg(r, s) = 0$$

with weights $(1, 2, 2)$ in $[Z : U : V]$. The projection to $[U : V]$ identifies its coarse space with $\mathbb{P}^1_{[r:s]} \times \mathbb{P}^1_{[U:V]}$: indeed, the second Veronese replaces Z by $T = Z^2$, and the equation eliminates T. Here Z has degree one along the first factor, so that T, Uf and Vg have the same degree. Along $Z = 0$ the transverse weighted chart has stabilizer of order two. It follows either by adjunction on the index cover, or directly from the cyclic quotient chart of the $(1, 2)$-weighted blowup, that the different is $D/2$. There is no other divisorial stabilizer. Since f and g are relatively prime, D is the graph of the morphism $[r : s] \mapsto [-g(r, s) : f(r, s)]$; it is smooth and has bidegree $(2, 1)$. The local weighted blowup of a smooth codimension-two centre with coprime transverse weights is purely log terminal. This also follows here from the quotient charts and the smoothness of D. □

Remark 105.3. If $c_0 = 0$, the line Λ passes through the vertex of the singular quadric. The quadric fibre of $|B|$ is then nonnormal. For example, blowing up $\{y = z = 0\}$ in the quadric $x^2 - yz = 0$ gives, in the chart $y = tz$, the surface $x^2 - tz^2 = 0$, singular along $x = z = 0$. Thus a proof using only sextic del Pezzo surfaces with Du Val singularities would omit these smooth threefolds. The weighted extraction above applies without alteration: in this case f is a square, while f and g remain relatively prime and D remains smooth.

Lemma 105.4. *Let $T = \mathrm{Bl}_\Lambda(\mathbb{P}^3) \times \mathbb{P}^1$, so that $X \subset T$ has class $2H + B$. Write the Cox coordinates of T as $r, z, s, a_0, a_1, b_0, b_1$, of respective divisor classes*

$$H, H, H - A, A, A, B, B.$$

Assign these coordinates weights $0, 1, 0, 0, 2, 0, 2$. Then the initial equation of X is

$$F_0 = b_0 z^2 + b_0 a_1 s(c_0 r + c_1 a_0 s) + b_1 g(r, a_0 s).$$

This equation is irreducible. The induced weight filtration on $H^0(X, -mK_X)$ is the filtration defined by G.

Proof. The initial equation follows from the normal form above, after homogenizing s by a_0. Regard F_0 as a polynomial of degree one in a_1. Its coefficient is $b_0 s(c_0 r + c_1 a_0 s)$, and its constant term is $b_0 z^2 + b_1 g(r, a_0 s)$. These polynomials are relatively prime: no factor of $s(c_0 r + c_1 a_0 s)$ divides the term $b_0 z^2$, and b_0 does not divide $b_1 g$. In the case $c_0 = 0$ the extra factor a_0 also does not divide the constant term. Gauss's lemma proves irreducibility.

For $m \geq 1$ the restriction sequence for $X \subset T$ is exact on sections in degree $m(H+A+B)$. Indeed, after subtracting $2H+B$ the relative degree is $m-2$. For $m=1$ the relative line bundle $\mathcal{O}_{\mathbb{P}^2}(-1)$ is acyclic; for $m \geq 2$ the pushforward is a direct sum of line bundles with nonnegative degrees on both base factors. Thus the relevant section spaces are the homogeneous pieces of the Cox ring modulo the equation of X.

Taking the initial form of a principal ideal gives the ideal (F_0). As its quotient is a domain, localization in a coordinate chart meeting ℓ is injective. In that chart the same filtration is the $(1,2)$-weight filtration just computed on the transverse parameters. Consequently a nonzero initial section cannot disappear after restriction to the weighted normal cone. This identifies the global weight filtration with ord_G, including all possible cancellations between sections. □

Lemma 105.5 (The weighted curve moment)**.** *For the extraction of Lemma 105.2, the weights on the product-graph coordinates are*

$$(H_0,H_1,H_2,H_3)=(0,0,1,2), \qquad (U_0,U_1)=(V_0,V_1)=(0,2).$$

Its two equations have degrees $(2,1,0)$ *and* $(1,0,1)$ *and both have weight* 2*. One has*

$$S_X(G)=\frac{243}{88}.$$

Proof. The initial product graph is the same variety as the initial Cox hypersurface in Lemma 105.4. Indeed, its plane-graph equation is homogeneous for these weights. Its two base centres are

$$\{z^2+w(c_0r+c_1s)=g(r,s)=0\}, \qquad \{s=w=0\}.$$

They are disjoint because $g(1,0)\neq 0$, and the first is a complete intersection. Thus the initial product graph is their integral blow-up graph, with the displayed regular sequence of two equations. This also verifies directly the integrality needed in the Koszul calculation. Write $h_m=h^0(X,-mK_X)$ and let T_m be the sum of the weights. The Koszul resolution gives

$$\begin{aligned} h_m &= \binom{m+3}{3}(m+1)^2-\left(\binom{m+1}{3}+\binom{m+2}{3}\right)m(m+1)+\binom{m}{3}m^2 \\ &= \frac{(2m+1)(11m^2+11m+6)}{6}. \end{aligned}$$

The average weight in $H^0(\mathbb{P}^3\times\mathbb{P}^1\times\mathbb{P}^1,\mathcal{O}(a,b,c))$ is $3a/4+b+c$. Put

$$N_f=\binom{m+1}{3}m(m+1), \qquad N_g=\binom{m+2}{3}m(m+1), \qquad N_{fg}=\binom{m}{3}m^2.$$

The shifts by the two equations therefore give

$$T_m=\frac{11m}{4}h_m+\frac{1}{2}N_f-\frac{1}{4}N_g-\frac{1}{4}N_{fg}=\frac{m(81m^3+118m^2+83m+22)}{8}.$$

Taking the limit of $T_m/(mh_m)$ proves the assertion. Integrality of the initial graph ensures that these are the actual valuation weights. □

Definition 105.6. For the exceptional divisor G, let

$$V_m^{\geq j}=\{s\in H^0(X,-mK_X):\mathrm{ord}_G(s)\geq j\}, \qquad W_{m,j}=V_m^{\geq j}/V_m^{\geq j+1}.$$

These are the actual refined spaces on G. Put $h_m=h^0(X,-mK_X)$. Their level-m basis boundaries are normalized by mh_m, summing over all j and a basis in each $W_{m,j}$. The different is $D/2$, so throughout the surface calculation

$$\delta(G,D/2;W)=\inf_F\frac{A_{G,D/2}(F)}{S(W;F)}.$$

For an odd weight the factor z denotes a rational section whose square cuts out D; its contribution to a basis boundary is $D/2$. This convention retains the different, which is an additional fixed boundary independent of m. More precisely, if a_G, b_G denote the two ruling classes, the weight-j space has rational divisor class $ma_G + (j/2)b_G$. Writing $j = 2n + \varepsilon$, its polynomial part has class $(m - \varepsilon)a_G + nb_G$, and its fixed half-divisor is $\varepsilon D/2$.

Lemma 105.7. *On the chart $a_0 = b_0 = 1$, the direct sum of the actual spaces at level m is spanned, modulo $z^2 + af + bg = 0$, by*

$$r^i z^j s^k a^\alpha b^\beta, \qquad i + j + k = m, \quad 0 \le \alpha \le m + k, \quad 0 \le \beta \le m.$$

The primary weight is $j + 2\alpha + 2\beta$. All primary weights between zero and $6m$ occur, and $W_{m,2m}$ contains the complete ample series $H^0(G, \mathcal{O}_G(m, m))$. Thus the bounded-support and ample-series hypotheses of Abban–Zhuang theory hold.

Proof. The Cox degrees in Lemma 105.4 give exactly these inequalities after setting $a_0 = b_0 = 1$. The same lemma proves that their span is the actual associated graded space. Its largest primary weight is $6m$. Every intermediate integral weight occurs, by taking products of the degree-one generators of weights $0, 1, 2, 3, 4, 5, 6$. Taking $j = 0$, $i + k = m$, and $\alpha + \beta = m$ gives all monomials of bidegree (m, m) on G. The ray $(m, 2m)$ is interior to the support, and $\mathcal{O}_G(1, 1)$ is ample. □

Example 105.8. For $m = 1$, the even pieces, after the square relation is used, have dimensions $2, 4, 3, 1$ in weights $0, 2, 4, 6$. The odd pieces have dimensions $1, 2, 1$ in weights $1, 3, 5$. Their total is 14, and their weighted sum is 38. Thus the first-level expected order is $19/7$; its limit, computed above, is $243/88$.

106 The refined estimate on the exceptional surface

Lemma 106.1 (The square case)**.** *Assume that the second initial equation and the generators are*

$$z^2 + as^2 + br^2 = 0, \qquad r^i z^j s^k a^\alpha b^\beta, \quad i + j + k = m, \quad 0 \le \alpha \le m + k, \quad 0 \le \beta \le m.$$

On the coarse exceptional surface $G = \mathbb{P}^1_{r,s} \times \mathbb{P}^1_{a,b}$, put $D = (as^2 + br^2 = 0)$. Then $\delta(G, \frac{1}{2}D; W) \ge 1$.

Proof. Use the second weights $(r, s, a, b, z) = (1, 0, 2, 0, 1)$. Put $A = as^2$, $B = br^2$ and write $j = 2h + \varepsilon$, where $\varepsilon \in \{0, 1\}$. The primary weight is $2N + \varepsilon$, where $N = \alpha + \beta + h$; the second weight is $Q = i + j + 2\alpha$. Set

$$R = Q - \varepsilon - 2N, \qquad S = m - Q, \qquad d = N - K - L,$$

$$K = \max\left\{0, \left\lceil \frac{Q - m}{2} \right\rceil, Q - 2m\right\}, \quad L = \max\left\{0, \left\lceil \frac{2N + \varepsilon - Q}{2} \right\rceil\right\}.$$

The corresponding space is zero unless $L \le m$ and $d \ge 0$. Otherwise it is exactly

$$z^\varepsilon r^R s^S A^K B^L \operatorname{Sym}^d\langle A, B\rangle.$$

Indeed, the original inequalities become $\alpha \ge K$, $\beta \ge L$, $\alpha + \beta \le N$, and $\beta \le m$. Fixing $\beta = L$ and varying α gives the basis $A^i(A + B)^{d-i}$ of the remaining binary space. The integers R, S may be negative, but the factors $A^K B^L$ make every displayed section regular. In particular, this computes the entire space, including all cancellations.

For the asymptotic sums, put $q = Q/m$, $n = N/m$,

$$k(q) = \max\{0, (q-1)/2, q-2\}, \quad l(n,q) = \max\{0, n - q/2\}, \quad d(n,q) = \min\{n, q/2\} - k(q).$$

The domain is $0 \le q \le 4$, $k(q) \le n \le q/2 + 1$. Splitting at $q = 1, 3$ and $n = q/2$ gives six elementary integrals. Their total mass is $\int d\,dn\,dq = 11/6$. Since the two parities have the same leading term, normalized moments are $\frac{6}{11}\int d(\cdots)\,dn\,dq$. The bounds for the coefficients of the invariant curves are as follows:

curve	integrand after the factor d	moment
$r = 0$	$q - 2n + 2l + d$	$21/44$
$s = 0$	$1 - q + 2k + d$	$23/44$
$a = 0$	$k + d/2$	$67/88$
$b = 0$	$l + d/2$	$109/176$
$A + \kappa B = 0$ $(\kappa \neq 0)$	$d/2$	$37/176$

The last row is the maximal average order of a basis of binary forms of degree d at one prescribed point. At D there is also the odd factor z, whose contribution is $O(1/m)$.

The curves $A + \kappa B = 0$ are smooth. They meet only at $P = (r = a = 0)$ and $P' = (s = b = 0)$, with contact order two. At P, blow up with weights $(1, 2)$ in (r, a); at P', use weights $(1, 2)$ in (s, b). The total moments of these valuations are

$$\frac{6}{11}\int qd\,dn\,dq = 2, \qquad \frac{6}{11}\int (1 + 2n - q)d\,dn\,dq = \frac{155}{88}.$$

These are moments of the whole basis divisor; one must not add the separate maxima in the table. The boundary $\frac{1}{2}D$ has value 1 for either weighted valuation, whose log discrepancy on G is 3. Thus the two required weighted inequalities hold. All curve coefficients are strictly below one, even after adding $\frac{1}{2}D$. The ordinary blow-up inequality follows from the weighted inequality and the bound one for the coefficient of the transverse coordinate line. The displayed blow-ups resolve the arrangement, so the resulting pairs are log canonical.

For precision at the equality at P, the sum of the second weights is exactly $2mh_m$ for every m. Indeed, the mean weights of the hyperplane, line-pencil, and quadric-pencil coordinates are respectively $1, 1, 0$. Both equations have weight 2, equal to the mean weights of their divisor classes. Every shifted term of the Koszul character therefore has mean $2m$, proving the identity. Thus there is no positive error at this corner. At the other corner and along curves all inequalities have a fixed positive margin for large m. Applying this to bases compatible with an arbitrary divisorial valuation and the second weight filtration, separately in each primary graded piece, and using semicontinuity of log canonical thresholds, proves the assertion. □

Corollary 106.2. *In the weighted curve construction, write the equation as*

$$z^2 + af(r,s) + bg(r,s) = 0, \qquad f = s(cr + ds), \qquad \gcd(f, g) = 1.$$

If either f or g is a square, the refined threshold is at least one.

Proof. Suppose first that f is a square. After rescaling, $f = s^2$. The coefficient of r^2 in g is nonzero by coprimality. Taking the maximum initial form for the weights $(r, s, a, b, z) = (1, 0, 2, 0, 1)$ gives $z^2 + as^2 + br^2$, after rescaling b. Suppose instead that g is a square. Take $g = r^2$. Coprimality gives $d \neq 0$. The minimum initial form for the same weights is $z^2 + ads^2 + br^2$; the omitted term $acrs$ has weight three. Rescale a.

In both cases the action preserves the Cox grading. The initial equation is prime, so localization introduces no extra initial relations. Its different remains the smooth curve of bidegree $(2, 1)$ with coefficient $1/2$. The preceding lemma applies to the complete initial series. Semicontinuity of log canonical thresholds, with bases chosen separately in each primary graded piece, proves the assertion. □

106.1 Four distinct marked points

Lemma 106.3 (The actual surface series)**.** *Suppose that $f = s(c_0 r + c_1 s)$ and g each have two distinct zeroes and have no common zero. On $G = \mathbb{P}^1_{[r:s]} \times \mathbb{P}^1_{[a:b]}$, let p_1, p_2 be the points $a = g = 0$, let p_3 be the point $b = s = 0$, and let p_4 be the point $b = c_0 r + c_1 s = 0$. Let $\tau : T \to G$ blow up these four points. Write A, B for the pullbacks of the classes of $\mathcal{O}_G(1,0), \mathcal{O}_G(0,1)$, and e_i for the exceptional curves.*

The even piece of primary weight $2n$ of the actual refined series is exactly

$$H^0\left(T, mA + nB - (n-m)_+(e_1+e_2+e_3) - (n-2m)_+ e_4\right).$$

For the odd piece the same assertion holds with mA replaced by $(m-1)A$, after removing the factor z.

Proof. A spanning monomial has the form

$$r^i z^{2h+\varepsilon} s^k a^\alpha b^\beta, \qquad i + 2h + \varepsilon + k = m, \quad 0 \le \alpha \le m + k, \quad 0 \le \beta \le m,$$

where $\varepsilon = 0$ or 1. Put $n = \alpha + \beta + h$ and replace z^{2h} by $(-af - bg)^h$. The remaining polynomial has bidegree $(m - \varepsilon, n)$. At either p_1 or p_2 its order is at least $\alpha + h = n - \beta \ge n - m$. At p_3 its order is at least $k + \beta + h = n - \alpha + k \ge n - m$. At p_4 its order is at least $\beta + h = n - \alpha \ge n - 2m$. These inequalities are preserved by taking linear combinations, proving the inclusions.

Here the inclusions are equalities, as can be seen without any asymptotic argument. For $\varepsilon \in \{0, 1\}$, the actual piece is contained in the direct sum

$$\bigoplus_{k=0}^{n} a^{n-k} b^k g^{(k-m)_+} s^{(n-m-k)_+} \operatorname{Sym}^{m-\varepsilon-2(k-m)_+-(n-m-k)_+}\langle r, s\rangle,$$

where a summand with negative exponent is zero. Indeed, in expanding $(af + bg)^h$, suppose that j factors contribute af. The final exponent of b is $k = \beta + h - j$. The coefficient contains g^{h-j} and s^{k_0+j}, where k_0 is the original exponent of s. Since $\beta \le m$ and $\alpha \le m + k_0$, these powers are at least $(k-m)_+$ and $(n-m-k)_+$ respectively.

Conversely, the displayed direct sum is the whole fat-point space. The two points on $a = 0$ impose the power of g, and the point $b = s = 0$ imposes the power of s. The fourth condition is automatic: if $k < n - 2m$, the required power of s already exceeds the horizontal degree. This description also makes sense when g has a double root, with its full scheme-theoretic power retained.

Put $a' = (k-m)_+$ and $b' = (n-m-k)_+$. For $a' = 0$ there are $m+1$ choices of k, and for $a' > 0$ there is one; the same statement holds for the other exponent. Summing dimensions over all n and both parities gives

$$\begin{aligned}(m+1)^2(2m+1) + (m+1)\frac{m(m-1)}{2} + (m+1)m^2 + \binom{m}{3} \\ = \frac{(2m+1)(11m^2+11m+6)}{6} = h^0(X, -mK_X).\end{aligned}$$

For example, the last term is $\sum_{a'=1}^{\lfloor m/2 \rfloor}(m-2a')^2 = \binom{m}{3}$. Since every actual piece is contained in its displayed space and the sums of their dimensions agree, all inclusions are equalities. In particular, this accounts for every possible cancellation among the original generators.

Set

$$D_t = A + tB - (t-1)_+(e_1+e_2+e_3) - (t-2)_+ e_4,$$

and put $F_1 = B - e_1 - e_2$, $F_2 = B - e_3 - e_4$, and $R = A - e_3$. The negative part N_t of $D_t = P_t + N_t$ and the square of its positive part are

t	N_t	P_t^2
$[0,1]$	0	$2t$
$[1,3/2]$	0	$2t - 3(t-1)^2$
$[3/2,2]$	$(t-3/2)F_1$	$5/2 - (t-1)^2$
$[2,5/2]$	$(t-3/2)F_1 + (t-2)F_2$	$11/2 - 2t$
$[5/2,3]$	$(2t-4)F_1 + (t-2)F_2 + (2t-5)R$	$2(3-t)^2$.

Here the negative curves in each indicated support have negative definite intersection matrix and P_t is orthogonal to that support. To check nefness it is enough, by linearity in each chamber, to check the endpoints. The only endpoint requiring explanation is

$$P_{5/2} = A + B - \frac{1}{2}(e_1 + e_2) - e_3.$$

For an irreducible curve $\Gamma = uA + vB - \sum m_i e_i$ different from the exceptional curves and the indicated fibres, intersection with F_1 gives $m_1 + m_2 \le u$, and intersection with the horizontal fibre through p_3 gives $m_3 \le v$. Hence

$$P_{5/2} \cdot \Gamma = u + v - \frac{1}{2}(m_1 + m_2) - m_3 \ge \frac{u}{2} \ge 0.$$

The excluded fibres and exceptional curves are checked directly. The other endpoints are A, $A + B$, $P_2 = P_{5/2} + \frac{1}{2}B$, and $P_{3/2} = P_2 + \frac{1}{2}e_3$; for the last one its intersection with e_3 is $1/2$, and all other intersections increase. Finally,

$$P_t = (3-t)(2A + 2B - e_1 - e_2 - 2e_3) \quad \text{for } 5/2 \le t \le 3.$$

The divisor $M = 2P_{5/2}$ is nef and $D_t \cdot M = 6 - 2t$ for $t \ge 2$. Thus no D_t with $t > 3$ is pseudoeffective.

Asymptotic Riemann–Roch in these finitely many rational chambers gives

$$h^0(T, mD_{n/m}) = \frac{m^2}{2} P_{n/m}^2 + O(m),$$

uniformly for $0 \le n \le 3m$. Replacing mA by $(m-1)A$ changes the total over n by $O(m^2)$. Direct integration yields

$$\int_0^3 P_t^2 \, dt = \frac{11}{3}, \qquad \frac{3}{11} \int_0^3 2t P_t^2 \, dt = \frac{243}{88}.$$

There are two parities. Their sum therefore has leading term $\frac{11}{3}m^3$, as claimed. □

Remark 106.4. The preceding lemma identifies the actual even and odd spaces, not merely their leading dimensions. For the integral formulas one may ignore the bounded change from mA to $(m-1)A$ and the odd factor z: these change normalized expected orders by $O(1/m)$ for a fixed divisorial valuation. The Zariski decompositions are taken on the surface T; no Zariski decomposition on the weighted threefold extraction is needed.

Example 106.5. For $m = 1$, the nonzero primary pieces have weights $0, 1, \dots, 6$ and dimensions

$$2,\ 1,\ 4,\ 2,\ 3,\ 1,\ 1.$$

For instance, the even piece with $n = 2$ is

$$a^2 s\,\mathbb{C} \ \oplus\ ab\langle r, s\rangle,$$

and the even piece with $n = 3$ is $a^2 bs\,\mathbb{C}$. The dimensions sum to 14, and their weighted sum is 38, agreeing with the Hilbert and weight formulas.

Lemma 106.6. *Let $G = \mathbb{P}^1 \times \mathbb{P}^1$, let $a = \mathcal{O}_G(1,0)$ and $b = \mathcal{O}_G(0,1)$, and let $D \subset G$ be a smooth curve of bidegree $(2,1)$. Suppose that p_1, p_2 lie on one fibre of the second projection, p_3, p_4 lie on another, and the four points are distinct points of D. Let $\rho : S \to G$ be their blowup, with exceptional curves e_i. Set*

$$v_i = a - e_i, \qquad h_1 = b - e_1 - e_2, \qquad h_2 = b - e_3 - e_4.$$

Then the negative curves on S are the eight (-1)-curves e_i, v_i and the two (-2)-curves h_1, h_2. Moreover,

$$\widetilde{D} = 2a + b - e_1 - e_2 - e_3 - e_4, \qquad \widetilde{D}^2 = 0.$$

The curve $\widetilde{D}$ meets e_i transversally at a point different from $e_i \cap v_i$ and $e_i \cap h_j$.

Proof. The restriction to D of the first projection is an isomorphism. Thus no two of the four points lie on a fibre of the first projection. The displayed intersection numbers follow from $a^2 = b^2 = 0$, $ab = 1$, $e_i^2 = -1$. To check the list, consider an irreducible curve other than the ten listed curves, of class $da + eb - \sum m_i e_i$. Intersection with h_1 and h_2 gives $m_1 + m_2 \le d$ and $m_3 + m_4 \le d$. Consequently

$$-K_S \cdot \Gamma = 2d + 2e - \sum m_i \ge 2e.$$

The pencil generated by the two products defining D becomes base-point-free on S, so $\widetilde{D}$ is nef. Hence $-K_S = \widetilde{D} + b$ is nef, and $(-K_S)^2 = 4$. Thus every negative curve is a (-1)-curve or a (-2)-curve. The inequality forces $e = 0$ for any additional negative curve. Its image would then be a fibre of the first projection, which gives one of the already listed v_i. This proves the assertion. We also record a direct nefness test used below. If $P = xa + yb - \sum z_i e_i$ has nonnegative intersection with these ten curves and with a, b, then

$$z_i \ge 0, \qquad y \ge z_i, \qquad x \ge z_1 + z_2, \qquad x \ge z_3 + z_4.$$

Thus $\sum z_i^2 \le y \sum z_i \le 2xy$, and P lies in the closure of the positive cone. The same argument shows that every irreducible curve other than the ten negative curves has nonnegative square and lies in that closure. The Hodge index theorem therefore gives $P\Gamma \ge 0$ for every such curve. This proves that P is nef without any further assertion about the Mori cone. Finally, each fibre of the second projection meets D in two distinct points and hence transversally. The first projection restricts isomorphically to D, so its fibres are transverse to D as well. Thus the three tangent directions at each p_i are pairwise distinct. □

Lemma 106.7. *Use the notation of Lemma 106.6, and put*

$$M(t) = (t-1)_+, \qquad N(t) = (t-2)_+, \qquad L_t = a + tb - M(t)(e_1+e_2+e_3) - N(t)e_4 \quad (0 \le t \le 3).$$

Consider the asymptotic multigraded series with these complete systems, normalized by $\int_0^3 \mathrm{vol}(L_t)\, dt = 11/3$. The moments of the exceptional flags are as follows. In the second column, the order includes the prescribed multiplicity $M(t)$ or $N(t)$ at the original point of G.

	$S(e_i)$	*unmarked point of* e_i	$e_i \cap v_i$	$e_i \cap h_j$
e_1, e_2	213/176	59/176	79/176	67/88
e_3	201/176	4/11	23/44	109/176
e_4	183/176	51/176	37/88	109/176

In particular, every valuation of G centred at one of p_1, p_2, p_3, p_4 has positive beta invariant for the pair consisting of $D/2$ and this multigraded series.

Proof. For $i = 1, 3, 4$ write

$$L_t - we_i = P_i(t,w) + R_i(t,w)$$

for its Zariski decomposition, and put $d_i(t,w) = P_i(t,w)e_i$. Let $r_i(t,w)$ and $s_i(t,w)$ be the coefficients of v_i and h_j in $R_i(t,w)$, respectively. No other component of the negative part meets e_i. The formulae for the moments are

$$\begin{aligned}
S(e_i) &= \frac{6}{11}\iint (w + c_i(t))d_i(t,w)\,dw\,dt,\\
S_{\text{free}}(e_i) &= \frac{6}{11}\iint \frac{d_i(t,w)^2}{2}\,dw\,dt,\\
S_{v_i}(e_i) &= \frac{6}{11}\iint \left(\frac{d_i(t,w)^2}{2} + r_i(t,w)d_i(t,w)\right)dw\,dt,\\
S_{h_j}(e_i) &= \frac{6}{11}\iint \left(\frac{d_i(t,w)^2}{2} + s_i(t,w)d_i(t,w)\right)dw\,dt,
\end{aligned}$$

where $c_1 = c_3 = M$ and $c_4 = N$. The integration is over the effective range $w \geq 0$, $0 \leq t \leq 3$; pieces with $d_i = 0$ make no contribution. All these integrals are integrals of quadratic polynomials over rational polygons. Their unnormalized values are

i	$\iint d_i$	$\iint (w+c_i)d_i$	$\iint d_i^2/2$	$\iint r_i d_i$	$\iint s_i d_i$
1	11/6	71/32	59/96	5/24	25/32
3	11/6	67/32	2/3	7/24	15/32
4	11/6	61/32	17/32	23/96	29/48

The following tables give the chamber polygons and the three linear functions d_i, r_i, s_i. Each polygon is the convex hull of the listed vertices (t,w); its boundary has measure zero and may belong to either adjacent chamber.

i	vertices of the chamber	d_i	r_i	s_i
1	$\{(0,0),(1,0),(1,1)\}$	w	0	0
1	$\{(0,0),(1,1),(0,1)\}$	t	$-t+w$	0
1	$\{(1,1),(1,2),(0,1)\}$	$t-w+1$	$-t+w$	$w-1$
1	$\{(1,0),(\frac{3}{2},0),(1,1)\}$	$t+w-1$	0	0
1	$\{(\frac{3}{2},0),(2,0),(2,1),(1,1)\}$	$\frac{w}{2}+\frac{1}{2}$	0	$t+\frac{w}{2}-\frac{3}{2}$
1	$\{(2,1),(1,2),(1,1)\}$	$2-w$	$w-1$	$t+w-2$
1	$\{(2,0),(\frac{5}{2},0),(2,1)\}$	$\frac{w}{2}+\frac{1}{2}$	0	$t+\frac{w}{2}-\frac{3}{2}$
1	$\{(\frac{5}{2},0),(3,0),(2,1)\}$	$3-t$	0	$2t+w-4$

i	vertices of the chamber	d_i	r_i	s_i
3	$\{(0,0),(1,0),(1,1)\}$	w	0	0
3	$\{(0,0),(1,1),(0,1)\}$	t	$-t+w$	0
3	$\{(1,1),(1,2),(0,1)\}$	$t-w+1$	$-t+w$	$w-1$
3	$\{(1,0),(\frac{3}{2},0),(\frac{3}{2},\frac{1}{2}),(1,1)\}$	$t+w-1$	0	0
3	$\{(\frac{3}{2},0),(2,0),(\frac{3}{2},\frac{1}{2})\}$	$t+w-1$	0	0
3	$\{(\frac{3}{2},1),(1,1),(\frac{3}{2},\frac{1}{2})\}$	1	0	$t+w-2$
3	$\{(1,2),(1,1),(\frac{3}{2},1)\}$	$2-w$	$w-1$	$t+w-2$
3	$\{(2,0),(2,\frac{1}{2}),(\frac{3}{2},1),(\frac{3}{2},\frac{1}{2})\}$	1	0	$t+w-2$
3	$\{(2,\frac{1}{2}),(2,1),(1,2),(\frac{3}{2},1)\}$	$-2t-2w+6$	$2t+2w-5$	$t+w-2$
3	$\{(2,0),(\frac{5}{2},0),(2,\frac{1}{2})\}$	1	0	$t+w-2$
3	$\{(\frac{5}{2},0),(3,0),(2,1),(2,\frac{1}{2})\}$	$-2t-2w+6$	$2t+2w-5$	$t+w-2$

i	vertices of the chamber	d_i	r_i	s_i
4	$\{(0,0),(1,0),(1,1)\}$	w	0	0
4	$\{(0,0),(1,1),(0,1)\}$	t	$-t+w$	0
4	$\{(1,1),(1,2),(0,1)\}$	$t-w+1$	$-t+w$	$w-1$
4	$\{(1,0),(\frac{3}{2},0),(\frac{3}{2},\frac{1}{2}),(1,1)\}$	w	0	0
4	$\{(\frac{3}{2},0),(2,0),(\frac{3}{2},\frac{1}{2})\}$	w	0	0
4	$\{(\frac{3}{2},\frac{3}{2}),(1,1),(\frac{3}{2},\frac{1}{2})\}$	$-\frac{t}{2}+\frac{w}{2}+1$	0	$\frac{t}{2}+\frac{w}{2}-1$
4	$\{(2,0),(2,\frac{3}{2}),(\frac{3}{2},\frac{3}{2}),(\frac{3}{2},\frac{1}{2})\}$	$-\frac{t}{2}+\frac{w}{2}+1$	0	$\frac{t}{2}+\frac{w}{2}-1$
4	$\{(1,2),(1,1),(\frac{3}{2},\frac{3}{2})\}$	$t-w+1$	$-t+w$	$w-1$
4	$\{(2,\frac{3}{2}),(2,2),(\frac{3}{2},\frac{3}{2})\}$	$-\frac{t}{2}-\frac{3w}{2}+4$	$2w-3$	$\frac{t}{2}+\frac{w}{2}-1$
4	$\{(2,2),(1,2),(\frac{3}{2},\frac{3}{2})\}$	$4-2w$	$2w-3$	$w-1$
4	$\{(2,0),(\frac{5}{2},0),(\frac{5}{2},1),(2,\frac{3}{2})\}$	$\frac{w}{2}$	0	$t+\frac{w}{2}-2$
4	$\{(\frac{5}{2},0),(3,0),(\frac{5}{2},1)\}$	$\frac{w}{2}$	0	$t+\frac{w}{2}-2$
4	$\{(2,2),(2,\frac{3}{2}),(\frac{5}{2},1)\}$	$-2t-\frac{3w}{2}+7$	$2t+2w-7$	$t+\frac{w}{2}-2$

These tables are obtained directly from the intersection matrix in Lemma 106.6: for a proposed negative support $C_1,\dots,C_k$, solve

$$\sum_j n_j(C_jC_l) = (L_t - we_i)C_l,$$

then impose $n_j \geq 0$ and $(L_t - we_i - \sum n_jC_j)C \geq 0$ for each of the ten negative curves. These inequalities give exactly the polygons in the tables. For example, the chamber $0 \leq w \leq t \leq 1$ has no negative part and $d_i = w$; it contributes $\int_0^1 \int_0^t w^2/2\,dw\,dt = 1/24$ to the free-point integral.

The log discrepancy of e_i over $(G, D/2)$ is $3/2$. At the point $\widetilde{D} \cap e_i$, adjunction gives boundary coefficient $1/2$, and the relevant refined moment is the free-point entry of the table, which is strictly smaller than $1/2$. At every other point of e_i the adjunction boundary is zero, and all refined moments are strictly smaller than one. The first moments are strictly smaller than $3/2$. The Abban–Zhuang inequality therefore proves the last assertion. □

Lemma 106.8. *For the same series,*

$$S(D) = \frac{37}{176}, \qquad S(\text{refinement along } D; P) = \frac{81}{88} \quad \text{if } P \notin \{p_1,p_2,p_3,p_4\}.$$

Thus all valuations centred at an unmarked point of D also have positive beta invariant for $(G, D/2)$ and the series.

Proof. Write $L_t - w\widetilde{D} = P(t,w) + R(t,w)$ and $d = P(t,w)\widetilde{D}$. The seven chamber polygons and their linear functions are as follows.

vertices of the chamber	d	$N_{e_1} = N_{e_2} = N_{e_3}$	N_{e_4}
$\{(0,0),(1,0),(1,\frac{1}{2}),(\frac{1}{2},\frac{1}{2})\}$	$2t-4w+1$	w	w
$\{(1,0),(\frac{3}{2},0),(\frac{3}{2},\frac{1}{2})\}$	$-t-w+4$	0	w
$\{(\frac{3}{2},0),(2,0),(\frac{3}{2},\frac{1}{2})\}$	$-t-w+4$	0	w
$\{(2,0),(2,\frac{1}{2}),(\frac{3}{2},\frac{1}{2})\}$	$-2t-2w+6$	0	$t+2w-2$
$\{(1,0),(\frac{3}{2},\frac{1}{2}),(1,\frac{1}{2})\}$	$2t-4w+1$	$-t+w+1$	w
$\{(2,0),(\frac{5}{2},0),(\frac{5}{2},\frac{1}{2}),(2,\frac{1}{2})\}$	$-2t-2w+6$	0	$2w$
$\{(\frac{5}{2},0),(3,0),(\frac{5}{2},\frac{1}{2})\}$	$-2t-2w+6$	0	$2w$

The only components of R meeting $\widetilde{D}$ are the exceptional curves e_i, so they produce no fixed part at an unmarked point. Direct integration gives

$$\iint d = \frac{11}{6}, \qquad \iint wd = \frac{37}{96}, \qquad \iint \frac{d^2}{2} = \frac{27}{16}.$$

Multiplying by $6/11$ gives the asserted moments. Since $A_{G,D/2}(D) = 1/2$, both terms in the resulting Abban–Zhuang minimum are strictly greater than one. □

106.2 The refinement away from the branch curve

We use the following notation for the surface calculation. Let $G = \mathbb{P}^1 \times \mathbb{P}^1$, let a, b be its two ruling classes, and let $D \in |2a + b|$ be the smooth branch curve. Let p_1, p_2 lie on one b-fibre and p_3, p_4 on another, all four on D. For the moment assume that they are distinct. The point p_3 corresponds to the intersection with the blown-up line, and p_4 to the vertex of the quadric cone; their multiplicities below are different. On their blow-up Y, denote the total exceptional classes by e_i, and put

$$\alpha = b - e_1 - e_2, \qquad \beta = b - e_3 - e_4, \qquad J_i = a - e_i.$$

The negative curves are the four e_i, the four J_i, and the two (-2)-curves α, β. The two pairs of points have disjoint sets of first coordinates, since the two quadratics defining D have no common root.

Here is a check that no negative curve is missing. The strict transform $2a+b-e_1-e_2-e_3-e_4$ is a pencil class, so its sum with b is the nef anticanonical class of Y, whose square is four. The sixteen possible (-1)-classes on a degree-four weak del Pezzo surface are e_i, $a - e_i$, $b - e_i$ and $a + b - e_i - e_j - e_k$. The classes $b - e_i$ split into a marked fibre and an exceptional curve. Each triple of indices contains one marked pair, so $a + b - e_i - e_j - e_k$ splits into that marked fibre and the vertical line through the third point. The possible additional roots $a - e_i - e_j$ are excluded by the distinct first coordinates. A curve of class $a+b-e_1-e_2-e_3-e_4$ is impossible: a $(1,1)$-curve containing two points on a horizontal fibre must contain that fibre, and its remaining vertical component cannot contain the other pair. The exceptional-difference classes are not effective because the four points are distinct. This leaves exactly the ten negative curves displayed above.

Put $r = (t-1)_+$, $s = (t-2)_+$, where $x_+ = \max\{x, 0\}$, and set

$$T_t = a + tb - r(e_1 + e_2 + e_3) - se_4 \qquad (0 \le t \le 3).$$

These are the complete-ideal divisors of the preceding refinement. For each parity of the original weight, the corresponding spaces are contained in their complete section spaces, with changes bounded independently of m in the integral coefficients. The exact dimension calculation gives total leading coefficient $11/3$. Since there are two parities and each surface section space has leading coefficient $P^2/2$, the normalization for each double integral below is $3/11$. The bounded changes contribute a term tending to zero to each normalized moment. Thus upper bounds for these complete section spaces give the required upper bounds for the actual refinement.

Lemma 106.9. *At every point $p \in G \setminus D$, the refinement satisfies*

$$\delta_p(G, \tfrac{1}{2}D; W) \ge \frac{88}{67} > 1.$$

Here the four marked points are assumed distinct.

Proof. Use the horizontal fibre through p. If it is different from the two marked fibres, its strict transform has class b. For the two marked fibres use α and β, respectively. Write $P(t,v)+N(t,v)$ for the Zariski decomposition of $T_t - vC$, where C is this flag curve. The following table gives the integrated curve and point moments. All entries already include the factor $3/11$.

C	curve moment	point	point moment
b	$115/264$	general	$31/66$
		J_1, J_2	$125/264$
		J_3	$23/44$
		J_4	$31/66$
α	$67/88$	general	$37/88$
		J_3	$23/44$
		J_4	$37/88$
β	$109/176$	general	$37/88$
		J_1, J_2	$79/176$

Here an entry J_i means the point where the flag meets J_i. Intersections with exceptional curves are omitted from the last two blocks because they lie over the marked points, all of which belong to D and are excluded from the statement.

For completeness, the chamber data for this calculation are given below. A polygon is specified by its vertices (t,v); its listed negative part determines the positive part as $P = T_t - vC - N$. On every listed polygon the coefficients of N are nonnegative, its support has negative definite intersection matrix, P is nef, and $P \cdot N = 0$. These assertions can be checked against the ten negative curves listed above. The polygons cover the big region; outside them the volume is zero. Integrate

$$P^2 \quad \text{for the curve moment,} \qquad (P\cdot C)^2 + 2(P\cdot C)\,\mathrm{ord}_p(N|_C) \quad \text{for the point moment.}$$

In the second expression a component equal to C is omitted from N: its chamber has $P\cdot C = 0$ and makes no point contribution.

The point moment bounds the coefficient at p of the normalized restricted basis divisor on C. Choose the finite-level bases simultaneously adapted to the tested valuation and to C, and apply inversion of adjunction as in Proposition 104.4. This bounds the quotient of expected vanishing by discrepancy by the larger of the curve and point moments. The branch curve is absent from a neighbourhood of p, so the discrepancy of $(G, \frac{1}{2}D)$ there is the ordinary surface discrepancy. The largest relevant entry is $67/88$. □

Chambers for $C = b$	Negative part N
$(0,0),(1,0)$ $(1,1)$	0
$(1,0),(\frac{3}{2},0)$ $(\frac{3}{2},1),(1,1)$	0
$(\frac{3}{2},0),(2,0)$ $(2,\frac{1}{2}),(\frac{3}{2},1)$	$\left(\frac{2t-3}{2}\right)\alpha$
$(2,\frac{1}{2}),(2,1)$ $(\frac{3}{2},1)$	$(2t+v-4)\,\alpha+(2t+2v-5)\,J_3$
$(\frac{4}{3},\frac{4}{3}),(1,1)$ $(\frac{3}{2},1)$	$(v-1)\,J_1+(v-1)\,J_2$ $+(v-1)\,J_3$
$(2,1),(\frac{4}{3},\frac{4}{3})$ $(\frac{3}{2},1)$	$(2t+v-4)\,\alpha+(v-1)\,J_1$ $+(v-1)\,J_2+(2t+2v-5)\,J_3$
$(2,0),(\frac{5}{2},0)$ $(2,\frac{1}{2})$	$\left(\frac{2t-3}{2}\right)\alpha+(t-2)\,\beta$
$(\frac{5}{2},0),(3,0)$ $(2,1),(2,\frac{1}{2})$	$(2t+v-4)\,\alpha+(t-2)\,\beta$ $+(2t+2v-5)\,J_3$

Chambers for $C = \alpha$	Negative part N
$(0,0),(1,0)$ $(1,1)$	$(v)\,e_1 + (v)\,e_2$
$(1,0),(\frac{3}{2},0)$ $(2,\frac{1}{2}),(2,1)$	0
$(\frac{3}{2},0),(2,0)$ $(2,\frac{1}{2})$	$\left(\frac{2t-2v-3}{2}\right)\alpha$
$(1,0),(2,1)$ $(1,1)$	$(-t+v+1)\,e_1 + (-t+v+1)\,e_2$
$(2,1),(2,2)$ $(1,1)$	$(-t+v+1)\,e_1 + (-t+v+1)\,e_2$ $+ (v-1)\,J_3$
$(2,1),(2,\frac{1}{2})$ $(\frac{5}{2},1)$	$(t-2)\,\beta$
$(2,0),(\frac{5}{2},0)$ $(\frac{5}{2},1),(2,\frac{1}{2})$	$\left(\frac{2t-2v-3}{2}\right)\alpha + (t-2)\,\beta$
$(3,2),(2,1)$ $(\frac{5}{2},1)$	$(t-2)\,\beta + (v-1)\,J_3$
$(\frac{5}{2},0),(3,0)$ $(3,2),(\frac{5}{2},1)$	$(2t-v-4)\,\alpha + (t-2)\,\beta$ $+ (2t-5)\,J_3$
$(3,2),(2,2)$ $(2,1)$	$(-t+v+1)\,e_1 + (-t+v+1)\,e_2$ $+ (t-2)\,\beta + (v-1)\,J_3$

Chambers for $C = \beta$	Negative part N
$(0,0),(1,0)$ $(1,1)$	$(v)\,e_3 + (v)\,e_4$
$(1,0),(\frac{3}{2},0)$ $(\frac{3}{2},\frac{1}{2})$	$(v)\,e_4$
$(1,0),(\frac{3}{2},\frac{1}{2})$ $(\frac{3}{2},1),(1,1)$	$(-t+v+1)\,e_3 + (v)\,e_4$
$(\frac{3}{2},0),(2,0)$ $(2,1),(\frac{3}{2},\frac{1}{2})$	$(v)\,e_4 + \left(\frac{2t-3}{2}\right)\alpha$
$(\frac{3}{2},\frac{1}{2}),(2,1)$ $(\frac{3}{2},1)$	$(-t+v+1)\,e_3 + (v)\,e_4$ $+\left(\frac{2t-3}{2}\right)\alpha$
$(\frac{3}{2},\frac{3}{2}),(1,1)$ $(\frac{3}{2},1)$	$(-t+v+1)\,e_3 + (v)\,e_4$ $+(v-1)\,J_1 + (v-1)\,J_2$
$(2,1),(2,\frac{3}{2})$ $(\frac{3}{2},\frac{3}{2}),(\frac{3}{2},1)$	$(-t+v+1)\,e_3 + (v)\,e_4$ $+\left(\frac{2t-3}{2}\right)\alpha + (v-1)\,J_1$ $+(v-1)\,J_2$
$(2,0),(\frac{5}{2},\frac{1}{2})$ $(\frac{5}{2},1),(2,1)$	$(-t+v+2)\,e_4 + \left(\frac{2t-3}{2}\right)\alpha$
$(2,0),(\frac{5}{2},0)$ $(\frac{5}{2},\frac{1}{2})$	$\left(\frac{2t-3}{2}\right)\alpha + (t-v-2)\,\beta$
$(\frac{5}{2},\frac{1}{2}),(3,1)$ $(\frac{5}{2},1)$	$(-t+v+2)\,e_4 + (2\,(t-2))\,\alpha$ $+(2t-5)\,J_3$
$(\frac{5}{2},0),(3,0)$ $(3,1),(\frac{5}{2},\frac{1}{2})$	$(2\,(t-2))\,\alpha + (t-v-2)\,\beta$ $+(2t-5)\,J_3$
$(\frac{5}{2},\frac{3}{2}),(2,1)$ $(\frac{5}{2},1)$	$(-t+v+2)\,e_4 + \left(\frac{2t-3}{2}\right)\alpha$ $+(v-1)\,J_1 + (v-1)\,J_2$
$(2,\frac{3}{2}),(2,1)$ $(\frac{5}{2},\frac{3}{2})$	$(-t+v+1)\,e_3 + (-t+v+2)\,e_4$ $+\left(\frac{2t-3}{2}\right)\alpha + (v-1)\,J_1$ $+(v-1)\,J_2$
$(3,1),(\frac{5}{2},\frac{3}{2})$ $(\frac{5}{2},1)$	$(-t+v+2)\,e_4 + (2\,(t-2))\,\alpha$ $+(v-1)\,J_1 + (v-1)\,J_2$ $+(2t-5)\,J_3$

Proposition 106.10. *For every double conic in a smooth member of Family 3.6,*

$$\delta(G, D/2; W) \geq 1.$$

If both binary quadratics f, g have distinct roots, then

$$\delta(G, D/2; W) \geq \frac{88}{81}.$$

Proof. If either quadratic is a square, apply Corollary 106.2. Otherwise the four marked points are distinct. Lemma 106.9 treats points outside D. Lemma 106.7 treats the four marked points, and Lemma 106.8 treats the remaining points of D. The smallest of the resulting ratios is $88/81$. These local inequalities include valuations with curve centres and therefore bound the global refined threshold. □

107 Completion of the proof for Family 3.6

Proof of Theorem 103.1. Suppose that $\beta_X(F) \leq 0$ for a prime divisor F over X. By Proposition 104.6, its centre is contained in a double conic 2ℓ. Take the weighted extraction $\sigma : Y \to X$ constructed above. It is plt, its different is $D/2$, and its actual refinement has threshold at least one by Proposition 106.10. Abban–Zhuang adjunction in its strict form gives

$$\beta_X(F) \geq \operatorname{ord}_F(G)\,(A_X(G) - S_X(G)) = \frac{21}{88}\operatorname{ord}_F(G) > 0.$$

The last inequality holds because $c_X(F) \subset \ell$, so the centre on Y is contained in the exceptional divisor. This is the required contradiction.

For clarity, the strict inequality retains more information than the minimum bound for local delta invariants. Choose a basis compatible with the F- and G-filtrations and restrict its moving part to G. The refined bound and inversion of adjunction give, after taking limits,

$$S_X(F) \leq A_X(F) - A_X(G)\operatorname{ord}_F(G) + S_X(G)\operatorname{ord}_F(G).$$

This is precisely the displayed positive remainder. Thus a refined threshold equal to one causes no gap in the argument. □

Remark 107.1 (The role of the section calculation). *No Zariski decomposition on the weighted threefold Y has been computed or assumed.* The prime initial Cox equation identifies the valuation filtration. Its character gives $S_X(G) = 243/88$, and the exact binary-space count identifies every refined section, including cancellations. The remaining Zariski decompositions are on the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at four points. When marked points coalesce, a separate degeneration keeps the different smooth and gives the threshold one. In particular, the nonnormal sextic fibre occurring when Λ passes through a quadric-cone vertex is included in the proof; it is never treated as a Du Val surface.

Part XV
Family 3.7: sextic fibres, quintic sections, and point extractions

108 Statement and method

Let $V \subset \mathbb{P}^2 \times \mathbb{P}^2$ be a smooth divisor of bidegree $(1,1)$, and let $C \subset V$ be a smooth complete intersection of two members of $|\mathcal{O}_V(1,1)|$. We consider the blow-up $\pi : X \to V$ of C. These are the smooth members of Family 3.7. The elliptic curve C and its two plane cubic embeddings control the singular fibres and the few points requiring extraction.

Theorem 108.1. *Every smooth member of Family 3.7 is K-stable.*

The proof uses Abban–Zhuang adjunction in three ways. The sextic del Pezzo pencil excludes all centres except those on the two lines through a fibre singularity. A smooth quintic del Pezzo section treats the open parts of these lines. Ordinary and weighted point extractions treat their remaining points. We compute the expected orders in separate lemmas, including the fixed parts of the surface refinements. The complete initial section algebras are established by Hilbert counts valid in every degree.

No Zariski decomposition on an extracted threefold is computed or assumed. The threefold decompositions below are on X itself. On the point blow-ups the actual section spaces replace such a calculation. Their further refinements are controlled by complete binary spaces, surface resolutions, or explicit monomial limits.

109 The pencil of sextic del Pezzo surfaces

Write

$$V = \{y^{\mathsf{t}}x = 0\} \subset \mathbb{P}^2_x \times \mathbb{P}^2_y, \qquad C = \{y^{\mathsf{t}}x = y^{\mathsf{t}}Ax = y^{\mathsf{t}}Bx = 0\}.$$

Here A and B are 3×3 matrices, and the three displayed bilinear forms are linearly independent. Put $N = \langle I, A, B\rangle$. We assume throughout this section that C is smooth. Let $\pi\colon X \to V$ be its blow-up, let E be the exceptional divisor, and let $\phi\colon X \to \mathbb{P}^1$ be the morphism defined by the pencil spanned by $y^{\mathsf{t}}Ax$ and $y^{\mathsf{t}}Bx$.

Lemma 109.1. *The curve C is connected of genus one. Both projections of C to $\mathbb{P}^2$ are embeddings onto smooth plane cubics.*

Proof. The Koszul resolution for three divisors of bidegree $(1,1)$ gives $H^0(C, \mathcal{O}_C) = \mathbb{C}$, and adjunction gives $K_C \sim 0$. Thus C is a smooth connected curve of genus one. For fixed x, its fibre under the first projection is the projectivized kernel of the matrix with rows x^{t}, $(Ax)^{\mathsf{t}}$ and $(Bx)^{\mathsf{t}}$. If this matrix had rank at most one at a point of the image, C would contain a projective line, contrary to irreducibility and genus one. Its rank is therefore exactly two everywhere over the image. Its kernel is consequently a line bundle locally on the determinant cubic. This identifies C, as a scheme, with that cubic. The second projection is treated in the same way. Alternatively, the degree of either image is $\mathcal{O}_C(1,0) = \mathcal{O}_C(0,1) = 3$, by intersection theory. □

Lemma 109.2. *The plane cubic*

$$D = \{\det(\lambda I + \mu A + \nu B) = 0\} \subset \mathbb{P}(N)$$

is smooth. Every nonzero matrix in N has rank at least two.

Proof. A rank-one matrix in N would give a bilinear form $l(x)m(y)$ vanishing identically on the irreducible curve C. Hence one of its plane cubic images would lie in a line, which is impossible by Lemma 109.1. A nonzero rank-zero matrix is impossible. Suppose that $[M]$ is a singular point of D. Then M has rank two. Choose generators x and y of its right and left kernels. The derivative of the determinant in a direction $T \in N$ is a nonzero constant times $y^{\mathsf{t}}Tx$. Thus $(x, y) \in C$. But both partial gradients of $y^{\mathsf{t}}Mx$ vanish at (x, y), since $Mx = 0$ and $y^{\mathsf{t}}M = 0$. This gives a nontrivial linear dependence among the gradients of the three defining equations of C, contradicting its smoothness. □

Lemma 109.3. *Every fibre of ϕ is a normal del Pezzo surface of degree six. It is either smooth, has precisely one A_1 singularity, or has precisely one A_2 singularity. Every singular point is disjoint from E. If n_1 and n_2 count the singular fibres of the last two types, then $n_1 + 2n_2 = 6$.*

Proof. A fibre is isomorphic to

$$S_M = \{y^{\mathsf{t}}x = y^{\mathsf{t}}Mx = 0\}, \qquad M = \mu A + \nu B,$$

since blowing up the Cartier divisor C on S_M changes neither S_M nor its scheme structure. The description below also verifies normality, so this observation does not presuppose it. No eigenvalue of M can have a two-dimensional eigenspace: otherwise $M - \lambda I$ would be a nonzero matrix in N of rank at most one. Consequently M is cyclic. Up to conjugation and an affine change $M \mapsto aM + bI$, its Jordan type is one of the following: three distinct eigenvalues; a block of size two and a distinct eigenvalue; a single block of size three.

The Jacobian criterion says that a singular point of S_M is a pair of right and left eigenvectors with the same eigenvalue, satisfying $y^{\mathsf{t}}x = 0$. Three distinct eigenvalues give no such pair. For the second Jordan type take

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

There is one singular point, namely $([1 : 0 : 0], [0 : 1 : 0])$. On the chart $x_0 = y_1 = 1$, put $a = x_1$, $b = x_2$, and $c = y_2$. Eliminating y_0 gives $bc(1 - a) = a^2$, which is an A_1 singularity. For the last Jordan type take

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The only singular point is $([1 : 0 : 0], [0 : 0 : 1])$. On the chart $x_0 = y_2 = 1$, put $a = x_1$, $c = y_1 - a$, and $d = x_2 - a^2$. Eliminating y_0 gives $cd = a^3$, which is an A_2 singularity. This proves normality and the asserted list. Adjunction gives $-K_{S_M} = \mathcal{O}_{S_M}(1, 1)$ and $K_{S_M}^2 = 6$. If a singular point of S_M lay on C, the two equations defining C inside the smooth threefold V would have dependent gradients. Thus all singular points avoid C, and their lifts avoid E.

Projection of the smooth cubic D from $[I] \notin D$ gives a finite map $D \to \mathbb{P}^1$ of degree three. Its fibre over $[\mu : \nu]$ records the eigenvalues of $\mu A + \nu B$. Ordinary ramification gives the second Jordan type, and ramification of index three gives the third. Riemann–Hurwitz therefore gives $n_1 + 2n_2 = 6$. □

Lemma 109.4. *For an A_1 fibre, its minimal resolution is the blow-up of $\mathbb{P}^2$ at a point p_1, a point p_2 infinitely near p_1, and an ordinary point p_3 off the corresponding tangent line. Writing l, e_1, e_2, e_3 for the usual total-transform classes, its (-2)-curve has class $e_1 - e_2$, and its four (-1)-curves have classes*

$$e_2, \quad e_3, \quad l - e_1 - e_2, \quad l - e_1 - e_3.$$

The first projection contracts the first two lines and the second projection contracts the last two. The singular point lies on exactly the images of e_2 and $l - e_1 - e_3$.

For an A_2 fibre, the three blow-up centres form a curvilinear cluster whose third point is off the strict transform of its tangent line. The (-2)-curves have classes $e_1 - e_2$ and $e_2 - e_3$. There are exactly two (-1)-curves, of classes e_3 and $l - e_1 - e_2$. Their images pass through the singular point, and each projection contracts one of them.

Proof. The first projection is the graph of the rational map $x \mapsto x \times Mx$, so its centres are the eigenscheme of M. In the A_1 normal form, near $[1 : 0 : 0]$ its ideal is (b, a^2), and the other centre is $[0 : 0 : 1]$. In the A_2 normal form its ideal is $(b - a^2, a^3)$. Successive ordinary blow-ups give the stated clusters. The curve classes and their intersections follow directly. To check completeness, a (-1)-curve on the resolution is either an exceptional curve or has class $dl - m_1e_1 - m_2e_2 - m_3e_3$ with nonnegative integers m_i. The equations $3d - \sum m_i = 1$ and $d^2 - \sum m_i^2 = -1$ give $d \leq 1$ by Cauchy–Schwarz. The degree-one possibilities are then exactly the lines specified above. The second projection and the incidences with the singularities follow either by transposition or by intersecting these classes with the (-2)-curves. □

Example 109.5. Take

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then

$$\det(\lambda I + \mu A + \nu B) = \lambda^3 + \mu\nu(\mu + \nu).$$

This cubic is smooth: its three partial derivatives have no common projective zero. The first projection identifies C with $x_0^2x_2 - x_0x_1^2 + x_1x_2^2 = 0$, which is also a smooth plane cubic. There are three singular fibres, over $\mu\nu(\mu + \nu) = 0$, and each has an A_2 singularity. Thus the worst singular-fibre case really occurs for smooth members of Family 3.7.

Lemma 109.6. *The automorphism group of X is finite.*

Proof. Put $H_1 = \pi^*\mathcal{O}_V(1, 0)$, $H_2 = \pi^*\mathcal{O}_V(0, 1)$, and $F = \phi^*\mathcal{O}_{\mathbb{P}^1}(1)$. These are independent nef classes. The exceptional fibre of π has intersections $(0, 0, 1)$ with them. The strict transform of a fibre of the first projection of V meeting C has intersections $(0, 1, 0)$: the two pencil sections restrict to linear polynomials with the same zero, so their ratio is constant after blowing up. The second projection similarly supplies a curve with intersections $(1, 0, 0)$. Hence the nef cone is generated by H_1, H_2, F. The connected automorphism group fixes these rays and their integral primitive classes. It therefore descends via both plane projections, and via π, to an automorphism of V preserving C. Its action on C preserves the degree-three line bundle $\mathcal{O}_C(1, 0)$. The group of automorphisms of an elliptic curve preserving a positive-degree line bundle is finite: its translation subgroup is contained in the kernel of the associated isogeny, and the subgroup fixing an origin is finite. The kernel of this action fixes both plane cubic images pointwise, hence is the identity on both planes and on X. Thus the connected automorphism group is trivial. Since the automorphism group of a Fano variety is an algebraic group of finite type, it is finite. □

Lemma 109.7. *Write $H = \pi^*\mathcal{O}_V(1, 1)$ and $F = \phi^*\mathcal{O}_{\mathbb{P}^1}(1)$. Then*

$$-K_X = H + F, \qquad (-K_X)^3 = 24, \qquad h^0(X, -mK_X) = 4m^3 + 6m^2 + 4m + 1 \quad (m \geq 0).$$

If $f, g \in H^0(V, \mathcal{O}_V(1, 1))$ generate the pencil, the fifteen anticanonical sections are the products of the eight bilinear sections on V with the two base coordinates, modulo the single relation defining the graph of the pencil.

Proof. The blow-up is the complete intersection of divisors of multidegrees $(1,1,0)$ and $(1,1,1)$ in $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^1$. Indeed, the ideal (f,g) is a regular sequence on V, so its blow-up is its usual incidence graph. Adjunction gives $-K_X = \mathcal{O}_X(1,1,1)$, which is ample. Its cube is 24 by intersection in this product. The Koszul resolution, twisted by $\mathcal{O}(m,m,m)$, has no higher cohomology contributing to its space of sections. It gives

$$h^0(X,-mK_X) = (m+1)\binom{m+2}{2}^2 - (2m+1)\binom{m+1}{2}^2 + m\binom{m}{2}^2 = 4m^3+6m^2+4m+1.$$

For $m = 0, 1$ the same formula follows directly, with the zero binomial terms omitted. Finally, $h^0(V,\mathcal{O}_V(1,1)) = 8$; the graph equation is the unique relation among its sixteen products with the base coordinates. This proves the last assertion. □

Lemma 109.8. *Put $H_1 = \pi^*\mathcal{O}_V(1,0)$, $H_2 = \pi^*\mathcal{O}_V(0,1)$, and $F = \phi^*\mathcal{O}_{\mathbb{P}^1}(1)$. The morphisms given by (F,H_i) are blow-ups of smooth elliptic curves in $\mathbb{P}^1 \times \mathbb{P}^2$. Their exceptional divisors are*

$$D_1 = F + 2H_1 - H_2 = 3H_1 - E, \qquad D_2 = F + 2H_2 - H_1 = 3H_2 - E.$$

The divisor D_i is also the strict transform of the inverse image of the plane cubic C_i under the ith projection of V.

Proof. Consider (ϕ, p_1). Over $([\mu:\nu], x)$ the remaining coordinate y satisfies $y^{\mathsf{t}}x = y^{\mathsf{t}}(\mu A + \nu B)x = 0$. There is a unique solution except along the eigencurve

$$Z_1 = \{([\mu:\nu], x) : x \times (\mu A + \nu B)x = 0\}.$$

The smooth determinant cubic in Lemma 109.2 maps isomorphically to Z_1 by taking the right kernel of $\lambda I + \mu A + \nu B$. The kernel is a line everywhere. Conversely, the eigenvalue is recovered regularly on a chart where a coordinate of x is nonzero. This proves that Z_1 is smooth and that the indicated map is a closed immersion.

The two-row matrix with rows x^{t} and $((\mu A + \nu B)x)^{\mathsf{t}}$ has rank one along Z_1 and rank two elsewhere. Locally a unit entry reduces its incidence equations to the usual graph of two generators of the ideal of Z_1. Since Z_1 is smooth of codimension two, this graph is its blow-up. This proves the assertion about the morphism. The canonical divisor formula gives

$$D_1 = (2F + 3H_1) - (-K_X) = F + 2H_1 - H_2.$$

Here $-K_X = H_1 + H_2 + F$.

The projection of Z_1 to the first plane is C_1: its determinant equation is $\det(x, Ax, Bx) = 0$, the equation of that plane cubic. Over $x \in C_1$, the two pencil equations restrict proportionally to the first-projection fibre, so its strict transform is contracted by (ϕ, p_1). The inverse image of C_1 in V contains C with multiplicity one. Its strict transform therefore has class $3H_1 - E$ and is the exceptional divisor just described. The second assertion is obtained by transposing the matrices. □

110 Fibre refinements and the remaining centres

Write $A = \pi^*\mathcal{O}_V(1,0)$, $B = \pi^*\mathcal{O}_V(0,1)$, $H = A + B$, $F = H - E$ and $L = -K_X = H + F$. A fibre is denoted by S, and $C_S = E|_S$. Thus C_S is the original smooth elliptic curve C, viewed as an anticanonical curve in the smooth locus of S.

Lemma 110.1. *One has*

$$L^3 = 24, \qquad S_X(F) = \frac{11}{16}, \qquad S_X(E) = \frac{3}{8}.$$

For every divisorial valuation v of a fibre S, the exact refinement satisfies

$$S(W^S;v)=\frac{15}{16}S_S(v)+\frac{1}{16}v(C_S).$$

Proof. The identities $H^3=H^2F=6$ and $F^2=0$ give $L^3=24$. The positive and fixed parts of $L-uF$ are

$$(P(u),N(u))=\begin{cases}(H+(1-u)F,0), & 0\le u\le 1,\\ ((2-u)H,(u-1)E), & 1\le u\le 2.\end{cases}$$

Indeed, H and F are nef. For $u>1$, the fibres of $E\to C$ force the fixed multiplicity $u-1$. Subtracting this multiple of E leaves the nef divisor $(2-u)H$. The intersection with HF shows that $u>2$ is not pseudoeffective. Therefore

$$S_X(F)=\frac{1}{24}\left(\int_0^1(24-18u)\,du+\int_1^2 6(2-u)^3\,du\right)=\frac{11}{16}.$$

Similarly $L-uE=(1-u)H+(1+u)F$ is nef for $0\le u\le 1$, and its intersection with HF is negative for $u>1$. Consequently

$$S_X(E)=\frac{1}{24}\int_0^1(24-36u+12u^3)\,du=\frac{3}{8}.$$

The restrictions of the positive parts to S are $-K_S$ for $u\le 1$ and $(2-u)(-K_S)$ for $u\ge 1$. The fixed part restricts to $(u-1)C_S$ in the latter interval. The complete moving systems occur in the restriction calculation: for $0<u<1$ this follows from vanishing, and for $1<u<2$ it follows after removing the fixed part from restriction of multiples of H to its member S on V. For this last assertion one may use the restriction sequence and Kodaira vanishing on the smooth index-two Fano threefold V. The endpoints do not affect the integrals. The coefficients are

$$\frac{3\cdot 6}{24}\left(1+\int_1^2(2-u)^3\,du\right)=\frac{15}{16},\qquad \frac{3\cdot 6}{24}\int_1^2(2-u)^2(u-1)\,du=\frac{1}{16}.$$

This proves the asserted equality for the actual restricted series. All decompositions in this lemma take place on X itself. □

Lemma 110.2. *If $\delta_p(S)\ge 1$, every prime divisor Q over X whose centre contains p and is contained in S satisfies*

$$\beta_X(Q)\ge\frac{5}{16}\operatorname{ord}_Q(S)>0.$$

If $p\notin E$, one has the stronger local estimate $\delta_p(X)\ge 16/15$.

Proof. The pair (S,C_S) is log canonical, so $v(C_S)\le A_S(v)$. Lemma 110.1 gives $S(W^S;v)\le A_S(v)$. The basis-divisor proof of Abban–Zhuang adjunction, retaining the coefficient of S, gives $\beta_X(Q)\ge(1-S_X(S))\operatorname{ord}_Q(S)$. For clarity, choose a basis simultaneously adapted to Q and S, write its basis divisor as $a_mS+D'_m$, and restrict D'_m to S. The refined threshold bound gives numbers $c_m\to 1$ for which $(S,c_mD'_m|_S)$ is log canonical near p. Inversion of adjunction then gives

$$A_X(Q)\ge c_mS_m(Q)+(1-c_ma_m)\operatorname{ord}_Q(S).$$

Pass to the limit. If $p\notin E$, all relevant valuations of S have zero order on C_S, and the refined threshold is at least $16/15$. The other Abban–Zhuang term is $16/11$. □

Lemma 110.3. *Let S be one of the sextic del Pezzo fibres occurring here. If S is smooth, then $\delta_p(S)\ge 1$ for every $p\in S$, and $\delta_p(S)\ge 6/5$ off the six lines. If S has an A_1 singularity, then $\delta_p(S)\ge 1$ off the two lines through the singular point. If S has an A_2 singularity, then $\delta_p(S)\ge 1$ off its two lines.*

Proof. We give the surface calculations to fix the values needed here. On the minimal resolution, write the anticanonical class as D. At a smooth point off all lines, blow up the point, with exceptional curve G. The pencil classes $h - e_i$ in the plane model give the following Zariski decomposition. For $0 \le v \le 2$ its positive part is $D - vG$. For $2 \le v \le 3$ its negative part is $(v-2)Z$, where Z has one of the following forms:

$$Z = B_1 + B_2 + B_3, \qquad Z = 2B_1 + C_1 + B_2, \qquad Z = 3B_1 + 2C_1 + C_2.$$

Here the B_i are the proper transforms of the distinct moving pencil members through the point and have square -1; the C_i are the intervening (-2)-curves. The three cases correspond to three distinct plane points, one infinitely near pair, and a chain of three infinitely near points. The curves B_i meet G in distinct points, and the C_i do not meet G. In each case the proposed positive part is nef: its intersection with every negative curve is nonnegative, and the displayed divisor has zero intersection with the support of the negative part. These negative curves generate the effective cone of the resulting weak quintic del Pezzo surface. Equivalently, the same verification uses the plane classes $h - e_i$ and their residual exceptional chains. The squares and intersections are

$$P(v)^2 = \begin{cases} 6 - v^2, & 0 \le v \le 2, \\ 2(3-v)^2, & 2 \le v \le 3, \end{cases} \qquad P(v)G = \begin{cases} v, & 0 \le v \le 2, \\ 2(3-v), & 2 \le v \le 3. \end{cases}$$

They give $S_S(G) = 5/3$. At any point $q \in G$, let k be the coefficient of the component of Z meeting q; put $k = 0$ when there is no such component. Then $k \le 3$ and

$$S(W_S^G; q) \le \frac{1}{6}\left(\int_0^2 v^2\,dv + \int_2^3 4(3-v)^2\,dv\right) + \frac{k}{3}\int_2^3 2(3-v)(v-2)\,dv = \frac{2}{3} + \frac{k}{9} \le 1.$$

On a smooth sextic surface $k \le 1$, so this bound is $7/9$ and Abban–Zhuang gives $\delta_p(S) \ge 6/5$.

On a smooth sextic surface, a line T has adjacent lines U, V. For $0 \le v \le 1$, $D - vT$ is nef. For $1 \le v \le 2$, the negative part is $(v-1)(U+V)$. Thus

$$P(v)^2 = \begin{cases} 6 - 2v - v^2, & 0 \le v \le 1, \\ (2-v)(4-v), & 1 \le v \le 2, \end{cases} \qquad P(v)T = \begin{cases} 1 + v, & 0 \le v \le 1, \\ 3 - v, & 1 \le v \le 2. \end{cases}$$

Consequently $S_S(T) = 1$, and the secondary invariant is $7/9$ away from $T \cap (U \cup V)$ and 1 at these two intersections. This proves the assertion on all smooth fibres.

For an A_1 fibre, let T be one of the two lines not through the node. On the resolution, T meets a (-1)-curve U meeting the exceptional (-2)-curve R. For $1 \le v \le 2$ the negative part is $(v-1)(2U+R)$; for $v \le 1$ it is zero. The same displayed formula for $P(v)^2$ holds, because $(2U+R)^2 = -2$. Thus $S_S(T) = 1$, and the secondary invariant is $7/9$ away from $T \cap U$. The latter point lies on a line through the node and is excluded from the assertion. This finishes the proof. □

Remark 110.4. The restriction to the two lines through the singularity is necessary for this particular fibre argument. On an A_1 fibre, either such line T has

$$P(v)^2 = \begin{cases} 6 - 2v - v^2/2, & 0 \le v \le 1, \\ 7 - 4v + v^2/2, & 1 \le v \le 2, \\ (3-v)^2, & 2 \le v \le 3, \end{cases} \qquad S_S(T) = \frac{11}{9}.$$

On an A_2 fibre either line has

$$P(v)^2 = \begin{cases} 6 - 2v - v^2/3, & 0 \le v \le 3/2, \\ (3-v)^2, & 3/2 \le v \le 3, \end{cases} \qquad S_S(T) = \frac{5}{4}.$$

These are obtained by solving the intersection equations against the (-2)-chain and then adjoining the adjacent (-1)-curves when their intersection becomes negative. They agree with the calculations in [12, Lemmas 4.5–4.6]. In particular the first refined moments are $55/48$ and $75/64$, respectively. They exceed one, so the fibre refinement alone does not prove the required result there.

Lemma 110.5. *Every prime divisor over X whose centre is a curve in E dominating the pencil base has positive beta invariant.*

Proof. First consider a point $p \in C_S$ on a smooth fibre. If p is off the six lines, Lemmas 110.1 and 110.3 give

$$\delta_p(X) \geq \min\left\{\frac{16}{11}, \frac{16(6/5)}{15+6/5}\right\} = \frac{32}{27} > 1.$$

If p lies on just one line T, refine once more by T. The first invariant is $15/16$. Since $C_S T = 1$, the intersection at p is transverse; the secondary invariant is at most

$$\frac{15}{16}\cdot\frac{7}{9} + \frac{1}{16} = \frac{19}{24}.$$

Hence $\delta_p(X) \geq 16/15$.

Only the corners of the hexagon of lines remain. There are finitely many pairs (S, p) with S smooth and $p \in C_S$ such a corner. To see this, view V as the incidence variety of points and lines in $\mathbb{P}^2$. The contact plane at (x, ℓ) is the span of the tangent directions of the two projection fibres. At a corner, these are the two line directions in S, so $T_p C$ lies in the contact plane. The induced morphism

$$T_C \longrightarrow \mathcal{O}_V(1,1)|_C$$

is not identically zero. Otherwise the second projection would send every point of the smooth plane cubic given by the first projection to its tangent line. This tangent-line map has degree six with respect to $\mathcal{O}_{(\mathbb{P}^2)^*}(1)$: the three quadratic partial derivatives of its smooth cubic equation have no common zero on the cubic. But the second projection of C has degree three, a contradiction. The displayed morphism therefore has only finitely many zeros. For each zero there is at most one smooth pencil member whose tangent plane is the contact plane, since the two pencil sections have independent differentials along the smooth complete intersection C.

A curve in E dominating the pencil base meets a general fibre at points avoiding this finite set. At any such point the preceding strict local bound excludes the curve as a nonpositive-beta centre. □

Proposition 110.6. *If Q is a prime divisor over X with $\beta_X(Q) \leq 0$, its centre is contained in one of the two lines through the singular point of an A_1 or A_2 fibre.*

Proof. A centre meeting a smooth fibre outside E is excluded by Lemma 110.2. A horizontal curve in E is excluded by Lemma 110.5. The divisor E itself has $S_X(E) = 3/8 < 1$. These observations exclude all horizontal centres. A divisorial centre in a fibre is the fibre itself and has $S_X(F) = 11/16 < 1$. Any remaining centre lies in a fibre and has dimension at most one. At every point outside the stated lines, Lemma 110.3 and the strict inequality in Lemma 110.2 exclude it. The singular point of either singular fibre lies on the stated lines. □

111 Quintic del Pezzo sections

Lemma 111.1. *The degree-three line bundles $L_1 = \mathcal{O}_C(1,0)$ and $L_2 = \mathcal{O}_C(0,1)$ are not isomorphic.*

Proof. Both plane embeddings are defined by their complete linear systems. If $L_1 \simeq L_2$, a projective change of coordinates identifies these embeddings. The bilinear equation of V then restricts on C to a quadratic form in the coordinates of one plane. A nonzero quadratic form cannot vanish on a smooth plane cubic. Therefore this quadratic form vanishes identically, so the matrix of the bilinear equation is skew-symmetric. A skew-symmetric matrix of size three is singular. This contradicts the smoothness of V. □

Lemma 111.2. *Put $H_1 = \pi^*\mathcal{O}_V(1,0)$ and $H_2 = \pi^*\mathcal{O}_V(0,1)$. For every point $p \in X$, a general member $T \in |H_1|$ containing p is a smooth del Pezzo surface of degree five. The same holds with H_1 and H_2 interchanged. More precisely, T is the blow-up of $\mathbb{P}^2_y$ at four points p_0, p_1, p_2, p_3 in general position, and*

$$H_1|_T = l - e_0, \qquad H_2|_T = l, \qquad F|_T = 2l - e_0 - e_1 - e_2 - e_3, \qquad E|_T = e_1 + e_2 + e_3.$$

Here l is the pull-back of a line and the e_i are the exceptional curves.

Proof. Let q be the first plane image of p. Write C_i for the two smooth plane cubic images of C. Choose a general line $L \subset \mathbb{P}^2_x$ through q. It meets C_1 transversely in three distinct points x_1, x_2, x_3. We may also require that its point $[L] \in \mathbb{P}^2_y$ does not belong to C_2: the pencil of lines through q is a line in $\mathbb{P}^2_y$, and cannot be contained in the smooth cubic C_2.

The surface $V_L = V \cap (L \times \mathbb{P}^2_y)$ is the incidence surface of points of L and lines of $\mathbb{P}^2_x$. Its second projection is the blow-up of $\mathbb{P}^2_y$ at $p_0 = [L]$. Indeed, for $y \neq [L]$ the unique point above y is $L \cap y$, whereas the fibre over $[L]$ is L. Let $c_i = (x_i, y_i)$ be the three points of $C \cap V_L$. They are distinct and avoid this exceptional curve, by the choice $[L] \notin C_2$. Since V_L meets C transversely, its inverse image T in the blow-up X is the blow-up of V_L at c_1, c_2, c_3. In particular it is smooth and is the blow-up of $\mathbb{P}^2_y$ at $p_0, p_1 = y_1, p_2 = y_2, p_3 = y_3$.

These four points are in general position. They are distinct because $C \to C_2$ is an embedding and $p_0 \notin C_2$. If p_0, p_i, p_j were collinear, the three corresponding lines L, y_i, y_j in $\mathbb{P}^2_x$ would be concurrent. As $x_i = L \cap y_i$ and $x_j = L \cap y_j$, this would give $x_i = x_j$, a contradiction. If p_1, p_2, p_3 were collinear, the divisor $c_1 + c_2 + c_3$ on C would belong to both $|L_1|$ and $|L_2|$. This contradicts Lemma 111.1. Thus T is a smooth del Pezzo surface of degree five. The displayed divisor classes follow from the two projections and $F = H_1 + H_2 - E$.

This construction includes points on either component of every singular conic fibre of $X \to \mathbb{P}^2_x$, as well as their intersection: the Cartier inverse image of every line through q contains the entire fibre over q. In particular there is no exception for $p \in E$. Transversality of L to C_1 at q, when $q \in C_1$, is compatible with all the preceding choices. □

Remark 111.3. For this surface T, put $D_1 = 3H_1 - E$ and $D_2 = 3H_2 - E$. The divisor $D_1|_T$ is the union of the three disjoint exceptional lines of classes $l - e_0 - e_i$, while

$$D_2|_T \sim 3l - e_1 - e_2 - e_3 = -K_T + e_0.$$

The conic bundle $T \to L$ has precisely three singular fibres, $e_i + (l - e_0 - e_i)$. Thus the choice of T does not remove points on the reducible conic fibres; it places all of them on an ordinary smooth del Pezzo surface with an explicit ten-line configuration.

Lemma 111.4. *For the general surface T chosen in Lemma 111.2, the curve $(3H_2 - E)|_T$ is smooth and irreducible, of class $3l - e_1 - e_2 - e_3$.*

Proof. The inverse image of C_2 in V under the second plane projection is a $\mathbb{P}^1$-bundle containing C as a section. Its strict transform D_2 in X is isomorphic to it, since blowing up a Cartier divisor on a smooth surface does not change the surface. For a point $y \in C_2$, the corresponding ruling fibre is the line represented by y in the first plane. The line L defining T is different from each of these lines, because $[L] \notin C_2$. It therefore meets each ruling fibre once. Thus $T \cap D_2$ is a section of this bundle, isomorphic to C_2. Its class was computed in Remark 111.3. □

Lemma 111.5. *For $D \in |A|$, one has $S_X(D) = 69/128$. The divisorial positive and negative parts of $L - uA$ are*

$$(P(u), N(u)) = \begin{cases} ((1-u)A + B + F, 0), & 0 \le u \le 1, \\ ((3-2u)B + (2-u)F, (u-1)(F + 2B - A)), & 1 \le u \le 3/2. \end{cases}$$

Proof. The divisors A, B, F are nef. The divisor $J = F + 2B - A$ is the exceptional divisor of the contraction to $\mathbb{P}^1 \times \mathbb{P}^2$ given by F, B. A fibre of J has intersection one with A and zero with B, F, so its intersection with J is -1. Thus $u > 1$ forces the indicated fixed multiplicity along J. Subtraction leaves the stated nef divisor until $u = 3/2$. For $u > 3/2$ one has $(L - uA)BF = 3 - 2u < 0$, excluding pseudoeffectivity. The volumes are $6u^2 - 27u + 24$ on $[0, 1]$ and $-12u^3 + 60u^2 - 99u + 54$ on $[1, 3/2]$. Their integral, divided by 24, is $69/128$. □

Remark 111.6. The preceding flag alone cannot settle points of the original exceptional divisor. In a smooth quintic section $D = \mathrm{Bl}_{p_0,p_1,p_2,p_3} \mathbb{P}^2$, write h for the line class and e_0, e_1, e_2, e_3 for the exceptional classes. Then $A|_D = h - e_0$, $B|_D = h$, $F|_D = 2h - \sum e_i$, and $E|_D = e_1 + e_2 + e_3$. Direct surface Zariski integration gives

$$S(W^D; e_i) = \frac{69}{64} > 1 \quad (i = 1, 2, 3), \qquad S(W^D; h - e_0 - e_i) = \frac{139}{128} > 1.$$

Consequently further refinement of this same surface flag cannot prove a local threshold exceeding one at a point of e_i. This is a limitation of the chosen refinement, not an instability statement about X.

Lemma 111.7. *The moving section spaces on D used below are complete in every sufficiently large divisible degree, for each rational u in the corresponding interval. In particular, the numerical moments are computed with the original normalization $(-K_X)^3 = 24$.*

Proof. Let $\rho_2 : X \longrightarrow Y = \mathbb{P}^1 \times \mathbb{P}^2$ be the birational morphism defined by F and B, with exceptional divisor D_2. Recall that $A = F + 2B - D_2$, so the image of a general $D \in |A|$ is an integral divisor of bidegree $(1, 2)$ containing the centre Z_2 of ρ_2. Moreover,

$$(F + B)|_D = 3l - e_0 - e_1 - e_2 - e_3 = -K_D.$$

The six sections of $\mathcal{O}_Y(1, 1)$ restrict independently to D: a relation would give a divisor of bidegree $(1, 1)$ containing its image of bidegree $(1, 2)$, which is impossible. Since $h^0(D, -K_D) = 6$ and $-K_D$ is very ample, these sections give the complete anticanonical embedding. Thus $\rho_2|_D$ is an isomorphism onto a smooth divisor of bidegree $(1, 2)$, and $D_2|_D$ is its smooth elliptic curve Z_2.

For rational $0 \leqslant u < 1$, the divisor $P_X(u) = (1 - u)A + B + F$ on X is ample. After clearing denominators, Serre vanishing applied to

$$0 \longrightarrow \mathcal{O}_X(mP_X(u) - D) \longrightarrow \mathcal{O}_X(mP_X(u)) \longrightarrow \mathcal{O}_D(mP(u)) \longrightarrow 0$$

gives surjectivity of restriction for all sufficiently large divisible m.

For $1 \leqslant u \leqslant 3/2$, removing the fixed divisor $m(u - 1)D_2$ identifies the global moving sections with the sections of $\mathcal{O}_Y(a, b)$, where

$$a = m(2 - u), \qquad b = m(3 - 2u).$$

Both integers are non-negative. As $D \subset Y$ has bidegree $(1, 2)$, the restriction sequence is

$$0 \longrightarrow \mathcal{O}_Y(a - 1, b - 2) \longrightarrow \mathcal{O}_Y(a, b) \longrightarrow \mathcal{O}_D(mP(u)) \longrightarrow 0.$$

Its first cohomology vanishes: $H^1(\mathbb{P}^1, \mathcal{O}(a-1)) = 0$ for $a \geqslant 0$, and $H^1(\mathbb{P}^2, \mathcal{O}(b-2)) = 0$. Thus the moving restrictions are complete also on this interval.

Refining on D by a flag curve gives the usual surface Zariski-decomposition bounds. One may enlarge the resulting spaces on the curve to complete series, since a basis compatible with the point filtration extends to a compatible basis of the larger space and all added vanishing orders are non-negative. This only increases the moments. Rounding errors vanish in the normalized limit. Finally, the fixed divisor removed on the second interval restricts to $m(u-1)C_2$; it is retained in the refined divisor and contributes $7(C_2 \cdot T)_p/384$ to the point moment. Omitting this contribution would give an incorrect calculation at points of C_2. □

Lemma 111.8. *Let D be the blow up of $\mathbb{P}^2$ at four points in general position, with classes l, e_0, e_1, e_2, e_3, and put $T = 2l - e_1 - e_2 - e_3$. Suppose that the refinement by D has moving divisors*

$$P(u) = \begin{cases} (4-u)l - (2-u)e_0 - e_1 - e_2 - e_3, & 0 \leqslant u \leqslant 1, \\ (7-4u)l - (2-u)(e_0 + e_1 + e_2 + e_3), & 1 \leqslant u \leqslant 3/2, \end{cases}$$

and fixed part $(u-1)C_2$ on the second interval, where C_2 is an irreducible curve of class $3l - e_1 - e_2 - e_3$. Let p be outside the ten (-1)-curves of D, and let $Q \in |l - e_0|$ contain p. With the threefold normalization $(-K_X)^3 = 24$, we have

$$S(W^D_{\bullet,\bullet}; Q) \leqslant \frac{245}{288}, \qquad S(W^{D,Q}_{\bullet,\bullet,\bullet}; p) \leqslant \frac{133}{144} + \frac{7}{384}(C_2 \cdot Q)_p \leqslant \frac{1127}{1152} < 1.$$

In particular, if $S_X(D) = 69/128$, then $\delta_p(X) \geqslant 1152/1127 > 1$.

Proof. The curve Q is a smooth rational curve, since the reducible fibres of $|l - e_0|$ are the unions $e_i + (l - e_0 - e_i)$, for $1 \leqslant i \leqslant 3$. Put

$$L_{ij} = l - e_i - e_j, \qquad R = L_{12} + L_{13} + L_{23},$$
$$a = 3 - 2u, \quad b = 2 - u, \quad k = 8 - 5u - 2v, \quad r = 2 - u - v.$$

The four curves $e_0, L_{12}, L_{13}, L_{23}$ are pairwise disjoint. The positive and negative parts $M(u,v), N(u,v)$ of $P(u) - vQ$ are as follows. The last column records $M(u,v) \cdot Q$.

range	$M(u,v)$	$N(u,v)$	$M(u,v) \cdot Q$
$0 \leqslant u \leqslant 1$, $0 \leqslant v \leqslant 2-u$	$P(u) - vQ$	0	2
$0 \leqslant u \leqslant 1$, $2-u \leqslant v \leqslant 5/2-u$	$(5-2u-2v)T$	$(v+u-2)(e_0+R)$	$2(5-2u-2v)$
$1 \leqslant u \leqslant 3/2$, $0 \leqslant v \leqslant a$	$P(u) - vQ$	0	$5-3u$
$1 \leqslant u \leqslant 4/3$, $a \leqslant v \leqslant b$	$kT - re_0$	$(v-a)R$	$2k-r$
$4/3 \leqslant u \leqslant 3/2$, $a \leqslant v \leqslant 6-4u$	$kT - re_0$	$(v-a)R$	$2k-r$
$1 \leqslant u \leqslant 4/3$, $b \leqslant v \leqslant 4-5u/2$	kT	$(v-a)R + (v-b)e_0$	$2k$

These are Zariski decompositions: the listed negative curves are disjoint, the positive part is orthogonal to each component of the negative part, and its intersection with every (-1)-curve is non-negative. Since the cone of curves is generated by the ten (-1)-curves, the positive part is nef. The upper endpoints are the pseudoeffective thresholds. At these endpoints the positive part is zero, except when $4/3 \leqslant u \leqslant 3/2$, where it is a multiple of the nef class $2l - e_0 - e_1 - e_2 - e_3$ of square zero. Increasing v then gives negative intersection with this nef class, or with T, as appropriate.

For completeness, the squares in the six rows are, respectively,

$$9 - 4u - 4v, \quad (5-2u-2v)^2, \quad 33 - 40u + 12u^2 - 2v(5-3u), \quad k^2 - r^2, \quad k^2 - r^2, \quad k^2.$$

Integrating these polynomials over their indicated ranges gives

$$\iint M(u,v)^2 \, dv \, du = \frac{245}{36}, \qquad \iint (M(u,v) \cdot Q)^2 \, dv \, du = \frac{133}{18},$$

where each integral means the sum over the six rows. Moreover,

$$\int_1^{3/2}\int_0^{t(u)} (u-1)(M(u,v)\cdot Q)\,dv\,du = \frac{7}{96}.$$

Here $t(u)$ is the upper endpoint in the table. Since p is outside the ten negative curves, $N(u,v)|_Q$ has zero multiplicity at p. The Abban–Zhuang formula, with factors $3/24$ and $6/24$, now gives the two asserted bounds. Finally, $C_2 \cdot Q = 3$, so $(C_2 \cdot Q)_p \leqslant 3$. Applying the flag estimate proves the assertion. □

Remark 111.9. If Q is smooth and p is allowed to lie on a negative curve, the complete point bound is

$$S(W^{D,Q}_{\bullet,\bullet,\bullet};p) \leqslant \frac{133}{144} + \frac{7}{384}(C_2\cdot Q)_p + \frac{13}{576}\operatorname{ord}_p(e_0|_Q) + \frac{29}{1152}\operatorname{ord}_p((L_{12}+L_{13}+L_{23})|_Q).$$

The last four intersections have multiplicity one whenever they occur. This gives $545/576$ at a point of e_0, since $C_2 \cdot e_0 = 0$. At a point of $L_{12} \cup L_{13} \cup L_{23}$ it gives at most $1135/1152$ if $(C_2 \cdot Q)_p \leqslant 2$. The upper bound becomes $289/288$ for contact three, so this last configuration requires another flag.

Lemma 111.10. *Let ℓ be a line through the singular point of an A_1 or A_2 fibre, and suppose that the first plane projection contracts ℓ. Put $D_2 = 3H_2 - E$. If $p \in \ell \setminus (E \cup D_2)$, there is a smooth degree-five del Pezzo surface $T \in |H_2|$ through p such that p is on none of its ten (-1)-curves.*

For an A_1 fibre, the support of $D_2 \cap \ell$ consists of its singular point and one further point, at which ℓ meets the other line contracted by the second plane projection. This further point is smooth on the fibre; it may also belong to E. For an A_2 fibre, $D_2 \cap \ell$ is supported only at its singular point, with multiplicity two. The same assertions hold after interchanging the two projections.

Proof. Write $\pi(p) = (x_0, y)$ and let $c_0 = (x_0, y_0)$ be the unique point of C above x_0. The line ℓ is the strict transform of the first-projection fibre over x_0. The divisor D_2 is the strict transform of the inverse image of the plane cubic C_2 under the second projection. Thus $p \notin E \cup D_2$ implies $y \notin C_2$.

Choose a general line $M \subset \mathbb{P}^2_y$ through y. The proof of Lemma 111.2, with the projections interchanged, identifies its inverse image T with the blow-up of $\mathbb{P}^2_x$ at four points $r = [M], x_1, x_2, x_3$, where $(x_i, y_i) = c_i \in C$ and $\{y_1, y_2, y_3\} = M \cap C_2$. We may take these four points in general position. We may also assume $r \neq x_0$ and $M \not\ni y_0$. Consequently p is on none of the four exceptional curves.

The line through r and x_i in $\mathbb{P}^2_x$ is precisely the line represented by y_i: both points lie on it. Thus p is on the strict transform of this line only if $y_i(x_0) = 0$. There are only finitely many such points y_i on C_2, and none equals y, since $y \notin C_2$. Avoiding the finitely many lines joining y to these points ensures that p lies on none of the three curves of classes $l - e_0 - e_i$.

Finally, suppose that infinitely many choices of M put p on a line through two of x_1, x_2, x_3. Projection of C_1 from x_0 gives the degree-two morphism defined by $|L_1 - c_0|$; let ι be its involution. Projection of C_2 from $y \notin C_2$ gives a degree-three morphism $g\colon C \to \mathbb{P}^1$. The supposed incidences give infinitely many points c with $g(c) = g(\iota(c))$. Therefore these two morphisms are identical, and g factors through the quotient by ι. Its degree would be even, a contradiction. Only finitely many further choices of M are excluded. This proves the first assertion.

For completeness, the exceptional points can be read directly from the Jordan forms. At an A_1 fibre take

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \ell = \{x = [1:0:0],\ y = [0:s:t]\}.$$

If $B = (b_{ij})$ defines the other pencil member, the plane cubic C_2 restricted to this line has equation, up to nonzero scalar,

$$st(b_{10}s + b_{20}t) = 0.$$

The last factor cuts out $C \cap \ell$. Removing it in the blow-up leaves $st = 0$. The point $[s:t] = [1:0]$ is the node, and $[s:t] = [0:1]$ is the smooth intersection with the good line contracted by the second projection. Smoothness of C ensures $b_{10} \neq 0$, so C misses the node. The other point may coincide with $C \cap \ell$, but the strict-transform computation still gives one residual intersection there. For an A_2 fibre take A to be the nilpotent block of size three. On the same line the equation is

$$s^2(b_{10}s + b_{20}t) = 0.$$

Here $b_{20} \neq 0$, since C misses the singular point $[s:t] = [0:1]$. After removing the base-curve factor, the intersection is twice this point. Transposing the matrices proves the symmetric assertions. □

112 The singular point of an A_1 fibre

Proposition 112.1. *Let p be the singular point of a fibre with an A_1 singularity. Then $\delta_p(X) \geq 144/121 > 1$.*

Lemma 112.2. *Let $\pi : Y \to X$ be the blow-up of p, and let G be its exceptional plane. There are homogeneous coordinates a, b, c on G such that the refined anticanonical section algebra is generated, in anticanonical degree one, by*

$$1, a, b, c, ab, b^2, bc, \qquad F, Fa, Fb, Fc, Fab, Fb^2, Fbc, F^2, \qquad F = ac - b^2.$$

The primary degrees are the ordinary polynomial degrees in a, b, c. For every m, a basis of its anticanonical degree-m part consists of

$$a^r b^d c^q F^s, \qquad rq = 0, \qquad n = \max\left\{r + q, \left\lceil \frac{r + d + q}{2} \right\rceil\right\} \leq m, \qquad s + n \leq 2m. \quad (37.A1)$$

Proof. Write the two bilinear forms defining the flag threefold and this fibre as

$$x_0y_0 + x_1y_1 + x_2y_2, \qquad x_0y_1 + x_2y_2.$$

The singular point is $([0:1:0],[1:0:0])$. In the chart $x_1 = y_0 = 1$ put $a = x_2$, $b = y_1$, $c = y_2$. The flag equation gives $x_0 = -b - ac$, and the fibre equation becomes

$$f = ac - b^2 - abc.$$

A basis of the restrictions of bilinear forms can be chosen with the following eight least homogeneous terms:

$$1, a, b, c, ab, b^2, bc, F.$$

Indeed one can use the actual functions $1, a, b, c, ab, ac - f, bc + ac^2, f$. The other generator g of the pencil is a unit at p, since its base curve misses every singular point of a fibre. Rescale

it so that $g(p) = 1$. The base coordinate is $t = f/g$. The anticanonical bundle is $H + \mathcal{O}_{\mathbb{P}^1}(1)$ in the incidence presentation of X. Its sections are products of the eight bilinear sections with $1, t$, modulo the one relation $f = tg$. Removing f from the first group gives precisely the fifteen least terms in the statement. Thus the displayed algebra is contained in the actual associated graded algebra.

We prove equality in every degree, rather than assuming generation of an associated graded ring in degree one. A monomial $a^r b^d c^q$ with $rq = 0$ requires exactly n factors among a, b, c, ab, b^2, bc, with n as in (37.A1). For example, when $q = 0$, pair as many a's as possible with b's, and use b^2 for the remaining pairs of b's. Its n factors may each carry one extra F, and the remaining $m - n$ slots may carry $1, F$, or F^2. This proves that every monomial in (37.A1) belongs to degree m. Conversely, reduce every factor ac by $ac = F + b^2$. For the first summand n decreases by at least one while s increases by one; for the second n does not increase. Hence both inequalities are preserved. Normal monomials are independent, since $ac - F - b^2$ has leading monomial ac and they are its standard monomials.

For $n \geq 1$ there are exactly $6n$ normal monomials of cost n. For $r = q = 0$ there are two. For each of the two branches $r > 0$ or $q > 0$, there are $2(n-1) + (n+1) = 3n - 1$. Consequently the number in degree m is

$$(2m+1) + \sum_{n=1}^{m} 6n(2m - n + 1) = 4m^3 + 6m^2 + 4m + 1 = h^0(X, -mK_X).$$

The inclusion of graded algebras is therefore equality in every degree. In particular these are the actual refined section spaces. □

Lemma 112.3. *Fix the primary degree j and the weight k for the action $(a, b, c) \mapsto (\lambda a, b, \lambda^{-1} c)$, and put $r = |k|$. The corresponding component is zero unless $r \leq m$ and $j \leq 4m$. Otherwise put*

$$l = \max\{0, \lceil j/2 \rceil - m\}, \qquad u = \min\{\lfloor (j - r)/2 \rfloor, 2m - r\}, \qquad D = u - l.$$

It is zero if $D < 0$. If $D \geq 0$, it is the complete binary space

$$a^{\max(k,0)} c^{\max(-k,0)} b^{j-r-2u} F^l \operatorname{Sym}^D \langle b^2, F \rangle. \tag{37.A2}$$

Its dimension is $D + 1$.

Proof. In (37.A1) the prescribed gradings give $q = 0$, $r = k$ when $k \geq 0$ (and the symmetric formula when $k < 0$), and $d = j - |k| - 2s$. The two inequalities in (37.A1) say exactly $l \leq s \leq u$. All integers in this interval occur. Factoring its two endpoints proves (37.A2). This also proves completeness; the argument does not replace a section space by its Newton polygon. □

Lemma 112.4. *Normalize sums of vanishing orders by mN_m, where $N_m = 4m^3 + 6m^2 + 4m + 1$. In the binary spaces (37.A2) the limiting fixed coefficients of a, c, b, F and the limiting half movable degree are, respectively,*

$$\alpha = \frac{5}{24}, \quad \alpha = \frac{5}{24}, \quad \beta = \frac{1}{32}, \quad \varphi = \frac{65}{192}, \quad h = \frac{67}{192}.$$

Moreover $S_X(G) = 121/48$.

Proof. Here are explicit finite sums for the calculation. Sum over $-m \leq k \leq m$, $|k| \leq j \leq 4m$, retaining only $u \geq l$ as in Lemma 112.3. Every summand has weight $D + 1$. The quantities to sum are

$$\max(k, 0), \quad \max(-k, 0), \quad j - |k| - 2u, \quad l, \quad D.$$

For even m the resulting sums, in this order, are

$$\begin{aligned} A_m = C_m &= \frac{m(m+1)(5m^2+5m+2)}{6}, \\ B_m &= \frac{m(m^3+18m^2+18m+8)}{8}, \\ Q_m &= \frac{m(65m^3+130m^2+100m+32)}{48}, \\ R_m &= \frac{m(67m^3+38m^2+8m-8)}{24}. \end{aligned}$$

These identities follow by substituting the displayed lower and upper endpoints and summing the arithmetic progressions. They also provide a direct calculation of all the constants: divide the first four sums by mN_m and the last by $2mN_m$. Floors and ceilings change only lower-order terms, so the same limits hold without the parity restriction. Finally the average ordinary degree is

$$2\alpha + \beta + 2\varphi + 4h = \frac{121}{48}.$$

By definition this is $S_X(G)$. The sum of the primary degrees is in fact

$$\frac{m(121m^3+170m^2+113m+28)}{12}$$

for every $m \geq 0$. To check this without a parity argument, group the normal monomials by the cost n in (37.A1). For $n \geq 1$, the $6n$ monomials $a^r b^d c^q$ have total ordinary degree

$$(4n-1) + 2\left((n-1)(4n-1) + \frac{3n(n+1)}{2}\right) = 11n^2 - 3n + 1.$$

The first term comes from $r = q = 0$; in either remaining branch, $1 \leq r < n$ contributes degrees $2n-1, 2n$, while $r = n$ contributes degrees $n, n+1, \ldots, 2n$ (and similarly for q). Writing $M = 2m - n$, summation over $0 \leq s \leq M$ consequently gives

$$2m(2m+1) + \sum_{n=1}^{m} \Big((2m-n+1)(11n^2-3n+1) + 6n(2m-n)(2m-n+1)\Big),$$

which evaluates to the stated polynomial. Equality of the normal algebra with the actual refinement was proved in Lemma 112.2, so these are the valuation weights. □

Lemma 112.5. *On $G = \mathbb{P}^2_{a,b,c}$ put $A_0 = (a = 0)$, $B_0 = (b = 0)$, $C_0 = (c = 0)$, and $T_\lambda = (ac - \lambda b^2 = 0)$, with $T_0 = A_0 + C_0$ and $T_\infty = 2B_0$. Suppose*

$$B = \alpha(A_0 + C_0) + \beta B_0 + \varphi T_1 + \sum \nu_\lambda T_\lambda, \qquad \sum \nu_\lambda = 2h, \quad 0 \leq \nu_\lambda \leq h.$$

For arbitrary nonnegative $\alpha, \beta, \varphi, h$, the pair (G, B) is log canonical if

$$\alpha + h, \quad \beta + 2h, \quad \varphi + h, \quad h \leq 1,$$

and

$$\alpha + \beta + \varphi + 3h \leq 2, \qquad 2\alpha + \beta + 2\varphi + 4h \leq 3.$$

Proof. The first four quantities bound the coefficients of all components. Away from $[0:0:1]$ and $[1:0:0]$ the support has simple normal crossings. At each of these two points the pencil is $a = \lambda b^2$ in local coordinates. Two successive ordinary blow-ups resolve all its members simultaneously. The exceptional valuations have weights $(1,1)$ and $(2,1)$, and log discrepancies

two and three. Their orders on B are at most $\alpha+\beta+\varphi+3h$ and exactly $2\alpha+\beta+2\varphi+4h$, respectively. After the second blow-up all distinct branches meet the last exceptional curve at different points. Thus the stated inequalities give a log resolution with all boundary coefficients at most one. This proves the claim for every valuation, including valuations above the exceptional curves. □

Proof of Proposition 112.1. For an arbitrary basis of a complete degree-D binary space, the sum of the vanishing orders at any prescribed point of $\mathbb{P}^1$ is at most $D(D+1)/2$, whereas its total degree is $D(D+1)$. Consequently arbitrary bases in the components (37.A2) give divisors of the form in Lemma 112.5. The limiting component bounds are

$$\frac{107}{192},\quad \frac{35}{48},\quad \frac{11}{16},\quad \frac{67}{192}.$$

The limiting exceptional orders are 13/8 and 121/48, whose ratios to the discrepancies are 13/16 and 121/144. The largest of all six ratios is 121/144. Hence for every $\tau < 144/121$ all sufficiently high-degree basis divisors in these components give log canonical pairs after multiplication by τ, uniformly over the choices of bases.

Now fix a divisorial valuation over G. In each already fixed primary-degree piece choose a basis compatible with its valuation filtration and the displayed torus-weight filtration; this uses only the two-filtration lemma. Its least-weight terms are bases in (37.A2). Degeneration under this one-parameter subgroup and lower semicontinuity of log canonical thresholds give the same bound for the original valuation-adapted basis. Passing to the limit gives

$$\delta(G;W^G)\geq\frac{144}{121}.$$

The ordinary point blow-up is a plt extraction and has zero different. Abban–Zhuang adjunction therefore gives

$$\delta_p(X)\geq\min\left\{\frac{3}{121/48},\frac{144}{121}\right\}=\frac{144}{121}.$$

No Zariski decomposition on Y has been used. The calculation used actual sections, an exact Hilbert count in every degree, and a resolution of a conic pencil on its exceptional plane. □

Example 112.6. In anticanonical degree one the primary-degree-two, torus-weight-zero component is $\langle b^2,F\rangle=\langle b^2,ac\rangle$. A basis adapted to T_1 is (b^2,F), whereas one adapted to A_0 is (b^2,ac). The two bases illustrate why the complete binary spaces must be retained: choosing just the original monomial basis would miss cancellation along A_0.

113 The singular point of an A_2 fibre

Lemma 113.1. *At the singular point of an A_2 fibre there are regular parameters A,b,C such that the weighted blow-up with weights $(3,2,3)$ has the fifteen anticanonical initial forms used below.*

Proof. Choose the flag and fibre equations

$$x_0y_0+x_1y_1+x_2y_2=0,\qquad x_0y_1+x_1y_2=0.$$

The singular point is $([0:0:1],[1:0:0])$. On $x_2=y_0=1$ set $a=x_1$, $b=y_1$, $c=y_2$. Then $x_0=-ab-c$ and the fibre equation is $(a-b)c-ab^2$. Put $A=a-b$ and $C=c-b^2$. These are regular parameters and the fibre equation becomes $F=AC-b^3$. The following eight actual bilinear sections form a basis:

$$1,\ A,\ b,\ C+b^2,\ Ab+b^2,\ bC+Ab^2+2b^3,$$
$$(C+Ab+2b^2)(C+b^2),\ F.$$

Their least terms for weights $(3,2,3)$ are $1, A, b, C, b^2, bC, C^2, F$. For example the sixth and seventh sections are $-x_0y_1$ and $-x_0y_2$. The other pencil generator g is a unit at this point; normalize $g(p) = 1$. In the incidence model the base coordinate is $t = F/g$. Anticanonical sections are products of the eight bilinear sections with $1, t$, modulo $F = tg$. Removing F from the first group gives seven sections and t times all eight. Their initial forms are exactly those displayed below. This proves their inclusion in the actual associated graded section algebra. The next lemma proves equality by an exact dimension count in every degree. □

113.1 The refined series on the weighted exceptional surface

Put $F = AC - b^3$. The anticanonical initial forms at the point under consideration are

$$1, A, b, C, b^2, bC, C^2, F, AF, bF, CF, b^2F, bCF, C^2F, F^2.$$

Here the weights of A, b, C are $3, 2, 3$, respectively. In particular, F has weight 6.

Lemma 113.2. *For every $m \geqslant 0$, a basis of the degree-m initial section space consists of*

$$A^a b^d C^c F^s, \qquad ac = 0, \quad n = a + \left\lceil \frac{d+c}{2} \right\rceil \leqslant m, \quad s + n \leqslant 2m,$$

where all four exponents are nonnegative integers. Its dimension is $4m^3 + 6m^2 + 4m + 1$.

Proof. First consider the seven forms $1, A, b, C, b^2, bC, C^2$. The least number of their nonconstant factors needed to write $A^a b^d C^c$ is $a + \lceil (d+c)/2 \rceil$. Thus a monomial satisfying the displayed conditions is a product of m of the fifteen initial forms: use n nonconstant factors, multiply any of these by F when necessary, and use $1, F, F^2$ for the remaining $m - n$ factors. The possible total powers of F are precisely $0, \dots, 2m - n$.

Conversely, reduce every product by $AC = F + b^3$. Replacing AC by b^3 preserves $a + \lceil (d+c)/2 \rceil$. Replacing it by F increases s by one and decreases this integer by at least one. Hence both inequalities are preserved. The resulting monomials have $ac = 0$. They are linearly independent: they are the normal monomials in $\mathbb{C}[A, b, C, F]/(AC - b^3 - F)$ for a monomial order with leading term AC.

For fixed $n \geqslant 1$, there are $4n + 1$ possibilities with $a = 0$, and $2n - 1$ with $a > 0$ and $c = 0$. There are therefore $6n$ possibilities. The total number is

$$2m + 1 + \sum_{n=1}^{m} 6n(2m - n + 1) = (2m + 1)(2m^2 + 2m + 1).$$

This equals $h^0(X, -mK_X)$, by Riemann–Roch and vanishing, since $(-K_X)^3 = 24$. Thus products of the displayed initial forms already have the dimension of the entire section space. In particular, the calculation identifies the actual initial algebra in every degree. □

Lemma 113.3. *With the normalization $1/(mh^0(X, -mK_X))$, the limiting sums of the exponents a, d, c, s in Lemma 113.2 are, respectively,*

$$\frac{5}{48}, \qquad \frac{5}{8}, \qquad \frac{5}{12}, \qquad \frac{11}{16}.$$

Consequently, $S_X(G) = 111/16$ for the weighted exceptional divisor G.

Proof. The two limiting regions are

$$P_A = \{(a, d, s) : a, d, s \geqslant 0,\ a + d/2 \leqslant 1,\ s + a + d/2 \leqslant 2\},$$
$$P_C = \{(c, d, s) : c, d, s \geqslant 0,\ (c + d)/2 \leqslant 1,\ s + (c + d)/2 \leqslant 2\}.$$

Their overlap contributes only $O(m^2)$ lattice points. Their volumes sum to 4. Integrating first in s gives

$$\int_{P_A} a = \frac{5}{12}, \qquad \int_{P_C} c = \frac{5}{3}, \qquad \int_{P_A} d + \int_{P_C} d = \frac{5}{2},$$
$$\int_{P_A} s + \int_{P_C} s = \frac{11}{4}.$$

Division by 4 gives the asserted moments. The order along G of a basis monomial is $3a + 2d + 3c + 6s$. Its mean is therefore

$$3 \cdot \frac{5}{48} + 2 \cdot \frac{5}{8} + 3 \cdot \frac{5}{12} + 6 \cdot \frac{11}{16} = \frac{111}{16}.$$

The normal basis is compatible with the weighted order, so this is $S_X(G)$. There is also an exact all-degree verification. Among the $6n$ monomials $A^a b^d C^c$ of cost $n \geq 1$, the branch $a = 0$ has $d + c = 2n - 1$ or $2n$ and total weight $20n^2$. The branch $a > 0$, $c = 0$ has total weight

$$\sum_{a=1}^{n-1} (8n - 2a - 2) + 3n = 7n^2 - 6n + 2.$$

Thus their combined weight is $27n^2 - 6n + 2$. Including powers of F, whose weight is six, gives

$$6m(2m+1) + \sum_{n=1}^{m} ((2m - n + 1)(27n^2 - 6n + 2) + 18n(2m - n)(2m - n + 1)) = \frac{m(111m^3 + 158m^2 + 105m + 26)}{4}.$$

Dividing by $m(4m^3 + 6m^2 + 4m + 1)$ gives the same limit. □

Lemma 113.4. *On the coarse exceptional surface*

$$G = \mathbb{P}(1, 1, 2)_{A,C,z}, \qquad z = b^3,$$

the different is $(2/3)\{z = 0\}$. *The refined series is preserved by the torus acting with opposite weights on* A, C *and fixing* z. *After decomposing into its weight spaces, every nonzero component is a fixed monomial times a complete binary series in* z, F. *The limiting fixed coefficients along* $A = 0, C = 0, F = 0$ *are*

$$\frac{5}{48}, \qquad \frac{5}{12}, \qquad \frac{23}{48}.$$

The limiting total degree of the binary series is $5/12$*; its normalized order along any prescribed member of the pencil* $AC = \lambda z$ *is at most* $5/24$.

Proof. The coarse description follows by taking the third Veronese of the weighted coordinate ring. The generic stabilizer on $b = 0$ has order three, giving the stated different.

For the monomials with $a = r \geqslant 0, c = 0$, write $d = j + 3k$, where $0 \leqslant j \leqslant 2$, and put $\ell = k + s$. Fixing the primary weighted degree and the torus weight fixes r, j, ℓ. The corresponding component is

$$A^r b^j F^{\ell - K} \operatorname{Sym}^K \langle z, F \rangle, \quad K = \min \left\{ \ell, \left\lfloor \frac{2m - 2r - j}{3} \right\rfloor, 4m - 2\ell - 2r - j \right\}.$$

Indeed, its normal monomials have exactly $0 \leqslant k \leqslant K$. For $a = 0, c = r > 0$ the same statement holds with C^r in place of A^r and

$$K = \min \left\{ \ell, \left\lfloor \frac{2m - r - j}{3} \right\rfloor, 4m - 2\ell - r - j \right\}.$$

Thus these are complete binary spaces, including at the ends of the ranges.

In a complete binary space of degree K, the sum of the orders of any basis at a fixed point of $\mathbb{P}^1$ is at most $0+1+\cdots+K$. The normal basis has mean $k=K/2$ in each block. Since $d=j+3k$ and $j\leqslant 2$, Lemma 113.3 gives mean $K/2=5/24$. The mean fixed exponent of F is consequently $11/16-5/24=23/48$. The other two fixed coefficients are the moments of a,c. The factor b^j has normalized coefficient tending to zero. □

Lemma 113.5. *For the refined series $W^{G}_{\bullet,\bullet}$ on G one has*

$$\delta\left(G,\frac{2}{3}\{z=0\};W^{G}_{\bullet,\bullet}\right)\geqslant\frac{64}{63}.$$

Proof. We explain the estimate for divisors obtained from torus-compatible bases. This suffices: fix one primary weighted-degree piece of the refined series. For any divisorial valuation choose a basis of this piece simultaneously compatible with its valuation filtration and the torus filtration. These are two filtrations on one vector space; the primary weighted degree has already been fixed. Repeat for each primary piece and take the torus limit. The initial bases remain bases of the displayed complete binary spaces. Lower semicontinuity of log canonical thresholds reduces the assertion to these limits.

Resolve the vertex of G and blow up the two smooth base points of the pencil $AC=\lambda z$. On this surface the strict transforms of all pencil members, the two coordinate lines, the different, and the exceptional curves have simple normal crossings. Distinct pencil members meet each exceptional curve over a base point at distinct points. The reducible member $AC=0$ has its two components separated by the resolution of the vertex.

The limiting expected orders and log discrepancies of the relevant prime divisors are as follows. The boundary is always $(2/3)\{z=0\}$.

divisor	expected order	log discrepancy
$A=0$	$5/16$	1
$C=0$	$5/8$	1
$z=0$	$5/24$	$1/3$
$F=0$	$11/16$	1
$AC=\lambda z,\quad \lambda\neq 0,1,\infty$	$\leqslant 5/24$	1
exceptional curve over $[0:1:0]$	1	$4/3$
exceptional curve over $[1:0:0]$	$21/16$	$4/3$
exceptional curve over $[0:0:1]$	$15/32$	1

The first five entries follow from Lemma 113.4; the member $F=0$ has its additional fixed coefficient $23/48$. At $[0:1:0]$ the orders of A,z,F are all one, while C is a unit. Hence the expected order is $5/48+(5/8)/3+11/16=1$. At $[1:0:0]$ it is $5/12+(5/8)/3+11/16=21/16$. The resolution of the $\frac{1}{2}(1,1)$ vertex has order $1/2$ on each of A,C and zero on z,F. The reducible pencil member contributes at most $5/24$, so its expected order is $(5/48+5/12)/2+5/24=15/32$.

Every ratio of the middle entry to the last is at most $63/64$. The simple normal crossings criterion therefore proves the claim. More precisely, for each real $t<64/63$, all the relevant coefficients satisfy the log canonical inequalities for all sufficiently large m. The preceding binary bound is uniform in the pencil member, so the same conclusion holds for every valuation-adapted basis. Passing to the asymptotic threshold and then letting t tend to $64/63$ gives the assertion. □

Corollary 113.6. *At the point under consideration,*

$$\delta_p(X)\geqslant\min\left\{\frac{8}{111/16},\frac{64}{63}\right\}=\frac{64}{63}>1.$$

Proof. The weighted blow-up is plt and $A_X(G) = 3 + 2 + 3 = 8$. Apply the Abban–Zhuang inequality using Lemmas 113.3 and 113.5. □

The calculation uses the full initial section algebra and a resolution of the exceptional surface. It does not use a Zariski decomposition on the weighted blow-up of the threefold.

114 Points on the original exceptional divisor

114.1 The point where a bad line meets the exceptional divisor

We give the point extraction needed at the intersection of a line on an A_2 fibre with the exceptional divisor of the original elliptic blow-up. The other ruling is obtained by interchanging the two planes.

Lemma 114.1. *Let S be an A_2 fibre, let ℓ be one of its two lines, and let $p = \ell \cap E$. There are regular parameters (a, t, z) on X at p with the following description. Put*

$$U = a^2 + abz, \qquad V = ab + b^2 z, \qquad H = \langle 1, a, b, z, az, bz, U, V \rangle.$$

The local equation of X is

$$b + U + tg = 0,$$

where $g \in H$, $g(p) = 0$, and the coefficient of z in g is nonzero. After rescaling t, we have

$$b = -Q + O((a, t, z)^3), \qquad Q = a^2 + \alpha at + tz$$

for some $\alpha \in \mathbb{C}$. A basis of the anticanonical sections is

$$1, a, z, az, bz, U, V, t, ta, tb, tz, taz, tbz, tU, tV.$$

Proof. On the incidence threefold use coordinates

$$x = (a, b, 1), \qquad y = (1, z, -a - bz).$$

Its bilinear sections form the space H. A regular nilpotent matrix may be chosen so that the second bilinear equation is $b + U = 0$. The line under consideration is $a = b = 0$; its singular endpoint is $z = \infty$. The unipotent centralizer acts transitively on the complement of that endpoint, so that the point under consideration may be put at $z = 0$. The other pencil generator restricts to a linear polynomial on the line. Its coefficient at z is nonzero: otherwise it would vanish at the singular endpoint, contrary to smoothness of the base curve. Its constant coefficient is zero because $p \in E$. The implicit function theorem applied to $b + U + tg$ gives the displayed quadratic term of b. The anticanonical sections are $H + tH$. Their unique linear relation is $b + U + tg = 0$, which eliminates b. □

Lemma 114.2. *Let M consist of the following fifteen lattice points, in the order shown:*

$$(0,0,0), (0,0,1), (0,1,0), (0,1,1), (0,1,2),$$
$$(0,2,1), (0,2,2), (1,0,0), (1,0,1), (1,1,0),$$
$$(1,1,1), (1,2,1), (2,0,0), (2,1,0), (3,0,0).$$

The semigroup generated by $(1, q)$, $q \in M$, has precisely

$$4m^3 + 6m^2 + 4m + 1$$

elements of degree m. Its degree-one convex body has volume 4 and barycentre

$$\left(\frac{41}{48}, \frac{5}{6}, \frac{35}{48}\right).$$

Proof. The convex hull is cut out by

$$a,t,z\ge 0,\quad t\le 2,\quad |t-z|\le 1,\quad a+t\le 3,$$
$$a-t+2z\le 3,\quad 2a+3z\le 6,\quad 2a+t+2z\le 6.$$

Here is an explicit unimodular triangulation, with the points indexed from 0 to 14. Every tetrahedron has vertex 10; its other three vertices are the following:

$$\begin{array}{cccccc}(0,1,2) & (0,1,7) & (0,2,7) & (1,2,3) & (1,3,4) & (1,4,8)\\ (1,7,8) & (2,3,5) & (2,5,11) & (2,7,9) & (2,9,11) & (3,4,6)\\ (3,5,6) & (4,6,14) & (4,8,14) & (5,6,11) & (6,11,14) & (7,8,12)\\ (7,9,13) & (7,12,13) & (8,12,14) & (9,11,13) & (11,13,14) & (12,13,14).\end{array}$$

The boundary triangles tile the boundary of the convex hull; taking cones from its interior point $(1,1,1)$ gives the claimed triangulation. Each determinant has absolute value one. In particular the polytope has the integer decomposition property. The numbers of vertices, edges, triangles, and tetrahedra are $15, 50, 60, 24$, respectively. Counting lattice points in the relative interiors of the dilated simplices gives, for $m\ge 1$,

$$15+50(m-1)+60\binom{m-1}{2}+24\binom{m-1}{3}=4m^3+6m^2+4m+1.$$

The tetrahedra have equal volume $1/6$. The barycentre is therefore the mean of their barycentres, which is the displayed vector. □

Lemma 114.3. *In the notation of Lemma 114.1, let $G\cong\mathbb{P}^2$ be the exceptional divisor of the ordinary blow-up of p. Then*

$$S_X(G)=\frac{29}{12},\qquad \delta_p(X)\ge\frac{48}{41}>1.$$

Proof. First take the lowest ordinary homogeneous parts in the parameters (a,t,z). Within each ordinary degree, refine by the lexicographic order on the exponents of a,t,z, taking the least exponent first. The fifteen sections in Lemma 114.1 can be adapted to these two filtrations. Their refined initial monomials are exactly those with exponent vectors in Lemma 114.2. Here are the only cancellations needed to see this. The leading terms of bz, tb, tbz, tV are respectively

$$-zQ,\quad -tQ,\quad -tzQ,\quad -atQ.$$

The sections U, tU have initial terms a^2, a^2t. Moreover $V+taz+\alpha tU$ has initial term $-a^3$. Subtracting the a^2t term from tb leaves a cubic whose least refined monomial is t^2z. The least refined monomials of the other three displayed expressions are tz^2, t^2z^2, at^2z. The remaining sections give $1, a, z, az, t, at, tz, atz, a^2, a^2t$. All fifteen initial monomials are distinct.

Products of these sections give all the semigroup monomials in every degree. Their number is, by Lemma 114.2, exactly

$$h^0(X,-mK_X)=4m^3+6m^2+4m+1.$$

Consequently they exhaust the actual refined section spaces in every degree. This is an equality of section spaces; a calculation of a convex hull alone would not suffice.

The ordinary order is the sum of the three exponents. Hence the barycentre calculation gives

$$S_X(G)=\frac{41}{48}+\frac{5}{6}+\frac{35}{48}=\frac{29}{12}.$$

We also explain the estimate on the exceptional plane. Fix any valuation on G. For each ordinary graded piece, choose a basis adapted both to that valuation and to the refining monomial filtration. For fixed m the latter filtration is induced by an integral one-parameter subgroup of the diagonal torus of G; its weights can be chosen to realize the required lexicographic order on all the monomials in question. The basis divisor degenerates to a divisor supported on the three coordinate lines. The asymptotic coefficients of those lines are

$$\frac{41}{48},\quad \frac{5}{6},\quad \frac{35}{48}.$$

The three lines form a simple normal crossing divisor on $\mathbb{P}^2$. Lower semicontinuity of log canonical thresholds therefore gives

$$\delta(G;W^{G}_{\bullet,\bullet})\geq \frac{48}{41}.$$

Abban–Zhuang adjunction and $A_X(G)=3$ now give

$$\delta_p(X)\geq \min\left\{\frac{3}{29/12},\frac{48}{41}\right\}=\frac{48}{41}.$$

We have used no Zariski decomposition on the extracted threefold. The complete section algebra and its two filtrations supply both $S_X(G)$ and the required restricted-series estimate. □

114.2 The exceptional points on the two lines of an A_1 fibre

Lemma 114.4. *Let Γ be one of the two lines through the singular point of an A_1 fibre, and put $p=\Gamma\cap E$. There are two cases. If p is not the intersection with another line on the fibre, its ordinary point extraction has the same refined section algebra as in Lemma 114.3. If p is such an intersection, there are regular parameters a,z,t in which its ordinary initial section space has a basis*

$$1,a,z,t,az,at,zt,\quad a^2(z+t),z^2t,azt,t^2(a+z),$$
$$taQ,az^2t,t^2z(a+z),\quad t^2(a+z)Q,\qquad Q=az+t(a+z).$$

Proof. It suffices to consider one projection; transposition interchanges the two lines. Write the flag and fibre equations as

$$x_0y_0+x_1y_1+x_2y_2=0,\qquad x_0y_1+x_2y_2=0.$$

The singular point is $([0:1:0],[1:0:0])$, and the line considered is $x=[0:1:0]$, $y_1=0$. Its point on the base curve has coordinates $y=[\rho:0:1]$ for some $\rho\in\mathbb{C}$. On $x_1=y_2=1$ use regular coordinates

$$a=x_0,\qquad z=y_0-\rho,\qquad h=y_1+\rho a.$$

Then $x_2=-az-h$. Put

$$R=az+h,\quad K=zR,\quad L=R(\rho a-h),\quad f=ah-\rho a^2-R.$$

The eight bilinear sections span

$$\langle 1,a,z,h,az,K,L,f\rangle.$$

The second pencil generator g vanishes at p and has linear term $z+\gamma a+\beta h$ after rescaling. Its z-coefficient is nonzero because its restriction to Γ has a simple zero at p. The blow-up chart is $f=tg$, and a,z,t are regular parameters.

Suppose first that $\rho \neq 0$. We may replace z by the regular coordinate $c = g$; its linear term on X is $z + \gamma a$. The quadratic terms of R, h are

$$R_2 = -\rho a^2 - tc, \qquad h_2 = (\gamma - \rho)a^2 - (a + t)c.$$

Use $R + f = ah - \rho a^2$ in place of h. The quadratic initial forms of $az, R + f, f$ span ac, a^2, tc. Moreover

$$K_3 = -(c - \gamma a)(\rho a^2 + tc), \qquad L_3 = -\rho a(\rho a^2 + tc).$$

Thus the eight bilinear initial forms can be chosen as

$$1, a, c, a^2, ac, aQ, cQ, tc, \qquad Q = \rho a^2 + tc.$$

Removing f from $H + tH$ gives seven of these forms and t times all eight. After rescaling a the resulting algebra is the one used in Lemma 114.3, with the coordinates renamed. In comparing the two displayed bases, use $tQ = \rho a^2 t + t^2 c$ to interchange tQ and $t^2 c$.

Suppose now that $\rho = 0$. The fibre contains the second line $z = h = 0$ through p. We must have $\gamma \neq 0$: otherwise g vanishes identically on this line, so the smooth elliptic base curve would contain it. A diagonal rescaling permits $\gamma = 1$ in the quadratic computation. The equation $f = tg$ gives

$$h = -Q + O((a, z, t)^3), \qquad Q = az + t(a + z).$$

Since $f = ah - R$, we can use

$$H = \langle 1, a, z, az, ah, K, L, f \rangle.$$

Its last four initial forms are, up to signs, $aQ, tz(a + z), t(a + z)Q, t(a + z)$. Take the seven sections other than f and t times all eight. Their cubic terms are row-reduced using

$$aQ - azt = a^2(z + t), \qquad tz(a + z) - azt = z^2 t.$$

Their quartic terms are row-reduced using

$$t(a + z)Q - taQ - t^2 z(a + z) = az^2 t.$$

These are precisely the fifteen homogeneous forms stated above. No coefficient of the terms of g beyond its linear part enters this calculation. □

Lemma 114.5. *Let M be the following fifteen vectors in coordinates (a, z, t), indexed from 0 to 14 in the indicated order:*

$$(0,0,0), (0,0,1), (0,1,0), (0,1,1), (0,2,1),$$
$$(1,0,0), (1,0,1), (1,0,2), (1,1,0), (1,1,1),$$
$$(1,1,2), (1,2,1), (2,1,0), (2,1,1), (2,1,2).$$

The semigroup generated by $(1, q)$, $q \in M$, has $4m^3 + 6m^2 + 4m + 1$ elements of degree m. Its convex body has volume four and barycentre

$$\left(\frac{85}{96}, \frac{27}{32}, \frac{43}{48}\right).$$

Proof. The convex hull is given by

$$a, z, t \geq 0, \quad t \leq 2, \quad a + z \leq 3, \quad a - z \leq 1, \quad z - t \leq 1, \quad t - a \leq 1, \quad z + t \leq 3.$$

The point indexed 9 is interior. Cone from this point over the following twenty-four boundary triangles:

$$\begin{array}{cccc} (0,1,2) & (0,1,6) & (0,2,8) & (0,5,6) \\ (0,5,8) & (1,2,3) & (1,3,10) & (1,6,7) \\ (1,7,10) & (2,3,4) & (2,4,8) & (3,4,10) \\ (4,8,11) & (4,10,11) & (5,6,12) & (5,8,12) \\ (6,7,13) & (6,12,13) & (7,10,14) & (7,13,14) \\ (8,11,12) & (10,11,14) & (11,12,13) & (11,13,14). \end{array}$$

On each facet the listed triangles form its triangulation. Every cone has determinant of absolute value one. This is therefore a unimodular triangulation, with face numbers $15, 50, 60, 24$. As in Lemma 114.2, relative-interior counting and the integer decomposition property give

$$15 + 50(m-1) + 60\binom{m-1}{2} + 24\binom{m-1}{3} = 4m^3 + 6m^2 + 4m + 1.$$

Averaging the barycentres of the twenty-four equal-volume tetrahedra gives the asserted vector. □

Proposition 114.6. *At the point p of Lemma 114.4,*

$$\delta_p(X) \geq \frac{48}{43} > 1.$$

If $\rho \neq 0$, the stronger bound $48/41$ holds.

Proof. The first case has already been identified with the algebra of Lemma 114.3. In the second case, refine the ordinary homogeneous initial forms by taking the least exponent of t, then the least exponent of z. Their initial monomials are

$$1, a, z, t, az, at, zt, a^2z, z^2t, azt, at^2, a^2zt, az^2t, azt^2, a^2zt^2.$$

These are exactly the points of Lemma 114.5. The fifteen distinct initial monomials and their products therefore exhaust the full associated graded section space in every degree, by its exact Hilbert count. The barycentre gives, for the ordinary exceptional plane G,

$$S_X(G) = \frac{85}{96} + \frac{27}{32} + \frac{43}{48} = \frac{21}{8}.$$

For every fixed m the refinement on its primary pieces is induced by an integral diagonal one-parameter subgroup: use weights $(0, 1, N)$ in (a, z, t) with N larger than all exponents occurring in degree m. Choose bases compatible with this filtration and any prescribed valuation filtration, and degenerate to their initial monomials. The limiting basis divisor is supported on the three coordinate lines of G, whose limiting coefficients are the three barycentre coordinates above. They form a simple normal crossing divisor. The largest coefficient is $43/48$. The same two-filtration and semicontinuity argument as in Lemma 114.3 proves

$$\delta(G; W^G) \geq 48/43.$$

Finally Abban–Zhuang adjunction gives

$$\delta_p(X) \geq \min\{3/(21/8), 48/43\} = 48/43.$$

Again no Zariski decomposition on the point blow-up is required. □

115 The smooth corner of an A_1 fibre

Proposition 115.1. *Suppose that $p \notin E$ is the intersection of a line through the singular point of an A_1 fibre with a line contracted by the opposite plane projection. Then*

$$\delta_p(X) \geqslant \frac{96}{95} > 1.$$

Lemma 115.2. *There are regular parameters u, a, b at p with the following property. Blow up p with weights $(1, 2, 1)$, and refine the weighted order by a one-parameter subgroup with weights $(h, 1, 0)$, where $h > 1$. The successive initial anticanonical forms include the fifteen monomials*

$$\begin{aligned} &1, u, b, a, ub, \\ &ua, ab, a^2, u^2b, uab, \\ &ab^2, a^2b, ua^2, uab^2, a^3. \end{aligned} \tag{37.C1}$$

The coefficient of every displayed monomial is nonzero for every smooth member of the family.

Proof. Take the flag equation and the fibre equation to be

$$x_0y_0 + x_1y_1 + x_2y_2 = 0, \qquad x_0y_1 + x_2y_2 = 0.$$

The point under consideration is $([0 : 1 : 0], [0 : 0 : 1])$. On the chart $x_1 = y_2 = 1$ put $u = x_0$, $a = x_2$, $b = y_0$. Then $y_1 = -a - ub$, and the fibre equation is

$$f = a(1 - u) - u^2b.$$

The bad line has $u = a = 0$, and its singular endpoint is $b = \infty$. Put $U = u(a + ub)$ and $V = a(a + ub)$. A basis of restricted bilinear forms is

$$\mathcal{H} = \langle 1, u, a, b, ub, ab, U, V \rangle.$$

For example, $U = -x_0y_1$ and $V = -x_2y_1$.

The other pencil generator g is a unit at p. After rescaling,

$$g = 1 + \alpha u + \beta b + \text{terms of weighted degree at least 2.}$$

Moreover $\beta \neq 0$: the restriction of g to the bad line is $1 + \beta b$, and its value at the singular endpoint is nonzero, since the base curve misses the singular points of the pencil. Multiplying the local anticanonical trivialization by the unit g, the section space becomes $g\mathcal{H} + f\mathcal{H}$.

In the order of (37.C1), choose the following fifteen sections:

$$\begin{aligned} &g, gu, gb, f, gub, \\ &fu, fb, fa, R, fub, \\ &S, fab, fU, T, fV, \end{aligned}$$

where

$$\begin{aligned} R &= gU - fu, \\ S &= gab - fb - (\alpha + 1)fub, \\ T &= gV - fa - fub - (\alpha + 1)fU - \beta fab. \end{aligned}$$

The weighted initial forms needed for the nontrivial reductions are:

$$\begin{aligned} \operatorname{in}(R) &= u^2b, \\ \operatorname{in}(S) &= \beta ab^2 + u^2b^2, \\ \operatorname{in}(fU) &= ua^2 + u^2ab, \\ \operatorname{in}(T) &= \beta uab^2 + u^2ab + u^3b^2, \\ \operatorname{in}(fV) &= a^3 + uba^2. \end{aligned}$$

Taking least weights for $(h,1,0)$ gives (37.C1), since $h>1$ and $\beta\neq 0$. Terms of higher weighted degree in g cannot affect these identities. This proves the assertion for arbitrary g, including all special values of its remaining coefficients. □

Lemma 115.3. *The monomials in (37.C1) generate the entire successive initial section algebra in every anticanonical degree. Their normalized limiting exponent sums in the order (u,a,b) are*

$$\frac{61}{96},\qquad \frac{95}{96},\qquad \frac{3}{4}.$$

Proof. Here is an integer certificate which proves the assertion in every degree. Number the exponent vectors in (37.C1) from v_0 to v_{14} in the displayed order. The following twenty-four quadruples specify unimodular tetrahedra:

$$\begin{array}{ccc}
(0,1,4,5) & (1,4,5,8) & (0,2,3,4)\\
(0,3,4,5) & (7,11,12,14) & (11,12,13,14)\\
(3,7,11,12) & (2,6,10,13) & (6,10,11,13)\\
(2,4,6,13) & (2,3,4,6) & (3,4,5,6)\\
(3,5,6,12) & (3,6,11,12) & (4,5,6,9)\\
(4,6,9,13) & (4,8,9,13) & (4,5,8,9)\\
(5,6,9,12) & (5,8,9,12) & (8,9,12,13)\\
(9,11,12,13) & (6,9,11,12) & (6,9,11,13).
\end{array}\tag{37.C2}$$

They form a simplicial complex. This can be checked without any geometric guess: lift the fifteen vectors to the respective heights

$$\begin{aligned}(h_0,\dots,h_7)&=(0,123,77,99,173,216,177,424),\\(h_8,\dots,h_{14})&=(483,308,516,495,521,582,906).\end{aligned}$$

For each quadruple, the affine function through its four lifted vertices is strictly below all eleven other lifted vertices. Consequently the tetrahedra are lower facets of one convex hull and meet along common faces. Each determinant $\det(v_j-v_i,v_k-v_i,v_l-v_i)$ has absolute value one. Their numbers of faces in dimensions $0,1,2,3$ are $15,50,60,24$.

Let Q be their union. Every lattice point in mQ is a sum of m vertices of one tetrahedron, by unimodularity. It therefore gives a product of m of the monomials (37.C1). Counting the relative interiors of all faces gives, for every $m\geqslant 1$,

$$\#(mQ\cap\mathbb{Z}^3)=15+50\binom{m-1}{1}+60\binom{m-1}{2}+24\binom{m-1}{3}=4m^3+6m^2+4m+1.$$

This is exactly $h^0(X,-mK_X)$. The inclusion supplied by Lemma 115.2 is therefore an equality in every degree. In particular, this argument does not infer generation of an initial algebra from computations in finitely many degrees.

Each tetrahedron has volume $1/6$. The sums of its four vertex coordinates, added over all tetrahedra in (37.C2), are $61,95,72$. It follows that

$$\operatorname{vol}(Q)=4,\qquad \int_Q(u,a,b)=\left(\frac{61}{24},\frac{95}{24},3\right).$$

Division by 4 gives the claimed moments. □

Proof of Proposition 115.1. The exceptional surface is $G=\mathbb{P}(1,2,1)$, the different is zero, and $A_X(G)=4$. By the preceding lemma,

$$S_X(G)=\frac{61}{96}+2\cdot\frac{95}{96}+\frac{3}{4}=\frac{323}{96}.$$

We justify the bound on the refined series for arbitrary valuations. Fix a primary weighted-degree piece and a divisorial valuation over G. Choose h sufficiently large that the weights $(h, 1, 0)$ separate all monomials in this piece. Choose a basis simultaneously compatible with the valuation filtration and this weight filtration. Only two filtrations on the already fixed primary piece are used. Lemma 115.3 identifies its limit with the corresponding monomial basis. Lower semicontinuity of log canonical thresholds applies to this one-parameter degeneration.

The limiting normalized basis divisor is supported on the three toric coordinate curves. Its coefficients tend to $61/96, 95/96, 3/4$. The toric boundary of $\mathbb{P}(1, 2, 1)$ is log canonical, including at its $\frac{1}{2}(1, 1)$ point. Hence the thresholds of these normalized basis divisors have lower limit at least $96/95$. The same bound holds for every valuation-adapted basis by the preceding degeneration. Thus

$$\delta(G; W^{G}_{\bullet,\bullet}) \geqslant \frac{96}{95}.$$

Abban–Zhuang adjunction now gives

$$\delta_p(X) \geqslant \min\left\{\frac{384}{323}, \frac{96}{95}\right\} = \frac{96}{95}.$$

The proof uses a weighted point extraction, actual section spaces, and an integer certificate for their refined initial algebra. It requires no Zariski decomposition on the extracted threefold. □

116 Proof for every smooth member

Proof of Theorem 108.1. Let Q be a prime divisor over X. Suppose that $\beta_X(Q) \leq 0$. Proposition 110.6 puts its centre on one of the two lines through the singular point of an A_1 or A_2 fibre. Denote this line by Γ and choose a closed point p of the centre.

If p is the singular point of the fibre, Proposition 112.1 and Corollary 113.6 give $\delta_p(X) > 1$, a contradiction. If $p \in E$, use Lemma 114.3 or Proposition 114.6. Their bounds are again strictly greater than one.

Assume that p is outside these two points. Interchange the plane projections so that the first contracts Γ. If $p \notin D_2$, Lemma 111.10 supplies a smooth quintic del Pezzo section through p on which p lies on none of the ten lines. Lemmas 111.5, 111.7 and 111.8, with the two projections interchanged, give $\delta_p(X) \geq 1152/1127 > 1$. If $p \in D_2$, Lemma 111.10 says that the fibre has type A_1 and p is its remaining smooth corner. Proposition 115.1 gives $\delta_p(X) > 1$ there. This is the last possibility.

In every case $A_X(Q) > S_X(Q)$, contrary to our assumption. Thus every prime divisor over X has positive beta invariant. The valuative criterion proves K-stability. □

Example 116.1. The pencil in Example 109.5 has three A_2 fibres. At each singular point the weighted extraction gives $64/63$. Each of the two lines has one point on E; its ordinary point extraction gives $48/41$. Every other point on these lines is treated by an opposite quintic section. Thus even this pencil, whose six ramification contributions are all concentrated at three points, is covered by the proof.

The extraction arguments do not infer a refined series from a volume alone. In each case, actual anticanonical sections supply independent initial forms, and the exact Hilbert count proves that they exhaust the associated graded space. The barycentres or binary-space sums then compute $S_X(G)$ with its original normalization. This is why the Abban–Zhuang inequalities apply without a Zariski decomposition on the extracted threefold.

Part XVI
Family 3.8

117 The two blow-ups and the quartic pencil

We work over $\mathbb{C}$. Let $U = \langle f, g\rangle$ be a two-dimensional subspace of $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4))$ without base points. Here $f = f(x, y)$ and $g = g(x, y)$ are binary quartics. Put

$$C_U = \{uf(x, y) + vg(x, y) = 0\} \subset \mathbb{P}^1_{u,v} \times \mathbb{P}^1_{x,y}.$$

The second projection identifies C_U with $\mathbb{P}^1$, and its first projection is the morphism

$$\eta_U \colon \mathbb{P}^1 \longrightarrow \mathbb{P}^1, \qquad [x : y] \longmapsto [-g(x, y) : f(x, y)].$$

It has degree four. We identify the second $\mathbb{P}^1$ with the conic $Q = \{XZ - Y^2 = 0\} \subset \mathbb{P}^2$ by $[x : y] \mapsto [x^2 : xy : y^2]$, and regard C_U as a curve in $Z = \mathbb{P}^1 \times \mathbb{P}^2$. Let

$$\pi \colon X_U = \mathrm{Bl}_{C_U}(Z) \longrightarrow Z.$$

These are precisely the smooth members of Family 3.8 [3, Sections 5.16 and 7.6]. We give the geometry that we use below.

Lemma 117.1. *The threefold X_U is smooth, Fano, and $(-K_{X_U})^3 = 24$. It also admits a morphism $\sigma \colon X_U \to Y_U$ which is the blow-up of a smooth conic fibre of a smooth divisor $Y_U \subset \mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(1, 2)$. The exceptional divisors of σ and π are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{F}_4$, respectively.*

Proof. Choose quadrics $F(X, Y, Z)$ and $G(X, Y, Z)$ restricting to f and g on Q, and write $q = XZ - Y^2$. Then C_U is the complete intersection

$$q = uF + vG = 0.$$

Since C_U is the graph of η_U, it is smooth. Thus X_U is smooth. The quadrics q, F, G have no common zero: a common zero would belong to Q and be a common zero of f and g.

In $\mathbb{P}^2_{a,b,c} \times \mathbb{P}^2_{X,Y,Z}$, put

$$Y_U = \{aq + bF + cG = 0\}, \qquad O = [1 : 0 : 0].$$

The second projection of Y_U is the projectivization of a rank-two vector bundle, since q, F, G have no common zero. In particular, Y_U is smooth. The fibre of its first projection over O is Q. Let $\mathbb{F}_1 = \mathrm{Bl}_O(\mathbb{P}^2)$, with ruling $\mathbb{F}_1 \to \mathbb{P}^1_{u,v}$ given by $[b : c]$. The proper transform of Y_U in $\mathbb{F}_1 \times \mathbb{P}^2$ is X_U. Indeed, writing $b = \lambda u$ and $c = \lambda v$, its equation is

$$aq + \lambda(uF + vG) = 0.$$

Over Z this is the usual incidence equation for the blow-up of the ideal $(q, uF + vG)$. Over Y_U it is the blow-up of its smooth fibre over O. Its exceptional divisor E is the strict transform of $\mathbb{P}^1 \times Q$, hence $E \cong \mathbb{P}^1 \times \mathbb{P}^1$.

Let H_1 be the pullback of a line from $\mathbb{P}^2_{a,b,c}$, let H_2 be the pullback of a line from $\mathbb{P}^2_{X,Y,Z}$, and let H_3 be the pullback of a point from $\mathbb{P}^1_{u,v}$. Adjunction in $\mathbb{F}_1 \times \mathbb{P}^2$ gives

$$-K_{X_U} = H_1 + H_2 + H_3.$$

The divisor $H_1 + H_3$ is ample on $\mathbb{F}_1$, and H_2 is ample on the second factor, so their sum is ample. On $\mathbb{F}_1$ we have $H_1^2 = H_1 H_3 = 1$ and $H_3^2 = 0$. Since $[X_U] = H_1 + 2H_2$, we obtain

$$(H_1 + H_2 + H_3)^3 (H_1 + 2H_2) = 6(H_1 + H_3)^2 + 3H_1(H_1 + H_3) = 24.$$

Finally, the two equations defining C_U have bidegrees $(0,2)$ and $(1,2)$. Their restrictions to $C_U \cong \mathbb{P}^1$ have degrees 4 and 8. Thus

$$N_{C_U/Z} \cong \mathcal{O}_{\mathbb{P}^1}(4) \oplus \mathcal{O}_{\mathbb{P}^1}(8),$$

so the exceptional divisor E' of π is $\mathbb{F}_4$. □

Lemma 117.2. *Two threefolds X_U and $X_{U'}$ are isomorphic if and only if the pencils U and U' belong to the same* PGL_2*-orbit.*

Proof. The backward implication follows by extending an automorphism of Q to $\mathbb{P}^2$ and changing the basis of the pencil. For the converse, $\rho(X_U) = 3$, and the nef cone is generated by H_1, H_2, H_3. To see the latter assertion, these divisors are nef, and the fibres of π, the fibres of σ, and the fibres of $X_U \to \mathbb{F}_1$ have intersection vectors proportional to $(1,0,0)$, $(0,0,1)$, and $(0,1,0)$, respectively. Consequently these are the three extremal rays of the cone of curves. The first two contractions are divisorial and the third is a conic bundle. Their exceptional divisors $\mathbb{F}_4$ and $\mathbb{P}^1 \times \mathbb{P}^1$ are not isomorphic. An isomorphism therefore preserves the contraction π and descends to an isomorphism of the pairs (Z, C_U) and $(Z, C_{U'})$. An automorphism of Z preserves its two projections and must preserve the conic which is the image of the centre in $\mathbb{P}^2$. Its action on this conic belongs to PGL_2; its action on the first factor merely changes the basis of the pencil. □

Lemma 117.3. *Let $\theta\colon X_U \to \mathbb{P}^1$ be induced by the first projection of Z. If the fibre of η_U over t has multiplicities $e_1, \dots, e_r$, then the singularities of $\theta^{-1}(t)$ are exactly one point of type $A_{e_i - 1}$ for each $e_i \geq 2$. They are disjoint from E. In particular, the partitions $2+1+1$, $2+2$, $3+1$, and 4 give A_1, $2A_1$, A_2, and A_3, respectively.*

Proof. Near a point of C_U, choose a coordinate z on Q, a normal coordinate w to Q, and a coordinate t on the first factor. The centre has ideal $(w, t - h(z))$. At a ramification point of index e, analytic coordinates give $h(z) = z^e$. In one blow-up chart the equation is $t = z^e + \lambda w$. The fibre $t = 0$ thus has equation $\lambda w + z^e = 0$. It is smooth for $e = 1$ and has type A_{e-1} otherwise. The other chart has coordinates (t, z, μ) and equation $w = \mu(t - z^e)$, so its fibres are smooth. The strict transform of $\mathbb{P}^1 \times Q$ lies in the latter chart and is given by $\mu = 0$. Thus it contains none of these singular points. □

118 Pencils, ramification, and geometric invariant theory

Write e_p for the ramification index of η_U at p, and put

$$R_U = \sum_{p \in \mathbb{P}^1} (e_p - 1)p.$$

By Riemann–Hurwitz, $\deg R_U = 6$. The Wronskian $W_U = f_x g_y - f_y g_x$ is a binary sextic defined up to a nonzero scalar. Its zero divisor is R_U: locally this follows by differentiating g/f, after choosing $f(p) \neq 0$.

We use the Plücker linearization on $\mathrm{Gr}(2, H^0(\mathbb{P}^1, \mathcal{O}(4)))$. The following statement concerns GIT. A separate argument is needed to obtain the corresponding assertion for the threefolds.

Proposition 118.1. *Every basepoint-free quartic pencil is GIT semistable. Such a pencil is GIT stable if and only if $e_p \le 3$ for every p. There is one GIT-polystable orbit which is not stable, represented by $U_0 = \langle x^4, y^4 \rangle$. Every other nonstable pencil is strictly GIT semistable and has a unique point of ramification index four.*

Proof. Fix $p = [1 : 0]$. Row reduction gives a basis of U with vanishing orders $(0, e_p)$ at p. For the one-parameter subgroup

$$\lambda(t)(x, y) = (t^{-1}x, ty),$$

the weight of $x^{4-i}y^i$ is $-4+2i$. Hence the least nonzero weight of a Plücker vector representing U is $-8 + 2e_p$. Every nontrivial one-parameter subgroup is conjugate to a positive multiple of one of these. In this convention Hilbert–Mumford semistability means that each least weight is nonpositive, and stability means that it is negative. Since $1 \le e_p \le 4$, this proves the first two assertions.

If $e_p = 4$, the pencil contains y^4, and a second generator has nonzero x^4 coefficient. The limit under λ is $\langle x^4, y^4 \rangle$. The orbit of this pencil consists of the spans of the fourth powers of two independent linear forms. Its boundary in the Grassmannian consists of tangent pencils, equivalent to $\langle x^4, x^3y \rangle$. This also follows by extending the secant map from unordered distinct pairs of points on the rational normal quartic to its diagonal. Such a tangent pencil is unstable: at $[0 : 1]$ its two vanishing orders are 3 and 4, giving least weight $-8 + 2(3 + 4) > 0$. Therefore the orbit of U_0 is closed in the semistable locus, and every nonstable polystable pencil is equivalent to it.

Finally, two points of index four contribute all six to $\deg R_U$. Their images are distinct, and coordinates on source and target then identify the map with $z \mapsto z^4$. Thus their pencil is U_0. □

Lemma 118.2. *The strictly GIT-semistable pencils have normal forms*

$$U_{a,b} = \langle x^4, y^4 + ax^2y^2 + bx^3y \rangle, \qquad (a, b) \ne (0, 0).$$

Two such pencils are equivalent if and only if $(a', b') = (\lambda^2 a, \lambda^3 b)$ for some $\lambda \ne 0$. Their isomorphism classes are therefore parametrized by $\mathbb{P}(2, 3)$.

Proof. Send the unique point of index four and its image to infinity. The resulting map is a polynomial of degree four. Multiplication by a constant, translation on the source, and translation on the target put it in the form $z^4 + az^2 + bz$, giving the displayed pencil. If this polynomial had another point of index four, its derivative would equal $4(z-r)^3$. Comparing the coefficient of z^2 gives $r = 0$, and then $a = b = 0$, which is excluded. Any equivalence between two normal forms must preserve the unique totally ramified point and its image. The source and target changes are therefore affine. The missing cubic term forces the source translation to vanish; comparison of leading and constant terms then leaves exactly the indicated weighted scaling. □

Lemma 118.3. *The pair $(\mathbb{P}^1, \frac{1}{4}R_U)$ is log Fano and*

$$\delta\left(\mathbb{P}^1, \frac{1}{4}R_U\right) = 4 - \max_{p \in \mathbb{P}^1}(e_p - 1).$$

In particular, its delta invariant is greater than one precisely when U is GIT stable.

Proof. Put $L = -(K_{\mathbb{P}^1} + \frac{1}{4}R_U)$; it has degree $1/2$. All coefficients of $\frac{1}{4}R_U$ are at most $3/4$, so the pair is klt. Every divisorial valuation of the function field of a smooth curve is proportional to ord_p for a point p. Writing $n_p = e_p - 1$, its log discrepancy is $1 - n_p/4$, while

$$S_L(\mathrm{ord}_p) = \frac{1}{\deg L}\int_0^{\deg L}(\deg L - t)\,dt = \frac{1}{4}.$$

Taking the infimum of the ratio of these two quantities proves the formula. The valuative criterion also shows that all these log curves are K-semistable. The usual criterion for weighted points on $\mathbb{P}^1$ says that equality gives a K-polystable pair only when $R_U = 3p + 3q$, with $p \neq q$; this is precisely the pencil U_0. This is the log-curve formulation in [3, Section 7.6]. □

Example 118.4. The pencil $\langle x^4, y^4 \rangle$ has $R_U = 3[1:0] + 3[0:1]$ and log-curve delta invariant one. For $\langle x^4, y^4 + x^2y^2 \rangle$, the map $z \mapsto z^4 + z^2$ has one point of index four at infinity and three simple ramification points. Its log-curve delta invariant is also one, but its pencil is not polystable.

Lemma 118.5. *Let T be the open subset of $\mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}(4)) \otimes \mathbb{C}^2)$ parametrizing the smooth graph curves above, and let h be its hyperplane class. For the universal family $\varphi\colon \mathcal{X} \to T$, use the normalization $\lambda_{\mathrm{CM}} = -\varphi_*((-K_{\mathcal{X}/T})^4)$. Then $\lambda_{\mathrm{CM}} = 32h$. After quotienting by the changes of basis of the pencil, this is sixteen times the Plücker class.*

Proof. On $\mathbb{P}^1 \times \mathbb{P}^1 \times T$, write A, B for the classes of the first and second factors. The universal centre has class $D = A + 4B + h$. Its normal bundle in $\mathbb{P}^1 \times \mathbb{P}^2 \times T$ has Chern classes

$$c_1 = A + 8B + h, \qquad c_2 = 4B(A + 4B + h).$$

The ambient anticanonical divisor restricts to $L = 2A + 6B$ on $\mathbb{P}^1 \times \mathbb{P}^1 \times T$. The pushforward formulas for a codimension-two blow-up give

$$\varphi_*((-K_{\mathcal{X}/T})^4) = \mathrm{pr}_{T*}(D(-6L^2 + 4Lc_1 - c_1^2 + c_2)).$$

Indeed, the pushforwards of the square, cube, and fourth power of the exceptional divisor are $-[\mathcal{C}]$, $-i_*c_1$, and $-i_*(c_1^2 - c_2)$, respectively. Since $A^2 = B^2 = 0$, only the coefficient of ABh contributes. Its values in DL^2, DLc_1, and $D(c_1^2 - c_2)$ are 24, 36, and 32. The required coefficient is therefore $-6\cdot 24 + 4\cdot 36 - 32 = -32$, proving the assertion. The Plücker coordinates are the 2×2 minors of the two coefficient vectors, hence have degree two. Its hyperplane class thus pulls back to $2h$, giving the last assertion. □

Remark 118.6. The CM calculation identifies the relevant GIT polarization. It does not show that every special test configuration of X_U comes from a degeneration of its quartic pencil. Accordingly, the threefold K-stability assertion still requires the local proof below.

119 The total ramification cases

Proposition 119.1. *Suppose that η_U has a point of ramification index four. Then X_U is K-polystable if and only if U is equivalent to $\langle x^4, y^4 \rangle$. Every other such threefold is strictly K-semistable. Thus these strictly K-semistable threefolds are parametrized, up to isomorphism, by $\mathbb{P}(2, 3)$.*

Proof. The threefold X_{U_0} is K-polystable by [3, Proposition 5.74 and Remark 5.75]. For completeness, we give the degeneration proving the other assertion, which is [3, Lemma 7.7] in different coordinates. By Lemma 118.2, we may write the centre as

$$ux^4 + v(y^4 + ax^2y^2 + bx^3y) = 0, \qquad (a, b) \neq (0, 0).$$

Over $\mathbb{A}^1_t$, replace its equation by

$$ux^4 + v(y^4 + t^2ax^2y^2 + t^3bx^3y) = 0.$$

These centres form a smooth family of graphs. Blowing them up gives a smooth family of Fano threefolds, and hence its relative anticanonical divisor is relatively ample. The action

$$([u:v],[x:y],t) \longmapsto ([s^4u:v],[s^{-1}x:y],st)$$

extends to the ambient product via the conic embedding and preserves this family. It provides its one-parameter equivariance, and all nonzero fibres are isomorphic to $X_{U_{a,b}}$. The central fibre is X_{U_0}, so this is a special test configuration after the anticanonical polarization is chosen.

On the central fibre the involution interchanging u, v and x, y conjugates the induced one-parameter subgroup to its inverse. Its Futaki invariant is therefore zero. The Donaldson–Futaki invariant of our test configuration is zero as well [3, Lemma 1.29]. It is not a product test configuration: the central pencil has two points of index four, while the general pencil has only one, so Lemma 117.2 shows that its central and general fibres are not isomorphic. Consequently the general fibre is not K-polystable. It is K-semistable by openness of K-semistability at the central fibre [3, Theorem 1.11], since all nonzero fibres are isomorphic. The last assertion follows from Lemmas 117.2 and 118.2. □

120 Anticanonical sections and the distinguished surface

For local calculations write $X = X_U$ and $C = C_U$. Let $h = H_1$, $f_1 = H_3$, and $H = H_2$. The ambient realization of Lemma 117.1 has equation

$$s_2 q = s_1 \Phi, \qquad \Phi = uF + vG,$$

in $\mathbb{F}_1 \times \mathbb{P}^2$. The homogeneous coordinates u, v, s_1, s_2 of $\mathbb{F}_1$ have classes $f_1, f_1, h - f_1, h$, respectively. The three plane coordinates x_0, x_1, x_2 have class H. The irrelevant loci on $\mathbb{F}_1$ are $u = v = 0$ and $s_1 = s_2 = 0$. The distinguished surface E is $(s_1 = 0)$, of class $h - f_1$.

Let E' be the exceptional divisor of π and let e be its defining section. The global sections $s_1 = q/e$ and $s_2 = \Phi/e$ satisfy

$$es_1 = q, \qquad es_2 = \Phi,$$

and their divisor classes are

$$[e] = 2H - h + f_1, \qquad [s_1] = h - f_1, \qquad [s_2] = h.$$

These identities are identities of actual global sections. They will suffice for the inclusions used in the refined series.

Lemma 120.1. *For every $m \geq 1$, ambient sections restrict surjectively onto $H^0(X_U, -mK_{X_U})$. More explicitly, this space is spanned by the restrictions of*

$$s_1^{m-r} s_2^r u^i v^{2m-r-i} x_0^a x_1^b x_2^c, \quad 0 \leq r \leq m, \quad 0 \leq i \leq 2m - r, \quad a + b + c = m.$$

Its dimension is $(2m + 1)(2m^2 + 2m + 1)$.

Proof. The anticanonical class is $h + f_1 + H$, while the hypersurface has class $h + 2H$. Its restriction sequence has kernel

$$\mathcal{O}_{\mathbb{F}_1 \times \mathbb{P}^2}((m-1)h + mf_1 + (m-2)H).$$

For $m = 1$, the factor $\mathcal{O}_{\mathbb{P}^2}(-1)$ has no cohomology. For $m \geq 2$, pushing the first factor down to $\mathbb{P}^1$ gives line bundles of degrees $2m - 1, 2m - 2, \ldots, m$, all nonnegative. Künneth's formula therefore gives zero first cohomology in either case, proving surjectivity. Comparing the two degrees on $\mathbb{F}_1$ gives precisely the displayed monomials.

The dimensions of the ambient space and the kernel are

$$\frac{(m+1)^2(3m+2)(m+2)}{4}, \qquad \frac{m^2(3m+1)(m-1)}{4},$$

respectively. Their difference is the asserted dimension. This proof uses the homogeneous coordinates of the ambient product and its hypersurface equation; it requires no assertion about generators of the whole Cox ring of X_U. □

120.1 A flag on E

The surface $E \cong \mathbb{P}^1 \times \mathbb{P}^1$ has two ruling classes α, β, where $f_1|_E = \alpha$ and $H|_E = 2\beta$. Thus $h|_E = 0$, $E|_E = -\alpha$, and $E'|_E \sim \alpha + 4\beta$.

We also need the following calculation for a particular singular central fibre. It remains valid when the graph curve in E degenerates to a reduced curve with one vertical component, provided that the flag ruling does not meet that component.

Lemma 120.2. *Let X be a smooth member, or the nodal central member obtained from the net*

$$\langle Y_0Y_2 - Y_1^2,\ Y_1Y_2,\ Y_0^2\rangle.$$

Let $p \in E$ and let $C \in |\beta|$ be the ruling through p. In the nodal case, suppose that C is disjoint from the vertical component of $E'|_E$. Then

$$S_X(E) = \frac{17}{24}, \qquad S(W^E; C) = \frac{11}{12}, \qquad S(W^{E,C}; p) \le \frac{7}{8}.$$

Consequently $\delta_p(X) \ge 12/11$.

Proof. For $0 \le u \le 3/2$, the decomposition of $-K_X - uE$ is

u	$P(u)$	$N(u)$
$[0,1]$	$(1-u)h + H + (1+u)f_1$	0
$[1,3/2]$	$(3-2u)H + 2f_1$	$(u-1)E'$.

The positive parts are nef, since h, H, f_1 are pullbacks of globally generated divisors. The second row is the fixed-part decomposition: on a general exceptional fibre of π, the original divisor has degree $1-u$, so every effective representative contains each component of E' with coefficient at least $u-1$. After this part is removed, the remainder is pulled back from $\mathbb{P}^1 \times \mathbb{P}^2$ and is nef. The same argument applies to the nodal member, whose centre is a reduced union of two smooth components. Their exceptional divisors are both generically ordinary blow-ups, so the coefficient bound applies to each. For $u > 3/2$ the residual plane degree $3-2u$ is negative, proving that the threshold is $3/2$. The common intersection numbers give

$$S_X(E) = \frac{1}{24}\left(\int_0^1 (24 - 12u - 6u^2)\,du + \int_1^{3/2} 6(3-2u)^2\,du\right) = \frac{17}{24}.$$

Write $P(u)|_E = a(u)\alpha + b(u)\beta$, where

$$(a(u), b(u)) = \begin{cases} (1+u, 2), & 0 \le u \le 1, \\ (2, 6-4u), & 1 \le u \le 3/2. \end{cases}$$

For $0 \le v \le b(u)$ the divisor $P(u)|_E - vC = a(u)\alpha + (b(u)-v)\beta$ is nef; it has volume $2a(u)(b(u)-v)$ and intersection $a(u)$ with C. Since C is not a component of $E'|_E$, there is no

primary fixed contribution. Therefore

$$\begin{aligned} S(W^E;C) &= \frac{1}{8}\int_0^{3/2}\int_0^{b(u)} 2a(u)(b(u)-v)\,dv\,du \\ &= \frac{1}{8}\left(\int_0^1 4(1+u)\,du + \int_1^{3/2} 2(6-4u)^2\,du\right) = \frac{11}{12}. \end{aligned}$$

The moving point contribution is

$$\frac{1}{8}\int_0^{3/2} a(u)^2 b(u)\,du = \frac{1}{8}\left(\int_0^1 2(1+u)^2\,du + \int_1^{3/2} 4(6-4u)\,du\right) = \frac{5}{6}.$$

The curve $E'|_E$ meets C transversally at one point. For a smooth member this is the graph property. For the nodal member its other component is disjoint from C by assumption, and its graph component has the same property. The only possible fixed point contribution is consequently

$$\frac{1}{4}\int_1^{3/2}\int_0^{6-4u} 2(u-1)\,dv\,du = \frac{1}{24}.$$

This proves the three claimed values and bounds. The nodal singularity is disjoint from E, so the same adjunction inequality applies there. Finally,

$$\delta_p(X) \geq \min\left\{\frac{24}{17}, \frac{12}{11}, \frac{8}{7}\right\} = \frac{12}{11}.$$

□

120.2 Adjunction for a curve extraction

We recall explicitly the local form of the refinement inequality that we use for curves. A local threshold always takes the infimum over divisors whose centres contain the specified point.

Lemma 120.3. *Let $\pi\colon Y \to X$ be a plt extraction of a smooth curve Γ in a smooth Fano threefold, with exceptional divisor G. Write*

$$(K_Y + G)|_G = K_G + \Delta_G.$$

Put $L = -K_X$, and let $W^G_{m,j}$ be the image of the restriction map from $H^0(Y, m\pi^*L - jG)$ to

$$H^0(G, (m\pi^*L - jG)|_G).$$

For $p \in \Gamma$ one has

$$\delta_p(X) \geq \min\left\{\frac{A_X(G)}{S_X(G)}, \inf_{q \in \pi^{-1}(p)\cap G} \delta_q(G, \Delta_G; W^G)\right\}.$$

The restriction of π^*L to G is retained in this definition. In particular, it need not be trivial for a curve extraction.

Proof. This is the local refinement inequality of [4, Theorem 3.3]. We recall its basis-divisor argument. Choose an anticanonical basis compatible with the order along G and with a valuation being tested. Removing the powers of a local equation of G gives bases of the spaces $W^G_{m,j}$. The pullback basis divisor has the form

$$S_m(G)G + B'_m,$$

where B'_m does not contain G and its restriction is a basis divisor of W^G with the usual normalization by $mh^0(X, mL)$. If a positive number c is smaller than both quantities on the right, the proof of [4, Lemma 2.21] gives

$$S_m(W^G; v) \le (1+\epsilon)S(W^G; v)$$

for all valuations of finite discrepancy and all sufficiently large m, with the bound on m independent of v. Thus the same strict inequalities hold for the basis thresholds simultaneously over the proper fibre $\pi^{-1}(p) \cap G$. Then adjunction makes $(G, \Delta_G + cB'_m|_G)$ log canonical over p. Inversion of adjunction applies to $(Y, G + cB'_m)$ there. The coefficient of G in the pullback of the corresponding pair on X is

$$1 - A_X(G) + cS_m(G) < 1.$$

Decreasing the coefficient of G therefore gives log canonicity of the original basis pair at p. Taking the infimum over the valuations and the limit of the basis thresholds proves the inequality. Equivalently, this is the same refinement argument as for a point extraction, with the whole fibre over p retained. □

The elementary decompositions along the distinguished surface and the del Pezzo fibres have already been displayed. In all point and curve extractions below, *we do not compute a Zariski decomposition on the extracted threefold.* We identify the actual refined sections, or embed them in complete series with the same asymptotic total dimension. The remaining decompositions and blow-ups occur on the exceptional surface.

121 Reduction to exceptional rulings and joining lines

We retain the notation of Section 117. In particular, E is the strict transform of $\mathbb{P}^1 \times Q$, and $\theta\colon X_U \to \mathbb{P}^1$ is the del Pezzo fibration. Suppose that every ramification index of η_U is at most three. For a ramification point $p \in C_U$, the fibre $\pi^{-1}(p)$ is a curve isomorphic to $\mathbb{P}^1$; we call it its exceptional ruling.

Proposition 121.1. *Let $\Sigma \subset X_U$ consist of the following curves:*

- *the exceptional rulings over ramification points of index two or three;*
- *in each fibre of type $2A_1$, the proper transform of the line joining the two points of the plane centre;*
- *in each fibre of type A_2, the proper transform of the line joining the point of multiplicity three to the remaining point.*

Then $\delta_p(X_U) \ge 24/23$ for every $p \notin E \cup \Sigma$. Moreover, $\delta_p(X_U) \ge 12/11$ for every $p \in E$.

We prove the assertion by surface flags and give their decompositions. The exceptional rulings in the statement contain the singular points of the fibres. Thus no singular point of a fibre is omitted from the remaining locus.

121.1 The resolved fibres

Let S be a singular fibre and let $\mu\colon \widetilde{S} \to S$ be its minimal resolution. The morphism to $\mathbb{P}^2$ is the successive blow-up of four points, with infinitely near points following the smooth conic Q. Write h for the pullback of a line and $e_1, \dots, e_4$ for the total transforms of the exceptional classes. Thus

$$h^2 = 1, \qquad e_i^2 = -1, \qquad he_i = e_ie_j = 0 \quad (i \ne j),$$

and put

$$Q_0 = 2h - e_1 - e_2 - e_3 - e_4, \qquad -K_{\widetilde{S}} = h + Q_0.$$

The distinguished smooth curve in the class Q_0 is the proper transform of $E \cap S$. We use T_i for an irreducible terminal exceptional curve of class e_i, write $R_{ij} = e_i - e_j$ for the indicated (-2)-curve, and write $L_{ij} = h - e_i - e_j$ whenever this class is represented by an irreducible curve.

Lemma 121.2. *The negative curves and reducible members of $|Q_0|$ are as follows. In the first column we record the multiplicities of the four-point plane centre.*

	(-2)-*curves*	(-1)-*curves*
$(2,1,1)$	R_{12}	$T_2, T_3, T_4, L_{12}, L_{13}, L_{14}, L_{34}$
$(2,2)$	R_{12}, R_{34}	$T_2, T_4, L_{12}, L_{13}, L_{34}$
$(3,1)$	R_{12}, R_{23}	T_3, T_4, L_{12}, L_{14}

The complete list of reducible members is

$(2,1,1)$	$L_{12} + L_{34}, \quad L_{13} + R_{12} + L_{14},$
$(2,2)$	$L_{12} + L_{34}, \quad 2L_{13} + R_{12} + R_{34},$
$(3,1)$	$L_{12} + R_{23} + R_{12} + L_{14}.$

The curves mapping to the remaining locus $\Sigma \cap S$ are, respectively, T_2; T_2, T_4, L_{13}; and T_3, L_{14}.

Proof. For a nonexceptional (-1)-curve of degree d, adjunction gives $\sum m_i = 3d - 1$ and $\sum m_i^2 = d^2 + 1$. Cauchy–Schwarz forces $d \leq 1$. Thus its class is $h - e_i - e_j$; an exceptional (-1)-curve has class e_i. Taking their nonnegative intersections with the effective (-2)-curves gives precisely the displayed irreducible curves. They are terminal exceptional curves, chords of Q, or its tangent lines. No line contains three of the four points, including their infinitely near conditions, since a line has intersection number two with the smooth conic. Thus the only irreducible (-2)-curves are those displayed.

The plane conics through the length-four centre form a pencil with no fixed component. Its base points are exactly those resolved by the four blow-ups, so $|Q_0|$ is base point free. A reducible plane member is a pair of lines, possibly equal. Since each line meets Q in degree two, the possible pairs are exactly the pairs of chords or tangent lines in the displayed list. Adding the exceptional components gives the asserted equalities on $\widetilde{S}$. The local equation from Lemma 117.3 identifies the terminal curves over multiple points with the exceptional rulings. □

121.2 The restricted divisors and the flag integrals

The threefold decomposition along S is

u	$P(u)$	$N(u)$
$[0,1]$	$H_1 + H_2 + (1-u)H_3$	0
$[1,2]$	$(2-u)H_1 + H_2$	$(u-1)E.$

In particular,

$$S_{X_U}(S) = \frac{1}{24}\left(\int_0^1 (24 - 15u)\,du + \int_1^2 3(2-u)(5-2u)\,du\right) = \frac{5}{6}.$$

These are the decompositions in [3, Lemma 5.77]. On $\widetilde{S}$ the positive part restricts to $h + Q_0$ for $0 \leq u \leq 1$ and to $h + (2-u)Q_0$ for $1 \leq u \leq 2$. Thus we may use the parameter $c \in [0,1]$ and the divisor

$$A_c = h + cQ_0.$$

The value $c = 1$ is counted once more, with weight one, for the whole interval $0 \le u \le 1$.

For a flag curve L, write $A_c - vL = P_L(c,v) + N_L(c,v)$ for its surface Zariski decomposition, and denote its pseudo-effective threshold by $\tau_L(c)$. For any integrand $F(c,v)$, put

$$\mathcal{I}_L(F) = \int_0^{\tau_L(1)} F(1,v)\,dv + \int_0^1 \int_0^{\tau_L(c)} F(c,v)\,dv\,dc.$$

If $p \in S \setminus (E \cup \operatorname{Sing}(S))$ and L is a smooth flag curve through its inverse image $\widetilde{p}$, then the surface flag quantities are

$$\begin{aligned} s_L &= \frac{1}{8}\mathcal{I}_L(P_L^2), \\ t_{L,\widetilde{p}} &= \frac{1}{8}\mathcal{I}_L((P_L \cdot L)^2) + \frac{1}{4}\mathcal{I}_L\left((P_L \cdot L)\operatorname{ord}_{\widetilde{p}}(N_L|_L)\right). \end{aligned}$$

Here L is not the distinguished member $E \cap S$. The fixed part $N(u)|_S = (u-1)(E \cap S)$ contributes nothing at $\widetilde{p}$, since $p \notin E$. Adjunction is crepant on $\widetilde{S}$, and near $\widetilde{p}$ the resolution is an isomorphism. The Abban–Zhuang flag inequality therefore gives

$$\delta_p(X_U) \ge \min\left\{\frac{6}{5}, \frac{1}{s_L}, \frac{1}{t_{L,\widetilde{p}}}\right\}.$$

Lemma 121.3. *Suppose that $\widetilde{p}$ lies on a smooth member $L \in |Q_0|$ and is not on a terminal exceptional curve over a multiple point. Then $s_L = 9/16$ and $t_{L,\widetilde{p}} \le 23/24$.*

Proof. Put $B = e_1 + e_2 + e_3 + e_4$, interpreted as an effective sum of total transforms. The decomposition is

v	$P_L(c,v)$	$N_L(c,v)$
$[0,c]$	$h + (c-v)Q_0$	0
$[c, c+\frac{1}{2}]$	$(1+2c-2v)h$	$(v-c)B$.

Both positive parts are nef, and the indicated exceptional support is negative definite. The threshold is $c + 1/2$. The volume and intersection on the first interval are $1 + 4c - 4v$ and 2; on the second they are $(1+2c-2v)^2$ and $2(1+2c-2v)$. Substitution in the preceding integrals gives

$$s_L = \frac{9}{16}, \qquad \frac{1}{8}\mathcal{I}_L((P_L \cdot L)^2) = \frac{11}{12}.$$

For a cluster of length r, its contribution to B is

$$R_{12} + 2R_{23} + \cdots + (r-1)R_{r-1,r} + rT_r,$$

with the indices shifted for other clusters. A smooth fibre of $|Q_0|$ is disjoint from all the (-2)-curves and meets each terminal curve transversally once. At its intersection with T_r the extra term is

$$\frac{r}{4}\left(\int_1^{3/2} 2(3-2v)(v-1)\,dv + \int_0^1 \int_c^{c+1/2} 2(1+2c-2v)(v-c)\,dv\,dc\right) = \frac{r}{24}.$$

At all other points it is zero. In the case under consideration $r = 1$ whenever such an intersection occurs. This gives the bound $11/12 + 1/24 = 23/24$. □

121.3 Flags on the reducible conics

We give all the further decompositions needed for the proof. Throughout the following tables $0 \le c \le 1$ and $P_L = A_c - vL - N_L$; hence specifying N_L specifies the positive part.

First, let $L = L_{12}$ in type $(2,1,1)$ or $(2,2)$, and put $B_L = R_{12} + 2T_2$ and $M = L_{34}$. The same table applies to $L = L_{34}$ in type $(2,1,1)$ with $B_L = T_3 + T_4$, $M = L_{12}$, and to $L = L_{34}$ in type $(2,2)$ with $B_L = R_{34} + 2T_4$, $M = L_{12}$:

v	N_L	$P_L \cdot L$
$[0, c]$	0	$1 + v$
$[c, 1]$	$(v-c)B_L$	$1 + 2c - v$
$[1, 1+c]$	$(v-c)B_L + (v-1)M$	$2(1 + c - v)$.

The respective values of P_L^2 are

$$1 + 4c - 2v - v^2, \quad 1 + 4c + 2c^2 - 2v - 4cv + v^2, \quad 2(1 + c - v)^2.$$

For $L = L_{13}$ in type $(2,1,1)$ the decomposition is

v	N_L	$P_L \cdot L$
$[0, c]$	$\frac{v}{2}R_{12}$	$1 + \frac{v}{2}$
$[c, 2c]$	$\frac{v}{2}R_{12} + (v-c)T_3$	$1 + c - \frac{v}{2}$
$[2c, 1+c]$	$(v-c)(R_{12} + T_3) + (v-2c)T_2$	$1 + 2c - v$.

The respective volumes are

$$1 + 4c - 2v - \frac{v^2}{2}, \quad 1 + 4c + c^2 - 2v - 2cv + \frac{v^2}{2}, \quad 1 + 4c + 3c^2 - 2v - 4cv + v^2.$$

For L_{14} interchange 3 and 4.

Finally, for $L = L_{12}$ in type $(3,1)$ we have

v	N_L	$P_L \cdot L$
$[0, \frac{3}{2}c]$	$\frac{v}{3}R_{12} + \frac{2v}{3}R_{23}$	$1 + \frac{v}{3}$
$[\frac{3}{2}c, 1+c]$	$(v-c)(R_{12} + 2R_{23}) + (2v-3c)T_3$	$1 + 2c - v$.

The two volumes are

$$1 + 4c - 2v - \frac{v^2}{3}, \qquad 1 + 4c + 3c^2 - 2v - 4cv + v^2.$$

These are Zariski decompositions: every coefficient of the displayed negative part is nonnegative in its stated range, its support is negative definite, and its components are orthogonal to the positive part. Intersecting with the complete list in Lemma 121.2 shows that the positive part is nef. The cone of curves of these weak del Pezzo surfaces is generated by those negative curves. At the right endpoint the positive part has square zero; the same intersection inequalities show that the class ceases to be pseudo-effective beyond it.

Lemma 121.4. *For the flag curves just listed, at a point outside E, the contracted (-2)-curves, and the curves mapping to Σ in Lemma 121.2, the following bounds hold:*

type	L	s_L	$\frac{1}{8}\mathcal{I}_L((P_L \cdot L)^2)$	$t_{L,\widetilde{p}}$ *at most*
$(2,1,1)$	L_{12}	$19/24$	$11/16$	$19/24$
$(2,1,1)$	L_{13}, L_{14}	$29/32$	$115/192$	$27/32$
$(2,1,1)$	L_{34}	$19/24$	$11/16$	$27/32$
$(2,2)$	L_{12}, L_{34}	$19/24$	$11/16$	$19/24$
$(3,1)$	L_{12}	$179/192$	$115/192$	$115/192$.

Proof. Integrating the displayed volume and intersection polynomials gives the third and fourth columns. We describe every allowed correction for the final column. For the first decomposition table, the correction at $L\cap M$ is $5/48$. At a point of $L\cap T_i$ where B_L contains T_i with coefficient one, it is $5/32$. Coefficient two gives $5/16$, but this occurs only on a remaining terminal curve and is excluded. For the second decomposition table, the only allowed correction is at $L_{13}\cap T_3$ (or $L_{14}\cap T_4$), and it is $47/192$. The intersection with R_{12} is excluded. In the final table, the only negative component meeting L_{12} is R_{23}, so there is no allowed correction. These intersections are distinct and transverse, as follows either from the blow-up description or from the chords and tangents to the smooth conic. Thus no two of the listed corrections occur at the same allowed point. Adding them to the fourth column proves the result. □

Proof of Proposition 121.1. The assertion on E is [3, Lemma 5.76]. If p lies outside E on a smooth fibre of θ, use [3, Lemma 5.77]. Otherwise resolve its singular fibre as above. Since $p\notin\Sigma$, the surface is smooth at p. If the member of $|Q_0|$ through $\widetilde{p}$ is smooth, Lemma 121.3 applies. If it is reducible, Lemma 121.2 and the exclusions defining Σ show that $\widetilde{p}$ lies on one of the curves of Lemma 121.4. In every case both s_L and $t_{L,\widetilde{p}}$ are at most $23/24$, so the flag inequality gives $\delta_p(X_U)\geq 24/23$. This proves the assertion and covers all points outside the stated remaining locus. □

122 The point of a fibre with ramification index two

Let p be the singular point of a fibre corresponding to a ramification point of index two. We use the ordinary blow-up of p.

Theorem 122.1. *One has $\delta_p(X)\geq 72/67$.*

Lemma 122.2 (The actual initial sections)**.** *The exceptional divisor of the ordinary blow-up of p is $G=\mathbb{P}^2$, and $A_X(G)=3$. There are homogeneous coordinates x,z,w on G such that a diagonal degeneration of the actual refined anticanonical series contains the algebra generated in degree one by*

$$\{1,x,z,w,wx,x^2\},\qquad t\{1,x,z,w,wx,x^2\},\qquad t^2\{1,x,z\},\qquad zw=x^2+t.$$

The primary degrees of x,z,w,t are $1,1,1,2$, respectively.

Proof. Write the conic as $YC-X^2=0$. On the chart $Y=u=1$, put $x=X/Y$, $c=C/Y$, $z=c-x^2$ and let $t=v/u$ be the target parameter. The centre is cut out by the conic and $F=tg(x,c)+f(x,c)$. Adding a multiple of the conic equation to f, and rescaling the target parameter, we may assume

$$f=z+R(x,c),\qquad R(x,c)=Ax^2+Bxc+Cc^2,\qquad A\neq 0,\qquad g(0,0)=1.$$

The inequality $A\neq 0$ expresses ramification index two. On the blow-up chart $F=za$, put $w=a-1$. Then x,z,w are regular parameters at p, and

$$tg(x,c)=zw-R(x,c),\qquad c=z+x^2.$$

Thus the initial form of t for the ordinary point valuation is $Q=zw-R(x,z)$.

The complete-intersection ideal sequence gives the fifteen anticanonical sections

$$t^i\{1,x,c\}\quad(0\leq i\leq 2),\qquad t^ia\{1,x,c\}\quad(0\leq i\leq 1).$$

Subtracting the first group from the second gives their initial basis

$$\{1,x,z,w,wx,wz\},\qquad Q\{1,x,z,w,wx,wz\},\qquad Q^2\{1,x,z\}.$$

These forms are independent; in particular, wz and Q are independent because $A \neq 0$.

Now give x, z, w the secondary weights $1, 2, 0$. The least secondary term of Q is $zw - Ax^2$. Replacing wz by $wz - Q = R(x, z)$ gives Ax^2, and replacing Qwz by $Qwz - Q^2$ gives $Ax^2(zw - Ax^2)$. After rescaling x, these are precisely the generators in the statement. Products give an inclusion in every degree. We do not assume generation of the anticanonical ring in degree one. □

Lemma 122.3 (The complete normal basis)**.** *In anticanonical degree m, a basis of the algebra above consists of $x^a z^b w^c t^s$, where a, b, c, s are nonnegative integers, $bc = 0$, and*

$$\begin{array}{c|c} c = 0 & a + 2b \leq 2m, \quad s \leq \min(2m, 3m - a - b), \\ b = 0,\ c \geq 1 & c \leq m, \quad a + c \leq 2m, \quad s \leq \min(2m - c, 3m - a - c). \end{array}$$

Its dimension is $(2m+1)(2m^2+2m+1)$. Consequently it is the whole secondary initial algebra of the actual refined series.

Proof. Straighten every factor zw by $zw = x^2 + t$. The only pairs of generators requiring this are z, w and z, wx. Their products become $x^2 + t$ and $x^3 + xt$, respectively. The capacities for powers of t show that each resulting term is a product of the same number of degree-one generators: z has capacity two, whereas w, wx and x^2 have capacity one. Each replacement reduces the number of zw pairs. The resulting monomials are independent, since zw is a leading monomial of the single relation.

If $c = 0$, use r factors x^2, $a - 2r$ factors x and b factors z. The smallest possible r is $\max(0, a + b - m)$. Such a distribution is possible exactly when $a + 2b \leq 2m$, and its capacity for t is $2m - r = \min(2m, 3m - a - b)$. If $c \geq 1$, use as many factors wx as possible, namely $\min(a, c)$. The smallest necessary number of factors x^2 is $\max(0, a - m)$. This is possible exactly when $c \leq m$ and $a + c \leq 2m$. The resulting capacity is $\min(2m - c, 3m - a - c)$. Every smaller nonnegative power of t can be distributed among these factors.

The two rows contain, respectively,

$$\frac{(m+1)(11m^2+16m+6)}{6}, \qquad \frac{m(13m^2+9m+2)}{6}$$

monomials. Their sum is $(2m+1)(2m^2+2m+1)$, which equals $h^0(X, -mK_X)$ by Riemann–Roch and Kodaira vanishing. Thus the inclusion of Lemma 122.2 is an equality in every degree. □

Lemma 122.4 (The expected order)**.** *One has $S_X(G) = 131/48$.*

Proof. The primary degree of $x^a z^b w^c t^s$ is $a + b + c + 2s$. In the first row of Lemma 122.3, put $b = m - n$. Then $0 \leq n \leq m$, $0 \leq a \leq 2n$ and $0 \leq s \leq U = \min(2m, 2m + n - a)$. In the second row put $c = m - n$. Then $0 \leq n < m$, $0 \leq a \leq m + n$ and $0 \leq s \leq U = \min(m + n, 2m + n - a)$. In either row the sum in s is

$$(U + 1)(a + m - n) + U(U + 1).$$

Split the first a-sum at n and the second at m. The sums over the two rows are therefore

$$\frac{m(m+1)(31m^2+43m+16)}{6}, \qquad \frac{m^2(23m^2+14m+3)}{4}.$$

Their sum is

$$\frac{m(131m^3+190m^2+127m+32)}{12}.$$

The calculation holds for every nonnegative integer m. The all-degree equality in Lemma 122.3, or equivalently Lemma 94.2, identifies these primary degrees with the orders of the actual sections. Dividing by $m(2m+1)(2m^2+2m+1)$ and taking the limit proves the assertion. □

Lemma 122.5 (The binary spaces)**.** *The secondary initial series decomposes into complete binary spaces*

$$x^A z^b w^c t^\ell \operatorname{Sym}^d(x^2, zw).$$

Their limiting fixed coefficients on $(x=0)$*,* $(z=0)$*,* $(w=0)$ *and* $(t=0)$ *are, respectively,*

$$a_0 = \frac{3}{64}, \qquad b_0 = \frac{5}{32}, \qquad c_0 = \frac{7}{32}, \qquad s_0 = \frac{197}{384}.$$

The total mass of the moving pencil is $2h_0$*, where* $h_0 = 41/128$*, and its coefficient at each pencil member is at most* h_0*.*

Proof. Let j and q be the primary and secondary degrees. Then $j = a+b+c+2s$, $q = a+2b+2s$, so $b = \max(q-j, 0)$ and $c = \max(j-q, 0)$. Put $k = j-b-c$. On the first row of the normal basis one has

$$\ell = \max\left(0, \left\lceil \frac{k+2b-2m}{2} \right\rceil, k+b-3m\right), \qquad u = \min(2m, \lfloor k/2 \rfloor),$$

and on the second row

$$\ell = \max\left(0, \left\lceil \frac{k+c-2m}{2} \right\rceil, k+c-3m\right), \qquad u = \min(2m-c, \lfloor k/2 \rfloor).$$

Set $A = k-2u$ and $d = u-\ell$. Whenever $d \geq 0$, the corresponding component is precisely the binary space in the statement, since $\langle x^2, t\rangle = \langle x^2, zw\rangle$.

For completeness, the following integrations give all the coefficients. Divide the indices by m, write $y = b/m$ or $y = c/m$, and let k denote k/m. In each row $0 \leq y \leq 1$. The limiting lower and upper bounds for s/m are

	range of k	L	U
z	$[0, 2-2y]$	0	$k/2$
z	$[2-2y, 4]$	$(k+2y-2)/2$	$k/2$
z	$[4, 5-y]$	$k+y-3$	2
w	$[0, 2-y]$	0	$k/2$
w	$[2-y, 4-2y]$	$(k+y-2)/2$	$k/2$
w	$[4-2y, 4-y]$	$(k+y-2)/2$	$2-y$
w	$[4-y, 5-2y]$	$k+y-3$	$2-y$.

Put $d = U-L$. Integration with respect to $dk\,dy$ gives

	$\int d$	$\int (k-2U+d)d$	$\int y d$	$\int L d$	$\int d^2/2$
z	$11/6$	$9/8$	$5/8$	$7/6$	$13/24$
w	$13/6$	$13/8$	$7/8$	$85/96$	$71/96$.

Divide the summed columns by four, since $mh^0(X, -mK_X) = 4m^4 + O(m^3)$. This gives $\alpha = 11/16$, $b_0 = 5/32$, $c_0 = 7/32$, $s_0 = 197/384$ and $h_0 = 41/128$, where $a_0 = \alpha - 2h_0 = 3/64$. The rounding and boundary terms contribute $O(m^3)$ to the sums. As a check,

$$a_0 + b_0 + c_0 + 2s_0 + 4h_0 = \frac{131}{48}.$$

Finally, a basis of a complete binary space of degree d has total zero multiplicity $d(d+1)$ on the pencil parameter line. Its total order at any prescribed point is at most $d(d+1)/2$. This proves the last assertion, including the two reducible members of the pencil. $\square$

Example 122.6. In degree one the primary degrees of the fifteen initial sections are 0, three copies of each of $1, 2, 3, 4$, and two copies of 5. Their sum is 40, agreeing with Lemma 122.4. In anticanonical degree two, the component $j = q = 2$ is $\langle x^2, t\rangle = \langle x^2, zw\rangle$. Thus the complete binary spaces include the cancellations along every member of the conic pencil.

Proof of Theorem 122.1. Consider the pencil $T_\lambda = (zw - \lambda x^2 = 0)$ on G. Its special members are $T_0 = (z = 0) + (w = 0)$ and $T_\infty = 2(x = 0)$. Lemma 122.5 bounds the coefficients of every basis divisor of the secondary initial series. The bounds on $(x = 0)$, $(z = 0)$, $(w = 0)$, T_1 and all other irreducible members are

$$\frac{11}{16}, \qquad \frac{61}{128}, \qquad \frac{69}{128}, \qquad \frac{5}{6}, \qquad \frac{41}{128}.$$

The pencil has base points $[0 : 0 : 1]$ and $[0 : 1 : 0]$. Blow up each twice, the second time at the common point of the strict transforms of the finite pencil members. The total boundary then has simple normal crossings. At the first base point the orders on the first and second exceptional curves are

$$a_0 + b_0 + s_0 + 2h_0 + \nu_\infty, \qquad a_0 + 2b_0 + 2s_0 + 4h_0 = \frac{8}{3},$$

where $0 \le \nu_\infty \le h_0$ is the moving coefficient of T_∞. At the second base point the orders are obtained by replacing b_0 with c_0, and the second order is $67/24$. Their log discrepancies are two and three, respectively. The first order is half the sum of the second order and $a_0 + 2\nu_\infty$, the coefficient of $(x = 0)$; hence its ratio is bounded by the larger of the second ratio and $11/16$. These blow-ups and the component bounds check all divisorial valuations. The largest ratio of order to discrepancy is $67/72$. It follows that for each $\tau < 72/67$, every such basis divisor has log-canonical threshold at least τ for all sufficiently large m.

For any divisorial valuation over G, choose in each primary piece a basis compatible with both that valuation and the secondary weight filtration. Two filtrations admit a common adapted basis. Diagonal degeneration gives a basis of the secondary initial series. Lower semicontinuity of log-canonical thresholds gives the same lower bound for the original basis divisor. Passing to the limit yields $\delta(G; W^G) \ge 72/67$. Abban–Zhuang adjunction and Lemma 122.4 give

$$\delta_p(X) \ge \min\left\{\frac{144}{131}, \frac{72}{67}\right\} = \frac{72}{67}.$$

We have not computed a Zariski decomposition on the blow-up of the threefold. The exact normal basis determines the actual refined series and its expected orders. All subsequent blow-ups occur on its exceptional plane, to resolve the conic pencil. □

123 The point of a fibre with ramification index three

Let p be the singular point of a fibre corresponding to a ramification point of index three. We estimate its local threshold using a weighted point extraction.

Theorem 123.1. *One has $\delta_p(X) \ge 384/383$.*

Lemma 123.2 (The actual initial sections)**.** *There are regular parameters x, z, w at p such that the weight-$(1, 2, 1)$ extraction has exceptional pair $(G, \Delta) = (\mathbb{P}(1, 2, 1), 0)$ and $A_X(G) = 4$. Put $t = zw - x^3$, $c = x^2 + z$ and $a = w + x$ on its exceptional surface. The actual refined anticanonical series contains the fifteen sections*

$$t^i\{1, x, c\} \quad (0 \le i \le 2), \qquad t^i a\{1, x, c\} \quad (0 \le i \le 1)$$

in anticanonical degree one. Under the secondary weights $(1,0,3)$ *on* (x,z,w)*, its initial algebra contains the algebra* $\mathcal{A}$ *generated in degree one by*

$$\{1,x,z,w,x^2,xz\},\qquad t\{1,x,z,w,x^2,xz\},\qquad t^2\{1,x,z\},\qquad zw=x^3+t.$$

Proof. Write the conic as $YC-X^2=0$. On the chart $Y=u=1$, put $x=X/Y$, $c=C/Y$, $z=c-x^2$ and $t=v/u$. The curve blown up is the complete intersection of the conic equation and a section of $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(1,2)$. After changing this second equation by a multiple of the conic equation, it has the form

$$F=tg(x,c)+xc+\gamma c^2,\qquad g(0,0)=1.$$

Indeed the corresponding quartic on the conic has a root of multiplicity three at $x=0$, and its leading coefficient can be normalized to one.

On the blow-up chart $F=za$, the functions x,z,a are regular parameters at p. Put $w=a-x$. With weights $(1,2,1)$ on x,z,w, the initial form of the original target parameter is $zw-x^3$; the initial forms of c and a are x^2+z and $w+x$. This also exhibits the A_2 singularity of the fibre. Weighted adjunction gives the exceptional pair and discrepancy asserted.

The complete-intersection ideal sequence identifies $H^0(X,-K_X)$ with the direct sum of the conic equation times $H^0(\mathcal{O}(2,1))$ and F times $H^0(\mathcal{O}(1,1))$. The kernel sheaf $\mathcal{O}(1,-1)$ has no cohomology, so this describes the whole section space and the sum is direct. Divide these sections by the exceptional equation z. Their initial forms are precisely the fifteen sections in the statement.

The polynomial $t=zw-x^3$ is homogeneous of secondary weight three. Replacing a by $a-x$ gives w, and similarly gives tw. The least secondary terms of c, xa and ca are z, x^2 and xz, respectively. Products of these actual initial sections prove the asserted inclusion. We have not assumed that the anticanonical ring is generated in degree one. □

Lemma 123.3 (The complete normal basis)**.** *A basis of* $\mathcal{A}_m$ *consists of* $x^az^bw^ct^s$*, where all exponents are nonnegative integers,* $bc=0$*, and*

$$\begin{array}{r|l} c=0 & b\le m,\quad a+b\le 2m,\quad s\le\min(2m,3m-a-b),\\ b=0,\ c\ge 1 & a+2c\le 2m,\quad s\le\min(2m-c,3m-a-2c).\end{array}$$

Its dimension is

$$N_m=(2m+1)(2m^2+2m+1)=h^0(X,-mK_X).$$

Thus $\mathcal{A}$ *is the entire secondary initial algebra of the actual refined series, in every degree.*

Proof. Straighten factors containing both z and w using $zw=x^3+t$. For $(zt^i)(wt^j)$, one has $i\le 2$, $j\le 1$; distribute the t-exponent of x^3t^{i+j} between x and x^2, with capacities two and one. The remaining term t^{i+j+1} is a product of two listed generators. For the additional generator xz, use

$$(xzt^i)(wt^j)=(x^2t^i)(x^2t^j)+xt^{i+j+1},\qquad i,j\le 1.$$

Again both terms are products of two generators. Each replacement reduces the number of zw pairs. The resulting monomials with $bc=0$ are independent, since zw is the leading monomial of the single relation for a suitable monomial order.

For $c=0$, use r factors x^2, q factors xz, $a-2r-q$ factors x and $b-q$ factors z, and fill the remaining places with 1. The smallest possible $r+q$ is $\max(0,a+b-m)$. Such a distribution exists exactly when $b\le m$ and $a+b\le 2m$. The total capacity for t is then $2m-r-q=\min(2m,3m-a-b)$, and every smaller nonnegative integer can be distributed. For $b=0$, use r factors x^2, $a-2r$ factors x, and c factors w. Taking $r=\max(0,a+c-m)$ gives the second row.

The first row contributes $(m+1)(5m^2+6m+2)/2$ monomials. The second contributes $m^2(3m+1)/2$. For example, the latter count is obtained by setting $n=m-c$ and summing $(2n+1)(m+n+1)-n(n+1)/2$ for $0\le n<m$. Their sum is N_m. Since $(-K_X)^3=24$, Riemann–Roch and Kodaira vanishing give the same value for $h^0(X,-mK_X)$. Taking an associated graded space preserves dimension. The inclusion of Lemma 123.2 is therefore an equality in every degree. □

Lemma 123.4 (The expected order of the extraction)**.** *For the exceptional divisor above, $S_X(G)=23/6$.*

Proof. The primary degree of a normal monomial is $a+2b+c+3s$. For the first row of Lemma 123.3, put $b=m-n$. The ranges are $0\le n\le m$, $0\le a\le m+n$ and $0\le s\le U=\min(2m,2m+n-a)$. Summation in s gives

$$(U+1)(a+2m-2n)+\frac{3}{2}U(U+1).$$

For the second row put $c=m-n$. The ranges are $0\le n<m$, $0\le a\le 2n$ and $0\le s\le U=\min(m+n,m+2n-a)$, and the sum in s is

$$(U+1)(a+m-n)+\frac{3}{2}U(U+1).$$

Splitting both a-sums at $a=n$ gives the respective totals

$$\frac{m(m+1)(125m^2+141m+46)}{12},\qquad \frac{m^2(59m^2+6m-5)}{12}.$$

Consequently, for every $m\ge 0$,

$$\sum(a+2b+c+3s)=\frac{m(2m+1)(46m^2+45m+23)}{6}.$$

The equality of algebras in Lemma 123.3 preserves primary degrees, as in Lemma 94.2. This is therefore the exact weight sum of the actual valuation filtration. Divide by mN_m and let m tend to infinity to obtain $23/6$. □

Lemma 123.5 (Binary spaces and their moments)**.** *In a primary-degree and secondary-weight component put*

$$b=\max((j-q)/2,0),\qquad c=\max((q-j)/2,0),\qquad k=q-3c.$$

If $c=0$, set

$$\ell=\max\left(0,\left\lceil\frac{k+b-2m}{3}\right\rceil,\left\lceil\frac{k+b-3m}{2}\right\rceil\right),\qquad u=\min(2m,\lfloor k/3\rfloor),$$

requiring $b\le m$. If $c\ge 1$, set

$$\ell=\max\left(0,\left\lceil\frac{k+2c-2m}{3}\right\rceil,\left\lceil\frac{k+2c-3m}{2}\right\rceil\right),\qquad u=\min(2m-c,\lfloor k/3\rfloor).$$

Whenever $u\ge\ell$, put $d=u-\ell$ and $A=k-3u$. The component is

$$x^Az^bw^ct^\ell\,\mathrm{Sym}^d(x^3,zw).$$

Normalize the sums of $(A+3d/2)(d+1)$, $b(d+1)$, $c(d+1)$, $\ell(d+1)$ and $d(d+1)/2$ by mN_m, and denote their limits by α,b_0,c_0,σ,h. Then

$$\alpha=\frac{65}{96},\qquad b_0=\frac{13}{48},\qquad c_0=\frac{11}{96},\qquad \sigma=\frac{11}{18},\qquad h=\frac{2}{9}.$$

Proof. The equations for the degrees give $j = a+2b+c+3s$, $q = a+3c+3s$. Thus $a = k-3s$, and substituting in the normal-basis inequalities gives $\ell \le s \le u$. Replacing t by $zw - x^3$ identifies their span with the stated complete binary space.

Here are the integrations giving the five limits. Write $y = b/m$ on the first branch and $y = c/m$ on the second, with $0 \le y \le 1$, and write k for k/m. The lower and upper bounds for s/m are as follows:

	range of k	L	U
z	$[0, 2-y]$	0	$k/3$
z	$[2-y, 5-y]$	$(k+y-2)/3$	$k/3$
z	$[5-y, 6]$	$(k+y-3)/2$	$k/3$
z	$[6, 7-y]$	$(k+y-3)/2$	2
w	$[0, 2-2y]$	0	$k/3$
w	$[2-2y, 5-2y]$	$(k+2y-2)/3$	$k/3$
w	$[5-2y, 6-3y]$	$(k+2y-3)/2$	$k/3$
w	$[6-3y, 7-4y]$	$(k+2y-3)/2$	$2-y$.

Put $d = U - L$. Integration with respect to $dk\,dy$ gives

	$\int d$	$\int (k - 3U + 3d/2)d$	$\int y d$	$\int L d$	$\int d^2/2$
z	$5/2$	$7/4$	$13/12$	$229/144$	$83/144$
w	$3/2$	$23/24$	$11/24$	$41/48$	$5/16$.

Divide the summed columns by four, with the third column contributing separately to b_0 and c_0. This proves the assertion, since $mN_m = 4m^4 + O(m^3)$ and the boundary and rounding terms contribute $O(m^3)$ to the weighted sums. The fixed x coefficient is $\alpha - 3h = 1/96$. As a check, the average weighted degree is

$$(\alpha - 3h) + 2b_0 + c_0 + 3\sigma + 6h = \frac{23}{6},$$

in agreement with Lemma 123.4. □

Example 123.6. For $m = 1$, $j = 3$ and $q = 1$, one has $b = 1$, $c = 0$, $k = 1$ and $\ell = u = 0$. The component is the single section xz. This section distinguishes the present calculation from the index-three calculation for Family 3.5. For $m = 2$, $j = q = 3$, the component is $\langle x^3, t\rangle = \langle x^3, zw\rangle$; thus the cancellations are included in the complete binary spaces.

Lemma 123.7 (The pencil on the exceptional surface)**.** *On $G = \mathbb{P}(1,2,1)$ put $X_0 = (x = 0)$, $Z_0 = (z = 0)$, $W_0 = (w = 0)$ and $T_\lambda = (zw - \lambda x^3 = 0)$, with $T_0 = Z_0 + W_0$ and $T_\infty = 3X_0$. Suppose*

$$B = aX_0 + bZ_0 + cW_0 + sT_1 + \sum \nu_\lambda T_\lambda, \qquad \sum \nu_\lambda = 2h, \qquad 0 \le \nu_\lambda \le h,$$

where all coefficients are nonnegative and only finitely many ν_λ are nonzero. Set $\alpha = a + 3h$. Then (G, B) is log canonical provided

$$\alpha,\ b+h,\ c+h,\ s+h,\ h \le 1,$$

and

$$\alpha + 3(c+s+h) \le 4, \qquad \alpha + 3(b+s+h) \le 4.$$

Proof. The first five inequalities bound the coefficients of all components. Away from the coordinate vertices the pencil members are smooth and disjoint. At $[1:0:0]$, only Z_0 and W_0 occur, and they cross transversely.

At $[0:1:0]$, take the index-two cover of the chart $z=1$. Its coordinates are x, w, with the involution $(x,w)\mapsto(-x,-w)$. This cover is quasi-étale, so log canonicity can be checked upstairs. The branches are $x=0$ and $w=\lambda x^3$. Put

$$u=a+3\nu_\infty, \qquad v=c+s+2h-\nu_\infty.$$

Three successive ordinary blow-ups separate the branches. Their exceptional valuations have weights $(1,i)$, for $i=1,2,3$, and have orders $u+iv$ on B and log discrepancies $1+i$. The component inequality gives $u\le 1$, whereas

$$u+3v=a+3c+3s+6h=\alpha+3(c+s+h)\le 4.$$

Linear interpolation gives $u+iv\le 1+i$ for $i=1,2$. After the third blow-up the distinct branches meet the last exceptional curve at distinct points. The total transform has simple normal crossings, with every coefficient at most one. This proves log canonicity at the index-two vertex.

The chart $w=1$ at $[0:0:1]$ is smooth. Its branches are $x=0$ and $z=\lambda x^3$. The same three blow-ups, with b in place of c, use the second corner inequality. This checks every point on a log resolution, and hence every divisorial valuation over G. □

Proof of Theorem 123.1. Use the pencil $T_\lambda=(zw-\lambda x^3=0)$ on G, with $T_0=(z=0)+(w=0)$ and $T_\infty=3(x=0)$. In a complete binary space of degree d, the sum of the orders of any basis at a prescribed point is at most $d(d+1)/2$. Consequently every basis divisor of the secondary initial series has the form in Lemma 123.7, with limiting coefficients

$$a=\frac{1}{96},\quad b=\frac{13}{48},\quad c=\frac{11}{96},\quad s=\frac{11}{18},\quad h=\frac{2}{9}.$$

The five component bounds in that lemma are

$$\frac{65}{96},\quad \frac{71}{144},\quad \frac{97}{288},\quad \frac{5}{6},\quad \frac{2}{9}.$$

The two corner orders are $169/48$ and $383/96$, and their log discrepancies are four. Thus the largest ratio of order to log discrepancy is $383/384$. For every $\tau<384/383$, all these basis divisors have log-canonical threshold at least τ for sufficiently large m. The three ordinary surface blow-ups at each base point of the pencil, described in Lemma 123.7, check all valuations over G, including those above its index-two point.

Fix any divisorial valuation over G. In each primary piece choose a basis compatible with this valuation and the secondary weight filtration. Two filtrations admit such a common adapted basis. Its diagonal degeneration gives a basis divisor of the initial series just considered. Lower semicontinuity of log-canonical thresholds gives the same bound for the original basis divisor. Taking limits proves $\delta(G;W^G)\ge 384/383$. Abban–Zhuang adjunction now gives

$$\delta_p(X)\ge \min\left\{\frac{A_X(G)}{S_X(G)},\delta(G;W^G)\right\}\ge \min\left\{\frac{24}{23},\frac{384}{383}\right\}=\frac{384}{383}.$$

No Zariski decomposition on the weighted blow-up of the threefold has been used. The actual sections, the exact normal basis in every degree and the complete binary spaces replace this calculation. The subsequent blow-ups occur only on the exceptional surface to resolve its pencil. □

124 The exceptional ruling over an index-two point

Theorem 124.1. *Let Γ be an exceptional ruling over a ramification point of index two. If $p\in\Gamma$ lies neither on E nor at the A_1 point of its del Pezzo fibre, then $\delta_p(X)\ge 9/8$.*

124.1 The refined series over an index-two exceptional ruling

Let Γ be the exceptional ruling over an index-two ramification point. In the weighted initial model of the blow-up we have

$$t = az - x^2, \qquad c = z + x^2.$$

The weighted extraction with transverse weights $(1,2)$ on (x,z) has log discrepancy three. Its exceptional surface has coarse space $G \cong \mathbb{F}_1$ and different $\frac{1}{2}D_x$, where $D_x = (x=0)$ is the positive section. Indeed the two normal directions of the ruling are $\mathcal{O}$ and $\mathcal{O}(-1)$, with weights one and two. The second Veronese identifies the coarse projective bundle with $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-1))$, and the stabilizer of order two occurs along $x = 0$.

Write ξ for the positive section class and B for a ruling. Use $X = x^2$, $Z = z$, and put $C = X + Z$, $T = aZ - X$. Here C, T, X denote both the sections and their zero divisors when there is no risk of confusion. The point $P = C \cap T$ lies over $a = -1$. The point Q is the intersection of D_x with the ruling at infinity.

Lemma 124.2. *For the exceptional divisor of this extraction,*

$$S_X(G) = \frac{8}{3}.$$

Proof. The anticanonical Hilbert polynomial and its total weight are

$$N_m = (2m+1)(2m^2+2m+1), \qquad W_m = \frac{m(2m+1)(16m^2+15m+8)}{3}.$$

Here the target affine coordinate has weight two and the three plane coordinates have weights $0, 1, 2$. Both generators of the centre have weight two. The resolution of its m-th ideal power has terms

$$\bigoplus_{i=0}^{m-1} H^0\big(\mathcal{O}(2m-i-1, m-2)\big) \longrightarrow \bigoplus_{i=0}^{m} H^0\big(\mathcal{O}(2m-i, m)\big).$$

The average weights in these summands are respectively $3m - i - 1$ and $3m - i$, including the weight-two shift in the first term. Their weighted dimension difference gives the displayed polynomial. The initial equations form a regular sequence, so the same resolution computes the associated graded spaces. Dividing W_m by mN_m and passing to the limit gives the assertion. □

Lemma 124.3 (A complete series dominating the refinement)**.** *Put $j = 2K + \epsilon$, where $\epsilon \in \{0,1\}$, and set*

$$n = m - \epsilon, \quad h = \max(0, K-n), \quad d = \min(K, n), \quad A = \max(0, 2d-n), \quad b = \max(0, K-m).$$

The piece of weight j of the refined series is contained in

$$x^{\epsilon} T^h H^0\left(G, \mathcal{O}_G(mB + d\xi) \otimes \mathcal{I}_{P,(1,2)}^{\geq A} \otimes \mathcal{I}_Q^b\right). \tag{124.1}$$

The factor x^{ϵ} records the bounded fractional fixed divisor $\epsilon D_x/2$. The weighted ideal at P uses weights $(1,2)$ on (C,T). The sum of the dimensions of the spaces in (124.1) is $4m^3 + O(m^2)$. Consequently their expected vanishing orders dominate those of the actual refined series asymptotically, for every valuation over G.

Proof. Let q be the conic equation and let F be a quadratic lift of $uf + vg$. These generate the ideal of the centre. Twisting the standard resolution of its m-th power by $\mathcal{O}(2m, 3m)$ shows that every section of $-mK_X$ is a sum of

$$q^{m-i} F^i H^0\big(\mathcal{O}(2m-i, m)\big), \qquad 0 \leq i \leq m.$$

There is no obstruction to taking global sections: the first cohomology of each term $\mathcal{O}(2m-i-1,m-2)$ is zero. Give the target affine coordinate weight two and the plane coordinates weights $0,1,2$. The initial forms of q,F are $Y_0Y_2-Y_1^2$ and $uY_1^2+vY_0^2$, a regular sequence. The initial complex of the ideal-power resolution is therefore again exact. Consequently the filtered resolution is strict and the associated graded section space is generated by the sections of this initial ideal power. This argument accounts for all cancellations, not just the initial forms of degree-one sections. It also gives the geometric valuation filtration. On $u=Y_0=1$, after inverting $z=c-x^2$, the initial equation is $a(c-x^2)=x^2+t$; substituting $t=az-x^2$ identifies its ring with $\mathbb{C}[a,x,z,z^{-1}]$. This is injective on the initial section spaces, so a nonzero global initial section cannot acquire extra order at the generic point of the exceptional surface. After division by q^m, its sections have the form

$$x^{\epsilon}a^{i}X^{r}C^{s}T^{l}, \qquad r+s+l=K, \quad 2r+s+\epsilon\le m, \quad 0\le i\le \min(m,2m-l).$$

These inequalities imply $l\ge h$ and weighted vanishing at least $2K-n$ at P. After removing T^h this is exactly the condition in (124.1). At Q, homogenizing a^i as a section of mB gives vanishing at least $m-i+r+s\ge K-m$. This proves the inclusion.

We compute the asymptotic dimension. Resolve the weighted point P by two ordinary blow-ups and blow up Q. Equivalently, start with $\mathbb{P}^2$ and blow up the point defining $\mathbb{F}_1$, then P, the point infinitely near P in the direction T, and finally Q. Write H,e_1,e_2,e_3,e_4 for the standard classes. Thus

$$\xi=H, \quad B=H-e_1, \quad T=H-e_2-e_3, \quad C=H-e_2-e_4, \quad V=H-e_1-e_4, \quad R=e_2-e_3.$$

Ignoring bounded rounding terms and writing $k=K/m$, the divisor after removing the fixed power of T is

$$D_k=\begin{cases}(k+1)H-e_1, & 0\le k\le 1/2,\\ (k+1)H-e_1-(k-1/2)(e_2+e_3), & 1/2\le k\le 1,\\ 2H-e_1-\frac{1}{2}(e_2+e_3)-(k-1)e_4, & 1\le k\le 3.\end{cases}$$

Its negative part is zero up to $k=2$, is $(k-2)V$ for $2\le k\le 5/2$, and is

$$(k-2)V+(2k-5)C+(k-5/2)R$$

for $5/2\le k\le 3$. The positive parts are nef: this follows by intersection with

$$e_1,e_3,e_4,R,H-e_1-e_2,V,T,C,$$

which are precisely the negative curves of this weak del Pezzo surface. The volumes on the five successive intervals are

$$k(k+2), \quad \frac{-2k^2+8k-1}{2}, \quad \frac{-2k^2+4k+3}{2}, \quad \frac{11-4k}{2}, \quad 2(k-3)^2.$$

Their integrals are $7/24, 23/24, 13/6, 1/2, 1/12$, whose sum is four. The two parity choices cancel the factor $1/2$ in the surface Hilbert polynomial. This proves the dimension assertion. As a check on both the grading and the normalization,

$$\frac{1}{4}\int_0^3 2k\,\mathrm{vol}(D_k)\,dk=\frac{8}{3},$$

in agreement with Lemma 124.2.

For a fixed valuation, extend a compatible basis of each actual piece to a compatible basis of the containing space in (124.1). All the vanishing orders are nonnegative. Thus the sum for

the smaller space is no greater than the sum for the larger space. Since their total dimensions have ratio tending to one, the same comparison holds for expected vanishing orders. Notice that finite-degree equality of the two spaces is not required.

We also record why the containing spaces are a multigraded series. Before factoring T^{h}, their class is $mB + (j/2)\xi$ and their conditions are the homogeneous inequalities

$$v_T \geq j/2 - m, \qquad v_{P,(1,2)} \geq j - m, \qquad v_Q \geq j/2 - m.$$

Here $v_T(x) = v_{P,(1,2)}(x) = 0$ and $v_Q(x) = 1/2$. The factor x^{ϵ} and the rounding of these inequalities give exactly h, A, b above. These conditions are preserved under multiplication. The support is bounded by $0 \leq j \leq 6m$; on any rational ray $0 < j/(2m) < 1/2$ the even pieces are the complete series of the ample class $mB + (j/2)\xi$. One may clear the denominator by a Veronese grading; the odd pieces have the same leading moments. Thus the multigraded Abban–Zhuang estimates apply to the containing series as well. □

Lemma 124.4 (The point P)**.** *The series in* (124.1) *satisfies*

$$\delta_P\left(G, \tfrac{1}{2}D_x; W\right) \geq \frac{6}{5}.$$

Proof. The point P does not lie on D_x. Extract it with weights $(1, 2)$ on (C, T). Its exceptional curve is the image of e_3 on the resolution above. Its log discrepancy is three. The exceptional curve meets the contracted curve R at a point q_R; the different there is $1/2$. Let q_T be its intersection with the strict transform of T.

For clarity we give the complete numerical computation. Write $P(k, v) + N(k, v)$ for the Zariski decomposition of $D_k - ve_3$. The negative parts on its ten chambers are listed below; each chamber is the convex hull of the indicated vertices in the (k, v)-plane. Put $U = H - e_1 - e_2$.

Vertices	Negative part
$(0,0), (1/2,0), (1/2,1)$	$(v/2)R$
$(0,0), (1/2,1), (1/2,3/2), (0,1)$	$(v-k)R + (v-2k)U$
$(1/2,3/2), (1/2,2), (0,1)$	$(v-k)R + (v-2k)U + (v-k-1)T$
$(1/2,0), (1,0), (1,1), (1/2,1)$	$(v/2)R$
$(1,1), (1/2,3/2), (1/2,1)$	$(v-1/2)R + (v-1)U$
$(1,1), (1,2), (1/2,2), (1/2,3/2)$	$(v-1/2)R + (v-1)U + (k+v-2)T$
$(1,0), (2,0), (2,1), (1,1)$	$(v/2)R$
$(2,0), (5/2,0), (2,1)$	$(v/2)R + (k-2)V$
$(1,2), (1,1), (2,1)$	$(v-1/2)R + (v-1)U + (v-1)T$
$(5/2,0), (3,0), (2,1)$	$(k+v-5/2)R + (k-2)V + (2k+v-5)C$

The support matrices are negative definite, and subtracting the listed parts gives a nef divisor on each chamber. Thus the table also verifies the decomposition directly.

The normalization is $1/4$. The fixed order along e_3 is $(2k-1)_+$. Integration gives

$$\frac{1}{4}\iint P(k,v)^2\,dv\,dk = \frac{295}{384}, \qquad \frac{1}{4}\int_0^3 (2k-1)_+ \operatorname{vol}(D_k)\,dk = \frac{649}{384}.$$

Their sum is $S(W; e_3) = 59/24$. The moving point moment is

$$\frac{1}{4}\iint (P(k,v)\cdot e_3)^2\,dv\,dk = \frac{43}{128}.$$

At q_R the additional fixed coefficient before contracting R is $N_R(k, v) + (k - 1/2)_+$; at q_T it is $N_T(k, v) + (k-1)_+$. The corresponding additional moments are $165/128$ and $191/384$. Hence the uncontracted point moments are $13/8$ and $5/6$.

When R is contracted, the pullback of the exceptional curve is $e_3 + \frac{1}{2}R$. Thus its refinement at q_R subtracts one half of the first expected order. The actual point moment there is

$$\frac{13}{8} - \frac{1}{2}\frac{59}{24} = \frac{19}{48}.$$

Taking account of the different, the flag bound is

$$\min\left\{\frac{3}{59/24}, \frac{1/2}{19/48}, \frac{1}{5/6}, \frac{1}{43/128}\right\} = \frac{6}{5}.$$

This proves the lemma. □

Lemma 124.5 (The fibre flags on the exceptional surface)**.** *In the notation above, let q lie over a base point different from 0 and ∞, and suppose $q \neq P$. Then*

$$\delta_q(G, D_x/2; W^G) \geq \frac{384}{329}.$$

Proof. We give both fibre calculations, since the fibre through P has a different strict transform. On the smooth surface R, write

$$B = h - e_1, \quad T = h - e_2 - e_3, \quad C = h - e_2 - e_4, \quad V = h - e_1 - e_4, \quad R_0 = e_2 - e_3.$$

Its negative curves are

$$e_1, e_3, e_4, h - e_1 - e_2, V, T, C, R_0.$$

The first seven have self-intersection -1, and $R_0^2 = -2$. They generate the effective cone. The different is half the strict transform of the curve $D_x = h - e_4$.

For the limiting weight $u = K/m$, put

$$D_u = \begin{cases} (u+1)h - e_1, & 0 \leq u \leq 1/2, \\ (u+1)h - e_1 - (u - 1/2)(e_2 + e_3), & 1/2 \leq u \leq 1, \\ 2h - e_1 - \frac{1}{2}(e_2 + e_3) - (u-1)e_4, & 1 \leq u \leq 3. \end{cases}$$

There is also the fixed divisor $(u-1)_+ T$, where $a_+ = \max(a, 0)$.

First take a fibre F whose base point differs from $0, \infty, -1$. Its strict transform has class B. For $D_u - vB$, put

$$r = (v-1)_+, \quad s = (v - \max(1, 2-u))_+, \quad c = (2u + 2v - 5)_+, \quad b = (u-2)_+.$$

The negative and positive parts are

$$N = re_1 + sT + bV + cC + \frac{c}{2}R_0, \qquad P = D_u - vB - N.$$

The threshold is

$$\tau(u) = \begin{cases} 1 + u, & 0 \leq u \leq 1/2, \\ 3/2, & 1/2 \leq u \leq 1, \\ 2 - u/2, & 1 \leq u \leq 2, \\ 3 - u, & 2 \leq u \leq 3. \end{cases}$$

These formulas follow by imposing zero intersection with the support of N. On each resulting chamber the support matrix is negative definite and P has nonnegative intersection with all eight negative curves. At $v = \tau(u)$ its square is zero; a nef ruling class excludes larger values.

Put $l = P \cdot B$. Integration gives

$$\frac{1}{4}\int_0^3 \int_0^{\tau(u)} P^2 \, dv \, du = \frac{69}{128}, \qquad \frac{1}{4}\int_0^3 \int_0^{\tau(u)} l^2 \, dv \, du = \frac{79}{192}.$$

The first number is $S(W^G; F)$ and the second bounds the moving point moment. The extra point contributions, obtained by integrating l times the fixed coefficient with factor $1/2$, are

point on F	fixed coefficient	extra contribution
$F \cap e_1$	r	$7/384$
$F \cap T$	$s + (u-1)_+$	$57/128$
$F \cap C$	c	$3/64$.

The other negative curves do not meet this fibre. Thus the point bounds are $55/128$, $329/384$, $11/24$ at these three points, and $79/192$ elsewhere. These points are distinct: any coincidence would lie over 0, ∞ or -1. The point $F \cap D_x$ is also distinct from them and has log discrepancy $1/2$ on the adjoint curve, whereas its moment is $79/192$. Therefore the flag inequality gives

$$\delta_q(G, D_x/2; W^G) \geq \min\left\{\frac{128}{69}, \frac{384}{329}, \frac{96}{79}\right\} = \frac{384}{329}.$$

Now let the base point be -1. The strict transform of its fibre is $F = h - e_1 - e_2$. Set $k_0 = \min((u - 1/2)_+, 1/2)$ and

$$r = (v-1)_+, \quad s = (v - 2k_0)_+, \quad c = (2u + v - 5)_+, \quad b = (u-2)_+.$$

This time the Zariski decomposition is

$$N = re_1 + se_3 + bV + cC + \frac{v + s + c}{2} R_0, \qquad P = D_u - vF - N,$$

with threshold $\tau(u) = 1 + u$ for $0 \leq u \leq 1$, and $\tau(u) = 3 - u$ for $1 \leq u \leq 3$. The same intersection checks verify this decomposition. Writing $l = P \cdot F$, one obtains

$$\frac{1}{4}\int\int P^2 \, dv \, du = \frac{67}{96}, \qquad \frac{1}{4}\int\int l^2 \, dv \, du = \frac{11}{32}, \qquad \frac{1}{2}\int\int lr \, dv \, du = \frac{1}{16}.$$

Thus the point moment is at most $13/32$ at $F \cap e_1$ and $11/32$ elsewhere, except at $F \cap R_0$, which maps to the excluded point P. The fixed curve T is disjoint from this strict transform. At $F \cap D_x$ the moment $11/32$ is below the adjoint log discrepancy $1/2$. Consequently

$$\delta_q(G, D_x/2; W^G) \geq \min\left\{\frac{96}{67}, \frac{32}{13}, \frac{16}{11}\right\} = \frac{96}{67} > \frac{384}{329}.$$

All displayed integrals are over $0 \leq u \leq 3$ and $0 \leq v \leq \tau(u)$. Splitting at the zeros of the affine functions inside $(\,)_+$ gives ordinary integrals of quadratic polynomials. The complete series of degree $ml + O(1)$ on the flag curve bounds the maximal point vanishing from above. Hence the computation applies also if a restriction map is not surjective; the rounding terms have lower order. □

Proof of Theorem 124.1. The excluded points on Γ have coordinates $a = \infty$ and $a = 0$, respectively. For every other base point, the fibre flags of Lemma 124.5 cover all points of the extracted surface except possibly P. At P use Lemma 124.4. The domination in Lemma 124.3 applies to every divisorial valuation, so these bounds hold for the actual refined series. Consequently its local thresholds over p are at least $384/329$. Lemma 120.3 and Lemma 124.2 now give

$$\delta_p(X) \geq \min\left\{\frac{3}{8/3}, \frac{384}{329}\right\} = \frac{9}{8}.$$

The only decompositions in this argument occur on the explicitly blown-up exceptional surface. □

125 The exceptional ruling over an index-three point

Lemma 125.1 (Specialization of a marked point)**.** *Let $\mathcal{X} \to \mathbb{A}^1$ be a normal $\mathbb{Q}$-Gorenstein family of klt Fano varieties, equipped with a $\mathbb{G}_m$-action covering multiplication on $\mathbb{A}^1$. Suppose that its restriction to $\mathbb{G}_m$ is equivariantly isomorphic to $X \times \mathbb{G}_m$, and that the orbit of $p \in X$ extends to a section with value $p_0 \in X_0$. Then*

$$\delta_p(X) \geq \delta_{p_0}(X_0).$$

Proof. For a positive integer m divisible enough to make $-mK$ Cartier, let $\delta_{m,p}$ be the infimum of the local log canonical thresholds at p of the m-basis divisors. Thus

$$\delta_{m,p}(X) = \inf_{p \in c_X(F)} \frac{A_X(F)}{S_m(F)}.$$

One has $\delta_{m,p} \to \delta_p$. Indeed, [6, Corollary 2.10] gives $S_m(F) \leq (1+\eta)S(F)$ for all valuations and all sufficiently large m, so $\delta_{m,p} \geq \delta_p/(1+\eta)$. Conversely, fix any divisor whose centre contains p and use $S_m(F) \to S(F)$, then take the infimum over these divisors. This gives the opposite limiting inequality. Relative vanishing and base change identify the spaces of anticanonical sections with a free module over $\mathbb{C}[s]$. The equivariant structure determines a finite filtration of $H^0(X, -mK_X)$; its associated graded is $H^0(X_0, -mK_{X_0})$. Fix a prime divisor over X whose centre contains p, and choose a basis compatible with both its valuation filtration and this finite filtration. Two finite filtrations always have a common compatible basis. Rescaling the resulting sections by powers of s gives a family of bases whose central values form a basis of $H^0(X_0, -mK_{X_0})$.

The corresponding basis divisors specialize flatly. Lower semicontinuity of log canonical thresholds along the given section shows that the threshold of the original basis divisor at p is at least the local m-basis threshold of X_0 at p_0. Since the original basis was adapted to the chosen valuation, taking the infimum over valuations gives the same inequality for the local m-basis thresholds. Passing to the limit as m tends to infinity proves the assertion. □

Lemma 125.2 (The exceptional fibre over an index-three point)**.** *Let $q \in C$ have ramification index three. Let ℓ be the fibre of the exceptional divisor of $X \to \mathbb{P}^1 \times \mathbb{P}^2$ over q, and let $p_* \in \ell$ be the A_2 point of the del Pezzo fibre. Then every $p \in \ell \setminus \{p_*\}$ satisfies*

$$\delta_p(X) \geq \frac{12}{11}.$$

Proof. Points on the distinguished surface E already have this bound. Choose coordinates in which

$$f = x^3y + \alpha x^4, \qquad g = y^4 + g_1xy^3 + g_2x^2y^2 + g_3x^3y + g_4x^4,$$

and the centre has equation $uf + vg = 0$. Consider the family

$$u(x^3y + s\alpha x^4) + v\left(y^4 + \sum_{i=1}^{4} s^i g_i x^i y^{4-i}\right) = 0.$$

For $s \neq 0$ the substitutions $x' = sx$, $y' = y$, $u' = u$, $v' = s^3v$ identify its member with the original curve. The family of blow-ups is equivalently the family of divisors of class $h + 2H$ in $\mathbb{F}_1 \times \mathbb{P}^2$ obtained from the net of quadrics. Here h is the pullback of a line from $\mathbb{P}^2$ and f_1 is a ruling of $\mathbb{F}_1$. Consequently its relative anticanonical class is the restriction of $h + f_1 + H$, which is ample.

The central net is

$$\langle Y_0Y_2 - Y_1^2,\ Y_1Y_2,\ Y_0^2 \rangle.$$

Its incidence threefold has just one ordinary double point, over $[Y_0 : Y_1 : Y_2] = [0 : 0 : 1]$ and the coefficient of Y_0^2. Indeed, on the corresponding affine chart its quadratic part is $as + bt + s^2$, a nondegenerate quadratic form in four variables. The blow-up of the smooth fibre of the first generator takes place away from this point. Thus the central member is a nodal Fano threefold. The distinguished surface remains smooth.

We record the marked point carefully. On $u = Y_0 = 1$ put $x = Y_1$, $c = Y_2$, $z = c - x^2$, and

$$a = \frac{xc + \alpha c^2 + tG(x, c)}{z}.$$

The point p_* has $a = 0$. The given point has a finite value $a = a_0 \neq 0$, since the value infinity is the point on E. Under the displayed isomorphism the original exceptional coordinate and the coordinate on the member over s satisfy $a' = sa$. Hence the marked point on that member has $a = a_0/s$ and specializes to the point of E over $x = t = 0$. In particular, it does not specialize to p_*.

The ruling of E through this limit lies over $x = 0$. The additional vertical component of the central fixed curve lies over $x = \infty$, so these two curves are disjoint. Lemma 120.2 therefore applies: its three expected orders are $17/24$, $11/12$ and at most $7/8$, and hence $\delta_{p_0}(X_0) \geq 12/11$. Applying the preceding specialization lemma finishes the proof. □

126 The joining line of a fibre of type (2,2)

Suppose a fibre has two ramification points of index two. Let Γ be its joining line. Its endpoints are the two A_1 points of the fibre. We treat the remaining points by extracting the whole curve with weights $(1, 2)$ in its normal directions.

Theorem 126.1. *For every point $p \in \Gamma$ different from its endpoints, $\delta_p(X) \geq 72/67$.*

Lemma 126.2 (The exceptional pair)**.** *The weighted extraction has discrepancy three. Its exceptional pair is*

$$(G, \Delta) = \left(\mathbb{P}^1 \times \mathbb{P}^1, \frac{1}{2}D\right),$$

where D is a smooth curve of bidegree $(2, 1)$.

Proof. Write the conic and graph as

$$q = UV - x^2 = 0, \qquad F = vx^2 + uG(U, x, V) = 0,$$

with $[u : v]$ the target coordinates. Put $G_0(U, V) = G(U, 0, V)$. Its coefficients of U^2 and V^2 are nonzero, since the graph has no base point at either ramification point. The joining line is the strict transform of $u = x = 0$.

Let e define the exceptional divisor of the original blow-up, and let $s_1 = q/e$, $s_2 = F/e$. These are global sections of classes E, $2H - E$, $T + 2H - E$, respectively, satisfying

$$es_1 = UV - x^2, \qquad es_2 = vx^2 + uG(U, x, V).$$

Give u, x, s_2 weights $2, 1, 2$, and all other displayed sections weight zero. These weights agree with the transverse valuation at the generic point of Γ. Their initial identities are

$$es_1 = UV, \qquad es_2 = vx^2 + uG_0(U, V).$$

We shall identify the entire refined series by an exact dimension count below; no assertion about generators of the whole Cox ring is needed.

Along Γ, the sections v and s_1 are units. Set both equal to one, so $e = UV$, and write $t = u$, $a = s_2$. The exceptional weighted directions satisfy

$$x^2 = UVa - tG_0(U, V),$$

with weights $1, 2, 2$ on x, a, t. Their coarse surface is $G = \mathbb{P}^1_{U,V} \times \mathbb{P}^1_{a,t}$. The generic index two along $x = 0$ gives the different $D/2$, where

$$D = (UVa - tG_0(U, V) = 0).$$

The polynomials UV and G_0 have no common zero, so D is the graph of a morphism of degree two and is smooth. On the open part of Γ, the normal coordinates are x, u; at the endpoints one uses x, s_2. Both descriptions give weights $(1, 2)$ and log discrepancy three. □

Put $T_0 = (t = 0)$, and let P, Q be the two points of $D \cap T_0$. They lie over the two endpoints of Γ.

Lemma 126.3 (The complete refined spaces)**.** *Write $j = 2K + \varepsilon$, where $\varepsilon \in \{0, 1\}$, and put*

$$n = m - \varepsilon, \quad h = \max(0, K - 2m), \quad d = n - 2h, \quad e_0 = K - h, \quad k = \max(0, K - m) - h.$$

The actual weight-j piece of the refined anticanonical series is

$$x^\varepsilon (UVa - tG_0)^h H^0\big(G, \mathcal{O}(d, e_0) \otimes \mathcal{I}_P^k \mathcal{I}_Q^k\big).$$

It is zero when $d < 0$. The factor x^ε records the fractional fixed divisor $\varepsilon D/2$ on the coarse surface.

Proof. Products of the global sections in the preceding proof of class $m(2T + 3H - E)$, after setting $v = s_1 = 1$, have initial forms

$$(UV)^\alpha x^\ell t^i a^c H^0(\mathbb{P}^1, \mathcal{O}(m - 2\alpha - \ell)),$$

where $c \le m + \alpha$, $i + c \le 2m$ and $\ell \le m - 2\alpha$. These statements follow directly by comparing the three divisor classes of the monomial. Write $\ell = \varepsilon + 2r$ and replace x^{2r} by $(UVa - tG_0)^r$. Then $K = i + c + r$.

First suppose $K \le 2m$. Terms with $r = 0$ already span the space in the statement. To see this, expand a section of $\mathcal{O}(n, K)$ as

$$\sum_i f_i(U, V) a^{K-i} t^i.$$

Vanishing to order k at both P, Q is equivalent to divisibility of f_i by $(UV)^{\max(k-i,0)}$. This is precisely the inequality $c \le m + \alpha$ above. Every term with $r > 0$ satisfies the same vanishing condition and adds nothing outside this space.

If $K > 2m$, the inequality $i + c \le 2m$ forces $r \ge K - 2m = h$. Remove the common factor $(UVa - tG_0)^h$. The remaining fibre degree is $2m$, the base degree is $n - 2h$, and the required vanishing at both points is m. The preceding argument applies again. Thus the displayed space is contained in the actual refined piece. We check equality by counting dimensions in all degrees.

For $d, e_0 \ge 0$ and $e_0 \ge k$, the binary expansion used above gives

$$h^0(\mathcal{O}(d, e_0) \otimes \mathcal{I}_P^k \mathcal{I}_Q^k) = (d + 1)(e_0 - k + 1) + r(d - r), \qquad r = \min(k, \lfloor d/2 \rfloor).$$

Combine the two parity rows, and put $n = \lfloor m/2 \rfloor$. Their dimensions at fixed K are

$$\begin{array}{c|c} 0 \le K \le m & (2m+1)(K+1) \\ K = m+k,\ 1 \le k \le m & (2m+1)(m+1) + r(2m-1-2r), \quad r = \min(k,n) \\ K = 2m+h,\ 1 \le h \le n & (2d+1)(m+1) + d(d-1)/2, \quad d = m-2h. \end{array}$$

Summing the three rows gives $(2m+1)(2m^2+2m+1)$, as one checks separately for $m = 2n$ and $m = 2n+1$. Riemann–Roch and Kodaira vanishing give the same value for $h^0(X, -mK_X)$. Since associated graded spaces preserve dimension, all the inclusions are equalities. □

Let $\pi : S \to G$ blow up P, Q. Denote the exceptional curves by E_1, E_2, and use B, T for the pullbacks of the two fibre classes, with B a fibre of the projection to Γ. Put

$$C = T - E_1 - E_2, \qquad \widetilde{D} = 2B + T - E_1 - E_2.$$

The curve C has self-intersection -2. The other negative curves are $E_1, E_2, B - E_1, B - E_2$; these five curves generate the effective cone. This also follows by applying the plane model of the blow-up of a quadric at two points on the same ruling line.

Lemma 126.4 (The first moment)**.** *For the weighted extraction, $S_X(G) = 67/24$.*

Proof. Put $u = K/m$. The limiting fixed divisor is $h\widetilde{D}$, and the remaining class is $L_u = dB + eT - k(E_1 + E_2)$, with

u	d	e	k	h
$[0,1]$	1	u	0	0
$[1,2]$	1	u	$u-1$	0
$[2,5/2]$	$5-2u$	2	1	$u-2$.

Its negative part is rC, where $r = \max(0, k - d/2)$, and its positive part has square

$$V(u) = 2de - 2k^2 + 2r^2.$$

Indeed its intersections with E_i, $B - E_i$ and C are $k - r$, $e - k$ and $d - 2k + 2r$, respectively, and are nonnegative.

There are two parity rows in Lemma 126.3. Their leading terms agree, and

$$\int_0^{5/2} V(u)\, du = 4,$$

which agrees with $h^0(X, -mK_X) = 4m^3 + O(m^2)$. The weight divided by m is $2u$. Consequently

$$S_X(G) = \frac{1}{4} \int_0^{5/2} 2uV(u)\, du = \frac{67}{24}.$$

All contributions of the parity factor are of lower order. □

Lemma 126.5 (Points outside the different)**.** *Let $q \in G$ lie over an interior point of Γ, and suppose $q \notin D$. Then $\delta_q(G, \Delta; W^G) \ge 6/5$.*

Proof. Use the fibre $F = B$ through q. It avoids P, Q. Subtract vF from L_u, and write $b = d - v$, with $0 \le b \le d$. Its negative part is $r_b C$, where $r_b = \max(0, k - b/2)$. The positive square and its degree on F are

$$V_F = 2be - 2k^2 + 2r_b^2, \qquad l = e - r_b.$$

The fixed divisor restricted to F is $r_b(F \cap T_0) + h(F \cap D)$. These two points are distinct. Therefore

$$S(W^G; F) = \frac{1}{4} \int \int_0^d V_F \, db \, du = \frac{11}{24}.$$

At a point outside D, the subsequent point moment is bounded by

$$\frac{1}{2} \int \int_0^d l \left(n_q + \frac{l}{2} \right) db \, du,$$

where $n_q = r_b$ at $F \cap T_0$ and is zero otherwise. These integrals are $5/6$ and $69/128$, respectively. For clarity, the chamber contributions, with the normalizing factors included, are

u	b	$S(W^G; F)$	$n_q = 0$	$n_q = r_b$
$[0, 1]$	$[0, 1]$	$1/8$	$1/12$	$1/12$
$[1, 3/2]$	$[0, 2u - 2]$	$3/64$	$11/128$	$7/64$
$[1, 3/2]$	$[2u - 2, 1]$	$3/32$	$11/128$	$11/128$
$[3/2, 2]$	$[0, 1]$	$7/48$	$19/96$	$67/192$
$[2, 5/2]$	$[0, 5 - 2u]$	$3/64$	$11/128$	$79/384$.

The point bounds follow by replacing the restricted series by the complete series of its degree; this gives an upper bound for its maximal total vanishing. Boundary and rounding terms have lower order. Since F has log discrepancy one and $\Delta|_F$ has no support at q, the surface flag inequality gives

$$\delta_q(G, \Delta; W^G) \geq \min\{24/11, 6/5, 128/69\} = 6/5.$$

□

Lemma 126.6 (Points on the different)**.** *For $q \in D \setminus \{P, Q\}$ one has $\delta_q(G, \Delta; W^G) \geq 192/175$.*

Proof. Use D as the first member of the flag. Its log discrepancy is $1/2$, and its adjoint boundary is zero. Set $\tau = \min(d/2, e)$. For $0 \leq v \leq \min(k, \tau)$, the negative part of $L_u - v\widetilde{D}$ remains rC, since $\widetilde{D} \cdot C = 0$. Its positive square and degree on $\widetilde{D}$ are

$$V_D = 2(d - 2v)(e - v) - 2(k - v)^2 + 2r^2, \qquad l_D = d + 2e - 2k - 4v.$$

If $k \leq v \leq \tau$, which is possible only when $r = 0$, the negative part is $(v - k)(E_1 + E_2)$, and

$$V_D = 2(d - 2v)(e - v), \qquad l_D = d + 2e - 4v.$$

The positive parts are nef by the intersections with the five negative curves listed above. The threshold is τ; at its endpoint the positive part has square zero, and beyond it a nef ruling class has negative intersection.

The negative curves meet $\widetilde{D}$ only at P, Q, with C disjoint from $\widetilde{D}$. Thus there is no fixed point contribution at q. The first moment and the following point moment satisfy

$$S(W^G; D) = \frac{1}{4} \int \left(hV(u) + \int_0^\tau V_D \, dv \right) du = \frac{11}{48},$$

$$S(W^{G,D}; q) \leq \frac{1}{4} \int \int_0^\tau l_D^2 \, dv \, du = \frac{175}{192}.$$

Here are the integrations, including the factor $1/4$:

u	v	$\int V_D/4$	$\int l_D^2/4$
$[0, 1/2]$	$[0, u]$	$1/128$	$7/192$
$[1/2, 1]$	$[0, 1/2]$	$7/192$	$1/6$
$[1, 3/2]$	$[0, u - 1]$	$3/64$	$17/96$
$[1, 3/2]$	$[u - 1, 1/2]$	$7/384$	$25/192$
$[3/2, 2]$	$[0, 1/2]$	$7/96$	$13/48$
$[2, 5/2]$	$[0, (5 - 2u)/2]$	$3/128$	$25/192$.

The fixed contribution $\frac{1}{4}\int hV(u)\,du$ is $3/128$. The complete restricted series again bounds the point moment from above. Applying the flag inequality gives

$$\delta_q(G,\Delta;W^G)\geq \min\left\{\frac{1/2}{11/48},\frac{192}{175}\right\}=\frac{192}{175}.$$

□

Proof of Theorem 126.1. Every point of G above p is covered by one of the preceding two lemmas. Abban–Zhuang adjunction for the weighted curve extraction therefore gives

$$\delta_p(X)\geq \min\left\{\frac{3}{67/24},\frac{192}{175}\right\}=\frac{72}{67}>1.$$

We have not computed a Zariski decomposition on the weighted extracted threefold. The actual refined spaces were identified from its graded section algebra. All divisor decompositions used afterwards are on the explicitly described surface S. □

127 The joining line in a fibre of type (3,1)

We consider the strict transform of the line joining the points of multiplicity three and one in the corresponding plane fibre. The two endpoints are treated separately. Here the base coordinates b_0,b_1 are both nonzero.

Theorem 127.1. *At every point of this open joining line one has $\delta_p(X)\geq 96/91$.*

Lemma 127.2 (The actual series on the exceptional surface)**.** *The exceptional surface of the ordinary blow-up of the joining line is $G=\mathbb{F}_1$, and $A_X(G)=2$. Write ξ for the pullback of a line under $\mathbb{F}_1\to\mathbb{P}^2$, and f for a fibre of its ruling. Use base coordinates b_0,b_1 and fibre coordinates a,T, whose divisor classes are ξ and $\xi-f$, respectively. Put*

$$t=b_1T,\qquad x_1=b_0a-G_0T,\qquad G_0=Ab_0^2+Bb_0b_1+Cb_1^2,\qquad AC\neq 0.$$

Let

$$P_A=(b_0=T=0),\qquad P_B=(b_1=a=0),\qquad P_C=(b_1=x_1=0),$$

and let S be the blow-up of these three points. The piece of primary degree j in anticanonical degree m is as follows, and is zero for $j>3m$. For $0\leq j\leq 2m$, it is the complete system

$$H^0\left(S,j\xi+mf-\max(j-m,0)(E_A+E_B+E_C)\right).$$

For $2m\leq j\leq 3m$, put $k=j-2m$ and $n=3m-j$. It is

$$(x_1t)^k\,H^0\left(S,(m+n)\xi+nf-nE_A-mE_B-nE_C\right).$$

In these formulas ξ,f also denote their pullbacks to S.

Proof. We first justify that these are the actual section spaces. Use the hypersurface realization $X\subset\mathbb{F}_1\times\mathbb{P}^2$ of class $h+2H$; its anticanonical class is $h+f+H$. The restriction map in every anticanonical degree $m\geq 1$ is surjective. Indeed, the kernel has class $((m-1)h+mf,(m-2)H)$ and its first cohomology vanishes: for $m=1$ the factor $\mathcal{O}_{\mathbb{P}^2}(-1)$ is acyclic, and for $m\geq 2$ the divisor $(m-1)h+mf$ on $\mathbb{F}_1$ is nef. Thus ambient sections give the whole anticanonical section space.

Write the ambient equation as $s_2q=s_1F$. The weights of u,x_1,s_2 for the ordinary curve extraction are one, and those of v,x_0,x_2,s_1 are zero. The initial ambient equation is

$$s_2x_0x_2=s_1(uG_0+vx_1x_2).$$

This polynomial is irreducible: regarded as a polynomial in s_2, its two coefficients are coprime, since $AC \neq 0$. Hence its initial ring is integral and the ambient weight filtration is the geometric valuation filtration. The homogeneous quotient therefore gives the actual initial section spaces, including all cancellations.

On the exceptional surface put $x_0 = b_0$, $x_2 = b_1$, $u = t$, $s_2 = a$ and trivialize $v = s_1 = 1$. The initial relation gives the displayed equation for x_1. The ambient monomials of anticanonical degree m show that the actual degree (m, j) space is spanned by

$$a^r t^i x_1^l b_0^b b_1^c, \qquad 0 \le r \le m, \quad 0 \le i \le 2m - r, \quad l + b + c = m, \quad r + i + l = j.$$

All these sections vanish to order at least $\max(j - m, 0)$ at P_A, P_B, P_C. If $j > 2m$, the inequalities force both $l \ge j - 2m$ and $i \ge j - 2m$. Factoring $(x_1 t)^k$ gives precisely the residual vanishing conditions in the statement. This proves containment in the asserted complete systems.

Here is a dimension check which also proves equality. Regard S as the blow-up of $\mathbb{P}^2$ at four points, one infinitely near the first. Let h, e_0, e_1, e_2, e_3 be the usual orthogonal basis, with $\xi = h$, $f = h - e_0$, and $E_A = e_1, E_B = e_2, E_C = e_3$. The negative curves are

$$e_0 - e_1, \quad h - e_0 - e_2 - e_3, \quad e_1, \quad e_2, \quad e_3, \quad h - e_0 - e_1.$$

The first two have square -2 and meet once; the others have square -1. This is a weak del Pezzo surface of degree five. Intersection with this list shows that every residual divisor in the statement is nef. Riemann–Roch and vanishing therefore give its dimension as

$$d_{m,j} = \begin{cases} (j+1)(2m+j+2)/2, & 0 \le j \le m, \\ -j^2 + 4jm - 3m^2/2 + 5m/2 + 1, & m \le j \le 2m, \\ m(2n+1) + (n+1)(n+2)/2, & 2m \le j \le 3m. \end{cases}$$

Their sum is $(2m+1)(2m^2+2m+1) = h^0(X, -mK_X)$. Taking an associated graded preserves dimension, so every containment above is an equality. □

Lemma 127.3 (The expected order of the joining line)**.** *For the exceptional divisor G one has $S_X(G) = 13/8$.*

Proof. The preceding dimensions give

$$\sum_{j=0}^{3m} j d_{m,j} = \frac{m(13m^3 + 18m^2 + 12m + 3)}{2}.$$

Divide by $m(2m+1)(2m^2+2m+1)$ and pass to the limit. □

Lemma 127.4 (The surface flag over an open base point)**.** *Let $F \subset G$ be a ruling fibre with $b_0 b_1 \neq 0$. For the actual refined series, its expected order is* $271/384$*. The subsequent point expected order is at most* $91/96$ *at every point of F.*

Proof. The blow-up $S \to G$ is an isomorphism near F. Normalize the primary degree by $u = j/m$. The nef residual divisors in Lemma 127.2 are

$$D_u = \begin{cases} (u+1)h - e_0, & 0 \le u \le 1, \\ (u+1)h - e_0 - (u-1)(e_1 + e_2 + e_3), & 1 \le u \le 2, \\ (7-2u)h - (3-u)(e_0 + e_1 + e_3) - e_2, & 2 \le u \le 3. \end{cases}$$

For $u \ge 2$ there is also the fixed divisor $(u-2)((x_1 = 0) + (t = 0))$. Put $D_u - vF = P(u, v) + N(u, v)$ for its Zariski decomposition on this surface. Write $p = P(u, v) \cdot F$ and

write n for the coefficient of $E = e_0 - e_1$ in $N(u,v)$. The following table gives all chambers. A polygon is specified by its vertices (u,v); boundaries may be assigned to either adjacent chamber.

vertices	p	n
$(0,0),(1,0),(1,1),(0,1)$	u	0
$(1,1),(1,2),(0,1)$	$u-v+1$	$v-1$
$(1,0),(2,0),(1,1)$	u	0
$(2,0),(3/2,3/2),(1,1)$	$u/2-v/2+1$	$(u+v-2)/2$
$(2,0),(2,3/2),(3/2,3/2)$	$2-2v/3$	$(3u+2v-6)/3$
$(1,2),(1,1),(3/2,3/2)$	$u-v+1$	$v-1$
$(2,3/2),(2,2),(1,2),(3/2,3/2)$	$4-2v$	$u+2v-4$
$(2,0),(3,0),(2,3/2)$	$4-u-2v/3$	$2v/3$
$(3,0),(2,2),(2,3/2)$	$10-3u-2v$	$2(u+v-3)$.

These formulas follow by imposing $P\cdot C = 0$ on the negative curves in the support and checking nonnegative intersections with the six curves listed above. The other components of N are vertical and are disjoint from F. The volume is determined explicitly by

$$P(u,v)^2 = D_u^2 - 2\int_0^v p(u,s)\,ds, \qquad D_u^2 = \begin{cases} u^2+2u, & 0\le u\le 1,\\ -2u^2+8u-3, & 1\le u\le 2,\\ u^2-10u+21, & 2\le u\le 3. \end{cases}$$

Set $k=\max(u-2,0)$. Integration over the displayed polygons gives

range of u	$\iint P^2$	$\iint p^2$	$\iint 2p(n+k)$	$\iint 2pk$
$[0,1]$	$11/12$	$5/12$	$1/12$	0
$[1,2]$	$55/16$	$11/4$	$7/6$	0
$[2,3]$	$31/24$	$3/2$	$5/3$	$3/4$.

The normalization is $1/8$, since $h^0(X,-mK_X) = 4m^3+O(m^2)$ and surface volumes are twice the leading coefficient of the dimension. Thus the fibre expected order is $271/384$, and the moving point contribution is $7/12$.

The only possible fixed contributions on F are at its intersections with $E=(T=0)$ and with $(x_1=0)$. These points are distinct when $b_0b_1\neq 0$. At the first point the additional contribution is $35/96$, and at the second it is $3/32$. Consequently the point expected orders are

$$\frac{91}{96}, \qquad \frac{65}{96}, \qquad \frac{7}{12},$$

respectively at those two points and elsewhere. This proves the assertion. □

Example 127.5. In anticanonical degree one the primary pieces have dimensions $2,5,6,2$. The final piece is $x_1 t\langle a,t\rangle$. In degree two their dimensions are $3,7,12,15,16,9,3$. In particular, the fixed factors for $j>2m$ cannot be omitted.

Proof of Theorem 127.1. Surface adjunction along the ruling fibre, followed by point adjunction, gives

$$\delta(G;W^G)\big|_{b_0b_1\neq 0} \ge \min\left\{\frac{384}{271},\frac{96}{91}\right\} = \frac{96}{91}.$$

Abban–Zhuang adjunction for the ordinary curve extraction now gives

$$\delta_p(X) \ge \min\left\{\frac{16}{13},\frac{96}{91}\right\} = \frac{96}{91}.$$

No Zariski decomposition on a modification of the threefold is used. The all-degree calculation identifies the actual refined series, including its fixed factors. The displayed Zariski decompositions take place only on the blown-up exceptional surface. □

127.1 The simple endpoint

The simple endpoint of the joining line is $b_0 = 0$. Indeed, under $[x_0 : x_1 : x_2] = [s^2 : st : t^2]$, the equation $x_1 x_2 = 0$ restricts to $st^3 = 0$. Only P_A lies on the fibre of G over this endpoint. We retain the notation of Lemma 127.2. Put

$$E = e_0 - e_1, \qquad F_0 = h - e_0 - e_2 - e_3, \qquad F_A = h - e_0 - e_1, \qquad C = 2h - e_0 - e_1 - e_3.$$

Here F_A is the strict transform of the fibre $b_0 = 0$, and C is the strict transform of $(x_1 = 0)$.

Lemma 127.6. *At the simple endpoint p of the joining line one has $\delta_p(X) \geq 96/91$.*

Proof. We calculate on the same surface S. For $u = j/m$, the full pullback of the refined series has nef moving class D_u from Lemma 127.4, and fixed part

$$(u-1)_+(e_1 + e_2 + e_3) + (u-2)_+(C + E + F_0),$$

up to bounded rounding terms. These exceptional fixed coefficients are essential for the calculation at P_A.

First consider a point of $b_0 = 0$ different from P_A. For $D_u - vF_A = P + N$, the negative parts are as follows. If $0 \leq u \leq 1$, put $s = (v-1)_+$; then

$$N = sE + (v+s)e_1.$$

If $1 \leq u \leq 2$, put $r = (v+1-u)_+$, $s = (v-1)_+$ and $t = (u+v-3)_+$; then

$$N = (s+t)E + tF_0 + (r+s+t)e_1.$$

If $2 \leq u \leq 3$, put $t = (u+v-3)_+$; then

$$N = 2tE + tF_0 + 3te_1.$$

The thresholds in these three ranges are $u+1$, 2 and $6-2u$, respectively. The nef classes h for $u \leq 1$ and $h - e_2$ for $u \geq 1$ exclude larger values of v. As before, these formulas are verified by zero intersection with the support, nonnegative intersection with the six negative curves, and negative definiteness of the support. Integration gives

$$\frac{1}{8}\int\int P^2 \, dv\, du = \frac{3}{4}, \qquad \frac{1}{8}\int\int (P \cdot F_A)^2 \, dv\, du = \frac{7}{12}.$$

The only other negative curve meeting F_A is e_1, and their intersection maps to P_A. None of C, E, F_0 meets F_A. Thus at every other point of the original fibre the flag bound is at least $\min(4/3, 12/7) = 4/3$.

It remains to consider P_A. Extract this point by the ordinary blow-up; its exceptional curve is e_1 and its log discrepancy over G is two. For $D_u - ve_1 = P + N$ we give the complete negative parts in a compact form. For $0 \leq u \leq 1$,

$$N = \frac{(v-1)_+}{2}E + (v-u)_+F_A.$$

For $1 \leq u \leq 2$, set $r = (u+v-2)_+$ and $s = (3u+v-6)_+$; then

$$N = \left(\frac{r}{2} + \frac{s}{6}\right)E + \frac{s}{3}F_0 + (v-1)_+F_A.$$

For $2 \leq u \leq 3$, set $s = (3u+v-9)_+$; then

$$N = \frac{2v+s}{3}E + \frac{v+2s}{3}F_0 + se_3 + (v-1)_+F_A.$$

The thresholds are $2u+1$, 3 and $7-2u$, respectively. The nef classes $2h-e_0-e_1$ for $u \le 1$ and $2h-e_0-e_1-e_2$ for $u \ge 1$ exclude larger values. These formulas comprise twelve chambers and pass the same intersection checks. Write $l = P \cdot e_1$ and put $k = (u-2)_+$. The integrals are

$$\frac{1}{8}\int\!\!\int P^2 = \frac{107}{96}, \qquad \frac{1}{8}\int_1^3 (u-1)D_u^2\,du = \frac{65}{96},$$
$$\frac{1}{8}\int\!\!\int l^2 = \frac{115}{288}, \qquad \frac{1}{4}\int\!\!\int l(N_E+k) = \frac{79}{144},$$
$$\frac{1}{4}\int\!\!\int lN_{F_A} = \frac{101}{288}, \qquad \frac{1}{4}\int\!\!\int lk = \frac{3}{32}.$$

All double integrals use the stated thresholds and measure $dv\,du$. For example, they are evaluated by splitting at the zeros of the affine functions inside $(\ )_+$. The first expected order, including the fixed coefficient of e_1, is therefore $43/24$.

The curves E, F_A, C meet e_1 at three distinct points. In local coordinates at P_A their tangent directions are $T=0$, $b_0=0$ and $b_0-\gamma T=0$, with $\gamma \neq 0$. The point moments at these three points and elsewhere are, respectively,

$$\frac{91}{96}, \qquad \frac{3}{4}, \qquad \frac{71}{144}, \qquad \frac{115}{288}.$$

There is no different for this ordinary point extraction. Consequently

$$\delta_{P_A}(G; W^G) \ge \min\left\{\frac{48}{43}, \frac{96}{91}, \frac{4}{3}, \frac{144}{71}, \frac{288}{115}\right\} = \frac{96}{91}.$$

This covers the whole exceptional fibre over the simple endpoint. The threefold adjunction bound is then

$$\delta_p(X) \ge \min\left\{\frac{16}{13}, \frac{96}{91}\right\} = \frac{96}{91}.$$

□

128 The complete classification

Theorem 128.1. *Let X_U be a smooth member of Family 3.8, described by the basepoint-free pencil $U = \langle f, g\rangle$ of binary quartics. Then the following assertions hold.*

1. *X_U is K-stable if and only if the degree-four morphism η_U has no totally ramified point.*
2. *There is precisely one K-polystable member which is not K-stable. It is*

$$\mathrm{Bl}_{\{ux^4+vy^4=0\}}(\mathbb{P}^1 \times \mathbb{P}^2),$$

   where the second $\mathbb{P}^1$ is embedded as the smooth conic $[x:y] \mapsto [x^2 : xy : y^2]$.
3. *Every other smooth member is strictly K-semistable. Its pencil has a unique totally ramified point and has the form*

$$\langle x^4, y^4 + ax^2y^2 + bx^3y\rangle, \qquad (a,b) \neq (0,0).$$

   The isomorphism classes of these members are parametrized by $\mathbb{P}(2,3)$, with coordinates $[a:b]$ of weights two and three.

In particular, X_U is K-polystable if and only if U is GIT-polystable. Equivalently, X_U is K-polystable if and only if $(\mathbb{P}^1, \frac{1}{4}R_U)$ is K-polystable. This confirms the conjecture in [3, Section 7.6].

Proof. Suppose first that every ramification index is at most three. Proposition 121.1 reduces the point estimates to the exceptional rulings over ramification points and the two kinds of joining lines. The singularities of the del Pezzo fibres are exactly those in Lemma 117.3. The estimates established above give the following complete list.

location of p	lower bound for $\delta_p(X_U)$
$p \in E$	12/11
$p \notin E \cup \Sigma$	24/23
A_1 point of a fibre	72/67
A_2 point of a fibre	384/383
index-two ruling, away from E and its singular point	9/8
index-three exceptional ruling, away from its singular point	12/11
joining line of type $(2,2)$, away from its endpoints	72/67
joining line of type $(3,1)$, away from its A_2 endpoint	96/91.

In the index-two row, its point on E may instead be assigned to the first row. The last row includes the endpoint over the simple root. Thus there is no omitted endpoint in this list. The respective local results are Theorems 122.1, 123.1, 124.1, 126.1, 127.1 and Lemmas 125.2 and 127.6.

All local thresholds include valuations with positive-dimensional centres through the point. It follows that

$$\delta(X_U) = \inf_{p \in X_U} \delta_p(X_U) \geq \frac{384}{383} > 1.$$

The valuative criterion proves K-stability.

If there is a totally ramified point, Proposition 119.1 proves K-semistability and gives exactly one K-polystable isomorphism class, namely X_{U_0}. This member is not K-stable: the transformations $[x : y] \mapsto [\lambda x : y]$, together with the corresponding target transformations, give a nontrivial torus in its automorphism group. The remaining members and their isomorphisms are precisely those of Lemma 118.2. Finally, Proposition 118.1 and Lemma 118.3 identify these same three strata for the pencils and the log curves. This proves all the assertions. □

Example 128.2. The pencil $\langle x^4 + y^4, x^2y^2 \rangle$ belongs to the K-stable locus. The pencil $\langle x^4, y^4 \rangle$ gives the unique K-polystable member with a positive-dimensional automorphism group. The pencils

$$\langle x^4, y^4 + x^2y^2 \rangle, \qquad \langle x^4, y^4 + x^3y \rangle$$

give strictly K-semistable members representing the two coordinate points of $\mathbb{P}(2,3)$. Their special degenerations to the polystable member were constructed above.

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