HOMEWORK ASSIGNMENT 4

MATH-GA 2210.001 ELEMENTARY NUMBER THEORY

The problem will be marked out of 20 points.

Exercise 1 ([1, II.4.1])**.** An infinite algebraic extension of a complete field K is never complete.

`Solution.` Let us prove this under an additional assumption that the infinite algebraic extension is separable over K. Let L be an infinite algebraic extension of K. Denote by $\bar{K}$ the algebraic closure of K. Without loss of generality, we may assume that $L \subset \bar{K}$. Recall that we have a valuation $|\quad|_K \colon K \to \mathbb{R}_{\geqslant 0}$. By [1, Theorem 4.8], this valuation can be extended in a unique way to $\bar{K}$, so that we have a valuation $|\quad|_{\bar{K}} \colon \bar{K} \to \mathbb{R}_{\geqslant 0}$. Its restriction to L is the unique extension $|\quad|_K$ to L, so we denote all these valuations simply by $|\quad|$.

We must prove that L is not complete with respect to $|\quad|$. Suppose that this is not the case. i.e. L is complete. Let us seed for a contradiction. By Ostrowksi's theorem (see [1, Theorem 4.2]), we may assume that $|\quad|$ is non-archimedian.

Choose infinitely many elements $x_0, x_1, \ldots, x_n, \ldots$ in L that are all linearly independent over K. We can find them because L is of infinite degree over K. Then there exists a nonzero sequence of elements $\{a_n\}$ of K such that the sequence

$$\Big\{|a_n x_n|\Big\}$$

is monotone decreasing to a limit of zero, because we can always find an $a_n \in K$ with a sufficiently small valuation to put $|a_i x_i|$ below any positive real number we like. Put

$$s_n = \sum_{i=0}^{n-1} a_i x_i,$$

for every $n \geqslant 1$. Then all $s_1, s_2, \ldots, s_n, \ldots$ are linearly independent over K.

Define d_n to be the smallest distance from s_n to any of its conjugates in $\bar{K}$ over K. Then we can assume that

$$\Big|a_n x_n\Big| < d_n$$

for all $n > 1$. Indeed, we can choose the sequences inductively such that a_1 and x_1 are arbitrary and $|a_n x_n| < d_n$ for all $n > 1$.

The sequence of partial sums $\{s_n\}$ is Cauchy, since the summands go to zero and $|\quad|$ is non-archimedian valuation. Thus, there exists a limit

$$\lim_{n\to\infty} s_n = s \in L,$$

because we assumed that L is complete with respect to $|\quad|$. Then

$$|s - s_n| = |\sum_{i=n}^{\infty} a_i x_i| \leqslant |a_n x_n| < d_n \tag{2}$$

for every $n > 1$. Thus, if s_n is separable over K, then $s_n \in K(s)$ by (2) and Krasner's lemma (see [1, Exercise II.6.2]). In particular, if L is separable over K, we get see that

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all $s_1, s_2, \ldots, s_n, \ldots$ are in $K(s)$, which is absurd, since s is algebraic over K and all $s_1, s_2, \ldots, s_n, \ldots$ are linearly independent over K.

Corollary 3. *The algebraic closure of $\mathbb{Q}_p$ is not complete.*

References

[1] J. Neukirch, *Algebraic Number Theory*, Springer, 1999.