WORST SINGULARITIES OF PLANE CURVES OF GIVEN DEGREE

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Abstract. I prove that $\frac{2}{d}$, $\frac{2d-3}{(d-1)^2}$, $\frac{2d-1}{d(d-1)}$, $\frac{2d-5}{d^2-3d+1}$ and $\frac{2d-3}{d(d-2)}$ are the smallest log canonical thresholds of reduced plane curves of degree $d \geqslant 3$. I describe reduced plane curves of degree d whose log canonical thresholds are these numbers. I prove that every reduced plane curve of degree $d \geqslant 4$ whose log canonical threshold is smaller than $\frac{5}{2d}$ is GIT-unstable for the action of the group $\mathrm{PGL}_3(\mathbb{C})$, and I describe GIT-semistable reduced plane curves with log canonical thresholds $\frac{5}{2d}$. I prove that $\frac{2}{d}$, $\frac{2d-3}{(d-1)^2}$, $\frac{2d-1}{d(d-1)}$, $\frac{2d-5}{d^2-3d+1}$ and $\frac{2d-3}{d(d-2)}$ are the smallest values of the α-invariant of Tian of smooth surfaces in $\mathbb{P}^3$ of degree $d \geqslant 3$.

All varieties are assumed to be algebraic, projective and defined over $\mathbb{C}$.

1. Introduction

Let C_d be a *reduced* plane curve in $\mathbb{P}^2$ of degree $d \geqslant 3$, and let P be a point in C_d. The curve C_d can have *any* given plane curve singularity at P provided that its degree d is *sufficiently* big. This naturally leads to

Question 1.1. Given a plane curve singularity, what is the minimal d such that there exists C_d having this singularity at P?

The best *general* answer to this question has been given by Greuel, Lossen and Shustin who proved

Theorem 1.2 ([12, Theorem 2])**.** For *every* topological type of plane curve singularity with Milnor number μ, there exists C_d of degree $d \leqslant 14\sqrt{\mu}$ that has this singularity at P.

For special types of singularities this result can be considerably improved (see, for example, [13]). Since C_d can have any *mild* singularity at P, it is natural to ask

Question 1.3. What is the *worst* singularity that C_d can have at P?

Denote by m_P the multiplicity of the curve C_d at the point P, and denote by $\mu(P)$ the Milnor number of the point P. If I use m_P to measure the singularity of C_d at the point P, then a union of d lines passing through P is an answer to Question 1.3, since $m_P \leqslant d$, and $m_P = d$ if and only if C_d is a union of d lines passing through P. If I use the Milnor number $\mu(P)$, then the answer would be the same, since $\mu(P) \leqslant (d-1)^2$, and $\mu(P) = (d-1)^2$ if and only if C_d is a union of d lines passing through P. Alternatively, I can use the number

$$\mathrm{lct}_P\big(\mathbb{P}^2, C_d\big) = \sup\Big\{\lambda \in \mathbb{Q} \ \Big|\ \text{the log pair } \big(\mathbb{P}^2, \lambda C_d\big) \text{ is log canonical at } P\Big\}$$

that is known as the *log canonical threshold* of the log pair $(\mathbb{P}^2, C_d)$ at the point P or the log canonical threshold of the curve C_d at the point P (see [9, Definition 6.34]). The smallest $\mathrm{lct}_P(\mathbb{P}^2, C_d)$ when P runs through all points in C_d is usually denoted by $\mathrm{lct}(\mathbb{P}^2, C_d)$. Note that

$$\frac{1}{m_P} \leqslant \mathrm{lct}_P\big(\mathbb{P}^2, C_d\big) \leqslant \frac{2}{m_P}.$$

This is well-known (see, for example, [18, Lemma 8.10] or [9, Exercise 6.18 and Lemma 6.35]). So, the smaller $\mathrm{lct}_P(\mathbb{P}^2, C_d)$, the worse singularity of the curve C_d at the point P is.

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Example 1.4 ([19, Proposition 2.2])**.** Suppose that C_d is given by $x_1^{n_1}x_2^{n_2}(x_1^{km_1}+x_2^{km_2})=0$ in an *analytic neighborhood* of the point P, where k, n_1, n_2, m_1 and m_2 are arbitrary non-negative integers. Then

$$\mathrm{lct}_P\big(\mathbb{P}^2,C_d\big)=\min\left\{\frac{1}{n_1},\frac{1}{n_2},\frac{\frac{1}{m_1}+\frac{1}{m_2}}{k+\frac{n_1}{m_1}+\frac{n_2}{m_2}}\right\}.$$

Log canonical thresholds of plane curves have been intensively studied (see, for example, [19], [1], [11], [16], [14], [21], [20], [10]). Surprisingly, they give the same answer to Question 1.3 by

Theorem 1.5 ([1, Theorem 4.1], [11, Theorem 0.2])**.** One has $\mathrm{lct}_P(\mathbb{P}^2,C_d)\geqslant\frac{2}{d}$. Moreover, $\mathrm{lct}(\mathbb{P}^2,C_d)=\frac{2}{d}$ if and only if C_d is a union of d lines that pass through P.

In this paper I want to address

Question 1.6. What is the *second worst* singularity that C_d can have at P?

To give a *reasonable* answer to this question, I have to disregard m_P by obvious reasons. Thus, I will use the numbers $\mu(P)$ and $\mathrm{lct}_P(\mathbb{P}^2,\mathbb{C}_d)$. For cubic curves, they give the same answer.

Example 1.7. Suppose that $d=3$, $m_P<3$ and P is a singular point of C_3. Then P is a singular point of type $\mathbb{A}_1$, $\mathbb{A}_2$ or $\mathbb{A}_3$. Moreover, if C_3 has singularity of type $\mathbb{A}_3$ at P, then $C_3=L+C_2$, where C_2 is a smooth conic, and L is a line tangent to C_2 at P. Furthermore, I have

$$\mu(P)=\begin{cases}1 \text{ if } C_3 \text{ has } \mathbb{A}_1 \text{ singularity at } P,\\ 2 \text{ if } C_3 \text{ has } \mathbb{A}_2 \text{ singularity at } P,\\ 3 \text{ if } C_3 \text{ has } \mathbb{A}_3 \text{ singularity at } P.\end{cases}$$

Similarly, I have

$$\mathrm{lct}_P\big(\mathbb{P}^2,C_3\big)=\begin{cases}1 \text{ if } C_3 \text{ has } \mathbb{A}_1 \text{ singularity at } P,\\ \dfrac{5}{6} \text{ if } C_3 \text{ has } \mathbb{A}_2 \text{ singularity at } P,\\ \dfrac{3}{4} \text{ if } C_3 \text{ has } \mathbb{A}_3 \text{ singularity at } P.\end{cases}$$

For quartic curves, the numbers $\mu(P)$ and $\mathrm{lct}_P(\mathbb{P}^2,\mathbb{C}_d)$ give different answers to Question 1.6.

Example 1.8. Suppose that $d=4$, $m_P<4$ and P is a singular point of C_4. Going through the list of all possible singularities that C_P can have at P (see, for example, [15]), I obtain

$$\mu(P)=\begin{cases}6 \text{ if } C_4 \text{ has } \mathbb{D}_6 \text{ singularity at } P,\\ 6 \text{ if } C_4 \text{ has } \mathbb{A}_6 \text{ singularity at } P,\\ 6 \text{ if } C_4 \text{ has } \mathbb{E}_6 \text{ singularity at } P,\\ 7 \text{ if } C_4 \text{ has } \mathbb{A}_7 \text{ singularity at } P,\\ 7 \text{ if } C_4 \text{ has } \mathbb{E}_7 \text{ singularity at } P,\end{cases}$$

and $\mu(P)<6$ in all remaining cases. Similarly, I get

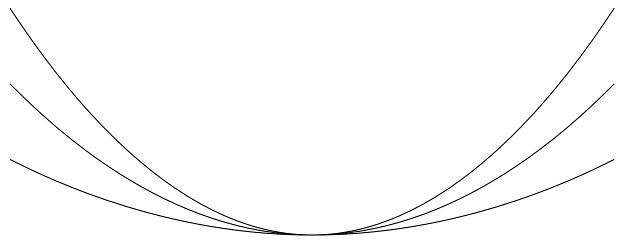
$$\mathrm{lct}_P\big(\mathbb{P}^2,C_4\big)=\begin{cases}\dfrac{5}{8} \text{ if } C_4 \text{ has } \mathbb{A}_7 \text{ singularity at } P,\\ \dfrac{5}{8} \text{ if } C_4 \text{ has } \mathbb{D}_5 \text{ singularity at } P,\\ \dfrac{3}{5} \text{ if } C_4 \text{ has } \mathbb{D}_6 \text{ singularity at } P,\\ \dfrac{7}{12} \text{ if } C_4 \text{ has } \mathbb{E}_6 \text{ singularity at } P,\\ \dfrac{5}{9} \text{ if } C_4 \text{ has } \mathbb{E}_7 \text{ singularity at } P,\end{cases}$$

and $\mathrm{lct}_P(\mathbb{P}^2,C_4)>\frac{5}{8}$ in all remaining cases.

Recently, Arkadiusz Płoski proved that $\mu(P) \leqslant (d-1)^2 - \lfloor\frac{d}{2}\rfloor$ provided that $m_P < d$. Moreover, he described C_d in the case when $\mu(P) = (d-1)^2 - \lfloor\frac{d}{2}\rfloor$. To present his description, I need

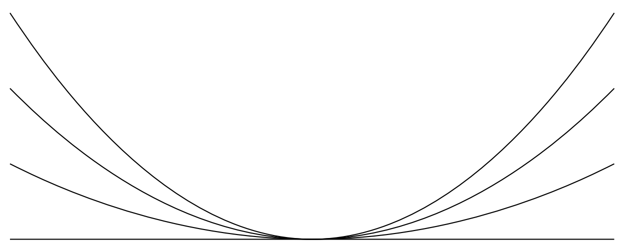
Definition 1.9. The curve C_d is an *even Płoski* curve if d is even, the curve C_d has $\frac{d}{2} \geqslant 2$ irreducible components that are smooth conics passing through P, and all irreducible components of C_d intersect each other pairwise at P with multiplicity 4.

Płoski curve of degree 6 looks like



Definition 1.10. The curve C_d is an *odd Płoski* curve if d is odd, the curve C_d has $\frac{d+1}{2} \geqslant 3$ irreducible components that all pass through P, $\frac{d-1}{2}$ irreducible component of the curve C_d are smooth conics that intersect each other pairwise at P with multiplicity 4, and the remaining irreducible component is a line in $\mathbb{P}^2$ that is tangent at P to all other irreducible components.

Płoski curve of degree 7 looks like



Each Płoski curve has unique singular point. If $d = 4$, then C_4 is a Płoski curve if and only if it has a singular point of type $\mathbb{A}_7$. Thus, if $d = 4$, then $\mu(P) = (d-1)^2 - \lfloor\frac{d}{2}\rfloor = 7$ if and only if either C_4 is a Płoski curve and P is its singular point or C_4 has singularity $\mathbb{E}_7$ at the point P (see Example 1.8). For $d \geqslant 5$, Płoski proved

Theorem 1.11 ([23, Theorem 1.4])**.** If $d \geqslant 5$, then $\mu(P) = (d-1)^2 - \lfloor\frac{d}{2}\rfloor$ if and only if C_d is a Płoski curve and P is its singular point.

This result gives a *very good* answer to Question 1.6. Surprisingly, the answer given by log canonical thresholds is very different. To describe it, I need

Definition 1.12. The curve C_d has singularity of type $\mathbb{T}_r$ (resp., $\mathbb{K}_r$, $\widetilde{\mathbb{T}}_r$, $\widetilde{\mathbb{K}}_r$) at the point P if the curve C_d can be given by $x_1^r = x_1x_2^r$ (resp., $x_1^r = x_2^{r+1}$, $x_2x_1^{r-1} = x_1x_2^r$, $x_2x_1^{r-1} = x_2^{r+1}$) in an *analytic neighborhood* of P.

The main purpose of this paper is to prove

Theorem 1.13. Suppose that $d \geqslant 4$ and $m_P < d$. If P is a singular point of the curve C_d of type $\mathbb{T}_{d-1}$, $\mathbb{K}_{d-1}$, $\widetilde{\mathbb{T}}_{d-1}$ or $\widetilde{\mathbb{K}}_{d-1}$, then

$$\mathrm{lct}_P\big(\mathbb{P}^2, C_d\big) = \begin{cases} \dfrac{2d-3}{(d-1)^2} & \text{if } C_d \text{ has } \mathbb{T}_{d-1} \text{ singularity at } P,\\ \dfrac{2d-1}{d(d-1)} & \text{if } C_d \text{ has } \mathbb{K}_{d-1} \text{ singularity at } P,\\ \dfrac{2d-5}{d^2-3d+1} & \text{if } C_d \text{ has } \widetilde{\mathbb{T}}_{d-1} \text{ singularity at } P,\\ \dfrac{2d-3}{d(d-2)} & \text{if } C_d \text{ has } \widetilde{\mathbb{K}}_{d-1} \text{ singularity at } P.\end{cases}$$

If P is not a singular point of the curve C_d of type $\mathbb{T}_{d-1}$, $\mathbb{K}_{d-1}$, $\widetilde{\mathbb{T}}_{d-1}$ or $\widetilde{\mathbb{K}}_{d-1}$, then either $\mathrm{lct}_P(\mathbb{P}^2, C_d) > \frac{2d-3}{d(d-2)}$, or $d=4$ and C_d is a Płoski quartic curve (in this case $\mathrm{lct}_P(\mathbb{P}^2, C_d) = \frac{5}{8}$).

This result fits well Examples 1.7 and 1.8, since $\mathbb{T}_2 = \mathbb{A}_3$, $\mathbb{K}_2 = \mathbb{A}_2$, $\widetilde{\mathbb{T}}_2 = \widetilde{\mathbb{K}}_2 = \mathbb{A}_1$, $\widetilde{\mathbb{K}}_3 = \mathbb{D}_5$, $\widetilde{\mathbb{T}}_3 = \mathbb{D}_6$, $\mathbb{K}_3 = \mathbb{E}_6$ and $\mathbb{T}_3 = \mathbb{E}_7$. Note that

$$\frac{2}{d} < \frac{2d-3}{(d-1)^2} < \frac{2d-1}{d(d-1)} < \frac{2d-5}{d^2-3d+1} < \frac{2d-3}{d(d-2)}$$

provided that $d \geqslant 4$. Thus, Theorem 1.13 describes the *five worst* singularities that C_d can have at the point P. In particular, it answers Question 1.6. Moreover, this answer is very explicit. Indeed, the curve C_d has singularity $\mathbb{T}_r$, $\mathbb{K}_r$, $\widetilde{\mathbb{T}}_r$ or $\widetilde{\mathbb{K}}_r$ at the point $[0:0:1]$ if and only if it can be given by

$$\alpha x^{d-1}z + \beta y x^{d-2} z = \gamma x y^{d-1} + \delta y^d + \sum_{i=2}^{d} a_i x^i y^{d-i},$$

where each a_i is a complex number, and

$$(\alpha, \beta, \gamma, \delta) = \begin{cases} (1,0,1,0) \text{ if } C_d \text{ has } \mathbb{T}_{d-1} \text{ singularity at } [0:0:1], \\ (1,0,0,1) \text{ if } C_d \text{ has } \mathbb{K}_{d-1} \text{ singularity at } [0:0:1], \\ (0,1,1,0) \text{ if } C_d \text{ has } \widetilde{\mathbb{T}}_{d-1} \text{ singularity at } [0:0:1], \\ (0,1,0,1) \text{ if } C \text{ has } \widetilde{\mathbb{K}}_{d-1} \text{ singularity at } [0:0:1]. \end{cases}$$

Remark 1.14. If C_d is a Płoski curve and P is its singular point, then it follows from Example 1.4 or from explicit computations that

$$\mathrm{lct}_P\big(\mathbb{P}^2, C_d\big) = \frac{5}{2d} > \frac{2d-3}{d(d-2)}$$

provided that $d \geqslant 5$. This shows that Theorems 1.11 and 1.13 gives *completely* different answers to Question 1.6.

The proof of Theorem 1.13 implies one result that is interesting on its own. To describe it, let me identify the curve C_d with a point in the space $|\mathcal{O}_{\mathbb{P}^2}(d)|$ that parameterizes all (not necessarily reduced) plane curves of degree d. Since the group $\mathrm{PGL}_3(\mathbb{C})$ acts on $|\mathcal{O}_{\mathbb{P}^2}(d)|$, it is natural to ask whether C_d is GIT-stable (resp., GIT-semistable) for this action or not. This question arises in many different problems (see, for example, [16], [14] and [20]). For small d, its answer is classical and immediately follows from the Hilbert–Mumford criterion (see, for example, [22, Chapter 2.1], [14, Proposition 10.4] or [16, Lemma 2.1]).

Example 1.15 ([22, Chapter 4.2])**.** If $d=3$, then C_3 is GIT-stable (resp., GIT-semistable) if and only if C_3 is smooth (resp., has at most $\mathbb{A}_1$ singularities). If $d=4$, then C_4 is GIT-stable (resp., GIT-semistable) if and only if C_4 has at most $\mathbb{A}_1$ and $\mathbb{A}_2$ singularities (resp., it has at most singular double points and C_4 is not a union of a cubic with an inflectional tangent line).

Paul Hacking, Hosung Kim and Yongnam Lee noticed that the log canonical threshold $\mathrm{lct}(\mathbb{P}^2, C_d)$ and GIT-stability of the curve C_d are closely related (cf. [10, Theorem 1.1]). In particular, they proved

Theorem 1.16 ([14, Propositions 10.2 and 10.4], [16, Theorem 2.3])**.** If $\mathrm{lct}(\mathbb{P}^2, C_d) \geqslant \frac{3}{d}$, then the curve C_d is GIT-semistable. If $d \geqslant 4$ and $\mathrm{lct}(\mathbb{P}^2, C_d) > \frac{3}{d}$, then the curve C_d is GIT-stable.

This gives a *sufficient* condition for the curve C_d to be GIT-stable (resp, GIT-semistable). However, this condition is not a *necessary* condition. Let me give two examples that illustrate this.

Example 1.17 ([31, p. 268], [14, Example 10.5])**.** Suppose that $d=5$, the quintic curve C_5 is given by

$$x^5+\Big(y^2-xz\Big)^2\Big(\frac{x}{4}+y+z\Big)=x^2\Big(y^2-xz\Big)\Big(x+2y\Big),$$

and $P=[0:0:1]$. Then C_5 is irreducible and has singularity $\mathbb{A}_{12}$ at the point P. In particular, it is rational. Furthermore, it is well-known that the curve C_5 is GIT-stable (see, for example, [22, Chapter 4.2]). On the other hand, it follows from Example 1.4 that

$$\mathrm{lct}\big(\mathbb{P}^2,C_5\big)=\mathrm{lct}_P\big(\mathbb{P}^2,C_5\big)=\frac{1}{2}+\frac{1}{13}=\frac{15}{26}<\frac{3}{5}.$$

Example 1.18. Suppose that C_d is a Płoski curve. Let P be its singular point, and let L be a general line in $\mathbb{P}^2$. Then

$$\mathrm{lct}\big(\mathbb{P}^2,C_d+L\big)=\mathrm{lct}\big(\mathbb{P}^2,C_d\big)=\mathrm{lct}_P\big(\mathbb{P}^2,C_d\big)=\frac{5}{2d}<\frac{3}{d}$$

by Remark 1.14. If d is even, then C_d is GIT-semistable, and C_d+L is GIT-stable. This follows from the Hilbert–Mumford criterion. Similarly, if d is odd, then C_d is GIT-unstable, and C_d+L is GIT-semistable.

If $m_P>\frac{2d}{3}$, then C_d is GIT-unstable by the Hilbert–Mumford criterion. In particular, if $d\geqslant 4$ and $\mathrm{lct}(\mathbb{P}^2,C_d)\leqslant\frac{2d-3}{d(d-2)}$, then C_d is GIT-unstable by Theorem 1.13 unless C_4 is a Płoski quartic curve. Arguing as in the proof of Theorem 1.13, this *necessary* condition can be considerably improved. In fact, I will give a combined proof of Theorem 1.13 and

Theorem 1.19. If $\mathrm{lct}(\mathbb{P}^2,C_d)<\frac{5}{2d}$, then C_d is GIT-unstable. Moreover, if $\mathrm{lct}(\mathbb{P}^2,C_d)\leqslant\frac{5}{2d}$, then C_d is not GIT-stable. Furthermore, if $\mathrm{lct}(\mathbb{P}^2,C_d)=\frac{5}{2d}$, then C_d is GIT-semistable if and only if C_d is an even Płoski curve.

Example 1.18 shows that this result is *sharp*. Now let me consider one application of Theorem 1.13. To describe it, I need

Definition 1.20 ([30, Appendix A], [5, Definition 1.20])**.** For a given smooth variety V equipped with an ample $\mathbb{Q}$-divisor H_V, let $\alpha_V^{H_V}:V\to\mathbb{R}_{\geqslant 0}$ be a function defined as

$$\alpha_V^{H_V}(O)=\sup\left\{\lambda\in\mathbb{Q}\ \left|\ \begin{array}{l}\text{the pair }\ (V,\lambda D_V)\ \text{ is log canonical at } O\\ \text{for every effective } \mathbb{Q}\text{-divisor } D_V\sim_{\mathbb{Q}}H_V\end{array}\right.\right\}.$$

Denote its infimum by $\alpha(V,H_V)$.

Let S_d be a smooth surface in $\mathbb{P}^3$ of degree $d\geqslant 3$, let H_{S_d} be its hyperplane section, let O be a point in S_d, and let T_O be the hyperplane section of S_d that is singular at O. Similar to $\mathrm{lct}_P(\mathbb{P}^2,C_d)$), I can define

$$\mathrm{lct}_O\big(S_d,T_O\big)=\sup\Big\{\lambda\in\mathbb{Q}\ \Big|\ \text{the log pair }\big(S_d,\lambda T_O\big)\text{ is log canonical at }O\Big\}.$$

Then $\alpha_{S_d}^{H_{S_d}}(O)\leqslant\mathrm{lct}_O(S_d,T_O)$ and T_O is reduced. In this paper I prove

Theorem 1.21. If $\alpha_{S_d}^{H_{S_d}}(O)<\frac{2d-3}{d(d-2)}$, then

$$\alpha_{S_d}^{H_{S_d}}(O)=\mathrm{lct}_O\big(S_d,T_O\big)\in\left\{\frac{2}{d},\frac{2d-3}{(d-1)^2},\frac{2d-1}{d(d-1)},\frac{2d-5}{d^2-3d+1}\right\}.$$

Corollary 1.22. If $\alpha(S_d,H_{S_d})<\frac{2d-3}{d(d-2)}$, then

$$\alpha\big(S_d,H_{S_d}\big)=\inf_{O\in S_d}\Big\{\mathrm{lct}_O\big(S_d,T_O\big)\Big\}\in\left\{\frac{2}{d},\frac{2d-3}{(d-1)^2},\frac{2d-1}{d(d-1)},\frac{2d-5}{d^2-3d+1}\right\}.$$

Corollary 1.23 ([5, Corollary 1.24])**.** Suppose that $d=3$. Then $\alpha_{S_3}^{H_{S_3}}(O)=\mathrm{lct}_O(S_3,T_O)$.

By [28, Theorem 2.1], this corollary implies that every smooth cubic surface in $\mathbb{P}^3$ *without Eckardt points* admits a Kähler–Einstein metric. In [29], Tian proved that *all* smooth cubic surfaces are Kähler–Einstein (see also [26] and [7]). This follows from his

Theorem 1.24 ([29, Main Theorem])**.** Smooth del Pezzo surface is Kähler–Einstein if and only if its automorphism group is reductive.

It should be pointed out that I cannot drop the condition $\alpha_{S_d}^{H_{S_d}}(O) < \frac{2d-3}{d(d-2)}$ in Theorem 1.21 for $d \geqslant 4$. For $d = 4$, this follows from

Example 1.25. Suppose that $d = 4$. Let S_4 be a quartic surface in $\mathbb{P}^3$ that is given by

$$t^3x + t^2yz + xyz(y + z) = 0,$$

and let O be the point $[0 : 0 : 0 : 1]$. Then S_4 is smooth, and T_O has singularity $\mathbb{A}_1$ at O, which implies that $\mathrm{lct}_O(S_4, T_O) = 1$. Let L_y be the line $x = y = 0$, let L_z be the line $x = z = 0$, and let C_2 be the conic $y + z = xt + yz = 0$. Then L_y, L_z and C_2 are contained in S_4, and $O = L_y \cap L_z \cap C_2$. Moreover,

$$L_y + L_z + \frac{1}{2}C_2 \sim 2H_{S_4},$$

because the divisor $2L_y + 2L_z + C_2$ is cut out on S_4 by $tx + yz = 0$. Furthermore, the log pair $(S_4, L_y + L_z + \frac{1}{2}C_2)$ is not log canonical at O. Thus, $\alpha_{S_4}^{H_{S_4}}(O) < 1$.

For $d \geqslant 5$, this follows from

Example 1.26. Suppose that $d \geqslant 5$ and T_O has $\mathbb{A}_1$ singularity at O. Then $\mathrm{lct}_O(S_d, T_O) = 1$. Let $f\colon \tilde{S}_d \to S_d$ be a blow up of the point O. Denote by E its exceptional curve. Then

$$\Big(f^*(H_{S_d}) - \frac{11}{5}E\Big)^2 = 5 - \frac{121}{25} > 0.$$

Hence, it follows from Riemann–Roch theorem there is an integer $n \geqslant 1$ such that the linear system $|f^*(5nH_{S_d}) - 11nE|$ is not empty. Pick a divisor $\tilde{D}$ in this linear system, and denote by D its image on S_d. Then $(S_d, \frac{1}{5n}D)$ is not log canonical at P, since $\mathrm{mult}_P(D) \geqslant 11n$. On the other hand, $\frac{1}{5n}D \sim_{\mathbb{Q}} H_{S_d}$ by construction. Hence, $\alpha_{S_d}^{H_d}(O) < 1$.

Cool and Coppens called the point O a *star point* in the case when T_O is a union of d lines that pass through O (see [8, Definition 2.2]). Theorems 1.13 and 1.21 imply

Corollary 1.27. If O is a star point on S_d, then $\alpha_{S_d}^{H_{S_d}}(O) = \frac{2}{d}$. Otherwise $\alpha_{S_d}^{H_{S_d}}(O) \geqslant \frac{2d-3}{(d-1)^2}$.

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2. Preliminaries

In this section, I consider results that will be used in the proof of Theorems 1.13 and 1.21. Let S be a smooth surface, let D be an effective non-zero $\mathbb{Q}$-divisor on the surface S, and let P be a point in the surface S. Put $D = \sum_{i=1}^{r} a_iC_i$, where each C_i is an irreducible curve on S, and each a_i is a non-negative rational number. Let me start recall

Definition 2.1 ([18, Definition 3.5], [9, § 6])**.** Let $\pi\colon \tilde{S} \to S$ be a birational morphism such that $\tilde{S}$ is smooth. Then π is a composition of blow ups of smooth points. For each C_i, denote by $\tilde{C}_i$ its proper transform on the surface $\tilde{S}$. Let $F_1, \ldots, F_n$ be π-exceptional curves. Then

$$K_{\tilde{S}} + \sum_{i=1}^{r} a_i\tilde{C}_i + \sum_{j=1}^{n} b_jF_j \sim_{\mathbb{Q}} \pi^*\big(K_S + D\big)$$

for some rational numbers $b_1, \ldots, b_n$. Suppose, in addition, that $\sum_{i=1}^{r} \tilde{C}_i + \sum_{j=1}^{n} F_j$ is a divisor with simple normal crossing at every point of $\cup_{j=1}^{n} F_j$. Then the log pair (S, D) is said to be *log canonical* at P if and only if the following two conditions are satisfied:

- $a_i \leqslant 1$ for every C_i such that $P \in C_i$,
- $b_j \leqslant 1$ for every F_j such that $\pi(F_j) = P$.

Similarly, the log pair (S, D) is said to be *Kawamata log terminal* at P if and only if $a_i < 1$ for every C_i such that $P \in C_i$, and $b_j < 1$ for every F_j such that $\pi(F_j) = P$.

Using just this definition, one can easily prove

Lemma 2.2. Suppose that $r = 3$, $P \in C_1 \cap C_2 \cap C_3$, the curves C_1, C_2 and C_3 are smooth at P, $a_1 < 1$, $a_2 < 1$ and $a_3 < 1$. Moreover, suppose that both curves C_1 and C_2 intersect the curve C_3 transversally at P. Furthermore, suppose that (S, D) is not Kawamata log terminal at P. Put $k = \mathrm{mult}_P(C_1 \cdot C_2)$. Then $k(a_1 + a_2) + a_3 \geqslant k + 1$.

Proof. Put $S_0 = S$ and consider a sequence of blow ups

$$S_k \xrightarrow{\pi_k} S_{k-1} \xrightarrow{\pi_{k-1}} \cdots \xrightarrow{\pi_3} S_2 \xrightarrow{\pi_2} S_1 \xrightarrow{\pi_1} S_0,$$

where each π_j is the blow up of the intersection point of the proper transforms of the curves C_1 and C_2 on the surface S_{j-1} that dominates P (such point exists, since $k = \mathrm{mult}_P(C_1 \cdot C_2)$). For each π_j, denote by E_j^k the proper transform of its exceptional curve on S_k. For each C_i, denote by C_i^k its proper transform on the surface S_k. Then

$$K_{S_k} + \sum_{i=1}^{n} a_i C_i^k + \sum_{j=1}^{k} \Big(j\big(a_1 + a_2\big) + a_3 - j \Big) E_j^k \sim_{\mathbb{Q}} (\pi_1 \circ \pi_2 \circ \cdots \circ \pi_k)^* \Big(K_S + D \Big),$$

and $\sum_{i=1}^{n} C_i^k + \sum_{j=1}^{k} E_j$ is a simple normal crossing divisor in every point of $\sum_{j=1}^{k} E_j$. Thus, it follows from Definition 2.1 that there exists $l \in \{1, \ldots, k\}$ such that $l(a_1 + a_2) + a_3 \geqslant l + 1$, because (S, D) is not Kawamata log terminal at P. If $l = k$, then I am done. So, I may assume that $l < k$. If $k(a_1 + a_2) + a_3 < k + 1$, then $a_1 + a_2 < 1 + \frac{1}{k} - a_3 \frac{1}{k}$, which implies that

$$l + 1 \leqslant l\big(a_1 + a_2\big) + a_3 < \left(l + \frac{l}{k} - a_3 \frac{l}{k} \right) + a_3 = l + \frac{l}{k} + a_3 \left(1 - \frac{l}{k} \right) \leqslant l + \frac{l}{k} + \left(1 - \frac{l}{k} \right) = l + 1,$$

because $a_3 < 1$. Thus, $k(a_1 + a_2) + a_3 \geqslant k + 1$. □

Corollary 2.3. Suppose that $r = 2$, $P \in C_1 \cap C_2$, the curves C_1 and C_2 are smooth at P, $a_1 < 1$ and $a_2 < 1$. Put $k = \mathrm{mult}_P(C_1 \cdot C_2)$. If (S, D) is not Kawamata log terminal at P, then $k(a_1 + a_2) \geqslant k + 1$.

The log pair (S, D) is called *log canonical* if it is log canonical at every point of S. Similarly, the log pair (S, D) is called *Kawamata log terminal* if it is Kawamata log terminal at every point of S.

Remark 2.4. Let R be any effective $\mathbb{Q}$-divisor on S such that $R \sim_{\mathbb{Q}} D$ and $R \neq D$. Put $D_\epsilon = (1 + \epsilon)D - \epsilon R$ for some rational number ϵ. Then $D_\epsilon \sim_{\mathbb{Q}} D$. Since $R \neq D$, there exists the greatest rational number ϵ_0 such that the divisor D_{ϵ_0} is effective. Then $\mathrm{Supp}(D_{\epsilon_0})$ does not contain at least one irreducible component of $\mathrm{Supp}(R)$. Moreover, if (S, D) is not log canonical at P, and (S, R) is log canonical at P, then (S, D_{ϵ_0}) is not log canonical at P by Definition 2.1, because

$$D = \frac{1}{1 + \epsilon_0} D_{\epsilon_0} + \frac{\epsilon_0}{1 + \epsilon_0} R$$

and $\frac{1}{1+\epsilon_0} + \frac{\epsilon_0}{1+\epsilon_0} = 1$. Similarly, if the log pair (S, D) is not Kawamata log terminal at P, and (S, R) is Kawamata log terminal at P, then (S, D_{ϵ_0}) is not Kawamata log terminal at P.

The following result is well-known and is very easy to prove.

Lemma 2.5 ([9, Exercise 6.18])**.** If (S, D) is not log canonical at P, then $\mathrm{mult}_P(D) > 1$.

Combining with

Lemma 2.6 ([24], [9, Lemma 5.36])**.** Suppose that S is a smooth surface in $\mathbb{P}^3$, and $D \sim_{\mathbb{Q}} H_S$, where H_S is a hyperplane section of S. Then each a_i does not exceed 1.

Lemma 2.5 gives

Corollary 2.7. Suppose that S is a smooth surface in $\mathbb{P}^3$, and $D \sim_{\mathbb{Q}} H_S$, where H_S is a hyperplane section of S. Then (S, D) is log canonical outside of finitely many points.

The following result is a special case of Shokurov's connectedness principle (see, for example, [9, Theorem 6.3.2]).

Lemma 2.8 ([27, Theorem 6.9])**.** If $-(K_S + D)$ is big and nef, then the locus where (S, D) is not Kawamata log terminal is connected.

Corollary 2.9. Let C_d be a reduced curve in $\mathbb{P}^2$ of degree d, let O and Q be two points in C_d such that $O \neq Q$. If $\mathrm{lct}_O(\mathbb{P}^2, C_d) < \frac{3}{d}$, then $\mathrm{lct}_Q(\mathbb{P}^2, C_d) \geqslant \frac{3}{d}$.

Let $\pi_1 \colon S_1 \to S$ be a blow up of the point P, and let E_1 be the π_1-exceptional curve. Denote by D^1 the proper transform of the divisor D on the surface S_1 via π_1. Then

$$K_{S_1} + D^1 + \Big(\mathrm{mult}_P(D) - 1\Big)E_1 \sim_{\mathbb{Q}} \pi_1^*\big(K_S + D\big).$$

Corollary 2.10. If $\mathrm{mult}_P(D) > 2$, then (S, D) is not log canonical at P. If $\mathrm{mult}_P(D) \geqslant 2$, then (S, D) is not Kawamata log terminal at P.

The log pair $(S_1, D^1 + (\mathrm{mult}_P(D) - 1)E_1)$ is called *the log pull back* of the log pair (S, D).

Remark 2.11. The log pair (S, D) is log canonical at P if and only if $(S_1, D^1 + (\mathrm{mult}_P(D) - 1)E_1)$ is log canonical at every point of the curve E_1. Similarly, the log pair (S, D) is Kawamata log terminal at P if and only if $(S_1, D^1 + (\mathrm{mult}_P(D) - 1)E_1)$ is Kawamata log terminal at every point of the curve E_1.

Let Z be an irreducible curve on S that contains P. Suppose that Z is smooth at P, and Z is not contained in $\mathrm{Supp}(D)$. Let μ be a non-negative rational number. The following result is a very special case of a much more general result known as *Inversion of Adjunction* (see, for example, [27, § 3.4] or [9, Theorem 6.29]).

Theorem 2.12 ([27, Corollary 3.12], [9, Exercise 6.31], [3, Theorem 7])**.** Suppose that the log pair $(S, \mu Z + D)$ is not log canonical at P and $\mu \leqslant 1$. Then $\mathrm{mult}_P(D \cdot Z) > 1$.

This result implies

Theorem 2.13. Suppose that $(S, \mu Z + D)$ is not Kawamata log terminal at P, and $(S, \mu Z + D)$ is Kawamata log terminal in a punctured neighborhood of the point P. Then $\mathrm{mult}_P(D \cdot Z) > 1$.

Proof. Since $(S, \mu Z + D)$ is Kawamata log terminal in a punctured neighborhood of the point P, I have $\mu < 1$. Then $(S, Z + D)$ is not log canonical at P, because $(S, \mu Z + D)$ is not Kawamata log terminal at P. Then $\mathrm{mult}_P(D \cdot Z) > 1$ by Theorem 2.12. ☐

Theorems 2.12 and 2.13 imply

Lemma 2.14. If (S, D) is not log canonical at P and $\mathrm{mult}_P(D) \leqslant 2$, then there exists a *unique* point in E_1 such that $(S_1, D^1 + (\mathrm{mult}_P(D) - 1)E_1)$ is not log canonical at it. Similarly, if (S, D) is not Kawamata log terminal at P, $\mathrm{mult}_P(D) < 2$, and (S, D) is Kawamata log terminal in a punctured neighborhood of the point P, then there exists a *unique* point in E_1 such that $(S_1, D^1 + (\mathrm{mult}_P(D) - 1)E_1)$ is not Kawamata log terminal at it.

Proof. If $\mathrm{mult}_P(D) \leqslant 2$ and $(S_1, D^1 + (\lambda \mathrm{mult}_P(D) - 1)E_1)$ is not log canonical at two distinct points P_1 and $\tilde{P}_1$, then

$$2 \geqslant \mathrm{mult}_P\big(D\big) = D^1 \cdot E_1 \geqslant \mathrm{mult}_{P_1}\Big(D^1 \cdot E_1\Big) + \mathrm{mult}_{\tilde{P}_1}\Big(D^1 \cdot E_1\Big) > 2$$

by Theorem 2.12. By Remark 2.11, this proves the first assertion. Similarly, I can prove the second assertion using Theorem 2.13 instead of Theorem 2.12. □

The following result can be proved similarly to the proof of Lemma 2.5. Let me show how to prove it using Theorem 2.13.

Lemma 2.15. Suppose that (S, D) is not Kawamata log terminal at P, and (S, D) is Kawamata log terminal in a punctured neighborhood of the point P, then $\mathrm{mult}_P(D) > 1$.

Proof. By Remark 2.11, the log pair $(S_1, D^1 + (\mathrm{mult}_P(D) - 1)E_1)$ is not Kawamata log terminal at some point $P_1 \in E_1$. Moreover, if $\mathrm{mult}_P(D) < 2$, then $(S_1, D^1 + (\mathrm{mult}_P(D) - 1)E_1)$ is Kawamata log terminal at a punctured neighborhood of the point P_1. Thus, if $\mathrm{mult}_P(D) \leqslant 1$, then $\mathrm{mult}_P\big(D\big) = D^1 \cdot E_1 > 1$ by Theorem 2.13, which is absurd. □

Let Z_1 and Z_2 be two irreducible curves on the surface S such that Z_1 and Z_2 are not contained in $\mathrm{Supp}(D)$. Suppose that $P \in Z_1 \cap Z_2$, the curves Z_1 and Z_2 are smooth at P, the curves Z_1 and Z_2 intersect each other transversally at P. Let μ_1 and μ_2 be non-negative rational numbers. A crucial role in the proofs of Theorems 1.13 and 1.21 is played by

Theorem 2.16 ([3, Theorem 13])**.** Suppose that the log pair $(S, \mu_1 Z_1 + \mu_2 Z_2 + D)$ is not log canonical at the point P, and $\mathrm{mult}_P(D) \leqslant 1$. Then either $\mathrm{mult}_P(D \cdot Z_1) > 2(1 - \mu_2)$ or $\mathrm{mult}_P(D \cdot Z_2) > 2(1 - \mu_1)$ (or both).

This result implies

Theorem 2.17. Suppose that $(S, \mu_1 Z_1 + \mu_2 Z_2 + D)$ is not Kawamata log terminal at P, and $\mathrm{mult}_P(D) < 1$. Then either $\mathrm{mult}_P(D \cdot Z_1) \geqslant 2(1 - \mu_2)$ or $\mathrm{mult}_P(D \cdot Z_2) \geqslant 2(1 - \mu_1)$ (or both).

Proof. Let λ be a rational number such that $\frac{1}{\mathrm{mult}_P(D)} \geqslant \lambda > 1$. Then $(S, D + \lambda\mu_1 Z_1 + \lambda\mu_2 Z_2)$ is not log canonical at P. Now it follows from Theorem 2.16 that either $\mathrm{mult}_P(D \cdot Z_1) > 2(1 - \lambda\mu_2)$ or $\mathrm{mult}_P(D \cdot Z_2) > 2(1 - \lambda\mu_1)$ (or both). Since I can choose λ to be as close to 1 as I wish, this implies that either $\mathrm{mult}_P(D \cdot Z_1) \geqslant 2(1 - \mu_2)$ or $\mathrm{mult}_P(D \cdot Z_2) \geqslant 2(1 - \mu_1)$ (or both). □

3. Reduced plane curves

The purpose of this section is to prove Theorems 1.13 and 1.19. Let C_d be a *reduced* plane curve in $\mathbb{P}^2$ of degree $d \geqslant 4$, and let P be a point in C_d. Put $m_0 = \mathrm{mult}_P(C_d)$.

Lemma 3.1. One has

$$\mathrm{lct}_P\big(\mathbb{P}^2, C_d\big) = \begin{cases} \dfrac{1}{2d} \text{ if } m_0 = d, \\ \dfrac{2d-3}{(d-1)^2} \text{ if } C_d \text{ has } \mathbb{T}_{d-1} \text{ singularity at } P, \\ \dfrac{2d-1}{d(d-1)} \text{ if } C_d \text{ has } \mathbb{K}_{d-1} \text{ singularity at } P, \\ \dfrac{2d-5}{d^2-3d+1} \text{ if } C_d \text{ has } \widetilde{\mathbb{T}}_{d-1} \text{ singularity at } P, \\ \dfrac{2d-3}{d(d-2)} \text{ if } C \text{ has } \widetilde{\mathbb{K}}_{d-1} \text{ singularity at } P. \end{cases}$$

Proof. The required assertion follows either from [18, Lemma 8.10] or from Example 1.4. Alternatively, one can easily prove it directly using only Definition 2.1. This is a good exercise. □

Put $\lambda_1=\frac{2d-3}{d(d-2)}$ and $\lambda_2=\frac{5}{2d}$. By Lemma 3.1, to prove Theorem 1.13, I have to show that if the log pair $(\mathbb{P}^2,\lambda_1 C_d)$ is not Kawamata log terminal, then one of the following assertions hold:

- $m_0=d$,
- C_d has singularity $\mathbb{T}_{d-1}$, $\mathbb{K}_{d-1}$, $\widetilde{\mathbb{T}}_{d-1}$ or $\widetilde{\mathbb{K}}_{d-1}$ at the point P,
- $d=4$ and C_4 is a Płoski curve (see Definition 1.9).

To prove Theorem 1.19, I have to show that if $(\mathbb{P}^2,\lambda_2 C_d)$ is not Kawamata log terminal, then either C_d is GIT-unstable or C_d is an even Płoski curve, which is GIT-semistable (see Example 1.18). In the rest of the section, I will do this simultaneously. Let me start with

Lemma 3.2. The following inequalities hold:

(i) $\lambda_1<\frac{2}{d-1}$,
(ii) $\lambda_1<\frac{2k+1}{kd}$ for every positive integer $k\leqslant d-3$,
(iii) if $d\geqslant 5$, then $\lambda_1<\frac{2k+1}{kd+1}$ for every positive integer $k\leqslant d-4$,
(iv) $\lambda_1<\frac{3}{d}$,
(v) $\lambda_1<\frac{2}{d-2}$,
(vi) $\lambda_1<\frac{6}{3d-4}$,
(vii) if $d\geqslant 5$, then $\lambda_1<\lambda_2$.

Proof. The equality $\frac{2}{d-1}=\lambda_1+\frac{d-3}{d(d-1)(d-2)}$ implies (i). Let k be positive integer. If $k=d-2$, then $\lambda_1=\frac{2k+1}{kd}$. This implies (ii), because $\frac{2k+1}{kd}=\frac{2}{d}+\frac{1}{kd}$ is a decreasing function on k for $k\geqslant 1$. Similarly, if $k=d-4$ and $d\geqslant 4$, then $\lambda_1=\frac{2k+1}{kd+1}-\frac{3}{d(d-2)(d^2-4d+1)}<\frac{2k+1}{kd+1}$. This implies (iii), since $\frac{2k+1}{kd+1}=\frac{2}{d}+\frac{d-2}{d(kd+1)}$ is a decreasing function on k for $k\geqslant 1$. The equality $\lambda_1=\frac{3}{d}-\frac{d-3}{d(d-2)}$ proves (iv). Note that (v) follows from (i). Since $\frac{6}{3d-4}>\frac{2}{d-1}$, (vi) also follows from (i). Finally, the equality $\lambda_1=\lambda_2-\frac{d-4}{2d(d-2)}$ implies (vii). $\square$

I may assume that $P=[0:0:1]$. Then C_d is given by $F_d(x,y,z)=0$, where $F_d(x,y,z)$ is a homogeneous polynomial of degree d. Put $x_1=\frac{x}{z}$, $x_2=\frac{y}{z}$ and $f_d(x_1,x_2)=F_d(x_1,x_2,1)$. Then

$$f_d(x_1,x_2)=\sum_{\substack{i\geqslant 0,j\geqslant 0,\\ m_0\leqslant i+j\leqslant d}}\epsilon_{ij}x_1^i x_2^j,$$

where each ϵ_{ij} is a complex number. For every positive integers a and b, define the weight of the polynomial $f_d(x_1,x_2)$ as

$$\mathrm{wt}_{(a,b)}\big(f_d(x_1,x_2)\big)=\min\Big\{ai+bj\ \Big|\ \epsilon_{ij}\neq 0\Big\}.$$

So, that $\mathrm{wt}_{(1,1)}\big(f_d(x_1,x_2)\big)=m_0$. Then the Hilbert–Mumford criterion implies

Lemma 3.3 ([16, Lemma 2.1])**.** Let a and b be positive integers. If C_d is GIT-stable, then

$$\mathrm{wt}_{(a,b)}\Big(f_d\big(x_1,x_2\big)\Big)<\frac{d}{3}\big(a+b\big).$$

Similarly, if C_d is GIT-semistable, then $\mathrm{wt}_{(a,b)}(f_d(x_1,x_2))\leqslant\frac{d}{3}(a+b)$.

This result can be used to give a *sufficient* condition for the curve C_d to be GIT-stabile (resp., GIT-semistabile) (for details, see [14, Proposition 10.4] and the proof of [16, Theorem 2.3]).

Corollary 3.4. If $m_0>\frac{2d}{3}$, then C_d is GIT-unstable.

Hence, to prove Theorem 1.13 and 1.19, I may assume that C_d is not a union of d lines passing through the point P. Suppose that

(**A**) either $(\mathbb{P}^2,\lambda_1 C_d)$ is not Kawamata log terminal,
(**B**) or $(\mathbb{P}^2,\lambda_2 C_d)$ is not Kawamata log terminal and C_d is GIT-semistable.

I will show that (**A**) implies that either C_d has singularity $\mathbb{T}_{d-1}$, $\mathbb{K}_{d-1}$, $\widetilde{\mathbb{T}}_{d-1}$ or $\widetilde{\mathbb{K}}_{d-1}$ at the point P, or C_d is a Płoski quartic curve. I will also show that (**B**) implies that C_d is an even Płoski curve. If (**A**) holds, let $\lambda = \lambda_1$. If (**B**) holds, let $\lambda = \lambda_2$.

Remark 3.5. If $d = 4$, then $\lambda_1 = \lambda_2$. If $d \geqslant 5$, then $\lambda_1 < \lambda_2$ by Lemma 3.2(vii). Since C_d is reduced and $\lambda < 1$, the log pair $(\mathbb{P}^2, \lambda C_d)$ is Kawamata log terminal outside of finitely many points. Thus, it is Kawamata log terminal outside of P by Lemma 2.8.

Let $f_1\colon S_1 \to \mathbb{P}^2$ be a blow up of the point P, and let E_1 be its exceptional curve. Denote by C_d^1 the proper transform on S_1 of the curve C_d. Put $m_0 = \mathrm{mult}_P(C_d)$. Then

$$K_{S_1} + \lambda C_d^1 + \Big(\lambda m_0 - 1\Big)E_1 \sim_{\mathbb{Q}} f_1^*\Big(K_{\mathbb{P}^2} + \lambda C_d\Big).$$

By Remark 2.11, the log pair $(S_1, \lambda C_d^1 + (\lambda m_0 - 1)E_1)$ is not Kawamata log terminal at some point $P_1 \in E_1$.

Lemma 3.6. One has $\lambda m_0 < 2$.

Proof. Since C_d is not a union of d lines passing through P, I have $m_0 \leqslant d-1$. By Lemma 3.2(i), (**A**) implies $\lambda m_0 < 2$, because $d \geqslant 4$. Similarly, it follows from (**B**) that $m_0 \leqslant \frac{2d}{3}$ by Lemma 3.4, which implies that $\lambda m_0 \leqslant \frac{10}{6} < 2$. □

Thus, the log pair $(S_1, \lambda C_d^1 + (\lambda m_0 - 1)E_1)$ is Kawamata log terminal outside of P_1 by Lemma 2.14. Put $m_1 = \mathrm{mult}_{P_1}(C_d^1)$.

Lemma 3.7. One has $m_0 + m_1 > \frac{2}{\lambda}$ and $m_1 \geqslant 1$.

Proof. The inequality $m_0 + m_1 > \frac{2}{\lambda}$ follows from Lemma 2.15. The inequality $m_1 \geqslant 1$ is obvious. Indeed, if $m_1 = 0$, then $(S_1, (\lambda m_0 - 1)E_1)$ is not Kawamata log terminal at P_1, which contradicts Lemma 3.6. □

Let L be the line in $\mathbb{P}^2$ whose proper transform on S_1 contains the point P_1. Such a line exists and it is unique. By a suitable change of coordinates, I may assume that L is given by $x = 0$.

Lemma 3.8. Suppose that C_d is GIT-semistable. Then $m_0 + m_1 \leqslant d$.

Proof. Note that $\mathrm{wt}_{(1,1)}(f_d(x_1, x_2)) = m_0$. For every $(a, b) \in \mathbb{N}^2$ different from $(1, 1)$, the number $\mathrm{wt}_{(a,b)}(f_d(x_1, x_2))$ depends on the choice of (global) coordinates (x, y, z). For instance, $\mathrm{wt}_{(1,2)}(f_d(x_1, x_2))$ is a sum of m_0 and the multiplicity of the curve C_d^1 at the point in E_1 that is cut out by the proper transform of the line given by $y = 0$. Similarly, $\mathrm{wt}_{(2,1)}(f_d(x_1, x_2))$ is a sum of m_0 and the multiplicity of the curve C_d^1 at the point in E_1 that is cut out by the proper transform of the line given by $x = 0$. Since I assumed that L is given by $x = 0$, I have

$$\mathrm{wt}_{(2,1)}\big(f_d(x_1, x_2)\big) = m_0 + m_1.$$

Thus, $m_0 + m_1 \leqslant d$ by Lemma 3.3, because C_d is GIT-semistable by assumption. □

Denote by L^1 the proper transform of the line L on the surface S_1.

Lemma 3.9. Suppose (**A**) and $m_0 = d-1$. Then C_d has singularity $\mathbb{K}_{d-1}$, $\widetilde{\mathbb{K}}_{d-1}$, $\mathbb{T}_{d-1}$ or $\widetilde{\mathbb{T}}_{d-1}$ at the point P.

Proof. I have $\lambda = \lambda_1$. Let me prove that

- if L is not an irreducible component of the curve C_d, then either C_d has singularity $\mathbb{K}_{d-1}$ at P, or C_d has singularity $\widetilde{\mathbb{K}}_{d-1}$ at P,
- if L is an irreducible component of the curve C_d, then either C_d has singularity $\mathbb{T}_{d-1}$ at P, or C_d has singularity $\widetilde{\mathbb{T}}_{d-1}$ at P.

Suppose that L is not an irreducible component of the curve C_d. Then $m_0+m_1\leqslant d$, because

$$d-1-m_0=C_d^1\cdot L^1\geqslant m_1.$$

Since $m_0=d-1$, this gives $m_1=1$, because $m_1\neq 0$ by Lemma 3.7. Then $P_1\in C_{d-1}^1$ and the curve C_{d-1}^1 is smooth at P_1. Put $k=\mathrm{mult}_{P_1}(C_d^1\cdot E_1)$. Applying Corollary 2.3 to the log pair $(S_1,\lambda_1C_d^1+(\lambda_1m_0-1)E_1)$ at the point P_1, I get

$$k\lambda_1m_0\geqslant k+1,$$

which gives $\lambda_1\geqslant\frac{2k+1}{kd}$. Then $k\geqslant d-2$ by Lemma 3.2(ii). Since

$$k\leqslant C_d^1\cdot E_1=m_0=d-1,$$

either $k=d-1$ or $k=d-2$. If $k=d-1$, then C_d has singularity $\mathbb{K}_{d-1}$ at P. If $k=d-2$, then C_d has singularity $\widetilde{\mathbb{K}}_{d-1}$ at the point P.

Thus, to complete the proof, I may assume that L is an irreducible component of the curve C_d. Then $C_d=L+C_{d-1}$, where C_{d-1} is a reduced curve in $\mathbb{P}^2$ of degree $d-1$ such that L is not its irreducible component. Denote by C_{d-1}^1 its proper transform on S_1. Put $n_0=\mathrm{mult}_P(C_{d-1})$ and $n_1=\mathrm{mult}_{P_1}(C_{d-1}^1)$. Then $n_0=m_0-1=d-2$ and $n_1=m_1-1$. Note that the log pair $(S_1,\lambda_1L^1+(\lambda_1m_0-1)E_1)$ is Kawamata log terminal at P by Lemma 3.6. This implies that $P_1\in C_{d-1}^1$. Hence, $n_1\geqslant 1$. One the other hand, I have

$$d-1-n_0=C_{d-1}^1\cdot L^1\geqslant n_1,$$

which implies that $n_0+n_1\leqslant d-1$. Then $n_1=1$, since $n_0=d-2$ and $n_1\neq 1$.

I have $P_1\in C_{d-1}^1$ and C_{d-1}^1 is smooth at P_1. Moreover, since

$$1=d-1-n_0=L^1\cdot C_{d-1}^1\geqslant n_1=1,$$

the curve C_{d-1}^1 intersects the curve L^1 transversally at the point P_1. Put $k=\mathrm{mult}_{P_1}(C_{d-1}^1\cdot E_1)$. Then $k\geqslant 1$. Applying Lemma 2.2 to the log pair $(S_1,\lambda_1C_{d-1}^1+\lambda_1L^1+(\lambda_1(n_0+1)-1)E_1)$ at the point P_1, I get

$$k\Big(\lambda_1(n_0+2)-1\Big)+\lambda_1\geqslant k+1.$$

Then $\lambda_1\geqslant\frac{2k+1}{kd+1}$. Then $k\geqslant d-3$ by Lemma 3.2(iii). Since

$$k\leqslant E_1\cdot C_{d-1}^1=n_0=d-2,$$

either $k=d-2$ or $k=d-3$. In the former case, P must be a singular point of type $\mathbb{T}_{d-1}$. In the latter case, P must be a singular point of type $\widetilde{\mathbb{T}}_{d-1}$. □

By Corollary 3.4 and Lemma 3.9, I may assume that $m_0\leqslant d-2$ to complete the proof of Theorems 1.13 and 1.19. Let me show that (**A**) implies that C_d is a Płoski quartic curve, and (**B**) implies that C_d is an even Płoski curve. In fact, to complete the proof of Theorems 1.13 and 1.19, it is enough to show that C_d is a Płoski curve (see Examples 1.18).

Lemma 3.10. Suppose (**A**). Then the line L is not an irreducible component of the curve C_d.

Proof. I have $\lambda=\lambda_1$. Suppose that L is an irreducible component of the curve C_d. Let me see for a contradiction. Put $C_d=L+C_{d-1}$, where C_{d-1} is a reduced curve in $\mathbb{P}^2$ of degree $d-1$ such that L is not its irreducible component. Denote by C_{d-1}^1 its proper transform on S_1. Put $n_0=\mathrm{mult}_P(C_{d-1})$ and $n_1=\mathrm{mult}_{P_1}(C_{d-1}^1)$. Then $(S_1,(\lambda_1(n_0+1)-1)E_1+\lambda_1L^1+\lambda_1C_{d-1}^1)$ is not Kawamata log terminal at P_1 and is Kawamata log terminal outside of the point P_1. In particular, $n_1\neq 0$, because $(S_1,(\lambda_1(n_0+1)-1)E_1+\lambda_1L^1)$ is Kawamata log terminal at P_1. On the other hand,

$$d-1-n_0=L^1\cdot C_{d-1}^1\geqslant n_1,$$

which implies that $n_0+n_1\leqslant d-1$. Furthermore, I have $n_0=m_0-1\leqslant d-3$.

Since $n_0 + n_1 \geqslant 2n_1$, I have $n_1 \leqslant \frac{d-1}{2}$. Then $\lambda n_1 < 1$ by Lemma 3.2(i). Thus, I can apply Theorem 2.17 to the log pair $(S_1, (\lambda_1(n_0+1)-1)E_1 + \lambda_1 L^1 + \lambda_1 C^1_{d-1})$ at the point P_1. This gives either

$$\lambda_1\big(d-1-n_0\big) = \lambda_1 C^1_{d-1}\cdot L^1 \geqslant 2\Big(2-\lambda_1\big(n_0+1\big)\Big)$$

or

$$\lambda_1 n_0 = \lambda_1 C^1_{d-1}\cdot E_1 \geqslant 2\big(1-\lambda_1\big)$$

(or both). In the former case, I have $\lambda_1(d+1+n_0) \geqslant 4$. In the latter case, I have $\lambda_1(n_0+2) > 2$. This implies $\lambda_1(d-1) \geqslant 2$ in both cases, since $n_0 \leqslant d-3$. But $\lambda_1(d-1) < 2$ by Lemma 3.2(i). $\square$

Let $f_2\colon S_2 \to S_1$ be a blow up of the point P_1, and let E_2 be its exceptional curve. Denote by C^2_d the proper transform on S_2 of the curve C_d, and denote by E^2_1 the proper transform on S_2 of the curve E_1. Then

$$K_{S_2} + \lambda C^2_d + \big(\lambda m_0 - 1\big)E^2_1 + \big(\lambda(m_0+m_1)-2\big)E_2 \sim_{\mathbb{Q}} f_2^*\Big(K_{S_1} + \lambda C^1_d + \big(\lambda m_0 - 1\big)E_1\Big).$$

By Remark 2.11, the log pair $(S_2, \lambda C^2_d + (\lambda m_0 - 1)E^2_1 + (\lambda(m_0+m_1)-2)E_2)$ is not Kawamata log terminal at some point $P_2 \in E_2$. Moreover, it is Kawamata log terminal outside of the point P_2 by Lemmas 2.14, since $\lambda(m_0+m_1) < 3$ by

Lemma 3.11. One has $m_0 + m_1 \leqslant d$.

Proof. By Lemma 3.8, (**B**) implies $m_0 + m_1 \leqslant d$. If L is not an irreducible component of the curve C_d, then

$$d - m_0 = C^1_d \cdot L^1 \geqslant m_1.$$

Thus, the assertion follows from Lemma 3.10. $\square$

Put $m_2 = \mathrm{mult}_{P_2}(C^2_d)$.

Lemma 3.12. One has $P_2 \neq E^2_1 \cap E_2$.

Proof. Suppose that $P_2 = E^2_1 \cap E_2$. Then

$$m_0 - m_1 = E^2_1 \cdot C^2_d \geqslant m_2,$$

which implies that $m_2 \leqslant \frac{m_0}{2}$, since $2m_2 \leqslant m_1 + m_2$. On the other hand, $m_0 \leqslant d-2$ by assumption. Thus, I have $m_2 \leqslant \frac{d-2}{2}$.

Suppose (**A**). Then $\lambda = \lambda_1$ and $\lambda_1 m_2 < 1$ by Lemma 3.2(v). Thus, I can apply Theorem 2.17 to the log pair $(S_2, \lambda_1 C^2_d + (\lambda_1 m_0 - 1)E^2_1 + (\lambda_1(m_0+m_1)-2)E_2)$. This gives either

$$\lambda_1\big(m_0 - m_1\big) = \lambda_1 C^2_d \cdot E^2_1 \geqslant 2\Big(3 - \lambda_1\big(m_0+m_1\big)\Big)$$

or

$$\lambda_1 m_1 = \lambda_1 C^2_d \cdot E_2 \geqslant 2\big(2 - \lambda_1 m_0\big)$$

(or both). The former inequality implies $\lambda_1(3m_0+m_1) \geqslant 6$. The latter inequality implies $\lambda_1(2m_0+m_1) \geqslant 4$. On the other hand, $m_0+m_1 \leqslant d$ by Lemma 3.11, and $m_0 \leqslant d-2$ by assumption. Thus, $3m_0+m_1 \leqslant 3d-4$ and $2m_0+m_1 \leqslant 2d-2$. Then $\lambda_1(3m_0+m_1) < 6$ by Lemma 3.2(vi), and $\lambda_1(2m_0+m_1) < 4$ by Lemma 3.2(i). The obtained contradiction shows (**A**) does not hold.

I see that (**B**) holds. Then $\lambda = \lambda_2$ and C_d is GIT-semistable by assumption. Moreover, arguing as in the proof of Lemma 3.8, I see that

$$\mathrm{wt}_{(3,2)}\Big(f_d\big(x_1,x_2\big)\Big) = 2m_0 + m_1 + m_2.$$

Thus, $2m_0 + m_1 + m_2 \leqslant \frac{5d}{3}$ by Lemma 3.3, because C_d is GIT-semistable by (**B**).

Let $f_3\colon S_3 \to S_2$ be a blow up of the point P_2, and let E_3 be its exceptional curve. Denote by C_d^3 the proper transform on S_3 of the curve C_d, denote by E_1^3 the proper transform on S_3 of the curve E_1, and denote by E_2^3 the proper transform on S_3 of the curve E_2. Then

$$K_{S_3} + \lambda_2 C_d^3 + \big(\lambda_2 m_0 - 1\big)E_1^3 + \big(\lambda_2(m_0+m_1) - 2\big)E_2^3 + \big(\lambda_2(2m_0+m_1+m_2) - 4\big)E_3 \sim_{\mathbb{Q}}$$
$$\sim_{\mathbb{Q}} f_3^*\Big(K_{S_2} + \lambda_2 C_d^2 + \big(\lambda_2 m_0 - 1\big)E_1^2 + \big(\lambda_2(m_0+m_1) - 2\big)E_2\Big).$$

Moreover, $\lambda_2(2m_0+m_1+m_2) - 4 < 1$, since $2m_0+m_1+m_2 < \frac{5d}{3}$. By Remark 2.11, the log pair $(S_3, \lambda_2 C_d^3 + (\lambda_2 m_0 - 1)E_1^3 + (\lambda_2(m_0+m_1) - 2)E_2^3 + (\lambda_2(2m_0+m_1+m_2) - 4)E_3)$ is not Kawamata log terminal at some point $P_3 \in E_3$ and is Kawamata log terminal outside of this point.

If $P_3 = E_1^3 \cap E_3$, then it follows from Theorem 2.13 that

$$\lambda_2\big(m_0 - m_1 - m_2\big) = \lambda_2 C_d^3 \cdot E_1^3 > 5 - \lambda_2\big(2m_0+m_1+m_2\big),$$

which implies that $m_0 > \frac{5}{3\lambda_2} = \frac{2d}{3}$, which is impossible by Corollary 3.4. If $P_3 = E_2^3 \cap E_3$, then it follows from Theorem 2.13 that

$$\lambda_2\big(m_1 - m_2\big) = \lambda_2 C_d^3 \cdot E_2^3 > 5 - \lambda_2\big(2m_0+m_1+m_2\big),$$

which implies that $m_0 + m_1 > \frac{5}{2\lambda_2} = d$, which is impossible by Corollary 3.11. Thus, I see that $P_3 \notin E_1^3 \cup E_2^3$. Then the log pair $(S_3, \lambda_2 C_d^3 + (\lambda_2(2m_0+m_1+m_2) - 4)E_3)$ is not Kawamata log terminal at P_3. Hence, Theorem 2.13 gives

$$\lambda_2 m_2 = \lambda_2 C_d^3 \cdot E_3 > 1,$$

which implies that $m_2 > \frac{1}{\lambda_2} = \frac{2d}{5}$. One the other hand, I proved earlier that $m_2 \leqslant \frac{m_0}{2}$. Thus, $m_0 > \frac{4d}{5}$, which is impossible by Corollary 3.4. The obtained contradiction completes the proof of the lemma. □

Denote by L^2 the proper transform of the line L on the surface S_2.

Lemma 3.13. One has $P_2 \neq L^2 \cap E_2$.

Proof. Suppose that $P_2 = L^2 \cap E_2$. If L is not an irreducible component of the curve C_d, then

$$d - m_0 - m_1 = L^2 \cap E_2 \geqslant m_2,$$

which implies that $m_0+m_1+m_2 \leqslant d$. Thus, if (**A**) holds, then $\lambda = \lambda_1$ and L is not an irreducible component of the curve C_d by Lemma 3.10, which implies that

$$\lambda_1 d \geqslant \lambda_1\big(m_0+m_1+m_2\big) > 3$$

by Lemma 2.15. On the other hand, $\lambda_1 d < 3$ by Lemma 3.2(iv). This shows that (**B**) holds.

Since $\lambda = \lambda_2 = \frac{5}{2d} < \frac{3}{d}$ and $\lambda_2(m_0+m_1+m_2) > 3$ by Lemma 2.15, I have $m_0+m_1+m_2 > d$. In particular, the line L must be an irreducible component of the curve C_d.

Put $C_d = L + C_{d-1}$, where C_{d-1} is a reduced curve in $\mathbb{P}^2$ of degree $d-1$ such that L is not its irreducible component. Denote by C_{d-1}^1 its proper transform on S_1, and denote by C_{d-1}^2 its proper transform on S_2. Put $n_0 = \mathrm{mult}_P(C_{d-1})$, $n_1 = \mathrm{mult}_{P_1}(C_{d-1}^1)$ and $n_2 = \mathrm{mult}_{P_2}(C_{d-1}^2)$. Then $(S_2, (\lambda_2(n_0+n_1+2) - 2)E_2 + \lambda_2 L^1 + \lambda_2 C_{d-1}^1)$ is not Kawamata log terminal at P_2 and is Kawamata log terminal outside of the point P_2. Then Theorem 2.13 implies

$$\lambda_2\big(d - 1 - n_0 - n_1\big) = \lambda_2 C_{d-1}^2 \cdot L^2 > 1 - \big(\lambda_2(n_0+n_1+2) - 2\big) = 3 - \lambda_2(n_0+n_1+2),$$

which implies that $\frac{5(d+1)}{2d} = \lambda_2(d+1) > 3$. Hence, $d = 4$. Then $\lambda = \lambda_2 = \frac{5}{8}$.

By Lemma 3.8, $n_0 + n_1 \leqslant 2$. Thus, $n_0 = n_1 = n_2 = 1$, since

$$\frac{5}{8}\big(n_0+n_1+n_2+3\big) = \lambda_2\big(m_0+m_1+m_2\big) > 3$$

by Lemma 2.15. Then C_3 is a irreducible cubic curve that is smooth at P, the line L is tangent to the curve C_3 at the point P, and P is an inflexion point of the cubic curve C_3. This implies that $\mathrm{lct}_P(\mathbb{P}^2, C_d) = \frac{2}{3}$. Since $\frac{2}{3} > \frac{5}{8} = \lambda_2$, the log pair $(\mathbb{P}^2, \lambda_2 C_d)$ must be Kawamata log terminal at the point P, which contradicts (**B**). $\square$

Let $f_3\colon S_3 \to S_2$ be a blow up of the point P_2, and let E_3 be its exceptional curve. Denote by C_d^3 the proper transform on S_3 of the curve C_d, and denote by E_2^3 the proper transform on S_3 of the curve E_2. Then

$$K_{S_3} + \lambda C_d^3 + \big(\lambda(m_0+m_1)-2\big)E_2^3 + \big(\lambda(m_0+m_1+m_2)-3\big)E_3 \sim_{\mathbb{Q}} \\ \sim_{\mathbb{Q}} f_3^*\Big(K_{S_2} + \lambda C_d^2 + \big(\lambda(m_0+m_1)-2\big)E_2\Big).$$

By Remark 2.11, the log pair $(S_3, \lambda C_d^3 + (\lambda(m_0+m_1)-2)E_2^3 + (\lambda(m_0+m_1+m_2)-3)E_3)$ is not Kawamata log terminal at some point $P_3 \in E_3$.

Lemma 3.14. One has $\lambda(m_0+m_1+m_2) \leqslant \lambda(m_0+2m_1) < 4$.

Proof. By Lemma 3.11, $m_0 + m_1 \leqslant d$. Since $2m_1 \leqslant m_0 + m_1$, I have $m_1 \leqslant \frac{d}{2}$. Then

$$\lambda\big(m_0+m_1+m_2\big) \leqslant \lambda\big(m_0+2m_1\big) \leqslant \lambda\frac{3d}{2} \leqslant \lambda_2\frac{3d}{2} = \frac{15}{4} < 4,$$

because $\lambda \leqslant \lambda_2$ and $m_2 \leqslant m_1$. $\square$

Thus, the log pair $(S_3, \lambda C_d^3 + (\lambda(m_0+m_1)-2)E_2^3 + (\lambda(m_0+m_1+m_2)-3)E_3)$ is Kawamata log terminal outside of the point P_3 by Lemma 2.14. Put $m_3 = \mathrm{mult}_{P_3}(C_d^3)$.

Lemma 3.15. One has $P_3 \neq E_2^3 \cap E_3$.

Proof. If $P_3 = E_2^3 \cap E_3$, then Theorem 2.13 gives

$$\lambda\big(m_1 - m_2\big) = \lambda C_d^3 \cdot E_2^3 > 1 - \Big(\lambda\big(m_0+m_1+m_2\big) - 3\Big) = 4 - \lambda\big(m_0+m_1+m_2\big),$$

which implies that $\lambda(m_0+2m_1) > 4$. But $\lambda(m_0+2m_1) < 4$ by Lemma 3.14. $\square$

Let $f_4\colon S_4 \to S_3$ be a blow up of the point P_3, and let E_4 be its exceptional curve. Denote by C_d^4 the proper transform on S_4 of the curve C_d, denote by E_3^4 the proper transform on S_4 of the curve E_3, and denote by L^4 the proper transform of the line L on the surface S_4. Then $(S_4, \lambda C_d^4 + (\lambda(m_0+m_1+m_2)-3)E_3^4 + (\lambda(m_0+m_1+m_2+m_3)-4)E_4)$ is not Kawamata log terminal at some point $P_4 \in E_4$ by Remark 2.11, because

$$K_{S_4} + \lambda C_d^4 + \big(\lambda(m_0+m_1+m_2)-3\big)E_3^4 + \big(\lambda(m_0+m_1+m_2+m_3)-4\big)E_4 \sim_{\mathbb{Q}} \\ \sim_{\mathbb{Q}} f_4^*\Big(K_{S_3} + \lambda C_d^3 + \big(\lambda(m_0+m_1+m_2)-3\big)E_3\Big).$$

Moreover, I have

$$2L^4 + E_1 + 2E_2 + E_3 \sim (f_1\circ f_2\circ f_3\circ f_4)^*\Big(\mathcal{O}_{\mathbb{P}^2}(2)\Big) - \\ - (f_2\circ f_3\circ f_4)^*\big(E_1\big) - (f_3\circ f_4)^*\big(E_2\big) - f_4^*\big(E_3\big) - E_4.$$

Lemma 3.16. The linear system $|2L^4 + E_1 + 2E_2 + E_3|$ is a pencil that does not have base points. Moreover, every divisor in $|2L^4+E_1+2E_2+E_3|$ that is different from $2L^4+E_1+2E_2+E_3$ is a smooth curve whose image on $\mathbb{P}^2$ is a smooth conic that is tangent to L at the point P.

Proof. All assertions follows from $P_2 \notin E_1^2 \cup L^2$ and $P_3 \notin E_2^3$. $\square$

Let C_2^4 be a general curve in $|2L^4+E_1+2E_2+E_3|$. Denote by C_2 its image on $\mathbb{P}^2$, and denote by $\mathcal{L}$ the pencil generated by $2L$ and C_2. Then P is the only base point of the pencil $\mathcal{L}$, and every conic in $\mathcal{L}$ except $2L$ and C_2 intersects C_2 at P with multiplicity 4 (cf. [5, Remark 1.14]).

Lemma 3.17. One has $m_0+m_1+m_2+m_3 \leqslant m_0+m_1+2m_2 \leqslant \frac{5}{\lambda}$. If $m_0+m_1+m_2+m_3 = \frac{5}{\lambda}$, then d is even and C_d is a union of $\frac{d}{2} \geqslant 2$ smooth conics in $\mathcal{L}$, where $d=4$ if (**A**) holds.

Proof. By Lemma 3.11, I have $m_2 + m_3 \leqslant 2m_2 \leqslant m_0 + m_1 \leqslant d$ by Lemma 3.11. This gives

$$m_0 + m_1 + m_2 + m_3 \leqslant m_0 + m_1 + 2m_2 \leqslant 2d = \frac{5}{\lambda_2} \leqslant \frac{5}{\lambda}.$$

To complete the proof, I may assume that $m_0+m_1+m_2+m_3 = \frac{5}{\lambda}$. Then all inequalities above must be equalities. Thus, I have $m_2 = m_3 = \frac{d}{2}$ and $\lambda_1 = \lambda_2$. In particular, if (**A**) holds, then $d = 4$, because $\lambda_1 < \lambda_2 = \frac{5}{2d}$ for $d \geqslant 5$ by Lemma 3.2(vii). Moreover, since $m_0 \geqslant m_1 \geqslant m_2 = \frac{d}{2}$ and $m_0 + m_1 \leqslant d$, I see that $m_0 = m_1 = \frac{d}{2}$. Thus, d is even and

$$C_d^4 \sim \frac{d}{2}\Big(2L^4 + E_1 + 2E_2 + E_3\Big),$$

where $d = 4$ if (**A**) holds. Since $|2L^4 + E_1 + 2E_2 + E_3|$ is a free pencil and C_d^4 is reduced, it follows from Lemma 3.16 that C_d^4 is a union of $\frac{d}{2}$ smooth curves in $|2L^4 + E_1 + 2E_2 + E_3|$. In particular, L^4 is not an irreducible component of C_d^4. Thus, the curve C_d is a union of $\frac{d}{2}$ smooth conics in $\mathcal{L}$, where $d = 4$ if (**A**) holds. □

Thus, if $m_0 + m_1 + m_2 + m_3 = \frac{5}{\lambda}$, then Theorems 1.13 and 1.19 are proved. Let me show that the inequality $m_0 + m_1 + m_2 + m_3 < \frac{5}{\lambda}$ is impossible. Suppose that it holds. Then the log pair $(S_4, \lambda C_d^4 + (\lambda(m_0 + m_1 + m_2) - 3)E_3^4 + (\lambda(m_0 + m_1 + m_2 + m_3) - 4)E_4)$ is Kawamata log terminal outside of the point P_4 by Lemma 2.14.

Lemma 3.18. One has $P_4 \neq E_3^4 \cap E_4$.

Proof. By Lemma 3.17, $m_0 + m_1 + 2m_2 \leqslant \frac{5}{\lambda}$. If $P_4 = E_3^4 \cap E_4$, then Theorem 2.13 gives

$$\lambda\big(m_2 - m_3\big) = \lambda C_d^4 \cdot E_3^4 > 5 - \lambda\big(m_0 + m_1 + m_2 + m_3\big),$$

which implies that $m_0 + m_1 + 2m_2 > \frac{5}{\lambda}$. This shows that $P_4 \neq E_3^4 \cap E_4$. □

Corollary 3.19. The log pair $(S_4, \lambda C_d^4 + (\lambda(m_0 + m_1 + m_2 + m_3) - 4)E_4)$ is not Kawamata log terminal at P_4 and is Kawamata log terminal outside of the point P_4.

Let Z^4 be the curve in $|2L^4 + E_1 + 2E_2 + E_3|$ that passes through the point P_4. Then Z^4 is a smooth irreducible curve by Lemma 3.13. Denote by Z the proper transform of this curve on $\mathbb{P}^2$. Then Z is a smooth conic in the pencil $\mathcal{L}$ by Lemma 3.16.

Lemma 3.20. The conic Z is not an irreducible component of the curve C_d.

Proof. Suppose that Z is an irreducible component of the curve C_d. Then $C_d = Z + C_{d-2}$, where C_{d-2} is a reduced curve in $\mathbb{P}^2$ of degree $d-2$ such that Z is not its irreducible component. Denote by C_{d-2}^1, C_{d-2}^2, C_{d-2}^3 and C_{d-2}^4 its proper transforms on the surfaces S_1, S_2, S_3 and S_4, respectively. Put $n_0 = \mathrm{mult}_P(C_{d-2})$, $n_1 = \mathrm{mult}_{P_1}(C_{d-2}^1)$, $n_2 = \mathrm{mult}_{P_2}(C_{d-2}^2)$, $n_3 = \mathrm{mult}_{P_3}(C_{d-2}^3)$ and $n_4 = \mathrm{mult}_{P_4}(C_{d-2}^4)$. Then $(S_4, \lambda C_{d-2}^4 + \lambda Z^4 + (\lambda(n_0 + n_1 + n_2 + n_3 + 4) - 4)E_4)$ is not Kawamata log terminal at P_4 and is Kawamata log terminal outside of the point P_4 by Corollary 3.19. Thus, applying Theorem 2.13, I get

$$\lambda\Big(2\big(d-2\big) - n_0 - n_1 - n_2 - n_3\Big) = \lambda C_{d-2}^4 \cdot Z^4 > 5 - \lambda\big(n_0 + n_1 + n_2 + n_3 + 4\big),$$

which implies that $\lambda > \frac{5}{2d}$. This is impossible, since $\lambda \leqslant \lambda_2 = \frac{5}{2d}$. □

Put $m_4 = \mathrm{mult}_{P_4}(C_d^4)$. Since Z is not an irreducible component of the curve C_d, I have

$$2d - \sum_{i=0}^{3} m_i = Z^4 \cdot C_d^4 \geqslant m_4,$$

which gives $\sum_{i=0}^{4} m_i \leqslant 2d$. On the other hand, $\sum_{i=0}^{4} m_i > \frac{5}{\lambda}$ by Lemma 2.15. Thus, I have $\lambda > \frac{5}{2d}$, which is impossible, since $\lambda \leqslant \lambda_2 = \frac{5}{2d}$. The obtained contradiction completes the proof of Theorems 1.13 and 1.19.

4. Smooth surfaces in $\mathbb{P}^3$

The purpose of this section is to prove Theorem 1.21. Let S be a smooth surface in $\mathbb{P}^3$ of degree $d \geqslant 3$, let H_S be its hyperplane section, let P be a point in S, let T_P be the hyperplane section of the surface S that is singular at P. Then T_P is reduced by Lemma 2.6. Put $\lambda = \frac{2d-3}{d(d-2)}$.

Proposition 4.1. Let D be any effective $\mathbb{Q}$-divisor on S such that $D \sim_{\mathbb{Q}} H_S$. Suppose that $\mathrm{Supp}(D)$ does not contain at least one irreducible component of the curve T_P. Then $(S, \lambda D)$ is log canonical at P.

If $d = 3$, then Proposition 4.1 is [5, Corollary 1.13] that implies two important results. It implies [26, Theorem 1.3], which implies that *all* smooth cubic surfaces are Kähler–Einstein by [7, Theorem 2]. By [17, Corollary 3.2], [5, Corollary 1.13] also implies that affine cones over smooth cubic surfaces do not admit effective actions of the additive group $\mathbb{G}_a$. On the other hand, Proposition 4.1 and Theorem 1.13 imply Theorem 1.21.

Proof of Theorem 1.21. Suppose that $\alpha_S^{H_S}(P) < \lambda$. Put $\mu = \mathrm{lct}_P(S, T_P)$. By Theorem 1.13, it is enough to show that $\alpha_S^{H_S}(P) \geqslant \mu$ in order to prove Theorem 1.21. Suppose that $\alpha_S^{H_S}(P) < \mu$. Then there exists an effective $\mathbb{Q}$-divisor D on the surface S such that $D \sim_{\mathbb{Q}} H_S$ and $(S, \lambda D)$ and $(S, \mu D)$ are not log canonical at the point P, since $\alpha_S^{H_S}(P) < \lambda$. Put

$$D_\epsilon = (1+\epsilon)D - \epsilon T_P$$

for some rational number ϵ. Since $T_P \neq D$, there exists the greatest rational number ϵ_0 such that the divisor D_{ϵ_0} is effective. Put $D' = D_{\epsilon_0}$. Then $\mathrm{Supp}(D')$ does not contain at least one irreducible component of $\mathrm{Supp}(T_P)$. Thus, the log pair $(S, \lambda D')$ is log canonical at P by Proposition 4.1. On the other hand, the log pair $(S, \mu T_P)$ is log canonical at P, which implies that $(S, \mu D')$ is not log canonical at P by Remark 2.4. Then $\mu > \lambda$. In particular, $(S, \lambda T_P)$ is log canonical at P. Then $(S, \lambda D')$ is not log canonical at P by Remark 2.4. The latter is impossible, since I already proved that $(S, \lambda D')$ is log canonical at P. □

In the remaining part of the section, I will prove Proposition 4.1. Note that I will do this *without* using [5, Corollary 1.13]. Let me start with

Lemma 4.2. The following assertions hold:

- (i) $\lambda \leqslant \frac{2}{d-1}$,
- (ii) if $d \geqslant 5$, then $\lambda \leqslant \frac{3}{d+1}$,
- (iii) if $d \geqslant 5$, then $\lambda \leqslant \frac{4}{d+3}$,
- (iv) If $d \geqslant 6$, then $\lambda \leqslant \frac{3}{d+2}$,
- (v) $\lambda \leqslant \frac{4}{d+1}$,
- (vi) $\lambda \leqslant \frac{3}{d}$.

Proof. The equality $\frac{2}{d-1} = \lambda + \frac{d-3}{d(d-1)(d-2)}$ implies (i), $\frac{4}{d+1} = \lambda + \frac{d^2-5d+3}{d(d+1)(d-2)}$ implies (ii), and $\frac{4}{d+3} = \lambda + \frac{2d^2-11d+9}{d(d+3)(d-2)}$ implies (iii). Similarly, (iv) follows from $\frac{3}{d+2} = \lambda + \frac{d^2-7d+6}{d(d^2-4)}$, (v) follows from $\frac{4}{d+1} = \lambda + \frac{2d^2-7d+3}{d(d+1)(d-2)}$, and (vi) follows from $\frac{3}{d} = \lambda + \frac{d-3}{d(d-2)}$. □

Let n be the number of irreducible components of the curve T_P. Put $T_P = T_1 + \cdots + T_n$, where each T_i is an irreducible curve. For every T_i, denote its degree by d_i, and put $t_i = \mathrm{mult}_P(T_i)$.

Lemma 4.3. Suppose that $n \geqslant 2$. Then $T_i \cdot T_i = -d_i(d - d_i - 1)$ for every T_i, and $T_i \cdot T_j = d_i d_j$ for every T_i and T_j such that $T_i \neq T_j$.

Proof. The curve T_P is cut out on S by a hyperplane $H \subset \mathbb{P}^2$. Then $H \cong \mathbb{P}^2$. Hence, for every T_i and T_j such that $T_i \neq T_j$, I have $(T_i \cdot T_j)_S = (T_i \cdot T_j)_H = d_i d_j$. In particular, I have

$$d_1 = T_P \cdot T_1 = T_1^2 + \sum_{i=2}^{n} T_i \cdot T_1 = T_1^2 + \sum_{i=2}^{n} d_i d_1 = T_1^2 + (d - d_1) d_1,$$

which gives $T_1\cdot T_1=-d_1(d-d_1-1)$. Similarly, I have $T_i\cdot T_i=-d_i(d-d_i-1)$ for every T_i. $\square$

Let D be any effective $\mathbb{Q}$-divisor on S such that $D\sim_{\mathbb{Q}} H_S$. Suppose that $\mathrm{Supp}(D)$ does not contain at least one irreducible component of the curve T_P. To prove Proposition 4.1, I must show that $(S,\lambda D)$ is log canonical at P. Suppose that this is not the case. Let me seek for a contradiction. Without loss of generality, I may assume that $\mathrm{Supp}(D)$ does not contain the curve T_n.

Lemma 4.4. Suppose that $n\geqslant 2$. Let k be a positive integer such that $k\leqslant n-1$. Write $D=\sum_{i=1}^{k}a_iT_i+\Delta$, where each a_i is a non-negative rational number, and Δ is an effective $\mathbb{Q}$-divisor on S whose support does not contain the curves $T_1,\ldots,T_k$. Put $k_0=\mathrm{mult}_P(\Delta)$. Then

$$\sum_{i=1}^{k}a_id_id_n\leqslant d_n-t_nk_0.$$

In particular, $\sum_{i=1}^{k}a_id_i\leqslant 1$ and each a_i does not exceed $\frac{1}{d_i}$.

Proof. Since T_n is not contained in $\mathrm{Supp}(D)$, it is not contained in $\mathrm{Supp}(\Delta)$. Then

$$d_n=T_n\cdot D=T_n\cdot\left(\sum_{i=1}^{n}a_iT_i+\Delta\right)=\sum_{i=1}^{n}a_id_id_n+T_n\cdot\Delta\geqslant\sum_{i=1}^{n}a_id_id_n+t_nk_0,$$

which implies the required inequality. $\square$

Put $m_0=\mathrm{mult}_P(D)$.

Lemma 4.5. Suppose that $P\in T_n$. Then $d_n>\frac{d-1}{2}$. If $n\geqslant 2$, then T_n is smooth at P.

Proof. Since T_n is not contained in the support of the divisor D, I have

$$d\geqslant d_n=T_n\cdot D\geqslant t_nm_0,$$

which implies that $m_0\leqslant\frac{d_n}{t_n}$. Since $m_0>\frac{1}{\lambda}$ by Lemma 2.5, I have $d_n>\frac{d-1}{2}$ by Lemma 4.2(i). Moreover, if $n\geqslant 2$ and $t_n\geqslant 2$, then it follows from Lemma 2.5 that

$$\frac{1}{\lambda}<m_0\leqslant\frac{d_n}{t_n}\leqslant\frac{d-1}{t_n}\leqslant\frac{d-1}{2},$$

which is impossible by Lemma 4.2(i). $\square$

Corollary 4.6. The point P is not a star point.

Now I am ready to use Theorem 2.16 to prove

Lemma 4.7. Suppose that $n\geqslant 3$ and P is contained in at least two irreducible components of the curve T_P that are different from T_n and that are both smooth at P. Then they are tangent to each other at P.

Proof. Without loss of generality, I may assume that $P\in T_1\cap T_2$ and $t_1=t_2=1$. I must show that T_1 and T_2 are tangent to each other at P. Suppose that this is not the case. Let me seek for a contradiction. Put $D=aT_1+bT_2+\Delta$, where a and b are non-negative rational numbers, and Δ is an effective $\mathbb{Q}$-divisor on the surface S whose support does not contain the curves T_1 and T_2. Then $ad_1+bd_2\leqslant 1$ by Lemma 4.4.

Put $k_0=\mathrm{mult}(\Delta)$. Then

$$d_1+ad_1\big(d-d_1-1\big)-bd_1d_2=\Delta\cdot T_1\geqslant k_0$$

by Lemma 4.3. Similarly, I have

$$d_2-ad_1d_2+bd_2\big(d-d_2-1\big)=\Delta\cdot T_2\geqslant k_0.$$

Adding these two inequalities together and using $ad_1+bd_2\leqslant 1$, I get

$$2k_0\leqslant d_1+d_2+\big(ad_1+ad_2\big)\big(d-d_1-d_2-1\big)\leqslant d_1+d_2+\big(d-d_1-d_2-1\big)=d-1.$$

Thus, $k_0 \leqslant \frac{1}{\lambda}$ by Lemma 4.2(i).

Since $\lambda k_0 \leqslant 1$, I can apply Theorem 2.16 to the log pair $(S, \lambda a T_1 + \lambda b T_2 + \lambda\Delta)$ at the point P. This gives either $\lambda\Delta \cdot T_1 > 2(1-\lambda b)$ or $\lambda\Delta \cdot T_2 > 2(1-\lambda a)$. Without loss of generality, I may assume that $\lambda\Delta \cdot T_2 > 2(1-\lambda a)$. Then

$$d_2 + bd_2\big(d - d_2 - 1\big) - ad_1d_2 = \Delta \cdot T_2 > \frac{2}{\lambda} - 2a. \tag{4.8}$$

Applying Theorem 2.13 to the log pair $(S, \lambda a T_1 + \lambda b T_2 + \lambda\Delta)$ and the curve T_1 at the point P, I get

$$d_1 + ad_1\big(d - d_1 - 1\big) = \Big(\lambda b T_2 + \lambda\Delta\Big) \cdot T_1 > \frac{1}{\lambda}.$$

Adding this inequality to (4.8), I get

$$d + 1 \geqslant d - 1 + 2a \geqslant d_1 + d_2 + \big(ad_1 + bd_2\big)\big(d - d_1 - d_2 - 1\big) + 2a > \frac{3}{\lambda},$$

because $ad_1 + bd_2 \leqslant 1$. Thus, it follows from Lemma 4.2(ii) that either $d = 3$ or $d = 4$.

If $d = 3$, then $n = 3$ and $d_1 = d_2 = d_3 = \lambda = 1$, which implies that $a + b > 1$ by (4.8). Since $ad_1 + bd_2 \leqslant 1$, I see that $d = 4$. Then $\lambda = \frac{5}{8}$ and $d_1 + d_2 \leqslant 3$. If $d_1 = d_1 = 1$, then (4.8) gives $2b + a > \frac{11}{5}$. If $d_1 = 1$ and $d_2 = 2$, then (4.8) gives $b > \frac{3}{5}$. If $d_1 = 2$ and $d_2 = 1$, then (4.8) gives $b > \frac{11}{5}$. All these three inequalities are inconsistent, because $ad_1 + bd_2 \leqslant 1$. The obtained contradiction completes the proof of the lemma. □

Note that every line contained in the surfaces S that passes through P must be an irreducible component of the curve T_P. Moreover, the curve T_n cannot be a line by Lemma 4.5. Therefore, Lemma 4.7 implies

Corollary 4.9. There exists at most one line in S that passes through P.

Corollary 4.10. One has $n < d$.

To apply Lemma 4.7, I need

Lemma 4.11. Suppose that $n \geqslant 3$ and P is contained in at least two irreducible components of the curve T_P that are different from T_n. Then these curves are both smooth at P.

Proof. Without loss of generality, I may assume that $P \in T_1 \cap T_2$ and $t_1 \leqslant t_2$. I have to show that $t_1 = t_2 = 1$. By Corollary 4.10, $d \neq 3$. If $d = 4$, then $n \leqslant 4$, and the curves T_1, T_2 and T_4 are either lines or conics. So, I may assume that $d \geqslant 5$. Put $D = aT_1 + bT_2 + \Delta$, where a and b are non-negative rational numbers, and Δ is an effective $\mathbb{Q}$-divisor on the surface S whose support does not contain the curves T_1 and T_2. Put $k_0 = \mathrm{mult}_P(\Delta)$. Then $m_0 = k_0 + at_1 + bt_2$. Moreover, $ad_1 + bd_2 \leqslant 1$ by Lemma 4.4. On the other hand, it follows from Lemma 4.3 that

$$d - 1 \geqslant d_1 + d_2 + \big(ad_1 + ad_2\big)\big(d - d_1 - d_2 - 1\big) = \Delta \cdot \Big(T_1 + T_2\Big) \geqslant k_0\big(t_1 + t_2\big),$$

because $ad_1 + bd_2 \leqslant 1$. Thus, $k_0 \leqslant \frac{d-1}{t_1+t_2}$. Thus, if $t_1 + t_2 \geqslant 4$, then

$$m_0 = k_0 + at_1 + bt_2 \leqslant k_0 + ad_1 + bd_2 \leqslant \frac{d-1}{t_1 + t_2} + ad_1 + bd_2 \leqslant \frac{d-1}{t_1+t_2} + 1 \leqslant \frac{d+3}{4}$$

because $ad_1 + bd_2 \leqslant 1$. Since $m_0 > \frac{1}{\lambda}$ by Lemma 2.5, the inequality $m_0 \leqslant \frac{d+3}{4}$ gives $\lambda > \frac{d+3}{4}$, which is impossible by Lemma 4.2(iii). Thus, $t_1 + t_2 \leqslant 3$. Since $t_1 \leqslant t_2$, I have $t_1 = 1$ and $t_2 \leqslant 2$.

To complete the proof of the lemma, I have to prove that $t_2 = 1$. Suppose $t_2 \neq 1$. Then $t_2 = 2$, since $t_1 + t_2 \leqslant 3$. Since $k_0 \leqslant \frac{d-1}{t_1+t_2} = \frac{d-1}{3}$ and $ad_1 + bd_2 \leqslant 1$, I have

$$m_0 = k_0 + at_1 + bt_2 \leqslant k_0 + ad_1 + bd_2 \leqslant \frac{d-1}{32} + ad_1 + bd_2 \leqslant \frac{d-1}{t_1+t_2} + 1 = \frac{d+2}{3}.$$

On the other hand, $m_0 > \frac{1}{\lambda}$ by Lemma 2.5. So, $\lambda > \frac{3}{d+2}$. Then $d = 5$ by Lemma 4.2(iv).

Since $d = 5$, I have $n = 3$, $d_1 = 1$, $d_2 = 3$ and $d_3 = 1$, because $t_1 = 1$ and $t_2 = 2$. Applying Theorem 2.13 to the log pair $(S, \lambda a T_1 + \lambda b T_2 + \lambda\Delta)$, I get

$$1 + 3a = d_1 + ad_1\big(d - d_1 - 1\big) = \Big(\lambda b T_2 + \lambda\Delta\Big) \cdot T_1 > \frac{1}{\lambda} = \frac{15}{7},$$

which gives $a > \frac{8}{21}$. On the other hand, $a + 3b \leqslant 1$, because $ad_1 + bd_2 \leqslant 1$. Since $m_0 > \frac{1}{\lambda} = \frac{15}{7}$ by Lemma 2.5, I see that

$$\frac{15}{7} - \frac{1}{9} = \frac{128}{63} > \frac{8 - 5a}{3} = \frac{3 - a + \frac{7(1-a)}{3}}{2} = \frac{3 - a + 7b}{2} = \frac{3 - 3a + 3b}{2} + a + 2b =$$
$$= \frac{\Delta \cdot T_2}{2} + a + 2b \geqslant \frac{\mathrm{mult}_P\Big(\Delta \cdot T_2\Big)}{2} + a + 2b \geqslant \frac{t_2 k_0}{2} + a + 2b = k_0 + a + 2b = m_0 > \frac{15}{7},$$

which is absurd. □

Now I am ready to prove

Lemma 4.12. One has $m_0 \leqslant \frac{d+1}{2}$.

Proof. Suppose that $m_0 > \frac{d+1}{2}$. Let me seek for a contradiction. If $n = 1$, then

$$d = T_n \cdot D \geqslant 2m_0,$$

which implies that $m_0 \leqslant \frac{d}{2}$. Thus, $n \geqslant 2$. Then either $t_n = 0$ or $t_n = 1$ by Lemma 4.5. Hence, there is an irreducible component of T_P that passes through P and is different from T_n, because T_P is singular at P. Without loss of generality, I may assume that $t_1 \geqslant 1$.

Put $D = aT_1 + \Omega$, where a is a non-negative rational number, and Ω is an effective $\mathbb{Q}$-divisor on the surface S whose support does not contain the curve T_1. Then $a \leqslant \frac{1}{d_1}$ by Lemma 4.4. Put $n_0 = \mathrm{mult}_P(\Omega)$. Then $m_0 = n_0 + at_1$.

By Lemma 4.4, $t_n n_0 \leqslant d_n - ad_1 d_n$. By Lemma 4.3, I have

$$d_1 + ad_1(d - d_1 - 1) = \Omega \cdot T_1 \geqslant t_1 n_0, \tag{4.13}$$

Adding these two inequalities, I get $(t_1 + t_n)n_0 \leqslant d_1 + d_n + ad_1(d - d_1 - d_n - 1)$. Hence, if $n \geqslant 3$ and $t_n = 1$, then

$$2n_0 \leqslant \big(t_1 + t_n\big)n_0 \leqslant d_1 + d_n + ad_1\big(d - d_1 - d_n - 1\big) \leqslant d - 1 \leqslant d - ad_1,$$

because $a \leqslant \frac{1}{d_1}$. Similarly, if $n = 2$ and $t_n = 1$, then

$$2n_0 \leqslant \big(t_1 + t_n\big)n_0 \leqslant d_1 + d_2 + ad_1\big(d - d_1 - d_2 - 1\big) = d_1 + d_2 - ad_1 = d - ad_1.$$

Thus, if $t_n = 1$, then $n_0 \leqslant \frac{d - ad_1}{2}$. On the other hand, if $n_0 \leqslant \frac{d-ad_1}{2}$, then

$$\frac{d+1}{2} < m_0 = n_0 + at_1 \leqslant n_0 + ad_1 \leqslant \frac{d - ad_1}{2} + ad_1 = \frac{d + ad_1}{2} \leqslant \frac{d+1}{2},$$

because $a \leqslant \frac{1}{d_1}$. This shows that $t_n = 0$.

If $t_1 \geqslant 2$, then it follows from (4.13) that

$$\frac{d+1}{2} < m_0 = n_0 + at_1 \leqslant n_0 + ad_1 \leqslant \frac{d_1 + ad_1(d - d_1 - 1)}{2} + ad_1 = \frac{d_1 + ad_1(d - d_1 + 1)}{2} \leqslant \frac{d+1}{2},$$

because $a \leqslant \frac{1}{d_1}$. This shows that $t_1 = 1$.

Since $t_1 = 1$ and $t_n = 0$, there exists an irreducible component of the curve T_P that passes through P and is different from T_1 and T_n. In particular, $n \geqslant 3$. Without loss of generality, I may assume that this irreducible component is T_2. Then T_2 is smooth at P by Lemma 4.11.

Put $D = aT_1 + bT_2 + \Delta$, where b is a non-negative rational number, and Δ is an effective $\mathbb{Q}$-divisor S whose support does not contain the curves T_1 and T_2. Put $k_0 = \mathrm{mult}_P(\Delta)$. Then $ad_1 + bd_2 \leqslant 1$ by Lemma 4.4. Thus, it follows from Lemma 4.3 that

$$2k_0 \leqslant \Delta \cdot \Big(T_1 + T_2\Big) = d_1 + d_2 + \big(ad_1 + ad_2\big)\big(d - d_1 - d_2 - 1\big) \leqslant d - 1,$$

which implies $k_0 \leqslant \frac{d-1}{2}$. Then

$$\frac{d+1}{2} < m_0 = k_0 + at_1 + bt_2 \leqslant k_0 + ad_1 + bd_2 \leqslant \frac{d-1}{2} + ad_1 + bd_2 \leqslant \frac{d-1}{2} + 1 = \frac{d+1}{2},$$

because $ad_1 + bd_2 \leqslant 1$. The obtained contradiction completes the proof of the lemma. $\square$

Let $f_1\colon S_1 \to S$ be a blow up of the point P, and let E_1 be its exceptional curve. Denote by D^1 the proper transform of the $\mathbb{Q}$-divisor D on the surface S_1. Then

$$K_{S_1} + \lambda D^1 + \big(\lambda m_0 - 1\big)E_1 \sim_{\mathbb{Q}} f_1^*\Big(K_S + \lambda D\Big),$$

which implies that $(S_1, \lambda D^1 + (\lambda m_0 - 1)E_1)$ is not log canonical at some point $P_1 \in E_1$.

Lemma 4.14. One has $\lambda m_0 \leqslant 2$.

Proof. By Lemma 4.12, $m_0 \leqslant \frac{d+1}{2}$. By Lemma 4.2(v), $\lambda \leqslant \frac{4}{d+1}$. This gives $\lambda m_0 \leqslant 2$. $\square$

Thus, the log pair $(S_1, \lambda D^1 + (\lambda m_0 - 1)E_1)$ is log canonical at every point of the curve E_1 that is different from P_1 by Lemma 2.14. Since $(S, \lambda D)$ is log canonical outside of finitely many points by Lemma 2.6, I see that the log pair $(S_1, \lambda D^1 + (\lambda m_0 - 1)E_1)$ is log canonical at a punctured neighborhood of the point P_1. Put $m_1 = \mathrm{mult}_{P_1}(D^1)$. Then Lemma 2.5 gives

Corollary 4.15. One has $m_0 + m_1 > \frac{2}{\lambda}$.

For each curve T_i, denote by T_i^1 its proper transform on S_1. Put $T_P^1 = \sum_{i=1}^n T_i^1$.

Lemma 4.16. One has $P_1 \notin T_P^1$.

Proof. Suppose that $P_1 \in T_P^1$. If T_P is irreducible, then $d - 2m_0 = T_P^1 \cdot D^1 \geqslant m_1$. On the other hand, if $m_1 + 2m_0 \leqslant d$, then

$$\frac{3}{\lambda} < m_1 + 2m_0 \leqslant d,$$

because $2m_0 \geqslant m_0 + m_1 > \frac{2}{\lambda}$ by Corollary 4.15. Thus, $n \geqslant 2$, because $\lambda \leqslant \frac{3}{d}$ by Lemma 4.2(vi). Similarly, $P_1 \notin T_n$. Indeed, if $P_1 \in T_n$, then

$$d - 1 - m_0 \geqslant d_n - m_0 = d_n - m_0 t_n = T_n^1 \cdot D^1 \geqslant m_1,$$

which is impossible, because $m_0 + m_1 > \frac{2}{\lambda}$ by Corollary 4.15, and $\lambda \leqslant \frac{2}{d-1}$ by Lemma 4.2(i).

Without loss of generality, I may assume that $P_1 \in T_1^1$. Put $D = aT_1 + \Omega$, where a is a non-negative rational number, and Ω is an effective $\mathbb{Q}$-divisor on S whose support does not contain the curve T_1. Put $n_0 = \mathrm{mult}_P(\Omega)$. Then $m_0 = n_0 + at_1$.

Denote by Ω^1 the proper transform of the $\mathbb{Q}$-divisor Ω on the surface S_1. Put $n_1 = \mathrm{mult}_{P_1}(\Omega^1)$ and $t_1^1 = \mathrm{mult}_{P_1}(T_1^1)$. Then $n_0 t_1 + n_1 t_1^1 \leqslant d_1 + ad_1(d - d_1 - 1)$, because

$$d_1 + ad_1\big(d - d_1 - 1\big) - n_0 t_1 = T_1^1 \cdot \Omega^1 \geqslant t_1^1 n_1.$$

Note that $t_1^1 \leqslant t_1$. Moreover, $a \leqslant \frac{1}{d_1}$ by Lemma 4.4. Thus, if $t_1^1 \geqslant 2$, then

$$2\big(n_0 + n_1\big) \leqslant t_1^1\big(n_0 + n_1\big) \leqslant n_0 t_1 + n_1 t_1^1 \leqslant d_1 + ad_1\big(d - d_1 - 1\big) \leqslant d_1 + \big(d - d_1 - 1\big) = d - 1,$$

which implies that $n_0 + n_1 \leqslant \frac{d-1}{2}$. Moreover, if $n_0 + n_1 \leqslant \frac{d-1}{2}$, then it follows from Corollary 4.15 that

$$\frac{d+3}{2} = 2 + \frac{d-1}{2} \geqslant 2ad_1 + \frac{d-1}{2} \geqslant 2at_1 + \frac{d-1}{2} \geqslant a\big(t_1 + t_1^1\big) + n_0 + n_1 = m_0 + m_1 > \frac{2}{\lambda}$$

which only possible if $d \leqslant 4$ by Lemma 4.2(iii). Thus, if $d \geqslant 5$, then $t_1^1 = 1$. Furthermore, if $d \leqslant 4$, then $d_1 \leqslant 3$, which implies that $t_1^1 \leqslant 1$. This shows that $t_1^1 = 1$ in all cases. Thus, the curve T_1^1 is smooth at P_1.

Applying Theorem 2.12 to the log pair $(S_1, \lambda\Omega^1 + \lambda a T_1^1 + (\lambda(n_0 + at_1) - 1)E_1)$ and the curve T_1^1 at the point P_1 gives

$$\lambda\big(d - 1 - n_0 t_1\big) \geqslant \lambda\Big(d_1 + ad_1\big(d - d_1 - 1\big) - n_0 t_1\Big) = \lambda\Omega^1 \cdot T_1^1 > 2 - \lambda\big(n_0 + at_1\big),$$

because $a \leqslant \frac{1}{d_1}$. Thus, I have $d-1+at_1-n_0(t_1-1) > \frac{2}{\lambda}$. But $m_0 = at_1+n_0 > \frac{1}{\lambda}$ by Lemma 2.5. Adding these inequalities, I get

$$d-1+2at_1-n_0(t_1-2) > \frac{3}{\lambda}. \tag{4.17}$$

If $t_1 \geqslant 2$, this gives

$$d+1 \geqslant d-1+2ad_1 \geqslant d-1+2at_1 \geqslant d-1+2at_1-n_0(t_1-2) > \frac{3}{\lambda}.$$

because $a \leqslant \frac{1}{d_1}$. One the other hand, if $d \geqslant 5$, then $\lambda \leqslant \frac{3}{d+1}$ by Lemma 4.2(ii). Thus, if $d \geqslant 5$, then $t_1 = 1$. Moreover, if $d = 3$, then $d_1 \leqslant 2$, which implies that $t_1 = 1$ as well. Furthermore, if $d = 4$ and $t_1 \neq 1$, then $d_1 = 3$, $t_1 = 2$, $\lambda = \frac{5}{8}$, which implies $\frac{1}{3} = \frac{1}{d_1} \geqslant a > \frac{9}{20}$ by (4.17). Hence, the curve T_1 is smooth at P.

Since $a \leqslant \frac{1}{d_1}$, I have

$$d-1-n_0 \geqslant d_1 + ad_1\big(d-d_1-1\big) - n_0 = \Omega^1 \cdot T_1^1 \geqslant n_1,$$

which implies that $n_1 \leqslant \frac{n_0+n_1}{2} \leqslant \frac{d-1}{2}$. Then $\lambda n_1 \leqslant 1$ by Lemma 4.2(i). Hence, I can apply Theorem 2.16 to the log pair $(S_1, \lambda\Omega^1 + \lambda a T_1^1 + (\lambda(n_0+at_1)-1)E_1)$ at the point P_1. This gives either

$$\Omega^1 \cdot T_1^1 > \frac{4}{\lambda} - 2(n_0+a)$$

or

$$\Omega^1 \cdot E_1 > \frac{2}{\lambda} - 2a$$

(or both). Since $a \leqslant \frac{1}{d_1}$, the former inequality gives

$$d-1-n_0 \geqslant d_1 + ad_1\big(d-d_1-1\big) - n_0 = \Omega^1 \cdot T_1^1 > \frac{4}{\lambda} - 2(n_0+a).$$

The latter inequality gives

$$n_0 = \lambda\Omega^1 \cdot E_1 > \frac{2}{\lambda} - 2a.$$

Thus, either $d-1+2a+n_0 > \frac{4}{\lambda}$ or $2a+n_0 > \frac{2}{\lambda}$ (or both).

If $t_n \geqslant 1$, then $d_n \neq 1$ by Lemma 4.5. Thus, if $t_n \geqslant 1$, then

$$d-1 \geqslant d_n \geqslant ad_1d_n + n_0 \geqslant 2a+n_0$$

by Lemma 4.4. Therefore, if $t_n \geqslant 1$, then $2(d-1) \geqslant d-1+2a+n_0 > \frac{4}{\lambda}$ or $d-1 \geqslant 2a+n_0 > \frac{2}{\lambda}$, because $d-1+2a+n_0 > \frac{4}{\lambda}$ or $2a+n_0 > \frac{2}{\lambda}$. In both cases, I get $\lambda > \frac{d-1}{2}$, which is impossible by Lemma 4.2(i). Thus, $t_n = 0$, so that $P \notin T_n$.

Since T_1 is smooth at P and $P \notin T_n$, there must be another irreducible component of T_P passing through P that is different from T_1 and T_n. In particular, $n \geqslant 3$. Then $d \geqslant 4$ by Corollary 4.10. Without loss of generality, I may assume that $P \in T_2$. Then T_2 is smooth at P by Lemma 4.11. Moreover, T_1 and T_2 must be tangent at P. This shows that $P_1 \in T_2^1$ as well.

Put $D = aT_1 + bT_2 + \Delta$, where b is a non-negative rational number, and Δ is an effective $\mathbb{Q}$-divisor on the surface S whose support does not contain T_1 and T_2. Put $k_0 = \mathrm{mult}_P(\Delta)$. Then $m_0 = k_0 + a + bt_2$, and $ad_1 + bd_2 \leqslant 1$ by Lemma 4.4. Denote by Δ^1 the proper transform of the $\mathbb{Q}$-divisor Δ on the surface S_1. Put $k_1 = \mathrm{mult}_{P_1}(\Delta^1)$. Then

$$d-1-2k_0 \geqslant d_1+d_2+\big(ad_1+ad_2\big)\big(d-d_1-d_2-1\big) - 2k_0 = \Delta^1 \cdot \Big(T_1^1 + T_2^1\Big) \geqslant 2k_1$$

because $ad_1+bd_2 \leqslant 1$ and $d-d_1-d_2-1 \geqslant 0$, since $n \geqslant 3$. This gives $k_0+k_1 \leqslant \frac{d-1}{2}$. On the other hand, I have $2a+2b+k_0+k_1 = m_0+m_1 > \frac{2}{\lambda}$ by Corollary 4.15. Thus,

$$\frac{d+3}{2} = 2 + \frac{d-1}{2} \geqslant 2\big(ad_1+bd_2\big) + \frac{d-1}{2} \geqslant 2a+2b+\frac{d-1}{2} \geqslant 2a+2b+k_0+k_1 > \frac{2}{\lambda}$$

because $ad_1+bd_2 \leqslant 1$. By Lemma 4.2(iii) this gives $d = 4$.

Since $d=4$ and $n\geqslant 3$, I have $n=3$ by Corollary 4.10. Without loss of generality, I may assume that $d_1\leqslant d_2$. By Corollary 4.9, there exists at most one line in S that passes through P. This shows that $d_1=1$, $d_2=2$ and $d_3=1$. Thus, T_1 and T_3 are lines, T_2 is a conic, T_1 is tangent to T_2 at P, and T_3 does not pass through P. In particular, the curves T_1^1 and T_1^2 intersect each other transversally at P_1.

By Lemma 4.3, $T_1\cdot T_1=T_2\cdot T_2=-2$ and $T_1\cdot T_2=2$. On the other hand, the log pair $(S_1,\lambda aT_1^1+\lambda bT_2^1+\lambda\Delta^1+(\lambda(a+b+k_0)-1)E_1)$ is not log canonical at P_1. Thus, applying Theorem 2.12 to this log pair and the curve T_1^1, I get

$$\lambda\big(1+2a-2b-k_0\big)=\lambda\Delta^1\cdot T_1^1>2-\lambda(a+b+k_0)-\lambda b,$$

which implies that $3a>\frac{2}{\lambda}-1=\frac{11}{5}$, because $\lambda=\frac{5}{8}$. Similarly, applying Theorem 2.12 to this log pair and the curve T_2^1, I get

$$\lambda\big(2-2a+2b-k_0\big)=\lambda\Delta^1\cdot T_2^1>2-\lambda(a+b+k_0)-\lambda a,$$

which implies that $3b>\frac{2}{\lambda}-2=\frac{6}{5}$. Hence, I have $a>\frac{11}{15}$ and $b>\frac{2}{5}$, which is impossible, since $a+2b=ad_1+bd_2\leqslant 1$. The obtained contradiction completes the proof of the lemma. ☐

Using Lemma 4.16, I can easily prove

Lemma 4.18. One has $\mathrm{mult}_P(T_P)=2$. Moreover, if the curve T_P is reducible, then $n=2$, $d_1\leqslant d_2$, $P\in T_1\cap T_2$, and both curves T_1 and T_2 are smooth at P.

Proof. If T_P is irreducible and $\mathrm{mult}_P(T_P)\geqslant 3$, then

$$d=T_P\cdot D\geqslant 3m_0,$$

which implies that $m_0\leqslant\frac{d}{3}$. On the other hand, I have $\frac{1}{\lambda}\geqslant\frac{d}{3}$ by Lemma 4.2(vi). Thus, if T_P is irreducible, then $\mathrm{mult}_P(T_P)=2$, because $m_0>\frac{1}{\lambda}$ by Lemma 2.5. Hence, I may assume that $n\geqslant 2$. Then $t_n=0$ or $t_n=1$ by Lemma 4.5. In particular, there exists an irreducible component of the curve T_P different from T_n that passes through P. Without loss of generality, I may assume that $P\in T_1$.

Put $D=aT_1+\Omega$, where a is a non-negative rational number, and Ω is an effective $\mathbb{Q}$-divisor on the surface S whose support does not contain the curve T_1. Put $n_0=\mathrm{mult}_P(\Omega)$. Denote by Ω^1 the proper transform of the $\mathbb{Q}$-divisor Ω on the surface S_1. Then $(S_1,\lambda\Omega^1+(\lambda(n_0+at_1)-1)E_1)$ is not log canonical at P_1, since $P_1\notin T_1^1$ by Lemma 4.16. In particular, it follows from Theorem 2.13 that

$$\lambda n_0=\lambda\Omega^1\cdot E_1>1,$$

which implies that $n_0>\frac{1}{\lambda}$. Thus, if $t_1\geqslant 2$, then it follows from Lemma 4.3 that

$$\frac{1}{\lambda}\geqslant\frac{d-1}{2}\geqslant\frac{d_1+ad_1(d-d_1-1)}{2}=\frac{\Omega\cdot T_1}{2}\geqslant\frac{t_1n_0}{2}\geqslant n_0>\frac{1}{\lambda},$$

because $a\leqslant\frac{1}{d_1}$ by Lemma 4.4, and $\lambda\leqslant\frac{2}{d-1}$ by Lemma 4.2(i). Thus, $t_1=1$. Similarly, if $P\in T_n$ and $n\geqslant 3$, then

$$\frac{2}{\lambda}\geqslant d-1\geqslant d_1+d_n+ad_1(d-d_1-d_n-1)=\Omega\cdot\Big(T_1+T_n\Big)\geqslant 2n_0>\frac{2}{\lambda}.$$

Thus, if $P\in T_n$, then $n=2$.

If $P\in T_n$, then $n=2$, and T_n is smooth at P. If $n=2$, then T_n must pass through P, because T_1 is smooth at P. Furthermore, if $n=2$, then $d_1\leqslant d_n$, because $d_n>\frac{d-1}{2}$ by Lemma 4.5. Therefore, the required assertions are proved in the case when $n=2$. Thus, I may assume that $n\geqslant 3$. In particular, $P\notin T_n$. Then every irreducible component of the curve T_P that contain P is smooth at P by Lemma 4.11. Hence, there should be at least two irreducible components of the curve T_P that pass through P. Since $P\notin T_n$, the point P is contained in an irreducible component of T_P that is different from T_1 and T_n. Without loss of generality, I may assume that $P\in T_2$.

Put $D = aT_1 + bT_2 + \Delta$, where b is a non-negative rational number, and Δ is an effective $\mathbb{Q}$-divisor on the surface S whose support does not contain T_1 and T_2. Put $k_0 = \mathrm{mult}_P(\Delta)$. Then $ad_1 + bd_2 \leqslant 1$ by Lemma 4.4. Thus, it follows from Lemma 4.3 that

$$2k_0 \leqslant \Delta\cdot\Big(T_1+T_2\Big) = d_1+d_2+\big(ad_1+ad_2\big)\big(d-d_1-d_2-1\big) \leqslant d_1+d_2+\big(d-d_1-d_2-1\big) = d-1$$

because $ad_1 + bd_2 \leqslant 1$. Hence, I have $k_0 \leqslant \frac{d-1}{2}$. Denote by Δ^1 the proper transform of the $\mathbb{Q}$-divisor Δ on the surface S_1. Then the log pair $(S_1, \lambda\Delta^1 + (\lambda(k_0 + a + b) - 1)E_1)$ is not log canonical at P_1, since $P_1 \notin T_1^1$ and $P_1 \notin T_2^1$ by Lemma 4.16. In particular, it follows from Theorem 2.12 that

$$\lambda k_0 = \lambda\Delta^1\cdot E_1 > 1,$$

which implies that $k_0 > \frac{1}{\lambda}$. This contradicts Lemma 4.2(i), because $k_0 \leqslant \frac{d-1}{2}$. The obtained contradiction completes the proof. $\square$

Later, I will need the following marginal

Lemma 4.19. Suppose that $d = 4$. Then $m_0 \leqslant \frac{11}{5}$.

Proof. If $n = 1$ or $d_1 = d_2 = n = 2$, then

$$2t_n \geqslant d_n = T_n\cdot D \geqslant t_n m_0,$$

which implies that $m_0 \leqslant 2$. Hence, I may assume that neither $n = 1$ nor $d_1 = d_2 = n = 2$. Then it follows from Lemma 4.18 that $n = 2$, $d_1 = 1$, $d_2 = 3$, $P \in T_1 \cap T_2$, and both curves T_1 and T_2 are smooth at P. Put $D = aT_1 + \Omega$, where a is a non-negative rational number, and Ω is an effective $\mathbb{Q}$-divisor whose support does not contain the line T_1. Put $n_0 = \mathrm{mult}_P(\Omega)$. Then $n_0 + 3a \leqslant 3$ by Lemma 4.4. Moreover, I have

$$1 + 2a = T_1\cdot\Omega \geqslant n_0.$$

The obtained inequalities give $m_0 = n_0 + a \leqslant \frac{11}{5}$. $\square$

Let $f_2\colon S_2 \to S_1$ be a blow up of the point P_1. Denote by E_2 the f_2-exceptional curve, denote by E_1^2 the proper transform of the curve E_1 on the surface S_2, and denote by D^2 the proper transform of the $\mathbb{Q}$-divisor D on the surface S_2. Then

$$K_{S_2} + \lambda D^2 + \big(\lambda m_0 - 1\big)E_1^2 + \Big(\lambda\big(m_0 + m_1\big) - 2\Big)E_2 \sim_{\mathbb{Q}} f_2^*\Big(K_{S_1} + \lambda D^1 + \big(\lambda m_0 - 1\big)E_1\Big).$$

By Remark 2.11, the log pair $(S_2, \lambda D^2 + (\lambda m_0 - 1)E_1^2 + (\lambda(m_0 + m_1) - 2)E_2)$ is not log canonical at some point $P_2 \in E_1$.

Lemma 4.20. One has $m_0 + m_1 \leqslant \frac{3}{\lambda}$.

Proof. Suppose that $m_0 + m_1 > \frac{3}{\lambda}$. Then $2m_0 \geqslant m_0 + m_1 > \frac{3}{\lambda}$. But $m_0 \leqslant \frac{d+1}{2}$ by Lemma 4.12. Then $\lambda > \frac{3}{d+1}$. Thus, $d \leqslant 4$ by Lemma 4.2(ii). If $d = 4$, then

$$\frac{22}{5} \geqslant 2m_0 \geqslant m_0 + m_1 > \frac{3}{\lambda} = \frac{24}{5}$$

by Lemma 4.19. Thus, $d = 3$. Then $\lambda = 1$. By Corollary 4.10, $n \leqslant 2$. If $n = 1$, then

$$3 = T_P\cdot D \geqslant 2m_0 \geqslant m_1 + m_0 > \frac{3}{\lambda} = 3,$$

which is absurd. Hence, $n = 2$. Then $d_1 = 1$ and $d_2 = 2$ by Lemma 4.5. Hence, $P \in T_1 \cap T_2$.

Put $D = aT_1 + \Omega$, where a is a non-negative rational number, and Ω is an effective $\mathbb{Q}$-divisor on S whose support does not contain the line T_1. Put $n_0 = \mathrm{mult}_P(\Omega)$. Then $m_0 = n_0 + a$, and $n_0 + 2a \leqslant 2$ by Lemma 4.4. Moreover, I have

$$1 + a = T_1\cdot\Omega \geqslant n_0,$$

which implies that $n_0 - a \leqslant 1$. Adding $n_0 - a \leqslant 1$ to $n_0 + 2a \leqslant 2$, I get

$$3 \geqslant 2n_0 + a = n_0 + m_0 = m_1 + m_0 > \frac{3}{\lambda} = 3,$$

because $P_1 \notin T_1^1$ by Lemma 4.16. $\square$

Thus, the log pair $(S_2, \lambda D^2+(\lambda m_0-1)E_1^2+(\lambda(m_0+m_1)-2)E_2)$ is log canonical at a punctured neighborhood of the point P. The log pair $(S_2, \lambda D^2 + (\lambda m_0 - 1)E_1^2 + (\lambda(m_0 + m_1) - 2)E_2)$ is log canonical at every point of the curve E_2 that is different from P_2 by Lemma 2.14.

Lemma 4.21. One has $P_2 \neq E_1^2 \cap E_2$.

Proof. Suppose that $P_2 = E_1^2 \cap E_2$. Then Theorem 2.12 gives

$$\lambda\big(m_0 - m_1\big) = \lambda D^2 \cdot E_1^2 > 3 - \lambda\big(m_0 + m_1\big),$$

which implies that $m_0 > \frac{3}{2\lambda}$. But $m_0 \leqslant \frac{d+1}{2}$ by Lemma 4.12. Therefore, $\lambda > \frac{3}{d+1}$, which implies that $d \leqslant 4$ by Lemma 4.2(ii). If $d = 4$, then $\frac{12}{5} = \frac{3}{2\lambda} < m_0 \leqslant \frac{11}{5}$ by Lemma 4.19. Thus, $d = 3$. Then $\lambda = 1$. By Corollary 4.10, $n \leqslant 2$. If $n = 1$, then

$$3 = T_P \cdot D \geqslant 2m_0 > \frac{3}{\lambda} = 3,$$

which is absurd. Hence, $n = 2$. Then $d_1 = 1$ and $d_2 = 2$ by Lemma 4.5. I have $P \in T_1 \cap T_2$.

Put $D = aT_1 + \Omega$, where a is a non-negative rational number, and Ω is an effective $\mathbb{Q}$-divisor on S whose support does not contain the line T_1. Put $n_0 = \mathrm{mult}_P(\Omega)$. Then $m_0 = n_0 + a$, and $n_0 + 2a \leqslant 2$ by Lemma 4.4, Then $2n_0 + a \leqslant 3$, because

$$1 + a = D \cdot \Omega \geqslant n_0.$$

Denote by Ω^1 the proper transform of the divisor Ω on S_1. Put $n_1 = \mathrm{mult}_{P_1}(\Omega^1)$. Then $n_1 = m_1$, since $P_1 \notin T_1^1$ by Lemma 4.16. Thus, the log pair $(S_2, (n_0+a-1)E_1^2+(n_0+n_1-a-2)E_2)$ is not log canonical at P_2. Applying Theorem 2.12 to this pair and the curve E_1^2, I get

$$n_0 - n_1 = \Omega^2 \cdot E_1^2 > 3 - n_0 - n_1 + a,$$

which implies that $2n_0 + a > 3$. But I already proved that $2n_0 + a \leqslant 3$. $\square$

Thus, the log pair $(S_2, \lambda D^2+(\lambda(m_0+m_1)-2)E_2)$ is not log canonical at P_2. Then Lemma 2.5 gives

Corollary 4.22. One has $m_0 + m_1 + m_2 > \frac{3}{\lambda}$.

Denote by T_P^2 the proper transform of the curve T_P on the surface S^2. Then

$$T_P^2 + E_1^2 \sim (f_1 \circ f_2)^*(\mathcal{O}_S(1)) - f_2^*(E_1) - E_2,$$

because $T_P^1 \sim f_1^*(\mathcal{O}_S(1)) - 2E_1$ by Lemma 4.18, and $P_1 \notin T_P^1$ by Lemma 4.16.

Lemma 4.23. The linear system $|T_P^2 + E_1^2|$ is a pencil that does not have base points in E_2.

Proof. Since $|T_P^1+E_1|$ is a two-dimensional linear system that does not have base points, $|T_P^2+E_1^2|$ is a pencil. Let C be a curve in $|T_P^1 + E_1|$ that passes through P_1 and is different from $T_P^1 + E_1$. Then C is smooth at P, since $P \in f_1(C)$ and $f_1(C)$ is a hyperplane section of the surface S that is different from T_P. Since $C \cdot E_1 = 1$, I see that $T_P^1 + E_1$ and C intersect transversally at P_1. Thus, the proper transform of the curve C on the surface S_2 is contained in $|T_P^1 + E_1|$ and have no common points with $T_P^2 + E_1^2$ in E_2. This shows that the pencil $|T_P^1 + E_1|$ does not have base points in E_2. $\square$

Since $|T_P^1+E_1|$ does not have base points in E_2, no curves in $|T_P^1+E_1|$ has E_2 as an irreducible component, because $(T_P^1 + E_1) \cdot E_2 = 1$. Moreover, the only divisor in $|T_P^1 + E_1|$ that contains E_1^2 as an irreducible component is $T_P^2 + E_1^2$.

Remark 4.24. Let C be a curve in $|T_P^1 + E_1|$. Then $P_1 \in f_2(C)$, and $f_1 \circ f_2(C)$ is a hyperplane section of the surface S that passes through P. In particular, the curve C is reduced by Lemma 2.6. Furthermore, if $C \neq T_P^2 + E_1^2$, then C is smooth at $C \cap E$, the curve $f_2(C)$ is smooth at P_1, and the curve $f_1 \circ f_2(C)$ is smooth at the point P.

Let Z^2 be the curve in $|T^1_P + E_1|$ that passes through the point P_2. Then $Z^2 \neq T^2_P + E^2_1$, because $P_2 \neq E^2_1 \cap E_2$ by Lemma 4.21. Then Z_2 is smooth at P_2. Put $Z = f_1 \circ f_2(Z^2)$ and $Z^1 = f_2(Z^2)$. Then $P \in Z$ and $P_1 \in Z^1$. Moreover, the curve Z is smooth at P, and the curve Z_1 is smooth at P_1. Furthermore, the curve Z is reduced by Lemma 2.6.

Lemma 4.25. The curve Z is reducible.

Proof. Suppose that Z is irreducible. Let me seek for a contradiction. Since Z is smooth at P, the log pair $(S, \lambda Z)$ is log canonical at P. Moreover, $Z \sim_{\mathbb{Q}} D$. Thus, it follows from Remark 2.4 that I may assume that $\mathrm{Supp}(D)$ does not contain the curve Z. Then

$$d - m_0 - m_1 = Z^2 \cdot D^2 \geqslant m_2,$$

which implies that $m_0+m_1+m_2 \leqslant d$. One the other hand, $m_0+m_1+m_2 > \frac{3}{\lambda}$ by Corollary 4.22. This gives $\lambda > \frac{3}{d}$, which is impossible by Lemma 4.2(vi). □

The log pair $(S, \lambda Z)$ is log canonical at P, because Z is smooth at P. Since $Z \sim_{\mathbb{Q}} D$, it follows from Remark 2.4 that I may assume that $\mathrm{Supp}(D)$ does not contain at least one irreducible component of the curve Z. Denote this irreducible component by $\bar{Z}$, and denote its degree in $\mathbb{P}^3$ by $\bar{d}$. Then $\bar{d} < d$.

Lemma 4.26. One has $P \notin \bar{Z}$.

Proof. Suppose that $P \in \bar{Z}$. Let me seek for a contradiction. Denote by $\bar{Z}^2$ the proper transform of the curve $\bar{Z}$ on the surface S_2. Then

$$d - m_0 - m_1 > \bar{d} - m_0 - m_1 = \bar{Z}^2 \cdot D^2 \geqslant m_2,$$

which implies that $m_0+m_1+m_2 < d$. One the other hand, $m_0+m_1+m_2 > \frac{3}{\lambda}$ by Corollary 4.22. This gives $\lambda > \frac{3}{d}$, which is impossible by Lemma 4.2(vi). □

Denote by $\hat{Z}$ the irreducible component of the curve Z that passes through P, denote its proper transform on the surface S_1 by $\hat{Z}^1$, and denote its proper transform on the surface S_2 by $\hat{Z}^2$. Then $\bar{Z} \neq \hat{Z}$, $P_1 \in \hat{Z}^1$ and $P_2 \in \hat{Z}^2$. Denote by $\hat{d}$ the degree of the curve $\hat{Z}$ in $\mathbb{P}^3$. Then $\hat{d} + \bar{d} \leqslant d$. Moreover, the intersection form of the curves $\hat{Z}$ and $\bar{Z}$ on the surface S is given by

Lemma 4.27. One has $\bar{Z} \cdot \bar{Z} = -\bar{d}(d - \bar{d} - 1)$, $\hat{Z} \cdot \hat{Z} = -\hat{d}(d - \hat{d} - 1)$ and $\bar{Z} \cdot \hat{Z} = \bar{d}\hat{d}$.

Proof. See the proof of Lemma 4.3. □

Put $D = a\hat{Z} + \Omega$, where a is a positive rational number, and Ω is an effective $\mathbb{Q}$-divisor on the surface S whose support does not contain the curve $\hat{Z}$. Denote by Ω^1 the proper transform of the divisor Ω on the surface S_1, and denote by Ω^2 the proper transform of the divisor Ω on the surface S_2. Put $n_0 = \mathrm{mult}_P(\Omega)$, $n_1 = \mathrm{mult}_{P_1}(\Omega^1)$ and $n_2 = \mathrm{mult}_{P_2}(\Omega^2)$. Then $m_0 = n_0 + a$, $m_1 = n_1 + a$ and $m_2 = n_2 + a$. Then the log pair $(S_2, \lambda a\hat{Z}^2 + \lambda\Omega^2 + (\lambda(n_0 + n_1 + 2a) - 2)E_2)$ is not log canonical at P_2, because $(S_2, \lambda D^2 + (\lambda(m_0 + m_1) - 2)E_2)$ is not log canonical at P_2. Thus, applying Theorem 2.12, I see that

$$\lambda\Big(\Omega \cdot \hat{Z} - n_0 - n_1\Big) = \lambda\Omega^2 \cdot Z^2 > 1 - \Big(\lambda\big(n_0 + n_1 + 2a\big) - 2\Big) = 3 - \lambda\big(n_0 + n_1 + 2a\big),$$

which implies that

$$\Omega \cdot \hat{Z} > \frac{3}{\lambda} - 2a. \tag{4.28}$$

On the other hand, I have

$$\bar{d} = D \cdot \bar{Z} = \Big(a\hat{Z} + \Omega\Big) \cdot \bar{Z} \geqslant a\hat{Z} \cdot \bar{Z} = a\hat{d}\bar{d}$$

by Lemma 4.27. This gives

$$a \leqslant \frac{1}{\hat{d}}. \tag{4.29}$$

Thus, it follows from (4.28), (4.29) and Lemma 4.27 that

$$\frac{3}{\lambda}-2\leqslant\frac{3}{\lambda}-2a<\Omega\cdot\hat{Z}=\hat{d}+a\hat{d}\Big(d-\hat{d}-1\Big)\leqslant d-1,$$

which implies that $\lambda>\frac{3}{d+1}$. Then $d\leqslant 4$ by Lemma 4.2(ii).

Lemma 4.30. One has $d\neq 4$.

Proof. Suppose that $d=4$. Then $\lambda=\frac{5}{8}$. By Lemma 4.25, $\hat{d}\leqslant 3$. By Lemma 4.16, $\hat{Z}$ is not a line, since every line passing through P must be an irreducible component of the curve T_P. Thus, either $\hat{Z}$ is a conic or $\hat{Z}$ is a plane cubic curve. If $\hat{Z}$ is a conic, then $\hat{Z}^2=-2$ and $a\leqslant\frac{1}{2}$ by (4.29). Thus, if $\hat{Z}$ is a conic, then

$$2+2a=\Omega\cdot\hat{Z}>\frac{3}{\lambda}-2a=\frac{24}{5}-2a,$$

which implies that $\frac{1}{2}\geqslant a>\frac{7}{10}$. This shows that $\hat{Z}$ is a plane cubic curve. Then $\hat{Z}^2=0$. Since $a\leqslant\frac{1}{3}$ by (4.29), I have

$$3=\Omega\cdot\hat{Z}>\frac{3}{\lambda}-2a=\frac{24}{5}-2a\geqslant\frac{24}{5}-\frac{2}{3}=\frac{62}{15},$$

which is absurd. □

Thus, I see that $d=3$. Then $\hat{Z}$ us either a line or a conic by Lemma 4.25. But every line passing through P must be an irreducible component of T_P. Since $\hat{Z}$ is not an irreducible component of T_P by Lemma 4.16, the curve $\hat{Z}$ must be a conic. Then $\hat{Z}\cdot\hat{Z}=0$. Therefore, it follows from (4.28) that

$$3-2a=\frac{3}{\lambda}-2a<\Omega\cdot\hat{Z}=\hat{d}+a\hat{d}\Big(d-\hat{d}-1\Big)=\hat{d}=2,$$

which implies that $a>\frac{1}{2}$. But $a\leqslant\frac{1}{\hat{d}}=\frac{1}{2}$ by (4.29). The obtained contradiction completes the proof of Theorem 1.21.

Appendix A. Log canonical thresholds of hypersurfaces

In this appendix, I will present some known results about hypersurfaces and pose one conjecture. Let V_d be a *reduced* hypersurface in $\mathbb{P}^n$ of degree d such that $d\geqslant n+1\geqslant 3$, and let P be a point in V_d. Put $m_P=\mathrm{mult}_P(V_d)$. The log canonical threshold of the log pair $(\mathbb{P}^n,V_d)$ at the point P is the number

$$\mathrm{lct}_P\big(\mathbb{P}^n,V_d\big)=\sup\Big\{\lambda\in\mathbb{Q}\ \Big|\ \text{the log pair }\big(\mathbb{P}^n,\lambda V_d\big)\text{ is log canonical at }P\Big\}.$$

Then $\frac{1}{m_P}\leqslant\mathrm{lct}_P(\mathbb{P}^n,V_d)\leqslant\frac{n}{m_P}$ by [18, Lemma 8.10]. Thus, if V_d is a cone with vertex in P, then $\mathrm{lct}_P(\mathbb{P}^n,V_d)\leqslant\frac{n}{d}$. Moreover, Tommaso de Fernex, Lawrence Ein and Mircea Mustaţă proved

Theorem A.1 ([11, Theorem 0.2])**.** Suppose that the log pair $(\mathbb{P}^n,\frac{n}{d}V_d)$ is Kawamata log terminal outside of the point P. Then $\mathrm{lct}_P(\mathbb{P}^n,V_d)\geqslant\frac{n}{d}$. If $\mathrm{lct}_P(\mathbb{P}^n,V_d)=\frac{n}{d}$, then V_d is a cone with vertex in P.

Let X_d be a smooth hypersurface in $\mathbb{P}^{n+1}$ of degree $d\geqslant n+1\geqslant 3$, and let T_O be the hyperplane section of X_d that is singular at O. Then T_O has isolated singularities (see, for example, [24]).

Definition A.2 ([8, Definition 2.2])**.** The point O is a *star point* if T_O is a cone with vertex in O.

If O is star point, then $\alpha_{X_d}^{H_{X_d}}(O)\leqslant\frac{n}{d}$ (see Definition 1.20). Moreover, $\alpha_{X_d}^{H_{X_d}}(O)\geqslant\frac{n}{d}$ by

Theorem A.3. Let D_X be an effective $\mathbb{Q}$-divisor on X_d such that $D_X\sim_{\mathbb{Q}}H_{X_d}$. Then $(X,\frac{n}{d}D_X)$ is not Kawamata log terminal at O if and only if $D_X=T_O$ and O is a star point.

Proof. Suppose that $(X, \frac{n}{d}D_X)$ is not Kawamata log terminal at the point O. By Theorems 1.13 and 1.21, I may assume that $n \geqslant 3$. Then $\mathrm{Pic}(X) = \mathbb{Z}[H_X]$. Hence, I may assume that $D_X = \frac{1}{m}D_m$ for a prime Weil divisor D_m on X such that $D \sim mH_X$, where $m \in \mathbb{N}$. Then $O \in D_m$, and it follows from [24] that $\mathrm{mult}_C(D_m) \leqslant m$ for every irreducible curve $C \subset X$. In particular, the log pair $(X, \frac{n-1}{dm}D_m)$ is Kawamata log terminal outside of finitely many points in X. On the other hand, there exists a *sufficiently general* linear projection $\gamma\colon X \to \mathbb{P}^{n-1}$ such that γ is etale in a neighborhood of the point O, the induced morphism $\gamma|_{D_m}\colon D_m \to \gamma(D_m)$ is birational and is an isomorphism in a neighborhood of the point O. Then $(\mathbb{P}^{n-1}, \frac{n-1}{dm}\gamma(D_m))$ is not Kawamata log terminal at $\gamma(O)$ and is Kawamata log terminal in a punctured neighborhood of the point $\gamma(O)$. Since $\gamma(D_m)$ is a hypersurface of degree dm, the divisor $-(K_{\mathbb{P}^{n-1}} - \frac{n-1}{dm}\gamma(D_m))$ is ample. Then the locus where $(\mathbb{P}^{n-1}, \frac{n-1}{dm}\gamma(D_m))$ is not Kawamata log terminal must be connected by the connectedness principle of Kollár–Shokurov (see, for example, [9, Theorem 6.3.2]). Hence, the log pair $(\mathbb{P}^{n-1}, \frac{n-1}{dm}\gamma(D_m))$ is Kawamata log terminal outside of $\gamma(O)$. By Theorem A.1, $\gamma(D_m)$ is a cone with vertex in $\gamma(O)$. This implies that D_m is a cone with vertex O. Then $D_m = T_O$, which implies that O is a star point. □

Thus, $\alpha(X_d, H_{X_d}) \geqslant \frac{n}{d}$. Moreover, if X_d contains a star point, then $\alpha(X_d, H_{X_d}) = \frac{n}{d}$. If $n = 2$, then Corollary 1.27 implies that $\alpha(X_d, H_{X_d}) > \frac{n}{d}$ if and only if X_d does not have star points. So, it is natural to expect

Conjecture A.4. If X_d does not contain star points, then $\alpha(X_d, H_{X_d}) > \frac{n}{d}$.

By [8, Theorem 2.10], X_d contains at most finitely many star points. If X_d is general, it does not contain star points at all. In particular, if Conjecture A.4 is true, then $\alpha(X_d, H_{X_d}) > \frac{n}{d}$ provided that X_d is general enough. If $d = n+1$, the latter is indeed true by

Theorem A.5 ([2, Theorem 1.7], [25, Theorem 2], [4, Theorem 1.1.5])**.** Suppose that X_d is a *general* hypersurface in $\mathbb{P}^{n+1}$ of degree $d = n+1 \geqslant 3$. If $n = 2$, then $\alpha(X_d, H_{X_d}) = \frac{3}{4}$. If $n = 3$, then $\alpha(X_d, H_{X_d}) \geqslant \frac{7}{9}$. If $n = 4$, then $\alpha(X_d, H_{X_d}) \geqslant \frac{5}{6}$. If $n \geqslant 5$, then $\alpha(X_d, H_{X_d}) = 1$.

By [28, Theorem 2.1] and [6, Theorem A.3], this result implies that every *general* hypersurface in $\mathbb{P}^{n+1}$ of degree $n+1 \geqslant 3$ admits a Kähler–Einstein metric. Similarly, Conjecture A.4 implies that every smooth hypersurface in $\mathbb{P}^{n+1}$ of degree $n+1 \geqslant 3$ without star points admits a Kähler–Einstein metric. Note that Conjecture A.4 follows Theorem A.3 and [30, Conjecture 5.3].

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