Lecture 6
Henselian fields

(1) Let $K \subset L$ be fields, $| \ |_K$ = valuation $K \to \mathbb{R}_{\geq 0}$

Suppose K is complete wrt $| \ |_K = | \ |$.

Put $O = \{x \in K : |x|_K \leq 1\}$, $I = \{x \in K : |x|_K < 1\}$.

Let $f(x) \in O[x]$. So $f = a_0 + a_1 x + \dots + a_n x^n$
$a_i \in O$.

Put $|f| = \max\{|a_0|, |a_1| \dots |a_n|\}$. Put $k = O/I$.

f is called primitive if $f \not\equiv 0 \mod I$.

Th (Hensel lemma) Suppose $f(x)$ is primitive.

Suppose $f = \bar{g}\bar{h} \mod I$, $\bar{g}$ & $\bar{h} \in k[x]$ relatively prime.

Then $f = g(x)h(x)$ for some $g, h \in O[x]$ s.t.

1) $\deg(g) = \deg(\bar{g})$
2) $g \equiv \bar{g} \mod I$
3) $h \equiv \bar{h} \mod I$

(2) Corollary: If f is irreducible & $a_0 a_n \neq 0$, $|f| = \max \begin{matrix} |a_0| \\ |a_n| \end{matrix}$

Corollary: If $K \subset L$ is finite, then $| \ |_K$ can be uniquely extended to $| \ |_L : L \to \mathbb{R}_{\geq 0}$. And $|x|_L = \sqrt[n]{|N_{L/K}(x)|}$
$n = [L:K]$

(3) We need uniquness of extensions.
We also need "algebraicity" of K.
Compromise: henselian fields.

Def: A henselian field is a field with a non-archimedean valuation v (exponensial one) s.t. the valuation ring $\{x \in K : v(x) \geq 0\}$ satisfies Hensel's lemma.

(4) $\mathcal{O}_X : \left(\mathbb{C}[x,y] / \langle y^2 = x(x-1)(x-2) \rangle\right)_{\langle x,y \rangle} =: \mathcal{O}$ local ring.

Put $K = \mathrm{FRAC}(\mathcal{O})$. Then $K = \mathbb{C}(x)\sqrt{x(x-1)(x-2)} \supset \mathbb{C}(x)$ deg $= 2$.

$v = v(x,y)$

Then $\hat{K}_v \cong \mathbb{C}((y))$ Too big.

$K \subset \hat{K}_v$ $\quad \underbrace{K \subset K_v}_{\text{all algebraic elements in } \hat{K}} \subset \hat{K}$, e.g. x.

(5) Let K be a henselian field wrt $|\,|$.
Let $K \subset L$ be an algebraic extension.
Then $|\,|$ can be uniquely extended to L.
If $[L:K] = n$, then $|x|_L = \sqrt[n]{|N_{L/K}(x)|}$.

(6) In fact, we have

Th: A non-archimedean valued field $(K, |\ |_K)$ is henselian if and only if the valuation $|\ |_K$ can be uniquely extended to any algebraic extension of K.

Proof: Let us assume corollary 2. $\boxed{\mathcal{O}, I, k \text{ as usual.}}$

$(\Leftarrow)$

LEMMA: $f(x) = a_0 + \cdots + a_n x^n \in \mathcal{O}[x]$ primitive + IRR. $a_0 a_n \neq 0$.

Put $\bar{f} = f \bmod I$, $\bar{a}_i = a_i \bmod I$.

Then either $\bar{f} = \bar{a}_0$ or $\deg \bar{f} = \deg f$ & $\bar{f} = \bar{a}\bar{\varphi}^m$

$\bar{\varphi}(x) \in k[x]$ IRR $\bar{a} \in k$.

Proof: $|f|_K = \max\{|a_0|_K, |a_n|_K\}$ by C2.

We may assume $a_n = 1$.

Let L be a splitting field of $f(x)$.

Extend $|\ |_K$ to $|\ |_L$. Define $\mathcal{O}_L, I_L, k_L$ as usual.

$\forall \sigma \in Gal(L|K)$, we have $|\sigma\alpha|_L = |\alpha|_L$ $\forall \alpha \in L$.

$\Rightarrow \sigma\mathcal{O}_L = \mathcal{O}_L$, $\sigma I_L = I_L$.

If α is a root of $f(x)$, then $\alpha \in \mathcal{O}_L$ $(\prod |\sigma\alpha|^{m_\alpha} = a_0)$.

Thus σ induces automorphism $\bar{\sigma}$ of k_L s.t. $\bar{\sigma}\bar{\alpha} = \bar{\sigma}\bar{\alpha}$.

Then $\bar{\sigma}$ conjugate zeroes of $\bar{f}(x)$. $\Rightarrow \bar{f} = \bar{a}\bar{\varphi}^m$

φ = minimal poly of $\bar{\alpha}/k$.

$a_n = 1 \Rightarrow \deg \bar{f} = \deg f$. □

Now $f(x) \in \mathcal{O}[x]$ any primitive. see book.

Put $f = f_1 \overset{IRR}{\cdots} f_r$. So $\bar{f} = \bar{f}_1 \cdots \bar{f}_r$. Then $\bar{f}_i = \bar{a}_i \bar{\varphi}_i^{m_i}$ $\Rightarrow$ henselian.

(7) "Hensel's lemma" $\underset{\text{"}\Longleftarrow\text{"}}{\Longrightarrow}$ Corollary 2.

Namely: Let $(K, |\,|)$ be a non-archimedean evalued field. Suppose that $|\,|$ can be uniquely extended to any algebraic extension. Then Corollar 2 holds:

Lemma: $f = a_0 + a_1 x + \cdots + a_n x^n \in K[x]$ IRR. $a_0 \neq 0$

Then $|f| = \max\{|a_0|, |a_n|\}$.

Proof:

(8) Consider its newton polygon: $(0, v(a_0))$ --- $(n, v(a_n))$

$v =$ exp valuatn $|\,|$

Let w be extension of v to L, where L is the splittin ext of $f(x)$.

Then $w(\alpha_1) = \cdots = w(\alpha_n) = m$

$\forall$ roots $\alpha_1 \ldots \alpha_n$ of $f(x)$.

(uniquness)

May assume $a_n = 1$.

$$v(a_n) = v(1) = 0$$

$$v(a_{n-1}) = w(\Sigma \alpha_i) \geq \min w(\alpha_i) = m$$

$$v(a_{n-2}) = w(\Sigma \alpha_i \alpha_j) \geq \min w(\alpha_i \alpha_j) = 2m$$

$$\vdots$$

$$v(a_0') = nm.$$

Done.

(9) $(K, |\,|_K)$ = henselian field.

$K \subset L$ extension of degree d.

Suppose (for simplicity) that $|\,|_K$ is discrete.

Put $\underbrace{O_K, I_K, k}_{|\,|_K}, \underbrace{O_L, I_L, k_L}_{|\,|_L}$ as always,

where $|\,|_L$ is the unique extension of $|\,|_K$ to L.

Then $I_K = \langle \varpi \rangle$ and $I_L = \langle \Pi \rangle$. Then $\varpi = \Pi^e \cdot a$ for some a = unit.

Def: e = ramification index $(w(L^*) : v(K^*))$

w = exp. val. of $|\,|_L$
v = exp. val of $|\,|_K$

Def: $k_K \subset k_L$ is an extension of finite degree. This degree is called inertia degree

(10) Th: Suppose $L \supset K$ is separable. Then d = (ramification index) $\times$ (inertia degree)

Th: In general $d \geqslant$ (ramification index) $\times$ (inertia degree).

VI

(Th) e = ramification index.

(11) Proof: Put f = inertia degree.

Let $\pi_i = \Pi^i \quad i = 0, \dots, e-1$.

Let $\omega_1 \dots \omega_f$ some lift to L of basis k_L/k_K

Consider $\omega_j \pi_i$ (ef of them)

Claim: they form a basis of L/K.

Proof:
Suppose NOT TRUE. $\sum a_{ij} \omega_j \pi_i = 0$ — not all $a_{ij} = 0$. $a_{ij} \in K$.

$$\text{Put } s_i = \sum_{j=1}^{f} a_{ij} \omega_j$$

Subclaim: If $s_i \neq 0$, then $w(s_i) \in v(K^*)$

Proof: $s_1 \neq 0$. $|a_{1k}| = \max \{ |a_{1j}| \}$.

$$\frac{1}{a_{1k}} \cdot s_1 = \sum_{j=1}^{f} \frac{a_{1j}}{a_{1k}} \omega_j \qquad \frac{a_{1j}}{a_{1k}} \in O_K$$

$\Downarrow$

$\frac{1}{a_{1k}} s_1 \not\equiv 0 \mod I_L$ $\Rightarrow$ it is a unit in O_L $\Rightarrow$ $w(s_i) = w(a_{iv})$. $\square$

$$0 = \sum_{i=0}^{e-1} s_i \pi_i \Rightarrow \exists \text{ 2 summands with the same } w(\ ).$$

$(w(x) \neq w(y) \Rightarrow w(x+y) = \min w(x)/w(y))$

$\Rightarrow$ wlog, $w(s_i \pi_i) = w(s_j \pi_j) \quad i \neq j \Rightarrow w(\pi_i) \equiv w(\pi_j) \mod v(K^*)$

$\Rightarrow$ linearly independent

(12) **Basis**

$$M = \sum_{i=0}^{e-1}\sum_{j=1}^{f} \mathcal{O}_K\, \omega_j \Pi^i \subseteq \mathcal{O}_L$$

$\mathcal{O}_K$-submodule

Let us show $M = \mathcal{O}_L$. Recall $\mathcal{O}_L$ = finitely generated $\mathcal{O}_K$-module.

integral closure of $\mathcal{O}_K$ in L

Put $N = \sum_{j=1}^{f} \mathcal{O}_K\, \omega_j$

$$\Rightarrow M = N + \Pi N + \cdots + \Pi^{e-1} N$$

Then $\mathcal{O}_L = N + \Pi \mathcal{O}_L$ $\quad (\forall \alpha \in \mathcal{O}_L,\ \alpha \equiv a_1\omega_1 + \cdots + a_f\omega_f \mod \Pi)$

$$\Rightarrow \mathcal{O}_L = \underbrace{N + \Pi N + \cdots + \Pi^{e-1} N}_{M} + \Pi^e \mathcal{O}_L$$

$\mathcal{O}_L = M + I\mathcal{O}_L \Rightarrow \mathcal{O}_L = M$ by Nakayama's Lemma. $\square$

(13) Nakayama's Lemma:

Lemma. A = local ring, m = max ideal.

$M \subseteq N$ f.g. A-modules. f.g.

Suppose $N = M + m \cdot N \Rightarrow M = N$

Corollary: $\mathcal{O}_L = M + I\mathcal{O}_L \Rightarrow M = \mathcal{O}_L$