

# Ricci flow on Kähler-Einstein surfaces

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**Abstract.** In this paper, we prove that if  $M$  is a Kähler-Einstein surface with positive scalar curvature, if the initial metric has nonnegative sectional curvature, and the curvature is positive somewhere, then the Kähler-Ricci flow converges to a Kähler-Einstein metric with constant bisectional curvature. In a subsequent paper [7], we prove the same result for general Kähler-Einstein manifolds in all dimension. This gives an affirmative answer to a long standing problem in Kähler Ricci flow: On a compact Kähler-Einstein manifold, does the Kähler-Ricci flow converge to a Kähler-Einstein metric if the initial metric has a positive bisectional curvature? Our main method is to find a set of new functionals which are essentially decreasing under the Kähler Ricci flow while they have uniform lower bounds. This property gives the crucial estimate we need to tackle this problem.

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## 1 Introduction

In the last two decades, the Ricci flow, introduced by R. Hamilton in [14], has been a subject of intense study. The Ricci flow provides an indispensable

tool of deforming Riemannian metrics towards canonical metrics, such as Einstein ones. It is hoped that by deforming a metric to a canonical metric, one can further understand geometric and topological structures of underlying manifolds. For instance, it was proved [14] that any closed 3-manifold of positive Ricci curvature is diffeomorphic to a spherical space form. We refer the readers to [17] for more information.

If the underlying manifold is a Kähler manifold, the Ricci flow preserves the Kähler class. It follows that the Ricci flow can be reduced to a fully nonlinear parabolic equation on functions (cf. Sect. 2 for details). Usually, this reduced flow is called the Kähler Ricci flow. Unlike the Ricci flow in the real case, it can be proved directly that the Kähler Ricci flow always has a global solution (cf. [4]). Following a similar calculation of Yau [30], Cao [4] proved that the solution converges to a Kähler-Einstein metric if the first Chern class of the underlying Kähler manifold is zero or negative. Consequently, he re-proved the famous Calabi-Yau theorem [30]. On the other hand, if the first Chern class of the underlying Kähler manifold is positive, the solution of a Kähler Ricci flow may not converge to any Kähler-Einstein metric. This is because there are compact Kähler manifolds with positive first Chern class which do not admit any Kähler-Einstein metrics (cf. [13], [27]). A natural and challenging problem is whether or not the Kähler Ricci flow on a compact Kähler-Einstein manifold converges to a Kähler-Einstein metric. It was proved by S. Bando [1] for 3-dimensional Kähler manifolds and by N. Mok [22] for higher dimensional Kähler manifolds that the positivity of bisectional curvature is preserved under the Kähler Ricci flow. A long standing problem in the study of the Ricci flow is whether or not the Kähler Ricci flow converges to a Kähler-Einstein metric if the initial metric has positive bisectional curvature. In view of the solution of the Frankel conjecture by S. Mori [23] and Siu-Yau [25], we suffice to study this problem on a Kähler manifold which is biholomorphic to  $\mathbb{C}P^n$ . Since  $\mathbb{C}P^n$  admits a Kähler-Einstein metric, the above problem can be restated as follows: on a compact Kähler-Einstein manifold, does the Kähler Ricci flow converge to a Kähler-Einstein metric? This problem was completely solved by R. Hamilton in the case of Riemann surfaces (cf. [16]). We also refer the readers to B. Chow's papers [9] for more developments on this problem. In this paper, we give an affirmative answer to this problem in dimension two.

**Theorem 1.1.** <sup>1</sup> *Let  $M$  be a Kähler-Einstein surface with positive scalar curvature. If the initial metric has nonnegative bisectional curvature and positive at least at one point, then the Kähler Ricci flow will converge exponentially fast to a Kähler-Einstein metric with constant bisectional curvature.*

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<sup>1</sup> In a subsequent paper [7], we will prove the same theorem for all dimensions. The proof for higher dimensions needs new ingredients. Both results were announced in [8].

**Corollary 1.2.** *The space of Kähler metrics with non-negative bisectional curvature (and positive at least at one point) is path-connected. Moreover, the space of metrics with non-negative curvature operator (and positive at least at one point) is also path-connected.*

*Remark 1.3.* Using the same arguments, we can also prove the version of our main theorem for Kähler orbifolds.

*Remark 1.4.* What we really need is that the Ricci curvature is positive for evolved metric under the Ricci flow. Since the condition on Ricci may not be preserved under the Ricci flow, in order to have the positivity of the Ricci curvature, we will use the fact that the positivity of the bisectional curvature is preserved.

*Remark 1.5.* We need the assumption on the existence of Kähler-Einstein metric because we will use a nonlinear inequality from [28]. Such an inequality is nothing but the Moser-Trudinger-Onofri type if the Kähler-Einstein manifold is the Riemann sphere.

The typical method in studying the Ricci flow depends on pointwise bounds of the curvature tensor by using its evolution equation as well as the blow-up analysis. In order to prevent formation of singularities, one blows up the solution of the Ricci flow to obtain profiles of singular solutions. Those profiles involve Ricci solitons and possibly more complicated singular models. Then one tries to exclude formation of singularities by checking that these solitons or models do not exist under appropriate global geometric conditions. It is a common sense that it is very difficult to detect how the global geometry affects those singular models even for a very simple manifold like  $\mathbb{C}P^2$ . The first step is to classify those singular models and hope to find their geometric information. Of course, it is already a very big task. There have been many exciting works on these (cf. [17]).

Our new contribution is to find a set of new functionals which are the Lagrangians of certain new curvature equations involving various symmetric functions of the Ricci curvature. We show that these functionals decrease essentially along the Kähler Ricci flow and have uniform lower bound. By computing their derivatives, we can obtain certain integral bounds on curvature of metrics along the flow.

For the readers' convenience, we will discuss more on these new functionals. Let  $M$  be a compact Kähler manifold with positive first Chern class  $c_1(M)$  and  $\omega$  be a fixed Kähler metric on  $M$  with the Kähler class  $c_1(M)$ . Consider the following expansion

$$(\omega + tRic(\omega))^n = \left( \sum_{k=0}^n \sigma_k(\omega)t^k \right) \omega^n, \tag{1.1}$$

where  $\sigma_k(\omega)$  is the  $k$ -th symmetric polynomial of the Ricci tensor  $Ric(\omega)$ . Then we say that a Kähler metric  $\omega$  is of extremal  $k$ -th symmetric Ricci

curvature ( $k = 0, 1, \dots, n$ ) if  $\sigma_k(\omega)$  satisfies

$$\Delta\sigma_k(\omega) - \frac{n - k}{k + 1}\sigma_{k+1}(\omega) = c_k, \tag{1.2}$$

where  $c_k$  is a constant determined by  $c_1(M)$  and the Kähler class  $[\omega]$ . Clearly, the extremal 0-symmetry means constant scalar curvature. When the first Chern class  $c_1(M)$  of  $M$  is positive and  $\omega$  represents  $c_1(M)$ , a Kähler metric with constant scalar curvature is of constant Ricci curvature and consequently, has extremal  $k$ -th symmetric Ricci curvature for all  $k$ . In general, a Kähler metric of constant scalar curvature may not have extremal  $k$ -th symmetric Ricci curvature for  $k > 1$ .

Our new functionals  $E_k$  are simply the Lagrangians of the above Ricci curvature equations (cf. Sect. 4 for details). When  $k = 0$ , the functional  $E_0$  is nothing but the K-energy of T. Mabuchi. We will prove that the derivative of each  $E_k$  along an orbit of automorphisms gives rise to a holomorphic invariant  $\mathfrak{S}_k$ , including the well-known Futaki invariant as a special one. When  $M$  admits a Kähler-Einstein metric, all these invariants  $\mathfrak{S}_k$  vanish, so the functionals  $E_k$  are invariant under the action of automorphisms.

Next we will prove that these  $E_k$  are bounded from below. This can be achieved by making use of a fully nonlinear inequality from [28] (cf. Sect. 5). But in order to apply this inequality, we have to adjust the fixed Kähler-Einstein metric so that the evolved Kähler metrics are centrally positioned with respect to the adjusted Kähler-Einstein metrics, that is, the Kähler potentials between the two evolved metrics are orthogonal to the first eigenspace of the evolved Kähler-Einstein metrics (cf. Sect. 6). It causes some extra difficulties in the proof of our main theorem (particularly in higher dimensions).

Next we will compute the derivatives of  $E_k$  along the Kähler-Ricci flow. Recall that the Kähler Ricci flow is given by

$$\frac{\partial\varphi}{\partial t} = \log \frac{(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\omega^n} + \varphi - h_\omega, \tag{1.3}$$

where  $h_\omega$  depends only  $\omega$ . The derivatives of these functionals are all bounded uniformly from above along the Kähler Ricci flow. Furthermore, we found that  $E_0$  and  $E_1$  decrease along the Kähler Ricci flow. These play a very important role in this and the subsequent paper. We can derive from these properties of  $E_k$  integral bounds on curvature, e.g. for almost all Kähler metrics  $\omega_{\varphi(t)}$  along the flow, we have

$$\int_M R(\omega_{\varphi(t)}) \text{Ric}(\omega_{\varphi(t)})^k \wedge \omega_{\varphi(t)}^{n-k} \leq C, \quad k = 1, \dots, n, \tag{1.4}$$

and

$$\int_M (R(\omega_{\varphi(t)}) - r)^2 \omega_{\varphi(t)}^n \rightarrow 0, \tag{1.5}$$

where  $R(\omega_{\varphi(t)})$  denotes the scalar curvature and  $r$  is the average scalar curvature.

In principle, one can then follow Hamilton’s arguments in the case of Riemann surfaces. But we need to do some changes since the sectional curvature may not be positive and we can not apply Klingenberg’s estimate on injectivity radius. We will generalize Klingenberg’s estimate to Kähler manifolds of positive bisectional curvature. Then, combining the above integral bounds on the curvature with Cao’s Harnack inequality and the generalization of Klingenberg’s estimate, we can bound the curvature uniformly along the Kähler Ricci flow in the case of Kähler-Einstein surfaces. Then it is quite routine to prove the convergence to the Kähler-Einstein metric. For higher dimensions, one has to develop new techniques in order to get the curvature bound. We will do it in a subsequent paper [7].

The organization of our paper is roughly as follows: In Sect. 2, we review briefly some basics in Kähler geometry and necessary information on the Kähler Ricci flow. In Sect. 3, we discuss two important energy functionals. In Sect. 4, we introduce a set of new functionals as we have briefly described in the above. In Sect. 5, we prove that these functionals are invariant on any Kähler-Einstein manifolds. In Sect. 6, we modify the evolved Kähler metrics to obtain desired integral estimates on the curvature. In Sects. 7, 8, 9, we will bound the scalar curvature uniformly along the Kähler Ricci flow. In Sect. 10, we prove the exponentially convergence. In Sect. 11, we make some concluding remarks and propose some open questions.

## 2 Basic Kähler geometry

### 2.1 Notations in Kähler geometry

Let  $M$  be an  $n$ -dimensional compact Kähler manifold. A Kähler metric can be given by its Kähler form  $\omega$  on  $M$ . In local coordinates  $z_1, \dots, z_n$ , this  $\omega$  is of the form

$$\omega = \sqrt{-1} \sum_{i,j=1}^n g_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}},$$

where  $\{g_{i\bar{j}}\}$  is a positive definite Hermitian matrix function. The Kähler condition requires that  $\omega$  is a closed positive (1,1)-form. In other words, the following holds

$$\frac{\partial g_{i\bar{k}}}{\partial z^j} = \frac{\partial g_{j\bar{k}}}{\partial z^i} \quad \text{and} \quad \frac{\partial g_{k\bar{i}}}{\partial z^{\bar{j}}} = \frac{\partial g_{k\bar{j}}}{\partial z^{\bar{i}}} \quad \forall i, j, k = 1, 2, \dots, n.$$

The Kähler metric corresponding to  $\omega$  is given by

$$\sqrt{-1} \sum_1^n g_{\alpha\bar{\beta}} dz^\alpha \otimes dz^{\bar{\beta}}.$$

For simplicity, in the following, we will often denote by  $\omega$  the corresponding Kähler metric. The Kähler class of  $\omega$  is its cohomology class  $[\omega]$  in  $H^2(M, \mathbb{R})$ . By the Hodge theorem, any other Kähler metric in the same Kähler class is of the form

$$\omega_\varphi = \omega + \sqrt{-1} \sum_{i,j=1}^n \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j} dz_i \wedge d\bar{z}_j > 0$$

for some real value function  $\varphi$  on  $M$ . The functional space in which we are interested (often referred as the space of Kähler potentials) is

$$\mathcal{P}(M, \omega) = \{\varphi \mid \omega_\varphi = \omega + \sqrt{-1} \partial\bar{\partial}\varphi > 0 \text{ on } M\}.$$

Given a Kähler metric  $\omega$ , its volume form is

$$\omega^n = (\sqrt{-1})^n \det(g_{i\bar{j}}) dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n.$$

Its Christoffel symbols are given by

$$\Gamma_{ij}^k = \sum_{l=1}^n g^{k\bar{l}} \frac{\partial g_{i\bar{l}}}{\partial z^j} \quad \text{and} \quad \Gamma_{i\bar{j}}^{\bar{k}} = \sum_{l=1}^n g^{\bar{k}l} \frac{\partial g_{i\bar{l}}}{\partial \bar{z}^j}, \quad \forall i, j, k = 1, 2, \dots, n.$$

The curvature tensor is

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 g_{i\bar{j}}}{\partial z^k \partial \bar{z}^l} + \sum_{p,q=1}^n g^{p\bar{q}} \frac{\partial g_{i\bar{q}}}{\partial z^k} \frac{\partial g_{p\bar{j}}}{\partial \bar{z}^l}, \quad \forall i, j, k, l = 1, 2, \dots, n.$$

We say that  $\omega$  is of nonnegative bisectional curvature if

$$R_{i\bar{j}k\bar{l}} v^j \bar{v}^i w^l \bar{w}^k \geq 0$$

for all non-zero vectors  $v$  and  $w$  in the holomorphic tangent bundle of  $M$ . The bisectional curvature and the curvature tensor can be mutually determined by each other (cf. the appendix for more information). The Ricci curvature of  $\omega$  is locally given by

$$R_{i\bar{j}} = -\frac{\partial^2 \log \det(g_{k\bar{l}})}{\partial z_i \partial \bar{z}_j}.$$

So its Ricci curvature form is

$$\text{Ric}(\omega) = \sqrt{-1} \sum_{i,j=1}^n R_{i\bar{j}}(\omega) dz^i \wedge d\bar{z}^j = -\sqrt{-1} \partial\bar{\partial} \log \det(g_{k\bar{l}}).$$

It is a real, closed (1,1)-form. Recall that  $[\omega]$  is a canonical Kähler class if this Ricci form is cohomologous to  $\lambda \omega$ , for some constant  $\lambda$

### 2.2 The Kähler Ricci flow

Now we assume that the first Chern class  $c_1(M)$  is positive. The Ricci flow (see for instance [14] and [15]) on a Kähler manifold  $M$  is of the form

$$\frac{\partial g_{i\bar{j}}}{\partial t} = g_{i\bar{j}} - R_{i\bar{j}}, \quad \forall i, j = 1, 2, \dots, n. \tag{2.1}$$

If we choose the initial Kähler metric  $\omega$  with  $c_1(M)$  as its Kähler class. Then the flow (2.1) preserves the Kähler class  $[\omega]$ . It follows that on the level of Kähler potentials, the Ricci flow becomes

$$\frac{\partial \varphi}{\partial t} = \log \frac{\omega_\varphi^n}{\omega^n} + \varphi - h_\omega, \tag{2.2}$$

where  $h_\omega$  is defined by

$$\text{Ric}(\omega) - \omega = \sqrt{-1} \partial \bar{\partial} h_\omega, \text{ and } \int_M (e^{h_\omega} - 1) \omega^n = 0.$$

As usual, the flow (2.2) is referred as the Kähler Ricci flow on  $M$ . Differentiating on both sides of equation (2.2) on  $t$ , we obtain

$$\frac{\partial}{\partial t} \frac{\partial \varphi}{\partial t} = \Delta_\varphi \frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial t},$$

where  $\Delta_\varphi$  is the Laplacian operator of the metric  $\omega_\varphi$ . Then it follows from the standard Maximum Principle

**Lemma 2.1.** *Along the Kähler Ricci flow (2.1),  $|\frac{\partial \varphi}{\partial t}|$  grows at most exponentially.*

In particular, the  $C^0$ -norm of  $\varphi$  can be bounded by a constant depending only  $t$ . Using this fact and following Yau’s calculation in [30], one can prove that for any initial metric with Kähler class  $c_1(M)$ , the evolution equation (2.2) has a global solution for all time  $0 \leq t < \infty$  (cf. [4]).

### 2.3 Preservation of nonnegative bisectional curvature

The Kähler Ricci flow induces an evolution equation on the bisectional curvature

$$\begin{aligned} \frac{\partial}{\partial t} R_{i\bar{j}k\bar{l}} &= \Delta R_{i\bar{j}k\bar{l}} + R_{i\bar{j}p\bar{q}} R_{q\bar{p}k\bar{l}} - R_{i\bar{p}k\bar{q}} R_{p\bar{j}q\bar{l}} + R_{i\bar{l}p\bar{q}} R_{q\bar{p}k\bar{j}} + R_{i\bar{j}k\bar{l}} \\ &\quad - \frac{1}{2} (R_{i\bar{p}} R_{p\bar{j}k\bar{l}} + R_{p\bar{j}} R_{i\bar{p}k\bar{l}} + R_{k\bar{p}} R_{i\bar{j}p\bar{l}} + R_{p\bar{l}} R_{i\bar{j}k\bar{p}}). \end{aligned}$$

Similarly, we have evolution equation for the Ricci tensor and the scalar curvature

$$\frac{\partial}{\partial t} R_{i\bar{j}} = \Delta R_{i\bar{j}} + R_{i\bar{k}} R_{i\bar{j}k\bar{l}} - R_{i\bar{k}} R_{k\bar{j}},$$

and

$$\frac{\partial}{\partial t} R = \Delta R + |\text{Ric}|^2 - R.$$

The following theorem was proved by S. Bando for 3-dimensional compact Kähler manifolds. This was later generalized to all dimensional Kähler manifolds by N. Mok in [22]. Their proof used Hamilton’s Maximum Principle for tensors. The proof for higher dimensions is quite intriguing.

**Theorem 2.2.** [1], [22] *Under the Kähler Ricci flow, if the initial metric has nonnegative bisectional curvature, then the evolved metrics also have nonnegative bisectional curvature. Furthermore, if the bisectional curvature of the initial metric is positive at least at one point, then the evolved metric has positive bisectional curvature at all points.*

Previously, R. Hamilton proved (by using his Maximum principle for tensors)

**Theorem 2.3.** *Under the Kähler Ricci flow, if the initial metric has nonnegative curvature operator, then the evolved metrics also have non-negative curvature operator. Furthermore, if the curvature operator of the initial metric is positive at least at one point, then the evolved metric has positive curvature operator at all points.*

It is still interesting to see if similar conclusion holds for sectional curvature, that is, if the initial metric has nonnegative sectional curvature, do evolved metrics along the Kähler Ricci flow have nonnegative sectional curvature? If so, our theorem will imply that there is no exotic Kähler metric with positive sectional curvature on complex projective spaces.

### 3 Generalized energy functionals

In this section, we will introduce some generalized energy functionals  $J_\omega$ ,  $F_\omega$  and  $\nu_\omega$ . The second functional was first used in [11], while the 3rd one was introduced by T. Mabuchi. These are all useful functionals in Kähler geometry. We then review some known properties of  $F_\omega$  and  $\nu_\omega$ , such as a) they both decrease under the Kähler Ricci flow; b) They are both invariant under automorphisms on any Kähler-Einstein manifolds.

### 3.1 A nonlinear inequality

Recall that the generalized energy:

$$J_\omega(\varphi) = \frac{1}{V} \sum_{i=0}^{n-1} \int_M \frac{i+1}{n+1} \sqrt{-1} \partial\varphi \wedge \bar{\partial}\varphi \wedge \omega^i \wedge \omega_\varphi^{n-1-i}. \tag{3.1}$$

where  $V = \int_M \omega^n = [\omega]^n([V])$  and  $\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$ . This is clearly a positive functional. When  $n = 1$ , it is just the standard Dirichlet energy

$$J_\omega(\varphi) = \frac{1}{2V} \int_M \sqrt{-1} \partial\varphi \wedge \bar{\partial}\varphi = \frac{1}{2V} \int_M |\partial\varphi|^2 \omega.$$

If  $n = 2$ , we have

$$J_\omega(\varphi) = \frac{1}{3V} \int_M \sqrt{-1} \partial\varphi \wedge \bar{\partial}\varphi \wedge \omega_\varphi + \frac{2}{3V} \int_M \sqrt{-1} \partial\varphi \wedge \bar{\partial}\varphi \wedge \omega.$$

Taking derivative of  $J_\omega$  along a path  $\varphi(t) \in \mathcal{P}(M, \omega)$ , we arrive at

$$\frac{d J_\omega(\varphi)}{d t} = -\frac{1}{V} \int_M \frac{\partial\varphi}{d t} (\omega_\varphi^n - \omega^n).$$

Alternatively, one can use this formula to define  $J_\omega$ . From this formula, one can see that  $J_\omega$  does not satisfy the cocycle condition. Recall that the functional  $F_\omega$  is defined by

$$F_\omega(\varphi) = J_\omega(\varphi) - \frac{1}{V} \int_M \varphi \omega^n - \log \left( \frac{1}{V} \int_M e^{h_\omega - \varphi} \right).$$

It satisfies the cocycle condition and its critical points are Kähler-Einstein metrics. If  $n = 1$ , then  $M = S^2$  and

$$F_\omega(\varphi) = \frac{1}{2V} \int_{S^2} |\partial\varphi|^2 - \frac{1}{V} \int_{S^2} \varphi \omega - \log \frac{1}{V} \int_{S^2} e^{h_\omega - \varphi} \omega.$$

This is precisely the functional in studying L. Nirenberg’s problem of prescribing the Gauss curvature on  $S^2$ .

Suppose that  $M$  has positive first Chern class and admits a Kähler-Einstein metric. Then there is a Kähler-Einstein metric  $\omega_1$  such that  $\text{Ric}(\omega_1) = \omega_1$ . We will denote by  $\Lambda_1$  the space of eigenfunctions with eigenvalue one if one is an eigenvalue of the Kähler-Einstein metric  $\omega_1$ . If one is not an eigenvalue, we simply put  $\Lambda_1$  to be  $\{0\}$ . By  $\phi \perp \Lambda_1$ , we mean  $\int_M \phi \psi \omega_1^n = 0$  for all  $\psi \in \Lambda_1$ . Note that if  $M$  admits no holomorphic vector fields, then  $\Lambda_1 = \{0\}$  and  $\phi \perp \Lambda_1$  is automatically true.

The following inequality plays an important role in our proof.

**Theorem 3.1.** (Tian) [27] Let  $M$  be given as above and  $\omega_1$  be any Kähler metric with  $c_1(M)$  as its Kähler class. Then there exist constants  $\delta = \delta(n)$  and  $c = c(M, \omega_1) \geq 0$  such that for any  $\phi \in \mathcal{H}$  which satisfies  $\phi \perp \Lambda_1$ , we have

$$F_{\omega_1}(\phi) \geq J_{\omega_1}(\phi)^\delta - c$$

which is the same as

$$\frac{1}{V} \int_M e^{-\phi} \omega_1^n \leq C e^{J_{\omega_1}(\phi) - \frac{1}{V} \int_M \phi \omega_1^n - J_{\omega_1}(\phi)^\delta}.$$

*Remark 3.2.* This inequality was first proved under an extra condition, which was removed later in [29].

*Remark 3.3.* Since the difference of  $J_\omega$  and  $J_{\omega_1}$  (resp.  $F_\omega$  and  $F_{\omega_1}$ ) is bounded by a constant depending only on  $\omega$  and  $\omega_1$ , the inequality in the above theorem holds irrelevant of choices of initial metrics.

Inspired by the work of Donaldson [12], T. Mabuchi introduced the K-energy.

**Definition 3.4.** (Mabuchi [21]) For any  $\varphi(t) \in \mathcal{P}$ , the derivative of the K-energy along this path  $\varphi(t)$  is:

$$\frac{d}{dt} v_\omega(\varphi(t)) = -\frac{1}{V} \int_M \frac{\partial \varphi}{\partial t} (R(\varphi(t)) - r) \omega_\varphi^n,$$

where  $r$  is the average value of the scalar curvature  $r = \frac{[c_1(M)] \cdot [\omega]^{n-1}}{[\omega]^n}$ .

It was found in [26] that the K-energy can be expressed as

$$\begin{aligned} v_\omega(\varphi) &= \frac{1}{V} \int_M \log \left( \frac{\omega_\varphi^n}{\omega^n} \right) \omega_\varphi^n - \frac{1}{V} \int_M h_\omega(\omega^n - \omega_\varphi^n) \\ &\quad - \frac{\sqrt{-1}}{V} \sum_{i=0}^{n-1} \int_M \partial \varphi \wedge \bar{\partial} \varphi \wedge \omega^i \wedge \omega_\varphi^{n-i-1}. \end{aligned} \tag{3.2}$$

It was also observed in [11] that

$$v_\omega(\varphi) \geq F_\omega(\varphi) - \frac{1}{V} \int_M h_\omega \omega^n.$$

Combining this with the above theorem, we get

**Corollary 3.5.** [27] Suppose that  $c_1(M) > 0$  and there is a Kähler-Einstein metric on  $M$ . Then for any Kähler metric  $\omega$  with  $c_1(M)$  as its Kähler class, there are constants  $\delta = \delta(n)$  and  $c = c(M, \omega) \geq 0$  such that for any  $\varphi \in \mathcal{P}(M, \omega)$  which satisfies  $\varphi \perp \Lambda_1$ , we have

$$v_\omega(\phi) \geq J_\omega(\phi)^\delta - c,$$

The following corollary will be crucial in our arguments.

**Corollary 3.6.** *Suppose that  $c_1(M) > 0$  and there is a Kähler-Einstein metric on  $M$ . Then for any function  $\varphi \in \mathcal{P}(M, \omega)$  perpendicular to  $\Lambda_1$ , we have*

$$\int_M \log \left( \frac{\omega_\varphi^n}{\omega^n} \right) \omega_\varphi^n \leq C(1 + \nu_\omega(\varphi))^{\frac{1}{3}},$$

where  $C$  is a constant depending only on  $M$  and  $\omega$ .

### 3.2 Monotonicity along the Kähler Ricci flow

First we collect two simple facts which were known to experts in the field for a while.

**Lemma 3.7.** *Under the Kähler Ricci flow,  $F_\omega$  decreases monotonely.*

*Proof.* Let  $c = \log \left( \frac{1}{V} \int_M e^{h_\omega - \varphi} \right)$ . Then

$$\begin{aligned} \frac{d}{dt} F_\omega(\varphi(t)) &= -\frac{1}{V} \int_M \frac{\partial \varphi}{\partial t} (\omega_\varphi^n - e^{h_\omega - \varphi - c} \omega^n) \\ &= -\frac{1}{V} \int_M \left( \log \frac{\omega_\varphi^n}{\omega^n} + \varphi - h_\omega \right) \left( \frac{\omega_\varphi^n}{\omega^n} - e^{h_\omega - \varphi - c} \right) \omega^n \\ &= -\frac{1}{V} \int_M \left( \log \frac{\omega_\varphi^n}{\omega^n} - (h_\omega - \varphi - c) \right) \left( \frac{\omega_\varphi^n}{\omega^n} - e^{h_\omega - \varphi - c} \right) \omega^n \\ &\leq 0. \end{aligned}$$

□

Similarly, we have

**Lemma 3.8.** *Under the Kähler Ricci flow, the K-energy  $\nu_\omega$  monotonely decreases!*

*Proof.* By the definition, we have

$$\begin{aligned} \frac{d}{dt} \nu_\omega(\varphi(t)) &= -\frac{1}{V} \int_M \frac{\partial \varphi}{\partial t} (R(\omega_{\varphi(t)}) - r) \omega_{\varphi_t}^n \\ &= \frac{1}{V} \int_M \left( \log \frac{\omega_{\varphi_t}^n}{\omega^n} + \varphi - h_\omega \right) \left( \Delta_{\varphi_t} \left( \log \frac{\omega_{\varphi_t}^n}{\omega^n} + \varphi - h_\omega \right) \right) \omega_\varphi^n \\ &\leq 0. \end{aligned}$$

The lemma follows.

□

*Remark 3.9.* The monotonicity of  $\nu_\omega$  and  $F_\omega$  along the continuous flow and the Ricci flow were known before from the works of Bando-Mabuchi [2], Ding-Tian [11], [10] and Cao H. D. .

Next we want to prove that  $F_\omega$  and  $\nu_\omega$  are both invariant under automorphisms on a Kähler-Einstein manifold.

Recall that the Futaki invariant  $f_M$  can be defined by (see [13])

$$f_M(\omega, X) = \int_M X(h_\omega)\omega^n,$$

where  $\omega$  is a Kähler metric with  $c_1(M)$  as its Kähler class and  $X$  is a holomorphic vector field on  $M$ . Futaki proved that the integral is independent of the choice of  $\omega$ , so it gives rise to a holomorphic invariant. If  $M$  admits a Kähler-Einstein metric, then  $f_M \equiv 0$ .

Let  $\Phi_t$  be a one-parameter group of automorphisms generated by  $\text{Re}(X)$ . Write  $\omega_t = \Phi_t^*\omega = \omega + \sqrt{-1}\partial\bar{\partial}\varphi_t$ . We can further normalize  $\varphi_t$  such that  $\int_M (e^{h_\omega - \varphi_t} - 1)\omega^n = 0$ . Then  $h_{\omega_t} = \Phi_t^*h_\omega$ . This implies that  $\dot{h}_{\omega_t} = \text{Re}(X)(h_{\omega_t})$ .

On the other hand, using the identity

$$\text{Ric}(\omega_t) - \omega_t = \text{Ric}(\omega) - \omega - \sqrt{-1}\partial\bar{\partial} \log\left(\frac{\omega_t^n}{\omega^n}\right) - \sqrt{-1}\partial\bar{\partial}\varphi_t,$$

we get

$$h_{\omega_t} = h_\omega - \log\left(\frac{\omega_t^n}{\omega^n}\right) - \varphi_t.$$

Differentiating it with respect to  $t$ , we have

$$\dot{h}_{\omega_t} = -\Delta_{\omega_t}\dot{\varphi}_t - \dot{\varphi}_t.$$

Combining all these, we arrive at

$$\frac{d}{dt}F_\omega(\varphi_t) = \frac{1}{V} \text{Re}(f_M(X)).$$

The following corollary is an immediate consequence of this.

**Lemma 3.10.** *The functional  $F_\omega$  is invariant under automorphisms if  $f_M \equiv 0$ . In particular, it is true if  $M$  is a Kähler-Einstein manifold.*

Similarly, we have

**Lemma 3.11.** *On a Kähler-Einstein manifold,  $\nu_\omega$  is invariant under automorphisms.*

We deduce the following from the above

**Proposition 3.12.** *Suppose that  $M$  admits a Kähler-Einstein metric. Let  $\varphi(t)$  ( $t > 0$ ) be a global solution of the Kähler Ricci flow and  $\Psi(t)$  be a family of automorphisms of  $M$ . Write*

$$\Psi_t^* \omega_{\varphi(t)} = \omega + \sqrt{-1} \partial \bar{\partial} \psi(t).$$

Then  $F_\omega(\psi_t)$  and  $v_\omega(\psi(t))$  are decreasing functions of  $t$ .

Combining this with Tian’s inequality last subsection, we get

**Corollary 3.13.** *Suppose that  $\omega_1$  is a Kähler-Einstein metric with  $\text{Ric}(\omega_1) = \omega_1$ . Let  $\varphi(t)$  ( $t > 0$ ) be a global solution of the Kähler Ricci flow and  $\Psi_t$  be a family of automorphisms of  $M$ . Write*

$$\Psi_t^* \omega_{\varphi(t)} = \omega + \sqrt{-1} \partial \bar{\partial} \psi(t).$$

If  $\psi(t)$  is perpendicular to the eigenspace of  $\omega_1$  with eigenvalue one, then

$$J_\omega(\psi(t)) \leq v_\omega(\varphi(0)) + c,$$

where  $c$  is a uniform constant.

### 3.3 Estimate on volume forms

The following is the main result of this subsection.

**Proposition 3.14.** *If  $\text{Ric}(\omega_\varphi) \geq 0$ , then there exists a uniform constant  $C$  such that*

$$\inf_M \left( \log \frac{\omega_\varphi^n}{\omega^n} \right) (x) \geq -4C e^{2(1 + \int_M \left( \log \frac{\omega_\varphi^n}{\omega^n} \right) \omega_\varphi^n)}.$$

*Proof.* Choose any constant  $c$  such that

$$\frac{1}{V} \int_M \log \frac{\omega_\varphi^n}{\omega^n} \omega_\varphi^n \leq c,$$

where  $V = \int_M \omega^n$ .

Put  $\epsilon$  to be  $e^{-2(1+c)}$ . Observe that

$$\left( \log \frac{\omega_\varphi^n}{\omega^n} \right) \omega_\varphi^n \geq -e^{-1} \omega^n,$$

we have

$$\begin{aligned}
 cV &\geq \int_{\epsilon\omega_{\varphi_i}^n > \omega^n} \left( \log \frac{\omega_\varphi^n}{\omega^n} \right) \omega_\varphi^n + \int_{\epsilon\omega_{\varphi_i}^n \leq \omega^n} \left( \log \frac{\omega_\varphi^n}{\omega^n} \right) \omega_\varphi^n \\
 &\geq \int_{\epsilon\omega_{\varphi_i}^n > \omega^n} \left( \log \frac{1}{\epsilon} \right) \omega_\varphi^n + \int_{\epsilon\omega_{\varphi_i}^n \leq \omega^n} (-e^{-1} \omega^n) \\
 &> 2(1+c) \int_{\epsilon\omega_{\varphi_i}^n > \omega^n} \omega_\varphi^n - V.
 \end{aligned}$$

It follows that

$$\int_{\epsilon\omega_{\varphi_i}^n > \omega^n} \omega_\varphi^n < \frac{V}{2},$$

and consequently,

$$\int_{\omega^n \leq 4\omega_{\varphi_i}^n} \omega^n \geq \epsilon \int_{\frac{\epsilon}{4}\omega^n \leq \epsilon\omega_{\varphi_i}^n \leq \omega^n} \omega_\varphi^n > \frac{\epsilon V}{4}.$$

The Ricci curvature being non-negative implies that

$$\omega + \sqrt{-1} \partial \bar{\partial} \left( h_\omega - \log \frac{\omega_\varphi^n}{\omega^n} \right) \geq 0.$$

Taking trace with respect to  $\omega$ , we get

$$n + \Delta \left( h_\omega - \log \frac{\omega_\varphi^n}{\omega^n} \right) \geq 0,$$

where  $\Delta$  denotes the Laplacian of  $\omega$ . Then by the Green formula, we have

$$\begin{aligned}
 &\left( h_\omega - \log \frac{\omega_\varphi^n}{\omega^n} \right) (x) \\
 &= \frac{1}{V} \int_M \left( h_\omega - \log \frac{\omega_\varphi^n}{\omega^n} \right) \omega^n - \frac{1}{V} \int_M \Delta \left( h_\omega - \log \frac{\omega_\varphi^n}{\omega^n} \right) G(x, y) \omega^n(y) \\
 &\leq \frac{1}{V} \int_M \left( h_\omega - \log \frac{\omega_\varphi^n}{\omega^n} \right) \omega^n - \frac{n}{V} \int_M G(x, y) \\
 &\leq \frac{1}{V} \int_M \left( h_\omega - \log \frac{\omega_\varphi^n}{\omega^n} \right) \omega^n + c',
 \end{aligned}$$

where  $G(x, y) \geq 0$  is a Green function of  $\omega$ . Note that we will always denote by  $c'$  a constant depending only on  $\omega$  in this proof.

It follows from the above inequalities that

$$\begin{aligned} & \inf_M \left( \log \frac{\omega_\varphi^n}{\omega^n} \right) \\ & \geq \frac{1}{V} \int_M \left( \log \frac{\omega_\varphi^n}{\omega^n} \right) \omega^n - c' \\ & \geq \inf_M \left( \log \frac{\omega_\varphi^n}{\omega^n} \right) \frac{1}{V} \int_{\omega^n \geq 4\omega_\varphi^n} \omega^n - \frac{\log 4}{V} \int_{\omega^n \leq 4\omega_\varphi^n} \omega^n - c' \\ & \geq \left( 1 - \frac{\epsilon}{4} \right) \inf_M \left( \log \frac{\omega_\varphi^n}{\omega^n} \right) - c'. \end{aligned}$$

Therefore, we have

$$\inf_M \left( \log \frac{\omega_\varphi^n}{\omega^n} \right) (x) \geq -4c'e^{2(1+c)}.$$

By the way we choose the constant  $c$  in the beginning of the proof, we have

$$\inf_M \left( \log \frac{\omega_\varphi^n}{\omega^n} \right) (x) \geq -4c'e^{2(1 + \int_M \left( \log \frac{\omega_\varphi^n}{\omega_\varphi^n} \right) \omega^n)}.$$

The proposition is proved. □

### 4 New functionals

In this section, we introduce a family of new functionals on the space of Kähler potentials  $\mathcal{P}(M, \omega)$ . We will show that their derivatives along the Kähler Ricci flow are bounded uniformly from above.

#### 4.1 Definition of functionals $E_k$

In this subsection, we introduce  $E_k$  for  $k = 0, 1, \dots, n$ .

**Definition 4.1.** For any  $k = 0, 1, \dots, n$ , we define a functional  $E_k^0$  on  $\mathcal{P}(M, \omega)$  by

$$E_{k,\omega}^0(\varphi) = \frac{1}{V} \int_M \left( \log \frac{\omega_\varphi^n}{\omega^n} - h_\omega \right) \left( \sum_{i=0}^k \text{Ric}(\omega_\varphi)^i \wedge \omega^{k-i} \right) \wedge \omega_\varphi^{n-k}.$$

If there is no possible confusion, we will often drop the subscript  $\omega$  in the following.

*Remark 4.2.* If  $k = n = 1$ , then

$$E_1^0 = \frac{1}{V} \int_M \left( \log \frac{\omega_\varphi}{\omega} - h_\omega \right) (R \omega_\varphi + 1 \cdot \omega).$$

This is analogous to the well-known Liouville energy on Riemann surfaces.

Next for each  $k = 0, 1, 2, \dots, n - 1$ , we will define  $J_{k,\omega}$  as follows: Let  $\varphi(t)$  ( $t \in [0, 1]$ ) be a path from 0 to  $\varphi$  in  $\mathcal{P}(M, \omega)$ , we define

$$J_{k,\omega}(\varphi) = -\frac{n-k}{V} \int_0^1 \int_M \frac{\partial \varphi}{\partial t} (\omega_\varphi^{k+1} - \omega^{k+1}) \wedge \omega_\varphi^{n-k-1} \wedge dt.$$

One can show that the integral on the right is independent of choices of the path. This is because  $\mathcal{P}(M, \omega)$  is simply-connected and its derivative has nothing to do with the path. Clearly, we have

$$\frac{dJ_{k,\omega}}{dt} = -\frac{n-k}{V} \int_M \frac{\partial \varphi}{\partial t} (\omega_\varphi^{k+1} - \omega^{k+1}) \wedge \omega_\varphi^{n-k-1},$$

For simplicity, we will often drop the subscript  $\omega$  in the following.

*Remark 4.3.* If  $k = n - 1$ , then

$$\frac{dJ_{n-1}}{dt} = -\frac{1}{V} \int_M \frac{\partial \varphi}{\partial t} (\omega_\varphi^n - \omega^n)$$

Thus  $J_{n-1,\omega} = J_\omega$  is just the generalized energy functional (cf. [11] and also Sect. 3.1).

**Proposition 4.4.** *For each  $k = 0, 1, \dots, n - 1$ , we have the following explicit formula for  $J_k$ :*

$$J_k(\varphi) = \frac{n-k}{V} \sum_{j=0}^{n-k-1} \sum_{i=0}^k \sum_{s=0}^{n-i-j-1} c_{sij} \int_M \sqrt{-1} \partial \varphi \wedge \bar{\partial} \varphi \wedge \omega_\varphi^s \wedge \omega^{n-1-s}, \tag{4.1}$$

where  $c_{sij}$  is

$$\frac{(-1)^{n-i-j-s-1}}{(n-i-j+1)} \binom{k+1}{i} \binom{n-k-1}{j} \binom{n-i-j-1}{s}$$

*Proof.* We will calculate  $J_k(\varphi)$  via a trivial path  $t\varphi \in \mathcal{P}(M, \omega)$  (the corresponding Kähler metrics are  $\omega + t\sqrt{-1}\partial\bar{\partial}\varphi$ ).

$$\begin{aligned}
 & J_k(\varphi) \\
 &= \frac{k-n}{V} \int_0^1 \int_M \varphi (\omega_{t\varphi}^{k+1} - \omega^{k+1}) \wedge \omega_{t\varphi}^{n-k-1} \wedge dt \\
 &= \frac{k-n}{V} \int_0^1 \int_M \varphi \sum_{i=0}^k \binom{k+1}{i} \omega^i \wedge (\sqrt{-1}\partial\bar{\partial}\varphi)^{k+1-i} t^{k+1-i} \\
 &\quad \wedge \sum_{j=0}^{n-k-1} \binom{n-k-1}{j} \omega^j \wedge (\sqrt{-1}\partial\bar{\partial}\varphi)^{n-k-1-j} t^{n-k-1-j} \wedge dt \\
 &= \frac{k-n}{V} \int_0^1 \int_M \varphi \sum_{i=0}^k \sum_{j=0}^{n-k-1} \binom{k+1}{i} \binom{n-k-1}{j} \omega^{i+j} \wedge (t\sqrt{-1}\partial\bar{\partial}\varphi)^{n-i-j} \wedge dt \\
 &= -\frac{n-k}{V} \int_M \varphi \sum_{j=0}^{n-k-1} \sum_{i=0}^k \frac{1}{n-i-j+1} \binom{k+1}{i} \binom{n-k-1}{j} \omega^{i+j} \wedge (\sqrt{-1}\partial\bar{\partial}\varphi)^{n-i-j} \\
 &= \sum_{i=0}^k \sum_{j=0}^{n-k-1} \binom{k+1}{i} \binom{n-k-1}{j} \frac{(n-k)\sqrt{-1}}{(n-i-j+1)V} \int_M \partial\varphi \wedge \bar{\partial}\varphi \wedge \omega^{i+j} \wedge (\omega_\varphi - \omega)^{n-i-j-1} \\
 &= \sum_{j=0}^{n-k-1} \sum_{i=0}^k \sum_{s=0}^{n-i-j-1} \frac{(n-k)(-1)^{n-i-j-s-1}}{(n-i-j+1)V} \binom{k+1}{i} \binom{n-k-1}{j} \binom{n-i-j-1}{s} \\
 &\quad \int_M \sqrt{-1} \partial\varphi \wedge \bar{\partial}\varphi \wedge \omega^{i+j} \wedge \omega_\varphi^s \wedge \omega^{n-s-i-j-1}.
 \end{aligned}$$

□

The following is an immediate corollary of Formula (4.1).

**Corollary 4.5.** *For each  $k$ , there is a uniform constant  $a_k$  such that for any  $\varphi \in \mathcal{P}(M, \omega)$ ,*

$$|J_{k,\omega}(\varphi)| \leq a_k \cdot J_\omega(\varphi).$$

This follows from Formula (4.1) and the explicit expression:

$$J_\omega(\varphi) = \frac{1}{V} \sum_{i=0}^{n-1} \int_M \frac{i+1}{n+1} \sqrt{-1} \partial\varphi \wedge \bar{\partial}\varphi \wedge \omega^i \wedge \omega_\varphi^{n-1-i}.$$

Now we simply define  $E_{k,\omega}$  ( $k = 0, 1, \dots, n$ ) by

$$E_{k,\omega}(\varphi) = E_{k,\omega}^0(\varphi) - J_{k,\omega}(\varphi),$$

where we set  $J_{n,\omega} = 0$ .

In the following, we will often write  $E_k$  for  $E_{k,\omega}$  if no confusion may occur.

### 4.2 Derivative of $E_k$

In this subsection, we derive a few basic properties of  $E_k$ .

**Theorem 4.6.** *For any  $k = 0, 1, \dots, n$ , we have<sup>2</sup>*

$$\begin{aligned} \frac{dE_k}{dt} &= \frac{k+1}{V} \int_M \Delta_\varphi \left( \frac{\partial\varphi}{\partial t} \right) \text{Ric}(\omega_\varphi)^k \wedge \omega_\varphi^{n-k} \\ &\quad - \frac{n-k}{V} \int_M \frac{\partial\varphi}{\partial t} (\text{Ric}(\omega_\varphi)^{k+1} - \omega_\varphi^{k+1}) \wedge \omega_\varphi^{n-k-1}. \end{aligned} \quad (4.2)$$

Here  $\{\varphi(t)\}$  is any path in  $\mathcal{P}(M, \omega)$ .

*Remark 4.7.* When  $k = 0$ , we have

$$\frac{dE_0}{dt} = \frac{n}{V} \int_M \frac{\partial\varphi}{\partial t} (\text{Ric}(\omega_\varphi) - \omega_\varphi) \wedge \omega_\varphi^{n-1}.$$

Thus  $E_0$  is a multiple of the well known K-energy function introduced by T. Mabuchi.

*Proof.* We suffice to compute the derivatives of  $E_k^0$  ( $k = 0, 1, \dots, n$ ). Put  $F = \log \frac{\omega_\varphi^n}{\omega^n} - h_\omega$ . It is clear that

$$\begin{aligned} \sqrt{-1} \partial \bar{\partial} F &= \text{Ric}(\omega) - \text{Ric}(\omega_\varphi) - \sqrt{-1} \partial \bar{\partial} h_\omega \\ &= \omega - \text{Ric}(\omega_\varphi) \end{aligned}$$

and

$$\frac{\partial \text{Ric}(\omega_\varphi)}{\partial t} = -\sqrt{-1} \partial \bar{\partial} \Delta_\varphi \left( \frac{\partial\varphi}{\partial t} \right).$$

---

<sup>2</sup> In a non canonical Kähler class, we need to modify the definition slightly since  $h_\omega$  is not defined there. For any  $k = 0, 1, \dots, n$ , we define

$$\begin{aligned} E_{k,\omega}(\varphi) &= \frac{1}{V} \int_M \log \frac{\omega_\varphi^n}{\omega^n} \left( \sum_{i=0}^k \text{Ric}(\omega_\varphi)^i \wedge \text{Ric}(\omega)^{k-i} \right) \wedge \omega_\varphi^{n-k} \\ &\quad - \frac{n-k}{V} \int_M \varphi (\text{Ric}(\omega)^{k+1} - \omega^{k+1}) \wedge \omega^{n-k-1} - J_{k,\omega}(\varphi). \end{aligned}$$

The second integral on the right is to offset the change from  $\omega$  to  $\text{Ric}(\omega)$  in the first term. The derivative of this functional is exactly same as in the canonical Kähler class. In other words, the Euler-Lagrange equation is not changed.

Thus,

$$\begin{aligned}
 & \frac{dE_k^0}{dt} \\
 &= \frac{1}{V} \int_M \Delta_\varphi \left( \frac{\partial\varphi}{\partial t} \right) \left( \sum_{i=0}^k \text{Ric}(\omega_\varphi)^i \wedge \omega^{k-i} \right) \wedge \omega_\varphi^{n-k} \\
 & \quad + \frac{1}{V} \int_M F \sum_{i=0}^k i \text{Ric}(\omega_\varphi)^{i-1} \wedge \omega^{k-i} \wedge \left( -\sqrt{-1} \partial \bar{\partial} \Delta_\varphi \frac{\partial\varphi}{\partial t} \right) \wedge \omega_\varphi^{n-k} \\
 & \quad + \frac{n-k}{V} \int_M F \left( \sum_{i=0}^k \text{Ric}(\omega_\varphi)^i \wedge \omega^{k-i} \right) \wedge \left( \sqrt{-1} \partial \bar{\partial} \left( \frac{\partial\varphi}{\partial t} \right) \right) \wedge \omega_\varphi^{n-k-1} \\
 &= \frac{1}{V} \int_M \Delta_\varphi \left( \frac{\partial\varphi}{\partial t} \right) \left( \sum_{i=0}^k \text{Ric}(\omega_\varphi)^i \wedge \omega^{k-i} \wedge \omega_\varphi^{n-k} \right. \\
 & \quad \left. - \sqrt{-1} \partial \bar{\partial} F \wedge \sum_{i=0}^k i \text{Ric}(\omega_\varphi)^{i-1} \wedge \omega^{k-i} \wedge \omega_\varphi^{n-k} \right) \\
 & \quad + \frac{n-k}{V} \int_M \frac{\partial\varphi}{\partial t} \left( \sum_{i=0}^k \text{Ric}(\omega_\varphi)^i \wedge \omega^{k-i} \right) \wedge \left( \sqrt{-1} \partial \bar{\partial} F \right) \wedge \omega_\varphi^{n-k-1}.
 \end{aligned}$$

Plugging  $\sqrt{-1} \partial \bar{\partial} F = \omega - \text{Ric}(\omega_\varphi)$ , we obtain

$$\begin{aligned}
 & \frac{dE_k^0}{dt} \\
 &= \frac{1}{V} \int_M \Delta_\varphi \left( \frac{\partial\varphi}{\partial t} \right) \left( \sum_{i=0}^k \text{Ric}(\omega_\varphi)^i \wedge \omega^{k-i} \wedge \omega_\varphi^{n-k} \right. \\
 & \quad \left. + (\text{Ric}(\omega_\varphi) - \omega) \wedge \sum_{i=0}^k i \text{Ric}(\omega_\varphi)^{i-1} \wedge \omega^{k-i} \wedge \omega_\varphi^{n-k} \right) \\
 & \quad + \frac{n-k}{V} \int_M \frac{\partial\varphi}{\partial t} \left( \sum_{i=0}^k \text{Ric}(\omega_\varphi)^i \wedge \omega^{k-i} \right) \wedge (\omega - \text{Ric}(\omega_\varphi)) \wedge \omega_\varphi^{n-k-1}.
 \end{aligned}$$

Now we recall a polynomial identity: For any two variables  $x, y$  and any integer  $k > 0$ , we have

$$\sum_{i=0}^k x^i y^{k-i} + (x - y) \sum_{i=0}^k i x^{i-1} y^{k-i} = (k + 1) x^k. \tag{4.3}$$

Applying this identity to the first integral above, we get

$$\begin{aligned} \frac{dE_k^0}{dt} &= \frac{k+1}{V} \int_M \Delta_\varphi \left( \frac{\partial\varphi}{\partial t} \right) \text{Ric}(\omega_\varphi)^k \wedge \omega_\varphi^{n-k} \\ &\quad + \frac{n-k}{V} \int_M \frac{\partial\varphi}{\partial t} (\omega^{k+1} - \text{Ric}(\omega_\varphi)^{k+1}) \wedge \omega_\varphi^{n-k-1}. \end{aligned}$$

The theorem follows from this and explicit expression of the derivative of  $J_k$ . □

From this theorem, we can show that all  $E_k$  satisfy a cocycle condition.

**Corollary 4.8.** *For each  $k = 0, 1, \dots, n$ , the functional  $E_{k,\omega}$  satisfies the following: For any  $\varphi$  and  $\psi$  in  $\mathcal{P}(M, \omega)$ ,*

$$E_{k,\omega}(\varphi) + E_{k,\omega_\varphi}(\psi - \varphi) = E_{k,\omega}(\psi).$$

Let us write down the Euler-Lagrange equation for the functional  $E_k(k = 0, 1, \dots, n)$ . Recall the expansion formula (1.1) in  $t$ :

$$(\omega_\varphi + t \text{Ric}(\omega_\varphi))^n = \left( \sum_{k=0}^n \sigma_k(\omega_\varphi) t^k \right) \omega_\varphi^n.$$

Clearly,  $\sigma_0(\omega_\varphi) = 1, \sigma_1(\omega_\varphi) = R(\omega_\varphi)$ , the scalar curvature of  $\omega_\varphi$ . In general,  $\sigma_k$  is a  $k$ -th symmetric polynomial of Ricci curvature. The Euler-Lagrange equation of  $E_k$  is

$$(k+1)\Delta_\varphi \sigma_k(\omega_\varphi) - (n-k)\sigma_{k+1}(\omega_\varphi) = c_k,$$

where  $\Delta_\varphi$  is the Laplacian of the metric  $\omega_\varphi$  and  $c_k$  is the constant

$$-(n-k)c_1(M)^{k+1} \cup [\omega]^{n-k-1}([M]).$$

Clearly, Kähler-Einstein metrics are solutions to the above equation for any  $k$ . If the Kähler class is canonical, one can show that for  $k = n$ , Kähler-Einstein metrics are the only solutions of the Euler-Lagrange equation with positive Ricci curvature. However, It is not clear what the critical points are in other Kähler classes. But it certainly merit further study of these equations.

**Proposition 4.9.** *Along the Kähler Ricci flow, we have*

$$\frac{dE_k}{dt} \leq -\frac{k+1}{V} \int_M (R(\omega_\varphi) - r) \text{Ric}(\omega_\varphi)^k \wedge \omega_\varphi^{n-k}. \tag{4.4}$$

When  $k = 0, 1$ , we have

$$\begin{aligned} \frac{dE_0}{dt} &= -\frac{n\sqrt{-1}}{V} \int_M \partial \frac{\partial\varphi}{\partial t} \wedge \bar{\partial} \frac{\partial\varphi}{\partial t} \omega_\varphi^{n-1} \leq 0, \\ \frac{dE_1}{dt} &\leq -\frac{2}{V} \int_M (R(\omega_\varphi) - r)^2 \omega_\varphi^n \leq 0. \end{aligned} \tag{4.5}$$

*In particular, both  $E_0$  and  $E_1$  are decreasing along the Kähler Ricci flow.*

*Proof.* Along the Kähler Ricci flow, we have

$$\Delta_\varphi \left( \frac{\partial \varphi}{\partial t} \right) = r - R(\omega_\varphi).$$

Here  $r$  is again the average of the scalar curvature  $R(\omega_\varphi)$ . We also have

$$\begin{aligned} \sqrt{-1} \partial \bar{\partial} \frac{\partial \varphi}{\partial t} &= \sqrt{-1} \partial \bar{\partial} \left( \log \frac{\omega_\varphi^n}{\omega^n} + \varphi - h_\omega \right) \\ &= -\text{Ric}(\omega_\varphi) + \left( \omega - \sqrt{-1} \partial \bar{\partial} h_\omega \right) + \sqrt{-1} \partial \bar{\partial} \varphi \\ &= -\text{Ric}(\omega_\varphi) + \omega + \sqrt{-1} \partial \bar{\partial} \varphi = \omega_\varphi - \text{Ric}(\omega_\varphi). \end{aligned}$$

Therefore,

$$\frac{dE_k}{dt} \tag{4.6}$$

$$\begin{aligned} &= -\frac{k+1}{V} \int_M (R(\omega_\varphi) - r) \text{Ric}(\omega_\varphi)^k \wedge \omega_\varphi^{n-k} \\ &\quad + \frac{n-k}{V} \int_M \frac{\partial \varphi}{\partial t} \sqrt{-1} \partial \bar{\partial} \left( \frac{\partial \varphi}{\partial t} \right) \wedge \sum_{i=0}^{k+1} \text{Ric}(\omega_\varphi)^{k+1-i} \wedge \omega_\varphi^{n-k+i-1} \tag{4.7} \end{aligned}$$

$$\leq -\frac{k+1}{V} \int_M (R(\omega_\varphi) - r) \text{Ric}(\omega_\varphi)^k \wedge \omega_\varphi^{n-k}. \tag{4.8}$$

□

The following is an easy corollary of the above, but it will be crucial in our proof.

**Theorem 4.10.** *Let  $\varphi(t)$  be the global solution of the Kähler Ricci flow. Then for any  $T > 0$ , we have*

$$\frac{k+1}{V} \int_0^T \int_M (R(\omega_\varphi) - r) \text{Ric}(\omega_\varphi)^k \wedge \omega_\varphi^{n-k} dt \leq E_k(\varphi(0)) - E_k(\varphi(T)).$$

When  $k = 1$ , this reduces to

$$\frac{2}{V} \int_0^T \int_M (R(\omega_\varphi) - r)^2 \omega_\varphi^n dt \leq E_1(\varphi(0)) - E_1(\varphi(T))$$

*In particular, if  $E_k(\varphi(t))$  is uniformly bounded from below, then for any sequence of positive numbers  $\epsilon_i$  with  $\lim_{i \rightarrow \infty} \epsilon_i = 0$ , there exists a sequence of  $t_i$  such that*

$$\sum_{k=0}^n \frac{k+1}{V} \int_M (R(\omega_{\varphi(t_i)}) - r) \text{Ric}(\omega_{\varphi(t_i)})^k \wedge \omega_{\varphi(t_i)}^{n-k} \leq \epsilon_i.$$

When  $k = 1$ , this becomes

$$\frac{1}{V} \int_M (R(\omega_{\varphi(t_i)}) - r)^2 \omega_{\varphi(t_i)}^n \leq \epsilon_i.$$

In order to have integral bounds of curvature from these inequalities, we need to bound these functionals  $E_k$  from below. The following provides a way of achieving it.

**Lemma 4.11.** *Let  $\varphi$  be in  $\mathcal{P}(M, \omega)$  such that  $\text{Ric}(\omega_\varphi) \geq 0$ . Then there is a uniform constant  $c = c(\omega)$  such that*

$$E_k(\varphi) \geq -e^{c(1+\max\{0, \nu_\omega(\varphi)\}+J_\omega(\varphi))}.$$

*Proof.* We will always denote by  $c$  a constant depending only on  $\omega$ . By the definition of  $E_k$  and Corollary 4.5, we have

$$E_k \geq \frac{1}{V} \int_M \left( \log \frac{\omega_\varphi^n}{\omega^n} \right) \left( \sum_{i=0}^k \text{Ric}(\omega_\varphi)^i \wedge \omega^{n-i} \right) - c(1 + J_\omega(\varphi)).$$

In particular, since  $E_0$  is just the K-energy, we have

$$\frac{1}{V} \int_M \left( \log \frac{\omega_\varphi^n}{\omega^n} \right) \omega_\varphi^n \leq \nu_\omega(\varphi) + c(1 + J_\omega(\varphi)).$$

Then the lemma follows from the above two inequalities and the volume estimate in Proposition 3.14. □

Because of the monotonicity of the K-energy along the Kähler Ricci flow, the K-energy  $\nu_\omega(\varphi)$  is bounded. Hence, in order to bound  $E_k$ , we suffice to bound the generalized energy  $J_\omega(\varphi)$  along the Kähler Ricci flow. The trouble is that  $J_\omega(\varphi)$  may not be bounded along the flow. We will bound  $J_\omega$  for modified Kähler Ricci flow, which turns out to be sufficient (cf. Sect. 6).

### 5 New holomorphic invariants

In this section, we want to show that on any Kähler-Einstein manifolds,  $E_k$  ( $k = 0, 1, \dots, n$ ) are invariant under automorphisms. First we want to show that the derivatives of  $E_k$  in the direction of holomorphic vector field give us holomorphic invariants of the Kähler class.

Let  $X$  be a holomorphic vector field and  $\omega$  be a Kähler metric. Then  $i_X \omega$  is a  $\bar{\partial}$ -closed (0,1) form, by the Hodge theorem, we can decompose  $i_X \omega$  into a parallel  $\alpha_X$  form plus  $\sqrt{-1} \bar{\partial} \theta_X$ , where  $\theta_X$  is some function. For simplicity, we will assume that  $\alpha_X = 0$ . This is automatically true if  $M$  is simply-connected. We will call that  $\theta_X$  is a potential of  $X$  with respect to  $\omega$ .

It is unique modulo addition of constants. Note that  $L_X(\omega) = \sqrt{-1}\partial\bar{\partial}\theta_X$ . Now we define  $\mathfrak{S}_k(X, \omega)$  for each  $k = 0, 1, \dots, n$  by

$$\begin{aligned} \mathfrak{S}_k(X, \omega) &= (n - k) \int_M \theta_X \omega^n \\ &+ \int_M ((k + 1)\Delta\theta_X \operatorname{Ric}(\omega)^k \wedge \omega^{n-k} - (n - k)\theta_X \operatorname{Ric}(\omega)^{k+1} \wedge \omega^{n-k-1}). \end{aligned}$$

Here and in the following,  $\Delta$  denotes the Laplacian of  $\omega$ . Clearly, the identity is unchanged if we replace  $\theta_X$  by  $\theta_X + c$  for any constant  $c$ .

The next theorem assures that the above integral gives rise to a holomorphic invariant.

**Theorem 5.1.** *The integral  $\mathfrak{S}_k(X, \omega)$  is independent of choices of Kähler metrics in the Kähler class  $[\omega]$ , that is,  $\mathfrak{S}_k(X, \omega) = \mathfrak{S}_k(X, \omega')$  so long as the Kähler forms  $\omega$  and  $\omega'$  represent the same Kähler class. Hence, the integral  $\mathfrak{S}_k(X, \omega)$  is a holomorphic invariant, which will be denoted by  $\mathfrak{S}_k(X, [\omega])$ .*

*Remark 5.2.* When  $k = 0$ , we have

$$\begin{aligned} \mathfrak{S}_0(X, \omega) &= \int_M \Delta\theta_X \omega^n + n \theta_X (\omega - \operatorname{Ric}(\omega)) \wedge \omega^{n-1} \\ &= -n \int_M \theta_X \Delta h_\omega \omega^n = n \int_M X(h_\omega) \omega^n. \end{aligned}$$

Thus  $\mathfrak{S}_0(X, [\omega])$  is a multiple of the Futaki invariant ([13]).

If  $[\omega]$  is a canonical Kähler class and there is a Kähler-Einstein metric on  $M$ , then we can choose  $\omega$  such that  $\operatorname{Ric}(\omega) = \omega$  and deduce

$$\mathfrak{S}_k(X, \omega) = (k + 1) \int_M \Delta\theta_X \omega^n = 0.$$

Therefore, we have

**Corollary 5.3.** *The above invariants  $\mathfrak{S}_k(X, c_1(M))$  all vanish for any holomorphic vector fields  $X$  on a compact Kähler-Einstein manifold. In particular, these invariants all vanish on  $\mathbb{C}P^n$ .*

Before we prove this theorem, we first use to show the invariance of  $E_k$  under automorphisms.

**Proposition 5.4.** *Let  $X$  be a holomorphic vector field and  $\{\Phi(t)\}_{|t|<\infty}$  be the one-parameter subgroup of automorphisms induced by  $\operatorname{Re}(X)$ . Then*

$$\frac{dE_k(\varphi_t)}{dt} = \frac{1}{V} \operatorname{Re}(\mathfrak{S}_k(X, \omega)),$$

where  $\varphi_t$  are the Kähler potentials of  $\Phi_t^*\omega$ , i.e.,  $\Phi_t^*\omega = \omega + \sqrt{-1}\partial\bar{\partial}\varphi_t$ .

*Proof.* Differentiating  $\Phi_t^* \omega = \omega + \sqrt{-1} \partial \bar{\partial} \varphi_t$ , we get

$$L_{Re(X)} \omega = \sqrt{-1} \partial \bar{\partial} \left( \frac{\partial \varphi_t}{\partial t} \right).$$

On the other hand, since  $L_X \omega = \sqrt{-1} \partial \bar{\partial} \theta_X$ , we have

$$\frac{\partial \varphi_t}{\partial t} = Re(\theta_X) + c,$$

where  $c$  is a constant. It follows

$$\begin{aligned} \frac{dE_k}{dt} &= \frac{1}{V} \int_M \left( (k+1) \Delta \left( \frac{\partial \varphi_t}{\partial t} \right) Ric(\omega)^k \wedge \omega^{n-k} \right. \\ &\quad \left. - (n-k) \frac{\partial \varphi_t}{\partial t} (Ric(\omega)^{k+1} - \omega^{k+1}) \wedge \omega^{n-k-1} \right) \\ &= \frac{1}{V} Re(\mathfrak{S}_k(X, \omega)). \end{aligned}$$

□

An immediate corollary is

**Corollary 5.5.** *On a Kähler-Einstein manifold  $M$  with  $c_1(M) = [\omega]$ , all functionals  $E_{k,\omega}$  ( $k = 0, 1, \dots, n$ ) are invariant under automorphisms of  $M$ .*

*Remark 5.6.* It also follows from the above proposition that  $E_k$  has a lower bound only if  $\mathfrak{S}_k(X, \omega) = 0$ .

The rest of this section is devoted to proving this theorem. We will follow the arguments in [28]. For this purpose, we first formulate  $\mathfrak{S}_k$  in terms of some particular forms, i.e., the Bott-Chern forms.

**Lemma 5.7.** *There exists a matrix  $(c_{ij})$  ( $1 \leq i, j \leq n+1$ ) such that*

$$\begin{aligned} \mathfrak{S}_{k-1}(X, \omega) &= -\frac{n-k+1+v_k}{n+1} \int_M (-\theta_X + \omega)^{n+1} \\ &\quad + \frac{1}{\binom{n+1}{k}} \sum_{i=1}^{n+1} c_{ik} \int_M (-\theta_X + \omega + i(\Delta\theta_X + Ric(\omega)))^{n+1}, \end{aligned}$$

where the matrix  $(c_{ij})$  is the inverse matrix of the well known Vandermonde

matrix <sup>3</sup> and

$$v_k = \frac{n+1}{\binom{n+1}{k}} \sum_{i=1}^{n+1} c_{ik}. \quad \text{where } k = 1, 2, \dots, n+1.$$

*Proof.* Consider

$$\begin{aligned} I_{pq} &= \int_M (-p \theta_X + q \Delta \theta_X) (q \operatorname{Ric}(\omega) + p \omega)^n \\ &= \frac{1}{n+1} \int_M (-p \theta_X + q \Delta \theta_X + q \operatorname{Ric}(\omega) + p \omega)^{n+1}. \end{aligned}$$

Note that the only forms of degree  $2n$  contribute to the above integral.

Expanding the integrand, we have

$$\begin{aligned} I_{pq} &= \int_M (-p \theta_X + q \Delta \theta_X) \left( \sum_{k=0}^n q^k p^{n-k} \binom{n}{k} \operatorname{Ric}(\omega)^k \wedge \omega^{n-k} \right) \end{aligned}$$

<sup>3</sup> There is an explicit way of finding  $c_{ij}$ , which we learned from E. Calabi. Let us define a sequence of polynomials of degree  $n+1$  by

$$f_i(x) = \sum_{j=1}^{n+1} c_{ij} x^j, \quad \forall i = 1, 2, \dots, n+1.$$

Since  $(c_{ij})$  is the inverse matrix of Vandermonde matrix:

$$\begin{pmatrix} 1 & 2 & \dots & n+1 \\ 1^2 & 2^2 & \dots & (n+1)^2 \\ 1^3 & 2^3 & \dots & (n+1)^3 \\ \vdots & \vdots & \ddots & \vdots \\ 1^{n+1} & 2^{n+1} & \dots & (n+1)^{n+1} \end{pmatrix},$$

we obtain

$$f_i(k) = \sum_{j=1}^{n+1} c_{ij} k^j = \delta_{kj}, \quad \forall i, k = 1, 2, \dots, n+1.$$

It follows that for each  $i = 1, 2, \dots, n+1$ ,

$$f_i(x) = \sum_{j=1}^{n+1} c_{ij} x^j = \frac{x \prod_{k \neq i, k=1}^{n+1} (x-k)}{i \prod_{k \neq i, k=1}^{n+1} (i-k)}.$$

Thus  $c_{ij}$  can be found.

$$\begin{aligned}
&= - \int_M \theta_X \left( \sum_{k=0}^n q^k p^{n-k+1} \binom{n}{k} \operatorname{Ric}(\omega)^k \wedge \omega^{n-k} \right) \\
&\quad + \int_M \Delta \theta_X \left( \sum_{k=0}^n q^{k+1} p^{n-k} \binom{n}{k} \operatorname{Ric}(\omega)^k \wedge \omega^{n-k} \right) \\
&= - \int_M \theta_X \left( \sum_{k=0}^n q^k p^{n-k+1} \binom{n}{k} \operatorname{Ric}(\omega)^k \wedge \omega^{n-k} \right) \\
&\quad + \int_M \Delta \theta_X \left( \sum_{k=1}^{n+1} q^k p^{n-k+1} \binom{n}{k-1} \operatorname{Ric}(\omega)^{k-1} \wedge \omega^{n-k+1} \right) \\
&= \sum_{k=0}^{n+1} q^k p^{n-k+1} \frac{n!}{k!(n-k+1)!} \\
&\quad \cdot \int_M (k \Delta \theta_X \operatorname{Ric}(\omega)^{k-1} \wedge \omega^{n-k+1} - (n-k+1) \theta_X \operatorname{Ric}(\omega)^k \wedge \omega^{n-k}).
\end{aligned}$$

Now set  $p = 1$  and observe ( $k = 0, 1, 2, \dots, n$ )

$$\begin{aligned}
&\mathfrak{S}_k(X, \omega) - (n-k) \int_M \theta_X \omega^n \\
&= \int_M ((k+1) \Delta \theta_X \operatorname{Ric}(\omega)^k \wedge \omega^{n-k} - (n-k) \theta_X \operatorname{Ric}(\omega)^{k+1} \wedge \omega^{n-k-1}).
\end{aligned}$$

Then

$$\begin{aligned}
I_{1q} &= - \int_M \theta_X \omega^n \\
&\quad + \frac{1}{(n+1)} \sum_{k=1}^{n+1} \binom{n+1}{k} \left( \mathfrak{S}_{k-1}(X, \omega) - (n-k+1) \int_M \theta_X \omega^n \right) q^k,
\end{aligned}$$

or equivalently,

$$\begin{aligned}
\frac{1}{(n+1)} \sum_{k=1}^{n+1} \binom{n+1}{k} \left( \mathfrak{S}_{k-1}(X, \omega) - (n-k+1) \int_M \theta_X \omega^n \right) q^k \\
= I_{1q} + \int_M \theta_X \omega^n.
\end{aligned}$$

Since  $(c_{ij})$  is the inverse matrix of the Vandermonde matrix, we have

$$\begin{aligned} & \frac{\binom{n+1}{k}}{(n+1)} \left( \mathfrak{S}_{k-1}(X, \omega) - (n-k+1) \int_M \theta_X \omega^n \right) \\ &= \sum_{i=1}^{n+1} c_{ik} \left( I_{1i} + \int_M \theta_X \omega^n \right) \\ &= \sum_{i=1}^{n+1} c_{ik} I_{1i} + \sum_{i=1}^{n+1} c_{ik} \int_M \theta_X \omega^n \\ &= \sum_{i=1}^{n+1} c_{ik} I_{1i} + v_k \frac{\binom{n+1}{k}}{(n+1)} \int_M \theta_X \omega^n. \end{aligned}$$

The lemma follows from this since

$$- \int_M \theta_X \omega^n = \frac{1}{n+1} \int_M (\theta_X + \omega)^{n+1}.$$

□

Now we continue the proof of Theorem 5.1. We suffice to prove the independence of  $I_{pq}$ . First we observe that

$$\sqrt{-1} \bar{\partial} \theta_X = i_X \omega \quad \text{and} \quad \sqrt{-1} \bar{\partial} \Delta \theta_X = -i_X \text{Ric}(\omega).$$

The second identity can be checked as follows: Suppose  $\omega =$

$$\sqrt{-1} \sum_{i,j=1}^n g_{i\bar{j}} dz_i \wedge d\bar{z}_j \text{ in local coordinates. Then}$$

$$\begin{aligned} i_X \text{Ric}(\omega) &= -\sqrt{-1} \bar{\partial} \left( X^i \frac{\partial}{\partial z_i} \log \det(g_{k\bar{l}}) \right) \\ &= -\sqrt{-1} \bar{\partial} \left( X^i g^{k\bar{l}} \frac{\partial g_{i\bar{l}}}{\partial z_k} \right) \\ &= -\sqrt{-1} \bar{\partial} \left( g^{k\bar{l}} \frac{\partial}{\partial z_k} (X^i g_{i\bar{l}}) - g^{k\bar{l}} g_{i\bar{l}} \frac{\partial X^i}{\partial z_k} \right) \\ &= -\sqrt{-1} \bar{\partial} \left( g^{k\bar{l}} \frac{\partial}{\partial z_k} (X^i g_{i\bar{l}}) \right) \\ &= -\sqrt{-1} \bar{\partial} \Delta_g \theta_X. \end{aligned}$$

Since the space of Kähler metrics is path-connected, it suffices to show that  $I_{pq}$  is invariant when we deform the Kähler potential along any path  $\varphi_t \in \mathcal{P}(M, \omega)$ . To emphasize the dependence on  $\omega_\varphi$ , we will denote by  $I_{pq}(\varphi)$  the integral

$$I_{pq}(\varphi) = \int_M (-p\theta_X + q\Delta\theta_X + q\text{Ric}(\omega_\varphi) + p\omega_\varphi)^{n+1}.$$

We need to show that

$$\frac{\partial I_{pq}}{\partial t}(\varphi_t) = 0.$$

Put  $\omega_t = \omega + \sqrt{-1} \partial \bar{\partial} \varphi_t$ . Then

$$\begin{aligned} i_X \omega_t &= \sqrt{-1} \bar{\partial} (\theta_X + X(\varphi_t)) \\ i_X \text{Ric}(\omega_t) &= -\sqrt{-1} \bar{\partial} \Delta_t (\theta_X + X(\varphi_t)), \end{aligned}$$

where  $\Delta_t$  is the Laplacian of  $\omega_t$ . For simplicity, we denote by  $\Psi_t$  the function

$$-p (\theta_X + X(\varphi_t)) + q \Delta_t (\theta_X + X(\varphi_t)).$$

Define

$$\alpha_t = -p \partial \left( \frac{\partial \varphi_t}{\partial t} \right) + q \partial \Delta_t \left( \frac{\partial \varphi_t}{\partial t} \right).$$

Using the identity

$$\text{Ric}(\omega_t) = \text{Ric}(\omega) - \sqrt{-1} \partial \bar{\partial} \log \left( \frac{\omega_t^n}{\omega^n} \right),$$

we can show

$$\sqrt{-1} \bar{\partial} \alpha_t = \frac{\partial}{\partial t} (p \omega_t + q \text{Ric}(\omega_t)).$$

On the other hand, we have

$$i_X \alpha_t = \frac{\partial \Psi_t}{\partial t}.$$

This can be seen as follows: Suppose that in local coordinates,

$$X = X^k \frac{\partial}{\partial z_k} \quad \text{and} \quad \omega_t = \sqrt{-1} \sum_{i,j=1}^n g_{i\bar{j}} dz_i \wedge d\bar{z}_j.$$

Then

$$\begin{aligned} \frac{\partial \Psi_t}{\partial t} &= -p X \left( \frac{\partial \varphi_t}{\partial t} \right) + q \frac{\partial}{\partial t} \left( g^{i\bar{j}} \frac{\partial^2}{\partial z_i \partial \bar{z}_j} (\theta_X + X(\varphi_t)) \right), \\ i_X \alpha_t &= -p X \left( \frac{\partial \varphi_t}{\partial t} \right) + q X \left( g^{i\bar{j}} \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \left( \frac{\partial \varphi_t}{\partial t} \right) \right). \end{aligned}$$

Notice that  $\sqrt{-1}\bar{\partial}(\theta_X + X(\varphi_t)) = i_X\omega_t$ . We then have

$$\begin{aligned} & \frac{\partial}{\partial t} \left( g^{i\bar{j}} \frac{\partial^2}{\partial z_i \partial \bar{z}_j} (\theta_X + X(\varphi_t)) \right) \\ &= g^{i\bar{j}} \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \left( X^k \frac{\partial}{\partial z_k} \left( \frac{\partial \varphi_t}{\partial t} \right) \right) + \frac{\partial g^{i\bar{j}}}{\partial t} \frac{\partial}{\partial z_i} \left( X^k g_{k\bar{j}} \right) \\ &= g^{i\bar{j}} \frac{\partial}{\partial z_i} \left( X^k \frac{\partial^2}{\partial z_k \partial \bar{z}_j} \left( \frac{\partial \varphi_t}{\partial t} \right) \right) - g^{i\bar{l}} \frac{\partial^2}{\partial z_m \partial \bar{z}_l} \left( \frac{\partial \varphi_t}{\partial t} \right) g^{m\bar{j}} \frac{\partial}{\partial z_i} \left( X^k g_{k\bar{j}} \right) \\ &= g^{i\bar{j}} X^k \frac{\partial^3}{\partial z_i \partial z_k \partial \bar{z}_j} \left( \frac{\partial \varphi_t}{\partial t} \right) - g^{i\bar{l}} \frac{\partial^2}{\partial z_m \partial \bar{z}_l} \left( \frac{\partial \varphi_t}{\partial t} \right) g^{m\bar{j}} X^k \frac{\partial g_{k\bar{j}}}{\partial z_i} \\ &= X^k \frac{\partial}{\partial z_k} \left( g^{i\bar{j}} \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \left( \frac{\partial \varphi_t}{\partial t} \right) \right). \end{aligned}$$

It follows that  $i_X\alpha_t = \frac{\partial \Psi_t}{\partial t}$ .

For simplicity, we will denote by  $R_t$  the curvature form  $p\omega_t + q\text{Ric}(\omega_t)$ . Then

$$\sqrt{-1}\bar{\partial}\Psi_t = -i_X R_t \quad \text{and} \quad \sqrt{-1}\bar{\partial}\alpha_t = \frac{\partial R_t}{\partial t}.$$

Hence, we have

$$\begin{aligned} & \sqrt{-1} \frac{\partial}{\partial t} I_{pq}(\varphi_t) \\ &= \int_M \left( \sqrt{-1} \frac{\partial \Psi_t}{\partial t} + \frac{\partial R_t}{\partial t} \right) \left( \sqrt{-1} \Psi_t + R_t \right)^n \\ &= \int_M \left( \sqrt{-1} i_X\alpha_t + \sqrt{-1} \bar{\partial}\alpha_t \right) \left( \sqrt{-1} \Psi_t + R_t \right)^n \\ &= \int_M \sqrt{-1} i_X\alpha_t \left( \sqrt{-1} \Psi_t + R_t \right)^n \\ &\quad + n \int_M \sqrt{-1} \alpha_t \wedge \bar{\partial} \left( \sqrt{-1} \Psi_t + R_t \right) \wedge \left( \sqrt{-1} \Psi_t + R_t \right)^{n-1} \\ &= \int_M \sqrt{-1} i_X\alpha_t \left( \sqrt{-1} \Psi_t + R_t \right)^n \\ &\quad - n \int_M \sqrt{-1} \alpha_t \wedge i_X \left( \sqrt{-1} \Psi_t + R_t \right) \wedge \left( \sqrt{-1} \Psi_t + R_t \right)^{n-1} \\ &= \int_M i_X \left( \sqrt{-1} \alpha_t \wedge \left( \sqrt{-1} \Psi_t + R_t \right)^n \right). \end{aligned}$$

Here we have used the second Bianchi identity:  $\bar{\partial}R(g_t) = 0$ . We also have used  $\bar{\partial}\psi_{X,t} = -i_X R(g_t) = -i_X(\psi_{X,t} + R(g_t))$ .

Put

$$\eta = \sqrt{-1} \alpha_t \wedge \left( \sqrt{-1} \Psi_t + R_t \right)^n.$$

We write it as  $g_0 + \dots + g_{2n}$  and  $i_X \eta = \beta_0 + \beta_1 + \dots + \beta_{2n}$ . The only term which contributes to the above integral is the  $\beta_{2n}$ , but  $\beta_{2n} = i_X g_{2n+1}$  and  $g_{2n+1} = 0$ . Therefore, the above integral is zero. Thus, the theorem is proved.

### 6 Modified Kähler Ricci flow

In the first subsection, we want to modify the Kähler Ricci flow by automorphisms so that the evolved Kähler form is centrally positioned with respect to a fixed Kähler-Einstein metric (see Definition 6.1 below). Our argument here essentially due to S. Bando and T. Mabuchi [2]. In the second subsection, we use this and Tian’s inequality [28] to derive a uniform lower bound on  $E_k$ . That in turn implies the desired integral estimate on curvature (Corollary 6.9).

#### 6.1 Modified Kähler form by automorphisms

As before, let  $\omega_1$  be a Kähler-Einstein metric in  $M$  such that  $\text{Ric}(\omega_1) = \omega_1$ . Let us first introduce the definition of “centrally positioned”:

**Definition 6.1.** Any Kähler form  $\omega_\varphi$  is called centrally positioned with respect to some Kähler-Einstein metric  $\omega_\rho = \omega + \sqrt{-1} \partial \bar{\partial} \rho$  if it satisfies the following:

$$\int_M (\varphi - \rho) \theta \omega_\rho^n = 0, \quad \forall \theta \in \Lambda_1(\omega_\rho). \tag{6.1}$$

We now introduce a well known functional in Kähler geometry first:

$$I(\omega_\varphi, \omega) = \frac{1}{V} \int_M \varphi (\omega^n - \omega_\varphi^n).$$

Note that this definition is symmetric with respect to  $\omega$  and  $\omega_\varphi$ . Alternatively, for any path  $\varphi(t) \in \mathcal{P}(M, \omega)$ , we have

$$\frac{dI}{dt} = \frac{1}{V} \int_M \frac{\partial \varphi}{\partial t} (\omega^n - \omega_\varphi^n) - \frac{1}{V} \int_M \varphi \Delta_\varphi \frac{\partial \varphi}{\partial t} \omega_\varphi^n.$$

Put

$$J(\omega_\varphi, \omega) = J_\omega(\varphi).$$

This implies that

$$\frac{d(I - J)}{dt} = -\frac{1}{V} \int_M \varphi \Delta_\varphi \frac{\partial \varphi}{\partial t} \omega_\varphi^n. \tag{6.2}$$

Now we consider a functional  $\Psi$  on  $\text{Aut}_r(M)$  by,

$$\Psi(\sigma) = (I - J)(\omega_\varphi, \sigma^* \omega_1) = (I - J)(\omega_\varphi, \omega_\rho) \tag{6.3}$$

for any  $\sigma \in \text{Aut}_r(M)$  and  $\sigma^* \omega_1 = \omega_\rho = \omega + \sqrt{-1} \partial \bar{\partial} \rho$ . If  $\sigma$  is a critical point in  $\in \text{Aut}_r(M)$ , then  $\omega_\rho$  is the desired Kähler-Einstein metric.

**Proposition 6.2.** *Let  $\omega_\rho$  be the minimal point of  $\Psi$ . For any  $u \in \Lambda_1(\omega_\rho)$ , we have*

$$\int_M (\rho - \varphi) u \omega_\rho^n = 0,$$

or equivalently

$$\rho - \varphi \perp \Lambda_1(\omega_\rho).$$

In other words,  $\omega_\rho$  is centrally positioned with respect to  $\omega_\rho$ .

Note that if  $\Lambda_1(M) = \emptyset$ , then this proposition hold trivially. Before we prove this proposition, we pause to establish the equivalence relation between the first eigenspace of  $\omega_1$ (or any Kähler-Einstein metric) and the space of holomorphic vector fields (denoted by  $\eta(M)$ ).

**Lemma 6.3.** *The first eigenvalue of  $\Delta_{\omega_1} \geq 1$ . Moreover, there is a 1-1 correspondence between the first eigenspace  $\Lambda_1$  of  $\omega_1$  and the space of holomorphic vector fields  $\eta(M)$ .*

The lemma is well-known. For the reader’s convenience, we outline its proof here.

*Proof.* Let  $\lambda_1$  be the first eigenvalues of  $\omega_1$  and  $u$  is any eigenfunction of  $\omega_1$  with eigenvalue  $\lambda_1$ , so  $\Delta_{\omega_1} u = -\lambda_1 u$ . Define a vector field  $X$  by  $i_X \omega_1 = \sqrt{-1} \bar{\partial} u$ . Then by a direct computation, we have

$$\int_M |\bar{\partial} X|^2 \omega_1^n = \lambda_1^2 \int_M u^2 \omega_1^n - \int_M |\partial u|^2 \omega_1^n.$$

This implies that

$$\begin{aligned} \lambda_1^2 \int_M u^2 \omega_1^n &= \int_M |\partial u|^2 \omega_1^n + \int_M |\bar{\partial} X|^2 \omega_1^n \\ &\geq \lambda_1 \int_M u^2 \omega_1^n + \int_M |\bar{\partial} X|^2 \omega_1^n \\ &\geq \lambda_1 \int_M u^2 \omega_1^n. \end{aligned}$$

Here we have used the variational characterization of  $\lambda_1$ . Thus  $\lambda_1 \geq 1$ . If the equality holds, i.e.,  $\lambda_1 = 1$ , we have  $\bar{\partial} X = 0$ . It follows that  $X$  is a holomorphic vector field.

Conversely, if  $X$  is a holomorphic vector field, we define  $u$  by  $i_X \omega_1 = \bar{\partial}u$  and  $\int_M u \omega_1^n = 0$ . Then a straightforward computation shows that

$$\bar{\partial}(\Delta_{\omega_1} u + u) = 0.$$

It follows that  $u$  is an eigenfunction with eigenvalue 1. So we have established the following identification

$$\eta(M) \simeq \{ \text{eigenfunctions of } \omega_1 \text{ with eigenvalue } 1 \}.$$

□

Next we return to the proof of Proposition 6.2.

*Proof.* Let  $\sigma_s$  be the one parameter subgroup generated by the real part of  $\bar{\partial}u$ , write

$$\begin{aligned} \omega_{\rho_s} &= \sigma_s^* \omega_\rho = \omega_\rho + \sqrt{-1} \partial \bar{\partial} (\rho_s - \rho) \\ &= \omega_\varphi + \sqrt{-1} \partial \bar{\partial} (\rho_s - \varphi). \end{aligned}$$

One can easily see that  $\frac{d\rho_s}{ds} |_{s=0} = u$  modulo constants. Denote the complex Laplacian operator of  $\omega_\rho$  by  $\Delta_\rho$ . Then,

$$\Delta_\rho u + u = 0, \quad \forall u \in \Lambda_1(\omega_\rho).$$

Computing the derivative of  $\Psi$  (see Formula 6.2) along this holomorphic path, we have

$$\begin{aligned} 0 &= \frac{d}{ds} \Psi(\sigma_s) |_{s=0} \\ &= -\frac{1}{V} \int_M (\rho - \varphi) \Delta_\rho \frac{d\rho_s}{ds} |_{s=0} \omega_\rho^n \\ &= -\frac{1}{V} \int_M (\rho - \varphi) \Delta_\rho u \omega_\rho^n \\ &= \frac{1}{V} \int_M (\rho - \varphi) u \omega_\rho^n. \end{aligned}$$

In other words,

$$\rho - \varphi \perp \Lambda_1(\omega_\rho).$$

□

The rest of the subsection is devoted to prove that there always exists a minimizer of  $\Psi$  in  $\text{Aut}_r(M)$ . Recall that  $\omega_1$  is a Kähler-Einstein metric, so  $\omega_\rho$  is also Kähler-Einstein metric:

$$\left( \omega_\varphi + \sqrt{-1} \partial \bar{\partial} (\rho(t) - \varphi(t)) \right)^n = \omega_\rho^n = e^{-(\rho-\varphi)+h_\varphi} \omega_\varphi^n, \tag{6.4}$$

where

$$\text{Ric}(\omega_\varphi) - \omega_\varphi = \sqrt{-1} \partial \bar{\partial} h_\varphi.$$

We shall normalize  $h_\varphi$  and  $\rho$  as

$$\frac{1}{V} \int_M e^{-(\rho-\varphi)+h_\varphi} \omega_\varphi^n = \frac{1}{V} \int_M e^{h_\varphi} \omega_\varphi^n = 1.$$

Therefore, we have

$$\sup_M (\rho - \varphi) \geq 0.$$

**Proposition 6.4.** *The following inequalities hold*

$$I - J \leq I \leq (n + 1)(I - J). \tag{6.5}$$

*Proof.* From the definition of  $I$ , we have

$$\begin{aligned} I(\omega_\varphi, \omega) &= \frac{1}{V} \int_M \varphi (\omega^n - \omega_\varphi^n) \\ &= \frac{1}{V} \int_M \varphi \wedge (-\sqrt{-1} \partial \bar{\partial} \varphi) \sum_{i=0}^{n-1} \omega^i \wedge \omega_\varphi^{n-i-1} \\ &= \sum_{i=0}^{n-1} \frac{1}{V} \int_M \sqrt{-1} \partial \varphi \wedge \bar{\partial} \varphi \wedge \omega^i \wedge \omega_\varphi^{n-i-1} \geq 0. \end{aligned}$$

Therefore,

$$I - J = \sum_{i=0}^{n-1} \frac{n-i}{n+1} \frac{1}{V} \int_M \sqrt{-1} \partial \varphi \wedge \bar{\partial} \varphi \wedge \omega^i \wedge \omega_\varphi^{n-i-1}.$$

This in turn implies that

$$I - J \leq I \leq (n + 1)(I - J).$$

□

Next we can prove that  $\Psi$  always achieve its minimum value in  $\text{Aut}_r(M)$ .

**Lemma 6.5.** *The minimal value of  $\Psi$  can be attained in  $\text{Aut}_r(M)$ . Moreover,  $\Psi$  is proper.*

*Proof.* Observe that

$$\Psi(\sigma) = (I - J)(\omega_\sigma, \omega_\rho) \geq \frac{1}{n + 1} \int_M (\rho - \varphi) (\omega_\sigma^n - \omega_\rho^n) \geq 0.$$

Put  $G_r = \{\sigma \in \text{Aut}_r(M) \mid \Psi(\sigma) \leq r\}$  and  $E_r = \{\rho \mid \sigma^* \omega_1 = \omega_\rho, \sigma \in G_r\}$ . Then,

$$\int_M (\rho - \varphi)(\omega_\varphi^n - \omega_\rho^n) \leq (n + 1)r, \quad \forall \rho \in E_r.$$

However,

$$\begin{aligned} -\frac{1}{V} \int_M (\rho - \varphi)\omega_\rho^n &= -\frac{1}{V} \int_M (\rho - \varphi)e^{-(\rho-\varphi)+h_\varphi} \omega_\varphi^n \\ &\geq -C_2 \int_M \rho e^{-\rho} \omega_\varphi^n - C_2' \\ &\geq -C_3. \end{aligned}$$

Therefore, we have

$$\int_M (\rho - \varphi)\omega_\varphi^n \leq C_3'.$$

Since  $\Delta_\varphi(\rho - \varphi) \geq -n$ , by the Green formula, we have

$$\begin{aligned} &\sup_M (\rho - \varphi) \\ &\leq \frac{1}{V} \int_M (\rho - \varphi)\omega_\varphi^n - \max_{x \in M} \left( \frac{1}{V} \int_M (G(x, \cdot) + C_4)\Delta_\varphi(\rho - \varphi)\omega_\varphi^n(y) \right) \\ &\leq \frac{1}{V} \int_M (\rho - \varphi)\omega_\varphi^n + nC_4, \end{aligned}$$

where  $G(x, y)$  is the Green function associated to  $\omega_\varphi$  satisfying  $G(x, \cdot) \geq 0$ . Therefore, there exists a uniform constant  $C$  such that

$$\sup_M (\rho - \varphi) \leq C.$$

On the other hand, we have

$$\begin{aligned} -\int_M (\rho - \varphi)\omega_\rho^n &\leq (n + 1)r - \int_M (\rho - \varphi)\omega_\varphi^n \\ &\leq (n + 1)r + nC_5 - \sup_M (\rho - \varphi) \\ &\leq C_6. \end{aligned}$$

Following Proposition 6.6 below, we can prove that there exists a constant  $C$  such that

$$\inf_M (\rho - \varphi) \geq -C.$$

Hence, by the  $C^2$  estimate of Yau [30] and  $C^3$  estimate of Calabi, we obtain

$$\|\rho - \varphi\|_{C^3} \leq C_7(r), \quad \forall \rho \in E_r.$$

Then  $E_r$  is compact in  $C^2$  topology, and so is  $G_r$ . In particular, the minimal value of  $\Psi$  can be attained.  $\square$

**Proposition 6.6.** *Let  $\omega_\rho$  be a Kähler-Einstein metric, then*

$$0 \leq -\inf_M(\rho - \varphi) \leq C \left( \frac{1}{V} \int_M (-(\rho - \varphi)) \omega_\rho^n + 1 \right).$$

The proposition is known (cf. [28]), we include its proof here for reader's convenience.

*Proof.* Denote by  $\Delta_\rho$  the Laplacian of  $\omega_\rho$ . Then, because  $\omega_\varphi + \partial\bar{\partial}(\rho - \varphi) > 0$ , we see that  $\omega_\varphi = \omega_\rho - \partial\bar{\partial}(\rho - \varphi) > 0$  and taking the trace of this latter expression with respect to  $\omega_\rho$ , we get

$$n - \Delta_\rho(\rho - \varphi) = \text{tr}_{\omega_\rho} \omega_\varphi > 0.$$

Defining now  $(\rho - \varphi)_-(x) = \max\{-(\rho - \varphi)(x), 1\} \geq 1$ , so that

$$(\rho - \varphi)_-^P (n - \Delta_\rho(\rho - \varphi)) \geq 0.$$

And integrating this, we get

$$\begin{aligned} 0 &\leq \frac{1}{V} \int_M (\rho - \varphi)_-^P (n - \Delta_\rho(\rho - \varphi)) \omega_\rho^n \\ &= \frac{n}{V} \int_M (\rho - \varphi)_-^P \omega_\rho^n + \frac{1}{V} \int_M \nabla_\rho(\rho - \varphi)_-^P \nabla_\rho(\rho - \varphi) \omega_\rho^n \\ &= \frac{n}{V} \int_M (\rho - \varphi)_-^P \omega_\rho^n + \frac{1}{V} \int_{\{(\rho - \varphi) \leq -1\}} \nabla_\rho(\rho - \varphi)_-^P \nabla_\rho(\rho - \varphi) \omega_\rho^n \\ &= \frac{n}{V} \int_M (\rho - \varphi)_-^P \omega_\rho^n + \frac{1}{V} \int_M \nabla_\rho(\rho - \varphi)_-^P \nabla_\rho(-(\rho - \varphi)_-) \omega_\rho^n \\ &= \frac{n}{V} \int_M (\rho - \varphi)_-^P \omega_\rho^n - \frac{1}{V} \frac{4p}{(p+1)^2} \int_M |\nabla_\rho(\rho - \varphi)_-^{\frac{p+1}{2}}|^2 \omega_\rho^n, \end{aligned}$$

which yields, using the fact that  $(\rho - \varphi)_- \geq 1$  and hence  $(\rho - \varphi)_-^P \leq (\rho - \varphi)_-^{p+1}$ ,

$$\frac{1}{V} \int_M |\nabla_\rho(\rho - \varphi)_-^{\frac{p+1}{2}}|^2 \omega_\rho^n \leq \frac{n(p+1)^2}{4pV} \int_M (\rho - \varphi)_-^{p+1} \omega_\rho^n.$$

Note that  $\omega_\rho$  is a Kähler Einstein metric which has a uniform Sobolev constant. Thus, we have

$$\frac{1}{V} \left( \int_M |(\rho - \varphi)_-|^{\frac{(p+1)n}{n-1}} \omega_\rho^n \right)^{\frac{n-1}{n}} \leq \frac{c(p+1)}{V} \int_M (\rho - \varphi)_-^{p+1} \omega_\rho^n.$$

Moser’s iteration will show us that

$$\sup_M (\rho - \varphi)_- = \lim_{p \rightarrow \infty} \|(\rho - \varphi)_-\|_{L^{p+1}(M, \omega_\rho)} \leq C \|(\rho - \varphi)_-\|_{L^2(M, \omega_\rho)}.$$

Recall that  $\lambda_1(\omega_\rho) \geq 1$ , so that the Poincaré inequality reads

$$\begin{aligned} \frac{1}{V} \int_M \left( (\rho - \varphi)_- - \frac{1}{V} \int_M (\rho - \varphi)_- \omega_\rho^n \right)^2 \omega_\rho^n &\leq \frac{1}{V} \int_M |\nabla(\rho - \varphi)_-|^2 \omega_\rho^n \\ &\leq \frac{C}{V} \int_M (\rho - \varphi)_- \omega_\rho^n, \end{aligned}$$

where we have set  $p = 1$  and used the same reasoning as before. This then implies that

$$\max\{-\inf_M (\rho - \varphi), 1\} = \sup_M (\rho - \varphi)_- \leq \frac{C}{V} \int_M (\rho - \varphi)_- \omega_\rho^n,$$

since  $\int_M e^{-h_\varphi + (\rho - \varphi)} \omega_\rho^n = V$ , we can easily deduce  $\int_{(\rho - \varphi) > 0} (\rho - \varphi) \omega_\rho^n \leq C$ .

Combining this together with the above, we get

$$-\inf_M (\rho - \varphi) \leq \frac{C}{V} \int_M (-\rho + \varphi) \omega_\rho^n + C,$$

which proves the proposition. □

### 6.2 Application to the Kähler Ricci flow

Let  $\varphi(t)$  be the global solution of the Kähler Ricci flow in the level of Kähler potentials. According to Lemma 6.5, there exists a one parameter family of Kähler Einstein metrics  $\omega_{\rho(t)} = \omega + \sqrt{-1} \partial \bar{\partial} \rho(t)$  such that  $\omega_{\varphi(t)}$  is centrally positioned with respect to  $\omega_{\rho(t)}$  for any  $t \geq 0$ . Suppose that  $\omega_{\varphi(0)}$  is already centrally positioned with the Kähler-Einstein metric  $\omega_1 = \omega + \sqrt{-1} \partial \bar{\partial} \rho(0)$ . Recall that  $E_{k, \omega}(\varphi)$  and  $v_\omega$  all satisfy the cocycle condition:

$$E_{k, \omega}(\varphi) + E_{k, \omega_\varphi}(\psi - \varphi) = E_{k, \omega}(\psi)$$

for any  $k = 0, 1, \dots, n$ . Note that  $v_\omega = E_{0, \omega}$ .

**Theorem 6.7.** *On a Kähler-Einstein manifold, the K-energy  $v_\omega$  is uniformly bounded from above and below along the Kähler Ricci flow. Moreover, there exist some uniform constants  $c, C, C'$  and  $C''$  such that*

$$\begin{aligned} |J_{k, \omega_{\rho(t)}}(\varphi(t) - \rho(t))| &\leq \{v_\omega(\varphi(t)) + C\}^{\frac{1}{\delta}}, \\ \log \frac{\omega_\varphi^n}{\omega_{\rho(t)}^n} &\geq -4C'' e^{2(v_\omega(\varphi(t)) + C)^{\frac{1}{\delta}} + C'}, \\ E_k(\varphi(t)) &\geq -e^{c(1 + \max\{0, v_\omega(\varphi(t))\} + v_\omega(\varphi(t)) + C)^{\frac{1}{\delta}}}. \end{aligned}$$

*Proof.* Since  $\omega_{\varphi(t)}$  is centrally positioned with respect to the Kähler-Einstein metric  $\omega_{\rho(t)}$ , Proposition 6.1 implies that

$$\varphi(t) - \rho(t) \perp \Lambda_1(\omega_{\rho(t)}).$$

Theorem 3.1 implies that K-energy is proper with respect to the evolved Kähler metric  $\omega_\varphi$  and the modified Kähler-Einstein metric  $\omega_{\rho(t)}$ . Thus,

$$v_{\omega_{\rho(t)}}(\varphi(t) - \rho(t)) \geq (J_{\omega_{\rho(t)}}(\varphi - \rho(t)))^\delta - c$$

for some uniform constant  $\delta > 0$  and  $c$ . Since the K energy satisfies the cocycle condition, we have

$$v_\omega(\varphi(t)) - v_{\omega_{\rho(t)}}(\varphi(t) - \rho(t)) = v_\omega(\rho(t)).$$

Lemma 3.8 implies that the K-energy monotonely decreases along the Kähler Ricci flow

$$v_\omega(\varphi(t)) \leq v_\omega(\varphi(0)), \quad \forall t < \infty.$$

Combining the three inequalities above, we arrive at

$$v_\omega(\varphi(0)) \geq v_\omega(\varphi(t)) \geq (J_{\omega_{\rho(t)}}(\varphi - \rho(t)))^\delta - c + v_\omega(\rho(t)).$$

Note that the K energy is invariant under automorphisms and the fact that  $\omega_{\rho(t)}$  is path connected with  $\omega_1$  via automorphisms, then we have

$$v_\omega(\rho(t)) = v_\omega(\rho(0)).$$

Thus

$$0 \leq J_{\omega_{\rho(t)}}(\varphi - \rho(t)) \leq (v_\omega(\varphi(t)) + C)^\frac{1}{\delta} \leq (v_\omega(\varphi(0)) + C)^\frac{1}{\delta}.$$

In particular, the K energy has a uniform up-bound and lower bound along the Kähler Ricci flow. Lemma 4.10 implies that  $E_{k,\omega_\rho}(\varphi(t) - \rho(t))$  are uniformly bounded from below. Now,

$$E_{k,\omega}(\varphi(t)) = E_{k,\omega_\rho}(\varphi(t) - \rho(t)) + E_{k,\omega}(\rho(t))$$

Similarly since  $E_k$  is invariant under automorphisms, we have

$$E_{k,\omega}(\rho(t)) = E_{k,\omega}(\rho(0)).$$

Thus

$$\begin{aligned} E_{k,\omega}(\varphi(t)) &= E_{k,\omega_\rho}(\rho(t) - \varphi(t)) + E_{k,\omega}(\rho(0)) \\ &\geq -e^{c(1+\max\{0, v_\omega(\varphi(t))\} + J_{\omega_\rho}(\varphi - \rho))} + E_{k,\omega}(\rho(0)) \\ &= -e^{c(1+\max\{0, v_\omega(\varphi(t))\} + (v_\omega(\varphi(t)) + C)^\frac{1}{\delta})} - C_1, \end{aligned}$$

where  $c, C$  and  $C_1$  are some uniform constant. It also implies that (from the explicit expression of the K energy (3.2)):

$$\int_M \left( \ln \frac{\omega_{\varphi(t)}^n}{\omega_{\rho}^n} \right) \omega_{\varphi(t)}^n \leq (v_{\omega}(\varphi(t)) + C)^{\frac{1}{\delta}} + C_2,$$

where  $C_2$  is some uniform constant. Proposition 3.14 then implies that  $\log \frac{\omega_{\varphi(t)}^n}{\omega_{\rho(t)}^n}$  is uniformly bounded from below:

$$\begin{aligned} \inf_M \left( \log \frac{\omega_{\varphi}^n}{\omega_{\rho}^n} \right) (x) &\geq -4C_3 e^{2(1 + \int_M \left( \log \frac{\omega_{\varphi}^n}{\omega_{\rho}^n} \right) \omega_{\rho}^n)} \\ &\geq -4C_3 e^{2(v_{\omega}(\varphi(t)) + C)^{\frac{1}{\delta}} + C'} \end{aligned}$$

where  $C, C_3$  and  $C'$  are some uniform constant. Corollary 4.5 shows that  $J_k (k = 0, 1, \dots, n - 2)$  are uniformly bounded from above and below.  $\square$

An immediate corollary is

**Corollary 6.8.** *The energy functional  $E_k (k = 0, 1, \dots, n)$  has a uniform lower bound from below along the Kähler Ricci flow.*

*Proof.* Since  $E_k$  is invariant under action of automorphisms. Thus

$$E_k(\varphi) = E_k(\tilde{\varphi}) \geq -C.$$

$\square$

Now combing Theorem 4.10 and this corollary, we arrive at the following important corollary:

**Corollary 6.9.** *For each  $k = 0, 1, \dots, n$ , there exists a uniform constant  $C$  such that the following holds (for any  $T \leq \infty$ ) along the Kähler Ricci flow:*

$$\int_0^T \frac{k + 1}{V} \int_M (R(\omega_{\varphi(t)}) - r) \text{Ric}(\omega_{\varphi(t)})^k \wedge \omega_{\varphi(t)}^{n-k} dt \leq C.$$

When  $k = 1$ , we have

$$\int_0^{\infty} \frac{1}{V} \int_M (R(\omega_{\varphi(t)}) - r)^2 \omega_{\varphi(t)}^n dt \leq C < \infty.$$

### 7 Injectivity radius

In 1959, Klingenberg proved that for any compact oriented, even dimensional manifold without boundary, if the sectional curvature is bounded in  $(0, 1]$ , then the injectivity radius is at least  $\pi$ . This theorem of Klingenberg does not apply to the evolved metrics in the Kähler Ricci flow since we do

not know if the positivity of the sectional curvature will be preserved. However, by Theorem 2.2, the bisectional curvature is positive along the Kähler Ricci flow if the initial metric has a positive bisectional curvature. Therefore, we need to adopt Klingenberg’s original theorem to our case. Namely, obtaining a similar estimate of the injectivity radius based on the positivity of the bisectional curvature only. Such a lemma is a natural extension of the original Klingenberg’s theorem to the Kähler setting.

**Lemma 7.1.** *Suppose that  $(M, g)$  is an orientable compact Kähler manifold with bisectional curvature bounded in  $(0, 1]$ . Then there exists some uniform constant  $\beta > 0$  such that the injectivity radius must be no less than  $\beta\pi^4$ .*

*Proof.* We follow the arguments in the proof of Klingenberg theorem (c.f. [6]). Since the bisectional curvature  $\leq 1$ , there exists a uniform constant  $\frac{1}{\beta^2}$  such that the sectional curvature is uniformly bounded from above by  $\frac{1}{\beta^2}$ . This follows that the conjugate radius is not shorter than  $\beta\pi$  :

$$conj\ rad_M \geq \beta\pi.$$

A lemma in [6] by Cheeger and Ebin asserts:

$$inj_M = \min\{\beta\pi, \frac{1}{2} \text{ the length of shortest closed geodesic}\}.$$

We want to prove the lemma by contradiction. If the injective radius  $< \beta\pi$ , then there exists a shortest closed geodesic which realizes this injectivity radius. Denote this shortest closed curve by  $c_0(t)$  ( $0 \leq t \leq 2 inj_M$ ) parameterized by the arc length. Suppose  $J$  is the underlying complex structure. The plane spanned by  $c_0(t)'$  and  $J(c_0(t)')$  is a holomorphic plane. Thus the sectional curvature of this plane must be strictly positive. Deform  $c_0$  on the direction of  $J(c_0(t)')$ . Since  $J(c_0(t)')$  is a parallel vector field along this closed geodesic, the second variation in this direction is strictly negative. Therefore, there exists a 1-parameter family of nearby closed curves  $c_s : \mathbb{R}/Z \rightarrow V, t \rightarrow exp_{c_0(t)}(sJ(c_0(t)'))$  which are strictly shorter than  $c_0$  provided  $s$  is small enough. Since the length of  $c_0$  equals to  $2 inj_M$ , the entire curve  $c_s$  ( $s > 0$ ) must be contained in the closed ball with radius  $\leq \frac{1}{2}L(c_s) < inj_M$ . Thus, one can lift the entire curve  $c_s(t)$  as a closed curve  $\tilde{c}_s$  in  $T_{c_s(0)}M$  such that  $\tilde{c}_s(0) = 0$ . Since everything occurs within the conjugate radius, by taking limit, we can lift up  $c_0(t)$  as a closed curve in  $T_{c_0(0)}M$ . That is a contradiction since the lifting of  $c_0$  is a straight line.  $\square$

### 8 Harnack inequality

Recall Cao’s Harnack inequality in the Kähler Ricci flow:

**Theorem 8.1.** [5] *Let  $g_{i\bar{j}}$  be the solution of the Kähler Ricci flow with positive bisectional curvature. Then for any  $x, y \in M$  and  $0 < t_1 < t_2 < \infty$ ,*

---

<sup>4</sup> According to Corollary 12.2, the best constant is  $\beta = \frac{1}{\sqrt{2}}$ .

the scalar curvature  $R$  satisfies the inequality:

$$R(x, t_1) \leq \frac{e^{t_2} - 1}{e^{t_1} - 1} e^{\frac{\Delta}{4}} R(y, t_2).$$

Here  $\Delta$  is defined as

$$\Delta = \Delta(x, y, t_1, t_2) = \inf_{\gamma} \int_{t_1}^{t_2} |\gamma'(s)|^2 ds$$

where the infimum is taken over all curves from  $x$  to  $y$ , where  $|\gamma'(s)|_s$  is the velocity of  $\gamma$  at time  $s$ .

The basic ideas<sup>5</sup> of the proof in [5] can be described as follows: If  $g$  is a Kähler Ricci soliton, we have

$$R_{i\bar{j}} - g_{i\bar{j}} = f_{,i\bar{j}} \tag{8.1}$$

and

$$f_{,ij} = 0, \quad \forall i, j = 1, 2, \dots, n.$$

Thus,  $X = f^i \frac{\partial}{\partial x^i}$  is a holomorphic vector field. Taking Laplacian of the soliton equation (8.1), we arrive at the following

$$\Delta R_{i\bar{j}} + R_{i\bar{j}k\bar{l}} R_{l\bar{k}} - R_{i\bar{j}} + R_{i\bar{j},k} f_{,\bar{k}} + R_{i\bar{j},\bar{k}} f_{,k} + R_{i\bar{j}k\bar{l}} f_{,i} f_{,\bar{k}} = 0.$$

Motivated by this identity for Ricci solitons, Cao introduced the following 2-tensor  $Q_{i\bar{j}}$  for any vector  $v \in T_x V$ ,

$$\begin{aligned} Q_{i\bar{j}} &= \Delta R_{i\bar{j}} + R_{i\bar{j}k\bar{l}} R_{l\bar{k}} - R_{i\bar{j}} + R_{i\bar{j},k} v^k + R_{i\bar{j},\bar{k}} v^{\bar{k}} + R_{i\bar{j}k\bar{l}} v^{\bar{l}} v^k + \frac{R_{i\bar{j}}}{1 - e^{-t}} \\ &= \frac{\partial}{\partial t} R_{i\bar{j}} + R_{i\bar{k}} R_{k\bar{j}} \\ &\quad - R_{i\bar{j}} + R_{i\bar{j},k} v^{\bar{k}} + R_{i\bar{j},\bar{k}} v^k + R_{i\bar{j}k\bar{l}} v^{\bar{l}} v^k + \frac{R_{i\bar{j}}}{1 - e^{-t}}. \end{aligned} \tag{8.2}$$

Clearly,  $Q$  is a positive tensor at  $t = 0$  and  $t = \infty$ . Through tedious but direct calculations, Cao proved that  $Q$  is positive for all the time and for all vectors  $v(x, t)$ . Taking trace on both side of (8.2), we obtain the following

$$\frac{\partial R}{\partial t} + R_{,k} v^k + R_{,\bar{k}} v^{\bar{k}} + R_{i\bar{j}} v^{\bar{i}} v^j + \frac{R}{1 - e^{-t}} > 0.$$

Let  $v_k = \frac{-R_{,k}}{R}$ , then

$$\frac{\partial R}{\partial t} - \frac{|D R|^2}{R} + \frac{R}{1 - e^{-t}} > 0.$$

Using this inequality and a similar argument of Li-Yau [20], Cao [5] proved the Harnack inequality for the scalar curvature of  $M$ .

<sup>5</sup> These types of arguments are due to R. Hamilton in the real case (cf. [17]).

### 9 Convergence by sequence in any $C^l$ norm

In this section, we want to show that for any sequence of metrics over the Kähler Ricci flow, there exists a subsequence which converges to a Kähler-Einstein metric with constant bisectional curvature. We first prove that the bisectional curvature and its derivatives are uniformly bounded in complex dimension 2 in the first subsection. In the second subsection, we then prove the convergence by sequences.

#### 9.1 Uniform curvature bound in complex dimension 2

In this subsection, we concentrate on complex dimension 2 and we will prove that the scalar curvature is uniformly bounded from above along the flow. One should note that only Lemma 9.3 need to be proved in complex dimension 2. All other theorems, lemmas hold for all dimensions.

**Lemma 9.1.** *In the Kähler Ricci flow with positive bisectional curvature, denote the maximal scalar curvature at time  $t$  as  $R_{max}(t)$ . Then*

$$R_{max}(t) \leq 2R_{max(t_0)}, \quad \forall t \in \left[ t_0, t_0 + \frac{1}{2R_{max}(t_0)} \right].$$

*Proof.* During time  $t \in [t_0, t_0 + \frac{1}{2R_{max}(t_0)}]$ , we have

$$\frac{d}{dt} R_{max} \leq R_{max}^2.$$

Thus,

$$R_{max}(t) \leq 2R_{max}(t_0), \quad \forall t \in \left[ t_0, t_0 + \frac{1}{2R_{max}(t)} \right].$$

□

By Theorem 4.10 and Corollary 6.9, for any fixed period  $T^6$ , we have

$$\begin{aligned} \int_0^\infty \int_M (R - r)^2 \omega_\varphi^n dt &= \sum_{n=0}^\infty \int_{nT}^{(n+1)T} \frac{1}{V} \int_M (R - r)^2 \omega_\varphi^n dt \\ &= \int_0^\infty \frac{1}{V} \int_M (R - r)^2 \omega_\varphi^n dt \\ &= \int_0^\infty \frac{1}{V} \int_M (R - r) \text{Ric} \wedge \omega_\varphi^{n-1} dt < \infty. \end{aligned}$$

Thus,

$$\lim_{n \rightarrow \infty} \int_{nT}^{(n+1)T} \frac{1}{V} \int_M (R - r)^2 \omega_\varphi^n dt = 0.$$

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<sup>6</sup> The value of  $T$  will be fixed later in the subsection when we prove Theorem 9.4.

This follows that  $\int_M (R - r)^2 \omega_{\varphi(t)}^n$  is small for almost all  $t$  large. In other words, at every interval of length  $T$ , there exists at least one time  $t$  such that this integral is small:

**Lemma 9.2.** *For any sequence  $s_i \rightarrow \infty$ , and for any fixed time period  $T$ , there exists  $t_i \rightarrow \infty$  and  $0 < s_i - t_i < T$  such that*

$$\lim_{t_i \rightarrow \infty} \frac{1}{V} \int_M (R - r)^2 \omega_{\varphi}^n = 0. \tag{9.1}$$

**Lemma 9.3.** *Over the Kähler Ricci flow on a Kähler surface, if  $t_i \rightarrow \infty$  satisfies the condition (9.1), then  $R_{\max}(t_i) = \max_{p \in V} R(p)(t_i)$  is uniformly bounded from above.*

*Proof.* Choose time  $\tau_i < t_i$  such that  $t_i - \tau_i = \frac{1}{2R_{\max}(\tau_i)}$ <sup>7</sup>. Such a  $\tau_i$  can always be chosen. Following Lemma 9.1, we have

$$R_{\max}(t) \leq 2R_{\max}(\tau_i), \quad \forall t \in [\tau_i, t_i].$$

Recall the flow equation,

$$\frac{\partial}{\partial t} g_{\alpha\bar{\beta}} = g_{\alpha\bar{\beta}} - R_{\alpha\bar{\beta}}.$$

Thus, the distance grows at most by a constant factor since  $R_{\alpha\bar{\beta}} > 0$  for all the time. For any fixed point  $p$ , we have

$$(g_{i\bar{j}})_{n \times n}(p, t) \leq (g_{i\bar{j}})_{n \times n}(p, \tau_i), \quad \forall t \in [\tau_i, t_i].$$

On the other hand,

$$\begin{aligned} \frac{\partial}{\partial t} g_{\alpha\bar{\beta}} &= g_{\alpha\bar{\beta}} - R_{\alpha\bar{\beta}} \\ &\geq -R_{\max}(t) g_{\alpha\bar{\beta}} \\ &\geq -2R_{\max}(\tau_i) g_{\alpha\bar{\beta}}, \quad \forall t \in [\tau_i, t_i]. \end{aligned}$$

Thus,

$$\begin{aligned} (g_{i\bar{j}})_{n \times n}(p, t_i) &\geq (g_{i\bar{j}})_{n \times n}(p, \tau_i) \cdot e^{-2R_{\max}(\tau_i)(t_i - \tau_i)} \\ &= (g_{i\bar{j}})_{n \times n}(p, \tau_i) e^{-1}. \end{aligned}$$

Therefore,

$$(g_{i\bar{j}})_{n \times n}(p, t) \leq 3 \cdot (g_{i\bar{j}})_{n \times n}(p, t_i), \quad \forall t \in [\tau_i, t_i].$$

If  $d(\xi, X)$  is the geodesic distance at time  $t_i$ , then

$$\Delta(\xi, \tau_i, X, t_i) \leq 9 \frac{d(\xi, X)^2}{t_i - \tau_i}.$$

---

<sup>7</sup> In Hamilton's paper, Hamilton choose  $\tau_i$  in a different way.

For all  $X$  in a ball around  $\xi$  of radius

$$\rho = \frac{\pi}{\sqrt{\frac{R_{max}(\tau_i)}{2}}} = \sqrt{\frac{2\pi^2}{R_{max}(\tau_i)}}$$

we have

$$\Delta(\xi, \tau_i, X, t_i) \leq 9 \frac{\left(\sqrt{\frac{2\pi^2}{R_{max}(\tau_i)}}\right)^2}{t_i - \tau_i} = 36\pi^2.$$

When  $t_i, \tau_i$  large enough, we have

$$\frac{e^{t_i} - 1}{e^{\tau_i} - 1} < 2.$$

Then the Harnack inequality gives

$$R(\xi, \tau_i) \leq 2e^{9\pi^2} R(X, t_i)$$

or

$$R(X, t_i) \geq \frac{1}{2} e^{-9\pi^2} R_{max}(\tau_i)$$

for all  $X$  in a ball around  $\xi$  of radius

$$\rho = \frac{\pi}{\sqrt{\frac{R_{max}(\tau_i)}{2}}}.$$

By Lemma 7.1, the injectivity radius of the evolved metric at time  $t_i$  is:

$$inj_M(t_i) \geq \frac{\beta\pi}{\sqrt{\frac{R_{max}(t_i)}{2n(n+1)}}} \geq \beta\rho \cdot \sqrt{\frac{n(n+1)}{2}}.$$

Set  $\rho_1 = \rho \cdot \sqrt{\frac{n(n+1)}{2}}$ . In complex dimension 2, if

$$\min_{X \in B_{\beta\rho_1}(\xi)} R(X, t_i) > 2r,$$

then

$$R(X, t_i) - r > \frac{1}{2} R(X, t_i), \quad \forall X \in B_{\beta\rho_1}(\xi).$$

Therefore, we have (at time  $t_i$ )

$$\int_{B_{\beta\rho_1}(\xi)} (R(X, t_i) - r)^2 \omega_{\varphi(t_i)}^n \geq \frac{1}{4} \int_{B_{\beta\rho_1}(\xi)} R(X, t_i)^2 \omega_{\varphi(t_i)}^n > C.$$

The last inequality follows from a volume comparison theorem (cf. [6]). This contradicts with the initial assumption (9.1). Thus,

$$\min_{X \in B_{\beta\rho_1}(\xi)} R(X, t_i) < 2r.$$

Again, by the Harnack inequality, we have

$$\begin{aligned} \max_{B_{\beta\rho}(\xi)} R(X, t_i) &\leq 2R_{max}(\tau_i) \\ &\leq 4e^{9\pi^2} \min_{B_{\beta\rho}(\xi)} R(t_i) \\ &< 8e^{9\pi^2} r. \end{aligned}$$

Then the scalar curvature must be uniformly bounded above for this sequence  $t_i \rightarrow \infty$ . □

Now combine Lemmas 9.1, 9.2 and 9.3, we can prove the following theorem:

**Theorem 9.4.** *In dimension 2 (or Lemma 9.3 holds), then  $R_{max}(t)$  is bounded from above uniformly along the Kähler Ricci flow.*

*Proof.* Choose  $T = \frac{1}{16r}e^{-9\pi^2}$  in Lemma 9.2. Let  $\{s_i\}, \{t_i\}$  be two sequences as in Lemma 9.2. Then, Lemma 9.3 implies that  $R_{max}(t_i)$  is uniformly bounded by a constant  $8e^{9\pi^2}r$ . Since  $s_i \leq t_i + T$ , Lemma 9.1 implies that  $R_{max}(s_i)$  is uniformly bounded by a constant  $16e^{9\pi^2}r$ . Since  $s_i$  is an arbitrary sequence of time, the maximal scalar curvature must be bounded from above uniformly. □

9.2 Convergence to Kähler-Einstein metrics by sequence

In this subsection, we want to show that for any integer  $l > 0$ , the Kähler Ricci flow converges to a Kähler-Einstein metric in any  $C^l$  norm. Note that the limit Kähler-Einstein metric may be different when extracting from a different sequences. We will defer to the next section to prove that the limit metric is in fact unique. Let us first recall a theorem by W.X. Shi [24] (which we have restated in our setting):

**Theorem 9.5.** [24] *Let  $(M, g_0)$  be a Kähler metric in  $M^n$  with bounded sectional curvature satisfying:*

$$|R_{i\bar{j}k\bar{l}}|^2 \leq k_0, \quad \forall i, j, k, l = 1, 2, \dots, n.$$

*Then there exists a constant  $T(n, k_0)$  which depends only on  $n$  and  $k_0$  such that the evolution equation*

$$\begin{aligned} \frac{\partial g_{i\bar{j}}}{\partial t} &= g_{i\bar{j}} - R_{i\bar{j}} && \text{on } M, \\ g_{i\bar{j}}(x, 0) &= g_{0i\bar{j}}(x), && \forall x \in M \end{aligned}$$

has a smooth solution in  $0 \leq t \leq T(n, k_0)$ , and satisfies the following estimates: For any integer  $m \geq 0$ , there exists constants  $c_m > 0$  depending only on  $n, m$ , and  $k_0$  such that

$$\sup_{x \in M} |\nabla^m R_{i\bar{j}k\bar{l}}| \leq \frac{c_m}{t^m}, \quad \forall 0 \leq t \leq T(n, k_0).$$

In particular, there exists a constant  $c$  such that

$$\frac{1}{c} g_{i\bar{j}}(x) \leq \tilde{g}_{i\bar{j}}(x) \leq c g_{i\bar{j}}(x)$$

where  $\tilde{g}_{i\bar{j}}(x) = g_{i\bar{j}}(x, T)$ .

Combining Shi’s theorem with Theorem 9.4, we arrive at the following

**Theorem 9.6.** *The following statements hold along the Kähler Ricci flow:*

1. *The injectivity radius has a uniform positive lower bound, and the diameter has a uniform upper bound.*
2. *The bisectional curvature and all its derivatives are uniformly bounded from above over the Kähler Ricci flow. In particular, the scalar curvature has a uniform upper bound and positive lower bound.*
3.  $\lim_{t \rightarrow \infty} (R - r) = 0$ .
4. *For any integer  $l > 0$ , and for any time sequence  $t_i \rightarrow \infty$ , there exists a subsequence of  $\{t_i\}$  (still using the same notation) such that the evolved Kähler metrics converges to a Kähler-Einstein metric with constant bisectional curvature in  $C^l$  norm.*

*Proof.* By Theorem 9.4, the bisectional curvature  $R$  is uniformly bounded. Lemma 7.1 then implies that the injectivity radius has a uniform positive lower bound, which in turns implies that the Sobolev constant has a uniform upper bound. Since the volume is fixed along the Kähler Ricci flow, the diameters are bounded uniformly from above. Repeatedly applying the theorem of Shi, we can show that all the derivatives of the sectional curvatures are uniformly bounded over the entire flow. In particular, any sequence of metrics over time must have a subsequence which converges to a limit metric in  $C^l$  for any fixed integer  $l$ .

Next we want to show that  $\lim_{t \rightarrow \infty} (R(t) - r) = 0$ . We just need to show this for an arbitrary sequence  $s_i \rightarrow \infty$ . Lemma 9.2 implies that there exists another sequence of time  $t_i \rightarrow \infty$  such that

$$\lim_{i \rightarrow \infty} \int_M (R(\omega_{\varphi(t_i)} - r)^2 \omega_{\varphi(t_i)}^2) = 0, \quad \text{where } t_i \leq s_i \leq t_i + T.$$

Combining this with the earlier result of convergence in any  $C^l$  norm, we arrive at

$$\lim_{i \rightarrow \infty} (R(\omega_{\varphi(t_i)} - r) = 0.$$

Thus  $\omega_{\varphi(t_i)}$  converges to a Kähler-Einstein metric as  $t_i \rightarrow \infty$ . Note that all of the  $l$ -th derivatives of the evolved metrics are controlled for any integer  $l \geq 0$ . Consider the sequence of the Kähler Ricci flow from  $t_i$  to  $t_i + T$ . This sequence of Ricci flow with fixed length  $T$  converges strongly to the Kähler Ricci flow of limit metrics. Since the limit of  $\omega_{\varphi(t_i)}$  is a Kähler-Einstein metric, the limiting Ricci flow must be trivial and all of the limits of sequences of flow from  $t_i$  to  $t_i + T$  are Kähler Einstein metrics. In particular, since  $t_i \leq s_i \leq t_i + T$ , we show that  $\lim_{i \rightarrow \infty} (R(\omega_{\varphi(s_i)} - r) = 0$ . Since  $\{s_i\}$  is a sequence chosen randomly, we then have

$$\lim_{t \rightarrow \infty} (R(\omega_{\varphi(t)} - r) = 0.$$

In other words, the limit metric of any sequence along the Kähler Ricci flow must be of constant scalar curvature. Consequently, the limit metric of any sequence must be a Kähler-Einstein metric. Moreover, in  $\mathbb{C}P^n$ , this in turn implies that the limit metric has constant bisectional curvature. In summary, we have

$$\lim_{t \rightarrow \infty} \left( R_{i\bar{j}} - \frac{1}{n} R g_{i\bar{j}} \right) = 0,$$

and

$$\lim_{t \rightarrow \infty} \left( R_{i\bar{j}k\bar{l}} - \frac{1}{n(n+1)} R (g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}}) \right) = 0.$$

□

### 10 Exponential convergence

In the previous section, we prove that the Kähler Ricci flow converges to Kähler-Einstein metrics by sequences. Although limit metrics (from different time sequences) might be isometric to each other, but certainly not necessarily unique. We want to show that the limit is unique and the Kähler Ricci flow converges exponentially to this metric. In the 1st subsection, we explain how to initialize the Kähler potential at time  $t = 0$  in order to have convergence on the Kähler potential level. In the second subsection, we prove that Kähler Ricci flow converges exponentially fast to a unique Kähler-Einstein metric.

#### 10.1 Normalization of initial value

Consider the Ricci flow on the Kähler potential level,

$$\frac{\partial \varphi}{\partial t} = \log \frac{\omega_{\varphi}^n}{\omega^n} + \varphi - h_{\omega}. \tag{10.1}$$

Define

$$c(t) = \frac{1}{V} \int_M \frac{\partial \varphi}{\partial t} \omega_\varphi^n.$$

We have the following lemma

**Lemma 10.1.** *Set the initial value of  $\varphi$  at time 0 so that*

$$c(0) = \frac{1}{V} \int_0^\infty e^{-t} \int_M \left| \nabla \frac{\partial \varphi}{\partial t} \right|_\varphi^2 \omega_\varphi^n dt < C.$$

*This normalization is appropriate when the K energy has a uniform lower bound along the Kähler Ricci flow. Then,  $c(t) > 0$  for all time  $t$ . We have*

$$\lim_{t \rightarrow \infty} c(t) = \lim_{t \rightarrow \infty} \frac{1}{V} \int_M \frac{\partial \varphi}{\partial t} \omega_\varphi^n = 0.$$

*Proof.* A simple calculation yields

$$c'(t) = c(t) - \frac{1}{V} \int_M \left| \nabla \frac{\partial \varphi}{\partial t} \right|_\varphi^2 \omega_\varphi^n.$$

Define

$$\epsilon(t) = \frac{1}{V} \int_M \left| \nabla \frac{\partial \varphi}{\partial t} \right|_\varphi^2 \omega_\varphi^n.$$

Since the K energy has a lower bound along the Kähler Ricci flow, we have

$$\int_0^\infty \epsilon(t) dt = \frac{1}{V} \int_0^\infty \int_M \left| \nabla \frac{\partial \varphi}{\partial t} \right|_\varphi^2 \omega_\varphi^n dt < C$$

for some constant  $C$ . Now, we normalize our initial value of  $c(t)$  as

$$\begin{aligned} c(0) &= \frac{1}{V} \int_0^\infty \epsilon(t) e^{-t} dt \\ &= \frac{1}{V} \int_0^\infty \int_M \left| \nabla \frac{\partial \varphi}{\partial t} \right|_\varphi^2 \omega_\varphi^n e^{-t} dt \\ &\leq \frac{1}{V} \int_0^\infty \int_M \left| \nabla \frac{\partial \varphi}{\partial t} \right|_\varphi^2 \omega_\varphi^n dt \\ &= \int_0^\infty \frac{dv}{dt} dt = v(0) - v(\infty) < C. \end{aligned}$$

This shows that our initial setting is correct. From the equation for  $c(t)$ , we have

$$(e^{-t} c(t))' = -\epsilon(t) e^{-t}.$$

Integrating this equation from 0 to  $t$ , we have

$$\begin{aligned} e^{-t}c(t) &= c(0) - \int_0^t \epsilon(\tau)e^{-\tau}d\tau \\ &= \int_0^\infty \epsilon(t)e^{-t}dt - \int_0^t \epsilon(\tau)e^{-\tau}d\tau \\ &= \int_t^\infty \epsilon(\tau)e^{-\tau}d\tau. \end{aligned}$$

Thus

$$\begin{aligned} c(t) &= e^t \int_t^\infty \epsilon(\tau)e^{-\tau}d\tau \\ &= \int_t^\infty \epsilon(\tau)e^{-(\tau-t)}d\tau \\ &\leq \int_t^\infty \epsilon(\tau)d\tau \rightarrow 0. \end{aligned} \tag{10.2}$$

Note that  $c(t) > 0$  for all time. In conclusion, we have

$$\lim_{t \rightarrow \infty} c(t) = \lim_{t \rightarrow \infty} \int_M \frac{\partial \varphi}{\partial t} \omega_\varphi^n = 0. \tag{10.3}$$

□

### 10.2 Exponential convergence

In this subsection, we assume that the evolved Kähler metrics  $\omega_{\varphi(t)}$  converge to a Kähler-Einstein metric in at least  $C^3$ -norm. We then show that the flow must converge to a unique Kähler-Einstein metric exponentially fast.

Recall that the Kähler Ricci flow equation:

$$\frac{\partial \varphi}{\partial t} = \ln \frac{\omega_\varphi^n}{\omega^n} + \varphi - h_\omega.$$

Since the evolved Kähler metrics converge to Kähler-Einstein metrics by sequences in any  $C^k$  norm, then we have the following

1. Modulo constants, we have

$$\lim_{t \rightarrow \infty} \frac{\partial \varphi}{\partial t} = \lim_{t \rightarrow \infty} \left( \ln \frac{\omega_\varphi^n}{\omega^n} + \varphi - h_\omega \right) = 0.$$

This together with the normalization of the initial value (see Lemma 10.1), we have

$$\lim_{t \rightarrow \infty} \frac{\partial \varphi}{\partial t} = \lim_{t \rightarrow \infty} \left( \ln \frac{\omega_\varphi^n}{\omega^n} + \varphi - h_\omega \right) = 0.$$

2. The eigenspace of  $\omega_{\varphi(t)}$  converges to the eigenspace of a Kähler-Einstein metric. Notice that in a fixed Kähler class, all Kähler-Einstein metrics are isometric to each other so that they have the same spectrum.
3. The eigenvalues of  $\omega_{\varphi(t)}$  converge to the eigenvalues of some Kähler Einstein metrics. Note that the second eigenvalue of a Kähler Einstein metric is strictly bigger than 1.

**Proposition 10.2.** *There exists a positive number  $\alpha > 0$  and constant  $C > 0$  such that*

$$\int_M \left( \frac{\partial \varphi}{\partial t} - c(t) \right)^2 \omega_{\varphi}^n \leq C e^{-\alpha t}.$$

Moreover, for every integer  $l > 0$ , there exists a constant  $C_l$  such that

$$\int_M |D^l \left( \frac{\partial \varphi(t)}{\partial t} - c(t) \right)|_{\varphi}^2 \omega_{\varphi(t)}^n \leq C_l e^{-\alpha t}.$$

First we want to prove a corollary of this proposition

**Corollary 10.3.** *There exists a uniform constant  $C$  such that*

$$0 < c(t) \leq C e^{-\alpha t}, \quad \forall t > 0.$$

*Proof.* Recall that

$$\epsilon(t) = \frac{1}{V} \int_M \left| \partial \left( \frac{\partial \varphi(t)}{\partial t} - c(t) \right) \right|_{\varphi}^2 \omega_{\varphi(t)}^n \leq C_1 e^{-\alpha t},$$

where  $C_1$  is some uniform constant. Plugging this into Formula (10.2), we obtain

$$c(t) = \int_t^{\infty} \epsilon(\tau) e^{-(\tau-t)} d\tau \leq \frac{C_1}{1 + \alpha} e^t e^{-(1+\alpha)t} = \frac{C_1}{1 + \alpha} e^{-\alpha t}.$$

□

Next we return to prove Proposition 10.2.

*Proof.* Differentiating the Kähler Ricci flow with respect to time  $t$ ,

$$\frac{\partial^2 \varphi}{\partial t^2} = \Delta_{\varphi} \frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial t}. \tag{10.4}$$

Put  $\mu(t) = \int_M \left( \frac{\partial\varphi}{\partial t} - c(t) \right)^2 \omega_\varphi^n$ . Then

$$\begin{aligned} \frac{d\mu(t)}{dt} &= 2 \int_M \left( \frac{\partial\varphi}{\partial t} - c(t) \right) \left( \frac{\partial^2\varphi}{\partial t^2} - c'(t) \right) \omega_\varphi^n + \int_M \left( \frac{\partial\varphi}{\partial t} - c(t) \right)^2 \Delta_\varphi \frac{\partial\varphi}{\partial t} \omega_\varphi^n \\ &= 2 \int_M \left( \frac{\partial\varphi}{\partial t} - c(t) \right) \left( \Delta_\varphi \frac{\partial\varphi}{\partial t} + \frac{\partial\varphi}{\partial t} - c(t) \right) \omega_\varphi^n + \int_M \left( \frac{\partial\varphi}{\partial t} - c(t) \right)^2 \Delta_\varphi \frac{\partial\varphi}{\partial t} \omega_\varphi^n \\ &= -2 \int_M \left( 1 + \frac{\partial\varphi}{\partial t} - c(t) \right) |\nabla \left( \frac{\partial\varphi}{\partial t} - c(t) \right)|_\varphi^2 \omega_\varphi^n + 2 \int_M \left( \frac{\partial\varphi}{\partial t} - c(t) \right)^2 \omega_\varphi^n. \end{aligned}$$

Here we have used the fact  $\int_M \left( \frac{\partial\varphi}{\partial t} - c(t) \right) \omega_\varphi^n = 0$  twice. Since  $\lim_{t \rightarrow \infty} \frac{\partial\varphi}{\partial t} = \lim_{t \rightarrow \infty} c(t) = 0$  for any  $\epsilon > 0$ , and for  $t$  large enough, we have

$$\begin{aligned} \frac{d\mu(t)}{dt} &\leq -2(1 - \epsilon) \int_M |\nabla \left( \frac{\partial\varphi}{\partial t} - c(t) \right)|_\varphi^2 \omega_\varphi^n \\ &\quad + 2 \int_M \left( \frac{\partial\varphi}{\partial t} - c(t) \right)^2 \omega_\varphi^n. \end{aligned} \tag{10.5}$$

If the first eigenvalue of  $\omega_{\varphi(t)}$  converges to 1, it appears to be quite difficult from the previous inequality to derive any control on  $\frac{d\mu(t)}{dt}$ . Denote the first, second eigenvalue of a Kähler Einstein metric as  $\lambda_1 < \lambda_2$ . Lemma 6.3 implies that  $\lambda_1 \geq 1$  and the equality holds if and only if the space of holomorphic vector fields  $\eta(M)$  is non-trivial. In case of  $\eta(M) = 0$ , we have  $\lambda_1 > 1$ . For  $t$  large enough, all eigenvalues of  $\omega_{\varphi(t)}$  will be bigger than  $\frac{\lambda_1 + 1}{2} > 1$ . Therefore,

$$\int_M |\nabla \left( \frac{\partial\varphi}{\partial t} - c(t) \right)|_\varphi^2 \omega_\varphi^n \geq \frac{\lambda_1 + 1}{2} \int_M \left( \frac{\partial\varphi}{\partial t} - c(t) \right)^2 \omega_\varphi^n.$$

Plugging this into inequality (10.5), we obtain

$$\begin{aligned} \frac{d\mu(t)}{dt} &\leq -2(1 - \epsilon) \frac{\lambda_1 + 1}{2} \int_M \left( \frac{\partial\varphi}{\partial t} - c(t) \right)^2 \omega_\varphi^n + 2 \int_M \left( \frac{\partial\varphi}{\partial t} - c(t) \right)^2 \omega_\varphi^n \\ &\leq -\alpha \int_M \left( \frac{\partial\varphi}{\partial t} - c(t) \right)^2 \omega_\varphi^n = -\alpha \mu(t), \end{aligned}$$

where  $\alpha = 2(1 - \epsilon)\frac{\lambda_1+1}{2} - 2$ . Choose  $\epsilon > 0$  to be small enough, we have  $\alpha > 0$ . It is straightforward to prove that there exists a uniform constant  $C$  such that

$$\mu(t) = \int_M \left( \frac{\partial\varphi}{\partial t} - c(t) \right)^2 \omega_\varphi^n \leq C e^{-\alpha t}.$$

On the other hand, if  $\eta(M) \neq 0$ , then  $\lambda_1 = 1$  and the first eigenvalue of  $\omega_{\varphi(t)}$  converges to 1. The inequality (10.5) gives us little control of the growth of  $\mu(t)$ . However, the Futaki invariant comes to rescue: Let  $X$  be any holomorphic vector field, then (in a Kähler-Einstein manifold)

$$\begin{aligned} 0 &= f_M(X, \omega_\varphi) \\ &= \int_M X \left( \ln \frac{\omega_\varphi^n}{\omega^n} + \varphi - h_\omega \right) \omega_\varphi^n \\ &= \int_M X \left( \frac{\partial\varphi}{\partial t} - c(t) \right) \omega_\varphi^n = - \int_M \Delta_\varphi \theta_X \cdot \left( \frac{\partial\varphi}{\partial t} - c(t) \right) \omega_\varphi^n, \end{aligned}$$

where  $L_X(\omega_\varphi) = \sqrt{-1} \partial \bar{\partial} \theta_X$  as defined in Sect. 6. If  $\omega_\varphi$  were already a Kähler-Einstein metric, then the above inequality would imply that

$$\int_M \Delta_\varphi(\theta_X) \cdot \left( \frac{\partial\varphi}{\partial t} - c(t) \right) \omega_\varphi^n = - \int_M \theta_X \cdot \left( \frac{\partial\varphi}{\partial t} - c(t) \right) \omega_\varphi^n = 0.$$

This in its turn would imply that  $\left( \frac{\partial\varphi}{\partial t} - c(t) \right)$  is perpendicular to the first eigenspace of  $\omega_\varphi$ . And that would give us the desired estimate from the inequality (10.5). Unfortunately,  $\omega_\varphi$  is not a Kähler-Einstein metric. However,  $\omega_{\varphi(t)}$  is at least  $C^3$  close to a Kähler-Einstein metric as  $t \rightarrow \infty$ ; and this shall be sufficient to derive the exponential convergence. Note that the eigenvalues of  $\omega_{\varphi(t)}$  shall converges to the eigenvalues of Kähler Einstein metrics. For any fixed  $\epsilon > 0$ , and for  $t$  large enough, the eigenvalues of  $\omega_{\varphi(t)}$  must be either in  $(1 - \epsilon, 1 + \epsilon)$  or are strictly bigger than  $\frac{\lambda_2+1}{2} > 1 + \epsilon$ . Denote the set of all eigenspaces of  $\omega_{\varphi(t)}$  whose eigenvalues are between  $(1 - \epsilon, 1 + \epsilon)$  as  $\Lambda_{\text{small}}(\omega_\varphi)$ . Then  $\Lambda_{\text{small}}(\omega_\varphi)$  converges to the first eigenspace of some Kähler Einstein metrics. Moreover, by Lemma 6.3,  $\{\Delta_{\varphi(t)}\theta_X \mid X \in \eta(M)\}$  converges to the first eigenspace of the limit Kähler-Einstein metric space. Thus,  $\{\Delta_{\varphi(t)}\theta_X \mid X \in \eta(M)\}$  is essentially  $\Lambda_{\text{small}}(\omega_{\varphi(t)})$ , where possible error terms become as small as needed when  $t \rightarrow \infty$ . In other words, the vanishing of Futaki invariant implies that the projection of  $\frac{\partial\varphi}{\partial t} - c(t)$  into the eigenspace  $\Lambda_{\text{small}}(\omega_\varphi)$  is very small (compare to the size of  $\frac{\partial\varphi}{\partial t} - c(t)$ ). Namely, we have

$$\frac{\partial\varphi}{\partial t} - c(t) = \varrho + \varrho^\perp,$$

where  $\varrho \in \Lambda_{\text{small}}(\omega_\varphi)$  and  $\varrho^\perp \perp \Lambda_{\text{small}}(\omega_\varphi)$ . For  $t$  large enough, we have

$$\int_M \varrho^2 \omega_\varphi^n \leq \epsilon \int_M \left( \frac{\partial \varphi}{\partial t} - c(t) \right)^2 \omega_\varphi^n,$$

and

$$\int_M \varrho^{\perp 2} \omega_\varphi^n \geq (1 - \epsilon) \int_M \left( \frac{\partial \varphi}{\partial t} - c(t) \right)^2 \omega_\varphi^n.$$

Notice that the eigenvalue of  $\omega_{\varphi(t)}$  corresponds to  $\Lambda_{\text{small}}(\omega_{\varphi(t)})^\perp$  are always bigger than  $\frac{\lambda_2 + 1}{2} > 1$  when  $t$  large enough. Therefore,

$$\begin{aligned} \int_M \left| \nabla \left( \frac{\partial \varphi}{\partial t} - c(t) \right) \right|^2 \omega_\varphi^n &\geq \frac{\lambda_2 + 1}{2} \int_M \varrho^{\perp 2} \omega_\varphi^n \\ &\geq (1 - \epsilon) \frac{\lambda_2 + 1}{2} \int_M \left( \frac{\partial \varphi}{\partial t} - c(t) \right)^2 \omega_\varphi^n. \end{aligned}$$

Plugging this into inequality (10.5), we obtain

$$\begin{aligned} \frac{d\mu(t)}{dt} &\leq -2(1 - \epsilon)(1 - \epsilon) \frac{\lambda_2 + 1}{2} \int_M \left( \frac{\partial \varphi}{\partial t} - c(t) \right)^2 \omega_\varphi^n + 2 \int_M \left( \frac{\partial \varphi}{\partial t} - c(t) \right)^2 \omega_\varphi^n \\ &\leq -\alpha \int_M \left( \frac{\partial \varphi}{\partial t} - c(t) \right)^2 \omega_\varphi^n = -\alpha \mu(t), \end{aligned}$$

where  $\alpha = 2(1 - \epsilon)^2 \frac{\lambda_2 + 1}{2} - 2$ . Again, we choose  $\epsilon > 0$  to be small enough, we have  $\alpha > 0$ . It is straightforward to prove that there exists a uniform constant  $C$  such that

$$\mu(t) = \int_M \left( \frac{\partial \varphi}{\partial t} - c(t) \right)^2 \omega_\varphi^n \leq C e^{-\alpha t}.$$

This proves the first part of Proposition 10.2. Next we want to prove the exponential convergence for all derivatives. For any integer  $l > 0$ , consider the  $L^2$  norm of  $l$ -th derivatives ( $l \geq 1$ ) of  $\left( \frac{\partial \varphi}{\partial t} - c(t) \right)$  :

$$\begin{aligned} \mu_l(t) &= \int_M \left| D^l \left( \frac{\partial \varphi(t)}{\partial t} - c(t) \right) \right|_{\varphi(t)}^2 \omega_{\varphi(t)}^n \\ &= \int_M \left| D^l \frac{\partial \varphi(t)}{\partial t} \right|_{\varphi(t)}^2 \omega_{\varphi(t)}^n. \end{aligned}$$

Since  $\lim_{t \rightarrow \infty} \left( \frac{\partial \varphi}{\partial t} - c(t) \right) = 0$  and since the Kähler Ricci flow converges to some Kähler-Einstein metrics in any  $C^k$  norm (Theorem 9.6) for any integer  $k > 0$ , we have

$$\lim_{t \rightarrow \infty} \mu_l(t) = 0.$$

Finally, we want to show that it is exponentially decay along the Kähler Ricci flow. Using the equation (10.4), and the fact that all the derivatives of curvature are uniformly bounded, we have ( $l \geq 1$ )

$$\begin{aligned} & \frac{d\mu_l(t)}{dt} \\ &= \int_M \frac{\partial}{\partial t} |D^l \frac{\partial\varphi(t)}{\partial t}|_\varphi^2 \omega_{\varphi(t)}^n + \int_M |D^l \frac{\partial\varphi(t)}{\partial t}|_\varphi^2 \Delta_{\varphi(t)} \frac{\partial\varphi(t)}{\partial t} \omega_{\varphi(t)}^n \\ &\leq -2 \int_M |D^{l+1} \frac{\partial\varphi(t)}{\partial t}|_\varphi^2 \omega_{\varphi(t)}^n + c(n, l) \int_M |D^l \frac{\partial\varphi(t)}{\partial t}|_\varphi^2 \omega_{\varphi(t)}^n \\ &\leq -2 \int_M |D^{l+1} \frac{\partial\varphi(t)}{\partial t}|_\varphi^2 \omega_{\varphi(t)}^n \\ &\quad + c(n, l) \left( \epsilon \int_M |D^{l+1} \frac{\partial\varphi(t)}{\partial t}|_\varphi^2 \omega_{\varphi(t)}^n + c(\epsilon) \int_M \left( \frac{\partial\varphi(t)}{\partial t} - c(t) \right)^2 \omega_{\varphi(t)}^n \right) \\ &= -(2 - c(n, l)\epsilon) \int_M |D^{l+1} \frac{\partial\varphi(t)}{\partial t}|_\varphi^2 \omega_{\varphi(t)}^n \\ &\quad + c(n, l)c(\epsilon) \int_M \left( \frac{\partial\varphi(t)}{\partial t} - c(t) \right)^2 \omega_{\varphi(t)}^n. \end{aligned}$$

In the first inequality, we use integration by parts and the fact that all of the  $l$ -th derivatives of the metrics are uniformly bounded. In the second to the last inequality, we have used an interpolation formula where  $C(\epsilon)$  is the interpolation constant. Choose  $\epsilon$  to be small enough so that

$$2 - c(n, l)\epsilon > 0.$$

Then, we have

$$\frac{d\mu_l(t)}{dt} \leq c(n, l)c(\epsilon) \int_M \left( \frac{\partial\varphi(t)}{\partial t} - c(t) \right)^2 \omega_{\varphi(t)}^n \leq C_l e^{-\alpha t}.$$

Here  $C_l = c(n, l)c(\epsilon)C < \infty$ . Integrating the above inequality from  $t$  to  $\infty$ , we arrive at the desired estimates:

$$\mu_l(t) = \int_M |D^l \left( \frac{\partial\varphi(t)}{\partial t} - c(t) \right)|_\varphi^2 \omega_{\varphi(t)}^n \leq C_l e^{-\alpha t},$$

where we have used the fact that  $\lim_{t \rightarrow \infty} \mu_l(t) = 0$ . □

Note that  $\omega_{\varphi(t)}$  have uniform positive lower bound on injective radius and a uniform positive bound for Sobolev constant. Combining the above inequality and Sobolev embedding theorem, we arrive at

$$|D^l \left( \frac{\partial\varphi(t)}{\partial t} - c(t) \right)|_\varphi^2 \leq c_l e^{-\alpha t}, \tag{10.6}$$

where  $c_l$  is another set of uniform constants. In particular when  $l = 0$ , we have

$$\left| \frac{\partial\varphi(t)}{\partial t} - c(t) \right| < c_0 e^{-\alpha t}.$$

Combining this inequality with Corollary 10.3, we have

$$\left| \frac{\partial\varphi(t)}{\partial t} \right| < c_0 e^{-\alpha t}$$

where  $c_0$  might be some new constant. Thus there exists a unique Kähler potential  $\varphi(\infty)$  such that

$$|\varphi(t) - \varphi(\infty)| \leq c_0 e^{-\alpha t}.$$

From here, we can easily obtain that  $\omega_{\varphi(t)} (0 \leq t \leq \infty)$  are mutually equivalent, i.e., there exists a uniform constant  $c > 1$  such that

$$\frac{1}{c} \omega_{\varphi(\infty)} \leq \omega_{\varphi(t)} \leq c \omega_{\varphi(\infty)}, \quad \forall t \geq 0.$$

Here  $\varphi(\infty)$  is the unique Kähler Einstein metric (arisen from the limit of the Kähler Ricci flow). Combining this with inequalities (10.6), we can easily imply that

$$\left| D^l \left( \frac{\partial\varphi(t)}{\partial t} - c(t) \right) \right|_{\varphi(\infty)}^2 \leq c_l e^{-\alpha t},$$

Thus,  $\varphi(t)$  converges exponentially fast to a unique Kähler-Einstein metric in  $\mathcal{P}(M, \omega)$  in any  $C^l$  norm. We then prove the following proposition

**Proposition 10.4.** *For any integer  $l > 0$ ,  $\frac{\partial\varphi}{\partial t}$  converges exponentially fast to 0 in any  $C^l$  norm. Furthermore, the Kähler Ricci flow converges exponentially fast to a unique Kähler Einstein metric on any Kähler-Einstein surfaces.*

### 11 Concluding remarks

Now we prove our main Theorem 1.1 and Corollary 1.2.

*Proof.* Theorem 1.1 follows from Proposition 10.4. Next we want to prove Corollary 1.2. For any Kähler metric in the canonical Kähler class such that it has non-negative bisectional curvature on  $M$  but positive bisectional curvature at least at one point, we apply the Kähler Ricci flow to this metric. According to Theorem 2.2, the bisectional curvature of the evolved metric is strictly positive. By our Theorem 1.1, the Kähler Ricci flow converges exponentially to a unique Kähler Einstein metric with constant positive

bisectional curvature. Thus, any Kähler metric with nonnegative bisectional curvature on  $M$  and positive at least at one point is path connected to a Kähler-Einstein metric with positive bisectional curvature. Note that all the Kähler-Einstein metrics are path connected by automorphisms [3]. Therefore, the space of all Kähler metrics with nonnegative bisectional curvature on  $M$  and positive at least at one point, is path connected. Similarly, using Theorem 2.3 and our Theorem 1.1, we can show that all of the Kähler metrics with nonnegative curvature operator on  $M$  and positive at least at one point is path connected. Note that the nonnegative curvature operator implies the nonnegative bisectional curvature.  $\square$

*Remark 11.1.* Combining our Main theorem 1.1 and Theorem 2.2, 2.3, we can easily generalize Corollary 1.2 to the case that the bisectional curvature (or curvature operator) is only assumed to be non-negative.

Next we want to propose some future problems.

*Question 11.2.* As our Remark 1.4 indicates, what we really need is the positivity of Ricci curvature along the Kähler Ricci flow. However, it is not expected that the positivity of Ricci curvature is preserved under the Kähler Ricci flow except on Riemann surfaces. The positivity of bisectional curvature is a technical assumption to assure the positivity of Ricci curvature. It is very interesting to extend Theorem 1.1 to metrics without the assumption on bisectional curvature.

*Question 11.3.* Our main theorem can be extended to the Kähler orbifold case. By an old theorem of Berger, any Kähler-Einstein metric on a Kähler manifold with positive bisectional curvature has constant bisectional curvature. One should be able to use the Kähler Ricci flow to classify all Kähler-Einstein orbifolds with positive bisectional curvature<sup>8</sup>.

*Question 11.4.* Is the positivity of the sectional curvature preserved under Kähler Ricci flow?

## 12 Appendix: Sectional curvature and bisectional curvature

It is well known that in a Kähler manifold, these two types of curvature tensors (the sectional curvature and the bisectional curvature) determine each other uniquely. However, an explicit formula which express the sectional curvatures in terms of the bisectional curvature is not so well publicized in the literature, although the derivation of such a formula is long but rather elementary. However, such an explicit formula is critically needed for the sharp estimate of the injectivity radius in terms of the upper bound of the bisectional curvature. Thus, we will include this formula here for the reader's convenience. However, we will omit the proof here and refer the reader to [18] and [19] for details.

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<sup>8</sup> In our second paper [7], we prove that any Kähler-Einstein orbifold with positive bisectional curvature must be a global quotient of  $\mathbb{C}P^n$ .

We first explain some basic concepts of the sectional curvature and the bisectonal curvature. Let  $u, v, w, x$  be any four tangent vectors in  $M$ . Suppose  $R(u, v, w, x)$  is the Riemannian curvature tensor. Then

$$R(u, v, Jw, Jx) = R(u, v, w, x)$$

where  $J$  is the complex structure of  $M$ . Because of splitting  $T_C M = T^{1,0} M \oplus T^{0,1} M$  into  $\pm\sqrt{-1}$  eigenspaces of  $J$ , we can deduce that  $R(u, v, w, x) = 0$  unless  $w$  and  $x$  are of different type. We will use this property strongly in the this appendix. Suppose  $x \perp y$  are two unit tangent vectors of  $M$ . Denote the sectional curvature on the plane  $x, y$  as  $K(x, y)$ . Set now

$$u = \frac{1}{\sqrt{2}}(x - \sqrt{-1}Jx), \quad v = \frac{1}{\sqrt{2}}(y - \sqrt{-1}Jy).$$

If  $y \perp Jx$ , then

$$R(u, \bar{u}, v, \bar{v}) = R(x, y, y, x) + R(x, Jy, Jy, x). \tag{12.1}$$

If  $y = Jx$ , then

$$R(u, \bar{u}, v, \bar{v}) = R(x, Jx, Jx, x).$$

This means that the bisectonal curvature and the sectional curvature are the same on any holomorphic plane. Now, we seek a formula which expresses the sectional curvature in terms of the bisectonal curvature.

**Lemma 12.1.** *If  $w_1, w_2$  are two mutually perpdcular real vectors in  $TM$  such that the two complex planes spanned by  $w_1$  and  $w_2$  respectively are either perpendicular to each other or are identical, then the sectional curvature of the real plane spanned by these two vector fields is*

$$K(w_1, w_2) = \frac{1}{4} (R(A, \bar{A}, A, \bar{A}) - 2R(B, \bar{B}, A, \bar{A}) + R(B, \bar{B}, B, \bar{B}))$$

where  $A = \frac{1}{\sqrt{2}}(u_1 + u_2)$  and  $B = \frac{1}{\sqrt{2}}(u_1 - u_2)$  and

$$u_1 = \frac{1}{\sqrt{2}}(w_1 - \sqrt{-1}Jw_1), \quad \text{and} \quad u_2 = \frac{1}{\sqrt{2}}(-Jw_2 - \sqrt{-1}w_2).$$

An immediate corollary is

**Corollary 12.2.** *If the bisectonal curvature is less than 1, then the sectional curvature is less than 2.*

We omit the proof for both the lemma and corollary.

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