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4.—The Principal Chords of an Elliptic Quartic.* By W. L. Edge

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Synopsis

The curve Γ common to two quadric surfaces has 24 principal chords; they are the sides of six skew quadrilaterals each of which has for its two diagonals a pair of opposite edges of that tetrahedron S which is self-polar for both quadrics. These three pairs of quadrilaterals serve to identify the three pairs of quadrics through Γ that are mutually apolar.

The vertices of the quadrilaterals lie four on each edge of S. Both nodes of the plane projection of Γ from such a vertex are biflecnodes.

§ 1. A chord AB of a curve Γ in [3], i.e. in three-dimensional projective space, is *principal* when it lies in the osculating planes of Γ at both A and B. This property, of being simultaneously the join of two points and the intersection of two planes, is self-dual; the number

$$\tfrac{1}{2}(n-3)(n-4)+\tfrac{1}{2}(n'-3)(n'-4)-12p$$

of such chords is, correspondingly, given by a self-dual formula (Severi 1907, p. 86); here n is the *order* (number of points in an arbitrary plane), n' the *class* (number of osculating planes through an arbitrary point) and p the *genus* of Γ. There would be a reduction for singularities, but as Γ is not to have any this need not be considered.

When n is 4, Γ a quartic, A and B account for all intersections of either osculating plane with Γ; there are $\tfrac{1}{2}(n'-3)(n'-4)-12p$ principal chords. If Γ is elliptic, p is 1 and (§ 5 below) $n' = 12$; there are 24 principal chords. But the mere announcement of their number gives no inkling of their geometrical relations to one another. Some of these, as well as the number 24, can be perceived quickly with the aid of the standard parametric representation of Γ by elliptic functions, and perhaps it is only right to indicate this (see § 11). But the relevant geometry can be investigated by elementary techniques without using the elliptic functions, and some account of this procedure now follows.

Not the least intriguing feature of the affair, and a main impetus to the composition of this note, is that Enriques (1924, p. 284) encounters the 24 chords and the skew quadrilaterals of which they are the sides, and discusses them at some length, without, it would seem, even suspecting that the chords are principal. He does however, earlier (p. 219) in the same text, identify the three principal chords of the *rational* quartic ($n = 4$, $n' = 6$, $p = 0$).

§ 2. Γ is the base curve of a pencil of quadrics all of which have a common self-polar tetrahedron S; this we take as tetrahedron $XYZT$ of reference for homogeneous coordinates x, y, z, t. Γ is then the aggregate of points whose coordinates satisfy both equations

$$\left.\begin{aligned} x^2+y^2+z^2+t^2 &= 0 \\ ax^2+by^2+cz^2+dt^2 &= 0 \end{aligned}\right\} \qquad (2.1)$$

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where it is supposed, in order to forestall unwanted specialisation, that no two of a, b, c, d are equal.

It is apparent that if (x, y, z, t) is on Γ so is every point derived therefrom by any changes of sign of any coordinates; Γ is invariant under four harmonic homologies h_1, h_2, h_3, h_4 centred at the vertices of S. These homologies commute, the product of all four is the identity projectivity, and their products in pairs are the biaxial harmonic inversions in pairs of opposite edges of S. These three biaxial inversions applied to any point (ξ, η, ζ, τ) of Γ produce the *tetrad*

$$(\xi, \eta, \zeta, \tau), \quad (\xi, -\eta, -\zeta, \tau), \quad (-\xi, \eta, -\zeta, \tau), \quad (-\xi, -\eta, \zeta, \tau)$$

of points of Γ; this is transformed into a second tetrad by applying the homologies, all four of which produce the same new tetrad. The two tetrads compose an *octad* and form with the vertices of S a trio of desmic tetrahedra. The only exception is that a point of Γ in a face of S yields coincident tetrads; such a tetrad is a quadrangle whose diagonal points are vertices of S.

A point P of Γ is transposed by h_j with a second point P_j of Γ; PP_j passes through a vertex of S while P and P_j are harmonic with respect to this vertex and the intersection of PP_j with the opposite face of S. As P moves on Γ PP_j generates one of the four quadric cones K_j that contain Γ; the tangent plane of K_j along its generator PP_j is bitangent to Γ. In particular: the intersections of Γ with the faces of S are stalls of Γ, points where the osculating plane has 4-point intersection.

§ 3. The projection of Γ from a point O (other than a vertex of S) on to any plane π not passing through O is a plane quartic q; one avoids projecting from a vertex of S because the projection would then be a conic, covered twice, with four branch points. Any self-projectivity of Γ implies a (1, 1) correspondence between points of q, but this need not be subordinated to any projectivity in π. If, however, O is on an edge of S, say on XY, points of Γ that are harmonic inverses in Z and XYT are projected into points of q that are harmonic inverses in the point Z' and the line UT' in which π meets OZ and XYT, and there is an analogous conjunction between the harmonic inversion of Γ in T and XYZ with that of q in T' and UZ', the point and line in which π meets OT and XYZ. Indeed q now admits a 4-group of self-projectivities whose involutions are harmonic inversions in the sides and vertices of the triangle $UZ'T'$; if this is chosen for the triangle of reference in π the equation of q contains only even powers of the three homogeneous coordinates.

Through a point O of general position there passes one of the quadrics through Γ; the two generators of this quadric which pass through O are chords of Γ, and are the only chords of Γ through O; the projection q of Γ from O has two nodes. But if O is on XY the tangent plane of the quadric there is OZT because $XYZT$ is a self-polar tetrahedron; since both generators of the quadric are in the tangent plane they meet ZT: the two nodes of q are on $Z'T'$.

§ 4. If O, on XY, is $(\xi, \eta, 0, 0)$ the line joining it to (x, y, z, t) is given in terms of a parameter p by

$$(p\xi + x, p\eta + y, z, t).$$

Its intersections with the quadrics (2.1) correspond to the roots of the quadratics

$$p^2(\xi^2+\eta^2)+2p(\xi x+\eta y)+x^2+y^2+z^2+t^2 = 0,$$
$$p^2(a\xi^2+b\eta^2)+2p(a\xi x+b\eta y)+ax^2+by^2+cz^2+dt^2 = 0.$$

The line meets Γ when these two quadratics have a common root, so that the quartic cone of lines which join O to points of Γ is

$$\begin{vmatrix} \xi^2+\eta^2 & 2(\xi x+\eta y) & x^2+y^2+z^2+t^2 & 0 \\ 0 & \xi^2+\eta^2 & 2(\xi x+\eta y) & x^2+y^2+z^2+t^2 \\ 0 & a\xi^2+b\eta^2 & 2(a\xi x+b\eta y) & ax^2+by^2+cz^2+dt^2 \\ a\xi^2+b\eta^2 & 2(a\xi x+b\eta y) & ax^2+by^2+cz^2+dt^2 & 0 \end{vmatrix} = 0 \qquad (4.1)$$

The outcome of putting $x = y = 0$ in this equation is, simply,

$$\begin{vmatrix} \xi^2+\eta^2 & z^2+t^2 \\ a\xi^2+b\eta^2 & cz^2+dt^2 \end{vmatrix}^2 = 0;$$

indeed this is the outcome, whatever w, of putting $x = w\xi$ and $y = w\eta$ in (4.1). So the plane OZT meets the cone in two lines, both repeated; these lines are the nodal generators of the cone, the chords of Γ through O.

§ 5. If $f(\theta) \equiv (\theta-a)(\theta-b)(\theta-c)(\theta-d)$

and

$$s_k = \frac{a^k}{f'(a)} + \frac{b^k}{f'(b)} + \frac{c^k}{f'(c)} + \frac{d^k}{f'(d)}$$

then

$$s_0 = s_1 = s_2 = 0. \qquad (5.1)$$

Since equations (2.1) have two, and only two, linearly independent solutions for x^2, y^2, z^2, t^2 every point of Γ satisfies (cf. Salmon 1882, p. 195), for some value of u,

$$x^2 f'(a) = 1+au, \ y^2 f'(b) = 1+bu, \ z^2 f'(c) = 1+cu, \ t^2 f'(d) = 1+du. \qquad (5.2)$$

This is not, of course, a parametric representation of Γ; it parametrises the octads on Γ. The four octads that consist of two coincident tetrads answer to the values $-1/a$, $-1/b$, $-1/c$, $-1/d$ of u.

The equation of the plane osculating Γ at (ξ, η, ζ, τ) is (Baker 1933, p. 208)

$$f'(a)\xi^3 x + f'(b)\eta^3 y + f'(c)\zeta^3 z + f'(d)\tau^3 t = 0. \qquad (5.3)$$

Hence there are 12 osculating planes through a general point $(\xi', \eta', \zeta', \tau')$, their contacts being the points where Γ is cut by the cubic surface

$$f'(a)x^3\xi' + f'(b)y^3\eta' + f'(c)z^3\zeta' + f'(d)t^3\tau' = 0.$$

§ 6. If the join of $A(\xi_1, \eta_1, \zeta_1, \tau_1)$ and $B(\xi_2, \eta_2, \zeta_2, \tau_2)$ on Γ is a principal chord then

$$\Sigma f'(a)\xi_1^3\xi_2 = \Sigma f'(a)\xi_1\xi_2^3 = 0 \qquad (6.1)$$

where summations have the obvious range of four terms. But since A and B are both on Γ there exist u_1 and u_2 such that $\xi_1^2 f'(a) = 1+au_1$, $\xi_2^2 f'(a) = 1+au_2$, etc., and so

$$\Sigma(1+au_1)\xi_1\xi_2 = \Sigma(1+au_2)\xi_1\xi_2 = 0,$$

$$u_1\Sigma a\xi_1\xi_2 = -\Sigma\xi_1\xi_2 = u_2\Sigma a\xi_1\xi_2$$

so that

$$\textit{either} \quad u_1 = u_2 \quad \textit{or} \quad \Sigma\xi_1\xi_2 = \Sigma a\xi_1\xi_2 = 0.$$

But this last pair of conditions requires that A and B be conjugate for every quadric through Γ. Since A is on Γ, B is then on the tangent of Γ at A and so, being itself on

Γ, is the same point as A. One is therefore forced to conclude that $u_1 = u_2$; the two intersections of a principal chord with Γ belong to the same octad. Indeed they belong to the same tetrad. For two points of an octad that are one in each of its complementary tetrads are joined by a generator of a cone through Γ; this generator is not a principal chord, being the join of contacts with Γ of a bitangent plane.

Thus any principal chord is a transversal of two opposite edges of S. The harmonic homologies and their products, since they leave Γ invariant, transform any principal chord either into itself or into other principal chords; when a line l is transversal to two opposite edges and does not contain any vertex of S the self-projectivities of Γ, when acting on l, produce a skew quadrilateral whose diagonals are the edges of S to which l is transversal (cf. Enriques *loc. cit.*). So the 24 principal chords of Γ fall into six quadrilaterals, each pair of opposite edges of S being diagonals of two of them.

§ 7. If the principal chord meets XY and ZT then $\xi_2 = \xi_1$, $\eta_2 = \eta_1$, $\zeta_2 = -\zeta_1$, $\tau_2 = -\tau_1$, and so (6.1) requires that

$$f'(a)\xi^4 + f'(b)\eta^4 = f'(c)\zeta^4 + f'(d)\tau^4.$$

Since, by (5.1) and (5.2), the sum of these two equal sums is zero they must both be zero themselves; the two quadrilaterals of principal chords whose diagonals are XY and ZT correspond one to each root u of the equation

$$(1+au)^2/f'(a) + (1+bu)^2/f'(b) = 0;$$

this may also be written

$$(b-c)(b-d)(1+au)^2 = (a-c)(a-d)(1+bu)^2,$$

or, cancelling $a-b$,

$$a+b-c-d+2(ab-cd)u + \{ab(c+d) - cd(a+b)\}u^2 = 0.$$

Alternatively: the points on XY that are on principal chords of Γ have coordinates

$$(\xi, \eta, 0, 0), \quad (\xi, -\eta, 0, 0), \quad (\xi, i\eta, 0, 0), \quad (\xi, -i\eta, 0, 0)$$

where

$$(a-c)(a-d)\xi^4 = (b-c)(b-d)\eta^4.$$

The points $(\xi, \pm\eta, 0, 0)$ are vertices of one quadrilateral, while $(\xi, \pm i\eta, 0, 0)$ are vertices of the other. These two pairs of vertices are harmonic to each other in addition to both pairs being harmonic to X and Y.

§ 8. There is a quadric containing Γ and any one of its chords. Should this chord be principal the whole quadrilateral of principal chords to which it belongs is on the quadric. Thus six quadrics are identified.

Let the two quadrics through Γ and the quadrilaterals of principal chords whose diagonals are XY and ZT be named Q and Q'; XY and ZT are polar lines for both. But if $(\xi, \eta, 0, 0)$ is a vertex of the quadrilateral on Q its polar plane with respect to Q' joins ZT to the harmonic conjugate of $(\xi, \eta, 0, 0)$ with respect to the intersections of Q' with XY; and this harmonic conjugate is, as has just been noted, the other vertex on XY of the quadrilateral on Q. Hence Q, being circumscribed to a self-polar tetrahedron of Q', is outpolar to Q'. The same reasoning, with the roles of the two quadrilaterals transposed, shows Q' to be outpolar to Q. Thus Q and Q' are both outpolar and inpolar to one another; they are 'apolar both ways' and their invariants Θ and Θ' both zero.

That three such pairs of quadrics are present among the pencil through Γ is known (Todd 1947, p. 267); the above paragraph provides a geometrical characterisation for them.

That these three pairs of quadrics afford a combinantal covariant of the pencil is salient; that it is the covariant of Todd's example is quickly confirmed. For the quadric

$$(\lambda+a)x^2+(\lambda+b)y^2+(\lambda+c)z^2+(\lambda+d)t^2 = 0$$

contains the points $(\xi, \pm\eta, 0, 0)$ when

$$\lambda = \lambda_1 = -(a\xi^2+b\eta^2)/(\xi^2+\eta^2);$$

it contains the points $(\xi, \pm i\eta, 0, 0)$ when

$$\lambda = \lambda_2 = -(a\xi^2-b\eta^2)/(\xi^2-\eta^2).$$

Thus

$$\lambda_1\lambda_2 : \lambda_1+\lambda_2 : 1 = a^2\xi^4-b^2\eta^4 : -2(a\xi^4-b\eta^4) : \xi^4-\eta^4.$$

But if

$$\xi^4 : \eta^4 = (b-c)(b-d) : (a-c)(a-d)$$

it follows, on cancelling a factor $a-b$, that

$$\lambda_1\lambda_2 : \lambda_1+\lambda_2 : 1 = ab(c+d)-cd(a+b) : 2(ab-cd) : a+b-c-d$$

so that

$$2(\lambda_1\lambda_2+ab) = (\lambda_1+\lambda_2)(a+b)$$

and

$$2(\lambda_1\lambda_2+cd) = (\lambda_1+\lambda_2)(c+d);$$

the quadratic whose roots are λ_1, λ_2 is indeed harmonic both to that whose roots are a, b and to that whose roots are c, d.

§ 9. The projection q of Γ from O onto π has inflections at the 12 intersections of π with the joins of O to the contacts of the 12 osculating planes that pass through O; also q has nodes at the intersections of π with the two chords of Γ that pass through O. Should either of these chords be principal the corresponding node of q is a biflecnode, having an inflection on each branch. It has been seen that there are 24 points, four on each edge of S, where principal chords concur; hence there are 24 points O from which Γ is projected into a plane quartic with two biflecnodes. Moreover, since every such point O is on an edge of S, the biflecnodal curve is its own harmonic inverse in the vertices and opposite sides of a triangle—the triangle being indeed the diagonal triangle of the quadrilateral of tangents at the two biflecnodes.

The pencil of quadrics through Γ is spanned by any two of its members, in particular by a pair Q, Q'. But if (2.1) is such a pair one of the three involutions determined by pairing $-a$, $-b$, $-c$, $-d$ has 0, ∞ for foci; and conversely. So there is no loss of generality in taking $a+b = c+d = 0$ and defining Γ by

$$x^2+y^2+z^2+t^2 = a(x^2-y^2)+c(z^2-t^2) = 0.$$

Then $f(\theta) \equiv (\theta^2-a^2)(\theta^2-c^2)$, so that

$$f'(a) : f'(-a) : f'(c) : f'(-c) = a : -a : -c : c.$$

The intersections $(\xi, \eta, 0, 0)$ of XY with principal chords now satisfy $\xi^4 = \eta^4$; those of ZT with principal chords are $(0, 0, \zeta, \tau)$ where $\zeta^4 = \tau^4$.

That the join of (1, 1, 0, 0) and (0, 0, 1, 1) is a principal chord is readily seen; its intersections with Γ are $(1, 1, \pm i, \pm i)$ and the osculating planes there are, by (5.3),

$$a(x-y) = \pm ic(z-t). \tag{9.1}$$

Similarly, the join of $(1, i, 0, 0)$ and $(0, 0, 1, -i)$ meets Γ at $(c, ic, \pm ia, \pm a)$, the two osculating planes being

$$c^2(x+iy) \pm a^2(iz+t) = 0.$$

§ 10. Now let us project Γ from (1, 1, 0, 0) onto the plane $x = 0$. Earlier discussions enable us to say that the equation of the projection q will contain only even powers of y, z, t; that q has two biflecnodes on $y = 0$, namely at the projections $(0, 1, \pm 1)$ of the points $(0, 0, 1, \pm 1)$ on ZT. Also, by (9.1), the tangents at the biflecnode (0, 1, 1) are $ay = \pm ic(z-t)$; those at $(0, 1, -1)$ will be $ay = \pm ic(z+t)$.

The equation of q is found on putting

$$\eta = \xi, \quad b = -a, \quad d = -c, \quad x = 0$$

in the determinant (4.1); the immediate result is

$$\begin{vmatrix} 2\xi^2 & 2\xi y & y^2+z^2+t^2 \\ 0 & -2a\xi y & -ay^2+c(z^2-t^2) \\ -2a\xi y & -ay^2+c(z^2-t^2) & 0 \end{vmatrix} = 0,$$

leading to

$$a^2y^4+2a^2y^2(z^2+t^2)+c^2(z^2-t^2)^2 = 0, \tag{10.1}$$

the anticipated form of equation. Every line $y = \rho(z-t)$ through (0, 1, 1) meets q twice there, and in two further points which satisfy

$$a^2\rho^4(z-t)^2+2a^2\rho^2(z^2+t^2)+c^2(z+t)^2 = 0. \tag{10.2}$$

The condition for one of these two further points to be at (0, 1, 1) is, clearly,

$$a^2\rho^2+c^2 = 0,$$

in which event (10.2) is $a^2\rho^2(\rho^2+1)(z-t)^2 = 0$. The prohibition of the equality of any two of $a, -a, c, -c$ prevents ρ^2 from being -1. Hence the two lines $ay = \pm ic(z-t)$ have all their four intersections with q at (0, 1, 1). This characterises the biflecnode.

The curve (10.1) is not merely its own harmonic inverse in the triangle of reference; its equation is unchanged when z and t are transposed. This transposition is the effect of the harmonic inversion whose centre is the biflecnode (0, 1, 1) and axis the join $z+t = 0$ of the other biflecnode to (1, 0, 0). This inversion in the plane is the section, by $x = 0$, of the biaxial harmonic inversion whose axes in space are $x = y$, $z = t$ and $x = -y$, $z = -t$; the former of these contains the centre of projection.

§ 11. In the standard parametrisation of Γ by doubly-periodic functions the sum of the parameters of four coplanar points is a period, congruent to zero. If u, v are parameters of points transposed by one of the harmonic homologies then $u+v$ is congruent either to zero or to one of three mutually incongruent half-periods; the fact of $2(u+v)$ being a period in all four instances accords with u and v being contacts with a bitangent plane. In particular: any point for which u is a quarter-period is a stall of Γ, a point where the osculating plane has four-point intersection.

The biaxial harmonic inversions in pairs of opposite edges of S are products of pairs of the four harmonic homologies; so, if u and v are transposed by such a biaxial inversion, i.e. if they are parameters of points in the same tetrad, $u-v$ is a half-period but not a period.

For a principal chord $3u+v \equiv 0 \equiv u+3v$. Thus $8u = 3(3u+v)-(u+3v) \equiv 0$. But since u is not the parameter of a stall—for it would then be the same as v—it is not to be a quarter-period, and there are $8^2-4^2 = 48$ mutually incongruent values available for u, each to be paired with $-3u$. So there are 24 principal chords. Since

$$u-v = \tfrac{1}{2}\{(3u+v)-(u+3v)\}$$

is a non-zero half-period each principal chord is a transversal of opposite edges of S.

If the principal chord joining u and $-3u$ meets that joining u' and $-3u'$ the sum $-2(u+u')$ of the four parameters is a period, and so $u+u'$ a half-period; and conversely. Thus the join of u and u' passes through a vertex of S, while that of $-3u$ and $-3u'$ passes through this same vertex: $u+u' \equiv -3u-3u'$ because $4(u+u')$ is a period. The joins of u to $-3u'$ and of u' to $-3u$ concur at a second vertex of S; $u-3u'$ is a half-period because $2(u-3u') \equiv 2(u+u')$, and $u'-3u$ is the same half-period. And the two half-periods $u-3u'$ and $u+u'$ are not congruent, their difference $4u'$ not being a period.

Thus the quadrangle of points on Γ has two of its diagonal points at vertices of S, the third diagonal point being the intersection of the principal chords themselves. Since S is self-polar for every quadric through Γ this intersection is, by the harmonic property of the quadrangle, on the opposite edge of S.

Addendum

A referee has suggested that the brief allusion to Enriques should be amplified—in particular, it should be made clear beyond a peradventure that the chords which he encounters are, in fact, principal.

For Enriques is mistaken in saying that there are 'precisely 8' chords of Γ incident to XY and ZT; there is an infinity of such chords; the two chords through a point on either XY or ZT both meet the other line (cf. § 4 above). Such chords form skew quadrilaterals R all having XY and ZT for diagonals and their opposite vertices harmonic whether to X and Y or to Z and T. Each R has a pair of opposite sides in each regulus on a quadric through Γ. Conversely: any quadric, other than the four cones, through Γ meets each of XY and ZT in a distinct pair of points and the four joins of points in different pairs lie on this quadric and form an R. Should, however, this quadric be a cone and so have one of X, Y, Z, T for vertex, R consists of two generators, reckoned twice. The R generate a quartic scroll having XY and ZT for nodal lines; its eight torsal generators are those belonging to the four cones.

Each R may be paired with another: namely that whose vertices on XY are harmonic not only to X and Y themselves but also to the vertices of the first R; so there is an involution J among the R whose double members are the degenerate ones on the cones with vertices X and Y. A second involution J' is defined similarly by using the vertices on ZT; the double members of J' are those on the cones with vertices Z and T. J and J' are different: they have a single common pair, and this pair of R have their vertices harmonic both on XY and ZT. They are the only pair of R with this property and so are manifestly the pair described by Enriques.

If one considers not the R but the quadrics Q through Γ on which they lie one finds Q and Q' harmonic (in the pencil through Γ) both to the two cones with vertices X, Y and to the two cones with vertices Z, T; Q and Q' are the mutually apolar quadrics of § 8.

It was seen in § 4 that the joins of $(\xi, \pm\eta, 0, 0)$ to $(0, 0, \zeta, \tau)$ are chords of Γ when, and only when

$$\zeta^2\{(c-a)\xi^2+(c-b)\eta^2\} = \tau^2\{(a-d)\xi^2+(b-d)\eta^2\}.$$

The joins of $(\xi, \pm i\eta, 0, 0)$ to $(0, 0, \zeta, \tau)$ are chords of Γ when, and only when,

$$\zeta^2\{(c-a)\xi^2-(c-b)\eta^2\} = \tau^2\{(a-d)\xi^2-(b-d)\eta^2\}.$$

The two quadratics for ζ/τ are not harmonic unless

$$(c-a)(a-d)\xi^4 = (b-d)(c-b)\eta^4.$$

This is the criterion for a principal chord obtained in § 7.

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