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Factorization semigroups and irreducible components of the Hurwitz space

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Abstract. We introduce a natural structure of a semigroup (isomorphic to the factorization semigroup of the identity in the symmetric group) on the set of irreducible components of the Hurwitz space of coverings of marked degree d of $\mathbb{P}^1$ of fixed ramification types. We shall prove that this semigroup is finitely presented. We study the problem of when collections of ramification types uniquely determine the corresponding irreducible components of the Hurwitz space. In particular, we give a complete description of the set of irreducible components of the Hurwitz space of three-sheeted coverings of the projective line.

Keywords: semigroup, factorization of an element of a group, irreducible components of the Hurwitz space.

Introduction

The Hurwitz space $\mathrm{HUR}_d(\mathbb{P}^1)$ of coverings of degree d of the projective line $\mathbb{P}^1 := \mathbb{C}\mathbb{P}^1$ is usually investigated in the following way. One fixes the Galois group G of the coverings, the number b of branch points and the types of local monodromies (that is, b-tuples of conjugacy classes of G) and studies the set of sets of representatives of these conjugacy classes up to the so-called Hurwitz moves (see [1]–[9], for example). Similar objects (finite collections of elements of a group considered up to Hurwitz moves) arise naturally in other problems: describing the set of plane algebraic curves up to equisingular deformation or, more generally, describing the set of plane pseudo-holomorphic curves up to symplectic isotopy, describing the set of symplectic Lefschetz pencils up to diffeomorphisms, and so on (see [10]–[12], for example). (To obtain such elements in the case of plane algebraic and pseudo-holomorphic curves, one should choose a pencil of (pseudo-)lines giving a fibration over $\mathbb{P}^1$.) As was shown in [13], there is a natural semigroup structure on the sets of such collections considered up to Hurwitz moves, namely, the so-called factorization semigroups over groups. Moreover, if we consider such fibrations over the discs $D_R = \{z \in \mathbb{C} \mid |z| \leqslant R\}$ instead of the whole of $\mathbb{P}^1$, then this semigroup structure has a natural geometric meaning (see [13]).

In § 1 we give basic definitions and investigate properties of factorization semigroups over finite groups. In particular, we prove that the factorization semigroups

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of the identity are finitely presented. We also study the problem of when elements of factorization semigroups are uniquely determined by their type and product.

Factorization semigroups over symmetric groups $\mathcal{S}_d$ are treated in more detail in § 2. We shall prove a stabilization theorem and give a complete description of the factorization semigroup of the identity in $\mathcal{S}_3$.

In § 3 we introduce the natural structure of a semigroup (the factorization semigroup of the identity in a symmetric group) on the set of irreducible components of the Hurwitz space of coverings of marked degree d of $\mathbb{P}^1$ with fixed ramification types and show that this structure induces a semigroup structure on the set of irreducible components of the Hurwitz space HUR_d^G of Galois coverings of $\mathbb{P}^1$ with Galois group G having no outer automorphisms. The results obtained in §§ 1, 2 are applied to the problem of deciding when the irreducible components of $\mathrm{HUR}_d(\mathbb{P}^1)$ are uniquely determined by the sets of types of local monodromies of the coverings.

§ 1. Semigroups over groups

1.1. Factorization semigroups. A quadruple (S, G, α, λ), where S is a semigroup, G is a group and $\alpha\colon S \to G$, $\lambda\colon G \to \mathrm{Aut}(S)$ are homomorphisms, is called a *semigroup S over a group G* if the following equalities hold for all $s_1, s_2 \in S$:

$$s_1 \cdot s_2 = \rho(\alpha(s_1))(s_2) \cdot s_1 = s_2 \cdot \lambda(\alpha(s_2))(s_1),$$

where $\rho(g) = \lambda(g^{-1})$.

Let $(S_1, G_1, \alpha_1, \lambda_1)$ and $(S_2, G_2, \alpha_2, \lambda_2)$ be semigroups over G_1 and G_2. A pair (h_1, h_2) of homomorphisms $h_1\colon S_1 \to S_2$ and $h_2\colon G_1 \to G_2$ is called a *homomorphism of semigroups over groups* if

(i) $h_2 \circ \alpha_1 = \alpha_2 \circ h_1$,

(ii) $\lambda_2(h_2(g))(h_1(s)) = h_1(\lambda_1(g))(s)$ for all $s \in S_1$ and all $g \in G_1$.

The *factorization semigroups* defined below are our main examples of semigroups over groups.

Let $O \subset G$ be a subset of a group G invariant under inner automorphisms. We call the pair (G, O) an *equipped group*. With the set O we associate an alphabet $X = X_O = \{x_g \mid g \in O\}$. For each pair of letters $x_{g_1}, x_{g_2} \in X$, $g_1 \neq g_2$, we define relations $R_{g_1,g_2;l}$ and $R_{g_1,g_2;r}$ in the following way: $R_{g_1,g_2;l}$ takes the form

$$x_{g_1} \cdot x_{g_2} = x_{g_2} \cdot x_{g_2^{-1} g_1 g_2} \tag{1.1}$$

if $g_2 \neq \mathbf{1}$, and $x_{g_1} \cdot x_{\mathbf{1}} = x_{g_1}$ if $g_2 = \mathbf{1}$, and $R_{g_1,g_2;r}$ takes the form

$$x_{g_1} \cdot x_{g_2} = x_{g_1 g_2 g_1^{-1}} \cdot x_{g_1} \tag{1.2}$$

if $g_1 \neq \mathbf{1}$, and $x_{\mathbf{1}} \cdot x_{g_2} = x_{g_2}$ if $g_1 = \mathbf{1}$.

We put

$$\mathcal{R} = \bigl\{R_{g_1,g_2;r}, R_{g_1,g_2;l} \mid (g_1, g_2) \in O \times O,\ g_1 \neq g_2\bigr\}.$$

Using the set of relations $\mathcal{R}$, we define a semigroup

$$S(G, O) = \langle x_g \in X \mid R \in \mathcal{R}\rangle$$

and call it the *factorization semigroup* of G with factors in O.

We also define a homomorphism $\alpha\colon S(G,O)\to G$ by the formula $\alpha(x_g)=g$ on the generators $x_g\in X$ and call it the *product homomorphism*.

Furthermore, we define an action λ of G on X by the formula

$$x_a\in X\mapsto \lambda(g)(x_a)=x_{g^{-1}ag}\in X.$$

The set $\mathcal{R}$ of relations is easily seen to be preserved by λ. Therefore λ determines a homomorphism $\lambda\colon G\to \operatorname{Aut}(S(G,O))$ (the *conjugation action*). The action $\lambda(g)$ on $S(G,O)$ is called *simultaneous conjugation* by g. We put $\lambda_S=\lambda\circ\alpha$ and $\rho_S=\rho\circ\alpha$.

Assertion 1.1 ([11]). *For all $s_1,s_2\in S(G,O)$ we have*

$$s_1\cdot s_2=s_2\cdot\lambda_S(s_2)(s_1)=\rho_S(s_1)(s_2)\cdot s_1.$$

Assertion 1.1 yields that $(S(G,O),G,\alpha,\lambda)$ is a semigroup over G. When G is fixed, we abbreviate $S(G,O)$ to S_O. We write $x_{g_1}\cdot\ldots\cdot x_{g_n}$ for the element of S_O defined by a word $x_{g_1}\ldots x_{g_n}$.

Note that $S\colon (G,O)\mapsto (S(G,O),G,\alpha,\lambda)$ is a functor from the category of equipped groups to the category of semigroups over groups. In particular, if subsets $O_1\subset O_2$ of G are invariant under inner automorphisms of G, then the identity map $\mathrm{id}\colon G\to G$ determines an embedding $\mathrm{id}_{O_1,O_2}\colon S(G,O_1)\to S(G,O_2)$. Thus, for every group G, the semigroup $S_G=S(G,G)$ is a *universal factorization semigroup* for elements of G, which means that every semigroup S_O over G is canonically embedded in S_G by $\mathrm{id}_{O,G}$.

Let Γ be a subgroup of G. We put $S_{O,\Gamma}=\{s\in S_O\mid \alpha(s)\in\Gamma\}$. Clearly, $S_{O,\Gamma}$ is a subsemigroup of S_O and if Γ is a normal subgroup of G, then $S_{O,\Gamma}$ is a semigroup over G. An important example of such semigroups is given by $S_{O,\mathbf{1}}$ (with $\Gamma=\{\mathbf{1}\}$).

The group G acts on itself by inner automorphisms, that is, for every group G there is a natural homomorphism $h\colon G\to\operatorname{Aut}(G)$ (the action of the image $h(g)=a$ of an element g on G is given by $(g_1)a=g^{-1}g_1g$ for all $g_1\in G$). We easily see that the homomorphism h endows S_G with the structure of a semigroup over $A=\operatorname{Aut}(G)$, where the homomorphism $\alpha_A\colon S_G\to\operatorname{Aut}(G)$ is the composite $h\circ\alpha$ and an element $a\in\operatorname{Aut}(G)$ acts on S_G by the rule $x_g\mapsto x_{(g)a}$. The subsemigroup $S_{G,\mathbf{1}}$ is easily seen to be invariant under the action of $\operatorname{Aut}(G)$ on S_G. Hence the semigroup $S_{G,\mathbf{1}}$ can also be regarded as a semigroup over $\operatorname{Aut}(G)$.

With every element $s=x_{g_1}\cdot\ldots\cdot x_{g_n}\in S_O$, $g_i\neq\mathbf{1}$, we associate a number $\ln(s)=n$ called the *length* of s. The map $\ln\colon S_O\to\mathbb{Z}_{\geqslant 0}=\{\mathbf{a}\in\mathbb{Z}\mid \mathbf{a}\geqslant 0\}$ is easily seen to be a homomorphism of semigroups.

Given any element $s=x_{g_1}\cdot\ldots\cdot x_{g_n}\in S_O$, we write G_s for the subgroup of G generated by the images $\alpha(x_{g_1})=g_1,\ldots,\alpha(x_{g_n})=g_n$ of the factors $x_{g_1},\ldots,x_{g_n}$.

Assertion 1.2. *The subgroup G_s of G is well defined, that is, it is independent of the representation of s as a product of generators $x_{g_i}\in X_O$.*

The proofs of Assertion 1.2 and the next proposition are very simple and we omit them.

Proposition 1.1 ([11]). *Suppose that (G,O) is an equipped group and $s\in S_O$. Then the following assertions hold.*

1) *The kernel $\ker\lambda$ coincides with the centralizer C_O of G_O in G.*

2) *If $\alpha(s)$ belongs to the centre $Z(G_s)$ of G_s, then the action $\lambda(g)$ leaves the element $s\in S_O$ fixed for every $g\in G_s$.*

3) *If $\alpha(s\cdot x_g)$ belongs to the centre $Z(G_{s\cdot x_g})$ of $G_{s\cdot x_g}$, then $s\cdot x_g = x_g\cdot s$.*

4) *If $\alpha(s)=\mathbf{1}$, then $s\cdot s' = s'\cdot s$ for all $s'\in S_G$.*

Assertion 1.3. *For every equipped group (G,O), the semigroup $S_{O,\mathbf{1}}$ is contained in the centre of the semigroup S_G and, in particular, is commutative.*

Proof. This follows from Proposition 1.1, 4).

It is easy to see that if $g\in O$ is an element of order n, then $x_g^n\in S_{O,\mathbf{1}}$.

Lemma 1.1. *Let $s\in S_{O,\mathbf{1}}$ and $s_1\in S_O$ be such that $G_{s_1}=G_O$. Then*

$$s\cdot s_1 = \lambda(g)(s)\cdot s_1 \tag{1.3}$$

for all $g\in G_O$. In particular, if $C\subset O$ is a conjugacy class of elements of order n_C and $s\in S_O$ satisfies $G_s=G$, then for all $g_1,g_2\in C$ we have

$$x_{g_1}^{n_C}\cdot s = x_{g_2}^{n_C}\cdot s. \tag{1.4}$$

Proof. (1.4) is proved in [5]. The proof of (1.3) is similar.

For every subgroup H of a group G we put

$$S_O^H = S(G,O)^H = \{s\in S(G,O)\mid G_s = H\}$$

and $S_{O,\mathbf{1}}^H = S_{O,\mathbf{1}}\cap S_O^H$. Then the semigroup S_O^H (resp. $S_{O,\mathbf{1}}^H$) is easily seen to be isomorphic to $S(H,H\cap O)^H$ (resp. $S(H,H\cap O)_{\mathbf{1}}^H$). The isomorphism is induced by the embedding $H\hookrightarrow G$.

1.2. *C*-groups associated with equipped groups, and the type homomorphism. Let (G,O) be an equipped group with $\mathbf{1}\notin O$, and let the set O be a union of m conjugacy classes: $O=C_1\cup\cdots\cup C_m$.

A group $\widehat{G}_O$ generated by an alphabet $Y_O=\{y_g\mid g\in O\}$ (of so-called C-generators) and defined by the relations

$$y_{g_1}y_{g_2} = y_{g_2}y_{g_2^{-1}g_1g_2} = y_{g_1g_2g_1^{-1}}y_{g_1},\qquad y_{g_1},y_{g_2}\in Y_O, \tag{1.5}$$

is called the *C-group associated with* (G,O). Clearly, the maps $x_g\mapsto y_g$ and $y_g\mapsto g$ determine homomorphisms $\beta\colon S(G,O)\to\widehat{G}_O$ and $\gamma\colon\widehat{G}_O\to G$ with $\alpha=\gamma\circ\beta$. The elements of $\operatorname{Im}\beta$ are called *positive* elements of $\widehat{G}_O$.

A C-group $\widehat{G}_O$ associated with an equipped group (G,O) has properties similar to those of the semigroup S_O. For example, as in the case of factorization semigroups, it is easy to check that for arbitrary $\hat{g}\in\widehat{G}_O$ and $g_1\in O$ the relation

$$\hat{g}^{-1}y_{g_1}\hat{g} = y_{g^{-1}g_1g} \tag{1.6}$$

is a consequence of the relations (1.5), where $g=\gamma(\hat{g})$.

We denote the subset $\{y_g \mid g \in O\}$ of $\widehat{G}_O$ by $\widehat{O}$. The relations (1.5), (1.6) yield that $\widehat{O}$ is invariant under inner automorphisms of $\widehat{G}_O$.

Assertion 1.4. *Let (G, O) be an equipped group. Then the semigroups $S(G, O)$ and $S(\widehat{G}_O, \widehat{O})$ are naturally isomorphic.*

Proof. In view of (1.5), (1.6) it is easy to see that the map $\xi\colon S(\widehat{G}_O, \widehat{O}) \to S(G, O)$ given by $\xi(x_{y_g}) = x_g$ for $g \in O$, is an isomorphism of semigroups.

The following proposition is an immediate corollary of the relations (1.5), (1.6) (see [14], for example).

Proposition 1.2. *For every equipped group (G, O) we have*

$$Z(\widehat{G}_O) = \gamma^{-1}(Z(G_O)),$$

where $Z(G_O)$ and $Z(\widehat{G}_O)$ are the centres of G_O and $\widehat{G}_O$ respectively.

The first homology group $H_1(\widehat{G}_O, \mathbb{Z}) = \widehat{G}_O/[\widehat{G}_O, \widehat{G}_O]$ of $\widehat{G}_O$ is easily seen to be free Abelian of rank m. Let $\mathrm{ab}\colon \widehat{G}_O \to H_1(\widehat{G}_O, \mathbb{Z})$ be the natural epimorphism. The group $H_1(\widehat{G}_O, \mathbb{Z}) \simeq \mathbb{Z}^m$ is generated by the elements $\mathrm{ab}(y_{g_i}) = (0, \dots, 0, 1, 0, \dots, 0)$, where $g_i \in C_i$ (1 is in the ith place).

The homomorphism of semigroups $\tau = \mathrm{ab} \circ \beta\colon S(G, O) \to \mathbb{Z}^m_{\geqslant 0} \subset \mathbb{Z}^m$ is called the *type homomorphism*, and the image $\tau(s)$ of an element $s \in S(G, O)$ is called the *type* of s. If O consists of a single conjugacy class, then the homomorphism τ can (and will) be identified with the homomorphism $\mathrm{ln}\colon S(G, O) \to \mathbb{Z}_{\geqslant 0}$.

Lemma 1.2. *Every element $\hat{g}$ of the C-group $\widehat{G}_O$ associated with an equipped group (G, O), can be written as*

$$\hat{g} = \hat{g}_1 \hat{g}_2^{-1}, \tag{1.7}$$

where $\hat{g}_1$, $\hat{g}_2$ are positive elements. In particular, $\hat{g} \in \widehat{G}'_O = [\widehat{G}_O, \widehat{G}_O]$ if and only if $\mathrm{ab}(\hat{g}_1) = \mathrm{ab}(\hat{g}_2)$ in the representation (1.7).

Proof. Write $\hat{g}$ in the form $\hat{g} = y_{g_{i_1}}^{\varepsilon_1} \dots y_{g_{i_k}}^{\varepsilon_k}$, where $g_{i_j} \in O$ and $\varepsilon_j = \pm 1$. To prove the lemma, it suffices to note that $y_{g_2}^{-1} y_{g_1} = y_{g_2^{-1} g_1 g_2} y_{g_2}^{-1}$ for all $g_1, g_2 \in O$ in view of the relations (1.5).

Assertion 1.5. *Let (G, O) be an equipped group. The homomorphism $\beta\colon S_O \to \widehat{G}_O$ is an embedding if and only if $O \subset Z(G_O)$, that is, if and only if G_O is an Abelian group.*

Proof. Let $O = C_1 \cup \dots \cup C_m$ be the decomposition into a union of conjugacy classes. If $O \subset Z(G_O)$, then we easily see that $\widehat{G}_O \simeq \mathbb{Z}^{|O|}$, where the isomorphism is induced by the homomorphism ab. In this case one can identify the semigroup S_O with the semigroup $\mathbb{Z}^{|O|}_{\geqslant 0} \subset \mathbb{Z}^{|O|}$.

If $O \not\subset Z(G_O)$, then there is a conjugacy class $C_i \subset O$ consisting of at least two elements, say g_1 and g_2. Let n be their order in G. Then we easily see that $x_{g_1}^n \neq x_{g_2}^n$ in S_O. On the other hand, their images $y_{g_1}^n = \beta(x_{g_1}^n)$ and $y_{g_2}^n = \beta(x_{g_2}^n)$

coincide in $\widehat{G}_O$. Indeed, there is no loss of generality in assuming that $g_2 = g^{-1}g_1g$ for some $g \in G_O$. Consider the element $\hat{g} \in \gamma^{-1}(g)$. Then

$$\hat{g}^{-1}y_{g_1}^n\hat{g} = (\hat{g}^{-1}y_{g_1}\hat{g})^n = y_{g^{-1}g_1g}^n = y_{g_2}^n.$$

But $y_{g_1}^n$ and $y_{g_2}^n$ belong to $Z(\widehat{G}_O)$ by Proposition 1.2. Therefore $y_{g_1}^n = y_{g_2}^n$.

1.3. Hurwitz equivalence. As above, let O be a subset of G invariant under inner automorphisms. Consider the set

$$O^n = \{(g_1, \dots, g_n) \mid g_i \in O\}$$

of all ordered n-tuples in O and let Br_n be the braid group with n strings. We fix a set $\{a_1, \dots, a_{n-1}\}$ of so-called *standard* (or *Artin*) *generators* of Br_n, that is, generators subject to the relations

$$\begin{aligned} a_ia_{i+1}a_i &= a_{i+1}a_ia_{i+1}, \qquad 1 \leqslant i \leqslant n-1, \\ a_ia_k &= a_ka_i, \qquad |i-k| \geqslant 2. \end{aligned} \tag{1.8}$$

The group Br_n acts on O^n by the formula

$$\big((g_1, \dots, g_{i-1}, g_i, g_{i+1}, g_{i+2}, \dots, g_n)\big)a_i = (g_1, \dots, g_{i-1}, g_ig_{i+1}g_i^{-1}, g_i, g_{i+2}, \dots, g_n).$$

The actions of the standard generators $a_i \in \mathrm{Br}_n$ and their inverses on O^n are usually called *Hurwitz moves*. Two elements of O^n are said to be *Hurwitz equivalent* if one can be obtained from the other by a finite sequence of Hurwitz moves, that is, if they belong to the same orbit under the action of Br_n.

The following formula defines a natural map $\alpha\colon O^n \to G$ (the *product map*):

$$\alpha((g_1, \dots, g_n)) = g_1 \dots g_n.$$

The element $(g_1, \dots, g_n) \in O^n$ is called a *factorization of* $g = \alpha((g_1, \dots, g_n)) \in G$ *with factors in* O.

There is a natural map $\varphi\colon O^n \to S(G, O)$ sending $(g_1, \dots, g_n)$ to $s = x_{g_1} \cdot \ldots \cdot x_{g_n}$.

Assertion 1.6. *Two factorizations $y, z \in O^n$ are Hurwitz equivalent if and only if $\varphi(y) = \varphi(z)$.*

Proof. This is obvious.

Remark 1.1. In what follows we identify the classes of Hurwitz-equivalent factorizations in O with their images in $S(G, O)$ in accordance with Assertion 1.6.

We also define a *conjugation action* of G on O^n:

$$\lambda(g)((g_1, \dots, g_n)) = (g^{-1}g_1g, \dots, g^{-1}g_ng).$$

The map φ identifies this action with the conjugation action λ of G on $S(G, O)$ defined above.

We denote the set of all words in the alphabet $X = X_{O\setminus\{\mathbf{1}\}}$ by $W = W(O)$, and let W_n be the subset consisting of all words of length n. In what follows we identify the elements of O^n with elements of W_n via the formula $(g_1, \dots, g_n) \in O^n \leftrightarrow x_{g_1} \dots x_{g_n} \in W_n)$. We put

$$W(s) = \big\{w \in W \mid \varphi(w) = s \in S(G, O)\big\}.$$

1.4. Non-perforated subsemigroups of $\mathbb{Z}^m_{\geqslant 0}$. We shall use the following facts about subsemigroups of $\mathbb{Z}^m_{\geqslant 0}$.

A subsemigroup S of $\mathbb{Z}^m_{\geqslant 0}$ is said to be *non-perforated* if we have $\mathbf{a}+\mathbf{b}\in S$ for all $\mathbf{a}\in S$ and $\mathbf{b}\in\mathbb{Z}^m_{\geqslant 0}$. Note that if S_1 and S_2 are non-perforated subsemigroups, then so are $S_1\cup S_2$ and $S_1\cap S_2$. An element $\mathbf{a}$ of a non-perforated subsemigroup S is called an *origin* of S if there are no elements $\mathbf{b}\in S$ and $\mathbf{c}\in\mathbb{Z}^m_{\geqslant 0}\setminus\{\mathbf{0}\}$ such that $\mathbf{a}=\mathbf{b}+\mathbf{c}$. The set of all origins of a non-perforated subsemigroup S is denoted by $O(S)$. A non-perforated subsemigroup S with a single origin is said to be *prime*. If $\mathbf{a}$ is the origin of a prime non-perforated subsemigroup S, then we easily see that

$$S=F_{\mathbf{a}}=\{\mathbf{c}=\mathbf{a}+\mathbf{b}\in\mathbb{Z}^m_{\geqslant 0}\mid\mathbf{b}\in\mathbb{Z}^m_{\geqslant 0}\}.$$

Clearly, every non-perforated subsemigroup S can be written as a union of prime non-perforated subsemigroups, for example,

$$S=\bigcup_{\mathbf{a}\in S}F_{\mathbf{a}}.$$

Suppose that S is represented as the union of prime non-perforated subsemigroups over some subset A of S:

$$S=\bigcup_{\mathbf{a}\in A}F_{\mathbf{a}}. \tag{1.9}$$

We say that representation (1.9) is *minimal* if

$$S\neq\bigcup_{\mathbf{a}\in A\setminus\{\mathbf{a}_0\}}F_{\mathbf{a}}$$

for any $\mathbf{a}_0\in A$.

Assertion 1.7. *Every non-perforated subsemigroup $S\subset\mathbb{Z}^m_{\geqslant 0}$ has a unique minimal representation as a union of prime non-perforated subsemigroups, namely,*

$$S=\bigcup_{\mathbf{a}\in O(S)}F_{\mathbf{a}}.$$

Proof. It follows from the definition of an origin that if $S=\bigcup F_{\mathbf{a}_i}$ is a representation as a union of prime non-perforated subsemigroups and $\mathbf{a}$ is an origin of S, then $\mathbf{a}=\mathbf{a}_i$ for some i.

Assume that the set

$$C=S\setminus\bigcup_{\mathbf{a}\in O(S)}F_{\mathbf{a}}$$

is non-empty. Then there is an element $\mathbf{c}_0=(c_{1,0},\dots,c_{m,0})\in C$ such that $c_{m,0}=\min c_m$ for $(c_1,\dots,c_m)\in C$, $c_{m-1,0}=\min c_{m-1}$ for $(c_1,\dots,c_{m-1},c_{m,0})\in C$, $\dots$, $c_{1,0}=\min c_1$ for $(c_1,c_{2,0},\dots,c_{m,0})\in C$. Clearly, $\mathbf{c}_0$ is an origin of S.

Proposition 1.3. *Every ascending chain*

$$S_1\subset S_2\subset S_3\subset\cdots$$

of non-perforated subsemigroups of $\mathbb{Z}^m_{\geqslant 0}$ with $S_i\neq S_{i+1}$ is finite.

Proof. This is obvious for $m=1$. Let us use induction on m. Consider an ascending chain of non-perforated subsemigroups $S_1 \subset S_2 \subset S_3 \subset \cdots \subset \mathbb{Z}^m_{\geqslant 0}$, $m \geqslant 2$. Put $P_j = \{(z_1, \dots, z_m) \in \mathbb{Z}^m_{\geqslant 0} \mid z_m = j\}$ and $S_{i,j} = S_i \cap P_j$. Then $S_{i,j}$ may also be regarded as non-perforated subsemigroups of $\mathbb{Z}^{m-1}_{\geqslant 0}$ (if we 'forget' the last coordinate). By the inductive assumption, the ascending chains $S_{1,j} \subset S_{2,j} \subset S_{3,j} \subset \cdots$ stabilize for every j. We denote the first largest semigroups in these chains by $\overline{S}_j = S_{i(j),j}$.

Define a map $\mathrm{sh}\colon \mathbb{Z}^m_{\geqslant 0} \to \mathbb{Z}^m_{\geqslant 0}$ by the formula

$$\mathrm{sh}((z_1, \dots, z_{m-1}, z_m)) = (z_1, \dots, z_{m-1}, z_m + 1).$$

It follows from the definition of a non-perforated subsemigroup that $\mathrm{sh}\colon S_{i,j} \to S_{i,j+1}$ is an embedding. Therefore we can (and will) identify each $S_{i,j}$ with the subsemigroup $\mathrm{sh}^n(S_{i,j})$ of $S_{i,j+n}$. It also follows from the definition of a non-perforated subsemigroup that if $j_1 < j_2$, then $\overline{S}_{j_1} = S_{i(j_1),j_1} \subset \overline{S}_{j_2} = S_{i(j_2),j_2}$. As a result, we obtain an ascending chain of non-perforated subsemigroups

$$S_{i(0),0} \subset S_{i(1),1} \subset S_{i(2),2} \subset \cdots \subset \mathbb{Z}^{m-1}_{\geqslant 0}.$$

It must stabilize. We easily see that if $S_{i(j_0),j_0}$ is the largest semigroup, then $S_{i(j_0)} = S_{i(j_0)+1} = S_{i(j_0)+2} = \cdots$.

Corollary 1.1. *The set of origins $O(S)$ of a non-perforated subsemigroup $S \subset \mathbb{Z}^m_{\geqslant 0}$ is non-empty and finite.*

Proof. If the set $O(S) = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \dots\}$ is infinite, then by Assertion 1.7 we have an infinite ascending sequence

$$F_{\mathbf{a}_1} \subset F_{\mathbf{a}_1} \cup F_{\mathbf{a}_2} \subset F_{\mathbf{a}_1} \cup F_{\mathbf{a}_2} \cup F_{\mathbf{a}_3} \subset \cdots,$$

contrary to Proposition 1.3.

1.5. Finite presentability of some subsemigroups of $S(G, O)$. Let (G, O) be a finite equipped group. Then the semigroup S_O is finitely presented by definition. From a geometric point of view, the most interesting subsemigroups of S_G are $S_{O,\mathbf{1}}$ and $S^G_{O,\mathbf{1}} = \{s \in S_{O,\mathbf{1}} \mid G_s = G\}$. (Note that $S^G_{O,\mathbf{1}}$ is non-empty if and only if $G_O = G$.) In this subsection we show that the semigroups $S_{O,\mathbf{1}}$ are finitely presented, but $S^G_{O,\mathbf{1}}$ may not be finitely presented (or even finitely generated).

Let $N = |G|$ be the order of G and $\mathcal{C} = \{C_1, \dots, C_m\}$ the set of conjugacy classes of G such that $O = \coprod C_i$. Given $C \in \mathcal{C}$, we denote the order of any element $g \in C$ by $n_C = n_g$. In each class $C \in \mathcal{C}$ we choose and fix an element $g_C \in C$.

An obvious necessary condition for a subsemigroup S of S_O to be finitely generated is that the image $\tau(S)$ is a finitely generated semigroup, where $\tau\colon S_O \to \mathbb{Z}^m_{\geqslant 0}$ is the type homomorphism.

Theorem 1.1. *The factorization semigroup $S_{O,\mathbf{1}}$ over a finite group G is finitely presented.*

Proof. Let $O = C_1 \cup \cdots \cup C_m$ be the decomposition into the union of conjugacy classes and suppose that $\mathbf{1} \notin O$. We enumerate the elements of $O = \{g_1, \ldots, g_K\}$ in such a way that $g_i = g_{C_i}$ for $i = 1, \ldots, m$.

For every $g \in O$ we have $s_g = x_g^{n_g} \in S_{O,\mathbf{1}}$. Let $F = \{s_1, \ldots, s_M\}$ be the set of elements of $S_{O,\mathbf{1}}$ of length less than or equal to K^N, where $N = |G|$ and we also assume that $s_i = s_{g_i} = x_{g_i}^{n_{g_i}}$ for $i \leqslant K$. We shall prove that the elements $s_1, \ldots, s_M \in F$ generate the semigroup $S_{O,\mathbf{1}}$.

Lemma 1.3. *Every element $s \in S_{O,\mathbf{1}}$ of length $\ln(s) > K^N$ can be written as*

$$s = s_{i_1}^{n_1} \cdot \ldots \cdot s_{i_l}^{n_l} \cdot \overline{s},$$

where $1 \leqslant i_1 \leqslant \cdots \leqslant i_l \leqslant K$ and the element $\overline{s} \in S_{O,\mathbf{1}}$ satisfies $\ln(\overline{s}) \leqslant K^N$.

Proof. If $\ln(s) > K^N$, then any representation of s as a product $x_{g_1} \cdot \ldots \cdot x_{g_{\ln(s)}}$ contains at least N equal factors x_g for some $g \in O$. Since $n_g \leqslant N$, we can move n_g such factors to the left (using the relations (1.1)) and obtain that $s = s_g \cdot s'$, where $s' \in S_{O,\mathbf{1}}$ satisfies $\ln(s') < \ln(s)$.

It follows from Lemma 1.3 that $S_{O,\mathbf{1}}$ is generated by the elements $s \in S_{O,\mathbf{1}}$ of length $\ln(s) \leqslant K^N$, that is, $S_{O,\mathbf{1}}$ is finitely generated.

To show that $S_{O,\mathbf{1}}$ is finitely presented, we partition the set of all relations in the following way. The first set R_1 consists of relations of the form

$$s_i \cdot s_j = s_j \cdot s_i, \qquad s_i, s_j \in F.$$

Given any M-tuple $\mathbf{k} = (k_1, \ldots, k_M)$ of non-negative integers, we put $s_{\mathbf{k}} = s_1^{k_1} \cdot \ldots \cdot s_M^{k_M}$. Since R_1 has already been defined, we can assume that all other relations between the generators $s_1, \ldots, s_M$ in $S_{O,\mathbf{1}}$ are of the form

$$s_{\mathbf{k}_1} = s_{\mathbf{k}_2}. \tag{1.10}$$

Note that if we have a relation of the form (1.10), then $G_{s_{\mathbf{k}_1}} = G_{s_{\mathbf{k}_2}}$ and $\tau(s_{\mathbf{k}_1}) = \tau(s_{\mathbf{k}_2})$.

Consider the set $\overline{R}_2$ of all relations (1.10) for which $G_{s_{\mathbf{k}_1}}$ is a proper subgroup of G. By induction, we can assume that the semigroups $S(\Gamma, \overline{O})_{\mathbf{1}}$ are finitely presented for all equipped groups $(\Gamma, \overline{O})$ of order less than N. Since there are only finitely many proper subgroups of G and the embeddings $(G_{s_{\mathbf{k}_1}}, O \cap G_{s_{\mathbf{k}_1}}) \hookrightarrow (G, O)$ determine embeddings $S(G_{s_{\mathbf{k}_1}}, O \cap G_{s_{\mathbf{k}_1}})_{\mathbf{1}} \hookrightarrow S_{O,\mathbf{1}}$, it follows that there is a finite set of relations $R_2 \subset \overline{R}_2$ such that all the relations in $\overline{R}_2$ are consequences of those in $R_1 \cup R_2$.

Let R_3 be the set of all relations in $S_{O,\mathbf{1}}$ of the form $s_{\mathbf{k}_1} = s_{\mathbf{k}_2}$ which are not contained in $R_1 \cup R_2$ and satisfy $\ln(s_{\mathbf{k}_1}) \leqslant K^N$. Clearly, R_3 is a finite set.

For each element s_i of the set of generators of $S_{O,\mathbf{1}}$ with $i \geqslant K + 1$ we put

$$n_i = \min_n \{\ln(s_i^n) > K^N\} - 1.$$

The following lemma is a corollary of Lemma 1.3.

Lemma 1.4. *For every $i \geqslant K+1$ the element $s_i^{n_i+1}$ can be written as*

$$s_i^{n_i+1} = \left(\prod_{j=1}^{K} s_j^{a_j}\right) \cdot s_l \tag{1.11}$$

for some K-tuple $\mathbf{a} = (a_1, \ldots, a_K)$ of non-negative integers and some generator $s_l \in F$ with $l \geqslant K+1$.

We denote the set of all relations of the form (1.11) by R_4. This set is finite. Lemma 1.4 shows that by applying the relations in $R_1 \cup R_4$, we can write every element $s \in S_{O,\mathbf{1}}$ in the form $s = s_{\mathbf{k}}$, where $\mathbf{k} = (k_1, \ldots, k_M)$ satisfies the following condition: $k_i \leqslant n_i$ for $i \geqslant K+1$.

An element $s_{\mathbf{k}}$ is said to be Γ-*primitive* if $\mathbf{k} = (k_1, \ldots, k_M)$ satisfies $k_i \leqslant 1$ for $i \leqslant K$, $k_i \leqslant n_i$ for $i \geqslant K+1$ and $G_{s_{\mathbf{k}}} = \Gamma$. By Lemma 1.1, for every G-primitive element $s_{\mathbf{k}}$ we have the following relations in $S_{O,\mathbf{1}}$:

$$s_i \cdot s_{\mathbf{k}} = s_j \cdot s_{\mathbf{k}},$$

where $i \leqslant m$ and $j \leqslant K$ are such that $g_j \in C_i$. We denote the set of all such relations by R_5. Clearly, R_5 is a finite set.

Let $s \in S_{O,\mathbf{1}}$ be such that $G_s = G$. By applying relations in R_5 and arguing as above, we easily see that s can be written in the form

$$s = \left(\prod_{j=1}^{m} s_j^{a_j}\right) \cdot s_{\mathbf{k}}, \tag{1.12}$$

where $s_{\mathbf{k}}$ is a G-primitive element. Let $\overline{R}_6$ be the set of all relations in $S_{O,\mathbf{1}}$ of the form

$$\left(\prod_{j=1}^{m} s_j^{b_{j,1}}\right) \cdot s_{\mathbf{k}_1} = \left(\prod_{j=1}^{m} s_j^{b_{j,2}}\right) \cdot s_{\mathbf{k}_2}, \tag{1.13}$$

where $s_{\mathbf{k}_1}$ and $s_{\mathbf{k}_2}$ are G-primitive elements.

To complete the proof of the theorem, it suffices to show that all the relations in $\overline{R}_6$ are consequences of some finite set of relations R_6. Since there are only finitely many G-primitive elements, it suffices to show that all the relations (1.13) with fixed G-primitive elements $s_{\mathbf{k}_1}$ and $s_{\mathbf{k}_2}$ are consequences of a finite set of relations.

Note that if we have a relation of the form (1.13), then

$$(b_{1,1}n_{C_1}, \ldots, b_{m,1}n_{C_m}) + \tau(s_{\mathbf{k}_1}) = (b_{1,2}n_{C_1}, \ldots, b_{m,2}n_{C_m}) + \tau(s_{\mathbf{k}_2}).$$

Therefore if $\tau(s_{\mathbf{k}_j}) = (\alpha_{1,j}, \ldots, \alpha_{m,j})$, then $\alpha_{i,1} \equiv \alpha_{i,2} \pmod{n_{C_i}}$ for all i. We put $a_{i,1,0} = b_{i,1} - b_{i,2}$ if $\alpha_{i,2} \geqslant \alpha_{i,1}$ and $a_{i,1,0} = 0$ otherwise. Conversely, put $a_{i,2,0} = b_{i,2} - b_{i,1}$ if $\alpha_{i,1} \geqslant \alpha_{i,2}$ and $a_{i,2,0} = 0$ otherwise. We have

$$n_{C_i}a_{i,1,0} + \alpha_{i,1} = n_{C_i}a_{i,2,0} + \alpha_{i,2}$$

and the numbers $a_{i,1,0}$, $a_{i,2,0}$ are uniquely determined by $\alpha_{i,1}$, $\alpha_{i,2}$ and n_{C_i}. Moreover, if we put $a_{i,j} = b_{i,j} - a_{i,j,0}$, then $a_{i,1} = a_{i,2} \geqslant 0$ for $i = 1, \ldots, m$ and each

of the relations (1.13) can be rewritten in the form

$$\left(\prod_{j=1}^{m} s_j^{a_j}\right)\cdot\left(\prod_{j=1}^{m} s_j^{a_{j,1,0}}\right)\cdot s_{\mathbf{k}_1} = \left(\prod_{j=1}^{m} s_j^{a_j}\right)\cdot\left(\prod_{j=1}^{m} s_j^{a_{j,2,0}}\right)\cdot s_{\mathbf{k}_2}, \tag{1.14}$$

where $a_j = a_{j,1} = a_{j,2}$.

If (1.14) is a relation in $S_{O,\mathbf{1}}$, then

$$\left(\prod_{j=1}^{m} s_j^{a_j+b_j}\right)\cdot\left(\prod_{j=1}^{m} s_j^{a_{j,1,0}}\right)\cdot s_{\mathbf{k}_1} = \left(\prod_{j=1}^{m} s_j^{a_j+b_j}\right)\cdot\left(\prod_{j=1}^{m} s_j^{a_{j,2,0}}\right)\cdot s_{\mathbf{k}_2}$$

is also a relation for each $\mathbf{b} = (b_1,\dots,b_m) \in \mathbb{Z}^m_{\geqslant 0}$ and it is a consequence of (1.14).

The above considerations show that the set $\{(a_1,\dots,a_m)\}$ of exponents occurring in the relations (1.14) for fixed $s_{\mathbf{k}_1}$ and $s_{\mathbf{k}_2}$ forms a non-perforated subsemigroup $F_{s_{\mathbf{k}_1},s_{\mathbf{k}_2}}$ of $\mathbb{Z}^m_{\geqslant 0}$. The set $O(F_{s_{\mathbf{k}_1},s_{\mathbf{k}_2}})$ of its origins is finite by Lemma 1.1. It is easy to see that the relations (1.14) with fixed $s_{\mathbf{k}_1}$ and $s_{\mathbf{k}_2}$ are consequences of the relations corresponding to the origins of $F_{s_{\mathbf{k}_1},s_{\mathbf{k}_2}}$. Since there are only finitely many G-primitive elements, we obtain that all the relations in $\overline{R}_6$ are consequences of some finite subset $R_6 \subset \overline{R}_6$.

To complete the proof of the theorem, it suffices to note that all the relations are consequences of the relations belonging to the finite set $R_1 \cup \dots \cup R_6$.

Note that not all subsemigroups $S^G_{O,\mathbf{1}}$ of S_G are finitely generated. For example, let $G \simeq (\mathbb{Z}/2\mathbb{Z})^2$ be generated by two elements, say, g_1 and g_2. If $O = \{g_1, g_2\}$, then $S^G_{O,\mathbf{1}}$ is isomorphic to the semigroup

$$S = \big\{(a_1,a_2) \in \mathbb{Z}^2_{\geqslant 0} \mid a_1 > 0,\ a_2 > 0\big\},$$

which is not finitely generated.

Proposition 1.4. *Let (G,O) be a finite equipped group. Suppose that $O = C_1 \cup \dots \cup C_m$ is a union of conjugacy classes such that, for every i, the elements of C_i generate G. Then the subsemigroup $S^G_{O,\mathbf{1}}$ of S_G is finitely presented.*

Proof. We use the same notation as in the proof of Theorem 1.1. Consider the element

$$s_{C_i} = \prod_{g_l \in C_i} x_{g_l}^{n_{C_i}} = \prod_{g_l \in C_i} s_l.$$

We have $s_{C_i} \in S^G_{O,\mathbf{1}}$ since the elements $g_l \in C_i$ generate G.

As shown in the proof of Theorem 1.1, every element $s \in S^G_{O,\mathbf{1}}$ can be written in the form (1.12):

$$s = \left(\prod_{i=1}^{m} s_i^{a_i}\right)\cdot s_{\mathbf{k}},$$

where $s_{\mathbf{k}}$ is a G-primitive element of $S^G_{O,\mathbf{1}}$. If $a_i \geqslant |C_i|$, then, by Lemma 1.1,

$$s_i^{a_i}\cdot s_{\mathbf{k}} = s_{C_i}\cdot s_i^{a_i-|C_i|}\cdot s_{\mathbf{k}}.$$

Therefore every element $s \in S^G_{O,\mathbf{1}}$ can be written in the form

$$s = \left(\prod_{i=1}^{m} s_{C_i}^{b_i}\right) \cdot \left(\prod_{i=1}^{m} s_i^{a_i}\right) \cdot s_{\mathbf{k}}, \tag{1.15}$$

where $(b_1, \dots, b_k) \in \mathbb{Z}^k_{\geqslant 0}$, $0 \leqslant a_i < |C_i|$, and $s_{\mathbf{k}}$ is G-primitive. It follows that $S^G_{O,\mathbf{1}}$ is generated by the elements

$$\left(\prod_{i=1}^{m} s_i^{a_i}\right) \cdot s_{\mathbf{k}},$$

where $0 \leqslant a_i < |C_i|$ and $s_{\mathbf{k}}$ is G-primitive, along with the elements s_{C_i}, $i = 1, \dots, m$. Clearly, this set of generators is finite. To prove the finite presentability of $S^G_{O,\mathbf{1}}$, we note that all the relations between these generators are consequences of the commutation relations and the set of relations $\overline{R}_6$ (in the notation of the proof of Theorem 1.1). Therefore the end of the proof of the proposition coincides with the corresponding part of the proof of Theorem 1.1.

1.6. Stabilizing elements. If G is a finite Abelian group, then the type homomorphism $\tau\colon S_G \to \mathbb{Z}^{|G|-1}_{\geqslant 0}$ is obviously an isomorphism. If G is non-Abelian and $c(G)$ is the number of conjugacy classes of its elements $g \neq \mathbf{1}$, then the type homomorphism $\tau\colon S_G \to \mathbb{Z}^{c(G)}_{\geqslant 0}$ is surjective and non-injective, and one of the main problems is to describe the pre-images $\tau^{-1}(\mathbf{a})$ of elements $\mathbf{a} \in \mathbb{Z}^{c(G)}_{\geqslant 0}$ (in particular, to describe the set of all elements $\mathbf{a} \in \mathbb{Z}^{c(G)}_{\geqslant 0}$ such that every $s \in \tau^{-1}(\mathbf{a})$ is uniquely determined by its value $\alpha(s) \in G$).

Proposition 1.5. *Let $S^G_{O,\mathbf{1}}$ be as in Proposition 1.4. Then there is a constant $c = c(G, O)$ such that for every $\mathbf{a} \in \mathbb{Z}^m_{\geqslant 0}$ the number $|\tau^{-1}(\mathbf{a})|$ of pre-images of $\mathbf{a}$ under the homomorphism $\tau\colon S^G_{O,\mathbf{1}} \to \mathbb{Z}^m_{\geqslant 0}$ is less than c.*

Proof. As shown in the proof of Proposition 1.4, every element $s \in S^G_{O,\mathbf{1}}$ can be written in the form (1.15). Since the number of different expressions (1.15) having the same type is finite and bounded by a constant c independent of the types of these expressions, the proposition follows.

Note that Proposition 1.5 does not hold for the semigroup $S_{O,\mathbf{1}}$ (instead of $S^G_{O,\mathbf{1}}$); see Corollary 2.4, for example.

An element $s \in S(G, O)$ is said to be *stabilizing* if $s \cdot s_1 = s \cdot s_2$ for all $s_1, s_2 \in S(G, O)$ such that $\tau(s_1) = \tau(s_2)$ and $\alpha(s_1) = \alpha(s_2)$. The semigroup $S(G, O)$ is said to be *stable* if it has a stabilizing element.

Assertion 1.8. *If s is a stabilizing element of $S(G, O)$, then so is the element $s \cdot s_1$ for every $s_1 \in S(G, O)$. In particular, if $S(G, O)$ is stable, then there is a stabilizing element $s \in S(G, O)$ with $\alpha(s) = \mathbf{1}$.*

Proof. This is obvious.

The Conway–Parker theorem ([5], Appendix) gives a sufficient condition for the stability of S_G. To state this theorem, we recall that a *Schur covering group* R of a finite group G is a group of maximal order with the following property: R has a subgroup $M \subset R' \cap Z(R)$ such that $R/M \simeq G$, where $R' = [R, R]$ is

the commutator subgroup and $Z(R)$ is the centre of R. Such a group R always exists (but need not be unique). The group M is isomorphic to the Schur multiplier $M(G) = H^2(G, \mathbb{C}^*)$ of G. The Schur multiplier $M(G)$ is said to be *generated by commutators* if $M \cap \{g^{-1}h^{-1}gh \mid g, h \in R\}$ generates M.

Theorem 1.2 (Conway–Parker, [5]). *Suppose that G is a finite group and $O = G \setminus \mathbf{1} = C_i \cup \cdots \cup C_m$ is the decomposition into conjugacy classes. Put*

$$\overline{s} = \prod_{g \in G \setminus \{\mathbf{1}\}} x_g^{n_g} \in S_G,$$

where n_g is the order of g in G. Assume that the Schur multiplier $M(G)$ of the group G is generated by commutators. Then there is a constant $n = n(G)$ such that $\overline{s}^n$ is a stabilizing element of S_G.

We note that a Schur covering group G of a finite group H satisfies the hypotheses of the Conway–Parker theorem (see [5]).

In the next section we shall prove that the factorization semigroups $S_{\mathcal{S}_d}$ of the symmetric group $\mathcal{S}_d$ are also stable. On the other hand, there are many finite equipped groups (G, O) whose semigroups $S(G, O)$ are unstable.

Proposition 1.6. *Let $(H, \widetilde{O})$ be a finite equipped group such that*

(i) *the elements of $\widetilde{O}$ generate H,*

(ii) $H' \cap Z(H) \neq \mathbf{1}$,

(iii) $\tilde{g}_1 \tilde{g}_2^{-1} \notin Z(H) \setminus \{\mathbf{1}\}$ *for all $\tilde{g}_1, \tilde{g}_2 \in \widetilde{O}$.*

Let $f\colon H \to H/Z(H) = G$ be the natural epimorphism and put $O = f(\widetilde{O}) \subset G$. Then there are at least two elements $s_1, s_2 \in S^G_{O,\mathbf{1}}$ such that $\tau(s \cdot s_1) = \tau(s \cdot s_2)$ but $s \cdot s_1 \neq s \cdot s_2$ for all $s \in S^G_{O,\mathbf{1}}$. In particular, if $\widetilde{O}$ consists of a single conjugacy class of H, then there is a constant $N \in \mathbb{N}$ such that for every $t \in \tau(S^G_{O,\mathbf{1}}) \cap \mathbb{Z}_{\geqslant N}$ there are at least two elements $s_1, s_2 \in S^G_{O,\mathbf{1}}$ such that $\tau(s_1) = \tau(s_2) = t$ but $s_1 \neq s_2$.

Proof. By condition (i), the elements of O generate G. By (iii), the surjective map $f_{|\widetilde{O}}\colon \widetilde{O} \to O$ is a bijection and, putting $g_i = f(\tilde{g}_i)$ for $\tilde{g}_i \in \widetilde{O}$, we see that the equality $g_i^{-1} g_j g_i = g_k$ holds in G for some elements $g_i, g_j, g_k \in O$ if and only if the equality $\tilde{g}_i^{-1} \tilde{g}_j \tilde{g}_i = \tilde{g}_k$ holds in H. Hence the induced homomorphism $f_*\colon S_{\widetilde{O}} \to S_O$ (sending the generators $x_{\tilde{g}_i}$ of $S_{\widetilde{O}}$ to the generators x_{g_i} of S_O) is an isomorphism of semigroups. In particular, the restriction of f_* to $S^H_{\widetilde{O},Z(H)} = \{\tilde{s} \in S^H_{\widetilde{O}} \mid \alpha(\tilde{s}) \in Z(H)\}$ gives an isomorphism between $S^H_{\widetilde{O},Z(H)}$ and $S^G_{O,\mathbf{1}}$. In addition, f induces a surjective homomorphism $f_*\colon \widehat{H}_{\widetilde{O}} \to \widehat{G}_O$ of the C-groups associated with the equipped groups $(H, \widetilde{O})$ and (G, O) (f_* sends the generator $y_{\tilde{g}_i}$ of $\widehat{H}_{\widetilde{O}}$ to the generators y_{g_i} of $\widehat{G}_O$) such that the diagram

$$\begin{array}{ccccc} S_{\widetilde{O}} & \xrightarrow{\beta} & \widehat{H}_{\widetilde{O}} & \xrightarrow{\gamma} & H \\ {\scriptstyle f_*}\downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f} \\ S_O & \xrightarrow{\beta} & \widehat{G}_O & \xrightarrow{\gamma} & G \end{array}$$

is commutative and the induced homomorphism

$$f_{**}\colon H_1(\widehat{H}_{\widetilde{O}},\mathbb{Z})\to H_1(\widehat{G}_O,\mathbb{Z})$$

is an isomorphism compatible with the isomorphism $f_*\colon S_{\widetilde{O}}\to S_O$ (that is, if $s=f_*(\tilde{s})$, then $\tau(s)=f_{**}(\tau(\tilde{s}))$). Therefore, to prove the first part of the proposition, it suffices to establish the existence of elements $\tilde{s}_1,\tilde{s}_2\in S^H_{\widetilde{O},Z(H)}$ such that $\tau(\tilde{s}_1)=\tau(\tilde{s}_2)$, but $\alpha(\tilde{s}_1)\ne\alpha(\tilde{s}_2)$. Indeed, for such elements we have $\tau(\tilde{s}\cdot\tilde{s}_1)=\tau(\tilde{s}\cdot\tilde{s}_2)$ but $\alpha(\tilde{s}\cdot\tilde{s}_1)\ne\alpha(\tilde{s}\cdot\tilde{s}_2)$ for all $\tilde{s}\in S^H_{\widetilde{O},Z(H)}$. Therefore, in view of the isomorphism $f_*\colon S^H_{\widetilde{O},Z(H)}\xrightarrow{\simeq}S^G_{O,\mathbf{1}}$, the elements $s_1=f_*(\tilde{s}_1)$ and $s_2=f_*(\tilde{s}_2)$ are not equal to each other in the semigroup $S_{O,\mathbf{1}}$, but $\tau(s\cdot s_1)=\tau(s\cdot s_2)$ and $s\cdot s_1\ne s\cdot s_2$ for all elements $s\in S^G_{O,\mathbf{1}}$.

It follows from Proposition 1.2 that for every subgroup $\widehat{H}_1$ of $\widehat{H}_{\widetilde{O}}$ we have

$$\gamma(\widehat{H}_1\cap Z(\widehat{H}_{\widetilde{O}}))=\gamma(\widehat{H}_1)\cap Z(H).$$

In particular,

$$\gamma(\widehat{H}'_{\widetilde{O}}\cap Z(\widehat{H}_{\widetilde{O}}))=H'\cap Z(H).$$

Hence, by condition (ii) there is an element $\hat{h}\in\widehat{H}'_{\widetilde{O}}\cap Z(\widehat{H}_{\widetilde{O}})\setminus\{\mathbf{1}\}$. By Lemma 1.2 we have $\hat{h}=\hat{h}_1\hat{h}_2^{-1}$, where $\hat{h}_1=\beta(\hat{s}_1)$ and $\hat{h}_2=\beta(\hat{s}_2)$ for some $\hat{s}_1,\hat{s}_2\in S_{\widehat{O}}$ (that is, $\hat{h}_1$ and $\hat{h}_2$ are positive elements). Since $\hat{h}\in\widehat{H}'_{\widetilde{O}}$, we have $\mathrm{ab}(\hat{h}_1)=\mathrm{ab}(\hat{h}_2)$.

Every element of the finite group H can be expressed as a positive word in its generators. Therefore, by condition (i), one can find $\hat{s}\in S_{\widetilde{O}}$ and a positive element $\hat{g}=\beta(\hat{s})\in\widehat{H}_{\widetilde{O}}$ such that $\gamma(\hat{g})=\gamma(\hat{h}_2^{-1})$. We put $\hat{s}_0=\prod_{\tilde{g}_i\in\widetilde{O}}x_{\tilde{g}_i}^{n_i}\in S^H_{\widetilde{O},\mathbf{1}}$, where n_i is the order of $\tilde{g}_i$. Then $\tilde{s}_1=\hat{s}_0\cdot\hat{s}\cdot\hat{s}_1$ and $\tilde{s}_2=\hat{s}_0\cdot\hat{s}\cdot\hat{s}_2$ are the desired elements.

To prove the second part of the proposition, we choose elements $\overline{s}_1,\dots,\overline{s}_n$ generating the semigroup $S^G_{O,\mathbf{1}}$ (by Proposition 1.4, the semigroup $S^G_{O,\mathbf{1}}$ is finitely generated in the case when O consists of a single conjugacy class) and let s_1, s_2 be the elements whose existence was proved in the first part of the proof. We put $t_0=\tau(s_1)=\tau(s_2)$ and $t_i=\tau(\overline{s}_i)$ for $i=1,\dots,n$ and write $\mathrm{GCD}(t_1,\dots,t_n)=d$, $t_i=a_id$. Then the type $\tau(s)$ of any element of $S^G_{O,\mathbf{1}}$ is divisible by d. We claim that there is a constant $M\in\mathbb{N}$ such that for every $j\in\mathbb{N}$ one can find $s\in S^G_{O,\mathbf{1}}$ with $\tau(s)=(M+j)d$. Indeed, there are $q_1,\dots,q_n\in\mathbb{Z}$ such that

$$\sum_{i=1}^n q_ia_i=1. \tag{1.16}$$

Renumbering the elements $\overline{s}_i$, we can assume that $q_i=-p_i<0$ for $i\leqslant k$ and $q_i\geqslant 0$ for $i\geqslant k+1$. We put $M=a_1d\sum_{i=1}^k a_ip_i$ and consider the following elements for $j=0,1,\dots,a_1$:

$$s_{0,j}=\left(\prod_{i=1}^k\overline{s}_i^{(a_1-j)p_i}\right)\cdot\left(\prod_{i=k+1}^n\overline{s}_i^{jq_i}\right)\in S^G_{O,\mathbf{1}}.$$

We have

$$\tau(s_{0,j}) = da_1 \sum_{i=1}^{k} p_i a_i + dj\left(-\sum_{i=1}^{k} a_i p_i + \sum_{i=k+1}^{n} a_i q_i\right) = d(M+j)$$

for $0 \leqslant j \leqslant a_1$. Then $\tau(\overline{s}_1^m \cdot s_{0,j}) = d(ma_1 + M + j)$. Since

$$\bigl\{d(ma_1 + M + j) \mid m \geqslant 0,\ 0 \leqslant j \leqslant a_1\bigr\} = d\mathbb{N}_{\geqslant M},$$

we easily see from this that M has the required property: for every $j \in \mathbb{N}$ there is an element $s \in S^G_{O,\mathbf{1}}$ with $\tau(s) = (M+j)d$.

To complete the proof of the proposition, it remains to note that $N = M + t_0 = M + \tau(s_1)$ is the desired constant.

It is easy to give examples of groups H satisfying the hypotheses of Proposition 1.6. For example, let $H = \mathrm{SL}_{p-1}(\mathbb{Z}_p)$ be the group of $(p-1)\times(p-1)$ matrices with determinant 1 over the finite field $\mathbb{Z}_p$, $p \neq 2$. It is well known that $H' = H$ and the centre $Z(H)$ consists of scalar matrices and is cyclic of order $p-1$. For $i \neq j$ let $e_{i,j}$ be the matrix with all entries equal to zero except for the entry equal to one at the intersection of the ith row and jth column. We put $t_{i,j} = e + e_{i,j}$, where e is the identity matrix. It is well known that the matrices $t_{i,j}$ (the so-called transvections) are conjugate to each other and generate the group $H = \mathrm{SL}_{p-1}(\mathbb{Z}_p)$. Hence, if we consider the equipped group (G, O) with $G = \mathrm{PGL}_{p-1}(\mathbb{Z}_p)$ and O the set of transvections, then almost all elements of the semigroup $S^G_{O,\mathbf{1}}$ are not uniquely determined by their type. In other words, $S^G_{O,\mathbf{1}}$ (resp. S_O) is not a stable semigroup.

§ 2. Factorization semigroups over symmetric groups

2.1. Basic notation and definitions. Let $\mathcal{S}_d$ be the symmetric group acting on the set $\{1,\dots,d\} = I_d$. We recall that an element $\sigma = (i_1,\dots,i_k) \in \mathcal{S}_d$ sending i_1 to i_2, i_2 to i_3, …, i_{k-1} to i_k, i_k to i_1 and leaving the other elements of I_d fixed is called a *cyclic permutation of length* k. Cyclic permutations of length 2 are called *transpositions*. Every cyclic permutation $\sigma = (i_1,\dots,i_k)$ is a product of $k-1$ transpositions:

$$\sigma = (i_1, i_2)(i_2, i_3)\dots(i_{k-1}, i_k). \tag{2.1}$$

The factorization (2.1) of $\sigma = (i_1,\dots,i_k)$ is said to be *canonical* if $i_1 = \min_{1\leqslant j\leqslant k} i_j$.

It is well known that every permutation $\sigma \in \mathcal{S}_d$, $\sigma \neq \mathbf{1}$, can be represented as a product of cyclic permutations:

$$\sigma = (i_{1,1},\dots,i_{k_1,1})(i_{1,2},\dots,i_{k_2,2})\dots(i_{1,m},\dots,i_{k_m,m}), \tag{2.2}$$

where $k_1 \geqslant k_2 \geqslant \dots \geqslant k_m \geqslant 2$ and the sets $\{i_{1,j_1},\dots,i_{k_{j_1},j_1}\}$ and $\{i_{1,j_2},\dots,i_{k_{j_2},j_2}\}$ are always disjoint for $j_1 \neq j_2$. If σ is written in the form (2.2), then the ordered set $t(\sigma) = [k_1,\dots,k_m]$ is called the *type* of σ and the number $l_t(\sigma) = \sum_{i=1}^{m} k_i - m$ is called the *transposition length* of σ.

Note that for any $k_1 \geqslant k_2 \geqslant \dots \geqslant k_m \geqslant 2$ with $\sum k_j \leqslant d$ there is a permutation σ of type $[k_1, \dots, k_m]$ and it is well known that two permutations σ_1 and σ_2 are conjugate in $\mathcal{S}_d$ if and only if $t(\sigma_1) = t(\sigma_2)$. For a fixed type $t(\sigma) = [k_1, \dots, k_m]$ the permutation

$$(1, \dots, k_1)(k_1 + 1, \dots, k_1 + k_2) \dots \left(\sum_{i=1}^{m-1} k_i + 1, \dots, \sum_{i=1}^{m} k_i \right)$$

is called the *canonical representative* of type $t(\sigma)$. We say that *the type* $t(\sigma_1) = [k_{1,1}, \dots, k_{m_1,1}]$ *is greater than the type* $t(\sigma_2) = [k_{1,2}, \dots, k_{m_2,2}]$ if there is $l \geqslant 0$ such that $k_{1,i} = k_{2,i}$ for $i \leqslant l$ and $k_{1,l+1} > k_{2,l+1}$ (here $k_{j,i} = 0$ if $i > m_j$). We say that the *cyclic permutation* $\sigma_1 = (j_1, \dots, j_{k_1})$ *is greater than the cyclic permutation* $\sigma_2 = (l_1, \dots, l_{k_2})$ if either $t(\sigma_1) > t(\sigma_2)$ or $t(\sigma_1) = t(\sigma_2)$ and there is $r < k_1 = k_2$ such that $j_1 = l_1, \dots, j_r = l_r$ and $j_{r+1} > l_{r+1}$ in the canonical factorizations of σ_1 and σ_2. Finally, we say that a *permutation* σ_1 *is greater than* σ_2 if either $t(\sigma_1) > t(\sigma_2)$ or $t(\sigma_1) = t(\sigma_2)$ and there is l such that $\sigma_{1,j} = \sigma_{2,j}$ for $j < l$ and $\sigma_{1,l} > \sigma_{2,l}$ in the cyclic factorizations $\sigma_i = \sigma_{i,1} \dots \sigma_{i,m}$, $i = 1, 2$. We denote the set of all types of permutations $\sigma \in \mathcal{S}_d$ by $\mathcal{T} = \{t_1 < t_2 < \dots < t_N\}$.

By definition, the factorization semigroup $\Sigma_d = S(\mathcal{S}_d, \mathcal{S}_d)$ over the symmetric group $\mathcal{S}_d$ is generated by the alphabet $X = \{x_\sigma \mid \sigma \in \mathcal{S}_d\}$. Let $s = x_{\sigma_1} \cdot \ldots \cdot x_{\sigma_n}$ be an element of Σ_d. Using the relations (1.1) and (1.2), we may assume that $t(\sigma_1) \leqslant \dots \leqslant t(\sigma_n)$. Then the sum $\tau(s) = \sum_{i=1}^{N} a_i t_i$ is the *type* of s, where a_i is the number of factors x_{σ_j} occurring in s with $t(\sigma_j) = t_i$.

For a subgroup Γ of $\mathcal{S}_d$ we put $\Sigma_{d,\Gamma} = \{s \in \Sigma_d \mid \alpha(s) \in \Gamma\}$ and $\Sigma_d^\Gamma = \{s \in \Sigma_d \mid (\mathcal{S}_d)_s = \Gamma\}$.

If $J \subset I_d$ is a subset of I_d with $|J| = d_1 \leqslant d$, then the embedding $J \subset I_d$ determines embeddings $\mathcal{S}_{d_1} \subset \mathcal{S}_d$ and $\psi_J \colon \Sigma_{d_1} \hookrightarrow \Sigma_d$.

2.2. Decompositions into products of transpositions. We denote the set of transpositions in $\mathcal{S}_d$ by T_d. The subsemigroup S_{T_d} of Σ_d is generated by the elements $x_{(i,j)}$, $1 \leqslant i, j \leqslant d$, $i \neq j$, subject to the relations

$$\begin{gathered} x_{(i,j)} = x_{(j,i)} \qquad \forall \{i, j\}_{\mathrm{ord}} \subset I_d, \\ x_{(i_1,i_2)} \cdot x_{(i_1,i_3)} = x_{(i_2,i_3)} \cdot x_{(i_1,i_2)} = x_{(i_1,i_3)} \cdot x_{(i_2,i_3)} \qquad \forall \{i_1, i_2, i_3\}_{\mathrm{ord}} \subset I_d, \\ x_{(i_1,i_2)} \cdot x_{(i_3,i_4)} = x_{(i_3,i_4)} \cdot x_{(i_1,i_2)} \qquad \forall \{i_1, i_2, i_3, i_4\}_{\mathrm{ord}} \subset I_d \end{gathered} \tag{2.3}$$

(here $\{i_1, \dots, i_k\}_{\mathrm{ord}}$ means an ordered subset of I_d consisting of k elements, so that for every subset $\{i_1, \dots, i_k\}$ of I_d we have $k!$ ordered subsets $\{\sigma(i_1), \dots, \sigma(i_k)\}_{\mathrm{ord}}$, $\sigma \in \mathcal{S}_k$).

We put $S_{T_d,\mathbf{1}} = S_{T_d} \cap \Sigma_{d,\mathbf{1}}$. By Proposition 1.1, 4), the semigroup $\Sigma_{d,\mathbf{1}}$ is a subsemigroup of the centre of Σ_d. In particular, it is a commutative semigroup.

It is easy to see that the element $s_{(i,j)} = x_{i,j} \cdot x_{i,j} = x_{(i,j)}^2$ belongs to $S_{T_d,\mathbf{1}}$ for every subset $\{i, j\} \subset I_d$. The element

$$h_{d,g} = s_{(1,2)}^{g+1} \cdot s_{(2,3)} \cdot \ldots \cdot s_{(d-1,d)} \in S_{T_d,\mathbf{1}} \subset \Sigma_d$$

is called a *Hurwitz element of genus* g.

Lemma 2.1. *For every ordered subset* $\{j_1,\dots,j_{k+1}\}_{\mathrm{ord}} \subset I_d$ *and any* $i,\ 1\leqslant i\leqslant k$, *the element*

$$s = x_{(j_1,j_2)}\cdot x_{(j_2,j_3)}\cdot\ \dots\ \cdot x_{(j_{k-1},j_k)}\cdot x_{(j_i,j_{k+1})}\in S_{T_d}$$

is equal to the element

$$s_i = x_{(j_1,j_2)}\cdot\ \dots\ \cdot x_{(j_{i-1},j_i)}\cdot x_{(j_i,j_{k+1})}\cdot x_{(j_{k+1},j_{i+1})}\cdot x_{(j_{i+1},j_{i+2})}\cdot\ \dots\ \cdot x_{(j_{k-1},j_k)}.$$

Proof. By (2.3) we have the following equalities (at every step we underline the factors to be transformed and write the result of transformation in brackets):

$$\begin{aligned}
s &= x_{(j_1,j_2)}\cdot x_{(j_2,j_3)}\cdot\ \dots\ \cdot \underline{x_{(j_{i+1},j_{i+2})}\cdot\ \dots\ \cdot x_{(j_{k-1},j_k)}\cdot x_{(j_i,j_{k+1})}}\\
&= x_{(j_1,j_2)}\cdot\ \dots\ \cdot \underline{x_{(j_i,j_{i+1})}\cdot\big(x_{(j_i,j_{k+1})}}\cdot x_{(j_{i+1},j_{i+2})}\cdot\ \dots\ \cdot x_{(j_{k-1},j_k)}\big)\\
&= x_{(j_1,j_2)}\cdot\ \dots\ \cdot x_{(j_{i-1},j_i)}\big(\underline{\cdot x_{(j_{i+1},j_{k+1})}\cdot x_{(j_i,j_{i+1})}}\big)\cdot\ \dots\ \cdot x_{(j_{k-1},j_k)}\\
&= x_{(j_1,j_2)}\cdot\ \dots\ \cdot x_{(j_{i-1},j_i)}\cdot\big(x_{(j_i,j_{k+1})}\cdot x_{(j_{k+1},j_{i+1})}\big)\cdot x_{(j_{i+1},j_{i+2})}\cdot\ \dots\ \cdot x_{(j_{k-1},j_k)}.
\end{aligned}$$

The lemma is proved.

Lemma 2.2. *For every ordered subset* $\{j_1,\dots,j_k\}_{\mathrm{ord}} \subset I_d$ *and any* $i,\ 1\leqslant i\leqslant k$, *the element* $s = x_{(j_1,j_2)}\cdot x_{(j_2,j_3)}\cdot\ \dots\ \cdot x_{(j_{k-1},j_k)}\cdot x_{(j_i,j_k)}\in S_{T_d}$, *where* $k\leqslant d-1$, *is equal to the element*

$$s_i = x_{(j_1,j_2)}\cdot\ \dots\ \cdot x_{(j_{i-1},j_i)}\cdot x_{(j_{i+1},j_{i+2})}\cdot\ \dots\ \cdot x_{(j_{k-1},j_k)}\cdot x^2_{(j_i,j_{i+1})}.$$

Proof. By (2.3) we have

$$\begin{aligned}
s &= x_{(j_1,j_2)}\cdot x_{(j_2,j_3)}\cdot\ \dots\ \cdot \underline{x_{(j_{k-1},j_k)}\cdot x_{(j_i,j_k)}}\\
&= x_{(j_1,j_2)}\cdot x_{(j_2,j_3)}\cdot\ \dots\ \cdot \underline{x_{(j_{k-2},j_{k-1})}\cdot\big(x_{(j_i,j_{k-1})}}\cdot x_{(j_{k-1},j_k)}\big) = \cdots\\
\cdots &= x_{(j_1,j_2)}\cdot\ \dots\ \cdot x_{(j_{i-1},j_i)}\cdot x_{(j_i,j_{i+1})}\cdot\big(x_{(j_i,j_{i+1})}\cdot x_{(j_{i+1},j_{i+2})}\big)\cdot\ \dots\ \cdot x_{(j_{k-1},j_k)}\\
&= x_{(j_1,j_2)}\cdot\ \dots\ \cdot x_{(j_{i-1},j_i)}\cdot \underline{x^2_{(j_i,j_{i+1})}\cdot x_{(j_{i+1},j_{i+2})}}\cdot\ \dots\ \cdot x_{(j_{k-1},j_k)}\\
&= x_{(j_1,j_2)}\cdot\ \dots\ \cdot x_{(j_{i-1},j_i)}\cdot\big(x_{(j_{i+1},j_{i+2})}\cdot \underline{x^2_{(j_i,j_{i+1})}\big)\cdot x_{(j_{i+2},j_{i+3})}}\cdot\ \dots\ \cdot x_{(j_{k-1},j_k)} = \cdots\\
\cdots &= x_{(j_1,j_2)}\cdot\ \dots\ \cdot x_{(j_{i-1},j_i)}\cdot x_{(j_{i+1},j_{i+2})}\cdot\ \dots\ \cdot\big(x_{(j_{k-1},j_k)}\cdot x^2_{(j_i,j_{i+1})}\big) = s_i.
\end{aligned}$$

The lemma is proved.

Lemma 2.3. *The equalities*

$$x^2_{(i_1,i_2)}\cdot x_{(i_2,i_3)} = x_{(i_2,i_3)}\cdot x^2_{(i_1,i_3)} = x^2_{(i_1,i_3)}\cdot x_{(i_2,i_3)} = x_{(i_2,i_3)}\cdot x^2_{(i_1,i_2)}, \tag{2.4}$$

$$x^2_{(i_1,i_2)}\cdot x^2_{(i_2,i_3)} = x^2_{(i_1,i_2)}\cdot x^2_{(i_1,i_3)} = x^2_{(i_2,i_3)}\cdot x^2_{(i_1,i_3)} \tag{2.5}$$

hold for all ordered triples $\{i_1,i_2,i_3\}_{\mathrm{ord}}\subset I_d$. *The equalities*

$$x^2_{(i_1,i_2)}\cdot x^2_{(i_3,i_4)} = x^2_{(i_3,i_4)}\cdot x^2_{(i_1,i_2)} \tag{2.6}$$

hold for all ordered quadruples $\{i_1,i_2,i_3,i_4\}_{\mathrm{ord}}\subset I_d$.

Proof. We check only two of the three equalities (2.4) since the others are verified in a similar way. By (2.3) we have

$$x^2_{(i_1,i_2)} \cdot x_{(i_2,i_3)} = x_{(i_1,i_2)} \cdot \underline{x_{(i_1,i_2)} \cdot x_{(i_2,i_3)}} = \underline{x_{(i_1,i_2)} \cdot (x_{(i_2,i_3)}} \cdot x_{(i_1,i_3)})$$
$$= (x_{(i_2,i_3)} \cdot x_{(i_1,i_3)}) \cdot x_{(i_1,i_3)} = x_{(i_2,i_3)} \cdot x^2_{(i_1,i_3)}.$$

Similarly,

$$x^2_{(i_1,i_2)} \cdot x_{(i_2,i_3)} = x_{(i_1,i_2)} \cdot \underline{x_{(i_1,i_2)} \cdot x_{(i_2,i_3)}} = \underline{x_{(i_1,i_2)} \cdot (x_{(i_1,i_3)}} \cdot x_{(i_1,i_2)})$$
$$= (x_{(i_2,i_3)} \cdot x_{(i_1,i_2)}) \cdot x_{(i_1,i_2)} = x_{(i_2,i_3)} \cdot x^2_{(i_1,i_2)}.$$

The lemma is proved.

Lemma 2.3 yields the following lemma.

Lemma 2.4. *For every ordered subset* $\{j_1, \dots, j_{k+1}\}_{\mathrm{ord}} \subset I_d$ *and any* i, $1 \leqslant i \leqslant k$, *the element* $s_i = x_{(j_1,j_2)} \cdot x_{(j_2,j_3)} \cdot \ldots \cdot x_{(j_{k-1},j_k)} \cdot x^2_{(j_i,j_{k+1})} \in S_{T_d}$ *is equal to the element* $s_1 = x_{(j_1,j_2)} \cdot x_{(j_2,j_3)} \cdot \ldots \cdot x_{(j_{k-1},j_k)} \cdot x^2_{(j_1,j_{k+1})}$.

The following lemma is a particular case of Lemma 1.1.

Lemma 2.5. *For every ordered subset* $\{j_1, \dots, j_k\}_{\mathrm{ord}} \subset I_d$ *we have*

$$x^2_{(j_1,j_2)} \cdot x_{(j_1,j_2)} \cdot x_{(j_2,j_3)} \cdot \ldots \cdot x_{(j_{k-1},j_k)} = x^2_{(j_i,j_l)} \cdot x_{(j_1,j_2)} \cdot x_{(j_2,j_3)} \cdot \ldots \cdot x_{(j_{k-1},j_k)},$$

where $1 \leqslant i < l \leqslant k$.

With every word $w(\overline{x_{(i,j)}}) = x_{(i_1,j_1)} \dots x_{(i_m,j_m)} \in W = W(T_d)$ we associate a graph $\widetilde{\Gamma}_w$ consisting of d vertices v_i, $1 \leqslant i \leqslant d$, with edge set in one-to-one correspondence with the set of letters occurring in w. Two vertices v_i and v_j are connected by an edge if the letter $x_{(i,j)}$ occurs in w. In particular, the number of edges connecting v_i and v_j is equal to the number of occurrences of the letter $x_{(i,j)}$ in w. The edges of $\widetilde{\Gamma}_w$ are enumerated according to the position of the corresponding letter in w. We denote the set of isolated vertices of $\widetilde{\Gamma}_w$ by V_{iso}. (A vertex v_i is *isolated* if it is not connected by an edge to any other vertex of $\widetilde{\Gamma}_w$.) We put $\Gamma_w = \widetilde{\Gamma}_w \setminus V_{\mathrm{iso}}$.

Lemma 2.6. *Let* $w', w'' \in W(s) = \{w \in W \mid \varphi(w) = s\}$ *be two words representing an element* $s \in S_{T_d}$ *and let* $\Gamma_{w'} = \Gamma_{1,1} \sqcup \dots \sqcup \Gamma_{1,n_1}$ *and* $\Gamma_{w''} = \Gamma_{2,1} \sqcup \dots \sqcup \Gamma_{2,n_2}$ *be the representations of the graphs* $\Gamma_{w'}$ *and* $\Gamma_{w''}$ *as disjoint unions of their connected components. Then* $n_1 = n_2 := n_s$ *and there is a one-to-one correspondence between the connected components of* $\Gamma_{w'}$ *and* $\Gamma_{w''}$ *such that the corresponding graphs* $\Gamma_{1,i}$ *and* $\Gamma_{2,i}$, $i = 1, \dots, n_s$, *have the same set of vertices* $V(\Gamma_{1,i}) = V(\Gamma_{2,i}) := V_i(s)$. *Moreover, the element* s *is uniquely representable as a product,* $s = s_1 \cdot \ldots \cdot s_{n_s}$, *of pairwise-commuting factors* $s_i \in S_{T_d}$ *such that for every* i *and every word* $w_i \in W(s_i)$ *the graph* Γ_{w_i} *is connected and* $V(\Gamma_{w_i}) = V(\Gamma_{1,i})$.

Proof. This follows easily from the relations (2.3).

Proposition 2.1. *Suppose that the length of an element $s \in S_{T_d}$ is equal to $k \leqslant d-1$. Then the element $\alpha(s) \in \mathcal{S}_d$ is a cyclic permutation of length k if and only if s satisfies the condition*

$$(*) \quad \textit{there is a word } w \in W(s) \textit{ whose graph } \Gamma_w \textit{ is a tree.}$$

Moreover, every element s satisfying condition $(*)$ is uniquely determined by the cyclic permutation $\alpha(s)$.

Proof. We claim that if s satisfies condition $(*)$, then there are exactly $k = \ln(s)$ words $w_1, \dots, w_k \in W(s)$ whose graphs Γ_{w_i} are simple paths when traced along the edges according to their enumeration. Indeed, it is easy to see that Lemma 2.1 yields the existence of a word $w_1 = x_{(i_1,i_2)}x_{(i_2,i_3)} \dots x_{(i_{k-1},i_k)}$ whose graph Γ_{w_1} is a simple path. Let us show that if we move the letter $x_{(i_{k-1},i_k)}$ to the extreme left position, then the resulting word w_2 determines the same element s, and its graph Γ_{w_2} is also a simple path. Indeed, we have

$$\begin{aligned} s &= x_{(i_1,i_2)} \cdot \ldots \cdot \underline{x_{(i_{k-2},i_{k-1})} \cdot x_{(i_{k-1},i_k)}} \\ &= x_{(i_1,i_2)} \cdot \ldots \cdot \underline{x_{(i_{k-3},i_{k-2})} \cdot (x_{(i_{k-2},i_k)}} \cdot x_{(i_{k-2},i_{k-1})}) = \dots \\ &\dots = (x_{(i_1,i_k)} \cdot x_{(i_1,i_2)}) \cdot \ldots \cdot x_{(i_{k-2},i_{k-1})}. \end{aligned}$$

Repeating this transformation k times, we find the desired words $w_1, \dots, w_k$.

We see that $\alpha(s)=(i_1,i_2)\dots(i_{k-2},i_{k-1})(i_{k-1},i_k)$ is a cyclic permutation of length k. On the other hand, if $\sigma \in \mathcal{S}_d$ is a cyclic permutation of length k, then it can be represented as a product of $k-1$ transpositions: $\sigma = (i_1,i_2)\dots(i_{k-2},i_{k-1})(i_{k-1},i_k)$ and, clearly, $\alpha(s) = \sigma$ for $s = x_{(i_1,i_2)} \cdot \ldots \cdot x_{(i_{k-2},i_{k-1})} \cdot x_{(i_{k-1},i_k)}$ and the graph $\Gamma_{x_{(i_1,i_2)}\dots x_{(i_{k-2},i_{k-1})}x_{(i_{k-1},i_k)}}$ satisfies condition $(*)$.

If we fix the set $\{i_1, \dots, i_k\} \subset I_d$, then there are exactly $(k-1)!$ distinct cyclic permutations of length k in $\mathcal{S}_d$ that cyclically permute the elements of $\{i_1, \dots, i_k\}$. On the other hand, there are exactly $k!$ distinct simple paths connecting the vertices $v_{i_1}, \dots, v_{i_k}$. Hence the elements s satisfying condition $(*)$ are uniquely determined by the cyclic permutations $\alpha(s)$.

Lemma 2.7. *Suppose that $s = x^2_{(i_1,i_2)} \cdot x^2_{(i_3,i_4)} \cdot \ldots \cdot x^2_{(i_{2k-1},i_{2k})}$ is a product of squares of generators of S_{T_d} and the graph Γ_w of the word $w = x^2_{(i_1,i_2)}x^2_{(i_3,i_4)} \dots x^2_{(i_{2k-1},i_{2k})}$ is connected. Then*

$$s = \psi_{V(\Gamma_w)}(h_{d_1,k-d_1-1}),$$

where $d_1 = |V(\Gamma_w)|$ is the number of vertices of Γ_w and $\psi_{V(\Gamma_w)}(h_{d_1,k-d_1-1})$ is the image of the Hurwitz element of the semigroup $S_{T_{d_1},\mathbf{1}}$ of genus $k-d_1-1$ under the embedding $\psi_{V(\Gamma_w)}\colon \Sigma_{d_1} \hookrightarrow \Sigma_d$ induced by the embedding $V(\Gamma_w) \hookrightarrow I_d$.

Proof. Arguing as in the proofs of Lemmas 2.1–2.3, we immediately deduce the lemma from the connectedness of Γ_w and the relations (2.5), (2.6).

Lemma 2.8. *For every $s \in S_{T_d}$ the difference $\ln(s) - l_t(\alpha(s))$ is a non-negative even number and one can find an element $\widetilde{s} \in S_{T_d}$ and an element $\overline{s}$ represented as*

a product of squares of generators $x_{(i,j)}$ of S_{T_d} (and therefore belonging to $S_{T_d,\mathbf{1}}$) such that

(i) $s = \widetilde{s} \cdot \overline{s}$,
(ii) $\ln(\widetilde{s}) = l_t(\alpha(s))$,
(iii) $\alpha(\widetilde{s}) = \alpha(s)$.

Moreover, the element $\widetilde{s}$ is uniquely determined by conditions (i)–(iii).

Proof. Consider the graph Γ_w of a word $w \in W(s) = \{w \in W \mid \varphi(w) = s\}$. It splits into a disjoint union of connected components: $\Gamma_w = \Gamma_{w,1} \sqcup \dots \sqcup \Gamma_{w,l}$. By Lemma 2.6, the element s can be uniquely represented (up to the order of factors) as a product of pairwise commuting factors: $s = \varphi(w_1(\overline{x_{(i,j)}})) \cdot \ldots \cdot \varphi(w_l(\overline{x_{(i,j)}}))$, where the word $w_i(\overline{x_{(i,j)}})$ consists of the letters $x_{(i,j)}$ such that $\Gamma_{w_i} = \Gamma_{w,i}$. Let $s_i = \varphi(w_i) \in S_{T_d}$ be the element determined by the word w_i. We have $(\mathcal{S}_d)_{s_i} \cap (\mathcal{S}_d)_{s_j} = \mathbf{1}$ for $i \neq j$. In particular, $\alpha(s_i)$ and $\alpha(s_j)$ are commuting permutations that act non-trivially on the disjoint sets $V(\Gamma_{w,i})$ and $V(\Gamma_{w,j})$. Hence it suffices to prove the lemma for the elements $s = \varphi(w)$ with a connected graph Γ_w.

Let $s = \varphi(w)$ be such that Γ_w is connected. Using Lemma 2.1, we easily find a representation of s as a word in the letters $x_{(i,j)}$ such that

$$s = x_{(j_1,j_2)} \cdot \ldots \cdot x_{(j_{k-1},j_k)} \cdot s_1,$$

and the set $\{v_{j_1}, \dots, v_{j_{k_i}}\}$ consists of all vertices of Γ_w.

Let $x_{(j_a,j_b)}$ be the first factor of s_1 if $s_1 \neq x_{\mathbf{1}}$. Then (2.3) and Lemma 2.2 yield that s can be written as $s = s' \cdot x^2_{(j_a,j_b)}$. Note that $x^2_{(j_a,j_b)} \in S_{T_d,\mathbf{1}}$ and $\ln(s') = \ln(s) - 2 < \ln(s)$, that is, the element s can be written in the form $s = \widetilde{s}_1 \cdot \overline{s}_1$, where $\ln(\widetilde{s}_1) < \ln(s)$ and $\overline{s}_1 \in S_{T_d,\mathbf{1}}$. Moreover, $\alpha(\widetilde{s}_1) = \alpha(s)$ since $\overline{s}_1 \in S_{T_d,\mathbf{1}}$. Repeating the above arguments for $\widetilde{s}_1$, …, if necessary, we obtain that s can be written in the form $s = \widetilde{s} \cdot \overline{s}$, where $\overline{s} \in S_{T_d,\mathbf{1}}$ is a product of squares of elements $x_{(i,j)}$, and $\widetilde{s} = s_1 \cdot \ldots \cdot s_m \in S_{T_d}$; here the elements $s_i = x_{(j_{1,i},j_{2,i})} \cdot \ldots \cdot x_{(j_{k_i-1,i},j_{k_i,i})}$, $1 \leqslant i \leqslant m$, are such that the subsets $\{j_{1,i}, \dots, j_{k_i,i}\}$ and $\{j_{1,l}, \dots, j_{k_l,l}\}$ of I_d are disjoint for $i \neq l$. Therefore,

$$\alpha(s) = \alpha(\widetilde{s}) = (j_{k_1,1}, \dots, j_{1,1}) \dots (j_{k_m,m}, \dots, j_{1,m}),$$

and hence $\ln(\widetilde{s}) = l_t(\alpha(s))$.

By Proposition 2.1, the elements s_i are uniquely determined (up to a permutation) by their products $\alpha(s_i)$. The lemma is proved.

Proposition 2.2. *Let $s \in S_{T_d}$ be such that $\alpha(s) = (i_1, i_2, \dots, i_k)$ is a cyclic permutation and the set $V(s)$ of vertices of the graph Γ_w, $w \in W(s)$, coincides with the set $\{i_1, i_2, \dots, i_k\} \subset I_d$. Then*

$$s = x_{(i_1,i_2)} \cdot x_{(i_2,i_3)} \cdot \ldots \cdot x_{(i_{k-1},i_k)} \cdot x^{2n}_{(i_2,i_1)},$$

where $2n = \ln(s) - k + 1$.

Proof. This follows from Lemmas 2.8 and 2.5.

Proposition 2.3. *Let $s = \varphi(w) \in S_{T_d}$ be such that Γ_w is a connected graph and if $\alpha(s) = \prod_{j=1}^{m}(i_{1,j}, i_{2,j}, \dots, i_{k_j,j})$ is a factorization into a product of cycles, then either $m > 1$, or $m = 1$ and $V(\Gamma_w) \neq \{i_{1,1}, i_{2,1}, \dots, i_{k_1,1}\}$. Put*

$$J = \{i_{1,1}, \dots i_{1,m}\} \cup \left(V(\Gamma_w) \setminus \bigcup_{j=1}^{m} \{i_{1,j}, i_{2,j}, \dots, i_{k_j,j}\} \right) \subset I_d.$$

Then

$$s = \psi_J(h_{d_1,g}) \cdot \prod_{j=1}^{m} (x_{(i_{1,j},i_{2,j})} \cdot x_{(i_{2,j},i_{3,j})} \cdot \ldots \cdot x_{(i_{k_j-1,j},i_{k_j,j})}),$$

where $h_{d_1,g} \in \Sigma_{T_{d_1}}$ is a Hurwitz element of genus g, $d_1 = |J|$, $g = \frac{\ln(s)-d_1+1}{2}$ and the embedding $\psi_J \colon \Sigma_{d_1} \hookrightarrow \Sigma_d$ is induced by the embedding $J \rightharpoonup I_d$.

Proof. By Lemma 2.8, the element s can be written in the form

$$s = \widetilde{s} \cdot \prod_{j=1}^{m} (x_{(i_{1,j},i_{2,j})} \cdot x_{(i_{2,j},i_{3,j})} \cdot \ldots \cdot x_{(i_{k_j-1,j},i_{k_j,j})}), \tag{2.7}$$

where $\widetilde{s}$ is a product of squares of generators $x_{(a,b)}$ of S_{T_d}, and it follows from the hypotheses of the proposition that $\ln(\widetilde{s}) \neq 0$. Consider one of the factors $x^2_{(a,b)}$ occurring in the factorization of $\widetilde{s}$. If a (or b) belongs to one of the sets $\{i_{1,j}, \dots, i_{k_j,j}\}$, then Lemmas 2.4, 2.5 show that this factor can be replaced in (2.7) by $x^2_{(i_{1,j},b)}$ without changing the element s. Therefore we can assume that only the following four possibilities occur for every factor $x^2_{(a,b)}$ in the factorization of $\widetilde{s}$:

1) $\{a, b\} \subset V(\Gamma_w) \setminus \bigcup_{j=1}^{m} \{i_{1,j}, i_{2,j}, \dots, i_{k_j,j}\}$,
2) $a = i_{1,j}$ for some $j \in [1, m]$, $b \in V(\Gamma_w) \setminus \bigcup_{j=1}^{m} \{i_{1,j}, i_{2,j}, \dots, i_{k_j,j}\}$,
3) $\{a, b\} = \{i_{1,j_1}, i_{1,j_2}\}$ for some $j_1, j_2 \in [1, m]$,
4) $\{a, b\} = \{i_{1,j}, i_{2,j}\}$ for some $j \in [1, m]$.

Let $\widetilde{w}$ be the word representing the factorization of $\widetilde{s}$ described above. Since Γ_w is connected, it follows that $\Gamma_{\widetilde{w}}$ is also connected, $J \subset V(\Gamma_{\widetilde{w}})$ and, moreover, for every $j \in [1, m]$ there is $b \notin \{i_{1,j}, \dots, i_{k_j,j}\}$ such that $x^2_{(i_{1,j},b)}$ is a subword of $\widetilde{w}$. If the word $\widetilde{w}$ contains a subword $x^2_{(i_{1,j},i_{2,j})}$ for some j, then Lemma 2.3 yields the following equalities (we recall that the elements $x^2_{(a,b)}$ belong to the centre of S_{T_d}):

$$\begin{aligned} \underline{x^2_{(i_{1,j},i_{2,j})} \cdot x^2_{(i_{1,j},b)}} \cdot x_{(i_{1,j},i_{2,j})} &= (x^2_{(i_{1,j},b)} \cdot \underline{x^2_{(i_{2,j},b)}) \cdot x_{(i_{1,j},i_{2,j})}} \\ &= x^2_{(i_{1,j},b)} \cdot (x^2_{(i_{1,j},b)} \cdot x_{(i_{1,j},i_{2,j})}). \end{aligned}$$

Therefore we can assume that $V(\Gamma_{\widetilde{w}}) = J$ and $\Gamma_{\widetilde{w}}$ is connected. To complete the proof of the proposition it suffices to use Lemma 2.7.

The following theorem is a consequence of Propositions 2.2, 2.3.

Theorem 2.1 ([7], [8]). *Two elements $s_1, s_2 \in S_{T_d}^{\mathcal{S}_d}$ are equal to each other if and only if $\alpha(s_1) = \alpha(s_2)$ and $\ln(s_1) = \ln(s_2)$.*

Proposition 2.4. *If* $s \in S_{T_d}^{\mathcal{S}_d}$ *and* $\ln(s) \geqslant l_t(\alpha(s)) + 2(d-1)$, *then there is a factorization* $s = \widetilde{s} \cdot \overline{s}$, *where* $\widetilde{s} = h_{d,g}$ *with* $g = \frac{1}{2}(\ln(s) - l_t(\alpha(s))) - d + 1$, *and the element* $\overline{s}$ *satisfies* $\ln(\overline{s}) = l_t(\alpha(s))$, $\alpha(\overline{s}) = \alpha(s)$. *Moreover, the element* $\overline{s}$ *is uniquely determined by the product* $\alpha(s)$.

Proof. By Propositions 2.2 and 2.3, the element s can be represented as a product $s = \widetilde{s} \cdot \overline{s} \in S_{T_d}$, where $\widetilde{s} \in S_{T_d}^{\mathcal{S}_d}$ is a product of squares of elements $x_{(i,j)}$, and $\overline{s}$ is such that

$$\alpha(s) = \alpha(\overline{s}) = (j_{1,1}, \dots, j_{k_1,1}) \dots (j_{1,m}, \dots, j_{k_m,m})$$

and $\ln(\overline{s}) = l_t(\alpha(s))$. Note that $\ln(\widetilde{s}) \geqslant 2(d-1)$ since $\ln(\overline{s}) = l_t(\alpha(s))$ and $\ln(s) \geqslant l_t(\alpha(s)) + 2(d-1)$.

Consider the graphs $\Gamma_{\widetilde{w}}$, $\Gamma_{\overline{w}}$ and $\Gamma_{\widetilde{w}\overline{w}}$, where $\widetilde{w} \in W(\widetilde{s})$, $\overline{w} \in W(\overline{s})$ and $\widetilde{w}\overline{w} \in W(s)$. We claim that there is a factorization $s = \widetilde{s} \cdot \overline{s}$ such that $V_{\widetilde{s}} = I_d$ and $\Gamma_{\widetilde{w}}$ is connected. We have $V_s = I_d$ since $(\mathcal{S}_d)_s = \mathcal{S}_d$. Suppose that either $V_{\widetilde{s}} \neq I_d$ or $\Gamma_{\widetilde{w}}$ is not connected for some factorization $s = \widetilde{s} \cdot \overline{s}$, and write $\widetilde{s} = \varphi(\widetilde{w}(\overline{x_{(i,j)}^2}))$ and $\overline{s} = \varphi(\overline{w}(\overline{x_{(i,j)}}))$. Since $\ln(\widetilde{s}) \geqslant 2(d-1)$, it follows from Lemma 2.3 that there is a connected component Γ_1 of $\Gamma_{\widetilde{w}}$ such that for each pair of vertices $v_{i_1}, v_{i_2} \in \Gamma_1$ we can find a word $\widetilde{w} \in W(\widetilde{s})$ with $\widetilde{s} = (x_{(i_1,i_2)}^2)^2 \cdot \widetilde{s}'$. Next, since $V_s = I_d$, there is a pair of vertices $v_{i_0}, v_{i_2} \in V_{\overline{s}}$ such that $v_{i_0} \notin V_{\widetilde{s}}$, $v_{i_2} \in V_{\widetilde{s}}$ and $\overline{s} = \overline{s}' \cdot x_{(i_0,i_2)}$. By Lemma 2.3 we have

$$s = \widetilde{s} \cdot \overline{s} = \overline{s}' \cdot x_{(i_0,i_2)} \cdot x_{(i_1,i_2)}^2 \cdot x_{(i_1,i_2)}^2 \cdot \overline{s}' = \overline{s}' \cdot x_{(i_0,i_2)} \cdot x_{(i_0,i_1)}^2 \cdot x_{(i_1,i_2)}^2 \cdot \widetilde{s}' = \widetilde{s} \cdot \widetilde{s}_1,$$

where for the word $\widetilde{w}_1 \in W(\widetilde{s}_1)$ either $V_{\widetilde{s}_1} = V_{\widetilde{s}} \cup \{i_0\}$ and the number of connected components of $\Gamma_{\widetilde{w}_1}$ is equal to that of $\Gamma_{\widetilde{w}}$ while the number of vertices of one of its connected components is increased by one, or the number of connected components of $\Gamma_{\widetilde{w}_1}$ is strictly less than that of $\Gamma_{\widetilde{w}}$. Repeating such transformations several times, we obtain a factorization $s = \widetilde{s} \cdot \overline{s}$ such that $V_{\widetilde{s}} = I_d$ and $\Gamma_{\widetilde{w}_1}$ is connected. To complete the proof of the proposition, it now suffices to use Lemma 2.3.

Proposition 2.5. *There is a unique homomorphism* $r\colon \Sigma_d \to S_{T_d}$ *such that*

(i) $\alpha(r(x_\sigma)) = \sigma$ *for* $\sigma \in \mathcal{S}_d$,
(ii) $\ln(r(x_\sigma)) = l_t(\sigma)$,
(iii) $r_{|S_{T_d}} = \mathrm{Id}$.

Proof. Every element $\sigma \in \mathcal{S}_d$, $\sigma \neq \mathbf{1}$, can be written as a product of pairwise commuting cyclic permutations: $\sigma = \sigma_1 \dots \sigma_m$, and this factorization is unique up to a permutation of the factors. By Proposition 2.1, every cyclic permutation σ_i uniquely determines an element $s_i \in S_{T_d}$ such that $\ln(s_i) = k_i - 1$ and $\alpha(s_i) = \sigma_i$, where k_i is the length of σ_i and, therefore, the product $s(\sigma) = s_1 \cdot \ldots \cdot s_m \in S_{T_d}$ is uniquely determined by σ. It is easy to see that the map $\sigma \mapsto s(\sigma)$ determines the homomorphism $r\colon \Sigma_d \to S_{T_d}$ given by the formula $r(x_\sigma) = s(\sigma)$ on the generators of Σ_d. Clearly, $\ln_t(s) = \ln(r(s))$ and $r_{|S_{T_d}} = \mathrm{Id}$.

The homomorphism $r\colon \Sigma_d \to S_{T_d}$ defined in Proposition 2.5 is called the *regenerating* homomorphism. The number $\ln_t(s) = \ln(r(s))$ is called the *transposition length* of $s \in \Sigma_d$.

2.3. Decompositions of the identity into a product of transpositions. Consider the semigroup $S_{T_d,\mathbf{1}}$.

Theorem 2.2. *The semigroup $S_{T_d,\mathbf{1}}$ is commutative and is generated by the elements $s_{(i,j)} = x^2_{(i,j)}$, $\{i,j\} \subset I_d$, subject to the relations*

$$s_{(i_1,i_2)} \cdot s_{(i_2,i_3)} = s_{(i_1,i_2)} \cdot s_{(i_1,i_3)} = s_{(i_2,i_3)} \cdot s_{(i_1,i_3)} \tag{2.8}$$

for all ordered triples $\{i_1,i_2,i_3\}_{\mathrm{ord}} \subset I_d$ and

$$s_{(i_1,i_2)} \cdot s_{(i_3,i_4)} = s_{(i_3,i_4)} \cdot s_{(i_1,i_2)} \tag{2.9}$$

for all ordered quadruples $\{i_1,i_2,i_3,i_4\}_{\mathrm{ord}} \subset I_d$. Moreover, every element $s \in S_{T_d,\mathbf{1}}$ has a normal form: it can uniquely be written as

$$s = \bigl(s^{k_1}_{(i_{1,1},i_{2,1})} \cdot s_{(i_{2,1},i_{3,1})} \cdot \ldots \cdot s_{(i_{j_1-1,1},i_{j_1,1})}\bigr) \cdot \ldots \\ \ldots \cdot \bigl(s^{k_n}_{(i_{1,n},i_{2,n})} \cdot s_{(i_{2,n},i_{3,n})} \cdot \ldots \cdot s_{(i_{j_n-1,n},i_{j_n,n})}\bigr),$$

where $1 \leqslant i_{1,1} < i_{1,2} < \cdots < i_{1,n} \leqslant d-1$, $k_l \in \mathbb{N}$ for $l = 1,\ldots,n$, and the sets $M_l = \{i_{1,l} < i_{2,l} < \cdots < i_{j_l,l}\}$, $1 \leqslant l \leqslant n$, are subsets of I_d of cardinality $j_l \geqslant 2$ such that $M_{l_1} \cap M_{l_2} = \varnothing$ for $l_1 \neq l_2$.

Proof. It follows from Lemma 2.8 that $S_{T_d,\mathbf{1}}$ is generated by the elements $s_{(i,j)}$. Lemma 2.3 shows that the elements $s_{(i,j)}$ satisfy the relations (2.8) and (2.9).

As in the proof of Proposition 2.4, for every $s = s_{(j_1,j_2)} \cdot \ldots \cdot s_{(j_{m-1},j_m)}$ we consider the graph Γ_w, where w is a word in letters $s_{(i,j)}$ representing the element s. The graph Γ_w splits into a disjoint union of connected components: $\Gamma_w = \Gamma_{w,1} \sqcup \cdots \sqcup \Gamma_{w,n}$. It follows easily from (2.3) that $w = w_1(\overline{s_{(i,j)}}) \ldots w_n(\overline{s_{(i,j)}})$, where $w_l(\overline{s_{(i,j)}})$ is a word in letters $s_{(i,j)}$ such that $\Gamma_{w_l} = \Gamma_{w,l}$. Let $s_l \in S_{T_d,\mathbf{1}}$ be the element defined by the word w_l, that is, $s_l = \varphi(w_l)$.

It follows from (2.8) and (2.9) that every element s_l can uniquely be written as

$$s_l = s^{k_l}_{(i_{1,l},i_{2,l})} \cdot s_{(i_{2,l},i_{3,l})} \cdot \ldots \cdot s_{(i_{j_l-1,l},i_{j_l,l})}, \tag{2.10}$$

where the set $M_l = \{i_{1,l} < i_{2,l} < \cdots < i_{j_l,l}\}$, $1 \leqslant l \leqslant n$, is in one-to-one correspondence with the set of vertices of the connected component $\Gamma_{w,l}$ of Γ_w.

Remark 2.1. The element $s^{k_l}_{(i_{1,l},i_{2,l})} \cdot s_{(i_{2,l},i_{3,l})} \cdot \ldots \cdot s_{(i_{j_l-1,l},i_{j_l,l})}$ in (2.10) is the Hurwitz element h_{j_l,k_l-1} of the semigroup $S_{T_{j_l},\mathbf{1}}$ if we regard $S_{T_{j_l},\mathbf{1}}$ as a subsemigroup of $S_{T_d,\mathbf{1}}$ and the embedding is given by the natural embedding $M_l \hookrightarrow I_d$.

Proposition 2.6. *The Hurwitz element $h_{d,g}$ belongs to the centre of the semigroup Σ_d and is fixed under the action of $\mathcal{S}_m$ on Σ_d by conjugation. For h_{d,g_1}, h_{d,g_2} we have*

$$h_{d,g_1} \cdot h_{d,g_2} = h_{d,g_1+g_2+d-1}.$$

Proof. The first part of the proposition follows from Proposition 1.1 since, on one hand, $\alpha(h_{d,g}) = \mathbf{1}$ and the transpositions $(i,i+1)$, $i = 1,\ldots,d-1$, generate the group $(\mathcal{S}_d)_{h_{d,g}}$ and, on the other hand, they generate the whole group $\mathcal{S}_d$. The second part of the proposition follows from Proposition 2.4.

Moreover, as a corollary of Theorems 2.4, 2.2, we obtain that the Hurwitz element $h_{d,g}$ is uniquely determined in the semigroup S_{T_d} by its length and the following two conditions.

Corollary 2.1 (Clebsch–Hurwitz theorem [1]). *Suppose that an element $s \in S_{T_d}$ satisfies the conditions*

(i) $(\mathcal{S}_d)_s = \mathcal{S}_d$,

(ii) $\alpha(s) = \mathbf{1}$.

Then $\ln(s) \geqslant 2(d-1)$ and $s = h_{d,g}$, where $g = \frac{\ln(s)}{2} - d + 1$.

2.4. Factorizations in symmetric groups (general case). In this subsection we prove the following generalization of Proposition 2.4.

Theorem 2.3. *Suppose that $s = x_{\sigma_1} \cdot \ldots \cdot x_{\sigma_m} \cdot \overline{s} \in \Sigma_d$, where $\overline{s} \in S_{T_d}$. For $j = 1, \ldots, m$ denote the canonical representative of type $t(\sigma_j)$ by $\sigma_{j,0}$* (see the definitions in § 2.1) *and put*

$$\sigma = \sigma(s) = (\sigma_{1,0} \ldots \sigma_{m,0})^{-1} \alpha(s).$$

If $s \in \Sigma_d^{\mathcal{S}_d}$ and $\ln(\overline{s}) = k \geqslant 3(d-1)$, then

$$s = x_{\sigma_{1,0}} \cdot \ldots \cdot x_{\sigma_{m,0}} \cdot r(x_\sigma) \cdot h_{d,g},$$

where $g = \frac{k - \ln_t(x_\sigma)}{2} - d + 1$.

Proof. We claim that there is a factorization

$$s = x_{\sigma'_1} \cdot \ldots \cdot x_{\sigma'_m} \cdot x_{(i_1,j_1)} \cdot \ldots \cdot x_{(i_k,j_k)} = x_{\sigma'_1} \cdot \ldots \cdot x_{\sigma'_m} \cdot \overline{s}_1,$$

where $t(\sigma_i) = t(\sigma'_i)$ for $i = 1, \ldots, m$, the graph $\Gamma_{\overline{w}_1}$ associated with the word $\overline{w}_1 = x_{(i_1,j_1)} \ldots x_{(i_k,j_k)} \in W(\overline{s}_1)$ is connected, and the set $V_{\overline{s}_1}$ of its vertices coincides with I_d.

Indeed, take $w \in W(\overline{s})$ and suppose that either $V_{\overline{s}} \neq I_d$ or the graph $\Gamma_{\overline{w}}$ is not connected. Since $\ln(\overline{s}) \geqslant 3(d-1)$, there is a connected component Γ_1 of Γ_w having more edges than vertices. Then the proof of Proposition 2.4 shows that for any v_{i_1}, v_{i_2} in the set $V(\Gamma_1)$ of vertices of Γ_1 there is a word $w' \in W$ such that $\overline{s} = x^2_{(i_1,i_2)} \cdot \varphi(w')$ and the vertices in $V(\Gamma_1)$ belong to the same connected component of $\Gamma_{x^2_{i_1,i_2} w'}$. Next, since $(\mathcal{S}_d)_s = \mathcal{S}_d$, there is a permutation σ_l for some l, $1 \leqslant l \leqslant m$, such that $\sigma_l(i_1, i_2)\sigma_l^{-1} = (i_0, j_0)$, where either v_{i_0} or v_{j_0} (but not both) does not belong to $V(\Gamma_1)$. There is no loss of generality in assuming that $l = m$. We have

$$\begin{aligned} s &= x_{\sigma_1} \cdot \ldots \cdot x_{\sigma_m} \cdot \overline{s} = x_{\sigma_1} \cdot \ldots \cdot x_{\sigma_m} \cdot x^2_{(i_1,i_2)} \cdot \varphi(w') \\ &= x_{\sigma_1} \cdot \ldots \cdot x_{\sigma_{m-1}} \cdot x_{(i_0,j_0)} \cdot x_{\sigma_m} \cdot x_{(i_1,i_2)} \cdot \varphi(w') \\ &= x_{\sigma_1} \cdot \ldots \cdot x_{\sigma_{m-1}} \cdot \rho((i_0,j_0))(x_{\sigma_m}) \cdot x_{(i_0,j_0)} \cdot x_{(i_1,i_2)} \cdot \varphi(w') \\ &= x_{\sigma_1} \cdot \ldots \cdot x_{\sigma_{m-1}} \cdot \rho((i_0,j_0))(x_{\sigma_m}) \cdot \varphi(w''), \end{aligned}$$

where $w'' = x_{(i_0,j_0)} x_{(i_1,j_1)} w'$ is a word such that either the set of vertices of $\Gamma_{w''}$ strictly contains the set $V_{\overline{s}}$, or the number of connected components of $\Gamma_{w''}$ is strictly less than that of $\Gamma_{w'}$.

Repeating such transformations several times, we obtain a factorization of s of the form

$$s = x_{\sigma'_1} \cdot \ldots \cdot x_{\sigma'_m} \cdot \overline{s}_1,$$

where $\overline{s}_1 \in S_{T_d}$, $V_{\overline{s}_1} = I_d$, the graph $\Gamma_{\overline{w}_1}$ is connected and $t(\sigma'_j) = t(\sigma_j)$ for $j = 1, \ldots, m$. This factorization satisfies $(\mathcal{S}_d)_{\overline{s}_1} = \mathcal{S}_d$ and $\ln(\overline{s}_1) \geqslant 3(d-1)$.

To complete the proof of the theorem, we use induction on m. If $m = 0$ (that is, $s \in S_{T_d}$), then the theorem follows from Proposition 2.4.

Suppose that $m = 1$. By Proposition 2.4 we have $\overline{s}_1 = h_{d,0} \cdot \overline{s}'$ for some element $\overline{s}' \in S_{T_d}$.

Lemma 2.9. *Let $\{i_{1,1}, \ldots, i_{k_1,1}\} \sqcup \cdots \sqcup \{i_{1,n}, \ldots, i_{k_n,n}\}$ be any disjoint union of ordered subsets of I_d. Then the Hurwitz element $h_{d,0}$ can be represented as a product*

$$h_{d,0} = (x_{(i_{1,1},i_{2,1})} \cdot \ldots \cdot x_{(i_{k_1-1,1},i_{k_1,1})}) \cdot \ldots \cdot (x_{(i_{1,n},i_{2,n})} \cdot \ldots \cdot x_{(i_{k_n-1,n},i_{k_n,n})}) \cdot \overline{h},$$

where $\overline{h}$ is an element of $S_{T_d}^{\mathcal{S}_d}$.

Proof. The semigroup $S_{T_{d,\mathbf{1}}}$ is commutative and the Hurwitz element $h_{d,0}$ is invariant under the action of $\mathcal{S}_d$ by conjugation. Hence $h_{d,0}$ can be written in the form

$$h_{d,0} = (s_{(i_{1,1},i_{2,1})} \cdot \ldots \cdot s_{(i_{k_1-1,1},i_{k_1,1})}) \cdot \ldots \cdot (s_{(i_{1,n},i_{2,n})} \cdot \ldots \cdot s_{(i_{k_n-1,n},i_{k_n,n})}) \cdot \widetilde{h},$$

where $\widetilde{h}$ is an element of $S_{T_{d,\mathbf{1}}}$. We have

$$\begin{aligned} s_{(i_{1,j},i_{2,j})} \cdot \ldots \cdot s_{(i_{k_j-1,j},i_{k_j,j})} &= x^2_{(i_{1,j},i_{2,j})} \cdot \ldots \cdot x^2_{(i_{k_j-1,j},i_{k_j,j})} \\ &= x_{(i_{1,j},i_{2,j})} \cdot (x^2_{(i_{2,j},i_{3,j})} \cdot \ldots \cdot x^2_{(i_{k_j-1,j},i_{k_j,j})}) \cdot x_{(i_{1,j},i_{2,j})} = \ldots \\ &\cdots = (x_{(i_{1,j},i_{2,j})} \cdot \ldots \cdot x_{(i_{k_j-1,j},i_{k_j,j})}) \cdot (x_{(i_{k_j-1,j},i_{k_j,j})} \cdot \ldots \cdot x_{(i_{1,j},i_{2,j})}), \end{aligned}$$

and the elements $x_{(i_{l_1,j_1},i_{l_1+1,j_1})}$ and $x_{(i_{l_2,j_2},i_{l_2+1,j_2})}$ commute if $j_1 \neq j_2$. To complete the proof of the lemma, we note that $(\mathcal{S}_d)_{s_j} = (\mathcal{S}_d)_{\overline{s}_j}$, where $s_j = s_{(i_{1,j},i_{2,j})} \cdot \ldots \cdot s_{(i_{k_j-1,j},i_{k_j,j})}$ and $\overline{s}_j = x_{(i_{k_j-1,j},i_{k_j,j})} \cdot \ldots \cdot x_{(i_{1,j},i_{2,j})}$. Therefore $\overline{h} = (\prod \overline{s}_i) \cdot \widetilde{h} \in S_{T_d}^{\mathcal{S}_d}$ because $h_{d,0} \in S_{T_d}^{\mathcal{S}_d}$. The lemma is proved.

For the canonical representative $\sigma_{m,0}$ of type $t(\sigma_m)$ there is an element $\overline{\sigma}_m \in \mathcal{S}_d$ such that $\sigma_{m,0} = \overline{\sigma}_m^{-1} \sigma'_m \overline{\sigma}_m$. The permutation $\overline{\sigma}_m$ can be factorized into a product of cycles and each cycle can be factorized into a product of transpositions:

$$\overline{\sigma}_m = \big((i_{1,1}, i_{2,1}) \ldots (i_{k_1-1,1}, i_{k_1,1})\big) \ldots \big((i_{1,n}, i_{2,n}) \ldots (i_{k_n-1,n}, i_{k_n,n})\big).$$

Consider an element

$$r(x_{\overline{\sigma}_m}) = (x_{(i_{1,1},i_{2,1})} \cdot \ldots \cdot x_{(i_{k_1-1,1},i_{k_1,1})}) \cdot \ldots \cdot (x_{(i_{1,n},i_{2,n})} \cdot \ldots \cdot x_{(i_{k_n-1,n},i_{k_n,n})}) \in S_{T_d},$$

where r is the regenerating homomorphism. By Lemma 2.9,

$$h_{d,0} = r(x_{\overline{\sigma}_m}) \cdot \overline{h}_m,$$

where $\overline{h}_m$ satisfies $(\mathcal{S}_d)_{\overline{h}_m} = \mathcal{S}_d$. We have

$$\begin{aligned} s &= x_{\sigma'_m} \cdot h_{d,0} \cdot \overline{s}\,' = x_{\sigma'_m} \cdot r(x_{\overline{\sigma}_m}) \cdot \overline{h}_m \cdot \overline{s}\,' \\ &= r(x_{\overline{\sigma}_m}) \cdot x_{\sigma_{m,0}} \cdot \overline{h}_m \cdot \overline{s}\,' = x_{\sigma_{m,0}} \cdot r(x_{\overline{\sigma}'_m}) \cdot \overline{h}_m \cdot \overline{s}\,', \end{aligned}$$

where $x_{\overline{\sigma}'_m} = \lambda(\sigma_{m,0})(x_{\overline{\sigma}_m})$. The element $\overline{s}'_1 = r(x_{\overline{\sigma}'_m}) \cdot \overline{h}_m \cdot \overline{s}\,' \in S_{T_d}$ satisfies $\ln(\overline{s}'_1) = k$, $\alpha(\overline{s}'_1) = \sigma_{m,0}^{-1}\alpha(s)$ and $(\mathcal{S}_d)_{\overline{s}'_1} = \mathcal{S}_d$. Therefore, by Theorem 2.4, $\overline{s}'_1 = r(x_\sigma) \cdot h_{d,g}$, where $\sigma = \alpha(\overline{s}'_1) = \sigma_{m,0}^{-1}\alpha(s)$ and $g = \frac{k - \ln_t(x_\sigma)}{2} - d + 1$.

Assume that the theorem is true for all $m < m_0$ and consider an element

$$s = x_{\sigma_1} \cdot \ldots \cdot x_{\sigma_{m_0}} \cdot \overline{s}_1,$$

where $\overline{s}_1 \in S_{T_d}$ has length $k \geqslant 3(d-1)$ and satisfies $(\mathcal{S}_d)_{\overline{s}_1} = \mathcal{S}_d$. We have

$$\begin{aligned} s &= x_{\sigma_1} \cdot \ldots \cdot x_{\sigma_{m_0}} \cdot \overline{s}_1 = x_{\sigma'_2} \cdot \ldots \cdot x_{\sigma'_{m_0}} \cdot x_{\sigma_1} \cdot \overline{s}_1 \\ &= x_{\sigma'_2} \cdot \ldots \cdot x_{\sigma'_{m_0}} \cdot x_{\sigma_{1,0}} \cdot \overline{s}'_1 = x_{\sigma_{1,0}} \cdot x_{\sigma''_2} \cdot \ldots \cdot x_{\sigma''_{m_0}} \cdot \overline{s}'_1, \end{aligned}$$

where $\sigma'_j = \sigma_1\sigma_j\sigma_1^{-1}$ and $\sigma''_j = \sigma_{1,0}^{-1}\sigma'_j\sigma_{1,0}$ for $j = 2, \ldots, m$ and the element $\overline{s}'_1 \in S_d$ satisfies $\ln(\overline{s}'_1) = k$ and $(\mathcal{S}_d)_{\overline{s}'_1} = \mathcal{S}_d$. Therefore, by the inductive assumption, we have

$$s = x_{\sigma_{1,0}} \cdot (x_{\sigma''_2} \cdot \ldots \cdot x_{\sigma''_{m_0}} \cdot \overline{s}'_1) = x_{\sigma_{1,0}} \cdot (x_{\sigma_{2,0}} \cdot \ldots \cdot x_{\sigma_{m_0,0}} \cdot \overline{s}''_1),$$

where the element $\overline{s}''_1 \in S_d$ satisfies $\ln(\overline{s}''_1) = k$ and $(\mathcal{S}_d)_{\overline{s}''_1} = \mathcal{S}_d$. By Proposition 2.4 we have $\overline{s}''_1 = r(x_\sigma) \cdot h_{d,g}$, where $\sigma = \alpha(\overline{s}''_1) = (\sigma_{1,0} \ldots \sigma_{m,0})^{-1}\alpha(s)$ and $g = \frac{k - \ln_t(x_\sigma)}{2} - d + 1$. The theorem is proved.

Corollary 2.2. *Suppose that $s_i = x_{\sigma_{1,i}} \cdot \ldots \cdot x_{\sigma_{m,i}} \cdot \overline{s}_i$, $i = 1, 2$, are elements of $\Sigma_d^{\mathcal{S}_d}$, where the elements $\overline{s}_i \in S_{T_d}$ have length $\ln(\overline{s}_1) = \ln(\overline{s}_2) = k$. Suppose that $\alpha(s_1) = \alpha(s_2)$ and $\tau(s_1) = \tau(s_2)$. If $k \geqslant 3(d-1)$, then $s_1 = s_2$.*

Corollary 2.3. *The Hurwitz element $h_{d,[\frac{d}{2}]}$ is a stabilizing element of Σ_d. Hence the semigroup Σ_d is stable.*

The factorization of the identity in $\mathcal{S}_d$ is unique in the following case.

Theorem 2.4 [9]. *Let $s = s_1 \cdot s_2$ and $s' = s'_1 \cdot s'_2 \in \Sigma_{d,\mathbf{1}}$ be such that $s_1, s'_1 \in S_{T_d}$ and the groups $(\mathcal{S}_d)_s$ and $(\mathcal{S}_d)_{s'}$ act transitively on I_d. If $\tau(s) = \tau(s')$ and $\ln(s_2) = \ln(s'_2) \leqslant 2$, then $s = s'$.*

Nevertheless, the following example shows that Theorem 2.4 does not hold even for s, $s' \in \Sigma_{d,\mathbf{1}}^{\mathcal{S}_d}$ if $\ln(s_2) = \ln(s'_2) > 2$.

Example 2.1 [9]. Consider the permutations $\sigma_1 = \sigma'_1 = (1,2,3)(5,6,7,8)$, $\sigma_2 = (1,2)(3,4,5)$, $\sigma_3 = (\sigma_1\sigma_2)^{-1} = (8,7,6,5,4,2,3)$ and $\sigma'_2 = (7,8)(3,4,5)$, $\sigma'_3 = (\sigma'_1\sigma'_2)^{-1} = (8,6,5,4,2,1,3)$ in $\mathcal{S}_8$. Then the elements $s = x_{\sigma_1} \cdot x_{\sigma_2} \cdot x_{\sigma_3}$ and $s' = x_{\sigma'_1} \cdot x_{\sigma'_2} \cdot x_{\sigma'_3} \in \Sigma_{8,\mathbf{1}}^{\mathcal{S}_8}$ have the same type, but $s \neq s'$.

2.5. Factorizations in $\mathcal{S}_3$. Consider the semigroup $\Sigma_{3,\mathbf{1}} \subset \Sigma_3$. The semigroup Σ_3 is generated by the elements $x_{(1,2)}$, $x_{(1,3)}$, $x_{(2,3)}$, $x_{(1,2,3)}$ and $x_{(1,3,2)}$ subject to the relations

$$x_{(1,2)} \cdot x_{(1,3)} = x_{(2,3)} \cdot x_{(1,2)} = x_{(1,3)} \cdot x_{(2,3)}, \tag{2.11}$$

$$x_{(1,3)} \cdot x_{(1,2)} = x_{(2,3)} \cdot x_{(1,3)} = x_{(1,2)} \cdot x_{(2,3)}, \tag{2.12}$$

$$x_{(1,2)} \cdot x_{(1,2,3)} = x_{(1,3,2)} \cdot x_{(1,2)} = x_{(1,3)} \cdot x_{(1,3,2)} = x_{(1,2,3)} \cdot x_{(1,3)}, \tag{2.13}$$

$$x_{(1,2)} \cdot x_{(1,3,2)} = x_{(1,2,3)} \cdot x_{(1,2)} = x_{(2,3)} \cdot x_{(1,2,3)} = x_{(1,3,2)} \cdot x_{(2,3)}, \tag{2.14}$$

$$x_{(2,3)} \cdot x_{(1,3,2)} = x_{(1,2,3)} \cdot x_{(2,3)} = x_{(1,3)} \cdot x_{(1,2,3)} = x_{(1,3,2)} \cdot x_{(1,3)}, \tag{2.15}$$

$$x_{(1,2,3)} \cdot x_{(1,3,2)} = x_{(1,3,2)} \cdot x_{(1,2,3)}. \tag{2.16}$$

We put

$$s_1 = x_{(1,2)}^2, \qquad s_2 = x_{(2,3)}^2, \qquad s_3 = x_{(1,3)}^2, \qquad s_4 = x_{(1,2,3)} \cdot x_{(1,3,2)},$$
$$s_5 = x_{(1,2,3)} \cdot x_{(1,3)} \cdot x_{(2,3)}, \qquad s_6 = x_{(1,2,3)}^3, \qquad s_7 = x_{(1,3,2)}^3.$$

It is easy to see that $s_1, \dots, s_7 \in \Sigma_{3,\mathbf{1}}$.

Theorem 2.5. *The semigroup $\Sigma_{3,\mathbf{1}}$ has the following presentation*:

$$\begin{aligned}\Sigma_{3,\mathbf{1}} = \big\{ s_1, \dots, s_7 \,\big|\, & s_i \cdot s_j = s_j \cdot s_i,\ 1 \leqslant i, j \leqslant 7; \\ & s_i \cdot s_k = s_j \cdot s_k,\ 1 \leqslant i, j \leqslant 3,\ 4 \leqslant k \leqslant 7; \\ & s_i \cdot s_6 = s_i \cdot s_7,\ 1 \leqslant i \leqslant 3; \\ & s_1 \cdot s_2 = s_1 \cdot s_3 = s_2 \cdot s_3; \\ & s_4^3 = s_6 \cdot s_7;\ s_5^2 = s_1^2 \cdot s_4;\ s_5^3 = s_1^3 \cdot s_6; \\ & s_4 \cdot s_5 = s_1 \cdot s_6 = s_1 \cdot s_7 \big\}.\end{aligned}$$

Proof. First let us show that the elements $s_1, \dots, s_7$ generate $\Sigma_{3,\mathbf{1}}$. Indeed, suppose that every element $s \in \Sigma_{3,\mathbf{1}}$ of length $\ln(s) \leqslant k$ can be written as a word in $s_1, \dots, s_7$ and consider an element $s \in \Sigma_{3,\mathbf{1}}$ of length $\ln(s) = k+1$. Moving the factors $x_{(1,2,3)}$ and $x_{(1,3,2)}$ to the left, we can write every element $s \in \Sigma_{3,\mathbf{1}}$ in the form

$$s = x_{(1,2,3)}^a \cdot x_{(1,3,2)}^b \cdot s',$$

where a, b are non-negative integers and s' is a word in the letters $x_{(1,2)}$, $x_{(1,3)}$ and $x_{(2,3)}$.

By Lemmas 2.1 and 2.2, if $\ln(s') \geqslant 3$, then s' can be written in the form $s' = x_{(i,j)}^2 \cdot s''$. If $a \geqslant 3$, $b \geqslant 3$, or both a and b are positive, then we similarly have $s = s_i \cdot \widetilde{s}$, where i is either 6, 7, or 4 and $\widetilde{s} \in \Sigma_{3,\mathbf{1}}$, $\ln(\widetilde{s}) \leqslant k-1$. Thus we only need to consider the cases when $\ln(s') \leqslant 2$ and either $0 \leqslant a \leqslant 2$, $b = 0$, or $a = 0$, $0 \leqslant b \leqslant 2$. If $a = b = 0$, then it is clear that $s' = s_i$ for some $i = 1, 2, 3$ since $s = s' \in \Sigma_{3,\mathbf{1}}$.

Consider the case when $a = 1$ and $b = 0$, that is, $s = x_{(1,2,3)} \cdot s'$. Since $s \in \Sigma_{3,\mathbf{1}}$ and $\alpha(x_{(1,2,3)}) = (1,2,3)$, we have $\alpha(s') = (1,3,2)$. Therefore s' is equal to either $x_{(1,3)} \cdot x_{(2,3)}$, $x_{(2,3)} \cdot x_{(1,2)}$ or $x_{(1,2)} \cdot x_{(1,3)}$. But (2.11) shows that the last three elements are equal to each other and, in this case, $s = s_5$.

If $a=0$ and $b=1$, that is, $s=x_{(1,3,2)}\cdot s'$, then we similarly see that s' is equal to either $x_{(1,2)}\cdot x_{(2,3)}$, $x_{(2,3)}\cdot x_{(1,3)}$ or $x_{(1,3)}\cdot x_{(1,2)}$, and the last three elements are equal to each other by (2.12). Hence we obtain from (2.13) that

$$\begin{aligned} s &= x_{(1,3,2)}\cdot x_{(1,2)}\cdot x_{(2,3)} = x_{(1,2)}\cdot x_{(1,2,3)}\cdot x_{(2,3)}\\ &= x_{(1,2,3)}\cdot x_{(1,3)}\cdot x_{(2,3)} = s_5. \end{aligned}$$

If $a=2$ and $b=0$, that is, $s=x_{(1,2,3)}^2\cdot s'$, then we obtain that $\alpha(s')=(1,2,3)$, whence $s'=x_{(1,2)}\cdot x_{(2,3)}$. Therefore we see from (2.14) that

$$\begin{aligned} s &= x_{(1,2,3)}^2\cdot x_{(1,2)}\cdot x_{(2,3)} = x_{(1,2,3)}\cdot x_{(1,2)}\cdot x_{(1,3,2)}\cdot x_{(2,3)}\\ &= x_{(1,2,3)}\cdot x_{(1,3,2)}\cdot x_{(2,3)}\cdot x_{(2,3)} = s_4\cdot s_2. \end{aligned}$$

Finally, if $a=0$ and $b=2$, that is, $s=x_{(1,3,2)}^2\cdot s'$, then we have $\alpha(s')=(1,3,2)$, whence $s'=x_{(1,3)}\cdot x_{(2,3)}$. Therefore we see from (2.15) that

$$\begin{aligned} s &= x_{(1,3,2)}^2\cdot x_{(1,3)}\cdot x_{(2,3)} = x_{(1,3,2)}\cdot x_{(1,3)}\cdot x_{(1,2,3)}\cdot x_{(2,3)}\\ &= x_{(1,3,2)}\cdot x_{(1,2,3)}\cdot x_{(2,3)}\cdot x_{(2,3)} = s_4\cdot s_2. \end{aligned}$$

As a result, we obtain that $\Sigma_{3,\mathbf{1}}$ is generated by $s_1,\dots,s_7$.

We now wish to verify that the generators $s_1,\dots,s_7$ of $\Sigma_{3,\mathbf{1}}$ satisfy the relations listed in the statement of Theorem 2.5. Since this is done in a similar way in every case, we verify only one of them.

For example, we shall show that $s_4\cdot s_5=s_6\cdot s_1$. By (2.11)–(2.16), we have

$$\begin{aligned} s_4\cdot s_5 &= x_{(1,2,3)}\cdot \underline{x_{(1,3,2)}\cdot x_{(1,2,3)}}\cdot x_{(1,3)}\cdot x_{(2,3)}\\ &= x_{(1,2,3)}\cdot (x_{(1,2,3)}\cdot \underline{x_{(1,3,2)}})\cdot x_{(1,3)}\cdot x_{(2,3)}\\ &= x_{(1,2,3)}\cdot x_{(1,2,3)}\cdot (x_{(1,2,3)}\cdot x_{(2,3)})\cdot x_{(2,3)} = x_{(1,2,3)}\cdot x_{(1,2,3)}\cdot \underline{x_{(1,2,3)}\cdot x_{(2,3)}^2}\\ &= x_{(1,2,3)}\cdot x_{(1,2,3)}\cdot (\underline{x_{(1,3)}^2\cdot x_{(1,2,3)}}) = x_{(1,2,3)}\cdot x_{(1,2,3)}\cdot (x_{(1,2,3)}\cdot x_{(1,2)}^2)\\ &= s_6\cdot s_1. \end{aligned}$$

The fact that the relations listed in Theorem 2.5 are defining is a consequence of the following theorem.

Theorem 2.6. *Every element $s\in\Sigma_{3,\mathbf{1}}$, $s\neq\mathbf{1}$, has a normal form. Namely, it is equal to one and only one element in the following list*:

$$s=\begin{cases} s_i^n, & i=1,2,3,\ n\in\mathbb{N},\\ s_4^a\cdot s_6^m\cdot s_7^n, & 0\leqslant a\leqslant 2,\ m\geqslant 0,\ n\geqslant 0,\ a+m+n>0,\\ s_1^n\cdot s_2, & n\in\mathbb{N},\\ s_1^n\cdot s_6^m, & m,n\in\mathbb{N},\\ s_1^n\cdot s_6^m\cdot s_4, & m\geqslant 0,\ n>0,\\ s_1^n\cdot s_6^m\cdot s_5, & m\geqslant 0,\ n\geqslant 0. \end{cases}$$

Proof. If $s \notin \Sigma_{3,\mathbf{1}}^{\mathcal{S}_3}$, then s is clearly equal to either s_i^n, $i = 1, 2, 3$, or $s_4^a \cdot s_6^m \cdot s_7^n$.

Suppose that $s \in \Sigma_{3,\mathbf{1}}^{\mathcal{S}_3}$. If $s \in S_{T_3,\mathbf{1}}$, then $s = h_{3,g}$ for some g by the Clebsch–Hurwitz theorem.

Suppose that $s = s' \cdot s''$, where $s' = x_{(1,2,3)}^{k_1} \cdot x_{(1,3,2)}^{k_2}$ and $s'' \in S_{T_3}$. Using the relations (2.13)–(2.16), we can assume that $s' = x_{(1,2,3)}^{k}$ for $k = k_1 + k_2$. If $k \equiv 0 \pmod 3$, then the relations in Theorem 2.5 yield that $s = s_1^n \cdot s_6^m$. If $k \equiv 1 \pmod 3$, then $s' = s_6^m \cdot x_{(1,2,3)}$ and $x_{(1,2,3)} \cdot s'' \in \Sigma_{3,\mathbf{1}}$. By Theorem 2.5 we have $x_{(1,2,3)} \cdot s'' = s_5 \cdot s_1^n$ for some $n \geqslant 0$. If $k \equiv 2 \pmod 3$, then we similarly have $s' = s_6^m \cdot x_{(1,2,3)}^2$ and $x_{(1,2,3)}^2 \cdot s'' \in \Sigma_{3,\mathbf{1}}$. Using the relations (2.13)–(2.16), we get $x_{(1,2,3)}^2 \cdot s'' = x_{(1,2,3)} \cdot x_{(1,3,2)} \cdot s_1'' = s_4 \cdot s_1''$ for some $s_1'' \in S_{T_3,\mathbf{1}}$, and the relations in Theorem 2.5 yield that $s = s_1^n \cdot s_4 \cdot s_6^m$.

To complete the proof, we note that different normal forms determine different elements because they have different invariants G_s and $\tau(s) \in \mathbb{Z}^2_{\geqslant o}$.

Theorem 2.7. *Up to simultaneous conjugation, an element $\overline{s} \in \Sigma_3$ is equal either to s, where s is one of the elements of $\Sigma_{3,\mathbf{1}}$ described in Theorem 2.6, or to*

$$\overline{s} = \begin{cases} x_{(1,2)}^{2k+1}, & k \geqslant 0, \\ x_{(1,2,3)}^n \cdot x_{(1,3,2)}^m, & n > m,\ n - m \not\equiv 0 \pmod 3, \\ x_{(1,2)}^n \cdot x_{(2,3)}, & n \in \mathbb{N}, \\ x_{(1,2)}^n \cdot x_{(1,2,3)}^{3m} \cdot x_{(1,3,2)}^a, & n \in \mathbb{N},\ m \geqslant 0,\ a = 0, 1, 2, \\ & \text{and } a \neq 0 \text{ if } n \equiv 0 \pmod 2. \end{cases}$$

Remark 2.2. The elements s_1^n, s_2^n and s_3^n in Theorem 2.6 are conjugate to each other. The elements $s_4^a \cdot s_6^m \cdot s_7^n$ and $s_4^a \cdot s_6^n \cdot s_7^m$ are also conjugate.

Proof of Theorem 2.7. We consider the following cases separately.

1) $(\mathcal{S}_3)_s = \mathcal{S}_2$.

2) $(\mathcal{S}_3)_s = A_3$, where A_3 is the alternating group.

3) $s \in S_{T_3}$, $(\mathcal{S}_3)_s = \mathcal{S}_3$, and $\alpha(s)$ is either a transposition or a cyclic permutation of length 3.

4) $s \notin S_{T_3}$, $(\mathcal{S}_3)_s = \mathcal{S}_3$, and $\alpha(s)$ is either a transposition or a cyclic permutation of length 3.

In cases 1)–3) we easily see that, up to conjugation, s is respectively equal to $x_{(1,2)}^{2k+1}$, $x_{(1,2,3)}^n \cdot x_{(1,3,2)}^m$, $x_{(1,2)}^n \cdot x_{(2,3)}$.

In case 4) we have $s = s_1 \cdot s_2$, where $s_1 \in S_{T_d}$ and s_2 is represented by a word in the letters $x_{(1,2,3)}$ and $x_{(1,3,2)}$. By (2.13) and (2.14) we can assume that $s_1 = x_{(1,2)}^n$. We also have

$$x_{(1,2)} \cdot x_{(1,2,3)}^3 = x_{(1,3,2)}^3 \cdot x_{(1,2)} = x_{(1,2)} \cdot x_{(1,3,2)}^3.$$

Using these relations and (2.16), we obtain that $s = x_{(1,2)}^n \cdot x_\sigma^{3m} \cdot x_{\sigma^{-1}}^a$, where $\sigma = (1, 2, 3)$ or $\sigma = (1, 3, 2)$. To complete the proof, we note that $\lambda((1, 2))(x_\sigma) = x_{\sigma^{-1}}$.

Corollary 2.4. *Suppose that $(\mathcal{S}_3)_s = \mathcal{S}_2$ or $(\mathcal{S}_3)_s = \mathcal{S}_3$ for $s \in \Sigma_3$. Then s is uniquely determined (up to simultaneous conjugation) by its type $\tau(s)$ and the*

type $t(\alpha(s))$ *of its image* $\alpha(s) \in \mathcal{S}_3$. *Up to simultaneous conjugation, there are exactly* $[\frac{n}{6}]+1$ *different elements* $s \in \Sigma^{A_3}_{3,\mathbf{1}}$ *of length* $\ln(s) = n$ *if* $n \not\equiv 1 \pmod 6$. *If* $n \equiv 1 \pmod 6$, *then there are exactly* $[\frac{n}{6}]$ *different elements* $s \in \Sigma^{A_3}_{3,\mathbf{1}}$ *of length* $\ln(s) = n$. *If* $\alpha(s) \neq \mathbf{1}$, *then there are exactly* $m = -[\frac{-n}{3}]$ *different elements* $s \in \Sigma^{A_3}_3$ *of length* $\ln(s) = n$.

2.6. The Cayley embedding. It is well known that every finite group G can be embedded in some symmetric group. In particular, if $N = |G|$ is the order of G, then we have the Cayley embedding $c\colon G \hookrightarrow \mathrm{Sym}(G) \simeq \mathcal{S}_N$:

$$(g_1)\sigma_g = g_1 g, \qquad g, g_1 \in G, \quad c(g) = \sigma_g,$$

that is, G acts on itself by right multiplication. We identify the group G with its image $c(G)$ and denote the normalizer and centralizer of G in $\mathcal{S}_N$ by $N(G)$ and $C(G)$ respectively. Since $N(G)$ acts on G by conjugation, we have a natural homomorphism $a\colon N(G) \to \mathrm{Aut}(G)$.

Theorem 2.8. *Let* $c\colon G \hookrightarrow \mathrm{Sym}(G) \simeq \mathcal{S}_N$ *be the Cayley embedding of a finite group* G. *Then the natural homomorphism* $a\colon N(G) \to \mathrm{Aut}(G)$ *has the following properties*:

(i) a *is an epimorphism*,

(ii) $\ker a = C(G) \simeq G$,

(iii) *the group generated by* G *and* $C(G)$ *is isomorphic to the amalgamated direct product* $G \times_C G$, *where* C *is the centre of* G.

Proof. We regard an automorphism $f \in \mathrm{Aut}(G)$ as a permutation $\sigma_f \in \mathcal{S}_N$ of the elements of G:

$$(g)\sigma_f = f(g), \qquad g \in G.$$

We claim that $\sigma_f \in N(G)$. Indeed, for all $g_1 \in G$ we have

$$\begin{aligned}(g_1)\sigma_f^{-1}\sigma_g\sigma_f &= (f^{-1}(g_1))\sigma_g\sigma_f = (f^{-1}(g_1)g)\sigma_f \\ &= f(f^{-1}(g_1)g) = g_1 f(g) = (g_1)\sigma_{f(g)},\end{aligned}$$

that is, $\sigma_f^{-1}\sigma_g\sigma_f = \sigma_{f(g)} \in G$ for all $g \in G$. Hence $\sigma_f \in N(G)$ and, moreover, the conjugation of elements of G by σ_f determines an automorphism f of G. Therefore a is an epimorphism.

Clearly, $C(G) = \ker a$. Consider an element $\sigma \in C(G)$. We have $\sigma_g\sigma = \sigma\sigma_g$ for all $g \in G$. Therefore,

$$(g_1)\sigma_g\sigma = (g_1 g)\sigma = ((g_1)\sigma)\cdot g$$

for all $g_1, g \in G$. In particular, if we take $g_1 = \mathbf{1}$ and denote $(\mathbf{1})\sigma$ by g_σ, then we have

$$(\mathbf{1})\sigma_g\sigma = (g)\sigma = g_\sigma g$$

for all $g \in G$. The equality $(g)\sigma = g_\sigma g$ shows that σ acts on G as left multiplication by $g_\sigma \in G$. Clearly, the left and right multiplications by elements of G commute. Therefore $C(G) \simeq G$.

We recall that, by definition, G acts on itself by right multiplication. Hence we easily see that the group generated by G and $C(G)$ is isomorphic to the amalgamated direct product $G \times_C G$, where C is the centre of G.

Every embedding $G \hookrightarrow \mathcal{S}_d$ determines an embedding of semigroups $S(G,O) \hookrightarrow \Sigma_d$. Let $c\colon S_G = S(G,G) \hookrightarrow \Sigma_d$ be the embedding of semigroups induced by the Cayley embedding $c\colon G \to \mathcal{S}_N$. Theorem 2.8 has the following corollary.

Corollary 2.5. *The orbits of the conjugation action of $\mathcal{S}_N$ on Σ_N intersecting the semigroup $S(G,G)$ are in one-to-one correspondence with the orbits of the action of $\mathrm{Aut}(G)$ on $S(G,G)$.*

§ 3. Hurwitz spaces

3.1. Marked Riemann surfaces. Let $f\colon C \to D_R = \{z \in \mathbb{C} \mid |z| \leqslant R\}$ be a Riemann surface, that is, a finite proper continuous ramified covering of the disc $D_R = \{|z| \leqslant R\}$ (or the projective line $\mathbb{P}^1$ if $R = \infty$) of degree d branched at finitely many points of $D_R^0 = D_R \setminus \partial D_R = \{|z| < R\}$ (we do not assume that C is connected). Two coverings (C', f') and (C'', f'') of D_R are said to be *isomorphic* if there is an orientation-preserving homeomorphism $h\colon C' \to C''$ such that $f' = h \circ f''$. They are said to be *equivalent* if there are orientation-preserving homeomorphisms $\psi\colon D_R \to D_R$ and $\varphi\colon C' \to C''$ such that ψ leaves all points of the boundary ∂D_R fixed and $\psi \circ f' = f'' \circ \varphi$. We denote the set of equivalence classes of coverings of degree d over D_R with respect to this equivalence relation by $\mathcal{R}_{R,d}$.

Let $q_1, \dots, q_b \in D_R^0$ be the points over which f is ramified. We fix a point $o = o_R = e^{\frac{3}{2}\pi i} R \in \partial D_R$ (if $R = \infty$, then we assume by definition that $o_\infty = \infty = \mathbb{P}^1 \setminus \mathbb{C}$) and enumerate the points in $f^{-1}(o)$. This enumeration induces an ordering of the set $f^{-1}(o)$. Such coverings (C, f) with a fixed point $o \in D_R$ and a fixed ordering of $f^{-1}(o)$ are called *coverings with an ordered set of sheets* or *marked coverings*. We say that two marked coverings $(C', f')_m$ and $(C'', f'')_m$ are *equivalent* if there are orientation-preserving homeomorphisms $\psi\colon D_R \to D_R$ and $\varphi\colon C' \to C''$ with the following properties:

(i) ψ fixes the points of ∂D_R,

(ii) $\varphi(p'_i) = p''_i \in (f'')^{-1}(o)$ for each $p'_i \in (f')^{-1}(o)$, $i = 1, \dots, d$,

(iii) $\psi \circ f' = f'' \circ \varphi$.

We denote the set of equivalence classes of marked coverings of degree d over D_R with respect to this equivalence relation by $\mathcal{R}^m_{R,d}$. Renumbering the sheets determines an action of the symmetric group $\mathcal{S}_d$ on $\mathcal{R}^m_{R,d}$. It is easy to see that $\mathcal{R}_{R,d} = \mathcal{R}^m_{R,d}/\mathcal{S}_d$.

If $R_1 < R_2 < \infty$, then every ramified covering $f\colon C \to D_{R_1}$ can be extended to a ramified covering $\tilde{f}\colon \widetilde{C} \to D_{R_2}$ which is unramified over $D_{R_2} \setminus D_{R_1}$. Lifting the path

$$l(t) = e^{\frac{3}{2}\pi i}(R_2 t + (1-t)R_1) \subset D_{R_2} \setminus D^0_{R_1}, \qquad t \in [0,1],$$

to $\widetilde{C}$, we get d paths $\tilde{f}^{-1}(l(t))$ connecting the points of $f^{-1}(o_{R_1})$ with points of $f^{-1}(o_{R_2})$. If $(C, f)_m$ is a marked covering, then these paths transfer the ordering from $f^{-1}(o_{R_1})$ to $f^{-1}(o_{R_2})$. As a result, we obtain an isomorphism $i_{R_1,R_2}\colon \mathcal{R}^m_{R_1,d} \hookrightarrow \mathcal{R}^m_{R_2,d}$.

For every marked covering $(C, f)_m$ of the projective line $\mathbb{P}^1$ and every $R > 0$, we can similarly find an equivalent covering $(\overline{C}, \bar{f})_m$ whose branch points belong

to D_R^0. Consider the restriction $\tilde{f}$ of the covering $\bar{f}$ to $\widetilde{C} = \bar{f}^{-1}(D_R)$. Lifting the path

$$l(t) = e^{\frac{3}{2}\pi i} R/t \subset \mathbb{P}^1 \setminus D_R^0, \qquad t \in [0,1],$$

to $\overline{C}$, we get d paths $\bar{f}^{-1}(l(t))$ connecting the points of $f^{-1}(o_\infty)$ with points of $f^{-1}(o_R)$. They transfer the ordering from $\bar{f}^{-1}(o_\infty)$ to $\tilde{f}^{-1}(o_R)$. Clearly, the equivalence class of the resulting marked coverings $(\widetilde{C}, \tilde{f})_m$ is independent of the choice of a representative $(\overline{C}, \bar{f})_m$. Therefore we obtain an embedding $i_{\infty,R}\colon \mathcal{R}^m_{\infty,d} \hookrightarrow \mathcal{R}^m_{R,d}$. It is easy to see that $i_{\infty,R_2} = i_{R_1,R_2} \circ i_{\infty,R_1}$ for all $R_2 \geqslant R_1 > 0$.

3.2. Semigroups of marked coverings. A closed loop $\gamma \subset D_R \setminus \{q_1, \dots, q_b\}$ starting and ending at $o = o_R$ can be lifted to C by means of f. We get d paths starting and ending at the points of $f^{-1}(o)$. This lift of loops determines a homomorphism (the *monodromy of marked coverings*) $\mu\colon \pi_1(D_R \setminus \{q_1, \dots, q_b\}, o) \to \mathcal{S}_d$ to the symmetric group $\mathcal{S}_d$ (the monodromy sends the starting points of the lifted paths to the endpoints of the corresponding paths). Conversely, every homomorphism $\mu\colon \pi_1(D_R \setminus \{q_1, \dots, q_b\}, o) \to \mathcal{S}_d$ determines a marked covering $f\colon C \to D$ whose monodromy is μ.

The fundamental group $\pi_1(D_R \setminus \{q_1, \dots, q_b\}, o)$ is generated by loops $\gamma_1, \dots, \gamma_b$ of the following form. Each loop γ_i consists of a path l_i starting at o and ending at a point q_i' close to q_i followed by a circuit in the positive direction (with respect to the complex orientation on $\mathbb{C}$) along a circle Γ_i of small radius with centre at q_i, $q_i' \in \Gamma$, followed by a return to q_0 along the path l_i in the opposite direction. For $i \neq j$ the loops γ_i and γ_j have only one common point, namely, o. The product $\gamma_1 \dots \gamma_b$ is equal to ∂D_R in the group $\pi_1(D_R \setminus \{q_1, \dots, q_b\}, o)$. Such a set of generators is called a *good geometric base* of the group $\pi_1(D_R \setminus \{q_1, \dots, q_b\}, o)$. It is well known that if $R < \infty$, then $\gamma_1, \dots, \gamma_b$ are free generators of $\pi_1(D_R \setminus \{q_1, \dots, q_b\}, o)$, that is, $\pi_1(D_R \setminus \{q_1, \dots, q_b\}, o) = \langle \gamma_1, \dots, \gamma_b \rangle$. If $R = \infty$, then $\gamma_1, \dots, \gamma_b$ generate the group $\pi_1(\mathbb{P}^1 \setminus \{q_1, \dots, q_b\}, o)$ and are subject to the relation $\gamma_1 \dots \gamma_b = \mathbf{1}$.

If we choose a good geometric base $\gamma_1, \dots, \gamma_b$, then the monodromy μ is determined by the set of elements $\sigma_1 = \mu(\gamma_1), \dots, \sigma_n = \mu(\gamma_b) \in \mathcal{S}_d$, which are called *local monodromies*. The product $\sigma = \sigma_1 \dots \sigma_b = \mu(\partial D)$ is called the *global monodromy* of f. We easily see that if $R = \infty$, then the global monodromy is equal to $\mathbf{1}$.

The set $(\sigma_1, \dots, \sigma_b)$ depends on the choice of a good geometric base $\gamma_1, \dots, \gamma_b$. Any good geometric base may be obtained from $\gamma_1, \dots, \gamma_b$ by a finite sequence of Hurwitz moves. In other words, the braid group Br_b acts naturally on the set of good geometric bases of $\pi_1(D_R \setminus \{q_1, \dots, q_b\}, o)$ by means of Hurwitz moves [10]. Therefore if $(\sigma_1', \dots, \sigma_b')$ is the set corresponding to another good geometric base $\gamma_1', \dots, \gamma_b'$, then $(\sigma_1', \dots, \sigma_b')$ can be obtained from $(\sigma_1, \dots, \sigma_b)$ by a finite sequence of Hurwitz moves (see § 1.3).

Suppose that $R < \infty$. One can define the structure of a semigroup on $\mathcal{R}^m_{R,d}$ as follows. Let $(C_1, f_1)_m$ and $(C_2, f_2)_m$ be two marked coverings of degree d. We choose two continuous orientation-preserving embeddings $\varphi_j\colon D_R \to D_R$, $j = 1, 2$, of the disc D_R into itself which fix the point o and have the following properties.

(i) The image $\varphi_1(D_R) = \{u \in D_R \mid \operatorname{Re} u \geqslant 0\}$ is the right half-disc and $\varphi_1(\{u \in \partial D_R \mid \operatorname{Re} u \leqslant 0\}) = \{u \in D_R \mid \operatorname{Re} u = 0\}$ is the vertical diameter.

(ii) $\varphi_2(D_R) = \{u \in D_R \mid \operatorname{Re} u \leqslant 0\}$ is the left half-disc and $\varphi_2(\{u \in \partial D_R \mid \operatorname{Re} u \geqslant 0\}) = \{u \in D_R \mid \operatorname{Re} u = 0\}$.

We identify the points belonging to the sets $f_1^{-1}(o)$ and $f_2^{-1}(o)$ using the orderings of these sets. Then we identify, by continuity, the points belonging to the d paths $f_1^{-1}(\{u \in \partial D_R \mid \operatorname{Re} u \leqslant 0\})$ in C_1 with the points belonging to the d paths $f_2^{-1}(\{u \in \partial D_R \mid \operatorname{Re} u \geqslant 0\})$ in C_2 in such a way that the images of the identified points under $\varphi_1 \circ f_1$ and $\varphi_2 \circ f_2$ coincide. These identifications enable us to glue the surfaces C_1 and C_2 along these d paths. As a result, we obtain a marked covering $(C, f)_m$, where $f(q) = \varphi_1(f_1(q))$ if $q \in C_1$ and $f(q) = \varphi_2(f_2(q))$ if $q \in C_2$. The resulting covering $(C, f)_m$ is called the *product* of the marked coverings $(C_1, f_1)_m$ and $(C_2, f_2)_m$ (we write $(C, f)_m = (C_1, f_1)_m \cdot (C_2, f_2)_m$). We easily see that this notion of product makes $\mathcal{R}^m_{R,d}$ a non-commutative semigroup such that the maps i_{R_1,R_2} are isomorphisms of semigroups for all $R_1 \geqslant R_2 > 0$.

Clearly, the semigroup $\mathcal{R}^m_d = \mathcal{R}^m_{R,d}$ is generated by those marked coverings $(C, f)_m$ that are coverings of $D = D_R$ with only one branch point q_1. Such coverings are uniquely determined (up to equivalence) by their global monodromy $\sigma_f = \mu(\partial D) \in \mathcal{S}_d$, where $\mu = \mu_f$ is the monodromy of the marked covering $(C, f)_m$. Therefore the number of generators is equal to $d!$. Let x_{σ_f} be the generator of the semigroup $\mathcal{R}_d$ corresponding to a covering $(C, f)_m$ with a single branch point. A simple inspection shows that the generators x_σ satisfy the following defining relations in the semigroup $\mathcal{R}^m_d$:

$$x_{\sigma_1} \cdot x_{\sigma_2} = x_{\sigma_2} \cdot x_{(\sigma_2^{-1}\sigma_1\sigma_2)}, \qquad x_{\sigma_1} \cdot x_{\sigma_2} = x_{(\sigma_1\sigma_2\sigma_1^{-1})} \cdot x_{\sigma_1},$$

and $x_{\sigma_1} \cdot x_{\mathbf{1}} = x_{\sigma_1}$, $x_{\mathbf{1}} \cdot x_{\sigma_2} = x_{\sigma_2}$ for all $\sigma_1, \sigma_2 \in \mathcal{S}_d$.

If a marked covering $(C, f)_m$ is equal to $x_{\sigma_1} \cdot \ldots \cdot x_{\sigma_n}$ in $\mathcal{R}^m_d$, then it is easy to see that its global monodromy $\sigma_f = \mu(\partial D)$ is equal to $\sigma_1 \ldots \sigma_n$. Clearly, sending each marked covering to its global monodromy determines a homomorphism from $\mathcal{R}^m_d$ to the symmetric group $\mathcal{S}_d$. We denote this homomorphism by $\alpha\colon \mathcal{R}^m_d \to \mathcal{S}_d$.

A renumbering of the sheets of a marked covering determines an action of the group $\mathcal{S}_d$ on $\mathcal{R}^m_d$. Namely, an element $\sigma_0 \in \mathcal{S}_d$ acts on the generators x_σ by the rule $x_\sigma \mapsto x_{(\sigma_0^{-1}\sigma\sigma_0)}$. This action determines a homomorphism $\lambda\colon \mathcal{S}_d \to \operatorname{Aut}(\mathcal{R}^m_d)$. Thus we get the following proposition.

Proposition 3.1. *As a semigroup over $\mathcal{S}_d$, $\mathcal{R}^m_d$ is naturally isomorphic to Σ_d.*

In accordance with Proposition 3.1, the elements of Σ_d will be referred to as *monodromy factorizations* of coverings of degree d.

The kernel $\ker\alpha = \mathcal{R}^m_{d,\mathbf{1}} = \{(C, f)_m \in \mathcal{R}^m_d \mid \sigma_f = \mathbf{1}\}$ is easily seen to be a subsemigroup of $\mathcal{R}^m_d$ isomorphic to $\Sigma_{d,\mathbf{1}}$. If the disc D is embedded in $\mathbb{P}^1$, then the elements of $\mathcal{R}^m_{d,\mathbf{1}}$ are those marked coverings $f\colon C \to D$ that can be extended to marked coverings $\tilde{f}\colon \widetilde{C} \to \mathbb{CP}^1$ that are unramified over $\mathbb{P}^1 \setminus D$. We note that such an extension $\tilde{f}\colon \widetilde{C} \to \mathbb{CP}^1$ of a marked covering $f\colon C \to D$ with global monodromy $\mu_f(\partial D) = \mathbf{1}$ is unique up to equivalence.

The converse is also true: the image of $\mathcal{R}^m_{\infty,d}$ under the embedding $i_{\infty,R}$ coincides with $\mathcal{R}^m_{d,\mathbf{1}}$. In what follows we identify the set $\mathcal{R}^m_{\infty,d}$ with the semigroup $\mathcal{R}^m_{d,\mathbf{1}}$ by means of this isomorphism. As a result, we have the following proposition.

Proposition 3.2. *The set of equivalence classes of marked coverings of degree d over $\mathbb{P}^1$ possesses the natural structure of a semigroup isomorphic to $\Sigma_{d,\mathbf{1}}$.*

3.3. Hurwitz spaces of marked Riemann surfaces. In this subsection we describe the Hurwitz space $\mathrm{HUR}^m_d(D)$ of marked ramified degree d coverings of the disc $D = D_R$ up to isomorphism. The space $\mathrm{HUR}^m_d(D) = \bigsqcup_{b=0}^{\infty} \mathrm{HUR}^m_{d,b}(D)$ is the disjoint union of the spaces of coverings branched at b points, $b \in \mathbb{N}$.

As in [3], we consider the symmetric product of b copies of the open disc $D^0 = D \setminus \partial D$ and denote it by $D^{(b)}$. This is the complex manifold of dimension b obtained as the quotient of the Cartesian product $D^b = D^0 \times \cdots \times D^0$ (b factors) by the action of $\mathcal{S}_b$ that permutes the factors. We identify the points of $D^{(b)}$ with unordered b-tuples of points of D^0. Those b-tuples that contain less than b distinct points form the *discriminant locus* Δ of $D^{(b)}$.

Given a point $B_0 = \{q_{1,0}, \dots, q_{b,0}\} \in D^{(b)} \setminus \Delta$, we fix an ordering of the set $B_0 = \{q_{1,0}, \dots, q_{b,0}\} \subset D$ and choose a good geometric base $\gamma_1, \dots, \gamma_b$ in the group $\pi_1(D \setminus B_0, o)$. Then every word w in the set W_b of words of length b in the letters x_σ, $\sigma \in \mathcal{S}_d$, determines a marked covering $f = f_w \colon C \to D$ branched over B_0. Its monodromy μ is such that $\mu(\gamma_i) = \sigma_i$, where x_{σ_i} is the letter in the ith place of w.

The choice of a good geometric base enables us to choose the standard generators $a_1, \dots, a_{b-1}$ of the group $\pi_1(D^{(b)} \setminus \Delta, B_0) \simeq \mathrm{Br}_b$ in such a way that we get an action of the group Br_b on the set of words W_b (see § 1.3). In other words, this choice determines a homomorphism $\theta_{d,b,R} \colon \pi_1(D^{(b)} \setminus \Delta, B_0) \simeq \mathrm{Br}_b \to \mathcal{S}_N$, where $N = (d!)^b$.

The homomorphism $\theta_{d,b,R}$ enables us to define the space $\mathrm{HUR}^m_{d,b}(D)$ as the unbranched covering $h_{d,b,R} \colon \mathrm{HUR}^m_{d,b}(D) \to D^{(b)} \setminus \Delta$ associated with $\theta_{d,b,R}$. Indeed, if we fix a marked covering $f \colon C \to D$ whose monodromy μ satisfies $\mu(\gamma_i) = \sigma_i$, then every path $\delta(t)$, $0 \leqslant t \leqslant 1$, in $D^{(b)}$ starting at B_0 can be lifted to D and we get b paths $\delta_i(t)$ in D starting at the points $q_{1,0}, \dots, q_{b,0}$. These paths determine (up to isotopy) a continuous family of homeomorphisms $\overline{\delta}_t \colon D \setminus B_0 \to D \setminus \{\delta_1(t), \dots, \delta_b(t)\}$ leaving the points of ∂D fixed and satisfying $\overline{\delta}_0 = \mathrm{Id}$. This family of homeomorphisms determines a continuous family of marked coverings $f_t \colon C_t \to D$ branched at $\delta_1(t), \dots, \delta_b(t)$ and having monodromy μ_t with $\mu_t(\overline{\delta}_{t*}(\gamma_i)) = \sigma_i$. Clearly, if $\delta(t)$ is a closed path, then the b-tuple $(\mu_1(\gamma_1), \dots, \mu_1(\gamma_b))$ is Hurwitz-equivalent to the b-tuple $(\mu_0(\gamma_1), \dots, \mu_0(\gamma_b))$. It follows that *the points of the covering space $\mathrm{HUR}^m_{d,b}(D)$ of the covering $h_{d,b,R} \colon \mathrm{HUR}^m_{d,b}(D) \to D^{(b)} \setminus \Delta$ naturally parametrize all marked coverings of D of degree d branched at b points.* The degree of the covering $h_{d,b,R}$ is equal to $(d!)^b$. As a result, we obtain the following proposition.

Proposition 3.3. *The irreducible components of $\mathrm{HUR}^m_{d,b}(D)$ are in one-to-one correspondence with the elements s of the semigroup Σ_d of length $\ln(s) = b$. The set of irreducible components of $\mathrm{HUR}^m_d(D)$ has the natural structure of a semigroup isomorphic to $\mathcal{R}_d \simeq \Sigma_d$.*

If $R_2 \geqslant R_1 > 0$, then we have an embedding $D^{(b)}_{R_1} \hookrightarrow D^{(b)}_{R_2}$, and it is easy to see that the restriction of h_{d,b,R_2} to $h^{-1}_{d,b,R_2}(D^{(b)}_{R_1} \setminus \Delta)$ can be identified with the covering $h_{d,b,R_1}\colon \mathrm{HUR}^m_{d,b}(D_{R_1}) \to D^{(b)}_{R_1} \setminus \Delta$ by means of i_{R_1,R_2}.

In accordance with Proposition 3.3, we shall write $\mathrm{HUR}^m_{d,s}(D)$ for the irreducible component of $\mathrm{HUR}^m_{d,\ln(s)}(D)$ corresponding to an element $s \in \Sigma_d$. In particular, the global monodromy $\sigma_f = \mu(\partial D) = \alpha(s) \in \mathcal{S}_d$ is an invariant of the irreducible component $\mathrm{HUR}^m_{d,s}(D)$. We put

$$\mathrm{HUR}^m_{d,b,\sigma}(D) = \bigcup_{\substack{\alpha(s)=\sigma \\ \ln(s)=b}} \mathrm{HUR}^m_{d,s}(D).$$

It follows from the above considerations that

$$\mathrm{HUR}^m_{d,b}(\mathbb{P}^1) = \bigcup_{R>0} \mathrm{HUR}^m_{d,b,\mathbf{1}}(D_R).$$

For a fixed type t of elements $s \in \Sigma_d$ we put

$$\mathrm{HUR}^m_{d,t}(D) := \bigcup_{\tau(s)=t} \mathrm{HUR}^m_{d,s}(D)$$

and

$$\mathrm{HUR}^m_{d,t,\sigma}(D) = \mathrm{HUR}^m_{d,t}(D) \cap \mathrm{HUR}^m_{d,\sigma}(D).$$

As mentioned above, every marked covering $f\colon C \to D$ of degree d branched at the points $q_1, \dots, q_b$ determines (and is in turn determined by) the monodromy $\mu\colon \pi_1(D \setminus \{q_1, \dots, q_b\}) \to \mathcal{S}_d$. The image $\mu(\pi_1(D \setminus \{q_1, \dots, q_b\})) = \mathrm{Gal}(f) \subset \mathcal{S}_d$ is called the *Galois group* of the covering f. It is easy to see that $\mathrm{Gal}(f) = (\mathcal{S}_d)_s$ if f belongs to $\mathrm{HUR}^m_{d,s}(D)$. The covering space C of a marked covering $(C, f)_m$ is connected if and only if the Galois group $\mathrm{Gal}(f)$ acts transitively on the set $I_d = [1, d]$.

Let $\mathrm{HUR}^{m,G}_d(D)$ be the union of the irreducible components of $\mathrm{HUR}^m_d(D)$ consisting of the coverings with Galois group $\mathrm{Gal}(f) = G \subset \mathcal{S}_d$. We also put

$$\mathrm{HUR}^{m,G}_{d,t}(D) = \mathrm{HUR}^{m,G}_d(D) \cap \mathrm{HUR}^m_{d,t}(D),$$
$$\mathrm{HUR}^{m,G}_{d,t,\sigma}(D) = \mathrm{HUR}^{m,G}_{d,t}(D) \cap \mathrm{HUR}^m_{d,t,\sigma}(D).$$

By Corollary 2.2 we have the following theorem.

Theorem 3.1. *Suppose that the type t of a monodromy factorization contains k transpositions. If $k \geqslant 3(d-1)$, then each irreducible component of $\mathrm{HUR}^{m,\mathcal{S}_d}_{d,t}(D)$ is uniquely determined by the global monodromy $\sigma_f = \mu(\partial D) \in \mathcal{S}_d$ of a covering $(C, f)_m$ belonging to this irreducible component.*

3.4. Hurwitz spaces of (unmarked) coverings of the disc. To obtain the Hurwitz space $\mathrm{HUR}_{d,b}(D)$ of coverings of the disc $D = D_R$ of degree d branched over b points lying in D^0, we must identify all marked coverings of D that differ only in the enumeration of the sheets. Renumbering the sheets induces an action

of $\mathcal{S}_d$ on the marked fibres. We recall that the actions of Br_b and $\mathcal{S}_d$ on W_b commute. Therefore this action of $\mathcal{S}_d$ induces an action of $\mathcal{S}_d$ on $\mathrm{HUR}^m_{d,b}(D)$, and we obtain that the space $\mathrm{HUR}_{d,b}(D)$ is the quotient space with respect to this action: $\mathrm{HUR}_{d,b}(D) = \mathrm{HUR}^m_{d,b}(D)/\mathcal{S}_d$. This yields the following proposition.

Proposition 3.4. *The irreducible components of* $\mathrm{HUR}_{d,b}(D)$ *are in one-to-one correspondence with the orbits of the action of* $\mathcal{S}_d$ *by simultaneous conjugation on* $\Sigma_{d,b} = \{s \in \Sigma_d \mid \ln(s) = b\}$.

If $f\colon C \to D$ is an unmarked covering, then we can also define the Galois group as $\mathrm{Gal}(f) = (\mathcal{S}_d)_s$. However, in this case the subgroup $\mathrm{Gal}(f)$ of the symmetric group $\mathcal{S}_d$ is uniquely determined only up to inner automorphisms of $\mathcal{S}_d$.

In what follows we write $\mathrm{HUR}_{\cdot,\cdot,\cdot}(D)$ (resp. $\mathrm{HUR}^G_{\cdot,\cdot,\cdot}(D)$) for the images of the subspaces $\mathrm{HUR}^m_{\cdot,\cdot,\cdot}(D)$ (resp. $\mathrm{HUR}^{m,G}_{\cdot,\cdot,\cdot}(D)$) of the space $\mathrm{HUR}^m_{d,b}(D)$ under the canonical map

$$\mathrm{HUR}^m_{d,b}(D) \to \mathrm{HUR}_{d,b}(D) = \mathrm{HUR}^m_{d,b}(D)/\mathcal{S}_d.$$

In particular, we have $\mathrm{HUR}_{d,s_1}(D) = \mathrm{HUR}_{d,s_2}(D)$ if and only if there is a permutation $\sigma \in \mathcal{S}_d$ such that $\lambda(\sigma)(s_1) = s_2$.

Corollary 2.4 gives us a complete description of the irreducible components of $\mathrm{HUR}_{d,b}(D)$ in the case $d = 3$.

Corollary 3.1. *If* $G \simeq \mathcal{S}_2$ *or* $G \simeq \mathcal{S}_3$, *then the irreducible components of* $\mathrm{HUR}^G_{3,b}(D)$ *are uniquely determined by the monodromy factorization type and the type of the global monodromy. If the global monodromy is equal to* $\mathbf{1}$, *then the space* $\mathrm{HUR}^{A_3}_{3,b}(D)$ *consists of* $m = [\frac{b}{6}]+1$ *irreducible components when* $b \not\equiv 1 \pmod 6$ *and* $[\frac{b}{6}]$ *irreducible components when* $b \equiv 1 \pmod 6$. *The space* $\mathrm{HUR}^{A_3}_{3,b}(D)$ *consists of* $m = -[\frac{-b}{3}]$ *irreducible components if the global monodromy is not equal to* $\mathbf{1}$.

3.5. Hurwitz spaces of (unmarked) coverings of $\mathbb{P}^1$. In [3], the Hurwitz spaces $\mathrm{HUR}_{d,b}(\mathbb{P}^1)$ of coverings of $\mathbb{P}^1$ of degree d branched over b points were described as unramified coverings of the complement of the discriminant locus Δ in the symmetric product $\mathbb{P}^{(b)}$ of b copies of $\mathbb{P}^1$. The choices of a point $\infty \in \mathbb{P}^1$ and of an identification of $\mathbb{C}$ with $\mathbb{P}^1 \setminus \{\infty\}$ determines an embedding of $\mathrm{HUR}_{d,b}(D_\infty)$ in $\mathrm{HUR}_{d,b}(\mathbb{P}^1)$ as an open dense subset. Hence we get the following proposition.

Proposition 3.5. *The irreducible components of* $\mathrm{HUR}_{d,b}(\mathbb{P}^1)$ *are in one-to-one correspondence with orbits of the action of* $\mathcal{S}_d$ *by simultaneous conjugation on the set* $\Sigma_{d,\mathbf{1},\mathbf{b}} = \{s \in \Sigma_{d,\mathbf{1}} \mid \ln(s) = b\}$.

As in § 3.4, we can introduce the unions $\mathrm{HUR}_{\cdot,\cdot,\cdot}(\mathbb{P}^1)$ (resp. $\mathrm{HUR}^G_{\cdot,\cdot,\cdot}(\mathbb{P}^1)$) of the irreducible components of $\mathrm{HUR}_{d,b}(\mathbb{P}^1)$ for fixed elements of $\Sigma_{b,\mathbf{1}}$, fixed types of monodromy factorizations, fixed Galois groups, and so on.

The following theorem is a consequence of Proposition 1.1.

Theorem 3.2. *The set of irreducible components of the Hurwitz space* $\mathrm{HUR}^{\mathcal{S}_d}_d(\mathbb{P}^1)$ *has the natural structure of a semigroup* $\Sigma^{\mathcal{S}_d}_{d,\mathbf{1}} = \{s \in \Sigma_{d,\mathbf{1}} \mid (\mathcal{S}_d)_s = \mathcal{S}_d\}$.

Theorems 2.3, 2.4 and Corollary 2.4 give us the following theorems.

Theorem 3.3. *The space* $\mathrm{HUR}_{d,t}^{\mathcal{S}_d}(\mathbb{P}^1)$ *is irreducible if the monodromy representation type* t *contains at least* $3(d-1)$ *transpositions.*

Theorem 3.4 [9]. *Let* G *be a transitive subgroup of the symmetric group* $\mathcal{S}_d$. *The Hurwitz space* $\mathrm{HUR}_{d,t}^{G}(\mathbb{P}^1)$ *is irreducible if the monodromy factorization type* t *contains at least* $l-2$ *transpositions, where* l *is the length of* t *(in other words,* l *is the number of branch points of the covering).*

Theorem 3.5. *If* $G \simeq \mathcal{S}_2$ *or* $G \simeq \mathcal{S}_3$, *then the irreducible components of* $\mathrm{HUR}_{3,b}^{G}(\mathbb{P}^1)$ *are uniquely determined by their monodromy factorization type. The space* $\mathrm{HUR}_{3,b}^{A_3}(\mathbb{P}^1)$ *consists of* $m = [\frac{b}{6}]+1$ *irreducible components when* $b \not\equiv 1 \pmod 6$ *and* $[\frac{b}{6}]$ *irreducible components when* $b \equiv 1 \pmod 6$.

3.6. Hurwitz spaces of Galois coverings. Let $f\colon C \to \mathbb{P}^1$ be a Galois covering with Galois group $G = \mathrm{Gal}(C/\mathbb{P}^1)$, that is, G is the deck transformation group of f and the quotient space C/G coincides with $\mathbb{P}^1$. In this case we have $\deg f = |G|$ and if we fix a point $\infty \in \mathbb{P}^1$ over which f is not ramified and fix a point $e \in f^{-1}(\infty)$, then the action of G on $f^{-1}(\infty)$ determines an enumeration of the points of $f^{-1}(\infty)$ by the elements of G. If we enumerate the points of $f^{-1}(\infty)$ by the integers in the closed interval $I_{|G|} = [1, |G|]$, then these enumerations determine an embedding $G \hookrightarrow \mathcal{S}_{|G|}$. We easily see that this is the Cayley embedding. Hence the Hurwitz space $\mathrm{HUR}^{G}(\mathbb{P}^1)$ of Galois coverings with Galois group G can be identified with the space $\mathrm{HUR}_{|G|,\mathbf{1}}^{G}(\mathbb{P}^1)$. In particular, the natural map

$$\mathrm{HUR}_{|G|,\mathbf{1}}^{m,G}(\mathbb{P}^1) \to \mathrm{HUR}_{|G|,\mathbf{1}}^{G}(\mathbb{P}^1) = \mathrm{HUR}^{G}(\mathbb{P}^1) \tag{3.1}$$

is a surjective unramified morphism.

Theorem 3.6. *The irreducible components of* $\mathrm{HUR}^{G}(\mathbb{P}^1)$ *are in one-to-one correspondence with the orbits of the elements* $s \in S_G^G \subset S(G,G)$ *under the action of* $\mathrm{Aut}(G)$ *on* $S(G,G)$. *If* $\mathrm{Aut}(G) = G$, *then the set of irreducible components of* $\mathrm{HUR}^{G}(\mathbb{P}^1)$ *has the natural structure of a semigroup* $S_{G,\mathbf{1}}^{G}$.

Proof. The first part follows from Corollary 2.5.

To prove the second part, we note that the equality $\mathrm{Aut}(G) = G$ means that all automorphisms of G are inner. By Proposition 1.1 the elements of $S_{G,\mathbf{1}}^{G}$ are fixed under the action of G by simultaneous conjugation. Therefore, by Corollary 2.5, the natural map (3.1) is an isomorphism giving the desired structure of a semigroup on $\mathrm{HUR}^{G}(\mathbb{P}^1)$.

In particular, Theorem 3.6 and Corollary 2.4 yield the following theorem.

Theorem 3.7. *The irreducible components of the Hurwitz space* $\mathrm{HUR}^{\mathcal{S}_3}(\mathbb{P}^1)$ *of Galois coverings with Galois group* $G = \mathcal{S}_3$ *are uniquely determined by the monodromy factorization type of the coverings belonging to them.*

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