MIDTERM EXAM

MATH-GA 2210.001 ELEMENTARY NUMBER THEORY

Each problem will be marked out of 20 points.

Problem 1. Find the p-adic expansions for

(1) $\frac{2}{3}$ in $\mathbb{Q}_2$,
(2) $-\frac{1}{6}$ in $\mathbb{Q}_7$,
(3) $\frac{1}{10}$ in $\mathbb{Q}_{11}$,
(4) $\frac{1}{120}$ in $\mathbb{Q}_5$.

Problem 2. Prove that the the field $\mathbb{Q}_p$ does not have automorphisms except the identity.

Problem 3. Prove that

(1) the polynomial $x^2 - 5$ is irreducible in $\mathbb{Q}_7[x]$,
(2) the polynomial $x^p - x - 1$ is irreducible in $\mathbb{Q}_p[x]$ for any prime $p \geqslant 2$,
(3) the polynomial $x^4 + 4x^3 + 2x^2 + x - 6$ is reducible in $\mathbb{Q}_{11}[x]$,
(4) the polynomial $x^4 - x^3 - 2x^2 - 3x - 1$ is reducible in $\mathbb{Q}_5[x]$.

Problem 4. Show that for every $d \in \mathbb{N}$, there is a field K containing $\mathbb{Q}_p$ such that

$$[K : \mathbb{Q}_p] := \dim_{\mathbb{Q}_p}(K) = d.$$

Problem 5. Let K be the field of fractions of the ring

$$\mathbb{Q}_7[x]/(x^2 - 5),$$

where $(x^2 - 5)$ is the ideal in $\mathbb{Q}_7[x]$ generated by $x^2 - 5$. Identify $\mathbb{Q}_7$ with a subfield of K via natural homomorphisms

$$\mathbb{Q}_7 \hookrightarrow \mathbb{Q}_7[x] \to \mathbb{Q}_7[x]/(x^2 - 5) \hookrightarrow K$$

Prove that the p-adic valuation $|\quad|_7$ on $\mathbb{Q}_7$ can be extended to a valuation

$$|\quad| \colon K \to \mathbb{R}_{\geqslant 0}$$

of the field K in a unique way. Describe its valuation ring

$$\mathcal{O} := \Big\{x \in K \text{ such that } |x| \leqslant 1\Big\},$$

its unique maximal ideal $\mathfrak{I}$, its residue class field $\mathcal{O}/\mathfrak{I}$, and its multiplicative group K^*.

This take home exam is due on Wednesday 4 March 2015.