FINAL EXAM

MATH-GA 2210.001 ELEMENTARY NUMBER THEORY

Each problem will be marked out of 20 points.

Problem 1. Show the following:

(i) the sum
$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots$$
diverges in $\mathbb{Q}_2$,

(ii) the polynomial $x^2 + 2x - 4$ has a root in $\mathbb{Z}_{11}$.

Problem 2. Let a, b and c be integers such that $a \neq 0$. Prove that the polynomial
$$ax^2 + bx + c$$
has a root in $\mathbb{Z}_p$ for infinitely many primes p.

Problem 3. Let $\mathbb{F}$ be a field extension of $\mathbb{Q}_p$ of degree d. Do the following:

(i) show that $\mathbb{F}$ is a local field,

(ii) prove that $|\mathrm{Aut}(\mathbb{F})| \leqslant d$.

Problem 4. For every quadratic extension of $\mathbb{Q}_2$, $\mathbb{Q}_3$ and $\mathbb{Q}_5$, do the following:

(i) describe its multiplicative group,

(ii) decide whether it is ramified or not.

Problem 5. Describe all valuations of the following fields:

(i) $\mathbb{Q}(\mu_3)$, where μ_3 is a primitive third root of unity,

(ii) $\mathbb{Q}(\sqrt{2})$.

This take home exam is due on Wednesday 6 May 2015.