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# On Kähler-Einstein metrics on certain Kähler manifolds with $C_1(M) > 0$

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In this paper, we prove that there exists a Kähler-Einstein metric, abbreviated as K-E metric, on a $m$-dimension Fermat hypersurface with degree greater than $m-1$. In particular, a Fermat cubic surface admits such a K-E metric. By standard Implicit function theorem, it also implies that there are a lot of $m$-dimension hypersurfaces with degree $\geqq m$, which admit K-E metrics. The problem of K-E metric on a Kähler manifold with definite first Chern class was raised by Calabi [3] thirty years ago. The most important part of the problem was solved in the famous paper of Yau [13]. But the problem is still open in case that the background manifold has positive definite 1$^{st}$ Chern class. In fact, people only know a few examples of K-E manifolds with 1$^{st}$ Chern class positive. As for our knowledge, all of them have automorphism groups of positive dimension. The K-E manifolds shown here only have finite automorphisms.

The idea of the proof is to introduce a global holomorphic invariant $\alpha(M)$ on a Kähler manifold $M$ with $C_1(M) > 0$ and prove that if $\alpha(M) > \frac{m}{m+1}$, where $m = \dim M$, then $M$ admits a K-E metric (Theorem 2.1). Then we estimate the lower bound of $\alpha(M)$. In case that $M$ enjoys a group $G$ of symmetries, we can define $\alpha_G(M)$, similar to $\alpha(M)$, and have a version of Theorem 2.1 for $\alpha_G(M)$ (Theorem 4.1). It turns out that $\alpha_G(M) > \frac{m}{m+1}$ if $M$ is a hypersurface mentioned above.

The invariant $\alpha(M)$ (resp. $\alpha_G(M)$) plays a role in the study of K-E metric more and less same as the Moser-Trundinger constant does in the study of prescribed curvature problem on $S^2$. It would be an interesting problem to determine how large $\alpha(M)$ is. A local problem, which is relevant to $\alpha(M)$, was considered by Bombieri [2] and Skoda [11]. Precisely, they proved that given a plurisubharmonic function $\phi$, if the Lelong number of $\phi$ is small enough, then $\phi$ is locally integrable. $\alpha(M)$ is regarding to the properties of anti-canonical bundle of $M$ and the families of holomorphic curves of smaller degree with respect to the polarization given by $C_1(M)$. We guess that $\alpha(M)$ has a lower bound only depending on the dimension $m$. It is pointed out by Professor Yau that this will result in a upper bound of $(-K_M)^m$.

The organization of this paper is as follows. In § 1, we formulate briefly the problem of K-E metric and reduce it to solving a complex Monge-Ampére equation. We state without proof some theorems on the higher order estimates derivatives of the complex Monge-Ampére equation on $M$. They are slight modifications of some results in S.-T. Yau [13]. Because of those estimates, the existence of K-E metric on $M$ is reduced to the $C^0$-estimate of solutions of that complex Monge-Ampére equation. In § 2, the invariant $\alpha(M)$ is defined. We prove that $\alpha(M)>0$ and the Theorem 2.1, which provides a sufficient condition to assure the existence of K-E metric. In § 3, we give a lower bound of $\alpha(M)$ by considering the families of holomorphic curves in $M$. In particular, if $M=CP^2 \# n\overline{CP^2}$, $3\leqq n\leqq 8$, we prove $\alpha(M)\geqq \frac{1}{2}$. Unfortunately, so far we are unable to provide an example where $\alpha(M)>\frac{m}{m+1}$. We guess that $CP^2\# 8\overline{CP^2}$ is such an example for some good reasons. We would like to mention that Theorem 3.1 has its own interest, even though it is a corollary of Hörmander's $L^2$ estimates of the $\bar\partial$ operator. Theorem 3.1 suggests a possibility of understanding the limiting behavior of a sequence of solutions of the complex Monge-Ampére equations in § 1. Such a situation is quite same as that in the study of Yamabe's equation. The difference is that we don't have a local estimate here as good as there. In § 4, we consider Kähler manifolds with certain group symmetries. The constant $\alpha_G(M)$ is defined. We have a correspondence of Theorem 2.1, i.e. Theorem 4.1. Based on the same trick used in § 3, we give an estimate of $\alpha_G(M)$ and prove that if $M$ is a Fermat hypersurface of dimension $m$ and degree $\geqq m$, then $\alpha_G(M)>\frac{m}{m+1}$. It follows the main result.

In this paper, $M$ is always a Kähler manifold, $g$ is a Kähler metric, in local coordinates, $g=(g_{\alpha\bar\beta})$, where $(g_{\alpha\bar\beta})$ is a positive definite hermitian form. $\omega_g=\frac{\sqrt{-1}}{2}\sum_{\alpha,\beta=1}^{m} g_{\alpha\bar\beta}\, dz^\alpha\wedge d\bar z^\beta$. $\frac{1}{\pi}\,\omega_g\sim C_1(M)$ means that they are cohomological.

## § 1. Preliminaries

Let $(M,g)$ be a Kähler manifold with $C_1(M)>0, g$, as a Kähler class, represents the same cohomology class as Ricci curvature does. Then it is well known that the conjecture of Calabi can be reduced to solving the following complex Monge-Ampére equation

$$\begin{aligned}&\det\left(g_{i\bar j}+\frac{\partial^2\phi}{\partial z_i\,\partial\bar z_j}\right)=\det(g_{i\bar j})\,e^{F-\phi}\\&\left(g_{i\bar j}+\frac{\partial^2\phi}{\partial z_i\,\partial\bar z_j}\right)>0,\qquad \phi\in C^\infty(M,R)\end{aligned}\tag{$*$}$$

where $F\in C^\infty(M,R)$ is a given function.

In order to use the continuity method to solve (*), Aubin [1] introduced the following family of equations

$$\begin{aligned}&\det\left(g_{i\bar{j}}+\frac{\partial^2\phi}{\partial z_i\,\partial\bar{z}_j}\right)=\det(g_{i\bar{j}})\,e^{F-t\phi}\\&\left(g_{i\bar{j}}+\frac{\partial^2\phi}{\partial z_i\,\partial\bar{z}_j}\right)>0,\qquad \phi\in C^\infty(M,R).\end{aligned}\tag{$*$}_t$$

Define $S=\{t\in[0,1]\,|\,(*)_s$ is solvable for $s\in[0,t]\}$. By Yau's solution for Calabi conjecture in case $C_1=0$, $S$ is nonempty. Aubin [1] also proved that $S$ is open by an estimate of first eigenvalue of Kähler metric $\left(g_{i\bar{j}}+\frac{\partial^2\phi}{\partial z_i\,\partial\bar{z}_j}\right)dz^i\otimes d\bar{z}^j$. Hence, to prove $(*)$ solvable, it suffices to show that $S$ is closed, which is equivalent to a uniform $C^3$ estimate of solutions of $(*)_t$ by the standard theory of elliptic equation (cf. [4]).

**Theorem 1.1.** *Suppose that $\phi$ be the solution of $(*)_t$, then*

$$0<m+\Delta\phi\leqq C_1\exp\left(C\left(\phi-\left(1+\frac{t}{m-1}\right)\inf\phi\right)\right)$$

*where $C$ is the constant such that $C+\inf\limits_{i\neq l}R_{i\bar{i}l\bar{l}}>1$, $\{R_{i\bar{i}l\bar{l}}\}$ is the curvature tensor of $g$, $C_1$ depends only on $\sup\limits_{m}(-\Delta F)$, $\sup\limits_{M}|\inf\limits_{i\neq l}(R_{i\bar{i}l\bar{l}})|$, $C\cdot m$ and $\sup\limits_{M}F$.*

*Proof.* A slight modification of Yau's proof in [13].

**Theorem 1.2.** *Let $\phi$ be a solution of $(*)_t$, then there is an estimate of the derivatives $\phi_{ij\bar{k}}$ in terms of*

$$\sum_{i,\bar{j}}g_{i\bar{j}}\,dz^i\otimes d\bar{z}^j,\qquad \sup|F|,\qquad \sup|\nabla F|,\qquad \sup_M\sup_i|F_{i\bar{i}}|$$

*and*

$$\sup_M\sup_{i,j,k}|F_{i\bar{j}k}|\quad\textit{and}\quad\sup_M|\phi|.$$

*Proof.* Same as Yau did in [13].

*Remark.* One can use the integral method to obtain a $C^3$-*estimate* of $\phi$ only depending up to second derivatives of $F$. See [12].

By the above theorems, one sees that the closeness of $S$ follows from the $C^0$-*estimate* of solutions of $(*)_t$.

## §2. A sufficient condition for the existence of K-E metric

In case $m=1$, there is a famous inequality by Trudinger,

$$\int_M e^{\alpha\phi^2}\,dv_M\leqq\gamma\qquad\text{for each}\quad\phi\in C^2(M),$$

with

$$\int_M|\nabla\phi|^2\leqq1,\qquad\int_M\phi=0$$

where $\alpha$, $\gamma$ depend only on the geometry of $M$.

Moser proved that in case $M=S^2$, $\alpha=4\pi$ is the best constant s.t. the inequality holds and applied it to the study of prescribed Gauss curvature problem on $S^2$.

In the following, we introduce a similar constant on Kähler manifold $(M, g)$, where $g$ is the Kähler metric.

Define $$P(M,g)=\left\{\phi\in C^2(M,R)\,|\, g_{i\bar{j}}+\frac{\partial^2\phi}{\partial z_i\,\partial \bar{z}_j}\geqq 0,\ \sup_M \phi=0\right\}.$$

**Lemma 2.1.** *Let $B_R(0)$ be the ball of radius $R$ in $C^n$, centered at 0, $\lambda$, a fixed positive number, then for all plurisubharmonic function $\psi$ in $B_R$, with $\psi(0)\geqq -1$, $\psi(z)\leqq 0$ in $B_R$, one has*

$$\int_{|z|<r} e^{-\lambda\psi(z)}\,dx\leqq C,\qquad \text{where } r<\mathrm{R}e^{-\frac{\lambda}{2}} \tag{1}$$

*where $C$ depends on $m$, $\lambda$, $R$.*

*Proof.* It is a modification of the Lemma 4.4 in Hömander [6].

**Proposition 2.1.** *There exist two positive constants $\alpha$, $C$, depending only on $(M, g)$, such that*

$$\int_M e^{-\alpha\phi}\,dV_M\leqq C\qquad \text{for each } \phi\in P(M,g)$$

*where* $dV_M=\left(\frac{\sqrt{-1}}{2}\right)^m \det(g_{i\bar{j}})\,dz_1\wedge d\bar{z}_1\wedge\ldots\wedge dz_m\wedge d\bar{z}_m=\omega_g^m$.

*Proof.* Let $2r$ be the injective radius of $(M, g)$, $G(x, y)$ be the Green function of the Laplace operator $\Delta$ on $(M, g)$. May assume $\inf_{M\times M} G(x,y)=0$

$$\forall\,\phi\in P(M,g),\qquad \Delta\phi+m\geqq 0,\qquad \text{i.e. } -\Delta\phi\leqq m$$

$$\phi(x)=\frac{1}{V}\int_M \phi(y)\,dV_M(y)-\int_M G(x,y)\,\Delta\phi\,dV_M(y)$$

then

$$0=\sup_M\phi\leqq\frac{1}{M}\int_M\phi(y)\,dV_M(y)+\sup_{x\in M}\int_M G(x,y)(-\Delta\phi)\,dV_M(y)$$
$$\leqq\frac{1}{V}\int_M\phi(y)\,dV_M(y)+m\sup_{x\in M}\int_M G(x,y)\,dV_M(y).$$

Now we fix a $\frac{r}{4}$-*net* $\{x_1,\ldots,x_N\}$ of $M$, s.t. $M=\bigcup_{i=1}^N B_{\frac{r}{4}}(x_i)$, where $B_{\frac{r}{4}}(x_i)$ is the geodesic ball of $M$ at $x_i$ with radius $\frac{r}{4}$.

$\forall\, i$, *by* $\frac{1}{V}\int_M\phi(y)\,dV_M(y)\geqq -m\sup_{x\in M}\int_M G(x,y)\,dV_M(y)=-C_1$ and $\phi\leqq 0$

$$\sup_{B_{r/4}(x_i)}\phi(y)\geqq\frac{-VC_1}{\mathrm{Vol}(B_{\frac{r}{4}}(x_i))}. \tag{2}$$

Let $\psi_i$ be the Kähler potential of $(M, g)$ in $B_{2r}(x_i)$, such that $\psi_i(x_i)=0$, put $C_2 = \sup_i \sup_{x\in B_{\frac{3r}{2}}(x_i)} |\psi_i(x)|$, then

$$\psi_i(x)+\phi(x)\leqq C_2 \quad \text{in } B_{\frac{3r}{4}}(x_i),$$

by (2).

$$\exists\, y_i\in B_{\frac{r}{4}}(x_i), \quad \text{such that } \phi(y_i)\geqq \frac{-VC_1}{\operatorname{Vol}(B_{\frac{r}{4}}(x_i))}$$

put $\alpha = C_2 + \frac{\min_i \operatorname{Vol}(B_{\frac{r}{4}}(x_i))}{VC_1+1}$, by Lemma 2.1, one obtains

$$\int_{B_{r/2}(y_i)} e^{-\alpha(\psi_i(x)+\phi(x)-C_2)}\, dV_M \leqq C.$$

Since $B_{\frac{r}{4}}(x_i)\subset B_{\frac{r}{2}}(y_i)$, and $M=\bigcup_{i=1}^N B_{\frac{r}{4}}(x_i)$, it follows that

$$\int_M e^{-\alpha\phi}\, dV_M \leqq C, \qquad C \text{ depending only on } (M, g). \qquad \square$$

Now we associate a number to $(M, g)$. Define

$$\alpha(M, g)=\sup\{\alpha>0 \mid \exists\, C>0, \text{ s.t. (1) holds for all } \phi\in P\}>0.$$

One can easily deduce the following properties of $\alpha(M, g)$.

**Proposition 2.2.** (i) *$\alpha(M, g)=\alpha(M, g')$, if $g, g'$ are in the same Kähler class.*

(ii) *$\alpha$ is invariant under biholomorphic transformation, i.e. if $\Phi: N\to M$ biholomorphic, $\alpha(M, g)=\alpha(N, \Phi^* g)$.*

In case that $M$ has the first Chern-class $>0$, we take $g$ in the class given by Ricci curvature, then the above proposition says that $\alpha(M)=\alpha(M, g)$ is a holomorphic invariant. One interesting question is how large $\alpha(M)$ is, and how to estimate it from below.

*Example.* $M=CP^m$, $g=(m+1)$ multiple of Fubini-study metric, i.e. $(m+1)\,\partial\bar\partial \log(|z|^2)$, where $z=[z_0, \ldots, z_m]$ is the homogeneous coordinates. Then $\alpha(M)=\frac{1}{m+1}$.

The following theorem is the main result of this section. It provides a sufficient condition to assure the existence of K-E metric.

**Theorem 2.1.** *Let $(M, g)$ be a Kähler manifold, $\frac{1}{\pi}\omega_g$ represents the first Chern class. If $\alpha(M)>\frac{m}{m+1}$, then $M$ admits a Kähler Einstein metric.*

The rest of this section is devoted to the proof of this theorem.

First we introduce two functionals defined by Aubin [1],

$$I(\phi)=\frac{(\sqrt{-1})^m}{V}\int_M \phi(\omega_0^m-\omega^m),$$

$$J(\phi)=\int_0^1 \frac{I(s\phi)\,ds}{s}$$

where

$$\omega_0 = g_{\alpha\beta}\, dz^\alpha \wedge d\bar{z}^\beta,\ \omega = \omega_0 + \partial\bar{\partial}\phi,\ V = (\sqrt{-1})^m \int_M \omega_0^m,\ \phi \in P(M,g), \quad \text{then } \sqrt{-1}\,\omega \geqq 0,$$

$$\sqrt{-1}\,\omega_0 > 0.$$

**Lemma 2.2.** $\frac{m+1}{m} J(\phi) \leqq I(\phi) \leqq (m+1)\, J(\phi)$.

*Proof.*

$$J(\phi) = \frac{(\sqrt{-1})^m}{V} \int_0^1 ds \int_M \phi(\omega_0^m - (\omega_0 + s\partial\bar{\partial}\phi)^m)$$

$$= \frac{(\sqrt{-1})^m}{V} \int_0^1 s\, ds \int_M \partial\phi \wedge \bar{\partial}\phi \wedge \left(\sum_{k=0}^{m-1} \omega_0^{m-k-1} \wedge \omega^k \left(\sum_{j=0}^{m-1} \binom{j}{k} (1-s)^{j-k} s^k\right)\right)$$

where

$$\binom{j}{k} = 0 \quad \text{if } j<k; \qquad \binom{j}{k} = \frac{j!}{k!\,(j-k)!} \quad \text{if } j \geqq k,$$

$$\text{for } j \geqq k, \qquad \int_0^1 (r-s)^{j-k} s^{k+1}\, ds = \frac{(j-k)!\,(k+1)!}{(j+2)!}.$$

Hence

$$J(\phi) = \frac{(\sqrt{-1})^m}{V} \int_M \partial\phi \wedge \bar{\partial}\phi \wedge \sum_{k=0}^{m-1} \omega_0^{m-k-1} \wedge \omega^k \left(\sum_{j=k}^{m-1} \frac{k+1}{(j+2)(j+1)}\right)$$

$$= \frac{(\sqrt{-1})^m}{V} \int_M \partial\phi \wedge \bar{\partial}\phi \wedge \left(\sum_{k=0}^{m-1} \frac{m-k}{m+1}\, \omega_0^{m-k-1} \wedge \omega^k\right)$$

$$\frac{m}{m+1} I(\phi) = \frac{(\sqrt{-1})^m}{V} \int_M \partial\phi \wedge \bar{\partial}\phi \wedge \left(\frac{m}{m+1} \sum_{k=0}^{m-1} \omega_0^{m-k-1} \wedge \omega^k\right)$$

$$= J(\phi) + \frac{(\sqrt{-1})^m}{V} \int_M \partial\phi \wedge \bar{\partial}\phi \wedge \sum_{k-1}^{m-1} \frac{k}{m+1}\, \omega_0^{m-k-1} \wedge \omega^k \geqq J(\phi)$$

$$(m+1)\, J(\phi) = \frac{(\sqrt{-1})^m}{V} \int_M \partial\phi \wedge \bar{\partial}\phi \wedge \sum_{k=1}^{m-1} \frac{k}{m-1-k}\, \omega_0^{m-k-1} \wedge \omega^k + I(\phi) \geqq I(\phi).$$

**Lemma 2.3.** *Let $t \to \phi_t$ be a curve in $\mathring{P}(M,g)$, then*

$$\frac{d}{dt}(I(\phi_t) - J(\phi_t)) = -\frac{(\sqrt{-1})^m}{V} \int_M \phi_t(\Delta_{\phi_t} \dot{\phi}_t)(\omega_0 + \sqrt{-1}\,\partial\bar{\partial}\phi_t)^m$$

*where $\Delta_{\phi_t}$ is the Laplace operator of the metric* $\left(g_{\alpha\beta} + \frac{\partial^2\phi}{\partial z_\alpha \partial \bar{z}_\beta}\right)$, $\dot{\phi}_t = \frac{d\phi_t}{dt}$.

*Proof.* At $t = t_0$, by Taylor expansion, $\phi_t = \phi_{t_0} + \dot{\phi}_{t_0} \cdot (t - t_0) + o(|t - t_0|)$.
For simplicity, we assume that $\phi = \phi_{t_0}$, $\dot{\phi} = \dot{\phi}_{t_0}$.

$$\omega_t = \omega_0 + \partial\partial\phi_t,$$

by the above,

$$I(\phi_t)-J(\phi_t)=\frac{(\sqrt{-1})^m}{V}\int_M\left(\partial\phi_t\wedge\bar\partial\phi_t\wedge\sum_{k=0}^{m-1}\frac{k+1}{m+1}\,\omega_0^{m-k-1}\wedge\omega_t^k\right)$$

$$=I(\phi)-J(\phi)+\frac{(\sqrt{-1})^m}{V}\left[\int_M\partial\dot\phi\wedge\bar\partial\phi\wedge\sum_{k=0}^{m-1}\frac{k+1}{m+1}\,\omega_0^{m-k-1}\wedge\omega^k\right.$$

$$+\int_M\partial\phi\wedge\bar\partial\dot\phi\wedge\sum_{k=0}^{m-1}\frac{k+1}{m+1}\cdot\omega_0^{m-k-1}\wedge\omega^k$$

$$\left.+\int_M\partial\phi\wedge\bar\partial\phi\wedge\sum_{k=1}^{m-1}\frac{k(k+1)}{m+1}\,\omega_0^{m-k-1}\wedge\partial\bar\partial\dot\phi\wedge\omega^{k-1}\right]\cdot(t-t_0)$$

$$+o(|t-t_0|).$$

It follows that

$$\left.\frac{d(I(\phi_t)-J(\phi_t))}{dt}\right|_{t=t_0}$$

$$=\frac{(\sqrt{-1})^m}{V}\left[2\int_M\partial\phi\wedge\bar\partial\dot\phi\wedge\sum_{k=0}^{m-1}\frac{k+1}{m+1}\,\omega_0^{m-k-1}\wedge\omega^k\right.$$

$$\left.+\int_M\partial\phi\wedge\bar\partial\phi\wedge\sum_{k=1}^{m-1}\frac{k(k+1)}{m+1}\,\omega_0^{m-k-1}\wedge\omega^{k-1}\wedge\partial\bar\partial\dot\phi\right]$$

$$=\frac{(\sqrt{-1})^m}{V}\left[-\int_M\phi\partial\bar\partial\dot\phi\wedge\left(\sum_{k=0}^{m-1}\frac{2(k+1)}{m+1}\,\omega_0^{m-k-1}\wedge\omega^k\right)\right.$$

$$\left.-\int_M\phi\partial\bar\partial\dot\phi\wedge\sum_{k=1}^{m-1}\frac{k(k+1)}{m+1}\,(\omega_0^{m-k-1}\wedge\partial\bar\partial\phi\wedge\omega^{k-1})\right]$$

$$=\frac{-(\sqrt{-1})^m}{V}\int_M\phi\partial\bar\partial\dot\phi\wedge\left[2\sum_{k=0}^{m-1}\frac{k+1}{m+1}\,\omega_0^{m-k-1}\wedge\omega^k\right.$$

$$\left.+\sum_{k=1}^{m-1}\frac{k(k+1)}{m+1}\,(\omega_0^{m-k-1}\wedge\omega^k-\omega_0^{m-k}\wedge\omega^{k-1})\right]$$

$$=-\frac{(\sqrt{-1})^m}{V}\int_M\phi\partial\bar\partial\dot\phi\wedge\left[\sum_{k=0}^{m-1}\frac{2(k+1)}{m+1}\,\omega_0^{m-k-1}\wedge\omega^k\right.$$

$$+\sum_{k=1}^{m-2}\left(\frac{k(k+1)}{m+1}-\frac{(k+1)(k+2)}{m+1}\right)\omega_0^{m-k-1}\wedge\omega^k$$

$$\left.+\frac{m(m-1)}{m+1}\,\omega^{m-1}-\frac{2}{m+1}\,\omega_0^{m-1}\right]$$

$$=-\frac{m(\sqrt{-1})^m}{V}\int_M\phi\partial\bar\partial\dot\phi\wedge\omega^{m-1}$$

$$=-\frac{(\sqrt{-1})^m}{V}\int_M\phi\Delta_\phi\dot\phi\,\omega^m.$$

Now we suppose that $\phi_t$ be the solution of $(*)_t$ for $t\in S$, then

$$\phi_t \in \mathring{P}(M,g), \qquad \det\left(g_{ij}+\frac{\partial^2 \phi_t}{\partial z_i\, \partial \bar{z}_j}\right)=\det(g_{ij})\, e^{f-t\phi_t}.$$

Take the differential with respect to $t$ on both sides of the above equation, one obtains

$$\Delta_{\phi_t}\dot{\phi}_t = -t\dot{\phi}_t - \phi_t.$$

**Corollary.** *For the family $\{\phi_t\}$ of solutions of $(*)_t$, $I(\phi_t)-J(\phi_t)$ is an increasing function of $t\in S$.*

*Proof.* By Lemma 2.3,

$$\begin{aligned}\frac{d(I(\phi_t)-J(\phi_t))}{dt} &= -\frac{(\sqrt{-1})^m}{V}\int_M \phi_t \Delta_{\phi_t}\dot{\phi}_t\,\omega_t^m\\ &= \frac{(\sqrt{-1})^m}{V}\int_M (\Delta_{\phi_t}\dot{\phi}_t + t\dot{\phi}_t)(\Delta_{\phi_t}\dot{\phi}_t)\,\omega_t^m.\end{aligned}$$

We compute the Ricci curvature of the new Kähler metric $\left(g_{ij}+\frac{\partial^2 \phi_t}{\partial z_i\, \partial \bar{z}_j}\right)$.

$$\begin{aligned}\mathrm{Ric}(\omega_t) &= -\partial\bar{\partial}\log\det\left(g_{ij}+\frac{\partial^2 \phi_t}{\partial z_i\, \partial \bar{z}_j}\right)\\ &= -\partial\bar{\partial}\log\det(g_{ij}) - \partial\bar{\partial}F + t\partial\bar{\partial}\phi_t\\ &= \mathrm{Ric}(\omega_0) - \partial\bar{\partial}F + t\partial\bar{\partial}\phi_t = \omega_0 + t\partial\bar{\partial}\phi_t = t\omega_t + (1-t)\,\omega_0 > t\omega_t.\end{aligned}$$

By the well-known Bochner identity, one sees that the first eigenvalue of $\Delta_{\phi_t}$ is greater than $t$. Hence $-\Delta_{\phi_t}-t>0$. It follows that

$$\frac{d(I(\phi_t)-J(\phi_t))}{dt} \geqq 0.$$

**Proposition 2.3.** *Let $\phi_t$ be the solution of $(*)_t$, $t\in S$, such that $t\to\phi_t$ is a smooth family. Then*

(i) $\frac{(\sqrt{-1})^m}{V}\int_M(-\phi_t)\,\omega_t^m \leqq m \sup_M \phi_t + C$, *where $C$ is a constant depending only on $(M,g)$.*

(ii) $\forall\, \varepsilon>0$, $\exists$ *constant $C_\varepsilon$, such that*

$$\sup_M \phi_t \leqq (m+\varepsilon)\frac{(\sqrt{-1})^m}{V}\int_M(-\phi_t)\,\omega_t^m + C_\varepsilon.$$

*Proof.* By Lemma 2.3,

$$\frac{d(I(\phi_t)-J(\phi_t))}{dt} = -\frac{(\sqrt{-1})^m}{V}\int_M \phi_t(\Delta_{\phi_t}\dot{\phi}_t)\,\omega_t^m.$$

Since $\phi_t$ is the solution of $(*)_t$, $\Delta_{\phi_t}\dot{\phi}_t = -t\dot{\phi}_t - \phi_t$

$$\begin{aligned}\frac{d(I(\phi_t)-J(\phi_t))}{dt}&=\frac{(\sqrt{-1})^m}{V}\int_M \phi_t(\phi_t+t\dot{\phi}_t)\,\omega_t^m\\&=\frac{(\sqrt{-1})^m}{V}\int_M \phi_t(\phi_t+t\dot{\phi}_t)\,e^{F-t\phi_t}\,\omega_0^m\\&=\frac{(\sqrt{-1})^m}{V}\frac{d}{dt}\Big(\int_M(-\phi_t)\,e^{F-t\phi_t}\,\omega_0^m\Big)\\&\quad+\frac{(\sqrt{-1})^m}{V}\int\dot{\phi}_t\,e^{F-t\phi_t}\,\omega_0^m.\end{aligned}$$

From the identity, $\int_M \omega_0^m=\int_M \omega_t^m=\int_M e^{F-t\phi_t}\,\omega_0^m$, we obtain

$$\int_M (t\dot{\phi}_t+\phi_t)\,e^{F-t\phi_t}\,\omega_0^m=0,\qquad \text{i.e. } \int_M \dot{\phi}_t\,e^{F-t\phi_t}\,\omega_0^m=-\frac{1}{t}\int_M \phi_t\,\omega_t^m.$$

Hence

$$\frac{d(I(\phi_t)-J(\phi_t))}{dt}=\frac{1}{t}\frac{d}{dt}\left(\frac{t(\sqrt{-1})^m}{V}\int_M(-\phi_t)\,\omega_t^m\right)$$

$$\frac{d[t(I(\phi_t)-J(\phi_t))]}{dt}-(I(\phi_t)-J(\phi_t))=\frac{d}{dt}\left(\frac{t(\sqrt{-1})^m}{V}\int_M(-\phi_t)\,\omega_t^m\right).$$

Integrating it from 0 to $t$,

$$t(I(\phi_t)-J(\phi_t))-\int_0^t(I(\phi_s)-J(\phi_s))\,ds=t\,\frac{(\sqrt{-1})^m}{V}\int_M(-\phi_t)\,\omega_t^m.$$

Dividing $t$ on both sides,

$$\frac{(\sqrt{-1})^m}{V}\int(-\phi_t)\,\omega_t^m=(I(\phi_t)-J(\phi_t))-\frac{1}{t}\int_0^t(I(\phi_s)-J(\phi_s))\,ds$$

by Lemma 2.2, $\frac{1}{m+1}I(\phi_t)\leqq I(\phi_t)-J(\phi_t)\leqq\frac{m}{m+1}I(\phi_t)$. Since $I(\phi_t)-J(\phi_t)$ is increasing,

$$\begin{aligned}\frac{(\sqrt{-1})^m}{V}\int_M(-\phi_t)\,\omega_t^m&\leqq\frac{m}{m+1}I(\phi_t)-(I(\phi_0)-J(\phi_0))\\&=\frac{m}{m+1}\frac{(\sqrt{-1})^m}{V}\int_M\phi_t(\omega_0^m-\omega_t^m)-(I(\phi_0)-J(\phi_0))\end{aligned}$$

i.e.

$$\frac{(\sqrt{-1})^m}{V}\int_M(-\phi_t)\,\omega_t^m\leqq m\,\frac{(\sqrt{-1})^m}{V}\int_M\phi_t\,\omega_0^m-(I(\phi_0)-J(\phi_0)).$$

On the other hand, put $\varepsilon'=\frac{\varepsilon}{m-1+\varepsilon}$,

$$\begin{aligned}\frac{(\sqrt{-1})^m}{V}\int(-\phi_t)\,\omega_t^m&\geqq(1-\varepsilon')(I(\phi_t)-J(\phi_t))-\frac{1}{t}\int_0^{t-\varepsilon'}(I(\phi_s)-J(\phi_s))\\&\geqq(1-\varepsilon')\frac{1}{m+1}I(\phi_t)-\frac{1}{t}\int_0^{t-\varepsilon'}(I(\phi_s)-J(\phi_s))\,ds\end{aligned}$$

$$\therefore \frac{(\sqrt{-1})^m}{V} \int_M \phi_t \omega_0^m \leqq (m+\varepsilon) \frac{(\sqrt{-1})^m}{V} \int_M (-\phi_t)\, \omega_t^m + (I(\phi_{t-\varepsilon'}) - J(\phi_{t-\varepsilon'})).$$

Hence, in order to prove the proposition, we only need to show that

$$\sup_M \phi_t \leqq \frac{(\sqrt{-1})^m}{V} \int_M \phi_t \omega_0^m + C$$

which appeared in the proof of Proposition 2.1 simply as an application of Green formula.

**Lemma 2.4.** *Let* $g_{tij} = g_{ij} + \frac{\partial \phi_t}{\partial z_i \partial \bar{z}_j}$, *then for* $t \geqq \varepsilon > 0$, *there exists two constants* $C_1$, $C_2$, *depending on* $\varepsilon$, $V$, *such that* $\forall f \in C^1(M, R)$,

$$C_1 \left( \int_M |f|^{\frac{2m}{m-1}} dV_t \right)^{\frac{m-1}{m}} - C_2 \int_M |f|^2 dV_t \leqq \int_M |\nabla^t f|^2 dV_t.$$

*Proof.* As we said, $\mathrm{Ric}_{g_t} \geqq t \geqq \varepsilon > 0$, and the volume is fixed, then the lemma follows from a combination of results in Croke [7] and Li [8].

*The proof of Theorem 2.1.* It suffices to prove that there exists a sequence $\{t_i\}$, such that $t_i \to \bar{t} \in \bar{S} \backslash S$ as $i \to +\infty$, and $\|\phi_{t_i}\|_{C^0}$ is uniformly bounded.

May assume that $t_i \geqq \varepsilon > 0$, since $0 \in S$, $S$ is open.

By the assumption, $\exists\, \alpha$ between $\frac{m}{m+1}$ and $\alpha(M)$, such that

$$\int_M e^{-\alpha(\phi_{t_i} - \sup_M \phi_{t_i})} dV_M \leqq C$$

i.e.

$$\int_M e^{(1-\alpha)\phi_{t_i} - \alpha \sup_M \phi_{t_i} - F} dV_t \leqq C \quad \text{where } C \text{ is independent of } t_i.$$

By the concavity of log,

$$\int_M ((1-\alpha)\,\phi_{t_i} - \alpha \sup_M \phi_{t_i} - F) \frac{(\sqrt{-1}\,\omega_t)^m}{V}$$

$$\leqq \log \left( \int_M e^{(1-\alpha)\phi_{t_i} - \alpha \sup_M \phi_{t_i} - F} \frac{(\sqrt{-1}\,\omega_t)^m}{V} \right) \leqq \log C.$$

Hence, $\sup_M \phi_{t_i} \leqq \frac{1-\alpha}{\alpha} \int_M (-\phi_{t_i}) \frac{\omega_t^m}{V} + C$. By the Proposition 2.3,

$$\int_M (-\phi_{t_i}) \frac{(\sqrt{-1}\,\omega_t)^m}{V} \leqq m \sup_M \phi_{t_i} + C$$

$$\leqq m \frac{1-\alpha}{\alpha} \int_M (-\phi_{t_i}) \frac{(\sqrt{-1}\,\omega_t)^m}{V} + C$$

$$\alpha > \frac{m}{m+1}, \quad \therefore \frac{m(1-\alpha)}{\alpha} < 1, \quad \text{it follows that } \int_M (-\phi_{t_i}) \frac{dV_t}{V} \leqq C.$$

Proposition 2.3 also implies that $\sup_M \phi_{t_i} \leqq C$. It remains to show that $-\inf_M \phi_{t_i} \leqq C$.

For this, we use the standard iteration. Rewrite the equation

$$\det\left(g_{i\bar{j}}+\frac{\partial^2 \phi_t}{\partial z_i \partial \bar{z}_j}\right)=\det(g_{i\bar{j}})\, e^{F-t\phi_t}$$

as

$$g'^{i\bar{j}}\left(g_{i\bar{j}}+\frac{\partial^2 \phi_t}{\partial z_i \partial \bar{z}_j}\right)=m$$

where $(g'^{i\bar{j}})$ is the inverse of $\left(g_{i\bar{j}}+\frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j}\right)$. Hence, $\Delta'\phi_t \leqq m$, where $\Delta'=\Delta_{\phi_t}$, set $\psi=\max\{-\phi_t, 0\}$, then for $p>0$,

$$\frac{4p}{(p+1)^2}\int_M \left|\nabla' \psi^{\frac{p+1}{2}}\right|^2 dV_{t_i} \leqq m\int_M \psi^p\, dV_{t_i}.$$

By Lemma 2.4,

$$C_1\left(\int_M \psi^{(p+1)\frac{m}{m-1}}\, dV_{t_i}\right)^{\frac{m-1}{m}} \leqq \frac{m(p+1)^2}{4p}\int_M \psi^p\, dV_{t_i}+C_2\int_M \psi^{p+1}\, dV_{t_i} \qquad (4)$$

take $p_1=1$, $p_l=(p_{l-1}+1)\dfrac{m}{m-1}-1$ for $l\geqq 2$.

If there exist infinity number of $p_l$, s.t.

$$\left(\int_M \psi^{p_l+1}\, dV_{t_i}\right)^{\frac{1}{p_l+1}} \leqq \max\left\{\left(\int_M \phi^2\, dV_{t_i}\right)^{1/2}, 1\right\}$$

then $\sup_M \psi \leqq \max\{(\int_M \psi^2\, dV_{t_i})^{1/2}, 1\}$ by taking the limit on $p_l$.

So we may assume that

$$\exists\, l_0 \geqq 1, \text{ s.t. } \quad \forall\, l \geqq l_0, \quad \left(\int_M \psi^{p_l+1}\, dV_{t_i}\right)^{\frac{1}{p_l+1}} \geqq \max\left\{\left(\int_M \psi^2\, dV_{t_i}\right)^{1/2}, 1\right\}.$$

The inequality (4) implies that for $l \geqq l_0$

$$C_1\left(\int_M \psi^{p_{l+1}+1}\, dV_{t_i}\right)^{\frac{m-1}{m}} \leqq (mp_l(1+V)+C_2)\int_M \psi^{p_l+1}\, dV_{t_i}$$

i.e.

$$\left(\int_M \psi^{p_{l+1}+1}\, dV_{t_i}\right)^{\frac{1}{p_{l+1}+1}} \leqq (Cp_l)^{\frac{1}{p_l+1}}\left(\int_M \psi^{p_l+1}\, dV_{t_i}\right)^{\frac{1}{p_l+1}}$$

$$\sup_M \psi=\lim_{l\to\infty}\left(\int_M \psi^{p_{l+1}+1}\, dV_{t_i}\right)^{\frac{1}{p_{l+1}+1}} \leqq \prod_{l=l_0}^{\infty}(Cp_l)^{\frac{1}{p_l+1}}\left(\int_M \psi^{p_{l_0}+1}\, dV_{t_i}\right)^{\frac{1}{p_{l_0-1}+1}}$$

$$\prod_{l=l_0}^{\infty}(Cp_l)^{\frac{m}{p_l+1}} \leqq C^{\frac{1}{p_{l_0}+1}\sum_{k=0}^{\infty}\left(\frac{m-1}{m}\right)^k} \cdot e^{\frac{1}{p_{l_0}+1}\sum_{k=0}^{\infty}\left(\frac{m-1}{m}\right)^k\left(\log(p_{l_0}+1)-k\log\frac{m}{m-1}\right)}$$

is bounded and

$$\left(\int_M \psi^{p_{l_0}+1} dV_{t_i}\right)^{\frac{1}{p_{l_0}+1}} \leqq \left(\frac{m p_{l_0-1}}{C_1} \int_M \psi^{p_{l_0-1}} dV_{t_i} + \frac{C_2}{C_1} \int_M \psi^{p_{l_0-1}+1} dV_{t_i}\right)^{\frac{1}{p_{l_0-1}+1}}$$
$$\leqq \left(\frac{m p_{l_0-1}+C_2}{C_1} \int_M \psi^{p_{l_0-1}+1} dV_{t_i} + \frac{m p_{l_0-1} V}{C_1}\right)^{\frac{1}{p_{l_0-1}+1}}$$
$$\leqq \left(\frac{m p_{l_0-1}+C_2}{C_1}\right)^{\frac{1}{p_{l_0-1}+1}} \left(\left(\int_M \psi^{p_{l_0-1}+1} dV_{t_i}\right)^{\frac{1}{p_{l_0-1}+1}} + \left(\frac{m p_{l_0-1} V}{C_1}\right)^{\frac{1}{p_{l_0-1}+1}}\right)$$
$$\leqq C \max\left\{\left(\int_M \psi^2 dV_{t_i}\right)^{1/2}, 1\right\}.$$

Therefore, we always have

$$\sup_M \psi \leqq C \max\left\{\left(\int_M \psi^2 dV_{t_i}\right)^{1/2}, 1\right\}.$$

On the other hand, $\int_M |\nabla' \psi|^2 dV_{t_i} \leqq m \int_M \psi dV_{t_i}$.

The first eigenvalue of $(M, \Delta_t)$ is greater than the lower bound of Ric curvature, i.e. $\mathrm{Ric}(g_t) \geqq t \geqq \varepsilon > 0$. Hence

$$\int_M \psi^2 dV_{t_i} \leqq \left(\int_M \psi dV_{t_i}\right)^2 + \frac{m}{\varepsilon} \int_M \psi dV_i \leqq C.$$

It follows that $-\inf_M \phi_{t_i} = \sup_M \psi \leqq C$. □

## § 3. A lower bound of $\alpha(M)$

In this section, we fix a Kähler manifold $(M, g)$ with $C_1(M) > 0$ and $\frac{1}{\pi}\omega_g \sim C_1(M)$, although almost all of the discussions are available to the general Kähler manifold. First we want to study the limiting behavior of a sequence of functions in $P(M, g)$.

**Theorem 3.1.** *Let $\{\phi_i\}$ be a sequence of functions in $P(M, g)$, $\lambda$ be a positive number. Then there exist a subsequence $\{i_k\}$ of $\{i\}$ and a subvariety $S$ of $M$ with* $\dim S \leqq m-1$, *such that*

(i) $\forall z \in M - S$, $\exists r > 0$, $C > 0$, s.t.

$$\int_{B_r(z)} e^{-\lambda \phi_{i_k}(w)} dV_g(w) \leqq C \quad \textit{for all } k.$$

(ii) $$\forall z \in S, \lim_{k \to +\infty} \int_{B_r(z)} e^{-\lambda \phi_{i_k}(w)} dV_g(w) = +\infty \quad \textit{for all } r > 0.$$

*Proof.* We need the following proposition, which is basically the Theorem 5.2.4 in Hömander's book [6].

**Proposition 3.1.** *Let $U$ be a stein manifold, then there exists an exhausting function $\rho$ satisfying; for every plurisubharmonic function $\psi$ on $U$, $(1,0)$-form $h$ with $\int_U |h|^2 e^{-(\psi+\rho)} dV$ and $\bar\partial h=0$, there exists a function $u$ such that $\bar\partial u=h$*

$$\int_U |u|^2 e^{(-\psi+\rho)} dV_U \leqq \int_U |h|^2 e^{-(\psi+\rho)} dV_U.$$

We continue the proof of the theorem. Let $x_i \in M$, $\sup_M \phi_i(x)=\phi_i(x_i)$. Without losing generality, may assume that $x_i \to \bar{x} \in M$ as $i \to +\infty$. $U$ is a Zariski open neighborhood of $\bar{x}$ and $U$ is stein. Furthermore, we may assume all $x_i$ in $U$.

Let $\theta$ be the Kähler potential of $g$ in $U$ with $\theta(\bar{x})=-\frac{1}{2}$, i.e. $\partial\bar\partial\theta=\omega_g$. Choose $R_1>0$, s.t. $-1 \leqq \theta(\bar{x}) \leqq 0$ in $B_{2R_1}(\bar{x})$.

For $i$ large enough, $B_{\frac{3}{2}R_1}(x_i) \subset B_{2R_1}(\bar{x})$, $B_{r_1}(\bar{x}) \subset B_{\frac{3}{2}r_1}(x_i)$ where $r_1$ is given in Lemma 2.1.

By Lemma 2.1, there exists a constant $C$ independent of $i$,

$$\int_{B_{r_1}(\bar{x})} e^{-\lambda(\theta+\phi_i)} dV_U \leqq C \quad \text{for all } i. \tag{5}$$

Let $\eta$ be the cut-off function in $B_{r_1}(\bar{x})$.

Take $\alpha>0$, s.t. $(\alpha-\lambda)\theta+\eta\log|z|^m$ is plurisubharmonic in $U$, where $z$ is the local coordinate near $\bar{x}$ with $z=0$ at $\bar{x}$.

By (5), if $h=\bar\partial\eta$, then $\bar\partial h=0$ and

$$\int_U |h|^2 e^{-(\alpha\theta+\eta\log|z|^m+\lambda\phi_i+\rho)} dV_U \leqq C$$

where $C$ is independent of $i$.

By Proposition 3.1, $\exists\, u_i$, s.t. $\bar\partial u_i=h$ and

$$\int_U |u_i|^2 e^{-(\alpha\theta+\eta\log|z|^m+\lambda\phi_i+\rho)} dV_U \leqq \int_U |h|^2 e^{-(\alpha\theta+\eta\log|z|^m+\lambda\phi_i+\rho)} dV_U \leqq C.$$

Since $\phi_i \leqq 0$,

$$\int_U |u_i|^2 e^{-(\alpha\theta+\eta\log|z|^m+\rho)} dV_U \leqq C.$$

It follows that $u_i(\bar{x})=0$, $\forall\, i$.

Define $f_i=\eta-u_i$, then $\bar\partial f_i=0$ in $U$, $f_i(\bar{x})=1$.

$$\int_U |f_i|^2 e^{-(\alpha\theta+\rho)} dV_U \leqq C. \tag{6}$$

Moreover, by (5),

$$\int_U |f_i|^2 e^{-(\alpha\theta+\lambda\phi_i+\rho)} dV_U \leqq C. \tag{7}$$

Hence, there exists a subsequence $\{i_k\}$ of $\{i\}$, such that $f_{i_k} \to f$ in $L^2_{\mathrm{loc}}(U)$, then $\bar\partial f=0$, $f(\bar{x})=1$.

Put $S_1=\{z \in U \setminus f(z)=0\} \cup (M \setminus U)$, then $\dim S_1 \leqq m-1$, $S_1$ is a subvariety.

$$\forall\, z \in M\setminus S_1,\ z\in U,\ f(z)\neq 0, \quad \text{then } \exists\, x>0,\ k_0>0, \quad \text{s.t.}$$
$$\forall\, k \geqq k_0, \quad w \in B_r(z), \quad |f_{i_k}(w)| \geqq \tfrac{1}{2}|f(z)|>0.$$

By (7),

$$\int_{B_r(z)} e^{-\lambda\phi_{i_k}} dV_U \leqq \frac{2C}{|f(z)|^2} e^{\sup_{B_r(z)}(\alpha\theta+\rho)(w)} \quad \text{for } i \geqq i_0,$$

i.e. $z \in M$ satisfies the property stated in (i).

If $\exists\, z \in S_1$, s.t. (ii) does not hold at $z$, by taking the subsequence, we may assume that $\exists\, r>0$, $C>0$, s.t. $\int_{B_r(z)} e^{-\lambda\phi_{i_k}(z)} dV_U \leqq C$ for all $k$.

Replacing $\{i\}$ by $\{i_k\}$, repeat the above procedure, one finds a subvariety $S'_2$ s.t. it enjoys the same property as $S_1$ does and does not contain a point $z \in S_1$. Put $S_2 = S'_2 \cap S_1$, then $S_2$ is a subvariety. $S_2 \subsetneqq S_1$, and every point $z$ in $M - S_2$ satisfies (i). Continuing such arguments, one obtains a filtration of $S_1$ by subvarieties $S_N \subsetneqq S_{N-1} \subsetneqq \dots \subsetneqq S_2 \subsetneqq S_1$. Since the length of such a filtration must be finite, one will finally find the subvariety $S$ as required in the statement of the theorem.

*Remark.* This theorem suggests to us that even if the solutions $\phi_t$ of $(*)_t$ do not converge as $t \to \bar{t}$, $\phi_t - \sup_M \phi_t$ still converge outside a subvariety, then the limiting function would be a solution of the degenerate complex Monge-Ampére equation $\det\left(g_{i\bar{j}} + \frac{\partial^2 \phi}{\partial z_i\, \partial \bar{z}_j}\right) = 0$ and provide certain special structures on $M$, such as holomorphic foliations, etc. This situation is quite the same as that in the study of harmonic mappings and Yamabe problem (cf. [9, 10]). The difficulty here is that the local estimate of complex Monge-Ampére equation is missing. Moreover, the limiting function only satisfies a degenerate elliptic equation so that it is much harder to study its behavior.

**Lemma 3.1.** *Let $\beta > 0$. For each $\varepsilon > 0$, $\delta > 0$, $R > 0$, there exist $\gamma = \gamma(\varepsilon, R)$, $C = C(\delta, \beta)$, such that $\forall$ subharmonic function $\psi$ in $B_R(0) \subset \mathbb{C}^1$, satisfying $\psi \leqq 0$ and $\int_{|z|<R} \Delta\psi\, dz \leqq \beta$, where $dz$ stands for the volume form of $\mathbb{C}^1$.*

Then

$$\int_{|z| \leqq r} e^{-\left(\frac{4\pi}{\beta} - \delta\right)\psi(z)} dz \leqq C R^2 e^{-(1+\varepsilon)\left(\frac{4\pi}{\beta} - \delta\right)\psi(0)}.$$

*Proof.* Note the Laplace here is the real one, i.e. $\Delta\psi = 4\frac{\partial^2 \psi}{\partial z\, \partial \bar{z}}$. By Green formula,

$$2\pi\psi(z) = \int_{B_R(0)} \log\left(\frac{|z-\zeta|}{\left|R - \frac{z\bar{\zeta}}{R}\right|}\right) \Delta\psi(\zeta)\, d\zeta + \int_{\partial B_R(0)} \frac{R^2 - |z|^2}{R|z-\zeta|^2} \psi(\zeta)\, d\zeta.$$

In particular,

$$-2\pi\psi(0) = \int_{B_R(0)} \left(-\log\frac{|\zeta|}{R}\right) \Delta\psi(\zeta)\, d\zeta + \frac{1}{R} \int_{\partial B_R(0)} (-\psi(\zeta))\, d\zeta.$$

Since

$$\psi \leqq 0, \qquad \Delta\psi \geqq 0, \qquad 0 \leqq \frac{1}{2\pi R} \int_{\partial B_R(0)} (-\psi(\zeta))\, d\zeta \leqq -\psi(0).$$

Put

$$\mu=\frac{1}{2\pi}\int_{|\zeta|<R}(\Delta\psi)\cdot\left(\frac{4\pi}{\beta}-\delta\right)d\zeta=\left(\frac{2}{\beta}-\frac{\delta}{2\pi}\right)\int_{|\zeta|<R}\Delta\psi\,d\zeta\leqq 2-\frac{\delta\beta}{2\pi}<2.$$

By the convexity of exp,

$$\exp\left(-\frac{\left(\frac{4\pi}{\beta}-\delta\right)}{2\pi}\int_{|\zeta|<R}\log\frac{|z-\zeta|}{\left|R-\frac{z\bar\zeta}{R}\right|}\Delta\psi\,d\zeta\right)$$

$$=\exp\left(\int_{|\zeta|<R}-\mu\log\frac{|z-\zeta|}{\left|R-\frac{z\bar\zeta}{R}\right|}\frac{\left(\frac{4\pi}{\beta}-\delta\right)\Delta\psi\,d\zeta}{2\pi\mu}\right)$$

$$\leqq\frac{4\pi-\delta\beta}{2\pi\beta\mu}\int_{|\zeta|<R}\left(\frac{|z-\zeta|}{\left|R-\frac{z\bar\zeta}{R}\right|}\right)^{-\mu}\Delta\psi\,d\zeta.$$

Take $r=\dfrac{\varepsilon R}{1+\sqrt{1+\varepsilon}}$, then for $|z|<r$

$$\left|-\int_{|\zeta|=R}\frac{R^2-|z|^2}{R|\zeta-z|^2}\psi(\zeta)\,d\zeta\right|\leqq 2\pi(1+\varepsilon)\,\psi(0).$$

Therefore

$$\int_{|z|<r}e^{-\left(\frac{4\pi}{\beta}-\delta\right)\psi(z)}dz\leqq e^{-(1+\varepsilon)\left(\frac{4\pi}{\beta}-\delta\right)\psi(0)}\max_{|\zeta|\leqq R}\int_{|z|\leqq r}\left|\frac{z-r}{R-\frac{z\bar\zeta}{R}}\right|^{-\mu}dz$$

$$\leqq R^2e^{-(1+\varepsilon)\left(\frac{4\pi}{\beta}-\delta\right)\psi(0)}\max_{|\zeta|\leqq 1}\int_{|\zeta|\leqq 1}\left|\frac{z-\zeta}{1-z\bar\zeta}\right|^{-\mu}dz$$

$$=C(\mu)\,R^2e^{-(1+\varepsilon)\left(\frac{4\pi}{\beta}-\delta\right)\psi(0)}.$$

**Lemma 3.2.** $B_{R_1}^{m-1}\times B_{R_2}\subset C^{m-1}\times\mathbb{C}^1$. *Let*

$$S_\beta=\{\phi\in C^2(B_{R_1}^{m-1}\times B_{R_2})\mid\forall\, z\in B_{R_1}^{m-1}\cdot\phi_z=\phi(z,\cdot)$$
$$\textit{is subharmonic}, \phi\leqq 0, \int_{B_{R_2}}\Delta_w\phi_z(w)\,dw\leqq\beta\}.$$

*For each* $\varepsilon,\delta>0$, *there exist* $r_2=r_2(\varepsilon,R_2)>0$, $C=C(\delta,\beta)$, *such that* $\forall\,\phi\in S_\beta$,

$$\iint_{\substack{|z|<R_1\\|w|<r_2}}e^{-\left(\frac{4\pi}{\beta}-\delta\right)\phi(z,w)}dz\,dw\leqq\frac{CR_2^2}{r_2^2}\iint_{\substack{|z|<R_1\\r_2\leqq|w|\leqq 2r_2}}e^{-(1+\varepsilon)\left(\frac{4\pi}{\beta}-\delta\right)\phi(z,w)}dz\,dw.$$

*Proof.* Let $r$ be given in Lemma 3.1 for $R=\dfrac{R_2}{2}$. $r_2=\dfrac{1}{4}\min\left\{r,\dfrac{R}{4}\right\}$, then $\forall\,(z,w_0)\in B_{R_1}^{m-1}\times B_{R_2}$, $|w_0|<2r_2$, by the assumption on $\phi$ and Lemma 3.1,

$$\int_{|w-w_0|<r} e^{-\left(\frac{4\pi}{\beta}-\delta\right)\phi(z,w)}\,dw \leqq CR_2^2\, e^{-(1+\varepsilon)\left(\frac{4\pi}{\beta}-\delta\right)\phi(z,w_0)}$$

$$\therefore \int_{|w|<r_2} e^{-\left(\frac{4\pi}{\beta}-\delta\right)\phi(z,w)}\,dw \leqq CR_2^2\, e^{-(1+\varepsilon)\left(\frac{4\pi}{\beta}-\delta\right)\phi(z,w_0)}.$$

In particular

$$\pi r_2^2 \int_{|w|<r_2} e^{-\left(\frac{4\pi}{\beta}-\delta\right)\phi(z,w)}\,dw \leqq CR_2^2 \int_{r_2\leqq|w|\leqq 2r_2} e^{-(1+\varepsilon)\left(\frac{4\pi}{\beta}-\delta\right)\phi(z,w)}\,dw.$$

Integrating it on $z$, we are done.

**Theorem 3.2.** *Let the Kähler manifold $(M,g)$ have $N$ families of curves $\{C_\alpha^1\}$, $\{C_\alpha^2\},\ldots,\{C_\alpha^N\}$, where $\alpha\in CP^{m-1}$ is the parameter, and $N$ subvarieties $S_1,\ldots,S_N$ such that*

(i) $S_1\cap\ldots\cap S_N=\emptyset$,
(ii) $N-S_j=\bigcup_\alpha (C_\alpha^j\cap(M-S_j))$, $C_\alpha^j\cap C_\beta^j\cap(M-S_j)=\emptyset$ *and* $C_\alpha^j\cap(M-S_j)$ *is smooth for each* $\alpha$.
(iii) $\forall\, z\in M-\bigcup_i S_i$, $\{T_z C_{\alpha_i}^j \mid C_{\alpha_j}^j\in z\}$ *spans* $T_z M$; $\forall\, z\in S_i$, *either*

$$\{T_z C_{\alpha_j}^j \mid z\in C_{\alpha_j}^j\cap(M-S_j)\}$$

*spans $T_zM$, or there exists $C_{\alpha_j}^j$, s.t. $z\in C_{\alpha_j}^j\cap(M-S_j)$, $C_{\alpha_j}^j\cap S_i=\{$finite points$\}$.*
(iv) $\forall\, i,\alpha$, $4\,\mathrm{Vol}_g(C_\alpha^i)\leqq\beta$.

Then $\alpha(M)\geqq\dfrac{4\pi}{\beta}$.

*Proof.* Fix an arbitrary $\delta>0$. Set $\delta_1=\dfrac{\delta}{m}$. We will prove that

$$\int_M e^{-\left(\frac{4\pi}{\beta}-\delta\right)\phi}\,dV_M\leqq C,\qquad \forall\,\phi\in P(M,g) \tag{8}$$

where $C$ is independent of $\phi$. Clearly, it implies: $\alpha(M)\geqq\dfrac{4\pi}{\beta}$, since $\delta$ is arbitrary.

To prove (8) it suffices to show that for any sequence $\{\phi_i\}\subset P(M,g)$, there is a subsequence $\{\phi_{i_k}\}$ and a constant $C$ such that (8) holds for $\phi_{i_k}$.

Put $\delta_1=\dfrac{\delta}{m}$. Applying Theorem 3.1, one may assume that there be subvarieties $E_0,\ldots,E_m$, s.t. $\forall\, z\in E_l$, $\lim\limits_{k\to+\infty}\int_{B_r(z)} e^{-\left(\frac{4\pi}{\beta}-\delta_1 l\right)\phi_{i_k}(z)}\,dV_M=+\infty$ for all $r>0$

$$\forall\, z\in M\setminus E_l,\quad \exists\, r>0,\ C>0,\quad \text{s.t.} \int_{B_r(z)} e^{-\left(\frac{4\pi}{\beta}-\delta_1 l\right)\phi_{i_k}(z)}\,dV_M\leqq C\quad \text{for all } k.$$

Obviously $E_0\supseteq\ldots\supseteq E_m$, $\dim E_0\leqq m-1$. That (8) holds for all $\phi_{i_k}$ is equivalent to that $E_m=\emptyset$. Since $\dim E_0\leqq m-1$, it suffices to prove that $\dim E_{l-1}-\dim E_l\geqq 1$.

Take a smooth point $z_0 \in E_{l-1}$, by (i), $\exists j$, s.t. $z_0 \in M - S_j$, let $C^j_\alpha$ pass through $z_0$.

If $C^j_{\alpha_j}$ is transversal to $E_{l-1}$ at $z_0$, then by (ii) we can find a special coordinate chart $B^{m-1}_{R_1} \times B_{R_2} \subset M$ s.t.

$$z_0 = (0, 0), \quad E_{l-1} \cap (B^{m-1}_{R_1} \times B_{R_2}) \subset B^{m-1}_{R_1} \times \{0\}$$

and

$$\forall z \in B^{m-1}_{R_1}, \quad z \times B_{R_2} \subset C^j_{\alpha_z} \quad \text{for certain } \alpha_z \in CP^{m-1}.$$

Now

$$\int_{z \times B_{R_2}} (4 + \Delta_w \phi_{i_j}(z, w))\, dw = 2 \int_{z \times B_{R_2}} (2\omega_g + \partial\bar\partial \phi_{i_k})(w) \leqq 2 \int_{C^j_{\alpha_z}} (2\omega_g + \partial\bar\partial \phi_{i_k})(w)$$
$$= 4 \int_{C^j_{\alpha_z}} \omega_g = 4 \mathrm{Vol}_g(C^j_{\alpha_z}) \leqq \beta.$$

By Lemma 2.2 with $\varepsilon = \dfrac{\delta_1 \beta}{4\pi - l\delta_1 \beta}$, $\delta = l\delta_1$, one sees that $z_0 \notin E_l$. To show that $\dim E_{l-1} - \dim E_l \geqq 1$, it suffices to show that for each smooth point $z_0$ of $E_{l-1}$, there is a point $z$ close to $z_0$ such that $z \in E_{l-1} - E_l$. We assume that it doesn't hold and will derive a contradiction. By our assumption, there is a smooth point $z_0 \in E_{l-1}$, a neighborhood $U$ of $z_0$ in $E_{l-1}$, s.t. $U \subset E_l$. By the above arguments, $\{T_z C^j_{\alpha_j} \mid z \in C^j_{\alpha_j} \cap (M - S_j)\}$ cannot span $T_z M$ for every $z \in U$. Hence, $U \subset \bigcup_{i=1}^{N} S_i$. Let $z_0 \in S_i$, then $\exists\, C^j_{\alpha_{z_0}}$, s.t. $C^j_{\alpha_{z_0}} \cap S_i = \{$finite points, $z_0 \in C^j_{\alpha_{z_0}} \cap (M - S_j)$. Shrinking $U$ if necessary, we may assume that $U \subset M - S_j$, and $\forall z \in U$, $\exists\, C^j_{\alpha_z}$ passing through $z$ and intersecting $S_i$ at finite points. Since $U \subset E_l$, the above arguments imply that $C^j_{\alpha_z}$ is tangential to $E_{l-1}$ at $z \in U$, so $U \cap C^j_{\alpha_{z_0}} \subset E_{l-1}$. Since $C^j_{\alpha_{z_0}} \cap S_i = \{$finite points$\}$, $\exists\, z_1 \in E_{l-1} \cap U$, $z_1 \notin S_i$. Replacing $z_0$ by $z_1$, $U$ by $U \cap (M - S_i)$ and repeating the above arguments, we will find $z_2 \in U \cap E_{l-1}$, $z_2 \notin S_i \cup S_{i'}$, $(i \neq i')$. In this way, after finite times, we will finally find a point $z_N \in U \cup E_{l-1}$, $z_N \notin \bigcup_{i=1}^{N} S_i$. A contradiction. Therefore, $\dim E_{l-1} - \dim E_l \geqq 1$. We are done.

**Corollary 1.** *Under the assumptions of Theorem* 3.2 *and* $C_1(M)>0$, $\dfrac{1}{\pi}\omega_g$ *is cohomological to* $C_1(M)$, $(*)_t$ *is solvable for* $t < \dfrac{m+1}{m} \cdot \dfrac{4\pi}{\beta}$.

*Proof.* It follows from Theorem 3.2 and the proof of Theorem 2.1.

In case $m=2$, any irreducible Kähler manifold with $C_1>0$ must be of form $CP^2 \# n\overline{CP^2}$ $(n \leqq 8)$, i.e. the manifolds produced by blowing up $CP^2$ at $n$ generic points, where the "generic" actually means that no three points are colinear, and no six points are in one quadratic curve in $CP^2$. This is the consequence of classification theory of algebraic surfaces (Griffith and Harris [5]).

**Corollary 2.** *Let* $M = CP^2 \# n\overline{CP^2}$, $3 \leqq n \leqq 8$, *then* $\alpha(M) \geqq \frac{1}{2}$. *In particular,* $(*)_t$ *is solvable for* $t < \frac{3}{4}$.

*Proof.* Suppose that $M$ be the blowing-up of $CP^2$ at $x_1, \ldots, x_n$, and $F_1, \ldots, F_n$ be the exceptional divisors.

$\{C_\alpha^i\} = \{$quadratic image in $M$ of lines in $CP^2$ passing through $x_i\}$, $\quad S_i = \bigcup_{j \neq i} F_j$.

It is trivial to verify that assumptions (i), (ii), (iii) are satisfied.

Now $C_1(M) = p^*(3H) - [F_1] - \ldots - [F_n]$, where $p: M \to CP^2$ is the natural projection, $H$ is the hyperplane line bundle of $CP^2$.

Because

$$\frac{1}{\pi}\omega_g \sim C_1(M), \mathrm{Vol}_g(C_\alpha^i) = \int_{C_{\alpha_i}} \omega_g = \pi \int_{C_{\alpha_i}} C_1(M)$$
$$= \pi C_1(M) \cdot [C_\alpha^i] \qquad \text{(Griffith and Harris [5], p. 141)}$$
$$= \pi C_1(M) \cdot (p^*(H) - [F_i]) = 2\pi.$$

$\therefore \beta = 8\pi$. The corollary follows.

*Remark.* One can prove that outside a finite set of points in $CP^2 \# 8\overline{CP^2}$, for $\alpha < 1$, $e^{-\alpha\phi}$ has locally uniform bounds for each $\phi \in P(M, g)$. Moreover, one can locate that finite set. We know that $CP^2 \# 8\overline{CP^2}$ has a pensil of elliptic curves having intersection number one with each exceptional divisor, only singular curves in the pensil is either a rational curve with an ordinary node, or a rational curve with a cusp. The finite set consists of those cusps.

## § 4. Kähler-Einstein metrics on Fermat hypersurfaces

So far, we have not known an example with $\alpha(M) > \frac{m}{m+1}$, but if we restrict $\phi$ to a proper subset $P_s$ of $P(M, g)$ and define $\alpha_s(M)$ with respect to $P_s$ as we do for $P(M, g)$, $\alpha_s(M)$ might be greater than $\frac{m}{m+1}$. A natural subset $P_s$ is $P_G(M, g) = \{\phi \in P(M, g) \mid \phi$ is invariant under $G\}$, where $G$ is a compact subgroup in $\mathrm{Aut}(M)$. $\frac{1}{\pi}\omega_g \sim C_1(M)$, we may assume that g is invariant under $G$. Then we have

**Theorem 4.1.** *$(M, g)$, $G$ stated as above. If $\alpha_G(M) > \frac{m}{m+1}$, then $M$ admits a Kähler-Einstein metric.*

*Proof.* Same as Theorem 2.1.

The following theorem gives an estimate of $\alpha_G(M)$.

**Theorem 4.2.** *Let $(M, g)$, $G$ as above. Furthermore, assume that $(M, g)$ have $N$ families of curves $\{C_\alpha^1\}, \ldots, \{C_\alpha^N\}$, $\alpha \in CP^{m-1}$, and $N$ subvarieties $S_1, \ldots, S_N$ satisfying* (i), (ii), (iii) *in Theorem* 3.2 *and* (iv)′: *Let $G_j \subset G$ be the subgroup preserving the fibration of $M - S_j$ by $\{C_\alpha^j \cap (M - S_j)\}$, then $S_j$ is invariant under $G_j$,*

$$\frac{4\,\mathrm{Vol}_g(C_\alpha^j)}{\mathrm{ord}(G_j)} \leqq \beta \qquad \forall\, \alpha \in CP^{m-1}$$

*where* $\mathrm{ord}(G_j) = \min_{z \in M - S_j} \frac{|G_j|}{|\mathrm{Stab}_z \subset G_j|}$. *Then,* $\alpha_G(M) \geqq \frac{4\pi}{\beta}$.

*Proof.* Almost same as the proof of Theorem 3.2. We adapt the notations there, $z_0 \in E_{l-1}$, a smooth point. We may find $j$ such that $z_0 \notin S_j$, $z_0 \in C^j_{\alpha_0}$.

If $C^j_{\alpha_0}$ is transversal to $E_{l-1}$ at $z_0$, $B^{m-1}_{R_1} \times B_{R_2}$ is taken exactly as in the proof of Theorem 3.2.

Put $\mu = \frac{|G_j|}{|\mathrm{Stab}_{z_0}|}$, $\mu \geqq \mathrm{ord}\, G_j$. By (iv)′, one can choose $R_1$, $R_2$ so small that $\sigma(B^{m-1}_{R_1} \times B_{R_2}) \cap B^{m-1}_{R_1} \times B_{R_2} = \emptyset$ for at least $\mu$ elements $\sigma$ of $G_j$.

Since $\omega_g$, $\phi_{i_k}$ are invariant under $G$,

$$\forall z \in B^{m-1}_{R_1}, \qquad \int_{z \times B_{R_2}} (4 + \Delta_w \phi_{i_k})\, dw = \int_{z \times B_{R_2}} (2\omega_g + \partial\bar{\partial}\phi_{i_k})$$

$$\leqq \frac{2}{\mu} \int_{C^j_{\alpha_0}} (2\omega_g + \partial\bar{\partial}\phi_{i_k}) \leqq \frac{4}{\mathrm{ord}(G_j)} \int_{C^j_{\alpha_0}} \omega_g = \frac{4\,\mathrm{Vol}_g(C^j_{\alpha_0})}{\mathrm{ord}(G_j)} \leqq \beta.$$

By Lemma 3.2 with $\varepsilon$ by $\frac{\beta\delta_1}{4\pi - l\delta_1\beta}$, $\delta$ by $l\delta_1$, $z_0 \notin E_l$, the rest is same as in the proof of Theorem 3.2.

Now we consider Fermat hypersurfaces

$$X_{m,p} = \{[Z_0, \dots, Z_{m+1}] \in CP^{m+1} \mid z_0^p + z_1^p + \dots + z_{m+1}^p = 0\}, \qquad p \leqq m+1,$$

$g = (m+2-p)$ multiple of the restriction of Fubini-study metric of $CP^{m+1}$

i.e.

$$(m+2-p)\, \partial\bar{\partial} \log(|z_0|^2 + \dots + |z_{m+1}|^2)|_{X_{m,p}}.$$

$G$: the group generated by permutations

$$\sigma_{ij}: [z_0, \dots, z_i, \dots, z_j, \dots, z_{m+1}] \to [z_0, \dots, z_j, \dots, z_i, \dots, z_{m+1}]$$

and

$$\tau_k: [z_0, \dots, z_k, \dots, z_{m+1}] \to [z_0, \dots, e_p z_k, \dots, z_{m+1}]$$

where

$$e_p = \exp\left(\frac{2\pi\sqrt{-1}}{p}\right).$$

$$0 \leqq i, j \leqq m+1,$$

$$S_{ij} = X_{m,p} \cap \{[z_0, \dots, \overset{(i)}{0} \dots \overset{(j)}{0} \dots z_{m+1}] \in CP^{m+1},$$
$$[0, \dots, 1, \dots, e_p^{k+\frac{1}{2}}, 0 \dots 0],\ k = 0, 1, \dots, p-1\}.$$

$C^{ij}_\alpha =$ the closure of

$$\{[z_0 \dots z_i \dots z_j \dots z_{m+1}] \in X_{m,p} - S_{ij} \mid [z_0 \dots \hat{z}_i \dots \hat{z}_j \dots z_{m+1}] = \alpha \in CP^{m-1}\},$$

where $[\alpha_0, \dots, \alpha_{m-1}] = \alpha \in CP^{m-1}$.

Obviously,

$$C^{ij}_\alpha = \sigma_{0i} \cdot \sigma_{1j}(C^{01}_\alpha)$$

$$S_{ij} = \sigma_{0i} \cdot \sigma_{1j}(S_{01}), \qquad \dim S_{ij} = m-2.$$

We claim that $\{C^{ij}_\alpha\}$, $S_{ij}$ satisfy the assumptions (ii), (iii), (iv) of Theorem 4.2.

It is clear that $\bigcap_{i,j} S_{ij}=\emptyset$, i.e. (i) is satisfied. For (ii),

$$\begin{aligned} X_{m,p}-S_{ij}&=\{[z_0,\dots,z_{m+1}]\in X_{m,p}\,|\,[z_0,\dots,\hat{z}_i\dots\hat{z}_j\dots z_{m+1}]\\ &\qquad\in CP^{m-1},\ |z_i|^2+|z_j|^2\neq 0\}\\ &=\Big(\bigcup_{\alpha\in CP^{m-1}} C_\alpha^{ij}\Big)\cap(X_{m,p}-S_{ij}),\end{aligned}$$

$$C_\alpha^{ij}\cap C_\beta^{ij}=\{[0,\dots,0,\overset{(i)}{1},\dots,\overset{(j)}{e_p^{k+\frac{1}{2}}},\dots,0]\}\subset S_{ij},\qquad \alpha\neq\beta.$$

$$C_\alpha^{ij}=\{[\alpha_0 t,\dots,\alpha_{i-1}t,z_i,\dots,z_j,\alpha_j t,\dots,\alpha_{m-1}t]\,|\,(\alpha_0^p+\dots+\alpha_{m-1}^p)\,t^p+z_i^p+z_j^p=0\}.$$

Hence, if $\alpha_0^p+\dots+\alpha_{m-1}^p\neq 0$, $C_\alpha^{ij}$ is smooth. If

$$\alpha_0^p+\dots+\alpha_{m-1}^p=0,\qquad C_\alpha^{ij}=\bigcup_{k=1}^{p}\{[\alpha_0 t,\dots,\alpha_{i-1}t,z_i,\dots,z_j,\alpha_{j-1}t,\dots,\alpha_{m-1}t]\}$$

is a union of $p$ rational curves with a singular point

$$[\alpha_0,\alpha_i,\dots,\alpha_{i-1},\overset{(i)}{0},\alpha_i,\dots,\alpha_{j-2},\overset{(j)}{0},\alpha_{j-1}\dots\alpha_{m-1}]\in S_{ij}.$$

Therefore, (ii) is verified.

For (iii), take $[z_0,\dots,z_{m+1}]$ in $X_{m,p}$, may assume that

$$[z_0,\dots,z_{m+1}]=[1,z_1,\dots,z_i,0,\dots,0],$$

where $i\geqq 1$, $z_j\neq 0$ for $1\leqq j\leqq i$, in particular, $z_1\neq 0$, so $[z_0,\dots,z_{m+1}]\neq S_{1k}$ for $k\geqq 2$. Define $H_{0k}=\{[w_0,\dots,w_{m+1}]\in CP^{m+1}\,|\,w_k=z_k w_0\}$, for

$$k\geqq 2,\qquad C^{1k}_{[z_0,\hat{z}_1\dots\hat{z}_k\dots z_{m+1}]}=X_{m,p}\cap\Big(\bigcap_{\substack{l=2\\ l\neq k}}^{m+1} H_{0l}\Big),$$

$$\begin{aligned} X_{m,p}\cap\Big(\bigcap_{k=2}^{m+1} H_{0k}\Big)&=\{[t,z_1 s,z_2 t,\dots,z_{m+1}t]\}\cap X_{m,p}\\ &=\{t^p(1+z_2^p+\dots+z_{m+1}^p)+z_1^p s^p=0\}\\ &=\{z_1^p(s^p-t^p)=0\}\end{aligned}$$

by $z_1\neq 0$, $X_{m,p}\cap\Big(\bigcap_{k=2}^{m+1} H_{0k}\Big)=\{p$ finite points$\}$ and multiplicity at $[z_0,\dots,z_{m+1}]$ is one, so $(T_{[z_0,\dots,z_{m+1}]}X_{m,p})\cap\Big(\bigcap_{k=2}^{m+1} H_{0k}\Big)=\{0\}$, it follows that

$$\begin{aligned}&\operatorname*{span}_{2\leqq k\leqq m+1}\{T_{[z_0\dots z_{m+1}]}\,C^{1k}_{[z_0,\hat{z}_1\dots\hat{z}_k\dots z_{m+1}]}\}\\ &\quad=\operatorname*{span}_{2\leqq k\leqq m+1}\Big\{T_z X_{m,p}\cap\Big(\bigcap_{\substack{l=2\\ l\neq k}}^{m+1} H_{0l}\Big)\Big\}\\ &\quad=T_z X_{m,p}.\end{aligned}$$

Hence, (iii) is satisfied.

Now we consider (iv)'. $G_{ij}=\{\sigma_{ij}, \tau_i, \tau_j\}$, obviously, $G_{ij}$ preserves $C_\alpha^{ij}$ and $S_{ij}$. For the estimate of $\operatorname{ord}(G_{ij})$, because of symmetry, we may assume $i=0$, $j=1$. Take $z\in X_{m,p}-S_{01}$, $z=[z_0, \ldots, z_{m+1}]$. $\exists\, z_i \neq 0$, for $i\geqq 2$, we may assume $z_i=1$, then $\tau_0^k \tau_1^l [z_0, \ldots, z_{m+1}]=[z_0, \ldots, z_{m+1}]=z$ if and only if $k\equiv 0 \pmod p$, $l\equiv 0 \bmod (p)$ when $z_0 z_1 \neq 0$, hence $\frac{|G_{01}|}{|\mathrm{Stab}_z|}\geqq p^2$ for such $z$.

If $z_0=0$, $z_1=1$, then

$$\sigma_{01}^k \tau_1^l [z_0, \ldots, z_{m+1}]=z \quad \text{if and only if } k\equiv 0 \pmod 2,\ l\equiv 0 \pmod 3$$

hence

$$\frac{|G_{01}|}{|\mathrm{Stab}_z|}\geqq 2p.$$

If $z_1=0$, $z_0=1$, we have also $\frac{|G_{01}|}{|\mathrm{Stab}_z|}\geqq 2p$.

Therefore $\operatorname{ord}(G_{01})\geqq 2p$.

**Theorem 4.3.** *If $m+1\geqq p\geqq m$, then $X_{m,p}$ admits a Kähler-Einstein metric.*

*Proof.* By Theorem 4.1, we only need to show that $\alpha_G(X_{m,p})>\frac{m}{m+1}$.

Using the notations of above,

$$\begin{aligned}\mathrm{Vol}_g(C_\alpha^{ij}) &= \int_{C_\alpha^{ij}} \omega_g=\pi(C_1(X_{m,p})\cdot C_\alpha^{ij})\\ &=(m+2-p)\,p\pi\\ \therefore \beta &= \frac{4\,\mathrm{Vol}_g(C_\alpha^{ij})}{2p}\leqq\frac{4(m+2-p)\pi}{2}\end{aligned}$$

$\frac{m}{m+1}<\frac{4\pi}{\beta}=\frac{2}{m+2-p}$ is equivalent to say $p>m-\frac{2}{m}$ i.e. $p\geqq m$.

Now this theorem follows from Theorem 4.2. □

**Corollary.** *For $m+1\geqq p\geqq m$, there exists an open subset $U_{m,p}$ in the moduli space of $m$-dimension hypersurfaces with degree $p$ in $CP^{m+1}$, such that any $M\in U_{m,p}$ admits a K-E metric.*

*Proof.* It follows from the previous theorem and the application of Implicit function theorem to the equation (*) in Sect. 1.

Note that the existence of K-E metric on a $m$-dimensional hypersurface of degree $p\geqq m+2$ follows from [13].

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**Note added in proof**

Various estimates of the lower bound of the holomorphic invariant $\alpha(M)$ are given by S.T. Yau and me in a joint paper, which is to appear in Comm. in Math. Phys. These estimates are applied there to produce Kähler-Einstein metrics on complex surfaces with $C_1>0$, for example, we prove that there are Kähler-Einstein structures with $C_1>0$ on any manifold of differential type $CP^2 \# n\overline{CP^2}$ $(3\leqq n\leqq 8)$. We were also informed that Prof. Y.T. Siu had independently produced results on the existence of Kähler-Einstein metrics on certain Kähler manifolds with $C_1>0$. His approach in completely different from ours.

Note that the proof of Theorem 2.1 also implies: if $\alpha(M)$ has a lower bound depending only on the dimension of $M$, then there is a constant $C(m)$ such that each compact Kähler manifold with $C_1>0$ admits a Kähler metric with Ricci curvature $\geqq C(m)$. An upper bound of $C_1(M)^m$ will follow from this and a volume comparison. This is pointed out to us by Yau