A CANONICAL CURVE OF GENUS 7

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[Received 12 November 1965]

Introduction

1. One of the best known and most intensively studied of all algebraic curves is Klein's plane quartic with its group of 168 self-projectivities. It afforded, antedating it by some fifteen years, a ready support to Hurwitz's proof (**6**) that no algebraic curve of genus $p \geqslant 3$ can possess more than $84(p-1)$ birational self-transformations. That this maximum is actually attained for certain genera exceeding 3 has only recently been appreciated, and is now publicized by Macbeath's discovery (**7**) of a Hurwitz curve with $p = 7$: a discovery the more exciting because it has lain so near the surface for so long. The following paragraphs aim to construct this newly discovered Hurwitz curve by the elementary procedures of classical projective geometry. The ultimate objective is a curve admitting 504 self-transformations, but the first step towards it is the construction of a curve, of genus 7, admitting merely two: the identity and an involutory self-transformation. This group of order 2 is then successively amplified, by further specializing the curve, to groups of orders 4, 8, 56.

2. Special features of a curve of genus p are mirrored by projective peculiarities of its canonical model Γ^{2p-2} in $[p-1]$; one denotes, following long-established usage, projective n-dimensional space by $[n]$. The first restriction to which Γ^{12} will be subjected is to admit a harmonic inversion. Invariance of Γ^4 under an involutory projectivity in its plane is a phenomenon so frequent as hardly to call for comment: suffice it to say that, if Γ^4 is its own harmonic inverse in a point O and a line λ, four of its bitangents concur at O and their eight contacts are on a conic for which O and λ are pole and polar. Klein's curve admits 21 such inversions. For $p = 5$, Γ^8 is the base curve of a net of quadrics in [4]. One could demand ((**1**) 24) that a given plane π and line λ be polar spaces for every quadric of the net, thus producing a special Γ^8 that admits harmonic inversion in π and λ: the joins of points of Γ^8 that are paired in this inversion generate a scroll and meet π in points of a Γ^4. This Γ^8 has, as will Γ^{12} below have, the special attribute of being in $(2,1)$ correspondence with a curve of genus 3; but, while the [4] containing Γ^8 is spanned by a plane and line, the [6] containing Γ^{12} is spanned by a plane and solid. If, in [4], the quadrics of a net have a common self-polar simplex the resulting Γ^8 admits an elementary abelian group of 16 self-projectivities of which ten are inversions in planes and lines ((**4**) 486). A Hurwitz Γ^{12} admits 63 inversions in planes and solids.

A scroll R in [6], of order 10 and genus 3

3. Take, then, in [6] a plane π and a solid Σ skew to one another and, in π, a quartic q to be the canonical model for a sextic σ, of genus 3, in Σ. The joins of corresponding points on q and σ will generate a scroll R, of order 10, having q and σ for directrices; a curve being a directrix on a scroll when it is unisecant to the generators.

4. σ is to be the twisted sextic studied by F. Schur (**8**) and defined by him as the locus of points of concurrence of related planes belonging one to each of four projectively related stars. It is the residual intersection of two cubic surfaces through a twisted cubic γ. It is not on any quadric, but it lies on a web of cubic surfaces—a web being a system linearly dependent on any four linearly independent members. Any two of these cubic surfaces meet in σ and a twisted cubic γ having eight intersections with σ ((**8**) 14, 15). When a cubic surface is mapped, as by Clebsch, on a plane so that the maps of its plane sections are cubic curves through six points A_i $(i = 1, 2, 3, 4, 5, 6)$, the map of its complete intersection with another cubic surface is a curve of order 9 with the A_i as triple points. Should the map of γ be a line, that of σ is of order 8 with the A_i as triple points; should, alternatively, the map of γ be a quintic with the A_i as nodes, that of σ is a quartic passing through the A_i. In either event the map of σ has genus 3 and the two maps of σ and γ have eight intersections apart from the A_i. It is by appealing to this mapping that Schur proves ((**8**) 14) σ to have genus 3 or, as he puts it, to have seven apparent double points. It has, by the same token, three trisecants through each of its points.

There is, and it is a basic property of σ, a $(1, 1)$ correspondence between its points P and its trisecants t ((**8**) 18). If P is any point of σ, the planes of those triads that complete tetrads of the canonical series $\mathfrak{X}(\sigma)$ which include P have a common line t, and this is the trisecant associated with P. The terms *canonical set* and *canonical series* are not used by Schur and only came, with *canonical curve*, into currency somewhat later ((**9**) 118); but Schur's quadruplets '$4g$' are, nevertheless, the sets of $\mathfrak{X}(\sigma)$ and are cut on σ by the net of quadrics through the twisted cubic γ common to any two cubic surfaces through σ ((**8**) 17; (**2**) 475). Mention of the canonical series $\mathfrak{X}$ affords the opportunity to introduce the bicanonical series $\mathfrak{X}^{(2)}$. $\mathfrak{X}(q)$ is cut on the canonical model q by the lines, $\mathfrak{X}^{(2)}(q)$ by the conics, of π. $\mathfrak{X}^{(2)}(\sigma)$ is cut on σ by the quartic surfaces having γ for a double curve. These are scrolls, and each is generated by those chords of γ that belong to a linear complex. There is a unique chord of γ through an arbitrary point; if five points are chosen arbitrarily on σ the chords of γ through them determine a linear complex and, hence, the whole set of $\mathfrak{X}^{(2)}(\sigma)$ that includes the five points. This accords with $\mathfrak{X}^{(2)}$ having freedom 5 on a curve of genus 3, each set of eight being uniquely determined by any five of its members.

5. The canonical model for σ can be taken as a quartic curve q in π. The points of σ are in $(1, 1)$ correspondence with the points of q, and the joins g of pairs of corresponding points generate, in the [6] spanned by π and Σ, a scroll R of order $6+4 = 10$. Since the g answer one, and only one, to each point of σ or of q one can speak of the canonical series $\mathfrak{X}(R)$ and bicanonical series $\mathfrak{X}^{(2)}(R)$, and of canonical and bicanonical sets of g.

The generators g_1, g_2, g_3 through collinear points a_1, a_2, a_3 on q meet σ in points whose plane contains t_4, the trisecant of σ associated with the point on the generator g_4 through the fourth intersection a_4 of q with $a_1a_2a_3$. As g_1, g_2, g_3 all meet the line $a_1a_2a_3$ they span a [4]. The pencil of primes (i.e. hyperplanes) through this [4] cuts R in a pencil of directrices D, of order 7, all passing through a_4 and the intersections of σ with t_4. Two directrices of this pencil are composite: one consists of q and the generators through the intersections of σ with t_4, the other of σ and g_4; but the others are not composite. Such a curve, of order 7 and genus 3 with a trisecant t in [4], is known to complete, with t, the base-curve of a net of quadrics.

The web of quadrics containing R

6. Any quadric containing R contains q, and so the whole of π, and σ, and so the whole of Σ. It is enough, in order that it contain the whole of R, for it to contain one directrix in addition to q and σ, since then every g would be trisecant to the quadric and so lie on it. Since D lies on three linearly independent quadrics in [4] its postulation for quadrics is $15-3 = 12$. But, once a quadric has been constrained to contain both π and Σ, it has four assigned intersections with D and so needs to be made to contain only eight further points of D. Hence the postulation of R for quadrics is

$$6+10+8 = 24 = 28-4,$$

and R lies on the quadrics of a web W. It would lie on more quadrics if the 24 conditions imposed were not independent; but they are seen to be independent in certain instances and therefore are so in general.

Such an instance is afforded ((2) 473; (3) 163) by the twisted sextic

$$X : Y : Z : T = zx^2 : yz^2 : xy^2 : xyz$$

being in (1, 1) correspondence with Klein's quartic

$$yz^3 + zx^3 + xy^3 = 0. \tag{6.1}$$

Any quadric which contains both

$$\pi:\ X = Y = Z = T = 0 \quad \text{and} \quad \Sigma:\ x = y = z = 0$$

has an equation

$$(a_1x+a_2y+a_3z)X+(b_1x+b_2y+b_3z)Y+(c_1x+c_2y+c_3z)Z + (d_1x+d_2y+d_3z)T = 0 \tag{6.2}$$

and contains the whole of R when, and only when,

$$(a_1x+a_2y+a_3z)zx^2+(b_1x+b_2y+b_3z)yz^2+(c_1x+c_2y+c_3z)xy^2 + (d_1x+d_2y+d_3z)xyz$$

vanishes identically in virtue of (6.1). This identical vanishing demands that $a_1 = b_3 = c_2$ and

$$d_1+a_2 = d_2+c_3 = d_3+b_1 = 0 = a_3 = b_2 = c_1.$$

The quadrics through R are therefore those, and only those, linearly dependent on

$$xX+yZ+zY = 0, \quad xT = yX, \quad yT = zZ, \quad zT = xY.$$

Canonical curves on R admitting a harmonic inversion

7. The equation of any quadric in [6] is

$$\psi(x,y,z)+Q+\chi(X,Y,Z,T) = 0, \tag{7.1}$$

where ψ, χ are quadratic forms in the variables shown, and Q is a linear combination of the twelve products appearing in (6.2). Every quadric of W has an equation $Q = 0$ with appropriate coefficients. The conic $\psi(x,y,z) = 0$ in π cuts q in a set of $\mathfrak{X}^{(2)}(q)$. The same equation represents, on the other hand, the cone in [6] which projects this conic from the vertex Σ. Both interpretations of the equation will be used, the context at any time indicating which of the two is in question. Likewise for $\chi(X,Y,Z,T) = 0$; this can be the equation either of a cone with vertex π or of a quadric surface in Σ. One can always take a quadric through any set of $\mathfrak{X}^{(2)}(\sigma)$; it is to be supposed henceforward that this set of $\mathfrak{X}^{(2)}(\sigma)$ *corresponds exactly* with the set of $\mathfrak{X}^{(2)}(q)$ on $\psi = 0$ in that the two sets, of eight points on σ and eight points on q, are on *the same eight* g composing a set of $\mathfrak{X}^{(2)}(R)$. All these g lie both on the cone $\psi = 0$ and on the cone $\chi = 0$; they therefore all lie on (7.1) if, and only if, they lie on $Q = 0$. That they should so lie is assured by taking $Q = 0$ to belong to W. The intersection of R and (7.1) then consists of this set of $\mathfrak{X}^{(2)}(R)$ and a curve Γ, bisecant to the generators and of order $2.10-8 = 12$, skew to q but meeting σ four times. Γ is canonical: to verify this one has only to prove that Γ has the proper genus. Since every g is now on $Q = 0$ it meets (7.1) in the same points as it meets $\psi+\chi = 0$. But π and Σ are polar spaces for this last quadric, so that every g cuts Γ in points harmonic to the intersections of g with q and σ. In particular, Γ touches, at its intersections with σ, four g; and these are the only g tangents to Γ. The $(2, 1)$ correspondence between Γ and a curve, whether q or σ, of genus 3 therefore has four branch-points; the classical formula of Zeuthen ((**13**) 152; (**6**) 417; (**9**) 83; (**11**) 211),

$$\eta-\eta' = 2\alpha(p'-1)-2\alpha'(p-1) \tag{7.2}$$

gives, with $\alpha = 2$, $\alpha' = 1$, $\eta = 0$, $\eta' = 4$, $p' = 3$, the value 7 for p. Any quadric containing Γ but not the whole of R meets R in Γ and eight g which, meeting q at its intersections with a conic, form a set of $\mathfrak{X}^{(2)}(R)$.

8. If Γ is cut on R by the quadric $\psi+\chi = 0$ the pencil of quadrics $k\psi+\chi = 0$ cuts R, apart from the eight g of a definite set of $\mathfrak{X}^{(2)}(R)$, in a pencil of canonical curves, all touching one another at four points on σ but having no further common points. Two of these curves are composite: that for $k = 0$ consists of σ reckoned twice, that for $k = \infty$ of q, reckoned twice, and those four g that are common tangents to the curves.

Suppose that Γ is given. There are six linearly independent quadric surfaces $\chi = 0$ through its four intersections with σ; each of them cuts σ further in a set of $\mathfrak{X}^{(2)}(\sigma)$. Each cone $\chi = 0$ contains the eight g through the corresponding set of $\mathfrak{X}^{(2)}(\sigma)$; if $\psi = 0$ is the conic through the intersections of these g with q there is a definite constant k such that $k\psi+\chi = 0$ contains Γ. So one has a linear system of quadric primals,

$$\sum_{i=1}^{6} a_i(k_i\psi_i+\chi_i) = 0,$$

all containing Γ. But none contains R because for R, and so π, to be on the quadric, would require $\sum a_i k_i \psi_i \equiv 0$ whereas the conics $\psi_i = 0$, cutting out the complete $\mathfrak{X}^{(2)}(q)$, are linearly independent. Since there is a web of quadrics through R it follows, in accord with the standard theory ((**5**) 106), that Γ is on ten linearly independent quadrics.

Canonical curves admitting a four-group of self-projectivities

9. These curves Γ are, whereas a general canonical curve of genus 7 is not, invariant under the harmonic inversion $\mathscr{H}$ in a plane π and solid Σ. If one requires two such inversions $\mathscr{H}$, $\mathscr{H}'$ to commute one can achieve this by taking π' to join a point O of π to a line λ' of Σ while Σ' joins a line λ of π to a line λ'' of Σ: λ' and λ'' are, of course, skew, and λ does not pass through O. There are other ways of ensuring the commutation, but in the way presented above the product $\mathscr{H}'' = \mathscr{H}\mathscr{H}' = \mathscr{H}'\mathscr{H}$ is an inversion of the same kind, namely in the plane $\pi'' \equiv O\lambda''$ and the solid $\Sigma'' \equiv \lambda\lambda'$. The whole [6] is spanned by O, λ, λ', λ'', and the object now is to find curves Γ invariant not only under $\mathscr{H}$ but also under $\mathscr{H}'$ and $\mathscr{H}''$. If coordinates are used it will be natural to take the vertices of the simplex of reference to be O and two points on each of λ, λ', λ''.

Any generator of R joins a point p on q in π to a point P on σ in Σ; its transform, the same under $\mathscr{H}'$ as under $\mathscr{H}''$, joins the harmonic inverse of p in O and λ to the harmonic inverse of P in λ' and λ''. This join, too, has to be a chord of Γ and so, being transversal to π and Σ, a generator of R; its intersections with π and Σ are therefore on q and σ respectively. Thus q is its own harmonic inverse in O and λ, σ its own in λ' and λ''.

10. Schur sextics invariant under a biaxial harmonic inversion H in [3] occur as intersections of two cubic surfaces which are transposed by H and both contain a twisted cubic γ invariant under H. H pairs the points of γ in an involution I; the joins of these pairs of points form a regulus and, since all these joins meet both axes of H, these axes λ', λ'' belong to the complementary regulus and meet γ each in a single point. These points, F' on λ' and F'' on λ'', are the foci of I. A cubic surface G through γ meets λ' in F', A, B, and λ'' in F'', C, D; its transform G' also contains these points, so that G and G' meet in γ and a sextic σ through A, B, C, D. Since $\mathfrak{X}(\sigma)$ is cut on σ by the quadrics through γ, $ABCD$ is a canonical set; it is mapped on the canonical model q of σ by a set a, b, c, d of points on a line λ. Since A, B, C, D are the only invariant points on σ the only points on q that are fixed under the corresponding involutory transformation h are a, b, c, d.

Its chord FF' lies, with γ, on the quadrics of a pencil K; on each quadric of K is a regulus of joins of pairs of an involution on γ that includes the pair F, F'; each such involution shares one pair with I. Since γ, FF', and the join of this pair of I determine the quadric, and since γ and these two chords are all invariant under H, so is the quadric invariant, and so is the tetrad of $\mathfrak{X}(\sigma)$ that it determines. Thus every tetrad of the pencil $\mathfrak{p}$ in $\mathfrak{X}(\sigma)$ cut by a quadric of K is invariant; H transposes the four points in pairs. One quadric of K contains A; but A is its own transform under H so that this quadric touches σ at A, cutting out that tetrad of $\mathfrak{X}(\sigma)$ of which A is a double member. Likewise for those quadrics of K that contain, respectively, B, C, D. The lines, in the plane π of q, that cut on q the pencil $\mathfrak{p}'$, corresponding to $\mathfrak{p}$, of tetrads of $\mathfrak{X}(q)$ invariant under h include the tangents at a, b, c, d; so these concur, at O say, and the two remaining points of q on any one of these four of its tangents are transposed by h. If a quadric of K touches σ at a point P other than A, B, C, D, it also touches σ at $P' \equiv H(P)$; the corresponding tetrad of $\mathfrak{X}(q)$ is on a bitangent of q through O.

11. It could perhaps be said, although it is not relevant to the following discussion, that the chords joining points of σ to their transforms by H generate an elliptic quartic scroll ((**2**) 476; (**3**) 177) on which AB and CD are nodal lines. The special case of σ having for its canonical model a Klein quartic, and so lying on 21 such quartic scrolls, has been studied in full detail. Baker shows ((**2**) 477) incidentally that, if a quartic scroll S with nodal lines λ', λ'' is given, a web of Schur sextics, all self-inverse in λ' and λ'', is obtained on cutting S by quartic surfaces, all of them self-inverse, through λ', λ'', and six fixed generators of S. He notes, too, that λ' and λ'' are chords of these Schur curves.

12. When q is its own harmonic inverse in a point O and a line λ, and, consequently, σ its own in two skew lines λ', λ'', are there any curves Γ on R that are self-inverse not only in π and Σ but also in π' and Σ', and, therefore, also in π'' and Σ''? The equations in Σ of λ' can be taken as $X = T = 0$, those of λ'' as $Y = Z = 0$. In π, λ is $x = 0$ and O is the point $y = z = 0$.

A base for ternary quadratic polynomials $\psi(x,y,z)$ can consist of (i) four polynomials invariant under h and (ii) two that are changed in sign by h. When equated to zero the polynomials (i) provide those conics for which λ is the polar of O, while (ii) provide the line-pairs of which λ is one member while the other contains O.

A base for quaternary quadratic polynomials $\chi(X, Y, Z, T)$ can consist of (i) six polynomials invariant under H and (ii) four that are changed in sign by H. When equated to zero the polynomials (i) provide those quadrics for which λ' and λ'' are polar lines, while (ii) provide those quadrics of which λ' and λ'' are both generators.

If ψ is of type (i), the set of $\mathfrak{X}^{(2)}(R)$ on the cone $\psi = 0$ meets σ in eight points which, corresponding to points of q that are transposed in pairs by h, are transposed in pairs by H. Hence only four conditions are required for a quadric $\chi = 0$ of type (i) to contain them all. This leaves a pencil of quadric surfaces any one of which gives quadrics $k\psi + \chi = 0$ cutting R in curves Γ that are invariant under $\mathscr{H}'$. If, alternatively, ψ is of type (ii), the cone $\psi = 0$ meets R in the four generators aA, bB, cC, dD and in a second set of $\mathfrak{X}(R)$ whose members are transposed in pairs by $\mathscr{H}'$.

Every quadric $\chi = 0$ of type (ii) contains A, B, C, D; in order that it contain any other set of $\mathfrak{X}(\sigma)$ that is invariant under H only two conditions are necessary. This, too, leaves a pencil of quadric surfaces any one of which gives quadrics $k\psi+\chi = 0$ cutting R in curves Γ that are invariant under $\mathscr{H}'$.

Canonical curves admitting an elementary abelian group of eight self-projectivities

13. The curves Γ just constructed admit a four-group of self-projectivities consequent upon σ admitting a biaxial and q a central harmonic inversion. It will appear that there are curves Γ which admit an elementary abelian group not merely of four but of eight self-projectivities, and that there are, in $(1, 2)$ correspondence with Γ, curves σ and q each admitting three harmonic inversions: σ in the pairs of opposite edges of a tetrahedron U, q in the vertices and opposite sides of a triangle Δ. Since the product of the three inversions is the identity, one expects either two or none of them to change the signs of ψ and χ. In the latter event $\chi = 0$ has U for a self-polar tetrahedron, and $\psi = 0$ has Δ for a self-polar triangle; $\psi+\chi$, referred to the simplex $\mathscr{S}$ whose vertices are those of U and Δ, is the sum of seven squares. In the former event $\chi = 0$ contains two pairs of opposite edges of U, and $\psi = 0$ consists of two sides of Δ; $\psi+\chi$, referred to $\mathscr{S}$, is a sum of three binary products.

14. Curves σ that are invariant under the three harmonic inversions in pairs of opposite edges of a tetrahedron U are found by using the web of cubic surfaces through σ. There is one, and only one, such surface G through any three vertices of U; since all these vertices are, and σ is to be, invariant under the three inversions, so is G. If U is the tetrahedron of reference there is a single coordinate, T say, whose cube is present in the equation of G; the cubes of X, Y, Z are all absent. G is to be invariant when T and any one of X, Y, Z are left unchanged while the other two change sign; the left-hand side of its equation is unaltered and hence is a linear combination of multiples of

$$T^3, \quad TX^2, \quad TY^2, \quad TZ^2, \quad XYZ;$$

i.e. of the five monomials that occur in the expansion of

$$\begin{vmatrix} T & Z & Y \\ Z & T & X \\ Y & X & T \end{vmatrix}.$$

The other three analogous sets of five monomials also provide eligible cubic surfaces. But the four sets are those that furnish the expansion of the four determinants with three rows in

$$\begin{bmatrix} T & Z & Y & X \\ Z & T & X & Y \\ Y & X & T & Z \end{bmatrix}, \tag{14.1}$$

and a Schur sextic can be defined precisely in this fashion, as the set of points that reduce to 2 the rank of a matrix of three rows and four columns, whose elements are quaternary linear forms. If these twelve elements are numerical multiples of the coordinates as displayed in (14.1), the matrix identifies a curve σ that is its own harmonic inverse in each pair of opposite edges of U.

15. The $(1, 1)$ correspondence between such a curve σ and q is governed by the fact that when any two of X, Y, Z change sign simultaneously so do two of x, y, z. One now arranges bases for polynomials $\psi(x,y,z)$ and $\chi(X, Y,Z,T)$ as follows.

(i) Polynomials that are invariant under all three inversions. These, equated to zero, give conics for which Δ is self-polar, and quadric surfaces for which U is self-polar. If ψ is of type (i), the set of $\mathfrak{X}^{(2)}(R)$ on the cone $\psi = 0$ meets σ in eight points composed of two tetrads, each tetrad being derivable from any one of its members by the inversions in pairs of opposite edges of U. Only two conditions are necessary, when χ is also of type (i), for $\chi = 0$ to hold at all eight points. As there are four linearly independent polynomials χ of type (i), a pencil of quadric surfaces is available to be associated with $\psi = 0$ and provide quadrics cutting R in curves Γ admitting all the desired self-transformations.

(ii) Polynomials that are invariant under one inversion but change sign under the other two. This type can be subdivided into three subcategories according to which inversion of the three leaves the sign unchanged. It is enough to consider one of the three: so take

$$\psi \equiv yz; \quad \chi \text{ a linear combination of } YZ \text{ and } XT.$$

The line-pair $\psi = 0$ contains two sets of $\mathfrak{X}(q)$, and both the corresponding sets of $\mathfrak{X}(\sigma)$ are common to the whole pencil of quadrics $\chi = 0$; the three sets of $\mathfrak{X}(q)$ on the sides of Δ correspond to those of $\mathfrak{X}(\sigma)$ on the pairs of opposite edges of U.

16. A Schur sextic, invariant for a four-group whose involutions are biaxial harmonic inversions, has been encountered previously: indeed there were 14 such four-groups for which the curve was invariant. But each of these had, as axes for its involutions, not the edges of a tetrahedron but three mutually harmonic pairs of lines in a regulus ((**3**) 180).

Such four-groups are of the type designated azygetic by Study ((**12**) 457), whereas the one that concerns us is syzygetic. There seems to be no ready reference to any earlier occurrence of the curve invariant under a syzygetic four-group.

17. A curve invariant for an elementary abelian group G_2 of order 8 admits not merely one but seven four-groups of self-projectivities. They are derived from any one by using the powers of an automorphism, of period 7, of G_2; this permutes the seven four-groups, each of which includes three involutions, in the same way that a projectivity of period 7 permutes the seven lines, each of which consists of three points, of a 7-point plane ϖ.

Take a simplex of reference

$$\mathscr{S}:\ Y_0Y_1Y_2Y_3Y_4Y_5Y_6$$

for homogeneous coordinates y_i in [6]. Each plane face of $\mathscr{S}$ is opposite to a solid face, and any such pair of opposite faces can serve as fundamental spaces for a harmonic inversion. The geometry of ϖ shows how to choose seven plane faces of $\mathscr{S}$ so that any two share one, and only one, vertex; for each of the seven lines in ϖ consists of three points, and any two lines share one, and only one, point. There are cyclic projectivities in ϖ which permute the points and lines in single cycles: in any such projectivity every four consecutive points of the cycle involve three collinear points, but these are not themselves consecutive. In order, then, to be able to impose the cycle (0123456) on the points of ϖ suppose that, say, 013 is a line. The lines of ϖ are then

013, 124, 235, 346, 450, 561, 602,

and the quadrangles of points complementing them are

2456, 3560, 4601, 5012, 6123, 0234, 1345.

Any two quadrangles share two points, namely those points of ϖ on neither of the lines l, l' of which the quadrangles are complements. Note that these two points are on the third line l'' through the intersection of l and l'.

18. Now let the same seven digits denote vertices of $\mathscr{S}$. The harmonic inversion in the plane 013 and the opposite solid 2456 is imposed by the diagonal matrix

$$\operatorname{diag}(1, 1, -1, 1, -1, -1, -1);$$

the other six harmonic inversions by those matrices arising from this by cyclically permuting its diagonal elements. The product of any two, D and D', of these seven matrices is a third, D''. For there are, answering to two vertices shared by quadrangles, two places occupied by -1 that are the same in D as in D', so that $DD' = D'D$ is also a diagonal matrix D'' four of whose seven entries are -1; moreover, the three entries in D'' that are 1 also answer to collinear points in ϖ. So one obtains an elementary abelian group G_2 consisting of the identity projectivity and seven harmonic inversions. G_2 induces, in either fundamental space of any of its own inversions, a four-group of projectivities, whether this space be a plane or a solid. For instance, the biaxial inversion in the lines 01 and 46 in the solid 0146 is induced by two operations of G_2: by the inversion in 013 and 2456 as well as by that in 346 and 0125.

Canonical curves admitting a group of 56 self-projectivities

19. A curve Γ constructed to be invariant under G_2 now appears as one of a septet, all invariant under G_2 and obtained from any one by imposing powers of a projectivity $\mathscr{P}$ that permutes the vertices of $\mathscr{S}$ in the cycle (0123456). The possibility thus arises, should two, and so all, of the septet happen to coincide, of Γ admitting a group G_1 of 56 self-projectivities. Assume, then, that this occurs. Since the 21 edges of $\mathscr{S}$ lie three in each of the seven planes π, and since Γ is skew to π, Γ cannot meet any edge of $\mathscr{S}$. Its intersections with, say, 0146 therefore have coordinates

$$\begin{array}{ccccccc} p, & q, & 0, & 0, & r, & 0, & s, \\ -p, & -q, & 0, & 0, & r, & 0, & s, \\ p, & -q, & 0, & 0, & -r, & 0, & s, \\ -p, & q, & 0, & 0, & -r, & 0, & s, \end{array}$$

where $pqrs$ is not zero. The quadric Q,

$$\sum_{i=0}^{6} a_i y_i^2 + \sum_{i<j} a_{ij} y_i y_j = 0,$$

contains these four points when, and only when,

$$a_0 p^2 + a_1 q^2 + a_4 r^2 + a_6 s^2 = 0$$

and

$$a_{01}pq + a_{46}rs = a_{04}pr + a_{16}qs = a_{06}ps + a_{14}qr = 0.$$

If Q contains the whole of Γ the 24 further conditions derived from these by using the powers of (0123456) have also to be satisfied. In order that there should be an adequate number of independent quadrics through Γ one has to circumscribe the independence of these 28 conditions. Treat the 21 binary conditions separately, and then later attend to the seven quaternary ones.

20. The binary conditions fall into seven sets of three associated conditions. Each coefficient a_{ij} corresponds, by its two suffixes, to a line in ϖ, and each set of three associated conditions answers to three concurrent lines in ϖ. The digit for the point of concurrence is absent, and the coefficient corresponding to the other two digits for any one of the lines figures in two of the three conditions. For example, one set is

$$\left.\begin{aligned} a_{45}pq + a_{13}rs \qquad\qquad &= 0,\\ a_{13}qs + a_{26}pr &= 0,\\ a_{45}ps \qquad\qquad + a_{26}qr &= 0. \end{aligned}\right\} \tag{20.1}$$

If, then, these three a_{ij} are not all zero a certain determinant has to be; the ratios of the a_{ij} are then uniquely determined. Since the same determinant occurs with each of the seven associated sets, its vanishing permits seven linearly independent quadrics to contain the 28 points, and so the whole curve Γ. As none of p, q, r, s vanishes, the determinantal condition is, from (20.1),

$$\begin{vmatrix} q & r & . \\ . & q & p \\ s & . & q \end{vmatrix} = 0,$$

$$q^3 = -rsp. \tag{20.2}$$

Then

$$a_{45} : a_{13} : a_{26} = r^2s^2 : q^4 : s^2q^2,$$

together with the other six sets of relations with suffixes obtained from these by using (0123456).

21. It remains to deal with the seven quaternary conditions. Each of these involves four of the coefficients of the squares y_i^2; their determinant is the circulant

$$\begin{vmatrix} p^2 & q^2 & . & . & r^2 & . & s^2 \\ s^2 & p^2 & q^2 & . & . & r^2 & . \\ . & s^2 & p^2 & q^2 & . & . & r^2 \\ r^2 & . & s^2 & p^2 & q^2 & . & . \\ . & r^2 & . & s^2 & p^2 & q^2 & . \\ . & . & r^2 & . & s^2 & p^2 & q^2 \\ q^2 & . & . & r^2 & . & s^2 & p^2 \end{vmatrix} \equiv \prod_{j=0}^{6} (p^2 + \zeta^j q^2 + \zeta^{4j} r^2 + \zeta^{6j} s^2),$$

ζ being any seventh root of unity other than unity itself.

If

$$p^2 + \zeta^j q^2 + \zeta^{4j} r^2 + \zeta^{6j} s^2 = 0 \tag{21.1}$$

then Γ lies on the quadric $\sum_{i=0}^{6} \zeta^{ij} y_i^2 = 0$, and one can choose $p^2 : q^2 : r^2 : s^2$ uniquely so that this holds for any three integers j of which no two are congruent mod 7. If it is possible to take these integers so that $q^6 = r^2 s^2 p^2$, the signs of p, q, r, s can be arranged so that $q^3 = -rsp$, and then one has, as the classical theory lays down ((5) 106), ten linearly independent quadrics through Γ.

22. If a, b, c are the integers for which (21.1) holds the condition $q^6 = r^2 s^2 p^2$ is

$$-(\zeta^{b+c} + \zeta^{c+a} + \zeta^{a+b})^3 = \zeta^{5(a+b+c)}(\zeta^{2b} + \zeta^{2c})(\zeta^{2c} + \zeta^{2a})(\zeta^{2a} + \zeta^{2b})$$
$$\times \{(\zeta^b + \zeta^c)(\zeta^c + \zeta^a)(\zeta^a + \zeta^b)\}^2. \tag{22.1}$$

If none of the congruences (mod 7)

$$b + c \equiv 2a, \quad c + a \equiv 2b, \quad a + b \equiv 2c \tag{22.2}$$

holds, the seven integers

$$2a + b, \quad 2b + c, \quad 2c + a, \quad a + b + c, \quad a + 2b, \quad b + 2c, \quad c + 2a$$

are all incongruent. The right-hand side of (22.1) then reduces to $\zeta^{2(a+b+c)}$, and the equation does not hold because

$$\zeta^{b+c} + \zeta^{c+a} + \zeta^{a+b} + \zeta^{3(a+b+c)} \neq 0.$$

One then has to presume that a congruence (22.2) does hold; not more than one can hold for incongruent a, b, c, so suppose that

$$b + c \equiv 2a, \quad a \equiv -3(b + c).$$

Then, since

$$-\zeta^{b+c} - \zeta^{-3b-2c} - \zeta^{-2b-3c} = \zeta^{2b} + \zeta^{3b-c} + \zeta^{3c-b} + \zeta^{2c}$$
$$= \zeta^{-b-c}(\zeta^b + \zeta^c)(\zeta^{3b} + \zeta^{3c}),$$

the left-hand side of (22.1) is now

$$\zeta^{-3(b+c)}(\zeta^b + \zeta^c)^3(\zeta^{3b} + \zeta^{3c})^3,$$

while the right-hand side is found to be

$$\zeta^{b+c}(\zeta^b + \zeta^c)^4(\zeta^{2b} + \zeta^{2c})(\zeta^{3b} + \zeta^{3c})^4.$$

So (22.1) will be satisfied when $b + c \equiv 2a$ provided that

$$\zeta^{3(b+c)} = (\zeta^b + \zeta^c)(\zeta^{2b} + \zeta^{2c})(\zeta^{3b} + \zeta^{3c}),$$

and this is certainly true whenever b and c are incongruent. So one has a curve Γ on the quadrics

$$\sum \zeta^{bi} y_i^2 = 0, \quad \sum \zeta^{ci} y_i^2 = 0, \quad \sum \zeta^{-3(b+c)i} y_i^2 = 0, \tag{22.3}$$

as well as on seven more quadrics, linearly independent of these, whose equations are

$$r^2 s^2 y_{i+4} y_{i+5} + q^4 y_{i+1} y_{i+3} + s^2 q^2 y_{i+2} y_{i+6} = 0 \quad (i = 0, 1, \ldots, 6),$$

with suffixes reduced mod 7. This last equation, when the values for $p^2 : q^2 : r^2 : s^2$ are calculated from the three equations (21.1) with $j = a, b, c$, becomes

$$\zeta^{b+c}(\zeta^{4b} - \zeta^{4c}) y_{i+4} y_{i+5} + \zeta^{3(b+c)}(\zeta^b - \zeta^c) y_{i+1} y_{i+3} + (\zeta^{2b} - \zeta^{2c}) y_{i+2} y_{i+6} = 0.$$

Abbreviate this, for the moment, to

$$A y_{i+4} y_{i+5} + B y_{i+1} y_{i+3} + C y_{i+2} y_{i+6} = 0. \tag{22.4}$$

23. That the manifold common to the quadrics (22.4) is its own harmonic inverse, in, say, $\pi \equiv 013$ and $\Sigma \equiv 2456$ is clear. π and Σ both lie on those quadrics for which $i = 2, 4, 5, 6$; their equations may be displayed as

$$\left.\begin{aligned} A y_0 y_6 + B y_3 y_5 + C y_1 y_4 \qquad\qquad &= 0, \\ C y_3 y_6 + B y_0 y_5 \qquad\qquad + A y_1 y_2 &= 0, \\ B y_1 y_6 \qquad\qquad + C y_0 y_4 + A y_3 y_2 &= 0, \\ C y_1 y_5 + A y_3 y_4 + B y_0 y_2 &= 0. \end{aligned}\right\} \tag{23.1}$$

These four equations have no solution in y_2, y_4, y_5, y_6 unless

$$\begin{vmatrix} A y_0 & B y_3 & C y_1 & \cdot \\ C y_3 & B y_0 & \cdot & A y_1 \\ B y_1 & \cdot & C y_0 & A y_3 \\ \cdot & C y_1 & A y_3 & B y_0 \end{vmatrix} = 0,$$

which is the equation of the curve q in π. On the other hand, there is no solution in y_0, y_1, y_3 unless

$$\begin{bmatrix} A y_6 & B y_5 & C y_4 & B y_2 \\ C y_4 & A y_2 & B y_6 & C y_5 \\ B y_5 & C y_6 & A y_2 & A y_4 \end{bmatrix}$$

has rank 2, as with (14.1), restraining (y_2, y_4, y_5, y_6) to lie on a Schur sextic σ in Σ. The edges of the tetrahedron U whose vertices are those of $\mathscr{S}$ which lie in Σ are all chords of σ, which cuts each of them in points harmonic to the vertices that it joins. For example, the points $y_2 = y_4 = 0$, $ACy_6^2 = B^2 y_5^2$ are on σ.

The four quadrics (23.1) are the base of a web intersecting in a scroll R having q and σ for directrices; other quadrics through Γ have equations $\psi(y_0, y_1, y_3) + \chi(y_2, y_4, y_5, y_6) = 0$; e.g.

$$Ay_0y_1 + By_4y_6 + Cy_2y_5 = 0$$

and those derived herefrom by the permutation $(ABC)(013)(254)$.

The Hurwitz curve of genus 7

24. The curve Γ now obtained admits the group G_1 of 56 self-projectivities, whereas a Hurwitz curve admits 504. Indeed, as is clear on comparing (22.3) and (22.4) with ((**7**) 535–36) Macbeath's (11) and (15), a further specialization is available by taking $b + c \equiv 0$. It is this last stipulation that allows a further self-projectivity of period 3 and thereby amplifies G_1 to the group G of order 504. If one solves the three equations (21.1) for $j = 0, 1, -1$ determinantally one finds, if $\zeta = \exp(2\pi i/7)$, that

$$p^2 : q^2 : r^2 : s^2 = 2\cos\frac{8\pi}{7} : 1 : 2\cos\frac{4\pi}{7} : 2\cos\frac{2\pi}{7}$$

and, then,

$$a_{45} : a_{13} : a_{26} = r^2s^2 : q^4 : s^2q^2 = \sin\frac{8\pi}{7} : \sin\frac{2\pi}{7} : \sin\frac{4\pi}{7}.$$

The Hurwitz curve Γ is the assemblage of points common to Macbeath's ten quadrics

$$\left.\begin{aligned} \Sigma y_i^2 = 0, \quad \Sigma \zeta^i y_i^2 = 0, \quad \Sigma \zeta^{-i} y_i^2 = 0, \\ \sin\frac{8\pi}{7} y_{i+4}y_{i+5} + \sin\frac{2\pi}{7} y_{i+1}y_{i+3} + \sin\frac{4\pi}{7} y_{i+2}y_{i+6} = 0. \end{aligned}\right\} \qquad (24.1)$$

25. The projectivity $\mathscr{P}$ embedded G_2 as a normal subgroup in a group G_1, of order 56, having eight subgroups $\mathscr{C}_7$. But Γ, which is the assemblage of points common to all ten quadrics (24.1), admits a group G of 504 self-projectivities of which G_1 is only one of nine conjugate subgroups. G includes projectivities of period 3, each a product of two non-commuting involutions; two such involutions, named U and V by Macbeath ((**7**) 540–41), give the projectivity imposed by

$$VU = M = \begin{bmatrix} \cdot & \cdot & -\tau & \tau & \tau & \cdot & \tau \\ \cdot & -\tau & \tau & \tau & \cdot & \tau & \cdot \\ \tau & -\tau & -\tau & \cdot & -\tau & \cdot & \cdot \\ -\tau & -\tau & \cdot & -\tau & \cdot & \cdot & \tau \\ \tau & \cdot & \tau & \cdot & \cdot & -\tau & \tau \\ \cdot & -\tau & \cdot & \cdot & \tau & -\tau & -\tau \\ \tau & \cdot & \cdot & -\tau & \tau & \tau & \cdot \end{bmatrix},$$

where $2\tau = 1$. If

$$Y = (y_0, y_1, y_2, y_3, y_4, y_5, y_6)'$$

then

$$(I+M+M^2)Y = 2(y_0+y_4+y_6)(1,0,0,0,1,0,1)',$$

so that any point in Ξ, the prime $y_0+y_4+y_6 = 0$, is collinear with its two transforms by M and M^2, whereas any point not in Ξ spans, with its two transforms, a plane through the point X with coordinates $(1,0,0,0,1,0,1)$.

26. X is invariant, its coordinate vector being latent for M with latent root 1. Since

$$|M-\lambda I| \equiv (1-\lambda)(1+\lambda+\lambda^2)^3,$$

the other latent roots of M are ω $(= \exp(2\pi i/3))$ and ω^2, each occurring three times. The invariant points, in addition to X, are those of two planes. The plane ϖ whose points are invariant with latent root ω is found to have equations

$$\left.\begin{aligned} y_3 &= (\omega-\omega^2)y_0+y_2, \\ y_4 &= y_0-y_1-(\omega-\omega^2)y_2, \\ y_5 &= (\omega-\omega^2)(-y_0+y_1)-2y_2, \\ y_6 &= -2y_0+y_1+(\omega-\omega^2)y_2, \end{aligned}\right\} \tag{26.1}$$

while the equations for the other plane ϖ' appear on transposing ω and ω^2. But this is tantamount to changing the signs of y_0, y_1, y_4, y_6; so ϖ and ϖ' are transposed by the harmonic inversion in 235 and 0146. Indeed, if

$$D = \text{diag}(-1,-1,1,1,-1,1,-1)$$

then

$$DM = M'D = M^{-1}D,$$

so that D and M generate a dihedral group of order 6.

Both ϖ and ϖ' lie in Ξ, and indeed span it. They also lie on the quadric $\sum y_i^2 = 0$, and because all the matrices in this representation of G are orthogonal each subgroup $\mathscr{C}_3$ of G gives two planes of invariant points on this quadric. Since $M^3 = I$,

$$M(I+\omega M+\omega^2M^2) = \omega^2(I+\omega M+\omega^2M^2),$$

$$M(I+\omega^2M+\omega M^2) = \omega(I+\omega^2M+\omega M^2);$$

thus any point not in Ξ spans, with its two transforms, a plane which contains a point for which $MY = \omega^2Y$ and another point for which $MY = \omega Y$. Every plane spanned by a triad which is its own transform under M meets both ϖ and ϖ' as well as passing through X.

27. Ξ has a total of 12 intersections with Γ. Were any of these not invariant it would be one of a collinear triad; but Γ has no trisecants ((**5**) 107–8): any such would lie on every quadric through Γ, whereas Γ is the complete set of points common to them. Every intersection of Γ with Ξ is therefore either in ϖ or in ϖ'; moreover since, at any such intersection, the tangent of Γ is invariant it comprises collinear triads and so lies in Ξ: Ξ is a *contact prime* of Γ, touching it wherever it meets it. Any intersection of Γ with ϖ is accompanied by one with ϖ', the coordinates of either intersection being the complex conjugates of those of the other. The conclusion is that ϖ meets Γ in three points A, B, C, while ϖ' meets Γ in three points D, E, F.

Now let P be a point of Γ 'near' A; its transforms P', P'' are also near A and hence, since the plane $PP'P''$ always contains X, so does the osculating plane of Γ at A as, likewise, do those at B, C, D, E, F. A second triad Q, Q', Q'' on Γ spans a second plane through X; the planes $PP'P''$ and $QQ'Q''$ span a [4], so that the osculating [4] of Γ at A has six-point, not merely the common five-point, intersection. The [5] spanned by P, P', P'', Q, Q', Q'', A contains ϖ since it passes not merely through A itself but also through the two intersections of ϖ with the planes $PP'P''$ and $QQ'Q''$: it thus, being invariant under M, meets Γ in A, B, C, P, P', P'', Q, Q', Q'', and an invariant triad R, R', R''. As P and Q both approach A along Γ so do P', P'', Q' Q''; but R, R', R'' need not do so. The osculating prime of Γ at A thus has seven-point intersection: only 1 higher in multiplicity than the intersection with the osculating [4]. A is a point of superosculation, or Weierstrassian point, on the canonical curve; it counts, by C. Segre's rule ((**10**) 90), for two among the totality of Weierstrassian points on Γ, as, likewise, do B, C, D, E, F. As there are 28 subgroups $\mathscr{C}_3$ in G, all giving rise as above to six points each contributing two to the total of Weierstrassian points, one obtains the whole aggregate of $28.2.6 = 336$—this being the value of $p(p^2-1)$ when $p = 7$.

28. The plane of any triad that is its own transform under M meets, as has been shown, both ϖ and ϖ'; hence there are two plane curves, one in ϖ and one in ϖ', in (1, 1) correspondence with each other and in (1, 3) correspondence with Γ. This latter correspondence has, in either plane, six branch-points: in ϖ these are A, B, C, and the intersections of ϖ with the osculating planes at D, E, F; in ϖ' they are D, E, F, and the intersections of ϖ' with the osculating planes at A, B, C. Now each branch-point contributes, with all three correspondent points on Γ coinciding, two to the proper total of branch-points ((**11**) 213). So, if p

is the genus of these two plane curves, the Zeuthen formula (7.2) gives

$$12-0 = 2(7-1)-6(p-1),$$
$$p = 1.$$

Thus Γ is, and in 28 different ways, in (3, 1) correspondence with elliptic curves.

29. The projectivity $\mathscr{P}$ of period 7 that was introduced in § 19 can be defined as the one permuting the bounding primes of the simplex $\mathscr{S}$ in a definite cycle while leaving unmoved the unit prime $\sum y_i = 0$. Its united points are

$$(1, \zeta^j, \zeta^{2j}, \zeta^{3j}, \zeta^{-3j}, \zeta^{-2j}, \zeta^{-j}) \quad (j = 0, 1, 2, \ldots, 6),$$

vertices of a simplex whose seven bounding primes are, like the unit prime which is one of them, unmoved. Of its vertices two, and these two only, are on Γ; namely those with $j = 1$ and 6. Their coordinates are seen to satisfy the equations of all the quadrics through Γ. Call them I, J; and call IJ a *principal chord* of Γ: it is one of 36 such chords, one for each subgroup $\mathscr{C}_7$ of G. The osculating planes of Γ at I and J are both invariant under $\mathscr{P}$, and so, therefore, is the prime S_5 spanned by them. Since S_5 has (at least) three-point intersection with Γ both at I and at J, it meets Γ in (at most) six further points which, however, are too few to furnish a cycle for $\mathscr{P}$. So one presumes that S_5 is the osculating prime of Γ both at I and at J.

30. If P on σ, and p on q, correspond, the solid spanned by the tangent to σ at P and the tangent to q at p touches Γ at both its intersections with Pp. If EFG is a tritangent plane of σ the prime joining it to π contains not only the tangents to σ at E, F, G but also the tangents to q at the corresponding points e, f, g; this prime therefore touches Γ at six points—namely its intersections with Ee, Ff, Gg—and so is a *contact prime* of Γ. Since σ has 64 tritangent planes ((**1**) 43), one accounts for $63.64 = 4032$ contact primes of Γ which is known, by standard theory ((**1**) 41), to have $2^6(2^7-1) = 8128$. Of these, 28 are primes such as Ξ; of the other 8100 there remain 4068, and these include the 36 that osculate Γ at both ends of principal chords. The remaining 4032 will fall into eight sets of 504, the primes of each set being permuted among themselves by G.

31. A point of Γ is one of a set of 504, closed under G, unless it happens to be invariant for some projectivity in G, in which event the cardinal of the set to which it belongs is a factor of 504. It has now been shown that there are three sets of this specialized kind, numbering 72, 168, 252 points: namely the intersections of Γ with

(a) its 36 principal chords;

(b) the 56 planes, of points invariant for operations of period 3, containing cubic curves on scrolls of trisecant planes of Γ;

(c) the 63 solids containing Schur sextics on scrolls of chords of Γ.

These factors 7, 3, 2 are orders of subgroups of G which possess invariant points on Γ, and are predestined for a Hurwitz curve. For the Klein group of order 168 there are invariant sets of 24, 56, 84 points on the plane quartic: namely inflexions, contacts of bitangents, and sextactic points.

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