A Remark on the Nonrationality Problem
for Generic Cubic Fourfolds

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Abstract—It is proved that the nonrationality of a generic cubic fourfold follows from a conjecture on the nondecomposability in the direct sum of nontrivial polarized Hodge structures of the polarized Hodge structure on transcendental cycles on a projective surface.

Before formulating the main statement of the present note, recall some definitions and statements related to Hodge structures.

Let $\mathscr{H} = \{M, H^{p,n-p}, Q\}$ be a *polarized Hodge structure of weight* n, i.e., a free $\mathbb{Z}$-module M, a nondegenerate bilinear form $Q\colon M \times M \to \mathbb{Z}$, the vector space

$$M \otimes \mathbb{C} = \bigoplus_{p=a}^{b} H^{p,n-p}$$

presented as the direct sum of its complex subspaces $H^{p,n-p}$ for some $a, b \in \mathbb{Z}$, $a \leq b$, such that $H^{n-p,p} = \overline{H^{p,n-p}}$ and

$$Q(u,v) = (-1)^n Q(v,u),$$
$$Q(u,v) = 0 \qquad \text{for} \quad u \in H^{p,n-p}, \, v \in H^{p',n-p'}, \, p \neq n - p'.$$

A polarized Hodge structure $\mathscr{H} = \{M, H^{p,q}, Q\}$ is said to be *unimodular* if Q is an unimodular bilinear form, i.e., if (e_i) is a free basis of M, then the determinant of the matrix $A = (Q(e_i, e_j))$ is equal to ± 1.

Let

$$\mathscr{H}_1 = \{M_1, H_1^{p,q}, Q_1\} \qquad \text{and} \qquad \mathscr{H}_2 = \{M_2, H_2^{p,q}, Q_2\}$$

be two polarized Hodge structures of weight n. A $\mathbb{Z}$-homomorphism $f\colon M_1 \to M_2$ of $\mathbb{Z}$-modules is a *morphism of polarized Hodge structures* if

$$Q_2(f(u), f(v)) = Q_1(u,v) \qquad \text{for all } u, v \in M_1$$

and f induces a morphism of Hodge structures, i.e.,

$$f_{\mathbb{C}}(H_1^{p,q}) \subset H_2^{p,q} \quad \text{for all } p, q, \qquad \text{where} \quad f_{\mathbb{C}} = f \otimes \mathrm{Id}.$$

Note that if $f\colon M_1 \to M_2$ is a morphism of polarized Hodge structures, then f is an embedding, since Q_1 and Q_2 are nondegenerate bilinear forms. We say that $\mathscr{H}_1$ is a *polarized Hodge substructure* of $\mathscr{H}_2$ (and write $\mathscr{H}_1 \subset \mathscr{H}_2$) if the embedding $M_1 \subset M_2$ is a morphism of polarized Hodge structures. A polarized Hodge structure

$$\mathscr{H} = \{M_1 \oplus M_2, H^{p,q}, Q_1 \oplus Q_2\}$$

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is called the *direct sum of the polarized Hodge structures* $\mathscr{H}_1$ and $\mathscr{H}_2$ if the canonical embeddings $M_i \subset M_1 \oplus M_2$ are morphisms of polarized Hodge structures for $i = 1, 2$. In the case of a direct sum, we have

$$H^{p,q} = H_1^{p,q} \oplus H_2^{p,q} \qquad \text{for all } p, q.$$

We say that a polarized Hodge structure $\mathscr{H}$ is *nondecomposable* if it is not isomorphic to the direct sum of two nontrivial polarized Hodge structures.

Let X be a smooth projective manifold defined over the field $\mathbb{C}$, $\dim_{\mathbb{C}} X = n$. It is well known that one can associate to X a polarized Hodge structure $\mathscr{H}_X = \{M_X, H^{p,q}, Q\}$ of weight n, where

$$M_X = H^n(X, \mathbb{Z})/\{\text{torsion}\},$$

$H^{p,q} = H_X^{p,q} \subset H^n(X, \mathbb{C})$ are the spaces of harmonic (p,q)-forms on X and $Q = Q_X$ is the restriction to

$$M_X \subset M_X \otimes \mathbb{C} \simeq H^n(X, \mathbb{C})$$

of the bilinear form

$$Q(\phi, \psi) = \int_X \phi \wedge \psi.$$

It is well known that the lattice (M_X, Q_X) is unimodular. Denote by

$$h^{p,q} = h^{p,q}(X) = \dim H_X^{p,q} = \dim H^q(X, \Omega_X^p)$$

the Hodge numbers of X, where Ω_X^p is the sheaf of holomorphic p-forms on X.

Lemma 1. *Let $f \colon X \to V$ be a birational morphism of smooth projective manifolds of dimension $\dim_{\mathbb{C}} X = \dim_{\mathbb{C}} V = n$. Then the polarized Hodge structure $\mathscr{H}_X$ can be decomposed into the direct sum of the polarized Hodge structure $f^*(\mathscr{H}_V) \simeq \mathscr{H}_V$ and a polarized Hodge structure $\mathscr{H}_V^\perp$.*

Proof. This statement is well known in the particular case where $f = \sigma \colon X \to V$ is the monoidal transformation with nonsingular center $C \subset V$ ([1]–[3]; also see [4]). By induction, the validity of the lemma follows for any composition

$$f = \sigma_m \circ \cdots \circ \sigma_1 \colon X \to V$$

of monoidal transformations $\sigma_i \colon X_i \to X_{i-1}$ with nonsingular centers $\widetilde{C}_{i-1} \subset X_{i-1}$, where $X_0 = V$ and $X_m = Y$.

Now, let f be an arbitrary birational morphism. Since $\deg f = 1$, we have

$$\int_V \phi \wedge \psi = \int_X f^*(\phi) \wedge f^*(\psi),$$

i.e., f^* is an embedding of the lattice $M_V = H^n(V, \mathbb{Z})$ in the lattice $M_X = H^n(V, \mathbb{Z})$. Therefore, $f^* \colon \mathscr{H}_V \to \mathscr{H}_X$ is an embedding of the polarized Hodge structure $\mathscr{H}_V$. Since the lattices $f^*(M_V)$ and M_X are unimodular, there is a sublattice $M_V^\perp$ of M_X such that

$$M_X = f^*(M_V) \oplus M_V^\perp$$

is the direct sum of lattices.

Let us show that this decomposition in the direct sum of lattices induces the decomposition

$$\mathscr{H}_X = f^*(\mathscr{H}_V) \oplus \mathscr{H}_V^\perp$$

in the direct sum of polarized Hodge structures. For this, since the image of Hodge structure under a morphism of Hodge structures is a Hodge structure, it suffices to show that the natural projection

$$\mathrm{pr}^\perp \colon M_X \otimes \mathbb{C} \to M_V^\perp \otimes \mathbb{C}$$

is a morphism of Hodge structures. To show that $\mathrm{pr}^\perp$ is a morphism of Hodge structures, let us consider the birational map $f^{-1} \colon V \dashrightarrow X$. By the Hironaka theorem, there is a composition

$$\sigma = \sigma_m \circ \cdots \circ \sigma_1 \colon Y \to V$$

of monoidal transformations $\sigma_i\colon Y_i \to Y_{i-1}$ with nonsingular centers $\widetilde{C}_{i-1} \subset Y_{i-1}$, where $Y_0 = V$ and $Y_m = Y$, such that $g = f^{-1} \circ \sigma\colon Y \to X$ is a birational morphism. We obtain the decomposition
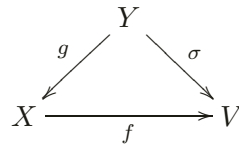
$$\mathscr{H}_Y = \sigma^*(\mathscr{H}_V) \oplus \bigoplus_{i=0}^{m-1} \widetilde{\mathscr{H}_i}$$

in a direct sum of polarized Hodge structures, where $\widetilde{\mathscr{H}_i}$ is the contribution in $\mathscr{H}_Y$ of the monoidal transformation σ_{i+1}. Note that the natural projection

$$\widetilde{\mathrm{pr}}_2\colon \mathscr{H}_Y \to \bigoplus_{i=0}^{m-1} \widetilde{\mathscr{H}_i}$$

is a morphism of polarized Hodge structures.

It follows from the commutative diagram

$$
\begin{array}{ccc}
 & Y & \\
{\scriptstyle g}\swarrow & & \searrow{\scriptstyle \sigma} \\
X & \xrightarrow{\ f\ } & V
\end{array}
$$

that $g^*\colon \mathscr{H}_X \to \mathscr{H}_Y$ is an embedding of polarized Hodge structures such that

$$g^*(f^*(\mathscr{H}_V)) = \sigma^*(\mathscr{H}_V) \qquad \text{and} \qquad g^*(M_V^\perp) \subset (\sigma^*(M_V))^\perp.$$

Therefore, we can identify $\mathscr{H}_X$ with its image $g^*(\mathscr{H}_X)$, $f^*(\mathscr{H}_V)$ with $\sigma^*(\mathscr{H}_V)$ and $M_V^\perp$ with $g^*(M_V^\perp)$. Under these identifications, the projection $\mathrm{pr}^\perp$ is identified with the restriction of $\widetilde{\mathrm{pr}}_2$ to $g^*(\mathscr{H}_X)$. Therefore, $\mathrm{pr}^\perp$ is a morphism of Hodge structures, being the composition of two morphisms of Hodge structures: namely, of the embedding g^* and the projection $\widetilde{\mathrm{pr}}_2$. $\qquad\square$

Let $\dim_{\mathbb{C}} X = 2k$. Believing in the Hodge Conjecture, the elements of $A_X = M_X \cap H^{k,k}$ will be called *algebraic*, and the module

$$T_X = \{\gamma \in M_X \mid Q(\gamma, \alpha) = 0 \text{ for all } \alpha \in A_X\}$$

will be called the module of *transcendental n-cycles on X*, $n = 2k$. It is easy to see that a polarized Hodge structure $\mathscr{H}_X$ induces the polarized Hodge structure

$$\mathscr{T}_X = \{T_X, H_T^{p,q}, Q_T\}, \qquad \text{where} \quad H_T^{p,q} = (T \otimes \mathbb{C}) \cap H_X^{p,q}$$

and Q_T is the restriction of Q to T_X.

If S is a smooth projective surface, then the form $Q = Q_S$ is symmetric unimodular and, by the index theorem, its signature is equal to $(2h^{2,0} + 1, h^{1,1} - 1)$. Note that the polarized Hodge structure $\mathscr{T}_S$ on the transcendental cycles on a smooth projective surface S is a birational invariant of the surface S.

Nondecomposability Conjecture. *The polarized Hodge structure $\mathscr{T}_S$ on the transcendental cycles on a smooth projective surface S is nondecomposable.*

Let $V \subset \mathbb{P}^5$ be a smooth cubic fourfold. It is known (see [5], [6]) that the moduli space of cubic fourfolds contains several families of rational cubics. Nevertheless, the following conjecture is well known.

Nonrationality Conjecture. *A generic cubic fourfold is nonrational.*

The aim of the present note is to show that the Nonrationality Conjecture follows from the Nondecomposability Conjecture. To formulate the precise statement, we fix one of the smooth cubic fourfolds V_0. For each cubic fourfold V, we can identify the lattice (M_V, Q_V) with the lattice (M_0, Q), where $M_0 = H^4(V_0, \mathbb{Z})$ and $Q = Q_{V_0}$. Let $\lambda = L^2 \in M_0$, where $L \in H^2(V_0, \mathbb{Z})$ is the class of the

hyperplane section of $V_0 \subset \mathbb{P}^5$. It is well known (see, e.g., [7]) that each smooth cubic fourfold V has the following Hodge numbers:

$$h^{4,0} = h^{0,4} = 0, \qquad h^{3,1} = h^{1,3} = 1, \qquad \text{and} \qquad h^{2,2} = 21.$$

Consider the polarized Hodge structures $\mathscr{H}_V = \{M_V, H^{p,q}, Q_V\}$ on the fourth cohomology groups of the smooth cubic fourfolds V. Since $h^{4,0} = 0$ and $h^{3,1} = 1$, the polarized Hodge structure

$$\mathscr{H}_V = \{M_0, H_V^{p,q}, Q\}$$

is defined by a nonzero element $\omega \in H_V^{3,1} \subset M_0 \otimes \mathbb{C}$. It follows from the Hodge–Riemann bilinear relations that

$$Q(\lambda, \omega) = Q(\omega, \omega) = 0 \qquad \text{and} \qquad Q(\omega, \overline{\omega}) < 0.$$

Therefore, the classifying space D of polarized Hodge structures of smooth cubic fourfolds coincides with

$$D = \{\omega \in \mathbb{P}^{22} \mid Q(\omega, \lambda) = Q(\omega, \omega) = 0, \ Q(\omega, \overline{\omega}) < 0\}.$$

The point $\omega(V) \in D$ corresponding to the polarized Hodge structure $\mathscr{H}_V = \{M_0, H^{p,q}(V), Q\}$ of a smooth cubic fourfold V is called the *periods* of V. By [8] and by the global Torelli theorem proved in [9], the set D_0 of periods of the smooth cubic fourfolds is an open subset of D.

Let

$$P = \{\mu \in M_0 \mid Q(\lambda, \mu) = 0\}$$

be the submodule of M_0 consisting of the primitive elements. For each endomorphism $A \in \text{End}(M_0)$ such that $A(P) \subset P$ and the restriction $A_{|P}$ of A to P is not proportional to the identity automorphism of P, denote

$$E_A = \{\omega \in D \subset \mathbb{P}^{22} \mid \omega \text{ is an eigenvector of } A\}.$$

Obviously, E_A is the intersection of D and a finite number of linear subspaces of $\mathbb{P}^{22}$ of codimension at least 2. Therefore, E_A is a proper closed analytic subvariety of D. Put $E = \bigcup E_A$. Then E is the union of countably many proper closed analytic subvarieties of D. Therefore, $D \setminus E$ is everywhere dense in D.

For each $\mu \in M_0$, μ is not proportional to λ, we put

$$B_\mu = \{\omega \in D \subset \mathbb{P}^{22} \mid Q(\mu, \omega) = 0\} \qquad \text{and} \qquad B = \bigcup B_\mu.$$

As above, B is the union of countably many proper closed analytic subvarieties of D. Therefore, $D_0 \setminus (E \cup B)$ is also everywhere dense in D_0.

Proposition 1. *If the Nondecomposability Conjecture is true, then a smooth cubic fourfold V is nonrational if its periods $\omega(V)$ are contained in $D_0 \setminus (E \cup B)$.*

Proof. Assume that the smooth cubic fourfold V is rational and its periods $\omega(V)$ are contained in $D_0 \setminus (E \cup B)$.

Note that the sublattice of transcendental elements T_V of the lattice M_V coincides with the sublattice of primitive elements P if the periods $\omega(V)$ belong to $D_0 \setminus B$.

Since V is rational, then there is a birational map $r \colon \mathbb{P}^4 \dashrightarrow V$. By the Hironaka theorem, there is a composition

$$\tau = \tau_n \circ \cdots \circ \tau_1 \colon X \to \mathbb{P}^4$$

of monoidal transformations $\tau_i \colon X_i \to X_{i-1}$ with nonsingular centers $C_{i-1} \subset X_{i-1}$, where $X_0 = \mathbb{P}^4$ and $X_n = X$, such that

$$f = r \circ \tau \colon X \to V$$

is a birational morphism.

Each τ_i induces an inclusion of unimodular polarized Hodge structures $\tau_i^* \colon \mathscr{H}_{X_{i-1}} \hookrightarrow \mathscr{H}_{X_i}$ such that

$$\mathscr{H}_{X_i} \simeq \tau_i^*(\mathscr{H}_{X_{i-1}}) \oplus (\tau_i^*(\mathscr{H}_{X_{i-1}}))^\perp.$$

In particular,

$$H^4(X_i, \mathbb{Z}) = \tau_i^*(H^4(X_{i-1}, \mathbb{Z})) \oplus (\tau_i^*(H^4(X_{i-1}, \mathbb{Z})))^\perp.$$

Moreover, if $\dim C_{i-1} \leq 1$, then

$$(\tau_i^*(H^4(X_{i-1}, \mathbb{Z})))^\perp \subset H^{2,2}_{X_i},$$

and if C_{i-1} is a smooth surface, then

$$(\tau_i^*(\mathscr{H}_{X_{i-1}}))^\perp \simeq -\mathscr{H}_{C_{i-1}}(-1),$$

where

$$-\mathscr{H}_{C_{i-1}}(-1) = (M_{C_{i-1}}, H^{p,q}_{i-1}, -Q_{C_{i-1}}) \qquad \text{and} \qquad H^{p,q}_{i-1} = H^{p-1,q-1}_{C_{i-1}}.$$

Therefore,

$$H^{3,1}_{X_i} = \tau_i^*(H^{3,1}_{X_{i-1}}) \oplus H^{3,1}_{i-1}, \qquad H^{4,0}_{X_i} = 0$$

and

$$Q_{X_i}(u, v) = 0 \qquad \text{for} \quad u \in \tau_i^*(H^{3,1}_{X_{i-1}} \oplus H^{1,3}_{X_{i-1}}), \quad v \in H^{3,1}_{i-1} \oplus H^{1,3}_{i-1}.$$

As a consequence, we obtain a decomposition of the unimodular polarized Hodge structure

$$\mathscr{H}_X = \tau^*(\mathscr{H}_{\mathbb{P}^4}) \oplus \bigoplus_{i=0}^{n-1} \mathscr{H}_i$$

into a direct sum of polarized Hodge structures, where $\mathscr{H}_i$ is the contribution in $\mathscr{H}_X$ of the $(i+1)$th monoidal transformation τ_{i+1}. By induction, we have $\mathscr{H}_i \simeq (\tau_i^*(\mathscr{H}_{X_{i-1}}))^\perp$ and, consequently,

$$\mathscr{T}_X \simeq - \oplus \mathscr{T}_{C_i}(-1),$$

where the sum is taken over all surfaces C_i with $p_g \geq 1$, since

$$H^4(\mathbb{P}^4, \mathbb{C}) = H^{2,2}_{\mathbb{P}^4}, \qquad h^{2,2}(\mathbb{P}^4) = 1, \qquad H^2(C_i, \mathbb{C}) = H^{1,1}_{C_i}$$

if the geometric genus p_g of the surface C_i is equal to zero.

The morphism f induces a morphism of Hodge structures $f^* \colon \mathscr{H}_V \to \mathscr{H}_X$. By Lemma 1, the Hodge structure $\mathscr{H}_X$ is decomposed into the direct sum $f^*(\mathscr{H}_V) \oplus \mathscr{H}_V^\perp$ of polarized Hodge structures. $\qquad\square$

Lemma 2. *Let V be a smooth cubic fourfold from Proposition 1, and let*

$$\tau = \tau_n \circ \cdots \circ \tau_1 \colon X \to \mathbb{P}^4 \qquad \text{and} \qquad f = r \circ \tau \colon X \to V$$

be the morphisms described above. Then there is an i_0 such that the polarized Hodge structure $\mathscr{T}_{C_{i_0}}$ on the transcendental cycles of the surface C_{i_0} can be decomposed into the direct sum of polarized Hodge structures $\mathscr{T}'_{C_{i_0}} \simeq -f^*(\mathscr{T}_V)(1)$ and $\mathscr{T}''_{C_{i_0}}$.

Proof. We have two decompositions

$$\mathscr{H}_X = f^*(\mathscr{H}_V) \oplus \mathscr{H}_V^\perp = \tau^*(\mathscr{H}_{\mathbb{P}^4}) \oplus \bigoplus_{i=0}^{n-1} \mathscr{H}_i$$

of the polarized Hodge structure $\mathscr{H}_X$. Consider a nonzero element $\omega \in f^*(H^{3,1}_V)$. It can be represented in the form

$$\omega = \sum_{i=0}^{n-1} \omega_i, \qquad \text{where} \quad \omega_i \in H^{3,1}_i.$$

Let i_0 be an index such that $\omega_{i_0} \neq 0$. Put $\omega_{i_0}^{\perp} = \sum_{i \neq i_0} \omega_i$, and let

$$\mathrm{pr}_{i_0} \colon \mathscr{H}_X \to \mathscr{H}_{i_0}, \quad \mathrm{pr}_{i_0}^{\perp} \colon \mathscr{H}_X \to \oplus_{i \neq i_0} \mathscr{H}_i, \quad \mathrm{pr} \colon \mathscr{H}_X \to f^*(\mathscr{H}_V), \quad \mathrm{pr}^{\perp} \colon \mathscr{H}_X \to \mathscr{H}_V^{\perp}$$

be the natural projections. Note that all these projections are defined over $\mathbb{Z}$. We have $\mathrm{pr}(\mathrm{pr}_{i_0}(\omega)) = a\omega$ for some $a \in \mathbb{C}$, since $\dim f^*(H_V^{3,1}) = 1$ and pr, pr_{i_0} are morphisms of Hodge structures.

Let us show that $a = 1$. Indeed, the restriction of $\mathrm{pr} \circ \mathrm{pr}_{i_0}$ to $f^*(M_V) \simeq M_V$ induces an endomorphism of M_V such that

$$\mathrm{pr} \circ \mathrm{pr}_{i_0}(f^*(P)) \subset f^*(P),$$

since $\mathrm{pr} \circ \mathrm{pr}_{i_0}$ is a morphism of Hodge structures, $M_V \cap H_V^{2,2} = \mathbb{Z}\lambda$ and $P = T_V$ by assumption. Therefore, $(\mathrm{pr} \circ \mathrm{pr}_{i_0})_{|f^*(P)} = a \cdot \mathrm{Id}$, since $\mathrm{pr}(\mathrm{pr}_{i_0}(\omega)) = a\omega$ (i.e., ω is an eigenvector of $(\mathrm{pr} \circ \mathrm{pr}_{i_0})_{|f^*(M_V)}$ and a is its eigenvalue, and, by assumption, $\omega(V) \in D_0 \setminus (E \cup B)$). Therefore, $a \in \mathbb{Z}$, since $(\mathrm{pr} \circ \mathrm{pr}_{i_0})_{|f^*(P)}$ is an endomorphism of $\mathbb{Z}$-module $f^*(P) \simeq P$. Let

$$\omega^{\perp} = \mathrm{pr}^{\perp}(\omega_{i_0}) \in H_X^{3,1}.$$

Then we have

$$\omega_{i_0} = a\omega + \omega^{\perp},$$
$$\omega_{i_0}^{\perp} = (1-a)\omega - \omega^{\perp}.$$

By Hodge–Riemann bilinear relations, $Q_X(\gamma, \overline{\gamma}) \leq 0$ for $\gamma \in H_X^{3,1}$ and $Q_X(\gamma, \overline{\gamma}) = 0$ if and only if $\gamma = 0$. Therefore, without loss of generality, we can assume that

$$Q_X(\omega, \overline{\omega}) = -1 \qquad \text{and} \qquad Q_X(\omega^{\perp}, \overline{\omega}^{\perp}) = b \leq 0.$$

We have $Q_X(\omega_{i_0}, \overline{\omega}_{i_0}) = -a^2 + b$ and, consequently, a and b cannot simultaneously be equal to zero, since $\omega_{i_0} \neq 0$. Besides, we have

$$Q_X(\omega, \omega^{\perp}) = Q_X(\omega, \overline{\omega}^{\perp}) = Q_X(\overline{\omega}, \omega^{\perp}) = Q_X(\overline{\omega}, \overline{\omega}^{\perp}) = Q_X(\omega_{i_0}, \overline{\omega}_{i_0}^{\perp}) = 0.$$

Therefore,

$$Q_X(\omega_{i_0}, \overline{\omega}_{i_0}^{\perp}) = Q_X(a\omega + \omega^{\perp}, (1-a)\overline{\omega} - \overline{\omega}^{\perp}) = a(1-a)(-1) - b = 0,$$

i.e., $a^2 - a - b = 0$, and hence

$$a = \frac{1 \pm \sqrt{1 + 4b}}{2}.$$

But, $a \in \mathbb{Z}$ and $b \leq 0$; therefore, $b = 0$, $a = 1$ and hence $\omega^{\perp} = 0$, since $Q_X(\omega^{\perp}, \overline{\omega}^{\perp}) = b = 0$. Therefore, $\omega \in H_{i_0}^{3,1}$. Besides, we showed that $(\mathrm{pr} \circ \mathrm{pr}_{i_0})_{|f^*(P)} = \mathrm{Id}$.

Let us show that $f^*(T_V) \subset T_{i_0}$. As above, we have two decompositions

$$\mathscr{T}_X = f^*(\mathscr{T}_V) \oplus f^*(\mathscr{T}_V)^{\perp} = \bigoplus_{i=0}^{n-1} \mathscr{T}_i$$

of the polarized Hodge structure $\mathscr{T}_X$. Each $\gamma \in f^*(T_V)$ can be written in the form

$$\gamma = \gamma_{i_0} + \gamma_{i_0}^{\perp}, \qquad \text{where} \quad \gamma_{i_0} = \mathrm{pr}_{i_0}(\gamma) \in T_{i_0}, \quad \gamma_{i_0}^{\perp} = \mathrm{pr}_{i_0}^{\perp}(\gamma) \in \oplus_{i \neq i_0} T_i.$$

Then, since $(\mathrm{pr} \circ \mathrm{pr}_{i_0})_{|f^*(P)} = \mathrm{Id}$ and $P = T_V$, we have

$$\gamma_{i_0} = \gamma + \gamma^{\perp},$$
$$\gamma_{i_0}^{\perp} = -\gamma^{\perp},$$

where

$$\gamma^{\perp} = \mathrm{pr}^{\perp}(\gamma_{i_0}) \in f^*(T_V)^{\perp}.$$

Denote by $p_{(3,1)}\colon M_X \otimes C \to H_X^{3,1}$ the natural projection. Then, by definition of transcendental cycles, $p_{(3,1)}(\gamma) \neq 0$ for each nonzero element $\gamma \in T_X$, since the Hodge number $h^{4,0}(X)$ is zero. In particular, $p_{(3,1)}(\gamma) = a_\gamma \omega$ for some $a_\gamma \in \mathbb{C}$, $a_\gamma \neq 0$, since $f^*(H_V^{3,1}) = \mathbb{C}\omega$. Therefore, $p_{(3,1)}(\gamma_{i_0}) = a_\gamma \omega$, since $\omega \in f^*(H_V^{3,1}) \subset H_{i_0}^{3,1}$, and hence $p_{(3,1)}(\gamma_{i_0}^\perp) = 0$. From this, we have $\gamma_{i_0}^\perp = 0$, i.e., $f^*(T_V) \subset T_{i_0}$.

As a consequence, we see that

$$\mathscr{T}_{i_0} = f^*(\mathscr{T}_V) \oplus \mathscr{T}_{i_0}'' \tag{1}$$

is a direct sum of polarized Hodge structures, where $\mathscr{T}_{i_0}''$ is a polarized Hodge substructure of $f^*(\mathscr{T}_V)^\perp$. To complete the proof of Lemma 2, recall that $\mathscr{T}_{i_0} \simeq -\mathscr{T}_{C_{i_0}}(-1)$, where $\mathscr{T}_{C_{i_0}}$ is the polarized Hodge structure on the transcendental 2-cycles on the smooth projective surface C_{i_0}. $\square$

Lemma 3. *There does not exist any smooth projective surface S such that $\mathscr{T}_V \simeq -\mathscr{T}_S(-1)$, where V is a smooth cubic fourfold whose periods $\omega(V)$ belong to $D_0 \setminus B$.*

Proof. If $\mathscr{T}_V \simeq -\mathscr{T}_S(-1)$ for some surface S, then its geometric genus is

$$p_g = h^{2,0}(S) = h^{3,1}(V) = 1$$

and $\operatorname{rk} T_S = \operatorname{rk} T_V = 22$. Since $p_g = 1$, the surface S is not a ruled surface. The lattice T_S is a birational invariant. Therefore, we can assume that S is a minimal model and its second Betti number satisfies

$$b_2(S) = h^{1,1}(S) + 2h^{2,0}(S) \geq 23,$$

because $\operatorname{rk} T_S = \operatorname{rk} T_V = 22$ and S should have also algebraic 2-cycles. Therefore, $h^{1,1}(S)$ should be greater than 20, since $h^{2,0}(S) = 1$.

On the other hand, it follows from the classification of algebraic surfaces that $K_S^2 \geq 0$, where K_S is the canonical class of S. Denote by

$$\chi(S) = 1 - h^{1,0}(S) + h^{2,0}(S)$$

the algebraic Euler characteristic of the surface S and by

$$e(S) = 2 + h_S^{1,1} + 2h^{2,0}(S) - 4h^{1,0}(S)$$

its topological Euler characteristic. Note that

$$h^{1,0}(S) = \dim H^0(S, \Omega_S^1) \geq 0.$$

By Noether's formula, we have

$$\chi(S) = \frac{1}{12}(K_S^2 + e(S)),$$

and hence

$$h^{1,1}(S) = 10 + 10h^{2,0}(S) - 8h^{1,0}(S) - K_S^2 = 20 - 8h^{1,0}(S) - K_S^2,$$

because $h^{2,0}(S) = 1$. Therefore, $h^{1,1}(S) \leq 20$. $\square$

It follows from Lemma 3 that, in the decomposition (1), the summand $\mathscr{T}_{i_0}''$ is nontrivial, which contradicts the Nondecomposability Conjecture.

ACKNOWLEDGMENTS

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