

## FINITE COLLINEATION GROUPS AND BIRATIONAL RIGIDITY

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ABSTRACT. We classify finite groups  $G$  in  $\mathrm{PGL}_4(\mathbb{C})$  such that  $\mathbb{P}^3$  is  $G$ -birationally rigid.

All varieties are assumed to be projective and defined over the field of complex numbers.

## 1. INTRODUCTION

The projective space  $\mathbb{P}^3$  is one of the most basic objects in geometry. Its biregular and birational symmetries have attracted attention of many mathematicians. Among them was Hans Frederick Blichfeldt, who classified in [3] all finite subgroups of

$$\mathrm{Aut}(\mathbb{P}^3) \cong \mathrm{PGL}_4(\mathbb{C})$$

back in 1905. In 1917, he wrote his *magnum opus* [4], which gave a detailed coverage of classification of finite subgroups in  $\mathrm{PGL}_3(\mathbb{C})$  and  $\mathrm{PGL}_4(\mathbb{C})$ . This book became a standard reference for the theory of finite collineation groups. We want to celebrate its centennial by enriching this classical theory with modern technique of birational geometry.

Finite subgroups of  $\mathrm{PGL}_4(\mathbb{C})$  are traditionally divided into two major types: transitive and intransitive. In geometric language, transitive groups do not fix any point in  $\mathbb{P}^3$  and do not leave any line invariant. Intransitive groups are those that are not transitive. Transitive groups are further subdivided into imprimitive and primitive groups. Imprimitive groups either leave a union of two skew lines invariant or have an orbit of length 4 (or both). All other finite subgroups in  $\mathrm{PGL}_4(\mathbb{C})$  are said to be primitive.

The structure of the group  $\mathrm{Bir}(\mathbb{P}^3)$ , known as the space Cremona group, is way more complicated than that of  $\mathrm{PGL}_4(\mathbb{C})$ . In particular, the classification of all finite subgroups in  $\mathrm{Bir}(\mathbb{P}^3)$  seems to be out of reach at the moment. There are just several sporadic known results in this direction. These include the classification of all finite *simple* subgroups in  $\mathrm{Bir}(\mathbb{P}^3)$  in [36], and some boundedness properties in [38]. The goal of this paper is to understand how the finite subgroups of  $\mathrm{PGL}_4(\mathbb{C})$  described by Blichfeldt behave as subgroups of the larger group  $\mathrm{Bir}(\mathbb{P}^3)$ .

Given a finite subgroup  $G$  in  $\mathrm{PGL}_4(\mathbb{C})$ , the most natural birational problem is to describe all  $G$ -birational maps from  $\mathbb{P}^3$  to other three-dimensional  $G$ -varieties. Minimal Model Program implies that to solve this problem it is enough to describe  $G$ -birational maps from  $\mathbb{P}^3$  to  $G$ -Mori fibre spaces (see [10, Definition 1.1.5]). In this paper we aim to describe all finite subgroups  $G \subset \mathrm{PGL}_4(\mathbb{C})$  such that the projective space  $\mathbb{P}^3$  cannot be  $G$ -birationally transformed into other  $G$ -Mori fiber spaces. In this case, we say that  $\mathbb{P}^3$  is  *$G$ -birationally rigid* (see [10, Definition 3.1.1]). This simply means that the following three conditions are satisfied:

- (1) There is no  $G$ -rational map  $\mathbb{P}^3 \dashrightarrow S$  whose general fiber is a rational curve.
- (2) There is no  $G$ -rational map  $\mathbb{P}^3 \dashrightarrow \mathbb{P}^1$  whose general fiber is a rational surface.

- (3) There is no  $G$ -birational map  $\mathbb{P}^3 \dashrightarrow X$  such that  $X$  is a Fano threefold with terminal singularities, the  $G$ -invariant class group of the threefold  $X$  is of rank 1, and  $X$  is not  $G$ -isomorphic to  $\mathbb{P}^3$  (equipped with the same action of the group  $G$ ).

The main result of our paper is the following theorem.

**Theorem 1.1.** *Let  $G$  be a finite subgroup in  $\mathrm{PGL}_4(\mathbb{C})$ . Then  $\mathbb{P}^3$  is  $G$ -birationally rigid if and only if  $G$  is a primitive group that is not isomorphic to  $\mathfrak{A}_5$  or  $\mathfrak{S}_5$ .*

It is a three-dimensional analogue of the following recent result of Sakovics.

**Theorem 1.2** ([40]). *Let  $G$  be a finite subgroup in  $\mathrm{PGL}_3(\mathbb{C})$ . Then  $\mathbb{P}^2$  is  $G$ -birationally rigid if and only if  $G$  is a transitive subgroup that is not isomorphic to  $\mathfrak{A}_4$  or  $\mathfrak{S}_4$ .*

In addition to Theorem 1.1, we provide a classification of finite subgroups  $G \subset \mathrm{PGL}_4(\mathbb{C})$  such that  $\mathbb{P}^3$  is  $G$ -birationally super-rigid (see [10, Definition 3.1.1] or §5 for a definition). This is done in Theorem 5.5.

One direction of Theorem 1.1 is easy to prove. Namely, if  $G$  is a finite intransitive subgroup in  $\mathrm{PGL}_4(\mathbb{C})$ , then  $\mathbb{P}^3$  is not  $G$ -birationally rigid. Indeed, if  $G$  fixes a point in  $\mathbb{P}^3$ , then the linear projection from this point  $\mathbb{P}^3 \dashrightarrow \mathbb{P}^2$  is a  $G$ -rational dominant map whose general fiber is  $\mathbb{P}^1$ . Likewise, if there is a  $G$ -invariant line in  $\mathbb{P}^3$ , then the linear projection from this line  $\mathbb{P}^3 \dashrightarrow \mathbb{P}^1$  is a  $G$ -rational dominant map whose general fiber is  $\mathbb{P}^2$ .

Similarly, if  $G$  is a finite imprimitive subgroup in  $\mathrm{PGL}_4(\mathbb{C})$ , then  $\mathbb{P}^3$  is not  $G$ -birationally rigid as well. If there is a  $G$ -invariant union of two skew lines in  $\mathbb{P}^3$ , then the blow up of  $\mathbb{P}^3$  along them is a  $G$ -equivariant  $\mathbb{P}^1$ -bundle over  $\mathbb{P}^1 \times \mathbb{P}^1$ . Likewise, if there exists a  $G$ -orbit in  $\mathbb{P}^3$  that consists of 4 points, but there is no  $G$ -invariant pair of skew lines, then  $\mathbb{P}^3$  is  $G$ -birational to a toric Fano threefold  $V_{24}$  with terminal  $\mathbb{Q}$ -factorial singularities such that the  $G$ -invariant class group of  $V_{24}$  is of rank 1. The detailed construction of the birational map  $\mathbb{P}^3 \dashrightarrow V_{24}$  is given in the proof of Proposition 5.1.

Thus, we are left with the case of finite primitive groups. Up to conjugation, there are finitely many such groups. They are listed in [4, Chapter VII]. Geometrically, these groups can be described as follows. First, there are primitive groups that leave a smooth quadric surface invariant. Next, there are seven sporadic examples: the group  $\mathfrak{A}_5$  that leaves a twisted cubic curve invariant, the group  $\mathfrak{S}_5$  that leaves a pair of two disjoint twisted cubic curves invariant, the Klein simple group  $\mathrm{PSL}_2(\mathbb{F}_7)$ , the group  $\mathfrak{A}_6$ , the group  $\mathfrak{S}_6$ , the group  $\mathfrak{A}_7$ , and the simple group of order 25920, which is known to be isomorphic to the group  $\mathrm{PSp}_4(\mathbb{F}_3)$ . Finally, there are primitive subgroups in  $\mathrm{PGL}_4(\mathbb{C})$  that contain the imprimitive group  $\mu_2^4$  as a normal subgroup. Some of them, like the group  $\mathfrak{A}_4 \times \mathfrak{A}_4$ , also leave a quadric surface invariant, so that this description is not exclusive (see the discussion in [4, §125]). As usual, we denote by  $\mu_n$  the cyclic group of order  $n$ .

Up to conjugation, there are two primitive subgroups in  $\mathrm{PGL}_4(\mathbb{C})$  isomorphic to  $\mathfrak{A}_5$ . One of them, leaves a quadric surface invariant. Its action on  $\mathbb{P}^3$  comes from the irreducible four-dimensional representation of the icosahedral group. Another one preserves a twisted cubic curve, so that its action on  $\mathbb{P}^3$  comes from an irreducible four-dimensional representation of binary icosahedral group. Similarly, there are two primitive subgroups in  $\mathrm{PGL}_4(\mathbb{C})$  isomorphic to  $\mathfrak{S}_5$ . One of them preserves a quadric surface, and its action on  $\mathbb{P}^3$  comes from an irreducible four-dimensional representation of the group  $\mathfrak{S}_5$ . Another one leaves invariant a pair of disjoint twisted cubic curves. Its action on  $\mathbb{P}^3$  comes from an irreducible four-dimensional representation of a central extension of the group  $\mathfrak{S}_5$ . In

all these cases, the projective space  $\mathbb{P}^3$  is not  $G$ -birationally rigid. This follows from the five explicit examples below.

**Example 1.3** ([43, 2.13.1], [35, §2], [10, Example 1.2.8]). Let  $G$  be a primitive finite subgroup  $\mathfrak{A}_5$  in  $\mathrm{PGL}_4(\mathbb{C})$  such that there exists a  $G$ -invariant quadric surface in  $\mathbb{P}^3$ . Then there is a unique  $G$ -invariant smooth cubic surface in  $\mathbb{P}^3$ , which is known as the *Clebsch cubic surface*. Denote this cubic surface by  $S$ . Then  $S$  contains a  $G$ -invariant irreducible smooth rational sextic curve. Denote it by  $\mathcal{C}$ . The surface  $S$  also contains a  $G$ -invariant curve that consists of six disjoint lines, which are bi-tangents of the curve  $\mathcal{C}$ . Denote this curve by  $\mathcal{L}_6$ . Let  $\pi: X \rightarrow \mathbb{P}^3$  be the blow-up of the rational sextic curve  $\mathcal{C}$ . Then there exists a  $G$ -commutative diagram (that is, a commutative diagram of  $G$ -rational maps, see [10, p. xix] for terminology)

$$\begin{array}{ccccc} & X & \overset{\chi}{\dashrightarrow} & Y & \\ \pi \swarrow & & \searrow \alpha & \swarrow \beta & \searrow \phi \\ \mathbb{P}^3 & & V_{14} & & V_{22} \end{array}$$

Here  $V_{14}$  is a Fano threefold with six isolated ordinary double points such that  $-K_{V_{14}}^3 = 14$ , and  $V_{22}$  is a smooth Fano threefold such that  $-K_{V_{22}}^3 = 22$  and  $\mathrm{Aut}(V_{22}) \cong \mathrm{PSL}_2(\mathbb{C})$ , which is known as the Mukai–Umemura threefold (see [34]). The morphism  $\alpha$  is the contraction of the proper transforms of the irreducible components of the curve  $\mathcal{L}_6$ , the rational map  $\chi$  is a composition of Atiyah flops in these 6 curves, the morphism  $\beta$  is a flopping contraction, and  $\phi$  is a contraction of the proper transform of the cubic surface  $S$  to a smooth point of the threefold  $V_{22}$ .

**Example 1.4** ([42, Application 1]). Let  $G$  be a primitive finite subgroup  $\mathfrak{A}_5$  in  $\mathrm{PGL}_4(\mathbb{C})$  such that  $\mathbb{P}^3$  contains a  $G$ -invariant twisted cubic curve  $\mathcal{C}$ . Let  $\pi: X \rightarrow \mathbb{P}^3$  be the blow-up of the curve  $\mathcal{C}$ . Then there exists a  $G$ -commutative diagram

$$\begin{array}{ccc} & X & \\ \pi \swarrow & & \searrow \phi \\ \mathbb{P}^3 & \overset{\rho}{\dashrightarrow} & \mathbb{P}^2 \end{array}$$

Here  $\phi$  is a  $\mathbb{P}^1$ -bundle whose fibers are proper transforms of secants of the curve  $\mathcal{C}$ , and the map  $\rho$  is given by the linear system of quadric surfaces that contain  $\mathcal{C}$ .

**Example 1.5** ([35, Proposition 4.7]). Let  $G$  be a primitive finite subgroup  $\mathfrak{S}_5$  in  $\mathrm{PGL}_4(\mathbb{C})$  such that there exists a  $G$ -invariant quadric surface in  $\mathbb{P}^3$ . Then  $\mathbb{P}^3$  contains a  $G$ -orbit  $\Sigma_5$  of length 5. Let  $\pi: X \rightarrow \mathbb{P}^3$  be the blow up of this orbit. Then there exists a  $G$ -commutative diagram

$$\begin{array}{ccccc} & X & \overset{\chi}{\dashrightarrow} & Y & \\ \pi \swarrow & & \searrow \alpha & \swarrow \beta & \searrow \phi \\ \mathbb{P}^3 & & V_3 & & S \end{array}$$

Here  $\alpha$  is the contraction of the proper transforms of the 10 lines in  $\mathbb{P}^3$  passing through pairs of points in  $\Sigma_5$ , the rational map  $\chi$  is a composition of Atiyah flops in these 10 curves, the morphism  $\beta$  is a flopping contraction, the morphism  $\phi$  is a  $\mathbb{P}^1$ -bundle, the

variety  $V_3$  is the Segre cubic hypersurface in  $\mathbb{P}^4$ , and  $S$  is the smooth del Pezzo surface of degree 5.

**Example 1.6.** Let  $G$  be a primitive finite subgroup  $\mathfrak{S}_5$  in  $\mathrm{PGL}_4(\mathbb{C})$  such that there exists a  $G$ -invariant quadric surface in  $\mathbb{P}^3$ . Then such quadric surface is smooth and unique. Denote it by  $Q$ . There is a unique  $G$ -invariant smooth cubic surface in  $\mathbb{P}^3$ , which is the Clebsch cubic surface. The complete intersection of this cubic surface and the quadric surface  $Q$  is a  $G$ -invariant irreducible smooth sextic curve of genus 4, which is known as the *Bring curve*. Let  $\pi: X \rightarrow \mathbb{P}^3$  be the blow up of this curve. Then there exists a  $G$ -commutative diagram

$$\begin{array}{ccc} & X & \\ \pi \swarrow & & \searrow \phi \\ \mathbb{P}^3 & \xleftarrow[\rho]{\quad} & V_3 \end{array}$$

Here  $V_3$  is a cubic hypersurface in  $\mathbb{P}^4$  that has one isolated ordinary double point, the morphism  $\phi$  is a contraction of the proper transform of the quadric  $Q$  to the singular point of the cubic  $V_3$ , and  $\rho$  is a linear projection from this point.

**Example 1.7.** Let  $G$  be a primitive finite subgroup  $\mathfrak{S}_5$  in  $\mathrm{PGL}_4(\mathbb{C})$  such that there exists a  $G$ -invariant pair of disjoint twisted cubic curves in  $\mathbb{P}^3$ . Let  $\pi: X \rightarrow \mathbb{P}^3$  be the blow up of the union of these twisted cubic curves. By Proposition 5.4, there exists a  $G$ -commutative diagram

$$\begin{array}{ccccc} & X & \xrightarrow{\quad \chi \quad} & Y & \\ \pi \swarrow & & \searrow \alpha & \swarrow \beta & \searrow \phi \\ \mathbb{P}^3 & & V_{12} & & \mathbb{P}^1 \times \mathbb{P}^1 \end{array}$$

such that  $V_{12}$  is a divisor of bi-degree  $(2, 2)$  in  $\mathbb{P}^2 \times \mathbb{P}^2$ , the morphisms  $\alpha$  and  $\beta$  are flopping contractions, the map  $\chi$  is a composition of ten Atiyah flops, and  $\phi$  is a  $\mathbb{P}^1$ -bundle.

The proof of the other direction of Theorem 1.1 is quite lengthy and requires a lot of computations. We use Blichfeldt's classification [4] to reduce the proof to the case of just two primitive groups. One of them is a primitive group  $\mu_2^4 \rtimes \mu_5$ , and another is  $\mathfrak{A}_4 \times \mathfrak{A}_4$ . Both of them contain the transitive subgroup  $\mu_2^4$  as a normal subgroup. Because of this, both groups have similar group-theoretical properties. However, there are geometrical discrepancies between the cases, so that we treat them separately.

Let us describe the structure of the paper. In §2, we present basic facts about the imprimitive group  $\mu_2^4$ . In §3, we study equivariant geometry of  $\mathbb{P}^3$  acted on by a primitive group  $\mathfrak{A}_4 \times \mathfrak{A}_4$ . Similarly, in §4, we do the same for the group  $\mu_2^4 \rtimes \mu_5$ . In particular, we construct a  $G$ -commutative diagram

$$\begin{array}{ccccc} & X & \xrightarrow{\quad \chi \quad} & Y & \\ \pi \swarrow & & \searrow \alpha & \swarrow \beta & \searrow \phi \\ \mathbb{P}^3 & & V_8 & & \mathbb{P}^3 \end{array}$$

Here  $G$  is a primitive subgroup  $\mu_2^4 \rtimes \mu_5$  in  $\mathrm{PGL}_4(\mathbb{C})$ , the morphisms  $\pi$  and  $\phi$  are blow ups of  $G$ -invariant smooth irreducible curves of degree 8 and genus 5, the variety  $V_8$  is a complete intersection of three quadrics in  $\mathbb{P}^6$  that has isolated terminal Gorenstein singularities,

the morphisms  $\alpha$  and  $\beta$  are small birational contractions, and  $\chi$  is a composition of flops. Finally, in §5, we prove our Theorem 1.1 using the results obtained in §3 and §4.

Throughout the paper we denote by  $\mu_n$  the cyclic group of order  $n$ . If  $G$  is a group and  $V$  is a variety acted on by  $G$ , we say that  $V$  is  $G$ -irreducible if  $G$  acts transitively on the set of irreducible components of  $V$ . If  $Z$  is a subvariety of  $V$ , we will sometimes abuse terminology and refer to the union of the images  $g(Z)$ ,  $g \in G$ , as the  $G$ -orbit of  $Z$ . By a  $G$ -commutative diagram we will mean a commutative diagram of  $G$ -rational maps, cf. Example 1.3.

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## 2. THE HEISENBERG GROUP

Let  $\varphi: \mathrm{SL}_4(\mathbb{C}) \rightarrow \mathrm{PGL}_4(\mathbb{C})$  be the natural projection. For every group  $G \subset \mathrm{SL}_4(\mathbb{C})$ , let us denote by  $\overline{G}$  the group  $\varphi(G)$ . Similarly, for every element  $g \in \mathrm{SL}_4(\mathbb{C})$ , we denote by  $\overline{g}$  its image  $\varphi(g)$ . Let

$$(2.1) \quad S_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$T_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

and denote by  $H$  the subgroup in  $\mathrm{SL}_4(\mathbb{C})$  that is generated by the matrices  $S_1, S_2, T_1, T_2$ . Then the center of  $H$  coincides with its commutator and consists of  $\pm I_4$ , where  $I_4$  is the identity matrix. Since  $|H| = 32$ , we have  $\overline{H} \cong \mu_2^4$ . We say that  $H$  is the Heisenberg group. The group  $\overline{H}$  is a transitive imprimitive subgroup of  $\mathrm{PGL}_4(\mathbb{C})$ .

**Lemma 2.2.** *The following assertions holds.*

- (i) *For every non-trivial element  $g \in \overline{H}$ , there exist two skew lines  $L_g$  and  $L'_g$  in  $\mathbb{P}^3$  that are pointwise fixed by  $g$ .*
- (ii) *For two distinct non-trivial elements  $g$  and  $h$  in  $\overline{H}$ , one has*

$$\{L_g, L'_g\} \cap \{L_h, L'_h\} = \emptyset.$$

- (iii) *No subgroup of order 8 in  $\overline{H}$  has a fixed point in  $\mathbb{P}^3$ .*

*Proof.* Assertions (i) and (ii) are easy to check by direct computations (cf. [20, §2.2]). Assertion (iii) follows from the transitivity of the subgroup  $\overline{H}$ .  $\square$

**Lemma 2.3.** *Let  $E$  be a rational or an elliptic curve. Then  $\overline{H}$  cannot act faithfully on  $E$ .*

*Proof.* If  $E$  is rational, the assertion follows from the classification of finite subgroups of the group  $\mathrm{PGL}_2(\mathbb{C})$ . Thus, we suppose that  $E$  is a (smooth) elliptic curve with a faithful action of  $\overline{H}$ . Then there is an exact sequence of groups

$$1 \rightarrow E(\mathbb{C}) \rightarrow \mathrm{Aut}(E) \rightarrow \mu_r \rightarrow 1,$$

where  $E(\mathbb{C})$  is the group of points of  $E$  acting on  $E$  by translations, and  $r \leq 6$ . We see that  $\overline{H} \cap E(\mathbb{C})$  contains a subgroup isomorphic to  $\mu_2^3$ . This is impossible, because there are just four 2-torsion points in  $E(\mathbb{C})$ .  $\square$

Denote by  $\mathbf{N}$  the normalizer of the group  $H$  in  $\mathrm{SL}_4(\mathbb{C})$ , and let

$$(2.4) \quad S = \frac{1+i}{\sqrt{2}} \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad T = \frac{1+i}{2} \begin{pmatrix} -i & 0 & 0 & i \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -i & i & 0 \end{pmatrix},$$

where  $i = \sqrt{-1}$ . One can check that both  $S$  and  $T$  are contained in  $\mathbf{N}$ , and  $S^4 = T^5 = -I_4$ . Let

$$A = \frac{1+i}{2} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & i & -i & 0 \\ i & 0 & 0 & -i \end{pmatrix}, \quad B = \frac{1+i}{2} \begin{pmatrix} -1 & i & 0 & 0 \\ 1 & i & 1 & 0 \\ 0 & 0 & i & -1 \\ 0 & 0 & -i & -1 \end{pmatrix}.$$

Then  $A$  and  $B$  are contained in  $\mathbf{N}$ , because  $A = T^4 \cdot S \cdot T \cdot S$  and

$$B = S \cdot T^4 \cdot S \cdot T \cdot S \cdot T^2 \cdot S \cdot T^3 \cdot S \cdot T \cdot S \cdot T^3 \cdot S \cdot T^3 \cdot S.$$

Similarly, let

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i & 0 & 0 \\ i & 1 & 0 & 0 \\ 0 & 0 & i & 1 \\ 0 & 0 & -1 & -i \end{pmatrix}.$$

Then  $R$  is also contained in  $\mathbf{N}$  (see [4, §124]).

It follows from [4, §124] that  $S$  and  $T$  together with  $H$  and  $\pm iI_4$  generate the group  $\mathbf{N}$ . Let  $\mathbb{H} = \langle H, \pm iI_4 \rangle$ . Then  $\overline{\mathbb{H}} = \overline{H}$ , and it follows from [4, §124] that there is an exact sequence

$$(2.5) \quad 1 \longrightarrow \mathbb{H} \longrightarrow \mathbf{N} \longrightarrow \mathfrak{S}_6 \longrightarrow 1.$$

Let  $\mathcal{U}_4$  be the vector space of all  $\mathbb{H}$ -invariant homogeneous polynomials of degree 4 in  $\mathbb{C}[x, y, z, w]$ . Recall from [20, §2.3] that  $\mathcal{U}_4$  is generated by the polynomials

$$x^4 + y^4 + z^4 + w^4, \quad x^2y^2 + z^2w^2, \quad x^2z^2 + y^2w^2, \quad x^2w^2 + y^2z^2, \quad xyzw.$$

This vector space is  $\mathbf{N}$ -invariant. Write

$$\begin{aligned}
 (2.6) \quad t_0 &= \frac{x^4 + y^4 + z^4 + w^4}{3} - 2(x^2y^2 + z^2w^2) - 2(x^2z^2 + y^2w^2) - 2(x^2w^2 + y^2z^2), \\
 t_1 &= \frac{x^4 + y^4 + z^4 + w^4}{3} - 2(x^2y^2 + z^2w^2) + 2(x^2z^2 + y^2w^2) + 2(x^2w^2 + y^2z^2), \\
 t_2 &= \frac{x^4 + y^4 + z^4 + w^4}{3} + 2(x^2y^2 + z^2w^2) - 2(x^2z^2 + y^2w^2) + 2(x^2w^2 + y^2z^2), \\
 t_3 &= \frac{x^4 + y^4 + z^4 + w^4}{3} + 2(x^2y^2 + z^2w^2) + 2(x^2z^2 + y^2w^2) - 2(x^2w^2 + y^2z^2), \\
 t_4 &= -\frac{2}{3}(x^4 + y^4 + z^4 + w^4) + 8xyzw, \\
 t_5 &= -\frac{2}{3}(x^4 + y^4 + z^4 + w^4) - 8xyzw.
 \end{aligned}$$

Note that  $t_0 + t_1 + t_2 + t_3 + t_4 + t_5 = 0$ , and  $t_0, t_1, t_2, t_3, t_4, t_5$  generate  $\mathcal{U}_4$ .

*Remark 2.7.* The representation of the group  $\mathbf{N}/\mathbb{H} \cong \mathfrak{S}_6$  in the vector space  $\mathcal{U}_4$  is the standard five-dimensional representation twisted by the sign character. Indeed, the matrix  $T$  acts on the polynomials  $t_0, \dots, t_5$  as the cycle  $(t_0 t_4 t_2 t_5 t_1)$ , and the matrix  $-S$  acts on  $t_0, \dots, t_5$  as the permutation  $(t_0 t_1)(t_2 t_3)(t_4 t_5)$ . Observe also that the matrix  $A$  acts on them as the permutation  $(t_0 t_5 t_2)(t_1 t_3 t_4)$ . Similarly, the matrix  $B$  acts as the permutation  $(t_0 t_2 t_5)(t_1 t_3 t_4)$ .

Let  $G_{80}$  be the subgroup in  $\mathrm{SL}_4(\mathbb{C})$  that is generated by  $\mathbb{H}$  and the matrix  $T$ , and let  $\overline{G}_{80}$  be its image in  $\mathrm{PGL}_4(\mathbb{C})$ . Then

$$\overline{G}_{80} \cong \mu_2^4 \rtimes \mu_5$$

by Remark 2.7. Similarly, let  $G_{320}$  be the subgroup in  $\mathrm{SL}_4(\mathbb{C})$  that is generated by  $\mathbb{H}$  and the matrices  $T$  and  $R$ , and let  $\overline{G}_{320}$  be its image in  $\mathrm{PGL}_4(\mathbb{C})$ . Then

$$\overline{G}_{320} \cong \mu_2^4 \rtimes (\mu_5 \rtimes \mu_4),$$

see [4, §124]. Moreover, it follows from (2.5) that there exists an exact sequence of groups

$$0 \longrightarrow \overline{\mathbf{H}} \longrightarrow \overline{G}_{320} \longrightarrow \mu_5 \rtimes \mu_4 \longrightarrow 0.$$

Let  $G_{160}$  be the subgroup in  $\mathrm{SL}_4(\mathbb{C})$  that is generated by  $\mathbb{H}$  and the matrices  $T$  and  $R^2$ , and let  $\overline{G}_{160}$  be its image in  $\mathrm{PGL}_4(\mathbb{C})$ . Then

$$\overline{G}_{160} \cong \mu_2^4 \rtimes (\mu_5 \rtimes \mu_2) \cong \mu_2^4 \rtimes D_{10},$$

where  $D_{10}$  is the dihedral group of order 10, see [4, §124]. There exists an exact sequence of groups

$$0 \longrightarrow \overline{\mathbf{H}} \longrightarrow \overline{G}_{160} \longrightarrow D_{10} \longrightarrow 0.$$

Furthermore, the group  $\overline{G}_{80}$  is a normal subgroup in both  $\overline{G}_{160}$  and  $\overline{G}_{320}$ , and the group  $\overline{G}_{160}$  is a normal subgroup in  $\overline{G}_{320}$ . Let  $G_{144}$  be the subgroup in  $\mathrm{SL}_4(\mathbb{C})$  that is generated by the subgroup  $\mathbb{H}$  together with the matrices  $A$  and  $B$ , and let  $\overline{G}_{144}$  be its image in  $\mathrm{PGL}_4(\mathbb{C})$ . Then  $G_{144}$  is a finite subgroup of order 576, and there exists an exact sequence of groups

$$0 \longrightarrow \overline{\mathbf{H}} \longrightarrow \overline{G}_{144} \longrightarrow \mu_3 \times \mu_3 \longrightarrow 0.$$

Moreover, one has

$$\overline{G}_{144} \cong \mu_2^4 \rtimes (\mu_3 \times \mu_3) \cong \mathfrak{A}_4 \times \mathfrak{A}_4$$

by Remark 2.7. The groups  $\overline{G}_{80}$ ,  $\overline{G}_{160}$ ,  $\overline{G}_{320}$ ,  $\overline{G}_{144}$  are primitive subgroups in  $\mathrm{PGL}_4(\mathbb{C})$ .

It is well known that  $\mathbb{P}^3$  contains exactly ten  $\overline{H}$ -invariant quadrics (see [20, §2.4]). The groups  $\overline{G}_{80}$ ,  $\overline{G}_{160}$ ,  $\overline{G}_{320}$  and  $\overline{G}_{144}$  naturally act on them, because  $\overline{H}$  is a normal subgroup of these four groups. For instance, the group  $\overline{G}_{80}$  splits them into two  $\overline{G}$ -orbits. One of them consists of the five quadrics

$$(2.8) \quad \begin{aligned} x^2 + y^2 + z^2 + w^2 = 0, \quad xw + yz = 0, \quad xz + yw = 0, \\ x^2 + y^2 - z^2 - w^2 = 0, \quad x^2 - y^2 - z^2 + w^2 = 0. \end{aligned}$$

Let us denote them by  $\mathcal{Q}_1$ ,  $\mathcal{Q}_2$ ,  $\mathcal{Q}_3$ ,  $\mathcal{Q}_4$  and  $\mathcal{Q}_5$ , respectively. The second  $\overline{G}_{80}$ -orbit consists of the five quadrics

$$(2.9) \quad x^2 - y^2 + z^2 - w^2 = 0, \quad xy + zw = 0, \quad xy - zw = 0, \quad xz - yw = 0, \quad xw - yz = 0.$$

We will denote them by  $\mathcal{Q}_6$ ,  $\mathcal{Q}_7$ ,  $\mathcal{Q}_8$ ,  $\mathcal{Q}_9$  and  $\mathcal{Q}_{10}$ , respectively. Observe that all these ten quadrics are smooth. Similarly, the only  $\overline{G}_{144}$ -invariant quadric in  $\mathbb{P}^3$  is the quadric  $\mathcal{Q}_3$ , i.e. the quadric given by  $xz + yw = 0$ . In fact, we can say more.

**Lemma 2.10.** *There are no  $\overline{G}_{80}$ -invariant surfaces in  $\mathbb{P}^3$  of degree 1, 2 and 3. Similarly, there are no  $\overline{G}_{144}$ -invariant surfaces in  $\mathbb{P}^3$  of degree 1 and 3. Finally, there is unique  $\overline{G}_{144}$ -invariant quadric surface in  $\mathbb{P}^3$ .*

*Proof.* This follows from explicit computations of symmetric powers of corresponding four-dimensional representations of the groups  $G_{80}$  and  $G_{144}$ . We used [21] to perform these computations.  $\square$

Recall from Lemma 2.2(i) that for every non-trivial element  $g \in \overline{H}$ , there exist two skew lines  $L_g$  and  $L'_g$  in  $\mathbb{P}^3$  that are pointwise fixed by  $g$ . The stabilizer of each of these lines in  $\overline{H}$  is  $\mu_2^3$ , and the kernel of its action on any of these lines is a subgroup  $\mu_2$  generated by  $g$ . In total, there are 30 lines in  $\mathbb{P}^3$  that are pointwise fixed by elements in  $\overline{H}$ . Denote the set of these lines by  $\mathbb{L}_{30}$ . Let us describe these lines explicitly.

The group  $\overline{H} \cong \mu_2^4$  acts on  $\mathcal{Q}_1 \cong \mathbb{P}^1 \times \mathbb{P}^1$  naturally. This means that the group  $\mu_2^2$  acts on  $\mathbb{P}^1$ , and the product action of the group  $\mu_2^2 \times \mu_2^2$  on  $\mathbb{P}^1 \times \mathbb{P}^1$  gives the action of the group  $\overline{H}$  on  $\mathcal{Q}_1$ . Since every subgroup  $\mu_2$  in  $\mu_2^2$  fixes exactly two points in  $\mathbb{P}^1$ , and there are three such subgroups in  $\mu_2^2$ , we see that there are exactly six pairs of skew lines in  $\mathcal{Q}_1$  that are  $\overline{H}$ -invariant. Three of these pairs lie in one family of lines on  $\mathcal{Q}_1$ , and the remaining three pairs lie in another family of lines on  $\mathcal{Q}_1$ . Let  $\ell$  be the line in  $\mathbb{P}^3$  that passes through the points  $[0 : i : 1 : 0]$  and  $[1 : 0 : 0 : -i]$ , and let  $\check{\ell}$  be the line in  $\mathbb{P}^3$  that passes through the points  $[0 : -i : 1 : 0]$  and  $[1 : 0 : 0 : i]$ . Then  $\ell$  and  $\check{\ell}$  are skew lines contained in  $\mathcal{Q}_1$ . Moreover, both of them are pointwise fixed by  $\overline{T}_1 \circ \overline{S}_2 \circ \overline{S}_1$  (see (2.1)), so that  $\ell$  and  $\check{\ell}$  are two lines in  $\mathbb{L}_{30}$ . Applying powers of  $\overline{T}$  to them, we obtain 8 more lines in  $\mathbb{L}_{30}$ . Here  $\overline{T}$  is an element of the group  $\overline{G}$  of order 5 defined in (2.4). To be precise, let

$$(2.11) \quad \begin{aligned} \ell_1 = \ell, \quad \ell_2 = \overline{T}(\ell_1), \quad \ell_3 = \overline{T}(\ell_2), \quad \ell_4 = \overline{T}(\ell_3), \quad \ell_5 = \overline{T}(\ell_4), \\ \check{\ell}_1 = \check{\ell}, \quad \check{\ell}_2 = \overline{T}(\check{\ell}_1), \quad \check{\ell}_3 = \overline{T}(\check{\ell}_2), \quad \check{\ell}_4 = \overline{T}(\check{\ell}_3), \quad \check{\ell}_5 = \overline{T}(\check{\ell}_4). \end{aligned}$$



Then the lines in (2.11) are contained in  $\mathbb{L}_{30}$ . Similarly, let  $\ell'$  be the line in  $\mathbb{P}^3$  that passes through the points  $[1 : 0 : -i : 0]$  and  $[0 : 1 : 0 : -i]$ , let  $\check{\ell}'$  be the line in  $\mathbb{P}^3$  that passes through the points  $[1 : 0 : i : 0]$  and  $[0 : 1 : 0 : i]$ , and let

$$(2.12) \quad \begin{aligned} \ell'_1 = \ell', \quad \ell'_2 = \overline{T}(\ell'_1), \quad \ell'_3 = \overline{T}(\ell'_2), \quad \ell'_4 = \overline{T}(\ell'_3), \quad \ell'_5 = \overline{T}(\ell'_4), \\ \check{\ell}'_1 = \check{\ell}', \quad \check{\ell}'_2 = \overline{T}(\check{\ell}'_1), \quad \check{\ell}'_3 = \overline{T}(\check{\ell}'_2), \quad \check{\ell}'_4 = \overline{T}(\check{\ell}'_3), \quad \check{\ell}'_5 = \overline{T}(\check{\ell}'_4). \end{aligned}$$

Then the lines in (2.12) are contained in  $\mathbb{L}_{30}$ . Finally, let  $\ell''$  be the line in  $\mathbb{P}^3$  that passes through the points  $[1 : i : 0 : 0]$  and  $[0 : 0 : 1 : -i]$ , let  $\check{\ell}''$  be the line in  $\mathbb{P}^3$  that passes through the points  $[1 : -i : 0 : 0]$  and  $[0 : 0 : 1 : i]$ , and let

$$(2.13) \quad \begin{aligned} \ell''_1 = \ell'', \quad \ell''_2 = \overline{T}(\ell''_1), \quad \ell''_3 = \overline{T}(\ell''_2), \quad \ell''_4 = \overline{T}(\ell''_3), \quad \ell''_5 = \overline{T}(\ell''_4), \\ \check{\ell}''_1 = \check{\ell}'', \quad \check{\ell}''_2 = \overline{T}(\check{\ell}''_1), \quad \check{\ell}''_3 = \overline{T}(\check{\ell}''_2), \quad \check{\ell}''_4 = \overline{T}(\check{\ell}''_3), \quad \check{\ell}''_5 = \overline{T}(\check{\ell}''_4). \end{aligned}$$

Then the lines in (2.13) are the remaining 10 lines in  $\mathbb{L}_{30}$ . By construction, the lines  $\ell$ ,  $\check{\ell}$ ,  $\ell'$ ,  $\check{\ell}'$ ,  $\ell''$  and  $\check{\ell}''$  are contained in the quadric surface  $\mathcal{Q}_1$ .

**Lemma 2.14.** *Each line among the 30 lines in  $\mathbb{L}_{30}$  is contained in exactly 4 of the quadric surfaces  $\mathcal{Q}_1, \dots, \mathcal{Q}_{10}$ . Vice versa, each of these quadrics contains exactly 12 such lines. For every quadric, these 12 lines contained in it are described by Table 1. Every two quadrics among  $\mathcal{Q}_1, \dots, \mathcal{Q}_{10}$  intersect by a quadruple of lines in  $\mathbb{L}_{30}$ .*

*Proof.* Direct computations. The last assertion can be also deduced from Lemma 2.3.  $\square$

Let us describe the intersection points of the 30 lines in  $\mathbb{L}_{30}$ . Namely, let  $\Sigma_{20}$  be the subset in  $\mathbb{P}^3$  that consists of 20 points

$$(2.15) \quad \begin{aligned} [i : 0 : 0 : 1], [0 : i : 1 : 0], [1 : 0 : 1 : 0], [0 : -1 : 0 : 1], [1 : i : -i : 1], \\ [1 : -i : i : 1], [0 : 0 : -1 : 1], [0 : 0 : 1 : 1], [-1 : i : -i : 1], [1 : i : i : 1], \\ [-i : 0 : 0 : 1], [0 : -i : 1 : 0], [1 : 0 : 1 : 0], [0 : 1 : 0 : 1], [-1 : i : i : 1], \\ [1 : -i : -i : 1], [-1 : 1 : 0 : 0], [1 : 1 : 0 : 0], [-1 : -i : i : 1], [1 : -i : -i : 1]. \end{aligned}$$

Similarly, let  $\Sigma'_{20}$  be the subset in  $\mathbb{P}^3$  that consists of 20 points

$$(2.16) \quad \begin{aligned} [-i : i : -1 : 1], [i : 0 : 1 : 0], [i : i : 1 : 1], [0 : i : 0 : 1], [-1 : 1 : -1 : 1], \\ [i : -i : -1 : 1], [1 : -1 : -1 : 1], [-1 : 0 : 0 : 1], [-1 : -1 : 1 : 1], [0 : -1 : 1 : 0], \\ [1 : 0 : 0 : 0], [1 : 0 : 0 : 1], [0 : 0 : 0 : 1], [-i : 0 : 1 : 0], [0 : 0 : 1 : 0], \\ [-i : -i : 1 : 1], [0 : -i : 0 : 1], [1 : 1 : 1 : 1], [0 : 1 : 1 : 0], [0 : 1 : 0 : 0]. \end{aligned}$$

Finally, let  $\Sigma''_{20}$  be the subset in  $\mathbb{P}^3$  that consists of 20 points

$$(2.17) \quad \begin{aligned} [i : -1 : i : 1], [0 : 0 : i : 1], [-i : 1 : i : 1], [-i : 1 : 0 : 0], [-i : -1 : i : 1], \\ [-i : i : 1 : 1], [-i : 1 : -i : 1], [i : -i : -1 : 1], [1 : -1 : 1 : 1], [i : -1 : -i : 1], \\ [1 : -1 : -1 : 1], [i : 1 : 0 : 0], [i : 1 : i : 1], [i : -i : 1 : 1], [-1 : 1 : 1 : 1], \\ [i : 1 : -i : 1], [0 : 0 : -i : 1], [i : -1 : -i : 1], [i : i : -1 : 1], [1 : 1 : -1 : 1]. \end{aligned}$$

Explicit computations show that  $\Sigma_{20}$ ,  $\Sigma'_{20}$  and  $\Sigma''_{20}$  are  $\overline{G}_{80}$ -orbits. This also follows from Lemma 4.13 below.

TABLE 1. Ten  $\bar{H}$ -invariant quadrics and thirty lines in them.

	$Q_1$	$Q_2$	$Q_3$	$Q_4$	$Q_5$	$Q_6$	$Q_7$	$Q_8$	$Q_9$	$Q_{10}$
$\ell_1$	+	−	−	−	+	−	+	−	+	−
$\ell_2$	−	−	−	+	+	−	−	+	−	+
$\ell_3$	−	−	+	+	−	+	+	−	−	−
$\ell_4$	−	+	+	−	−	−	−	−	+	+
$\ell_5$	+	+	−	−	−	+	−	+	−	−
$\check{\ell}_1$	+	−	−	−	+	−	+	−	+	−
$\check{\ell}_2$	−	−	−	+	+	−	−	+	−	+
$\check{\ell}_3$	−	−	+	+	−	+	+	−	−	−
$\check{\ell}_4$	−	+	+	−	−	−	−	−	+	+
$\check{\ell}_5$	+	+	−	−	−	+	−	+	−	−
$\ell'_1$	+	−	−	−	−	+	+	−	−	+
$\ell'_2$	−	−	−	−	+	+	−	−	+	+
$\ell'_3$	−	−	−	+	−	+	−	+	+	−
$\ell'_4$	−	−	+	−	−	−	+	+	+	−
$\ell'_5$	−	+	−	−	−	−	+	+	−	+
$\check{\ell}'_1$	+	−	−	−	−	+	+	−	−	+
$\check{\ell}'_2$	−	−	−	−	+	+	−	−	+	+
$\check{\ell}'_3$	−	−	−	+	−	+	−	+	+	−
$\check{\ell}'_4$	−	−	+	−	−	−	+	+	+	−
$\check{\ell}'_5$	−	+	−	−	−	−	+	+	−	+
$\ell''_1$	+	+	−	+	−	−	−	−	+	−
$\ell''_2$	+	−	+	−	+	−	−	+	−	−
$\ell''_3$	−	+	−	+	+	−	+	−	−	−
$\ell''_4$	+	−	+	+	−	−	−	−	−	+
$\ell''_5$	−	+	+	−	+	+	−	−	−	−
$\check{\ell}''_1$	+	+	−	+	−	−	−	−	+	−
$\check{\ell}''_2$	+	−	+	−	+	−	−	+	−	−
$\check{\ell}''_3$	−	+	−	+	+	−	+	−	−	−
$\check{\ell}''_4$	+	−	+	+	−	−	−	−	−	+
$\check{\ell}''_5$	−	+	+	−	+	+	−	−	−	−

**Lemma 2.18.** *The set  $\Sigma_{20} \cup \Sigma'_{20} \cup \Sigma''_{20}$  contains all intersection points of the lines from  $\mathbb{L}_{30}$ . Moreover, for every point  $P \in \Sigma_{20} \cup \Sigma'_{20} \cup \Sigma''_{20}$  there are exactly three lines from  $\mathbb{L}_{30}$  passing through  $P$ .*

*Proof.* Direct computations.  $\square$

Now we are going to describe  $\overline{G}_{80}$ -invariant quartic surfaces and  $\overline{G}_{144}$ -invariant quartic surfaces.

**Lemma 2.19.** *There are exactly five  $\overline{G}_{80}$ -invariant irreducible quartic surfaces in  $\mathbb{P}^3$ . Similarly, there are exactly four  $\overline{G}_{144}$ -invariant irreducible quartic surfaces in  $\mathbb{P}^3$ .*

*Proof.* It follows from Remark 2.7 that  $\mathcal{U}_4$  splits as a sum of distinct one-dimensional representations of  $G_{80}$  and  $G_{144}$ . Now the assertion follows from Lemma 2.10.  $\square$

Up to scaling, all homogeneous semi-invariants of the group  $G_{144}$  of degree 4 in  $\mathbb{C}[x, y, z, w]$  are

$$(2.20) \quad \begin{cases} p_0 = -\frac{1}{4}(t_0 + t_2 + t_5), \\ p_1 = -(\xi_3 t_0 + t_2 + \xi_3^2 t_5), \\ p_2 = (\xi_3 + 1)(t_0 + \xi_3^2 t_5 + \xi_3 t_2), \\ p_3 = (\xi_3 + 1)(t_1 + \xi_3^2 t_4 + \xi_3 t_3), \\ p_4 = -(\xi_3 t_1 + t_3 + \xi_3^2 t_4). \end{cases}$$

Here  $\xi_3$  is a primitive cubic root of unity. Observe that

$$p_0 = (wy + xz)^2$$

is the only homogeneous invariant of degree 4 of the group  $G_{144}$ . The remaining four polynomials  $p_1, p_2, p_3$  and  $p_4$  are semi-invariants, and they are irreducible. Thus, the four irreducible  $\overline{G}_{144}$ -invariant quartic surfaces in  $\mathbb{P}^3$  are given by  $p_1 = 0, p_2 = 0, p_3 = 0$  and  $p_4 = 0$ . We will study properties of these surfaces in §3.

The eigenvectors of the matrix  $T$  on the vector space of all  $\mathbb{H}$ -invariant homogeneous quartic polynomials in  $\mathbb{C}[x, y, z, w]$  are the polynomials

$$(2.21) \quad \begin{cases} q_0 = t_0 + t_1 + t_2 + t_3 + t_4, \\ q_1 = \xi_5^4 t_4 + \xi_5^3 t_2 + \xi_5^2 t_5 + \xi_5 t_1 + t_0, \\ q_2 = \xi_5^4 t_5 + \xi_5^3 t_4 + \xi_5^2 t_1 + \xi_5 t_2 + t_0, \\ q_3 = \xi_5^4 t_2 + \xi_5^3 t_1 + \xi_5^2 t_4 + \xi_5 t_5 + t_0, \\ q_4 = \xi_5^4 t_1 + \xi_5^3 t_5 + \xi_5^2 t_2 + \xi_5 t_4 + t_0. \end{cases}$$

Here  $\xi_5$  is a primitive fifth root of unity. Note that  $q_i$  is the eigenvector of  $T$  that corresponds to the eigenvalue  $\xi_5^i$ . The five  $\overline{G}_{80}$ -invariant quartic surfaces in  $\mathbb{P}^3$  are the quartic surfaces  $q_0 = 0, q_1 = 0, q_2 = 0, q_3 = 0$  and  $q_4 = 0$ . We will study properties of these surfaces in §4. The polynomial

$$q_0 = \frac{1}{2}(w^4 + y^4 + z^4 + x^4) + 3x^2 y^2 + 3z^2 w^2 + 3x^2 z^2 + 3y^2 w^2 - 3x^2 w^2 - 3y^2 z^2.$$

is the only  $G_{80}$ -invariant homogeneous quartic polynomial in  $\mathbb{C}[x, y, z, w]$ .

**Lemma 2.22.** *There are exactly four  $\overline{G}_{80}$ -invariant surfaces in  $\mathbb{P}^3$  of degree 6. Moreover, they are irreducible and reduced.*

*Proof.* One can find all irreducible representations of the group  $G_{80}$ . They can be described as follows: 10 one-dimensional representations, 10 four-dimensional representations, and 6 five-dimensional representations. Similarly, one can compute the sixth symmetric power of every four-dimensional representations. This computation imply that the sixth symmetric power of every four-dimensional representations contains exactly 4 different one-dimensional subrepresentations. Geometrically, this means that there are exactly four  $\overline{G}_{80}$ -invariant surfaces in  $\mathbb{P}^3$  of degree 6. Since  $\mathbb{P}^3$  does not contain  $\overline{G}_{80}$ -invariant surfaces of degree 1, 2 and 3 by Lemma 2.10, each  $\overline{G}_{80}$ -invariant surface of degree 6 must be irreducible and reduced. We used [21] to perform these computations.  $\square$

One can find explicit equations of the  $\overline{G}_{80}$ -invariant sextic surfaces in  $\mathbb{P}^3$ . However, we decided not to do this to keep the paper shorter.

### 3. THE GROUP OF ORDER 144

Let us use notation and assumptions of §2. In this section, we present basic facts about  $\overline{G}_{144}$ -orbits,  $\overline{G}_{144}$ -invariant curves and  $\overline{G}_{144}$ -invariant surfaces in  $\mathbb{P}^3$ .

Let us start with studying basic group-theoretic properties of the group  $\overline{G}_{144}$ . Recall that  $\overline{G}_{144}$  is isomorphic to

$$\mathfrak{A}_4 \times \mathfrak{A}_4 \cong \mu_2^4 \rtimes (\mu_3 \times \mu_3).$$

Note that the isomorphism  $\overline{G}_{144} \cong \mathfrak{A}_4 \times \mathfrak{A}_4$  is uniquely defined up to permutations of the factors and automorphisms of the group  $\mathfrak{A}_4$ . We will fix such an isomorphism until the end of this section. Denote by  $\mathfrak{A}_4^\Delta$  the *diagonal* subgroup in  $\overline{G}_{144}$ , i.e. the subgroup that consists of the elements  $(g, g)$ , where  $g \in \mathfrak{A}_4$ . Similarly, denote by  $\mathfrak{A}_4^\nabla$  the *twisted diagonal* subgroup in  $\overline{G}_{144}$ , i.e. the subgroup that consists of the elements  $(g, \sigma(g))$ , where  $g \in \mathfrak{A}_4$ , and  $\sigma$  is an outer automorphism of  $\mathfrak{A}_4$  that is given by conjugation with a fixed odd element of  $\mathfrak{S}_4$ . By a *basic* subgroup in  $\overline{G}_{144}$ , we will mean any subgroup contained in one of the factors. Similarly, we say that a subgroup in  $\overline{G}_{144}$  is of *product type* if it is a product of two basic subgroups.

**Lemma 3.1.** *Let  $\Gamma$  be a subgroup in  $\overline{G}_{144}$ . Then the following assertions holds.*

- (o) *If  $\Gamma$  is cyclic, then  $|\Gamma| \in \{1, 2, 3, 6\}$ .*
- (i) *The group  $\overline{H}$  contains all 2-subgroups of the group  $\overline{G}_{144}$ .*
- (ii) *If  $|\Gamma|$  is divisible by 16, then  $\Gamma$  contains the subgroup  $\overline{H}$ .*
- (iii) *If  $|\Gamma| = 9$ , then  $\Gamma \cong \mu_3^2$  and  $\Gamma$  is of product type.*
- (iv) *One has  $|\Gamma| \neq 18$ .*
- (v) *If  $|\Gamma| = 6$ , then  $\Gamma \cong \mu_6 \cong \mu_2 \times \mu_3$ , and  $\Gamma$  is of product type. Moreover, there are just two such subgroups up to conjugation. Furthermore, every subgroup in  $\overline{G}_{144}$  of order 12 that contains  $\Gamma$  is isomorphic to  $\mu_2^2 \times \mu_3$  and is of product type.*
- (vi) *If  $|\Gamma| = 12$  and  $\Gamma$  is not conjugate to  $\mathfrak{A}_4^\Delta$  and  $\mathfrak{A}_4^\nabla$ , then the group  $\Gamma$  contains a basic subgroup  $\mu_2^2$ .*
- (vii) *If  $\Gamma \neq \overline{G}_{144}$ , then  $|\Gamma| \leq 48$ .*
- (viii) *If  $|\Gamma| > 16$ , then there is a surjective homomorphism  $\Gamma \rightarrow \mathfrak{A}_4$ .*

*Proof.* Assertion (o) is obvious. The subgroup  $\overline{H}$  is the normal Sylow 2-subgroup of the group  $\overline{G}_{144}$ . This implies assertion (i). Assertion (ii) follows from (i) and Sylow theorems. Assertion (iii) also follows from Sylow theorems.

Let  $\pi_1: \overline{G}_{144} \rightarrow \mathfrak{A}_4$  and  $\pi_2: \overline{G}_{144} \rightarrow \mathfrak{A}_4$  be the two projections to the factors. If  $|\Gamma| = 18$ , then the subgroups  $\pi_1(\Gamma)$  and  $\pi_2(\Gamma)$  are proper subgroups, so that at least one of them has order greater than 4, which is impossible. This proves assertion (iv).

Similarly, if  $|\Gamma| = 6$ , then, up to permutation, the groups  $\pi_1(\Gamma)$  and  $\pi_2(\Gamma)$  are isomorphic to  $\mu_2$  and  $\mu_3$ , respectively. In this case, we have  $\Gamma \cong \mu_2 \times \mu_3 \cong \mu_6$ , and the conjugacy class of  $\Gamma$  is determined by the choice of  $\pi_i$  such that  $\pi_i(\Gamma) \cong \mu_2$ . Let  $\Upsilon$  be a subgroup in  $\overline{G}_{144}$  of order 12 that contains  $\Gamma$ . We may assume that  $\pi_1(\Gamma) \cong \mu_2$  and  $\pi_2(\Gamma) \cong \mu_3$ . Then  $\pi_2(\Gamma) \cong \mu_3$ , because  $\mu_3$  is a maximal proper subgroup in  $\mathfrak{A}_4$ . This easily implies assertion (v).

To prove assertion (vi), we suppose that  $|\Gamma| = 12$ . If  $\pi_1(\Gamma) = \mathfrak{A}_4$  and  $\pi_2(\Gamma) = \mathfrak{A}_4$ , then  $\Gamma$  is conjugate to either  $\mathfrak{A}_4^\Delta$  or  $\mathfrak{A}_4^\nabla$ . Thus, we may assume that  $\pi_1(\Gamma) \neq \mathfrak{A}_4$ . Then the kernel of  $\pi_1|_\Gamma$  contains a basic subgroup  $\mu_2^2$ .

To prove assertion (vii), we have to show that the index of  $\Gamma$  in  $\overline{G}_{144}$  is not 2. Suppose that it is. Then  $\Gamma$  is normal. Thus, the intersection of  $\Gamma$  with each factor in  $\overline{G}_{144} \cong \mathfrak{A}_4 \times \mathfrak{A}_4$  is either  $\mathfrak{A}_4$  or a subgroup of index 2 in  $\mathfrak{A}_4$ . Since  $\mathfrak{A}_4$  does not contain subgroups of index 2, we see that  $\Gamma$  contains both factors, so it is the whole group  $\overline{G}_{144}$ , which is absurd.

To prove assertion (viii), we suppose that  $|\Gamma| > 16$ . If  $\pi_1(\Gamma) = \mathfrak{A}_4$  or  $\pi_2(\Gamma) = \mathfrak{A}_4$ , then we are done. Otherwise, both  $\pi_1(\Gamma)$  and  $\pi_2(\Gamma)$  are proper subgroups of  $\mathfrak{A}_4$ , so that their orders are at most 4. This is impossible, because  $|\Gamma| > 16$ .  $\square$

To study  $\overline{G}_{144}$ -invariant curves in  $\mathbb{P}^3$ , we need the following result, which is a simple consequence of the Riemann–Hurwitz formula.

**Lemma 3.2.** *Let  $C$  be a smooth irreducible curve of genus  $g \leq 13$  with a faithful action of the group  $\overline{G}_{144}$ . Then  $g \in \{8, 13\}$ , and the curve  $C$  is not hyperelliptic. Furthermore, if  $\Omega \subset C$  is a  $\overline{G}_{144}$ -orbit, then  $|\Omega| \in \{24, 48, 72, 144\}$ . Finally, the possible numbers  $a_i$  of  $\overline{G}_{144}$ -orbits of length  $i \in \{24, 48, 72\}$  in  $C$  are contained in the following table:*

$g$	8	13	13
$a_{24}$	0	1	2
$a_{48}$	0	2	0
$a_{72}$	3	0	1

*Proof.* The assertion about the lengths of  $\overline{G}_{144}$ -orbits follows from Lemma 3.1(o), since the stabilizers in  $G$  of points in  $C$  are cyclic (see [10, Lemma 5.1.4]). By Lemma 2.3, the group  $\overline{G}_{144}$  cannot act faithfully on  $\mathbb{P}^1$  and on a smooth elliptic curve, so that  $g \geq 2$ .

Suppose that the curve  $C$  is hyperelliptic. Since the group  $\overline{G}_{144}$  does not contain normal subgroups of order 2, it does not contain the hyperelliptic involution of  $C$ , and we obtain a faithful action of the group  $\overline{G}_{144}$  on  $\mathbb{P}^1$ . The latter is impossible by Lemma 2.3.

Let  $\widehat{C} = C/\overline{G}_{144}$ . Then  $\widehat{C}$  is a smooth curve. Let  $\hat{g}$  be the genus of the curve  $\widehat{C}$ . Then the Riemann–Hurwitz formula gives

$$2g - 2 = 144(2\hat{g} - 2) + 72a_{72} + 96a_{48} + 120a_{24}.$$

Since  $a_k \geq 0$  and  $g \leq 13$ , one has  $\hat{g} = 0$ , so that

$$2g - 2 = -288 + 72a_{72} + 96a_{48} + 120a_{24},$$

Going through the possible values of  $g$ , and solving this equation case by case we obtain the required result.  $\square$

Denote by  $\mathcal{Q}$  the smooth quadric in  $\mathbb{P}^3$  that is given by

$$wy + xz = 0,$$

which is the quadric  $\mathcal{Q}_3$  in the notation of §2. Then  $\mathcal{Q}$  is  $\overline{G}_{144}$ -invariant. This quadric contains most of  $\overline{G}_{144}$ -orbits of small length in  $\mathbb{P}^3$  and most of  $\overline{G}_{144}$ -irreducible curves in  $\mathbb{P}^3$  that have small degree.

Observe that the action of the group  $\overline{G}_{144} \cong \mathfrak{A}_4 \times \mathfrak{A}_4$  on the surface  $\mathcal{Q} \cong \mathbb{P}^1 \times \mathbb{P}^1$  is just the product action given by the natural action of the group  $\mathfrak{A}_4$  on  $\mathbb{P}^1$ . The latter action is induced by a two-dimensional irreducible representation of the binary tetrahedral group  $2.\mathfrak{A}_4$ . Denote this representation by  $\mathbb{V}_2$ . Then there is a  $\overline{G}_{144}$ -equivariant identification

$$(3.3) \quad \mathbb{P}^3 \cong \mathbb{P}(\mathbb{V}_2 \boxtimes \mathbb{V}_2).$$

The points of the quadric  $\mathcal{Q}$  correspond to decomposable tensors in  $\mathbb{V}_2 \boxtimes \mathbb{V}_2$ .

Since  $\mathbb{P}^1$  contains two  $\mathfrak{A}_4$ -orbits of length 4 and one orbit of length 6, this gives four  $\overline{G}_{144}$ -irreducible curves in  $\mathcal{Q}$  that are unions of 4 disjoint lines, and two  $\overline{G}_{144}$ -irreducible curves in  $\mathcal{Q}$  that are unions of 6 disjoint lines. Denote the former four curves by  $\mathcal{L}_4^1, \mathcal{L}_4^2, \mathcal{L}_4^3$  and  $\mathcal{L}_4^4$ , and denote the latter two curves by  $\mathcal{L}_6^1$  and  $\mathcal{L}_6^2$ . Without loss of generality, we may assume that

$$\mathcal{L}_4^1 \cap \mathcal{L}_4^2 = \mathcal{L}_4^1 \cap \mathcal{L}_6^1 = \mathcal{L}_4^3 \cap \mathcal{L}_4^4 = \mathcal{L}_4^3 \cap \mathcal{L}_6^2 = \emptyset.$$

In other words, on  $\mathcal{Q} \cong \mathbb{P}^1 \times \mathbb{P}^1$ , the curves  $\mathcal{L}_4^1$  and  $\mathcal{L}_4^2$  have bi-degree  $(4, 0)$ , while the curves  $\mathcal{L}_4^3$  and  $\mathcal{L}_4^4$  have bi-degree  $(0, 4)$ . Similarly, the curve  $\mathcal{L}_6^1$  has bi-degree  $(6, 0)$ , and the curve  $\mathcal{L}_6^2$  has bi-degree  $(0, 6)$ . Let

$$\Sigma_{16}^1 = \mathcal{L}_4^1 \cap \mathcal{L}_4^3, \quad \Sigma_{16}^2 = \mathcal{L}_4^1 \cap \mathcal{L}_4^4, \quad \Sigma_{16}^3 = \mathcal{L}_4^2 \cap \mathcal{L}_4^3, \quad \Sigma_{16}^4 = \mathcal{L}_4^2 \cap \mathcal{L}_4^4.$$

Then  $\Sigma_{16}^1, \Sigma_{16}^2, \Sigma_{16}^3$  and  $\Sigma_{16}^4$  are  $\overline{G}_{144}$ -orbits of length 16. Similarly, let

$$\Sigma_{24}^1 = \mathcal{L}_4^1 \cap \mathcal{L}_6^2, \quad \Sigma_{24}^2 = \mathcal{L}_4^2 \cap \mathcal{L}_6^2, \quad \Sigma_{24}^3 = \mathcal{L}_4^3 \cap \mathcal{L}_6^1, \quad \Sigma_{24}^4 = \mathcal{L}_4^4 \cap \mathcal{L}_6^1.$$

Then the subsets  $\Sigma_{24}^1, \Sigma_{24}^2, \Sigma_{24}^3, \Sigma_{24}^4$  are  $\overline{G}_{144}$ -orbits of length 24. Finally, let  $\Sigma_{36} = \mathcal{L}_6^1 \cap \mathcal{L}_6^2$ . Then the subset  $\Sigma_{36}$  is a  $\overline{G}_{144}$ -orbit of length 36. We summarize the intersections of the curves  $\mathcal{L}_4^1, \mathcal{L}_4^2, \mathcal{L}_4^3, \mathcal{L}_4^4, \mathcal{L}_6^1$  and  $\mathcal{L}_6^2$  in the following table:

	$\mathcal{L}_4^1$	$\mathcal{L}_4^2$	$\mathcal{L}_4^3$	$\mathcal{L}_4^4$	$\mathcal{L}_6^1$	$\mathcal{L}_6^2$
$\mathcal{L}_4^1$	$\mathcal{L}_4^1$	$\emptyset$	$\Sigma_{16}^1$	$\Sigma_{16}^2$	$\emptyset$	$\Sigma_{24}^1$
$\mathcal{L}_4^2$	$\emptyset$	$\mathcal{L}_4^2$	$\Sigma_{16}^3$	$\Sigma_{16}^4$	$\emptyset$	$\Sigma_{24}^2$
$\mathcal{L}_4^3$	$\Sigma_{16}^1$	$\Sigma_{16}^3$	$\mathcal{L}_4^3$	$\emptyset$	$\Sigma_{24}^3$	$\emptyset$
$\mathcal{L}_4^4$	$\Sigma_{16}^2$	$\Sigma_{16}^4$	$\emptyset$	$\mathcal{L}_4^4$	$\Sigma_{24}^4$	$\emptyset$
$\mathcal{L}_6^1$	$\emptyset$	$\emptyset$	$\Sigma_{24}^3$	$\Sigma_{24}^4$	$\mathcal{L}_6^1$	$\Sigma_{36}$
$\mathcal{L}_6^2$	$\Sigma_{24}^1$	$\Sigma_{24}^2$	$\emptyset$	$\emptyset$	$\Sigma_{36}$	$\mathcal{L}_6^2$

**Lemma 3.4.** *Let  $\Sigma$  be a  $\overline{G}_{144}$ -orbit in  $\mathcal{Q}$  such that  $|\Sigma| \leq 36$ . Then  $\Sigma$  is one of the  $\overline{G}_{144}$ -orbits  $\Sigma_{16}^1, \Sigma_{16}^2, \Sigma_{16}^3, \Sigma_{16}^4, \Sigma_{24}^1, \Sigma_{24}^2, \Sigma_{24}^3, \Sigma_{24}^4, \Sigma_{36}$ .*

*Proof.* This follows from the fact that  $\mathbb{P}^1$  contains exactly two  $\mathfrak{A}_4$ -orbits of length 4, exactly one orbit of length 6, and the remaining  $\mathfrak{A}_4$ -orbits in  $\mathbb{P}^1$  are of length 12.  $\square$

By construction, the smooth quadric  $\mathcal{Q}$  contains the curves  $\mathcal{L}_4^1, \mathcal{L}_4^2, \mathcal{L}_4^3, \mathcal{L}_4^4, \mathcal{L}_6^1$  and  $\mathcal{L}_6^2$ . The same construction gives infinitely many  $\overline{G}_{144}$ -irreducible curves in  $\mathcal{Q}$  that are unions of 12 disjoint lines. Moreover, we have

**Lemma 3.5.** *Let  $C$  be a  $\overline{G}_{144}$ -irreducible curve in  $\mathcal{Q}$  such that  $\deg(C) \leq 23$ . Then either  $C$  is a disjoint union of 12 lines, or  $C$  is one of the curves  $\mathcal{L}_4^1, \mathcal{L}_4^2, \mathcal{L}_4^3, \mathcal{L}_4^4, \mathcal{L}_6^1, \mathcal{L}_6^2$ .*

*Proof.* The curve  $C$  is a curve of bi-degree  $(m, n)$  on  $\mathcal{Q} \cong \mathbb{P}^1 \times \mathbb{P}^1$  such that  $m+n \leq 23$ . The required assertion can be checked by a direct computation using the identification (3.3). However, we prefer to provide a more geometric proof.

Without loss of generality, we may assume that  $m \leq n$ . Then  $m \leq 11$ . Suppose that  $C$  is none of the curves  $\mathcal{L}_4^1, \mathcal{L}_4^2, \mathcal{L}_4^3, \mathcal{L}_4^4, \mathcal{L}_6^1, \mathcal{L}_6^2$ . Suppose also that  $C$  is not a disjoint union of 12 lines. Then  $m \geq 1$ . Let us seek for a contradiction.

Let  $P$  be a point in  $C$  that is not contained in the curves  $\mathcal{L}_4^1, \mathcal{L}_4^2, \mathcal{L}_4^3, \mathcal{L}_4^4, \mathcal{L}_6^1, \mathcal{L}_6^2$ . Let  $L$  be the line in  $\mathcal{Q}$  that is a curve of bi-degree  $(0, 1)$  that passes through  $P$ . Then  $L$  is not an irreducible component of the curve  $C$ , since  $C$  is  $\overline{G}_{144}$ -irreducible. Let  $\Gamma$  be the subgroup  $\mathfrak{A}_4 \times \text{Id}$  in  $\overline{G}_{144}$  (so that  $\Gamma$  is one of the two basic subgroups  $\mathfrak{A}_4$ ). Then  $L$  and  $C$  are  $\Gamma$ -invariant, and  $\Gamma$  acts faithfully on the line  $L$ . Hence, the intersection  $L \cap C$  is a union of  $\Gamma$ -orbits of length  $L \cdot C = m \leq 11$ . On the other hand, all such orbits are contained in the union  $\mathcal{L}_4^1 \cup \mathcal{L}_4^2 \cup \mathcal{L}_4^3 \cup \mathcal{L}_4^4 \cup \mathcal{L}_6^1 \cup \mathcal{L}_6^2$ , which gives us a contradiction.  $\square$

**Corollary 3.6.** *Let  $C$  be a  $\overline{G}_{144}$ -invariant curve in  $\mathbb{P}^3$  such that  $\deg(C) < 8$ . Then  $C$  is one of the curves  $\mathcal{L}_4^1, \mathcal{L}_4^2, \mathcal{L}_4^3, \mathcal{L}_4^4, \mathcal{L}_6^1, \mathcal{L}_6^2$ .*

*Proof.* If  $C$  is not contained in the quadric  $\mathcal{Q}$ , then

$$|C \cap \mathcal{Q}| \leq C \cdot \mathcal{Q} = 2\deg(C) \leq 14,$$

which is impossible by Lemma 3.4.  $\square$

**Corollary 3.7.** *Let  $C$  be a  $\overline{G}_{144}$ -irreducible curve in  $\mathcal{Q}$ , let  $\mathcal{D}$  be a (non-empty) mobile  $\overline{G}_{144}$ -invariant linear system on  $\mathbb{P}^3$ , let  $D$  be a general surface in  $\mathcal{D}$ , and let  $n$  be a positive integer such that  $\mathcal{D} \sim \mathcal{O}_{\mathbb{P}^3}(n)$ . Write*

$$D|_{\mathcal{Q}} = mC + \Delta,$$

*where  $m$  is a non-negative integer, and  $\Delta$  is an effective divisor on  $\mathcal{Q}$  whose support does not contain irreducible components of the curve  $C$ . Then  $m \leq \frac{n}{4}$ .*

*Proof.* Suppose that  $m > \frac{n}{4}$ . The curve  $C$  is a divisor of bi-degree  $(a, b)$  on  $\mathcal{Q} \cong \mathbb{P}^1 \times \mathbb{P}^1$ . But  $mC + \Delta$  is a divisor of bi-degree  $(n, n)$ , so that  $a < 4$  and  $b < 4$ , which is impossible by Lemma 3.5.  $\square$

**Corollary 3.8.** *Let  $C$  be a  $\overline{G}_{144}$ -irreducible curve in  $\mathcal{Q}$ , let  $\mathcal{D}$  be a (non-empty) mobile  $\overline{G}_{144}$ -invariant linear system on  $\mathbb{P}^3$ , and let  $n$  be a positive integer such that  $\mathcal{D} \sim \mathcal{O}_{\mathbb{P}^3}(n)$ . Then  $\text{mult}_C(\mathcal{D}) \leq \frac{n}{4}$ .*

Denote by  $\Sigma_{12}$  the subset in  $\mathbb{P}^3$  that consists of 12 points

$$[0 : 1 : 0 : 1], [0 : -1 : 0 : 1], [1 : i : i : 1], [1 : -i : -i : 1], [1 : -i : i : -1], [1 : i : -i : -1], \\ [1 : -1 : i : -i], [1 : -1 : -i : i], [1 : 1 : i : i], [1 : 1 : -i : -i], [1 : 0 : 1 : 0], [1 : 0 : -1 : 0].$$

Similarly, denote by  $\Sigma'_{12}$  the subset in  $\mathbb{P}^3$  that consists of 12 points

$$[1 : 1 : 1 : 1], [1 : -1 : -1 : 1], [1 : i : 1 : -i], [1 : -i : 1 : i], [1 : i : -1 : i], [1 : -i : -1 : -i], \\ [1 : 0 : i : 0], [1 : 0 : -i : 0], [0 : i : 0 : 1], [0 : -i : 0 : 1], [1 : 1 : -1 : -1], [1 : -1 : 1 : -1].$$

Explicit computations show that  $\Sigma_{12}$  and  $\Sigma'_{12}$  are  $\overline{G}_{144}$ -orbits. This also follows from Lemma 3.12 below.

**Lemma 3.9.** *The subsets  $\Sigma_{12}$  and  $\Sigma'_{12}$  are all  $\overline{G}_{144}$ -orbits in  $\mathbb{P}^3$  of length 12.*

*Proof.* Let  $\Upsilon$  be a basic subgroup  $\mu_2^2 \subset \overline{G}_{144}$ . Recall that the action of the group  $\Upsilon$  on  $\mathcal{Q} \cong \mathbb{P}^1 \times \mathbb{P}^1$  is a product action, with trivial group acting on one of the factors. Let  $\tilde{\Upsilon}$  be the preimage of  $\Upsilon$  via the canonical projection  $2\mathfrak{A}_4 \rightarrow \mathfrak{A}_4$ . Then the restriction of the two-dimensional representation  $\mathbb{V}_2$  to  $\tilde{\Upsilon}$  is an irreducible two-dimensional representations of the group  $\tilde{\Upsilon}$ . Using the identification (3.3), we see that the restriction of the four-dimensional representation  $\mathbb{V}_2 \boxtimes \mathbb{V}_2$  to the group  $\tilde{\Upsilon}$  splits as sum of two isomorphic irreducible two-dimensional representations. Thus,  $\Upsilon$  does not have fixed points in  $\mathbb{P}^3$ .

Let  $\Sigma$  be a  $\overline{G}_{144}$ -orbit in  $\mathbb{P}^3$  of length 12, and let  $\Gamma$  be the stabilizer in  $\overline{G}_{144}$  of a point in  $\Sigma$ . Then  $|\Gamma| = 12$ . We have seen that  $\Gamma$  does not contain  $\Upsilon$ . Thus, it follows from Lemma 3.1(vi) that  $\Gamma$  is conjugate to either  $\mathfrak{A}_4^\Delta$  or  $\mathfrak{A}_4^\nabla$ . Using the identification (3.3) one more time, we see that each of the latter groups has a unique fixed point in  $\mathbb{P}^3$ . Since we already know two  $\overline{G}_{144}$ -orbits in  $\mathbb{P}^3$  of length 12, namely  $\Sigma_{12}$  and  $\Sigma'_{12}$ , the assertion follows.  $\square$

Now we are ready to describe  $\overline{G}_{144}$ -orbits in  $\mathbb{P}^3$  of small length.

**Lemma 3.10.** *Let  $\Sigma$  be a  $\overline{G}_{144}$ -orbit in  $\mathbb{P}^3$  such that  $|\Sigma| \leq 35$ . Then  $\Sigma$  is one of the  $\overline{G}_{144}$ -orbits  $\Sigma_{12}$ ,  $\Sigma'_{12}$ ,  $\Sigma_{16}^1$ ,  $\Sigma_{16}^2$ ,  $\Sigma_{16}^3$ ,  $\Sigma_{16}^4$ ,  $\Sigma_{24}^1$ ,  $\Sigma_{24}^2$ ,  $\Sigma_{24}^3$ ,  $\Sigma_{24}^4$ .*

*Proof.* One has  $|\Sigma| > 4$ , because the group  $\overline{G}_{144}$  is primitive. This implies that

$$|\Sigma| \in \{6, 8, 9, 12, 16, 18, 24\}.$$

Let  $\Gamma$  be a stabilizer in  $\overline{G}_{144}$  of a point in  $\Sigma$ . If  $|\Sigma| \in \{6, 9, 18\}$ , then  $|\Gamma|$  is divisible by 8, so that  $\Gamma$  contains a subgroup  $\mu_2^3 \subset \overline{H}$  by Sylow theorem and Lemma 3.1(i). This is impossible by Lemma 2.2(iii). If  $|\Sigma| = 8$ , then  $|\Gamma| = 18$ , which is impossible by Lemma 3.1(iv). If  $|\Sigma| = 12$ , then the required assertion follows from Lemma 3.9.

Suppose that  $|\Sigma| = 16$ . Then  $|\Gamma| = 9$ , so that  $\Gamma \cong \mu_3^2$  by Lemma 3.1(iii). Recall that the action of the group  $\Gamma \cong \mu_3 \times \mu_3$  on  $\mathcal{Q} \cong \mathbb{P}^1 \times \mathbb{P}^1$  is a product action. Let  $\tilde{\mu}_3$  be the preimage of  $\mu_3$  via the canonical projection  $2\mathfrak{A}_4 \rightarrow \mathfrak{A}_4$ . Then the restriction of the two-dimensional representation  $\mathbb{V}_2$  to  $\tilde{\mu}_3$  splits as a sum of two distinct one-dimensional representations of the group  $\tilde{\mu}_3$ . Using the identification (3.3), we see that the restriction of the four-dimensional representation  $\mathbb{V}_2 \boxtimes \mathbb{V}_2$  to the group  $\tilde{\mu}_3 \times \tilde{\mu}_3$  splits as sum of four pairwise non-isomorphic one-dimensional subrepresentations, which are generated by decomposable tensors. Since decomposable tensors correspond to the points of the



quadric  $\mathcal{Q}$ , we see that  $\Sigma \subset \mathcal{Q}$ , so that  $\Sigma$  is one of the  $\overline{G}_{144}$ -orbits  $\Sigma_{16}^1, \Sigma_{16}^2, \Sigma_{16}^3, \Sigma_{16}^4$  by Lemma 3.4.

Suppose that  $|\Sigma| = 24$ . Then  $\Gamma \cong \mu_6 \cong \mu_2 \times \mu_3$  by Lemma 3.1(v). Now arguing as in the previous case, we see that  $\Sigma$  is one of the  $\overline{G}_{144}$ -orbits  $\Sigma_{24}^1, \Sigma_{24}^2, \Sigma_{24}^3, \Sigma_{24}^4$ .  $\square$

Let  $p_1, p_2, p_3$  and  $p_4$  be the quartic polynomials in (2.20). Denote by  $S_1, S_2, S_3, S_4$  the quartic surfaces  $\mathbb{P}^3$  that are given by  $p_1 = 0, p_2 = 0, p_3 = 0$  and  $p_4 = 0$ , respectively. Then  $S_1, S_2, S_3, S_4$  are all  $\overline{G}_{144}$ -invariant irreducible quartic surfaces in  $\mathbb{P}^3$  by Lemma 2.19. Moreover, explicit computations imply that

$$(3.11) \quad \Sigma_{12} \in S_1 \cap S_2, \quad \Sigma'_{12} \in S_3 \cap S_4, \quad \Sigma_{12} \not\subset S_3 \cup S_4, \quad \Sigma'_{12} \not\subset S_1 \cup S_2.$$

In fact, we can say more.

**Lemma 3.12.** *One has  $\text{Sing}(S_1) = \text{Sing}(S_2) = \Sigma_{12}$  and  $\text{Sing}(S_3) = \text{Sing}(S_4) = \Sigma'_{12}$ . Moreover, all singular points of the surfaces  $S_1, S_2, S_3$  and  $S_4$  are ordinary double points.*

*Proof.* It is enough to show the required assertions for the surface  $S_1$ , since the proof is identical in the remaining cases. We already know that  $S_1$  is irreducible. Taking a general plane section of the surface  $S_1$ , we see that either  $S_1$  has isolated singularities, or  $\text{Sing}(S_1)$  contains a  $\overline{G}_{144}$ -invariant curve of degree at most 3. The latter is impossible by Corollary 3.6. Thus, the surface  $S_1$  has at most isolated singularities.

To see that  $S_1$  is singular at every point of the  $\overline{G}_{144}$ -orbit  $\Sigma_{12}$ , one can simply take partial derivatives of the polynomial  $p_1$ , and plug the point  $[0 : 1 : 0 : 1]$  into them. Alternatively, one can use the fact that the stabilizer in  $\overline{G}_{144}$  of every point in  $\Sigma_{12}$  is isomorphic to  $\mathfrak{A}_4$ . Since the group  $\mathfrak{A}_4$  does not have faithful two-dimensional representations, the surface  $S_1$  must be singular at every point of  $\Sigma_{12}$  by [10, Lemma 4.4.1].

Thus, we see that  $S_1$  has isolated singularities and  $\Sigma_{12} \subset \text{Sing}(S_1)$ . Moreover, the surface  $S_1$  cannot have more than two non-Du Val singular points by [44, Theorem 1]. Thus, the surface  $S_1$  has at most Du Val singularities by Lemma 3.10. Then the minimal resolution of  $S_1$  is a smooth  $K3$  surface. Since the rank of the Picard group of a smooth  $K3$  surface is at most 20, we see that  $\text{Sing}(S_1) = \Sigma_{12}$  and every singular point of  $S_1$  is an ordinary double point.  $\square$

**Corollary 3.13.** *Let  $\Sigma$  be one of the  $\overline{G}_{144}$ -orbits  $\Sigma_{12}$  or  $\Sigma'_{12}$  and let  $\mathcal{M}$  be a linear system that consists of cubic surfaces in  $\mathbb{P}^3$  containing  $\Sigma$ . Then  $\Sigma$  is the base locus of the linear system  $\mathcal{M}$ .*

*Proof.* By Lemma 3.12, we have  $\Sigma = \text{Sing}(S_i)$  for some  $i \in \{1, 2, 3, 4\}$ . Recall that the surface  $S_i$  is given by the equation  $p_i(x, y, z, w) = 0$ , where  $p_i$  is the quartic polynomial in (2.20). Thus, the linear system  $\mathcal{M}$  contains the linear system that consists of cubic surfaces

$$\lambda_0 \frac{\partial p_i}{\partial x} + \lambda_1 \frac{\partial p_i}{\partial y} + \lambda_2 \frac{\partial p_i}{\partial z} + \lambda_3 \frac{\partial p_i}{\partial w} = 0,$$

where  $[\lambda_0 : \lambda_1 : \lambda_2 : \lambda_3] \in \mathbb{P}^3$ . The base locus of the latter system is  $\text{Sing}(S_i) = \Sigma$ .  $\square$

*Remark 3.14.* Let  $H$  be the plane section of the quadric  $\mathcal{Q}$ . Then there is an exact sequences of  $G$ -representations

$$0 \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^3}(4)) \longrightarrow H^0(\mathcal{O}_{\mathcal{Q}}(4H)) \longrightarrow 0.$$

On the other hand, the divisors  $\mathcal{L}_4^1 + \mathcal{L}_4^3$ ,  $\mathcal{L}_4^1 + \mathcal{L}_4^4$ ,  $\mathcal{L}_4^2 + \mathcal{L}_4^3$ ,  $\mathcal{L}_4^2 + \mathcal{L}_4^4$  are contained in the linear system  $|\mathcal{O}_{\mathcal{Q}}(4H)|$ . Thus, each intersection among  $S_1 \cap \mathcal{Q}$ ,  $S_2 \cap \mathcal{Q}$ ,  $S_3 \cap \mathcal{Q}$  and  $S_4 \cap \mathcal{Q}$  is one of the unions  $\mathcal{L}_4^1 \cup \mathcal{L}_4^3$ ,  $\mathcal{L}_4^1 \cup \mathcal{L}_4^4$ ,  $\mathcal{L}_4^2 \cup \mathcal{L}_4^3$  and  $\mathcal{L}_4^2 \cup \mathcal{L}_4^4$ .

Using this remark, we see that for every  $\overline{G}_{144}$ -orbit among  $\Sigma_{16}^1$ ,  $\Sigma_{16}^2$ ,  $\Sigma_{16}^3$  and  $\Sigma_{16}^4$ , there exists exactly one quartic surface among  $S_1$ ,  $S_2$ ,  $S_3$  and  $S_4$  that does not contain it. This implies

**Lemma 3.15.** *Let  $C$  be a  $\overline{G}_{144}$ -invariant curve in  $\mathbb{P}^3$  of degree less than 12. Then  $C \subset \mathcal{Q}$ .*

*Proof.* We may assume that  $C$  is  $\overline{G}_{144}$ -irreducible. Suppose that  $C$  is not contained in the quadric  $\mathcal{Q}$ . Let us seek for a contradiction. Observe that

$$C \cdot \mathcal{Q} = 2 \cdot \deg(C) \leq 22.$$

Thus, it follows from Lemma 3.10 that  $\deg(C) = 8$ , and the intersection  $C \cap \mathcal{Q}$  is one of the orbits  $\Sigma_{16}^1$ ,  $\Sigma_{16}^2$ ,  $\Sigma_{16}^3$  or  $\Sigma_{16}^4$ . Then the curve  $C$  does not contain other  $\overline{G}_{144}$ -orbits of length 16 in  $\mathbb{P}^3$ , because all of them lie in  $\mathcal{Q}$ . Let  $S_k$  be the surface among  $S_1$ ,  $S_2$ ,  $S_3$  and  $S_4$  that does not contain  $C \cap \mathcal{Q}$ . Then  $S_k$  does not contain the curve  $C$ . On the other hand, we have

$$C \cdot S_k = 4 \cdot \deg(C) = 32.$$

Hence, the intersection  $C \cap S_k$  is one of the  $\overline{G}_{144}$ -orbits  $\Sigma_{16}^1$ ,  $\Sigma_{16}^2$ ,  $\Sigma_{16}^3$ ,  $\Sigma_{16}^4$  by Lemma 3.10. This is impossible, since  $S_k$  does not contain the  $\overline{G}_{144}$ -orbit  $C \cap \mathcal{Q}$ , and  $C$  does not contain other  $\overline{G}_{144}$ -orbits of length 16.  $\square$

**Lemma 3.16.** *Let  $C$  be a reducible  $\overline{G}_{144}$ -irreducible curve in  $\mathbb{P}^3$  of degree less than 16. Then  $C \subset \mathcal{Q}$ .*

*Proof.* Suppose that  $C$  is not contained in the quadric  $\mathcal{Q}$ . Then  $\deg(C) \geq 12$  by Lemma 3.15. Moreover, since

$$30 \geq 2 \cdot \deg(C) = C \cdot \mathcal{Q},$$

we see that  $\deg(C) = 12$  by Lemma 3.4.

Let  $C = C_1 + \dots + C_r$ , where  $C_i$  is an irreducible curve of degree  $d$ . Denote by  $\Gamma$  the stabilizer in  $\overline{G}_{144}$  of the curve  $C_1$ . Since  $rd = \deg(C) = 12$  and  $\overline{G}_{144}$  does not have subgroups of index 2 by Lemma 3.1(vii), we see that either  $r = 3$  and  $d = 4$ , or  $r = 4$  and  $d = 3$ , or  $r = 6$  and  $d = 2$ , or  $r = 12$  and  $d = 1$ . If  $d \neq 1$ , then  $C_1$  is not contained in a plane, because there are no  $\overline{G}_{144}$ -orbits in  $\mathbb{P}^3$  of length at most 6 by Lemma 3.10. In particular, the curve  $C_1$  is not a conic, so that  $r \neq 6$ . We also conclude that  $\Gamma$  acts faithfully on  $C_1$  and on its normalization in the case when  $d \neq 1$ . Thus, the curve  $C$  is either a union of 3 irreducible quartic curves, or a union of 4 cubic curves, or a union of 12 lines. Let us deal with each of these cases one by one.

Suppose that  $r = 3$  and  $d = 4$ . Then  $|\Gamma| = 48$ , so that  $\Gamma$  contains the subgroup  $\overline{H}$  by Lemma 3.1(ii). On the other hand, the curve  $C_1$  must be either rational or elliptic, because  $d = 4$  and  $C_1$  is not contained in a plane. This is impossible by Lemma 2.3.

Suppose that  $r = 4$  and  $d = 3$ . Then  $C_1$  is a twisted cubic curve. One has  $|\Gamma| = 36$ , so that  $\Gamma$  has a surjective homomorphism to  $\mathfrak{A}_4$  by Lemma 3.1(viii). This is impossible by the classification of finite subgroups in  $\mathrm{PGL}_2(\mathbb{C})$ .

We see that  $r = 12$  and  $d = 1$ , so that  $C_1$  is a line and  $|\Gamma| = 12$ . Since  $C \cdot \mathcal{Q} = 24$ , we conclude from Lemma 3.4 that  $C \cap \mathcal{Q}$  is one of the  $\overline{G}_{144}$ -orbits  $\Sigma_{24}^1$ ,  $\Sigma_{24}^2$ ,  $\Sigma_{24}^3$  or  $\Sigma_{24}^4$ .

Without loss of generality, we may assume that  $C \cap \mathcal{Q} = \Sigma_{24}^1$ . We see that every irreducible component of the curve  $C$  contains exactly two points in  $\Sigma_{24}^1$ , because every line  $C_i$  can intersect  $\mathcal{Q}$  by at most two points. Write  $C_1 \cap \Sigma_{24}^1 = \{P, Q\}$ , and denote the stabilizer of the point  $P$  in  $\overline{G}_{144}$  by  $\Gamma_P$ . Then  $|\Gamma_P| = 6$ , so that  $\Gamma_P \cong \mu_6$  by Lemma 3.1(v). On the other hand, the stabilizer of this point in  $\Gamma$  has index at most 2. This means that this stabilizer is the group  $\Gamma_P$ , so that  $\Gamma_P \subset \Gamma$ . Now using Lemma 3.1(v) again, we see that  $\Gamma \cong \mu_2^2 \times \mu_3$  and  $\Gamma$  is of product type. Pick an element  $\gamma \in \Gamma$  such that  $\gamma \notin \Gamma_P$  and the order of  $\gamma$  is two. Recall that

$$\Sigma_{24}^1 = \mathcal{L}_4^1 \cap \mathcal{L}_6^2.$$

Then  $\gamma$  preserves the irreducible component of  $\mathcal{L}_4^1$  that contains the point  $P$ . Denote it by  $L$ . This means that  $Q = \gamma(P) \in L$ , so that the lines  $L$  and  $C_1$  coincide, which is impossible, since  $C_1$  is not contained in  $\mathcal{Q}$  by assumption.  $\square$

Let us describe the intersections  $S_i \cap S_j$ ,  $1 \leq i < j \leq 4$ .

**Lemma 3.17.** *Both  $S_1 \cap S_2$  and  $S_3 \cap S_4$  are  $\overline{G}_{144}$ -irreducible curves that are unions of 16 lines.*

*Proof.* It is enough to show that  $S_1 \cap S_2$  is a  $\overline{G}_{144}$ -irreducible curve that is a union of 16 lines, since the proof is identical in the remaining case. Write

$$S_2|_{S_1} = \sum_{i=1}^r n_i Z_i,$$

where each  $Z_i$  is a  $\overline{G}_{144}$ -irreducible curve, each  $n_i$  is a positive integer, and  $r \geq 1$ . We may assume that  $\deg(Z_i) \leq \deg(Z_j)$  for  $i \leq j$ . Then

$$16 = \sum_{i=1}^r n_i \deg(Z_i).$$

By Remark 3.14, the intersection  $S_1 \cap S_2 \cap \mathcal{Q}$  cannot contain two curves among  $\mathcal{L}_4^1$ ,  $\mathcal{L}_4^2$ ,  $\mathcal{L}_4^3$  and  $\mathcal{L}_4^4$ . Thus, it follows from Lemmas 3.5 and 3.15 that one of the following cases holds:

- either  $r = 1$  and  $n_1 = 1$ ,
- or  $r = 2$ ,  $n_1 = n_2 = 1$ ,  $\deg(Z_2) = 12$ , and  $Z_1$  is one of the curves  $\mathcal{L}_4^1$ ,  $\mathcal{L}_4^2$ ,  $\mathcal{L}_4^3$ ,  $\mathcal{L}_4^4$ .

Let  $f: \tilde{S}_1 \rightarrow S_1$  be the minimal resolution of singularities of the surface  $S_1$ . Then the action of the group  $\overline{G}_{144}$  lifts to the surface  $\tilde{S}_1$ . Denote by  $E_1, \dots, E_{12}$  the exceptional curves of the birational morphism  $f$ . Let  $E = E_1 + \dots + E_{12}$ , and let  $H$  be a plane section of the surface  $S_1$ . Denote by  $\tilde{Z}_i$  the proper transform of the curve  $Z_i$  on the surface  $\tilde{S}_1$ . Then

$$\sum_{i=1}^r \tilde{Z}_i \sim f^*(4H) - mE$$

for some none-negative integer  $m$ . Moreover, we have  $m \geq 2$ , since  $S_2$  is singular at every point of  $\Sigma_{12}$ . On the other hand, since  $\tilde{S}_1$  is a smooth  $K3$  surface, we have

$$\tilde{Z}_i^2 \geq -2 \cdot \deg(Z_i),$$

because the curve  $\tilde{Z}_i$  cannot consist of more than  $\deg(Z_i)$  irreducible components.

If  $r = 2$ , then  $\tilde{Z}_1^2 = -8$  and  $\tilde{Z}_1 \cdot E = 0$ , since  $Z_1$  is a disjoint union of 4 lines that does not contain  $\Sigma_{12}$ . In this case, we have

$$-24 = -2 \cdot \deg(Z_2) \leq \tilde{Z}_2^2 = \left(f^*(4H) - mE - \tilde{Z}_1\right)^2 = 64 - 24m^2 - 40 \leq -72,$$

which is absurd.

Thus, we see that  $r = 1$ , so that  $S_1 \cap S_2 = Z_1$  is a  $\overline{G}_{144}$ -irreducible curve of degree 16. This gives

$$-32 = -2 \cdot \deg(Z_1) \leq \tilde{Z}_1^2 = \left(f^*(4H) - mE\right)^2 = 64 - 24m^2 \leq -32,$$

so that  $\tilde{Z}_1^2 = -2 \cdot \deg(Z_1)$ . This easily implies that  $Z_1$  is a union of  $\deg(Z_1) = 16$  lines, cf. the proof of [10, Lemma 10.1.1].  $\square$

As we already mentioned in Remark 3.14, each of the intersections  $S_1 \cap \mathcal{Q}$ ,  $S_2 \cap \mathcal{Q}$ ,  $S_3 \cap \mathcal{Q}$  and  $S_4 \cap \mathcal{Q}$  is one of the unions  $\mathcal{L}_4^1 \cup \mathcal{L}_4^3$ ,  $\mathcal{L}_4^1 \cup \mathcal{L}_4^4$ ,  $\mathcal{L}_4^2 \cup \mathcal{L}_4^3$  or  $\mathcal{L}_4^2 \cup \mathcal{L}_4^4$ . On the other hand, we just proved that the intersections  $S_1 \cap S_2$  and  $S_3 \cap S_4$  are  $\overline{G}_{144}$ -irreducible curves. Thus, without loss of generality, we may assume that

$$(3.18) \quad S_1 \cap \mathcal{Q} = \mathcal{L}_4^2 \cup \mathcal{L}_4^4, \quad S_2 \cap \mathcal{Q} = \mathcal{L}_4^1 \cup \mathcal{L}_4^3, \quad S_3 \cap \mathcal{Q} = \mathcal{L}_4^1 \cup \mathcal{L}_4^4, \quad S_4 \cap \mathcal{Q} = \mathcal{L}_4^2 \cup \mathcal{L}_4^3.$$

Using this, we can determine which  $\overline{G}_{144}$ -orbit  $\Sigma_{16}^1, \Sigma_{16}^2, \Sigma_{16}^3, \Sigma_{16}^4$  is contained in which surface  $S_1, S_2, S_3$  and  $S_4$ . This is summarized in the following table.

	$S_1$	$S_2$	$S_3$	$S_4$
$\Sigma_{16}^1$	−	+	+	+
$\Sigma_{16}^2$	+	+	+	−
$\Sigma_{16}^3$	+	+	−	+
$\Sigma_{16}^4$	+	−	+	+

Observe that the intersection  $S_1 \cap S_3$  contains the curve  $\mathcal{L}_4^4$ , the intersection  $S_1 \cap S_4$  contains the curve  $\mathcal{L}_4^2$ , the intersection  $S_2 \cap S_3$  contains the curve  $\mathcal{L}_4^1$ , and the intersection  $S_2 \cap S_4$  contains the curve  $\mathcal{L}_4^3$ . This gives

**Lemma 3.19.** *One has*

$$S_1 \cap S_3 = \mathcal{L}_4^4 \cup C_{12}^4, \quad S_1 \cap S_4 = \mathcal{L}_4^2 \cup C_{12}^2, \quad S_2 \cap S_3 = \mathcal{L}_4^1 \cup C_{12}^1, \quad S_2 \cap S_4 = \mathcal{L}_4^3 \cup C_{12}^3,$$

where  $C_{12}^1, C_{12}^2, C_{12}^3$  and  $C_{12}^4$  are distinct  $\overline{G}_{144}$ -invariant irreducible smooth curves in  $\mathbb{P}^3$  of degree 12 and genus 13.

*Proof.* It is enough to prove that  $S_1 \cap S_3 = \mathcal{L}_4^4 \cup C_{12}^4$ , where  $C_{12}^4$  is a  $\overline{G}_{144}$ -invariant irreducible smooth curve of degree 12 and genus 13. Let  $H$  be a plane section of the surface  $S_1$ . Then it follows from Lemma 3.15 and (3.18) that

$$S_3|_{S_1} = \mathcal{L}_4^4 + Z \sim 4H,$$

where  $Z$  is a  $\overline{G}_{144}$ -irreducible curve on the surface  $S_1$ . Note that the curve  $Z$  is contained in the smooth locus of  $S_1$ , because  $S_1 \cap S_3$  does not contain  $\Sigma_{12}$  by (3.11). Thus, we have

$$-8 + \mathcal{L}_4^4 \cdot Z = \mathcal{L}_4^4 \cdot (\mathcal{L}_4^4 + Z) = 4H \cdot \mathcal{L}_4^4 = 16,$$

so that  $\mathcal{L}_4^4 \cdot Z = 24$  on the surface  $S_1$ . This gives

$$Z^2 + 24 = Z^2 + \mathcal{L}_4^4 \cdot Z = Z \cdot (\mathcal{L}_4^4 + Z) = 4H \cdot Z = 48,$$

so that  $Z^2 = 24$ .

If  $Z$  is reducible, then  $Z$  is contained in the quadric  $\mathcal{Q}$  by Lemma 3.16, which is impossible, because  $S_1 \cap \mathcal{Q} = \mathcal{L}_4^2 \cup \mathcal{L}_4^4$  by (3.18). Thus, we see that  $Z$  is irreducible. Then it has arithmetic genus 13, so that it must be smooth by Lemmas 3.2 and 3.10.  $\square$

**Corollary 3.20.** *Let  $C$  be one of the curves  $C_{12}^1, C_{12}^2, C_{12}^3, C_{12}^4$ , and let  $\mathcal{M}$  be the linear system of surfaces in  $\mathbb{P}^3$  of degree 6 that contains  $C$ . Then the base locus of  $\mathcal{M}$  consists of the curve  $C$ .*

*Proof.* Without loss of generality, we may assume that  $C = C_{12}^4$ . Then  $C$  is contained in  $S_1$  and

$$S_3|_{S_1} = \mathcal{L}_4^4 + C.$$

by Lemma 3.19. Note that both curves  $\mathcal{L}_4^4$  and  $C$  are contained in the smooth locus of the surface  $S_1$  by (3.11) and Lemma 3.12. Let  $H$  be a plane section of  $S_1$ , and let  $D = 2H + \mathcal{L}_4^4$ . Then  $D^2 = 24$  and  $D$  is nef, since  $D \cdot \mathcal{L}_4^4 = 0$ . Using the Riemann–Roch formula and Nadel vanishing theorem (see [30, Theorem 9.4.8]), we see that  $h^0(\mathcal{O}_{S_1}(D)) = 14$ . This implies that the linear system  $|D|$  does not have fixed curves. Indeed, if it does, then its fixed part must be the curve  $\mathcal{L}_4^4$ , so that

$$14 = h^0(\mathcal{O}_{S_1}(D)) = h^0(\mathcal{O}_{S_1}(2H)) = 10,$$

which is absurd. Thus  $|D|$  does not have base points by [39, Corollary 3.2]. Therefore, the linear system  $|6H - C|$  is base point free. Since  $S_1$  is projectively normal, we immediately obtain the required assertions.  $\square$

**Corollary 3.21.** *Let  $\mathcal{D}$  be a (non-empty) mobile  $\overline{G}_{80}$ -invariant linear system on  $\mathbb{P}^3$ , let  $n$  be a positive integer such that  $\mathcal{D} \sim \mathcal{O}_{\mathbb{P}^3}(n)$ , and let  $C$  be one of the curves  $C_{12}^1, C_{12}^2, C_{12}^3$  or  $C_{12}^4$ . Then  $\text{mult}_C(\mathcal{D}) \leq \frac{n}{4}$ .*

*Proof.* Without loss of generality, we may assume that  $C = C_{12}^4$ . Then  $C$  is contained in the surface  $S_1$  by Lemma 3.19, and  $S_3|_{S_1} = \mathcal{L}_4^4 + C$ . Let  $D$  be a general surface in  $\mathcal{D}$ . Then

$$D|_{S_1} = aC + b\mathcal{L}_4^4 + \Delta,$$

where  $a$  and  $b$  are non-negative integers such that  $a \geq \text{mult}_C(\mathcal{D})$ , and  $\Delta$  is an effective divisor on  $S_1$  whose support contains neither  $C$  nor any irreducible component of the curve  $\mathcal{L}_4^4$ . Then

$$aC + b\mathcal{L}_4^4 + \Delta \sim_{\mathbb{Q}} \frac{n}{4}(\mathcal{L}_4^4 + C).$$

If  $a > \frac{n}{4}$ , then  $b < \frac{n}{4}$ , since we have

$$\left(\frac{n}{4} - b\right)\mathcal{L}_4^4 \sim_{\mathbb{Q}} \left(a - \frac{n}{4}\right)C + \Delta.$$

Therefore, if  $a > \frac{n}{4}$ , then

$$0 > -8\left(\frac{n}{4} - b\right) = \left(\frac{n}{4} - b\right)\mathcal{L}_4^4 \cdot \mathcal{L}_4^4 = \left(a - \frac{n}{4}\right)\mathcal{L}_4^4 \cdot C + \mathcal{L}_4^4 \cdot \Delta \geq 0,$$

which is absurd. This shows that  $\text{mult}_C(\mathcal{D}) \leq a \leq \frac{n}{4}$  as required.  $\square$

Now we are ready to prove

**Theorem 3.22.** *Let  $C$  be  $\overline{G}_{144}$ -irreducible curve in  $\mathbb{P}^3$  such that  $\deg(C) \leq 15$ . Then one of the following cases holds:*

- $\deg(C) = 4$ , and  $C$  is one of the curves  $\mathcal{L}_4^1, \mathcal{L}_4^2, \mathcal{L}_4^3, \mathcal{L}_4^4$ ;
- $\deg(C) = 6$ , and  $C$  is one of the curves  $\mathcal{L}_6^1$  or  $\mathcal{L}_6^2$ ;
- $\deg(C) = 12$ , and  $C$  is one of the smooth irreducible curves  $C_{12}^1, C_{12}^2, C_{12}^3, C_{12}^4$ ;
- $\deg(C) = 12$ , and  $C$  is a union of 12 disjoint lines contained in the quadric  $\mathcal{Q}$ .

*Proof.* By Lemma 3.5, we may assume that  $C$  is not contained in the quadric  $\mathcal{Q}$ . Then  $\deg(C) \geq 12$  by Lemma 3.15, and  $C$  is irreducible by Lemma 3.16. Moreover, since

$$30 \geq 2 \cdot \deg(C) = C \cdot \mathcal{Q},$$

we see that  $\deg(C) = 12$  by Lemma 3.4. Let us show that  $C$  is one of the smooth irreducible curves  $C_{12}^1, C_{12}^2, C_{12}^3$  or  $C_{12}^4$ .

Since  $C \cdot \mathcal{Q} = 24$ , it follows from Lemma 3.4 that  $C \cap \mathcal{Q}$  is a  $\overline{G}_{144}$ -orbit of length 24, and  $C$  intersects  $\mathcal{Q}$  transversally at the points of  $C \cap \mathcal{Q}$ . In particular, the curve  $C$  is smooth at these points. Recall from Lemma 3.10 that all  $\overline{G}_{144}$ -orbits of length 24 in  $\mathbb{P}^3$  and all  $\overline{G}_{144}$ -orbits of length 16 in  $\mathbb{P}^3$  are contained in  $\mathcal{Q}$ . Thus, the curve  $C$  does not contain  $\overline{G}_{144}$ -orbits of length 24 that are different from  $C \cap \mathcal{Q}$ , and it does not contain  $\overline{G}_{144}$ -orbits of length 16. Without loss of generality, we may assume that

$$C \cap \mathcal{Q} = \Sigma_{24}^1 = \mathcal{L}_4^1 \cap \mathcal{L}_6^2$$

Recall that  $\Sigma_{24}^1$  is contained in  $S_2 \cap S_3$ , because  $S_2 \cap \mathcal{Q} = \mathcal{L}_4^1 \cup \mathcal{L}_4^3$  and  $S_3 \cap \mathcal{Q} = \mathcal{L}_4^1 \cup \mathcal{L}_4^4$ , see (3.18). Since  $S_2 \cdot C = S_3 \cdot C = 48$ , it follows from Lemma 3.10 and (3.11) that one of the following cases holds:

- the curve  $C$  is contained in one of the surfaces  $S_2$  and  $S_3$ ,
- both  $S_2$  and  $S_3$  are tangent to  $C$  at the points of  $\Sigma_{24}^1$ , and  $S_2 \cap C = S_3 \cap C = \Sigma_{24}^1$ ,
- $S_2 \cap C = \Sigma_{24}^1 \cup \Sigma_{12}$  or  $S_3 \cap C = \Sigma_{24}^1 \cup \Sigma_{12}'$ .

Suppose that  $S_2 \cap C = \Sigma_{24}^1 \cup \Sigma_{12}$ . Since  $C$  is irreducible, it must be singular at every point of  $\Sigma_{12}$  by Lemma 3.2. Then

$$48 = S_2 \cdot C \geq |\Sigma_{24}^1| \text{mult}_{\Sigma_{24}^1}(S_2) + |\Sigma_{12}| \text{mult}_{\Sigma_{12}}(S_2) \text{mult}_{\Sigma_{12}}(C) \geq 72,$$

which is absurd. Thus, we have  $S_2 \cap C \neq \Sigma_{24}^1 \cup \Sigma_{12}$ . Similarly, we see that the intersection  $S_3 \cap C$  is not the union  $\Sigma_{24}^1 \cup \Sigma_{12}'$ .

Suppose that the surfaces  $S_2$  and  $S_3$  are tangent to the curve  $C$  at the points of  $\Sigma_{24}^1$ , and

$$S_2 \cap C = S_3 \cap C = \Sigma_{24}^1.$$

Let  $\mathcal{P}$  be the pencil generated by  $S_2$  and  $S_3$ . Then  $\mathcal{P}$  is  $\overline{G}_{144}$ -invariant. Moreover, every surface in this pencil either contains  $C$  or is tangent to the curve  $C$  at the points of  $\Sigma_{24}^1$  (here we include the case when the surface is singular at some points of  $\Sigma_{24}^1$ ). This implies that there exists a surface  $S$  in  $\mathcal{P}$  that contains  $C$ . Indeed, let  $P$  be a point in  $C$  that is not contained in  $\Sigma_{24}^1$ , and let  $S$  be a surface in  $\mathcal{P}$  that passes through  $P$ . If  $C$  is not contained in  $S$ , then we obtain contradictory inequalities

$$48 = S \cdot C \geq \sum_{Q \in \Sigma_{24}^1} \text{mult}_Q(S \cdot C) + \text{mult}_P(S \cdot C) \geq 2|\Sigma_{24}^1| + 1 = 49,$$

because  $S$  is tangent to  $C$  at the points of  $\Sigma_{24}^1$ . Thus, we see that the curve  $C$  is contained in  $S$ . Moreover, if  $S$  is not  $\overline{G}_{144}$ -invariant, then there exists  $\bar{g} \in \overline{G}_{144}$  such that  $\bar{g}(S) \neq S$ , so that  $\bar{g}(S) \in \mathcal{P}$ , and the surfaces  $S$  and  $\bar{g}(S)$  generate the whole pencil  $\mathcal{P}$ . This implies that  $C$  is contained in  $S_2 \cap S_3$ , which is not the case by assumption.

We are left with the case when  $C$  is contained in one of the surfaces  $S_2$  and  $S_3$ . Without loss of generality, we may assume that  $C \subset S_2$ . Let us show that the curve  $C$  is also contained in  $S_3$ . This would imply that  $C = C_{12}^1$  by Lemma 3.19.

Let  $f: \tilde{S}_2 \rightarrow S_2$  be the minimal resolution of singularities of the surface  $S_2$ . Then the action of the group  $\overline{G}_{144}$  lifts to the surface  $\tilde{S}_2$ . Denote by  $E_1, \dots, E_{12}$  the  $f$ -exceptional curves. Let  $E = E_1 + \dots + E_{12}$ , and let  $H$  be a plane section of the surface  $S_2$ . Denote by  $\tilde{C}$  the proper transform of the curve  $C$  on the surface  $\tilde{S}_2$ . Recall from Lemma 3.19 that  $S_2 \cap S_3 = \mathcal{L}_4^1 \cup C_{12}^1$  and  $S_2 \cap S_4 = \mathcal{L}_4^3 \cup C_{12}^3$ . Denote by  $\tilde{C}_{12}^1$  and  $\tilde{C}_{12}^3$  the proper transforms on the surface  $\tilde{S}_2$  of the curves  $C_{12}^1$  and  $C_{12}^3$ , respectively. Similarly, denote by  $\tilde{\mathcal{L}}_4^1$  and  $\tilde{\mathcal{L}}_4^3$  the proper transforms on the surface  $\tilde{S}_2$  of the curves  $\mathcal{L}_4^1$  and  $\mathcal{L}_4^3$ , respectively. Since

$$\text{Sing}(S_2) = \Sigma_{12} \not\subset S_3 \cup S_4$$

by (3.11) and Lemma 3.12, we have

$$\tilde{\mathcal{L}}_4^1 + \tilde{C}_{12}^1 \sim \tilde{\mathcal{L}}_4^3 + \tilde{C}_{12}^3 \sim f^*(4H).$$

Moreover, we have  $\tilde{\mathcal{L}}_4^1 + \tilde{\mathcal{L}}_4^3 \sim f^*(2H)$ , so that  $\tilde{C} \cdot (\tilde{\mathcal{L}}_4^1 + \tilde{\mathcal{L}}_4^3) = 24$ . Using Lemma 3.4 and keeping in mind that the curve  $C$  contains the  $\overline{G}_{144}$ -orbit  $\Sigma_{24}^1 = \mathcal{L}_4^1 \cap \mathcal{L}_6^2$ , we conclude that  $\tilde{C} \cdot \tilde{\mathcal{L}}_4^1 = 24$  and  $\tilde{C} \cdot \tilde{\mathcal{L}}_4^3 = 0$ . Thus

$$48 = \tilde{C} \cdot (\tilde{\mathcal{L}}_4^1 + \tilde{C}_{12}^1) = 24 + \tilde{C} \cdot \tilde{C}_{12}^1,$$

so that  $\tilde{C} \cdot \tilde{C}_{12}^1 = 24$ . Therefore, it follows from Hodge index theorem that

$$\begin{vmatrix} \tilde{C}^2 & \tilde{C} \cdot \tilde{C}_{12}^1 \\ \tilde{C} \cdot \tilde{C}_{12}^1 & \tilde{C}_{12}^1 \cdot \tilde{C}_{12}^1 \end{vmatrix} = \begin{vmatrix} \tilde{C}^2 & 24 \\ 24 & 24 \end{vmatrix} \leq 0.$$

This implies that either  $\tilde{C}^2 \leq 22$  or  $\tilde{C}^2 = 24$ . Hence  $\tilde{C}$  is smooth by Lemma 3.2, so that  $\tilde{C}$  is a smooth curve of genus  $\frac{\tilde{C}^2}{2} + 1$ . Therefore, it follows from Lemma 3.2 that  $\tilde{C}^2 = 24$ , because  $\tilde{C}$  contains  $\overline{G}_{144}$ -orbits of length 24, for instance, the  $\overline{G}_{144}$ -orbit  $\tilde{C} \cap \tilde{\mathcal{L}}_4^1$ . Thus, it follows from Hodge index theorem that the curves  $\tilde{C}$  and  $\tilde{C}_{12}^1$  are numerically equivalent on  $\tilde{S}_2$ , so that  $\tilde{C} \sim \tilde{C}_{12}^1$ . Hence  $\tilde{C} \cdot E = 0$ , so that  $C$  does not contain the singular locus of the surface  $S_2$ , which implies that

$$C + \mathcal{L}_4^1 \sim 4H.$$

Since  $S_2$  is projectively normal, we see that  $C = C_{12}^1$  by Lemma 3.19.  $\square$

#### 4. THE GROUP OF ORDER 80

Let us use notation and assumptions of §2. In this section, we present basic facts about  $\overline{G}_{80}$ -orbits,  $\overline{G}_{80}$ -invariant curves and  $\overline{G}_{80}$ -invariant surfaces in  $\mathbb{P}^3$ . Let us start with the following very easy group-theoretic result.

**Lemma 4.1.** *Let  $\Gamma$  be a proper non-trivial subgroup of the group  $\overline{G}_{80}$ . Then  $\Gamma$  is isomorphic to one of the following groups:  $\mu_2$ ,  $\mu_2^2$ ,  $\mu_2^3$ ,  $\mu_2^4$ , or  $\mu_5$ . Moreover, a subgroup of  $\overline{G}_{80}$  isomorphic to  $\mu_2^4$  is unique, and a subgroup isomorphic to  $\mu_5$  is unique up to conjugation. Furthermore, the only non-trivial proper normal subgroup of  $\overline{G}_{80}$  is  $\overline{H} \cong \mu_2^4$ .*

*Proof.* Sylow theorems imply that the order of every non-trivial element of  $\overline{G}_{80}$  equals either 2 or 5. If every non-trivial element of  $\Gamma$  has order 2, then  $\Gamma$  is contained in the Sylow 2-subgroup of  $\overline{G}_{80}$ , which is exactly  $\overline{H}$ . If every non-trivial element of  $\Gamma$  has order 5, then  $\Gamma \cong \mu_5$ . The uniqueness assertions for subgroups isomorphic to  $\mu_2^4$  and  $\mu_5$  also follow from Sylow theorems.

Suppose that  $\Gamma$  contains both elements of order 2 and elements of order 5. Then it is isomorphic to  $\Gamma' \rtimes \mu_5$ , where  $\Gamma'$  is a non-trivial proper subgroup of the group  $\overline{H}$  that is invariant under the action of

$$\overline{G}_{80}/\overline{H} \cong \mu_5 \subset \mathrm{GL}_4(\mathbf{F}_2)$$

on  $\overline{H}$ . However, the group  $\mu_5$  is not a subgroup of  $\mathrm{GL}_k(\mathbf{F}_2)$  for  $k \leq 3$ . This implies that the action of  $\mu_5$  on  $\mathbf{F}_2^4$  is irreducible, so that  $\overline{H}$  has no non-trivial proper  $\mu_5$ -invariant subgroups. Therefore, we see that  $\Gamma = \overline{G}_{80}$ .

Now we suppose that  $\Gamma$  is a non-trivial normal subgroup of  $\overline{G}_{80}$ . Then  $\Gamma$  is not isomorphic to  $\mu_5$ . Also we know from the above arguments that it cannot be a proper subgroup of  $\mu_2^4$ . Thus, if  $\Gamma \not\cong \mu_2^4$ , then  $\Gamma$  must contain both elements of order 2 and elements of order 5. This implies that  $\Gamma = \overline{G}_{80}$ .  $\square$

**Corollary 4.2.** *Let  $\overline{G}$  be a primitive finite subgroup in  $\mathrm{PGL}_4(\mathbb{C})$  that is isomorphic to  $\overline{G}_{80}$ . Then  $\overline{G}$  is conjugate to  $\overline{G}_{80}$ .*

*Proof.* It follows from [4, Chapter VII] that  $\overline{G}$  is conjugate to  $\overline{G}_{80}$  provided that  $\overline{G}$  is primitive. Suppose  $\overline{G}$  is not primitive. Let us seek for a contradiction. To start with, suppose that  $\overline{G}$  is transitive. Then either there exists a  $\overline{G}$ -orbit of length 4, or there exists a  $\overline{G}$ -invariant pair of skew lines. In the former case, we have a non-trivial homomorphism  $\overline{G} \rightarrow \mathfrak{S}_4$ . In the latter case, we have a non-trivial homomorphism  $\overline{G} \rightarrow \mathfrak{S}_2$ . None of this is possible by Lemma 4.1.

Thus, we see that  $\overline{G}$  is intransitive. Then either there exists a  $\overline{G}$ -invariant point  $P$ , or there exist a  $\overline{G}$ -invariant line  $L$ . In the former case, the group  $\overline{G}$  acts faithfully on the tangent space  $T_P(\mathbb{P}^3) \cong \mathbb{C}^3$  by [10, Lemma 4.4.1], which is impossible, because  $\mathrm{GL}_3(\mathbb{C})$  does not contain subgroups isomorphic to  $\mu_2^4$ . In the latter case, we obtain a homomorphism

$$\overline{G} \rightarrow \mathrm{Aut}(L) \cong \mathrm{PGL}_2(\mathbb{C}).$$

We conclude from Lemma 4.1 and the classification of finite subgroups in  $\mathrm{PGL}_2(\mathbb{C})$  that the kernel of this homomorphism contains the subgroup in  $\overline{G}$  isomorphic to  $\mu_2^4$ , so that this subgroup must fix every point in  $L$ , which is again impossible by [10, Lemma 4.4.1].  $\square$

To study  $\overline{G}_{80}$ -invariant curves in  $\mathbb{P}^3$ , we need the following result, which follows from the Riemann–Hurwitz formula.

**Lemma 4.3.** *Let  $C$  be a smooth irreducible curve of genus  $g \leq 19$  with a faithful action of the group  $\overline{G}_{80}$ . Then  $g \in \{5, 13, 17\}$ , and  $C$  is not hyperelliptic. Furthermore, if  $\Omega \subset C$  is a  $\overline{G}_{80}$ -orbit, then  $|\Omega| \in \{16, 40, 80\}$ . Finally, the possible numbers  $a_i$  of  $\overline{G}_{80}$ -orbits of length  $i \in \{16, 40\}$  in  $C$  are contained in the following table:*



$g$	5	13	17
$a_{16}$	2	1	3
$a_{40}$	1	3	0

*Proof.* The assertion about the lengths of  $\overline{G}_{80}$ -orbits follows from Lemma 4.1, since the stabilizers in  $G$  of points in  $X$  must be cyclic (see [10, Lemma 5.1.4]). Moreover, the group  $\overline{G}_{80}$  cannot act faithfully on  $\mathbb{P}^1$  and on a smooth elliptic curve by Lemma 2.3. Hence, one has  $g \geq 2$ .

Suppose that the curve  $C$  is hyperelliptic. Since the group  $\overline{G}_{80}$  does not contain normal subgroups of order 2 by Lemma 4.1, it does not contain the hyperelliptic involution of  $C$ , and we obtain a faithful action of the group  $\overline{G}_{80}$  on  $\mathbb{P}^1$ . We already proved that this is impossible by Lemma 2.3.

Let  $\widehat{C} = C/\overline{G}_{80}$ . Then  $\widehat{C}$  is a smooth curve. Let  $\hat{g}$  be the genus of the curve  $\widehat{C}$ . Then the Riemann–Hurwitz formula gives

$$2g - 2 = 80(2\hat{g} - 2) + 40a_{40} + 64a_{16}.$$

Since  $a_k \geq 0$  and  $g \leq 19$ , one has  $\hat{g} = 0$ , so that

$$2g - 2 = -160 + 40a_{40} + 64a_{16},$$

Going through the possible values of  $g$ , and solving this equation case by case we obtain the required result.  $\square$

**Lemma 4.4.** *There are exactly four  $\overline{G}_{80}$ -orbits of length 16 in  $\mathbb{P}^3$ .*

*Proof.* A subgroup  $\mu_5 \subset \overline{G}_{80}$  has a fixed point in  $\mathbb{P}^3$ , so that the length of its  $\overline{G}_{80}$ -orbit is 16 by Lemma 4.1. This shows that there is at least one  $\overline{G}_{80}$ -orbit of length 16 in  $\mathbb{P}^3$ .

Now let  $\Gamma$  be a stabilizer of a point in a  $\overline{G}_{80}$ -orbit of length 16. Then  $\Gamma \cong \mu_5$ . By Lemma 4.1, the subgroup  $\Gamma$  is conjugate to the subgroup generated by the image  $T$  in  $\mathrm{PGL}_4(\mathbb{C})$ . On the other hand, the eigenvalues of the matrix  $T$  are four distinct primitive fifth roots of unity. Since the normalizer of  $\Gamma$  in  $\overline{G}_{80}$  coincides with  $\Gamma$  by Lemma 4.1, we conclude that there are exactly four  $\overline{G}_{80}$ -orbits of length 16 in  $\mathbb{P}^3$ .  $\square$

Let us denote the  $\overline{G}_{80}$ -orbits of length 16 in  $\mathbb{P}^3$  by  $\Sigma_{16}^1, \Sigma_{16}^2, \Sigma_{16}^3$  and  $\Sigma_{16}^4$ . Computing the null-spaces of the matrices  $T - \xi_5^i I_4$ ,  $1 \leq i \leq 4$ , one can find these orbits explicitly. In particular, we may assume that

$$\begin{aligned} [(-1+i)\xi_5^3 + (-1+i)\xi_5^2 - \xi_5 - 1 : 1 + (1+i)\xi_5^2 + \xi_5 : 1 + (1+i)\xi_5^3 + \xi_5 : \xi_5 - 1] &\in \Sigma_{16}^1, \\ [(1+i)\xi_5^3 + (1-i)\xi_5^2 + i\xi_5 + 1 : -2\xi_5^3 + (-1-i)\xi_5^2 - \xi_5 - 2 - i : (-1+i)\xi_5^3 + i\xi_5 + i : \xi_5 - 1] &\in \Sigma_{16}^2, \\ [-2i\xi_5^3 + (1-i)\xi_5^2 - i\xi_5 + 1 - 2i : (1-i)\xi_5^3 - (1+i)\xi_5^2 + \xi_5 - i : (1-i)\xi_5^3 - i\xi_5 - i : \xi_5 - 1] &\in \Sigma_{16}^3, \end{aligned}$$

and

$$[(-1+i)\xi_5^2 + i\xi_5 + i : (1+i)\xi_5^3 + (1+i)\xi_5^2 + i\xi_5 + i : -1 + (-1-i)\xi_5^3 - \xi_5 : \xi_5 - 1] \in \Sigma_{16}^4.$$

Recall from §2, that there are three  $\overline{G}_{80}$ -orbits  $\Sigma_{20}, \Sigma'_{20}$  and  $\Sigma''_{20}$  of length 20 in  $\mathbb{P}^3$  described by (2.15), (2.16) and (2.17), respectively.

**Lemma 4.5.** *The orbits  $\Sigma_{20}, \Sigma'_{20}$  and  $\Sigma''_{20}$  are all  $\overline{G}_{80}$ -orbits of length 20 in  $\mathbb{P}^3$ .*

*Proof.* Let  $\Sigma$  be a  $\overline{G}_{80}$ -orbit of length 20, let  $P$  be a point in  $\Sigma$ , and let  $\Gamma$  be the stabilizer in  $\overline{G}_{80}$  of the point  $P$ . Then  $\Gamma \cong \mu_2^2$  by Lemma 4.1. Let  $\gamma_1, \gamma_2$  and  $\gamma_3$  be three non-trivial elements in  $\Gamma$ . Then each of them fixes pointwise two skew lines in  $\mathbb{P}^3$  by Lemma 2.2(i). One of the lines from each pair must pass through the point  $P$ , so that  $P \in \Sigma_{20} \cup \Sigma'_{20} \cup \Sigma''_{20}$  by Lemma 2.18.  $\square$

To describe  $\overline{G}_{80}$ -orbits in  $\mathbb{P}^3$  of length 40, recall from Lemma 2.2(i) that there are 30 lines in  $\mathbb{P}^3$  such that each of them is pointwise fixed by some non-trivial element in  $\overline{H}$ . These lines have been described in (2.11), (2.12) and (2.13). Let

$$\mathcal{L}_{10} = \sum_{i=1}^5 \ell_i + \sum_{i=1}^5 \check{\ell}_i.$$

Similarly, let

$$\mathcal{L}'_{10} = \sum_{i=1}^5 \ell'_i + \sum_{i=1}^5 \check{\ell}'_i.$$

Finally, we let

$$\mathcal{L}''_{10} = \sum_{i=1}^5 \ell''_i + \sum_{i=1}^5 \check{\ell}''_i.$$

Then  $\mathcal{L}_{10}, \mathcal{L}'_{10}, \mathcal{L}''_{10}$  are  $\overline{G}_{80}$ -irreducible curves by construction. Moreover, the union  $\mathcal{L}_{10} \cup \mathcal{L}'_{10} \cup \mathcal{L}''_{10}$  contains all  $\overline{G}_{80}$ -orbits in  $\mathbb{P}^3$  of length 40. Furthermore, we have

**Lemma 4.6.** *The curve  $\mathcal{L}_{10}$  is a disjoint union of ten lines,  $\mathcal{L}'_{10}$  and  $\mathcal{L}''_{10}$  are nodal unions of ten lines,  $\text{Sing}(\mathcal{L}'_{10}) = \Sigma'_{20} \not\subset \mathcal{L}''_{10}$  and  $\text{Sing}(\mathcal{L}''_{10}) = \Sigma''_{20} \not\subset \mathcal{L}'_{10}$ . Moreover, one has*

$$\mathcal{L}_{10} \cap \mathcal{L}'_{10} \cap \mathcal{L}''_{10} = \mathcal{L}'_{10} \cap \mathcal{L}''_{10} = \Sigma_{20}, \quad \mathcal{L}_{10} \cap \mathcal{L}'_{10} = \Sigma_{20} \cup \Sigma'_{20}, \quad \mathcal{L}_{10} \cap \mathcal{L}''_{10} = \Sigma_{20} \cup \Sigma''_{20},$$

and  $\Sigma_{20} \cup \Sigma'_{20} \cup \Sigma''_{20} \subset \mathcal{L}_{10}$ . Furthermore, the  $\overline{G}_{80}$ -orbit of every point in

$$\left( \mathcal{L}_{10} \cup \mathcal{L}'_{10} \cup \mathcal{L}''_{10} \right) \setminus \left( \Sigma_{20} \cup \Sigma'_{20} \cup \Sigma''_{20} \right)$$

consists of 40 points, and every  $\overline{G}_{80}$ -orbit in  $\mathbb{P}^3$  of length 40 is contained in  $\mathcal{L}_{10} \cup \mathcal{L}'_{10} \cup \mathcal{L}''_{10}$ .

*Proof.* It is straightforward to check using explicit equations of the lines of  $\mathcal{L}_{10}$  that they are pairwise disjoint. Similarly, we see that  $\mathcal{L}_{10}$  contains  $\Sigma_{20}, \Sigma'_{20}$  and  $\Sigma''_{20}$ . Likewise, the curve  $\mathcal{L}'_{10}$  contains  $\Sigma_{20}$  and  $\Sigma'_{20}$ , and it does not contain  $\Sigma''_{20}$ . Moreover, for every point in  $\Sigma'_{20}$ , there are exactly two among the ten lines of  $\mathcal{L}'_{10}$  that pass through this point, and all these ten lines are pairwise disjoint away from  $\Sigma'_{20}$ . This means that the curve  $\mathcal{L}'_{10}$  is nodal and  $\text{Sing}(\mathcal{L}'_{10}) = \Sigma'_{20}$ . In fact, one can also check that

$$\ell'_i \cap \ell'_j \neq \emptyset \iff \check{\ell}'_i \cap \check{\ell}'_j \neq \emptyset \iff \ell'_i \cap \check{\ell}'_j \neq \emptyset \iff j - i = \pm 1 \pmod{5}.$$

Similarly, we see that  $\mathcal{L}''_{10}$  contains  $\Sigma_{20} \cup \Sigma''_{20}$ , and it does not contain  $\Sigma'_{20}$ . Furthermore, for every point in  $\Sigma''_{20}$ , there are exactly two lines among the ten lines of  $\mathcal{L}''_{10}$  that pass through this point, the lines of  $\mathcal{L}''_{10}$  do not intersect each other in other points, and

$$\ell''_i \cap \ell''_j \neq \emptyset \iff \check{\ell}''_i \cap \check{\ell}''_j \neq \emptyset \iff \ell''_i \cap \check{\ell}''_j \neq \emptyset \iff j - i = \pm 2 \pmod{5}.$$

In particular, the curve  $\mathcal{L}''_{10}$  is nodal, and  $\text{Sing}(\mathcal{L}''_{10}) = \Sigma''_{20}$ .

Explicit computations show that

$$\mathcal{L}'_{10} \cap \mathcal{L}''_{10} = \Sigma_{20}, \quad \mathcal{L}_{10} \cap \mathcal{L}'_{10} = \Sigma_{20} \cup \Sigma'_{20}, \quad \mathcal{L}_{10} \cap \mathcal{L}''_{10} = \Sigma_{20} \cup \Sigma''_{20}.$$

By construction, the curves  $\mathcal{L}_{10}$ ,  $\mathcal{L}'_{10}$  and  $\mathcal{L}''_{10}$  contain all  $\overline{G}_{80}$ -orbits in  $\mathbb{P}^3$  of length 40. Vice versa, let  $P$  be a point in  $\mathcal{L}_{10} \cup \mathcal{L}'_{10} \cup \mathcal{L}''_{10}$  such that  $P \notin \Sigma_{20} \cup \Sigma'_{20} \cup \Sigma''_{20}$ , and let  $\Sigma$  be its  $\overline{G}_{80}$ -orbit. Then  $|\Sigma| \neq 80$  by construction. Thus, we have  $|\Sigma| = 40$  by Lemmas 4.1 and 4.5.  $\square$

**Corollary 4.7.** *Let  $\Sigma$  be a  $\overline{G}_{80}$ -orbit in  $\mathbb{P}^3$  such that  $|\Sigma| < |\overline{G}_{80}| = 80$ . Then one of the following possibilities holds:*

- either  $|\Sigma| = 40$  and  $\Sigma \subset \mathcal{L}_{10} \cup \mathcal{L}'_{10} \cup \mathcal{L}''_{10}$ ,
- or  $|\Sigma| = 20$  and  $\Sigma$  is one of the  $\overline{G}_{80}$ -orbits  $\Sigma_{20}$ ,  $\Sigma'_{20}$ ,  $\Sigma''_{20}$ ,
- or  $|\Sigma| = 16$  and  $\Sigma$  is one of the  $\overline{G}_{80}$ -orbits  $\Sigma_{16}^1$ ,  $\Sigma_{16}^2$ ,  $\Sigma_{16}^3$ ,  $\Sigma_{16}^4$ .

*Proof.* Let  $\Gamma$  be a stabilizer of a point in  $\Sigma$ . If  $\Gamma \cong \mu_2$ , then  $|\Sigma| = 40$ , and the assertion follows from Lemma 4.6. If  $\Gamma \cong \mu_2^2$ , then  $|\Sigma| = 20$ , and the assertion follows from Lemma 4.5. If  $\Gamma \cong \mu_5$ , then  $|\Sigma| = 16$ , and the assertion follows from Lemma 4.4. Otherwise, the group  $\Gamma$  contains a subgroup  $\mu_2^3 \subset \overline{H}$  by Lemma 4.1, which is impossible by Lemma 2.2(iii).  $\square$

Now we are ready to prove

**Lemma 4.8.** *Let  $C$  be reducible  $\overline{G}_{80}$ -invariant curve in  $\mathbb{P}^3$  such that  $\deg(C) \leq 15$ . Then  $C$  is one of the curves  $\mathcal{L}_{10}$ ,  $\mathcal{L}'_{10}$  or  $\mathcal{L}''_{10}$ .*

*Proof.* We may assume that the curve  $C$  is  $\overline{G}_{80}$ -irreducible. Write  $C = C_1 + \dots + C_r$ , where each  $C_i$  is an irreducible curve. Then  $r \geq 2$  by assumption. Let  $d$  be the degree of the curve  $C_1$ . Then

$$15 \geq \deg(C) = dr.$$

Let  $\Gamma$  be the stabilizer in  $\overline{G}_{80}$  of the curve  $C_1$ . Then there exists an exact sequence of groups

$$1 \rightarrow \Gamma_{C_1} \rightarrow \Gamma \rightarrow \text{Aut}(C_1),$$

where  $\Gamma_{C_1}$  fixes the curve  $C_1$  pointwise. By Lemma 4.1, the group  $\Gamma$  is one of the groups  $\mu_2$ ,  $\mu_2^2$ ,  $\mu_2^3$ ,  $\mu_2^4$  or  $\mu_5$ . Since  $r|\Gamma| = |\overline{G}_{80}| = 80$  and  $15 > dr$ , we see that either  $r = 5$  or  $10$ . Moreover, if  $r = 5$ , then  $d \leq 3$ . If  $r = 5$  and  $C_1$  is rational, then  $\Gamma \cong \mu_2^4$ , which implies that  $\Gamma_{C_1}$  is one of the groups  $\mu_2^2$ ,  $\mu_2^3$  or  $\mu_2^4$ , because the group  $\mu_2^3$  does not act faithfully on  $\mathbb{P}^1$ . In this case, the  $\overline{G}_{80}$ -orbit of a general point of  $C_1$  consists of at most 16 points, which is impossible by Corollary 4.7. Therefore, if  $r = 5$ , then  $C_1$  is a plane cubic curve, which implies that there exists a  $\overline{G}_{80}$ -orbit in  $\mathbb{P}^3$  of length 5, because  $C_1$  is contained in a unique plane. The later is again impossible by Corollary 4.7.

Thus, we see that  $r \neq 5$ , so that  $r = 10$ , which implies that  $d = 1$ , so that  $C_1$  is a line. In this case  $\Gamma \cong \mu_2^3$  and  $\Gamma_{C_1} \cong \mu_2$ , because  $\mu_2^3$  cannot act faithfully on  $\mathbb{P}^1$ , and  $\mathbb{P}^3$  does not contain  $\overline{G}_{80}$ -orbits of length less than 16 by Corollary 4.7. Then  $C_1$  is one of two lines that is pointwise fixed by  $\Gamma_{C_1}$ . Now it follows from Lemma 4.6 that  $C_1$  is an irreducible component of one of the curves  $\mathcal{L}_{10}$ ,  $\mathcal{L}'_{10}$ ,  $\mathcal{L}''_{10}$ .  $\square$

**Corollary 4.9.** *There are no  $\overline{G}_{80}$ -invariant curves in  $\mathbb{P}^3$  of degree less than 5.*

*Proof.* Let  $C$  be a  $\overline{G}_{80}$ -invariant curve in  $\mathbb{P}^3$  of degree less 5. Then  $C$  is irreducible by Lemma 4.8. Thus the genus of its normalization is at most 3, which is impossible by Lemma 4.3.  $\square$

Using Lemma 2.14, we obtain the following two technical results.

**Lemma 4.10.** *Let  $\mathcal{M}$  (respectively,  $\mathcal{M}'$ ,  $\mathcal{M}''$ ) be the linear system of surfaces in  $\mathbb{P}^3$  of degree 6 (respectively, 4, 4) that contains  $\mathcal{L}_{10}$  (respectively,  $\mathcal{L}'_{10}$ ,  $\mathcal{L}''_{10}$ ). Then the base locus of  $\mathcal{M}$  (respectively,  $\mathcal{M}'$ ,  $\mathcal{M}''$ ) does not contain curves except for  $\mathcal{L}_{10}$  (respectively,  $\mathcal{L}'_{10}$ ,  $\mathcal{L}''_{10}$ ).*

*Proof.* Recall from §2, that  $\mathbb{P}^3$  contains ten  $\overline{H}$ -invariant quadrics  $\mathcal{Q}_i$ , where  $1 \leq i \leq 10$ , which are defined by equations (2.8) and (2.9). For every  $i < j < k$ , define the reducible sextic surface

$$T_{i,j,k} = \mathcal{Q}_i + \mathcal{Q}_j + \mathcal{Q}_k.$$

By Lemma 2.14, the curve  $\mathcal{L}_{10}$  is contained in  $T_{1,2,4}$ ,  $T_{3,5,6}$ ,  $T_{2,4,5}$ ,  $T_{2,3,5}$ ,  $T_{7,8,9}$ ,  $T_{1,6,10}$  and  $T_{6,9,10}$ , so that these surfaces are contained in the linear system  $\mathcal{M}$ . Using Lemma 2.14, one can check that

$$T_{1,2,4} \cap T_{3,5,6} = \mathcal{L}_{10} \cup \ell'_1 \cup \check{\ell}'_1 \cup \ell'_3 \cup \check{\ell}'_3 \cup \ell''_2 \cup \check{\ell}''_2 \cup \ell''_3 \cup \check{\ell}''_3 \cup \ell''_4 \cup \check{\ell}''_4 \cup \ell''_5 \cup \check{\ell}''_5.$$

On the other hand, the sextic surface  $T_{2,4,5}$  does not contain the lines  $\ell'_1$  and  $\check{\ell}'_1$ , and the sextic surface  $T_{2,3,5}$  does not contain the lines  $\ell'_3$  and  $\check{\ell}'_3$ . This also follows from Lemma 2.14. Similarly, the sextic surface  $T_{7,8,9}$  does not contain the lines  $\ell''_4$ ,  $\check{\ell}''_4$ ,  $\ell''_5$  and  $\check{\ell}''_5$ , the sextic surface  $T_{1,6,10}$  does not contain the lines  $\ell''_3$  and  $\check{\ell}''_3$ , and the sextic surface  $T_{6,9,10}$  does not contain the lines  $\ell''_2$  and  $\check{\ell}''_2$ . Thus, the intersection of the above seven sextic surfaces does not contain other curves except for  $\mathcal{L}_{10}$ , so that the same holds for the base locus of  $\mathcal{M}$ .

For every  $i < j$ , define the reducible quartic surface

$$R_{i,j} = \mathcal{Q}_i + \mathcal{Q}_j.$$

By Lemma 2.14, the curve  $\mathcal{L}'_{10}$  is contained in  $R_{6,7}$ ,  $R_{8,10}$  and  $R_{9,10}$ , so that these surfaces are contained in the linear system  $\mathcal{M}'$ . Using Lemma 2.14 one more time, we see that

$$R_{6,7} \cap R_{8,10} = \mathcal{L}'_{10} \cup \ell_5 \cup \check{\ell}_5.$$

However, the surface  $R_{9,10}$  does not contain the lines  $\ell_5$  and  $\check{\ell}_5$ . This shows that the base locus of  $\mathcal{M}'$  does not contain curves except for  $\mathcal{L}'_{10}$ .

Similarly, the quartics  $R_{1,2}$ ,  $R_{3,4}$  and  $R_{4,5}$  contain the curve  $\mathcal{L}''_{10}$ . Using Lemma 2.14, one can check that

$$R_{1,2} \cap R_{3,4} = \mathcal{L}''_{10} \cup \ell_4 \cup \check{\ell}_4.$$

However, the surface  $R_{4,5}$  does not contain the lines  $\ell_4$  and  $\check{\ell}_4$ , which implies that the base locus of  $\mathcal{M}''$  does not contain curves except for  $\mathcal{L}''_{10}$ .  $\square$

**Lemma 4.11.** *Let  $\mathcal{D}$  be a (non-empty) mobile  $\overline{G}_{80}$ -invariant linear system on  $\mathbb{P}^3$ , and let  $n$  be a positive integer such that  $\mathcal{D} \sim \mathcal{O}_{\mathbb{P}^3}(n)$ . Then*

$$\max \left\{ \text{mult}_{\mathcal{L}_{10}}(\mathcal{D}), \text{mult}_{\mathcal{L}'_{10}}(\mathcal{D}), \text{mult}_{\mathcal{L}''_{10}}(\mathcal{D}) \right\} \leq \frac{n}{4}.$$

*Proof.* Let us use the notation of the proof of Lemma 4.6. Let  $D$  be a general surface in  $\mathcal{D}$ . Since the lines  $\ell_1$ ,  $\ell_5$ ,  $\check{\ell}_1$  and  $\check{\ell}_5$  are contained in the quadric surface  $\mathcal{Q}_1$  by Lemma 2.14, we have

$$D|_{\mathcal{Q}_1} = m(\ell_1 + \ell_5 + \check{\ell}_1 + \check{\ell}_5) + \Delta,$$

where  $m$  is a non-negative integer such that  $m \geq \text{mult}_{\mathcal{L}_{10}}(\mathcal{D})$ , and  $\Delta$  is an effective divisor on  $\mathcal{Q}_1$  whose support does not contain the lines  $\ell_1$ ,  $\ell_5$ ,  $\check{\ell}_1$  and  $\check{\ell}_5$ . By Lemma 4.6, the lines  $\ell_1$ ,  $\ell_5$ ,  $\check{\ell}_1$  and  $\check{\ell}_5$  are disjoint. Thus, we may assume that these lines are divisors

of bi-degree  $(1, 0)$  on  $\mathcal{Q}_1 \cong \mathbb{P}^1 \times \mathbb{P}^1$ . Let  $\ell$  be a general line in  $\mathcal{Q}_1$  that is a divisor of bi-degree  $(0, 1)$ . Then

$$n = D \cdot \ell = m(\ell_1 + \ell_5 + \check{\ell}_1 + \check{\ell}_5) \cdot \ell + \Delta \cdot \ell \geq m(\ell_1 + \ell_5 + \check{\ell}_1 + \check{\ell}_5) \cdot \ell = 4m,$$

so that  $\text{mult}_{\mathcal{L}_{10}}(\mathcal{D}) \leq m \leq \frac{n}{4}$  as required. Similarly, it follows from Lemma 2.14 that the quadric  $\mathcal{Q}_1$  also contains the lines  $\ell''_1, \ell''_2, \ell''_4, \check{\ell}''_1, \check{\ell}''_2$  and  $\check{\ell}''_4$ . Moreover, it follows from the proof of Lemma 4.6 that the four lines  $\ell''_1, \ell''_2, \check{\ell}''_1$  and  $\check{\ell}''_2$  are disjoint. Arguing as in the previous case, we see that  $\text{mult}_{\mathcal{L}''_{10}}(\mathcal{D}) \leq \frac{n}{4}$ . Finally, observe that it also follows from Lemma 2.14 that the quadric  $\mathcal{Q}_6$  contains the lines  $\ell'_1, \ell'_2, \ell'_3, \check{\ell}'_1, \check{\ell}'_2$  and  $\check{\ell}'_3$ . Furthermore, in the proof of Lemma 4.6 we established that the four lines  $\ell'_1, \ell'_3, \check{\ell}'_1$  and  $\check{\ell}'_3$  are disjoint, which implies that  $\text{mult}_{\mathcal{L}'_{10}}(\mathcal{D}) \leq \frac{n}{4}$ .  $\square$

*Remark 4.12.* As we mentioned in the proof of Lemma 4.6, the lines  $\ell'_1, \ell'_2, \ell'_3, \ell'_4$  and  $\ell'_5$  form a “wheel” in  $\mathbb{P}^3$ , the lines  $\check{\ell}'_1, \check{\ell}'_2, \check{\ell}'_3, \check{\ell}'_4$  and  $\check{\ell}'_5$  also form a “wheel”, and these two wheels intersect by ten points in  $\Sigma'_{20}$ . Using Lemma 4.10, one can check that there exists a  $\overline{G}_{80}$ -commutative diagram

$$\begin{array}{ccc} & W & \\ \pi \swarrow & & \searrow \alpha \\ \mathbb{P}^3 & \xleftarrow{\quad \rho \quad} & X_4 \end{array}$$

Here  $\pi$  is the blow up of the curve  $\mathcal{L}'_{10}$ , the morphism  $\alpha$  is given by the linear system  $| - K_W |$ , the map  $\rho$  is given by the linear system of quartic surfaces in  $\mathbb{P}^3$  that contain  $\mathcal{L}'_{10}$ , and  $X_4$  is a quartic threefold in  $\mathbb{P}^4$ . This configuration of ten lines in  $\mathbb{P}^3$  has been studied by Todd in [41] under generality assumption (and without group action). He constructed the same commutative diagram with  $\alpha$  being a small birational morphism. However, the position of our 10 lines is not general from this point of view. Namely, it follows from the proof of Lemma 4.11 that the map  $\alpha$  contracts the proper transforms of the quadrics  $\mathcal{Q}_6, \mathcal{Q}_7, \mathcal{Q}_8, \mathcal{Q}_9$  and  $\mathcal{Q}_{10}$ , so that our  $\overline{G}_{80}$ -commutative diagram is a *bad  $\overline{G}_{80}$ -Sarkisov link*. One can consider this as a geometrical meaning of Lemma 4.11. Similarly, the lines of  $\mathcal{L}''_{10}$  form the same configuration in  $\mathbb{P}^3$ , which results in a similar  $\overline{G}_{80}$ -commutative diagram; the only difference in this case is that  $\alpha$  contracts the proper transforms of the quadrics  $\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3, \mathcal{Q}_4$  and  $\mathcal{Q}_5$ .

Denote by  $S_0, S_1, S_2, S_3$  and  $S_4$  the quartic surfaces in  $\mathbb{P}^3$  that are given by the equations  $q_0 = 0, q_1 = 0, q_2 = 0, q_3 = 0$  and  $q_4 = 0$ , respectively. Here  $q_0, q_1, q_2, q_3$  and  $q_4$  are quartic polynomials in (2.21). Then  $S_0, S_1, S_2, S_3$  and  $S_4$  are all  $\overline{G}_{80}$ -invariant quartic surfaces in  $\mathbb{P}^3$  by Lemma 2.10.

**Lemma 4.13.** *The polynomials  $q_0, q_1, q_2, q_3$  and  $q_4$  are irreducible. Moreover, the surface  $S_0$  is smooth, and the surfaces  $S_1, S_2, S_3$  and  $S_4$  have isolated ordinary double points. Furthermore, we have*

$$\text{Sing}(S_1) = \Sigma_{16}^1, \quad \text{Sing}(S_2) = \Sigma_{16}^2, \quad \text{Sing}(S_3) = \Sigma_{16}^3, \quad \text{Sing}(S_4) = \Sigma_{16}^4.$$

*Proof.* We can deduce all required assertions using the explicit formulas of the polynomials  $q_0, q_1, q_2, q_3$  and  $q_4$ . However, most of these assertions follow easily from general facts. Namely, the irreducibility of the polynomials  $q_0, q_1, q_2, q_3$  and  $q_4$  follows from Lemma 2.10. Now taking the general plane section of the surface  $S_i$ , we see that either it has isolated

singularities, or its singular locus contains a  $\overline{G}_{80}$ -curve of degree less than 4. The latter is impossible by Corollary 4.9. Thus, the surfaces  $S_0$ ,  $S_1$ ,  $S_2$ ,  $S_3$  and  $S_4$  have isolated singularities.

By [44, Theorem 1], the surface  $S_i$  cannot have more than two non-Du Val singular points. Thus, the surface  $S_i$  has at most Du Val singularities by Corollary 4.7. Then its minimal resolution of singularities is a smooth  $K3$  surface. Now using Corollary 4.7 and the fact that the rank of the Picard group of a smooth  $K3$  surface is at most 20, we see that either  $S_i$  is smooth, or  $S_i$  has isolated ordinary double points, and its singular locus is one of the  $\overline{G}_{80}$ -orbits  $\Sigma_{16}^1$ ,  $\Sigma_{16}^2$ ,  $\Sigma_{16}^3$  or  $\Sigma_{16}^4$ . Taking partial derivatives of the polynomial  $q_0$  at the points of the set  $\Sigma_{16}^1 \cup \Sigma_{16}^2 \cup \Sigma_{16}^3 \cup \Sigma_{16}^4$ , we see that  $S_0$  is smooth. Similarly, we see that  $\text{Sing}(S_i) = \Sigma_{16}^i$  for  $1 \leq i \leq 4$ .  $\square$

**Corollary 4.14.** *Let  $\Sigma$  be one of the  $\overline{G}_{80}$ -orbits  $\Sigma_{16}^1$ ,  $\Sigma_{16}^2$ ,  $\Sigma_{16}^3$ ,  $\Sigma_{16}^4$ , and let  $\mathcal{M}$  be the linear system that consists of cubic surfaces in  $\mathbb{P}^3$  containing  $\Sigma$ . Then  $\Sigma$  is the base locus of the linear system  $\mathcal{M}$ .*

*Proof.* See the proof of Corollary 3.13.  $\square$

The  $\overline{G}_{80}$ -orbits  $\Sigma_{20}$ ,  $\Sigma'_{20}$  and  $\Sigma''_{20}$  are not contained in any of the quartic surfaces  $S_0$ ,  $S_1$ ,  $S_2$ ,  $S_3$  and  $S_4$ . By Corollary 4.7, this shows that each curve among  $\mathcal{L}_{10}$ ,  $\mathcal{L}'_{10}$ ,  $\mathcal{L}''_{10}$  intersects each surface among  $S_0$ ,  $S_1$ ,  $S_2$ ,  $S_3$  and  $S_4$  transversally in a  $\overline{G}_{80}$ -orbit of length 40. Thus, all these surfaces contain exactly three  $\overline{G}_{80}$ -orbits of length 40 by Lemma 4.6. Observe also that  $S_0$  contains all  $\overline{G}_{80}$ -orbits  $\Sigma_{16}^1$ ,  $\Sigma_{16}^2$ ,  $\Sigma_{16}^3$ ,  $\Sigma_{16}^4$ . This gives the following

**Corollary 4.15.** *The group  $\text{Pic}(S_0)^{\overline{G}_{80}}$  is generated by the plane section of  $S_0$ .*

*Proof.* By [33, Lemma 2.1], the action of the group  $\overline{G}_{80}$  on the  $K3$  surface  $S_0$  is symplectic. Thus, the surface  $S_0/\overline{G}_{80}$  is a singular  $K3$  surface. By construction, it has four Du Val singular points of type  $A_4$  and three Du Val singular points of type  $A_1$ . The minimal resolution of singularities of this surface is a smooth  $K3$  surface. Since the rank of the Picard group of a smooth  $K3$  surface cannot exceed 20, we see that the rank of the Picard group of the surface  $S_0/\overline{G}_{80}$  must be 1. This shows that the rank of the group  $\text{Pic}(S_0)^{\overline{G}_{80}}$  is also 1. Since  $\text{Pic}(S_0)$  has no torsion and the intersection form of curves on the surface  $S_0$  is even, we see that the group  $\text{Pic}(S_0)^{\overline{G}_{80}}$  is generated by the plane section of  $S_0$ .  $\square$

Table 2 shows which  $\overline{G}_{80}$ -orbit  $\Sigma_{16}^1$ ,  $\Sigma_{16}^2$ ,  $\Sigma_{16}^3$  and  $\Sigma_{16}^4$  is contained in which surface  $S_1$ ,  $S_2$ ,  $S_3$  and  $S_4$ . In particular, we see that  $\Sigma_{16}^j \subset S_i$  if and only if  $j \neq 2i \pmod{5}$ .

Now we are ready to prove

**Theorem 4.16.** *Let  $C$  be an arbitrary  $\overline{G}_{80}$ -irreducible curve in  $\mathbb{P}^3$  such that  $\deg(C) \leq 15$ . Then  $C \not\subset S_0$  and one of the following cases holds:*

- $\deg(C) = 10$ , and  $C$  is one of the curves  $\mathcal{L}_{10}$ ,  $\mathcal{L}'_{10}$ ,  $\mathcal{L}''_{10}$ ,
- $\deg(C) = 12$ , and  $C$  is an irreducible smooth curve of genus 5 that is contained in exactly one surface among  $S_1$ ,  $S_2$ ,  $S_3$ ,  $S_4$ ,
- $\deg(C) = 8$ , and  $C$  is an irreducible smooth curve of genus 5 such that either  $C = S_1 \cap S_4$  or  $C = S_2 \cap S_3$ .

*Proof.* Recall from Corollary 4.15 that the group  $\text{Pic}(S_0)^{\overline{G}_{80}}$  is generated by the plane section  $H_{S_0}$  of the surface  $S_0$ . Since there is an exact sequence of  $\overline{G}_{80}$ -representations

$$0 \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^3}(n-4)) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^3}(n)) \longrightarrow H^0(\mathcal{O}_{S_0}(nH_{S_0})) \longrightarrow 0$$

TABLE 2. Incidence between  $S_i$  and  $\Sigma_{16}^j$ .

	$S_1$	$S_2$	$S_3$	$S_4$
$\Sigma_{16}^1$	$\text{Sing}(S_1)$	+	−	+
$\Sigma_{16}^2$	−	$\text{Sing}(S_2)$	+	+
$\Sigma_{16}^3$	+	+	$\text{Sing}(S_3)$	−
$\Sigma_{16}^4$	+	−	+	$\text{Sing}(S_4)$

for every  $n \geq 1$ , we see that  $S_0$  does not contain  $C$  by Lemma 2.10.

By Lemma 4.8, we may assume that  $C$  is irreducible. Let  $d = \deg(C)$ . Then

$$4d = S_0 \cdot C = 16a + 40b$$

for some non-negative integers  $a$  and  $b$ , because  $\Sigma_{20}$ ,  $\Sigma'_{20}$  and  $\Sigma''_{20}$  are not contained in  $S_0$ . Keeping in mind that  $d \geq 5$  by Corollary 4.9, we have the following possibilities:

$d$	$a$	$b$
8	2	0
10	0	1
12	3	0
14	1	1

Suppose that  $d = 14$ . Since  $a = 1$  and  $S_0$  contains all  $\overline{G}_{80}$ -orbits of length 16 in  $\mathbb{P}^3$ , we see that the curve  $C$  contains a unique  $\overline{G}_{80}$ -orbit of length 16. Without loss of generality, we may assume that  $C$  contains  $\Sigma_{16}^1$ . Then  $C \not\subset S_3$ , because  $S_3$  does not contain  $\Sigma_{16}^1$ . By Corollary 4.7, this implies that

$$56 = 4d = S_3 \cdot C$$

is divisible by 40, which is absurd. This shows that  $d \neq 14$ , so that  $d \leq 12$ .

Pick four general points  $O_1, O_2, O_3$  and  $O_4$  in the curve  $C$ . Since  $C$  is irreducible, their stabilizers in  $\overline{H}$  are trivial. In particular, the  $\overline{H}$ -orbits of these points consist of 16 different points. On the other hand, the vector space generated by the quartic polynomials

$$q_0(x, y, z, w), q_1(x, y, z, w), q_2(x, y, z, w), q_3(x, y, z, w), q_4(x, y, z, w),$$

contains all  $H$ -invariant polynomials of degree 4. Keeping in mind the dimension of this space, we see that there exists  $[\lambda_0 : \lambda_1 : \lambda_2 : \lambda_3 : \lambda_4] \in \mathbb{P}^4$  such that the polynomial

$$\lambda_0 q_0(x, y, z, w) + \lambda_1 q_1(x, y, z, w) + \lambda_2 q_2(x, y, z, w) + \lambda_3 q_3(x, y, z, w) + \lambda_4 q_4(x, y, z, w)$$

vanishes at the points  $O_1, O_2, O_3$  and  $O_4$ . This polynomials defines a (possibly reducible or non-reduced)  $\overline{H}$ -invariant quartic surface  $S$  in  $\mathbb{P}^3$ . Then  $S$  contains  $O_1, O_2, O_3$  and  $O_4$  together with their  $\overline{H}$ -orbits. This implies that

$$4d = S \cdot C \geq 64$$

provided that the curve  $C$  is not contained in  $S$ . Since  $d \leq 12$ , we see that  $C \subset S$ . Thus, the vector space of all  $H$ -invariant quartic polynomials in  $\mathbb{C}[x, y, z, w]$  that vanish along  $C$  is at least one-dimensional. Since  $H$  is a normal subgroup in  $G_{80}$ , the group

$$G_{80}/H \cong \mu_5$$

acts on this space. Keeping in mind that  $\mu_5$  is abelian, we see that there exists at least one  $G_{80}$ -invariant quartic polynomial in  $\mathbb{C}[x, y, z, w]$  that vanishes along the curve  $C$ . Moreover, if  $d \leq 10$ , then the rank-nullity theorem implies that the vector space consisting of all  $H$ -invariant quartic polynomials in  $\mathbb{C}[x, y, z, w]$  that vanish along  $C$  is at least two-dimensional, because we can repeat the same arguments with three general points in  $C$  instead of four. Therefore, if  $d \leq 10$ , then there exists at least two  $G_{80}$ -invariant quartic polynomials in  $\mathbb{C}[x, y, z, w]$  that vanishes at the curve  $C$ . Now using Lemma 4.13, we see that  $C$  is contained in one of the surfaces  $S_1, S_2, S_3$  or  $S_4$ . Moreover, if  $d \leq 10$ , then  $C$  is contained in at least two surfaces among  $S_1, S_2, S_3, S_4$ .

We see that there exists  $t \in \{1, 2, 3, 4\}$  such that  $C$  is contained in  $S_t$ . Denote by  $H$  the plane section of the surface  $S_t$ . If  $d \neq 12$ , then there is  $k \in \{1, 2, 3, 4\}$  such that  $C$  is contained in the surfaces  $S_k$  and  $k \neq t$ . In this case, we have

$$S_k|_{S_t} = C + Z,$$

where  $Z$  is a  $\overline{G}_{80}$ -invariant effective divisor such that  $H \cdot Z = 16 - d$ . On the other hand, we already proved that  $\mathbb{P}^3$  does not contain  $\overline{G}_{80}$ -invariant curves of degree less than 8. This shows that  $d \neq 10$ , so that either  $d = 8$  or  $d = 12$ .

Recall that  $\Sigma_{16}^t$  is the singular locus of the surface  $S_t$ . Let  $f: \tilde{S}_t \rightarrow S_t$  be the minimal resolution of singularities of the surface  $S_t$ . Then the action of the group  $\overline{G}_{80}$  lifts to the surface  $\tilde{S}_t$ , because  $f$  is the minimal resolution of singularities. Denote by  $E_1, \dots, E_{16}$  the  $f$ -exceptional curves, and denote by  $\tilde{C}$  the proper transform of the curve  $C$  on the surface  $\tilde{S}_t$ . Let  $E = E_1 + \dots + E_{16}$  and  $\tilde{H} = f^*(H)$ . Then

$$\tilde{C} \sim_{\mathbb{Q}} f^*(C) - \frac{m}{2}E$$

for some non-negative integer number  $m$ . By Hodge index theorem, one has

$$0 \geq \begin{vmatrix} \tilde{H}^2 & \tilde{H} \cdot (\tilde{C} + \frac{m}{2}E) \\ \tilde{H} \cdot (\tilde{C} + \frac{m}{2}E) & (\tilde{C} + \frac{m}{2}E)^2 \end{vmatrix} = \begin{vmatrix} 4 & d \\ d & (\tilde{C} + \frac{m}{2}E)^2 \end{vmatrix} = 4(\tilde{C}^2 + 8m^2) - d^2.$$

This implies

$$\tilde{C}^2 \leq \frac{d^2}{4} - 8m^2.$$

Let  $g$  be the genus of the normalization of the curve  $\tilde{C}$ , and let  $p_a(\tilde{C})$  be its arithmetic genus. Then

$$\begin{aligned} g &\leq p_a(\tilde{C}) - |\text{Sing}(\tilde{C})| = \frac{\tilde{C}^2}{2} + 1 - |\text{Sing}(\tilde{C})| \leq \\ &\leq \frac{d^2}{8} - 4m^2 + 1 - |\text{Sing}(\tilde{C})| \leq \\ &\leq \frac{144}{8} - 4m^2 + 1 - |\text{Sing}(\tilde{C})| = 19 - 4m^2 - |\text{Sing}(\tilde{C})|. \end{aligned}$$



On the other hand, it follows from Lemma 4.3 that  $g \geq 5$ . Thus, we see that the curve  $\tilde{C}$  is smooth, one has

$$g = p_a(\tilde{C}) \leq 19 - 4m^2,$$

and  $m \leq 1$ , because  $\mathbb{P}^3$  does not have  $\overline{G}_{80}$ -orbits of length less than 16 by Corollary 4.7. In particular, since  $m \leq 1$ , we see that the curve  $C$  is also smooth. Moreover, since  $g \leq 19$ , we have  $g \in \{5, 13, 17\}$  by Lemma 4.3.

Suppose that  $g = 17$ . Then  $C$  contains three  $\overline{G}_{80}$ -orbits of length 16 by Lemma 4.3. Thus, the curve  $C$  contains every  $\overline{G}_{80}$ -orbit  $\Sigma_{16}^i$  except for  $\Sigma_{16}^j$  such that  $j = 2t \bmod 5$ , because these are the only  $\overline{G}_{80}$ -orbits of length 16 contained in the surface  $S_t$ . Since  $C$  contains  $\Sigma_{16}^t = \text{Sing}(S_t)$ , we have  $m \geq 1$ , which implies that  $m = 1$ , because we proved already that  $m \leq 1$ . Since we also proved that  $g \leq 19 - 4m^2$ , we have

$$17 = g \leq 19 - 4m^2 = 15,$$

which is absurd. This shows that  $g \neq 17$ .

Suppose that  $g = 13$ . Then  $C$  contains one  $\overline{G}_{80}$ -orbit of length 16 by Lemma 4.3. Recall that there exists  $r \in \{1, 2, 3, 4\}$  such that the surface  $S_r$  that does not contain this  $\overline{G}_{80}$ -orbit. Then  $C \not\subset S_r$ , so that  $S_r \cdot C$  must be a multiple of 40 by Lemma 4.3. Since

$$S_r \cdot C = 4d,$$

we see that  $d$  is divisible by 10. This is impossible, because we already proved that either  $d = 8$  or  $d = 12$ . This shows that  $g \neq 13$ .

We see that  $C$  is a smooth curve of degree  $d \in \{8, 12\}$  and genus 5 that is contained in  $S_t$  such that  $t \neq 0$ . In particular, we see that there are no  $\overline{G}_{80}$ -invariant curves in  $\mathbb{P}^3$  of degree less than 8. Thus, if  $d = 12$ , then  $S_t$  is the only surface among  $S_1, S_2, S_3$  and  $S_4$  that contains  $C$ . Therefore, to complete the proof of the lemma, we may assume that  $d = 8$ . We have to show that either  $C = S_1 \cap S_4$  or  $C = S_2 \cap S_3$ .

Recall that  $C$  is also contained in a quartic surface  $S_k$  such that  $k \neq t$ . Then

$$(4.17) \quad 4H \sim S_k|_{S_t} = C + Z,$$

where  $Z$  is a  $\overline{G}_{80}$ -invariant effective divisor such that  $H \cdot Z = 8$ . Since we already proved that  $\mathbb{P}^3$  does not contain  $\overline{G}_{80}$ -invariant curves of degree less than 8, we see that  $Z$  is an irreducible curve of degree 8. Moreover, we already proved that every  $\overline{G}_{80}$ -invariant curve in  $\mathbb{P}^3$  of degree 8 is a smooth curve of genus 5. In particular, we see that  $Z$  is a smooth curve of genus 5. Note that, a priori, we may have the case when  $Z = C$ .

Suppose that both curves  $C$  and  $Z$  do not contain the singular locus of the surface  $S_t$ . Then  $C^2 = Z^2 = 8$  by the adjunction formula, which gives

$$32 = 4H \cdot C = C \cdot (C + Z) = C^2 + C \cdot Z = 8 + C \cdot Z,$$

which implies that  $C \cdot Z = 24$ . The latter is impossible. Indeed, if we have  $C = Z$ , then  $C \cdot Z = C^2 = 8$ . Similarly, if  $C \neq Z$ , then

$$C \cdot Z = 16\alpha + 40\beta$$

for some non-negative integers  $\alpha$  and  $\beta$ , because  $C$  only contains  $\overline{G}_{80}$ -orbits of length 16, 40 and 80 by Lemma 4.3. Thus, we see that  $C \cdot Z \neq 24$ . This shows that either  $C$  or  $Z$  contains  $\Sigma_{16}^t = \text{Sing}(S_t)$ .

If  $\Sigma_{16}^t \subset C$ , then  $\Sigma_{16}^t \subset Z$  by (4.17), because both  $C$  and  $Z$  are smooth. Similarly, if  $\Sigma_{16}^t \subset Z$ , then  $\Sigma_{16}^t \subset C$ . Thus, we see that  $\Sigma_{16}^t \subset C \cap Z$ . Applying the same arguments to the surface  $S_k$ , we see that

$$\text{Sing}(S_k) = \Sigma_{16}^k \subset C \cap Z,$$

which implies that  $k + t = 5$  (see Table 2).

Denote by  $\tilde{Z}$  the proper transform of the curve  $Z$  on the surface  $\tilde{S}_t$ . Then

$$\tilde{C} + \tilde{Z} \sim f^*(4H) - E.$$

This follows from the fact that  $C + Z$  is a Cartier divisor at every point of  $\Sigma_{16}^t$ , because both  $C$  and  $Z$  are smooth and  $\Sigma_{16}^t \subset C \cap Z$ . As above, the adjunction formula gives  $\tilde{C}^2 = \tilde{Z}^2 = 8$ . Then

$$8 + \tilde{C} \cdot \tilde{Z} = \tilde{C}^2 + \tilde{C} \cdot \tilde{Z} = \tilde{C} \cdot (\tilde{C} + \tilde{Z}) = \tilde{C} \cdot (f^*(4H) - E) = 32 - \tilde{C} \cdot E = 16,$$

so that  $\tilde{C} \cdot \tilde{Z} = 8$ . Keeping in mind that  $\tilde{C}$  only contains  $\overline{G}_{80}$ -orbits of length 16, 40, 80, we see that  $\tilde{C} = \tilde{Z}$ . Then  $C = Z$ , so that  $S_k \cap S_t = C$  and  $k + t = 5$  as required. This completes the proof of Theorem 4.16.  $\square$

It should be pointed out that Theorem 4.16 does not assert that the projective space  $\mathbb{P}^3$  contains  $\overline{G}_{80}$ -irreducible curves of degree 8 and 12.

*Remark 4.18.* The  $\overline{G}_{80}$ -invariant curves of degree 8 in  $\mathbb{P}^3$  do exist. In fact, there are exactly two of them. Indeed, one can show that the ideal in  $\mathbb{C}[x, y, z, w]$  generated by the polynomials

$$\begin{aligned} q_1(x, y, z, w), \quad q_4(x, y, z, w), \quad \frac{\partial q_1}{\partial x} \frac{\partial q_4}{\partial y} - \frac{\partial q_4}{\partial x} \frac{\partial q_1}{\partial y}, \quad \frac{\partial q_1}{\partial x} \frac{\partial q_4}{\partial z} - \frac{\partial q_4}{\partial x} \frac{\partial q_1}{\partial z}, \\ \frac{\partial q_1}{\partial x} \frac{\partial q_4}{\partial w} - \frac{\partial q_4}{\partial x} \frac{\partial q_1}{\partial w}, \quad \frac{\partial q_1}{\partial y} \frac{\partial q_4}{\partial z} - \frac{\partial q_4}{\partial y} \frac{\partial q_1}{\partial z}, \quad \frac{\partial q_1}{\partial y} \frac{\partial q_4}{\partial w} - \frac{\partial q_4}{\partial y} \frac{\partial q_1}{\partial w}, \quad \frac{\partial q_1}{\partial z} \frac{\partial q_4}{\partial w} - \frac{\partial q_4}{\partial z} \frac{\partial q_1}{\partial w} \end{aligned}$$

defines a one-dimensional subscheme in  $\mathbb{P}^3$ . This shows that the scheme-theoretic intersection  $S_1 \cdot S_4$  is not reduced. By Theorem 4.16, this implies that the set-theoretic intersection  $S_1 \cap S_4$  is a  $\overline{G}_{80}$ -invariant irreducible smooth curve of genus 5. Similarly, we see that  $S_2 \cap S_3$  is another  $\overline{G}_{80}$ -invariant irreducible smooth curve of genus 5. Note that the curves  $S_1 \cap S_4$  and  $S_2 \cap S_3$  are projectively equivalent, since

$$\overline{R}(S_3) = S_1, \quad \overline{R}(S_4) = S_3, \quad \overline{R}(S_2) = S_4, \quad \overline{R}(S_1) = S_2.$$

This also implies that both of these curves are  $\overline{G}_{160}$ -invariant, and they are swapped by the group  $\overline{G}_{320}$ . In particular, there exists a smooth irreducible curve of genus 5 with an action of the group  $\overline{G}_{160} \cong \mu_2^4 \rtimes D_{10}$ . This curve is well-known: it is a complete intersection in  $\mathbb{P}^4$  that is given by

$$(4.19) \quad \begin{cases} x_0^2 + x_1^2 + x_3^2 + x_4^2 + x_5^2 = 0, \\ x_0^2 + \xi_5 x_1^2 + \xi_5^2 x_3^2 + \xi_5^3 x_4^2 + \xi_5^4 x_5^2 = 0, \\ \xi_5^4 x_0^2 + \xi_5^3 x_1^2 + \xi_5^2 x_3^2 + \xi_5 x_4^2 + x_5^2 = 0, \end{cases}$$

where  $\xi_5$  is a primitive fifth root of unity. This was proved by Joseph McKelvey in [31], who also proved that  $\mu_2^4 \rtimes D_{10}$  is the full automorphism group of the curve (4.19), see also [29, 22]. The curve given by (4.19) is a special case of the so-called Humbert curves, which have been discovered by Humbert in [23], and have been studied by Edge in [19].

*Remark 4.20.* Let  $C$  be a  $\overline{G}_{80}$ -invariant curve of degree 8 in  $\mathbb{P}^3$ . Then  $C$  is a smooth curve of genus 5 by Theorem 4.16. Thus, it contains unique  $\overline{G}_{80}$ -orbit of length 40 by Lemma 4.3. We claim that this orbit is the intersection  $\mathcal{L}_{10} \cap C$ . To show this, let us describe the intersections  $\ell_1 \cap S_1$ ,  $\ell_1 \cap S_2$ ,  $\ell_1 \cap S_3$  and  $\ell_1 \cap S_4$ , where  $\ell_1$  is the irreducible component of the curve  $\mathcal{L}_{10}$  defined in (2.11). Note that  $\ell_1$  is the line in  $\mathbb{P}^3$  that passes through the points  $[0 : i : 1 : 0]$  and  $[1 : 0 : 0 : -i]$ , so that its parametric equation is

$$[\mu : \lambda i : \lambda : -\mu i],$$

where  $[\lambda : \mu] \in \mathbb{P}^1$ . To find the intersections  $\ell_1 \cap S_1$ ,  $\ell_1 \cap S_2$ ,  $\ell_1 \cap S_3$  and  $\ell_1 \cap S_4$ , we may assume that  $\lambda = 1$ , because  $[0 : i : 1 : 0]$  is not contained in  $S_1 \cup S_2 \cup S_3 \cup S_4$ . Substituting the point  $[\mu : i : 1 : -\mu i]$  into the polynomials  $q_1, q_2, q_3, q_4$  defined in (2.21), we see that both intersections  $\ell_1 \cap S_1$  and  $\ell_1 \cap S_4$  are given by

$$\mu^4 - (4\xi_5^3 + 4\xi_5^2 + 2)\mu^2 + 1 = 0,$$

and both intersections  $\ell_1 \cap S_2$  and  $\ell_1 \cap S_3$  are given by

$$\mu^4 + (4\xi_5^3 + 4\xi_5^2 + 2)\mu^2 + 1 = 0.$$

Thus, we see that  $\ell_1 \cap S_1 = \ell_1 \cap S_4$  and  $\ell_1 \cap S_2 = \ell_1 \cap S_3$ . Since  $C$  is one of the curves  $S_1 \cap S_4$  or  $S_2 \cap S_3$  by Theorem 4.16, the intersection  $\mathcal{L}_{10} \cap C$  is the unique  $\overline{G}_{80}$ -orbit in  $C$  of length 40.

We will show later that  $\mathbb{P}^3$  contains  $\overline{G}_{80}$ -invariant curves of degree 12 (see Lemma 4.26).

*Remark 4.21.* Let  $C$  be a  $\overline{G}_{80}$ -invariant curve in  $\mathbb{P}^3$  of degree 12. By Theorem 4.16, the curve  $C$  is smooth, its genus is 5, and  $C \subset S_t$ , where  $S_t$  is one of the surfaces  $S_1, S_2, S_3$  and  $S_4$ . By Remark 4.18, the surface  $S_t$  contains an irreducible  $\overline{G}_{80}$ -invariant curve  $Z$  such that  $2Z = S_k|_{S_t}$ , where  $k + t = 5$ . If  $C$  does not contain  $\text{Sing}(S_t)$ , then  $Z \cdot C = 24$ , which is impossible, because the only  $\overline{G}_{80}$ -orbits contained in  $C$  are of lengths 16, 40 and 80 by Lemma 4.3. Thus, we see that  $C$  contains the singular locus of the surface  $S_t$ .

The first consequence of Theorem 4.16 is

**Corollary 4.22.** *Let  $C$  be an irreducible  $\overline{G}_{80}$ -invariant curve in  $\mathbb{P}^3$  of degree 12, let  $\mathcal{D}$  be a mobile  $\overline{G}_{80}$ -invariant linear system on  $\mathbb{P}^3$ , let  $n$  be a positive integer such that  $\mathcal{D} \sim \mathcal{O}_{\mathbb{P}^3}(n)$ . Then  $\text{mult}_C(\mathcal{D}) \leq \frac{n}{4}$ .*

*Proof.* By Theorem 4.16, the curve  $C$  is a smooth curve of genus 5 that is contained in a surface among  $S_1, S_2, S_3, S_4$ . Denote this surface by  $S$ , and denote by  $H$  its general plane section. Let  $D$  be a general surface in  $\mathcal{D}$ . Then

$$D|_S = mC + \Delta \sim nH,$$

where  $m$  is a non-negative integer such that  $m \geq \text{mult}_C(\mathcal{D})$ , and  $\Delta$  is an effective divisor on  $S$  such that  $C \not\subset \text{Supp}(\Delta)$ .

Recall from Lemma 4.13, that  $\text{Sing}(S)$  is one of the  $\overline{G}_{80}$ -orbits  $\Sigma_{16}^1, \Sigma_{16}^2, \Sigma_{16}^3, \Sigma_{16}^4$ , and  $S$  has isolated ordinary double points. Let  $f: \tilde{S} \rightarrow S$  be the minimal resolution of singularities of the surface  $S$ . Then the action of the group  $\overline{G}_{80}$  lifts to the surface  $\tilde{S}$ . Denote by  $E_1, \dots, E_{16}$  the  $f$ -exceptional curves, denote by  $\tilde{C}$  the proper transform of the curve  $C$  on the surface  $\tilde{S}$ , and denote by  $\tilde{\Delta}$  the proper transform of the divisor  $\Delta$  on the surface  $\tilde{S}$ . Let  $E = E_1 + \dots + E_{16}$ . Then  $E^2 = -32$  and

$$m\tilde{C} + \tilde{\Delta} \sim_{\mathbb{Q}} f^*(nH) - \epsilon E$$

for some non-negative rational number  $\epsilon$ . Moreover, the curve  $\tilde{C}$  is a smooth irreducible curve of genus 5, so that  $\tilde{C}^2 = 8$  by the adjunction formula.

By Remark 4.21, the curve  $C$  contains  $\text{Sing}(S)$ . Then  $\tilde{C} \cdot E = 16$ . Let  $T$  be the divisor  $f^*(6H) - \tilde{C} - E$ . Then  $T \cdot E = 16$  and  $T^2 = 8$ , so that

$$\begin{aligned} h^0(\mathcal{O}_{\tilde{S}}(T)) &= 6 + h^1(\mathcal{O}_{\tilde{S}}(T)) - h^2(\mathcal{O}_{\tilde{S}}(T)) = \\ &= 6 + h^1(\mathcal{O}_{\tilde{S}}(T)) + h^0(\mathcal{O}_{\tilde{S}}(-T)) = 6 + h^1(\mathcal{O}_{\tilde{S}}(T)) \geq 6 \end{aligned}$$

by the Riemann–Roch formula and Serre duality, since  $T \cdot f^*(H) = 12 > 0$ .

We claim that the linear system  $|T|$  does not have base curves. Indeed, suppose it does. Then  $T \sim F + M$ , where  $F$  is the fixed part of the linear system  $|T|$ , and  $M$  is its mobile part. Then  $M \cdot f^*(H) \geq 3$ , because  $S$  is not uniruled. This gives

$$0 \leq f(F) \cdot H = F \cdot f^*(H) = 12 - M \cdot f^*(H) \leq 9.$$

Since  $F$  is  $\overline{G}_{80}$ -invariant, it follows from Theorem 4.16 that either  $F$  is contracted by  $f$ , or  $f(F)$  is an irreducible smooth curve of degree 8 and genus 5. In the former case we have  $F = rE$  for some integer  $r \geq 1$ , so that

$$M^2 = (T - rE)^2 = 6 - 2rT \cdot E - 32r^2 \leq 6 - 32r^2 < 0,$$

which is absurd. Thus, we see that  $f(F)$  is an irreducible smooth curve of degree 8 and genus 5. Denote its proper transform on  $\tilde{S}$  by  $\tilde{C}_8$ . Then  $\tilde{C}_8 \subset \text{Supp}(F)$  and  $\tilde{C}_8^2 = 8$ . Hence, it follows from the Riemann–Roch formula and Serre duality that

$$\begin{aligned} h^0(\mathcal{O}_{\tilde{S}}(F)) &\geq h^0(\mathcal{O}_{\tilde{S}}(\tilde{C}_8)) = 6 + h^1(\mathcal{O}_{\tilde{S}}(\tilde{C}_8)) - h^2(\mathcal{O}_{\tilde{S}}(\tilde{C}_8)) = \\ &= 6 + h^1(\mathcal{O}_{\tilde{S}}(\tilde{C}_8)) + h^0(\mathcal{O}_{\tilde{S}}(-\tilde{C}_8)) = 6 + h^1(\mathcal{O}_{\tilde{S}}(\tilde{C}_8)) \geq 6, \end{aligned}$$

which is also absurd, since  $F$  is the fixed part of the linear system  $|T|$ . Thus, we see that the linear system  $|T|$  does not have base curves.

Let  $Z$  be a general curve in  $|T|$ . Then  $Z \cdot E = 16$  and

$$Z \cdot \tilde{C} = \tilde{C} \cdot (6f^*(H) - \tilde{C} - E) = 48.$$

On the other hand, the divisor  $Z$  is nef, so that  $Z \cdot \tilde{\Delta} \geq 0$ . Therefore, we have

$$0 \leq Z \cdot \tilde{\Delta} = Z \cdot (f^*(nH) - m\tilde{C} - \epsilon E) = 12n - mZ \cdot \tilde{C} - \epsilon Z \cdot E \leq 12n - 48m,$$

which implies that  $\text{mult}_C(\mathcal{D}) \leq m \leq \frac{n}{4}$ .  $\square$

**Corollary 4.23.** *Let  $C$  be a  $\overline{G}_{80}$ -invariant curve of degree 12 in  $\mathbb{P}^3$ , and let  $\mathcal{M}$  be the linear system of surfaces in  $\mathbb{P}^3$  of degree 6 that contains  $C$ . Then the base locus of  $\mathcal{M}$  does not have curves that are different from  $C$ .*

*Proof.* The required assertion easily follows from the proof of Corollary 4.22. Namely, let us use notation of the proof of Corollary 4.22. Then the linear system  $|f^*(6H) - \tilde{C} - E|$  is free from base curves. This was shown in the proof of Corollary 4.22. In particular, the linear system  $|6H - C|$  does not have base points. Since the quartic surface  $S$  is projectively normal, we immediately obtain the required assertions.  $\square$

Now let us prove that  $\overline{G}_{80}$ -invariant curves of degree 12 in  $\mathbb{P}^3$  do exist.

**Lemma 4.24.** *Each surface among  $S_1, S_2, S_3, S_4$  contains at least one  $\overline{G}_{80}$ -invariant curve of degree 12.*

*Proof.* Let  $C$  be a smooth curve of genus 5 that admits a faithful action of the group  $\overline{G}_{80}$ . Then it follows from [17, Proposition 2.2] that there exists an exact sequence of groups

$$(4.25) \quad 0 \longrightarrow \mathrm{Hom}(\overline{G}_{80}, \mathbb{C}^*) \longrightarrow \mathrm{Pic}(\overline{G}_{80}, C) \longrightarrow \mathrm{Pic}(C)^{\overline{G}_{80}} \longrightarrow H^2(\overline{G}_{80}, \mathbb{C}^*) \longrightarrow 0,$$

where  $\mathrm{Pic}(\overline{G}_{80}, C)$  stands for the group of  $\overline{G}_{80}$ -equivariant line bundles on  $C$ . One can compute that  $\mathrm{Hom}(\overline{G}_{80}, \mathbb{C}^*) \cong \mu_5$  and  $H^2(\overline{G}_{80}, \mathbb{C}^*) \cong \mu_2^2$ . Moreover, one has  $C/\overline{G}_{80} \cong \mathbb{P}^1$ , so that it follows from [17, (2.2)] that

$$\mathrm{Pic}(\overline{G}_{80}, C) \cong \mathbb{Z} \oplus \mu_5.$$

Furthermore, let  $\kappa$  be a positive generator of the free part of  $\mathrm{Pic}(\overline{G}_{80}, C)$ . Then it follows from [17, (2.1)] that  $\deg(\kappa) = 8$ , so that we may assume that  $\kappa = K_C$ . Therefore, there is an exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathrm{Pic}(C)^{\overline{G}_{80}} \longrightarrow \mu_2^2 \longrightarrow 0.$$

Thus, either  $\mathrm{Pic}(C)^{\overline{G}_{80}} \cong \mathbb{Z} \oplus \mu_2$  or  $\mathrm{Pic}(C)^{\overline{G}_{80}} \cong \mathbb{Z} \oplus \mu_2^2$ . In both cases, there exists a  $\overline{G}_{80}$ -invariant 2-torsion line bundle on  $C$ . In the former case, such 2-torsion is unique, and there exists a  $\overline{G}_{80}$ -invariant line bundle  $\theta \in \mathrm{Pic}(C)$  such that  $2\theta \sim K_C$ , i.e. the divisor  $\theta$  is a  $\overline{G}_{80}$ -invariant theta-characteristic. In the latter case, there are exactly three non-trivial  $\overline{G}_{80}$ -invariant 2-torsion line bundles on  $C$ , and there are no  $\overline{G}_{80}$ -invariant theta-characteristics.

Let  $D$  be a 2-torsion divisor on  $C$  such that its class in  $\mathrm{Pic}(C)$  is  $\overline{G}_{80}$ -invariant. Then

$$h^0(\mathcal{O}_C(K_C + D)) = h^0(\mathcal{O}_C(-D)) + 4 = 4$$

by the Riemann–Roch formula. Moreover, the linear system  $|K_C + D|$  does not have base points, because the degree of the divisor  $K_C + D$  is 8, and  $C$  does not contain  $\overline{G}_{80}$ -orbits of length less than 16 by Lemma 4.3. So, the linear system  $|K_C + D|$  gives a  $\overline{G}_{80}$ -equivariant morphism  $\sigma: C \rightarrow \mathbb{P}^3$ . Note that the action of the group  $\overline{G}_{80}$  on  $\mathbb{P}^3$  is given by the projectivization of the  $\overline{G}_{80}$ -representation  $H^0(\mathcal{O}_C(K_C + D))^\vee$ . By Corollary 4.2, we may assume that  $\sigma(C)$  is a  $\overline{G}_{80}$ -invariant curve of degree at most 8. Then the degree of the curve  $\sigma(C)$  is 8 by Theorem 4.16. In particular, this gives another proof that  $\mathbb{P}^3$  contains a  $\overline{G}_{80}$ -invariant curve of degree 8 (see Remark 4.18). Note that Theorem 4.16 also implies that the curve  $\sigma(C)$  is smooth, and either  $\sigma(C) = S_1 \cap S_4$  or  $\sigma(C) = S_2 \cap S_3$ . Hence the morphism  $\sigma$  induces an isomorphism  $C \cong \sigma(C)$ . Since the curves  $S_1 \cap S_4$  and  $S_2 \cap S_3$  are projectively equivalent by Remark 4.18, we see that the class of the divisor  $D$  in  $\mathrm{Pic}(C)$  is uniquely determined.

We see that there exists a unique  $\overline{G}_{80}$ -invariant 2-torsion line bundle on  $C$ , which implies that there exists a  $\overline{G}_{80}$ -invariant theta-characteristic  $\theta \in \mathrm{Pic}(C)$ . In fact, there are exactly two such theta-characteristics: one is  $\theta$ , and another one is  $\theta + D$ . Arguing as above, we see that both linear systems  $|K_C + \theta|$  and  $|K_C + D + \theta|$  are free from base points and define  $\overline{G}_{80}$ -equivariant morphisms  $C \rightarrow \mathbb{P}^7$ , where  $\mathbb{P}^7$  are the projectivizations of  $H^0(\mathcal{O}_C(K_C + \theta))^\vee$  and  $H^0(\mathcal{O}_C(K_C + D + \theta))^\vee$ , which are eight-dimensional representations of central extensions of the group  $\overline{G}_{80}$  given by the images of  $K_C + \theta$  and  $K_C + D + \theta$  in

$$H^2(\overline{G}_{80}, \mathbb{C}^*) \cong \mu_2^2$$

in (4.25), respectively. One can show that these eight-dimensional representations split as sums of two four-dimensional representations, so that there exists a  $\overline{G}_{80}$ -equivariant map  $\varsigma: C \rightarrow \mathbb{P}^3$  that is given by any  $\overline{G}_{80}$ -invariant three-dimensional linear subsystem in  $|K_C + \theta|$  or  $|K_C + D + \theta|$ . Since the degree of both divisors  $K_C + \theta$  and  $K_C + D + \theta$  is 12, this linear subsystem does not have base points. By Corollary 4.2, we may assume that the curve  $\varsigma(C)$  is a  $\overline{G}_{80}$ -invariant curve whose degree divides 12. Then  $\varsigma(C)$  is a smooth curve of degree 12 and genus 5 by Theorem 4.16.

By Theorem 4.16, the curve  $\varsigma(C)$  is contained in a unique quartic surface among  $S_1, S_2, S_3$  and  $S_4$ . Let  $\overline{R}$  be the element in  $\mathrm{PGL}_4(\mathbb{C})$  defined in §2, cf. Remark 4.18. Then

$$\overline{R}(S_3) = S_1, \quad \overline{R}(S_4) = S_3, \quad \overline{R}(S_2) = S_4, \quad \overline{R}(S_1) = S_2.$$

This shows that each surface among  $S_1, S_2, S_3, S_4$  contains a unique curve among  $\varsigma(C), \overline{R}(\varsigma(C)), \overline{R}^2(\varsigma(C))$  and  $\overline{R}^3(\varsigma(C))$ .  $\square$

Now we are going to describe all  $\overline{G}_{80}$ -invariant curves of degree 12 in  $\mathbb{P}^3$ .

**Lemma 4.26.** *There are exactly eight  $\overline{G}_{80}$ -invariant curves of degree 12 in  $\mathbb{P}^3$ . Moreover, each quartic surface among  $S_1, S_2, S_3, S_4$  contains exactly two such curves. Furthermore, the union of these two curves is cut out on the quartic surface by a  $\overline{G}_{80}$ -invariant sextic surface in  $\mathbb{P}^3$ .*

*Proof.* By Theorem 4.16, it is enough to show that  $S_1$  contains exactly two  $\overline{G}_{80}$ -invariant curves of degree 12, and the union of these two curves is cut out by a  $\overline{G}_{80}$ -invariant sextic surface in  $\mathbb{P}^3$ . In the remaining cases, the proof is the same.

By Lemma 4.24, the surface  $S_1$  contains a  $\overline{G}_{80}$ -invariant curve of degree 12. Denote this curve by  $C_{12}$ , and consider the exact sequence of  $G_{80}$ -representations

$$\begin{aligned} 0 \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^3}(6) \otimes \mathcal{I}_{C_{12}}) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^3}(6)) \longrightarrow \\ \longrightarrow H^0(\mathcal{O}_{C_{12}}(9K_C)) \longrightarrow H^1(\mathcal{O}_{\mathbb{P}^3}(6) \otimes \mathcal{I}_{C_{12}}) \longrightarrow 0, \end{aligned}$$

where  $\mathcal{I}_{C_{12}}$  is the ideal sheaf of the curve  $C_{12}$ . By Lemma 2.22, the vector space  $H^0(\mathcal{O}_{\mathbb{P}^3}(6))$  contains exactly four one-dimensional subrepresentations of  $G_{80}$ , and  $H^0(\mathcal{O}_{C_{12}}(9K_C))$  contains exactly three one-dimensional subrepresentations of the group  $G_{80}$ , which correspond to three  $\overline{G}_{80}$ -invariant effective divisors on  $C$  of degree 72. Hence, there is a  $\overline{G}_{80}$ -invariant sextic surface  $R$  in  $\mathbb{P}^3$  that contains the curve  $C_{12}$ . By Lemma 2.22, the surface  $R$  is irreducible and reduced, so that Theorem 4.16 implies that

$$R|_{S_1} = C_{12} + Z_{12},$$

where  $Z_{12}$  is an irreducible smooth  $\overline{G}_{80}$ -invariant curve of degree 12 and genus 5.

By Remark 4.21, both curves  $C_{12}$  and  $Z_{12}$  contain the singular locus of the surface  $S_1$ . Let  $f: \tilde{S}_1 \rightarrow S_1$  be the minimal resolution of singularities of the surface  $S_1$ . Then the action of the group  $\overline{G}_{80}$  lifts to the surface  $\tilde{S}_1$ . Denote by  $E_1, \dots, E_{16}$  the exceptional curves of the birational morphism  $f$ . Denote by  $\tilde{C}_{12}$  and  $\tilde{Z}_{12}$  the proper transforms of the curves  $C_{12}$  and  $Z_{12}$  on the surface  $\tilde{S}_1$ , respectively. Let  $E = E_1 + \dots + E_{16}$  and let  $H$  be a plane section of the surface  $S_1$ . Then

$$\tilde{C}_{12} + \tilde{Z}_{12} \sim f^*(6H) - E,$$

because both curves  $C_{12}$  and  $Z_{12}$  are smooth. Then  $\tilde{C}_{12} \cdot \tilde{Z}_{12} = 48$ , which implies, in particular, that  $\tilde{C}_{12} \neq \tilde{Z}_{12}$ , because  $\tilde{C}_{12} \cdot \tilde{C}_{12} = \tilde{Z}_{12} \cdot \tilde{Z}_{12} = 8$  by the adjunction formula.

To complete the proof, we have to show that  $S_1$  does not contain  $\overline{G}_{80}$ -invariant curves of degree 12 that are distinct from  $C_{12}$  and  $Z_{12}$ . Suppose that this is not the case, i.e. the surface  $S_1$  contains a  $\overline{G}_{80}$ -invariant curve  $Z$  of degree 12 such that  $Z$  is distinct from the curves  $C_{12}$  and  $Z_{12}$ . Let us seek for a contradiction.

Let  $\tilde{Z}$  be the proper transform of the curve  $Z$  on  $\tilde{S}_1$  via  $f$ . Then  $\tilde{Z} \cdot (\tilde{C}_{12} + \tilde{Z}_{12}) = 56$ . Since the only  $\overline{G}_{80}$ -orbits in  $\tilde{Z}$  are of lengths 16, 40 and 80 by Lemma 4.3, we see that either  $\tilde{Z} \cdot \tilde{C}_{12} = 16$  and  $\tilde{Z} \cdot \tilde{Z}_{12} = 40$ , or  $\tilde{Z} \cdot \tilde{C}_{12} = 40$  and  $\tilde{Z} \cdot \tilde{Z}_{12} = 16$ . Without loss of generality, we may assume that  $\tilde{Z} \cdot \tilde{C}_{12} = 16$  and  $\tilde{Z} \cdot \tilde{Z}_{12} = 40$ . Then

$$\left(\tilde{C}_{12} + f^*(2H) - \tilde{Z}\right)^2 = 0.$$

Thus, it follows from the Riemann–Roch formula and Serre duality that

$$\begin{aligned} h^0\left(\mathcal{O}_{\tilde{S}}(\tilde{C}_{12} + f^*(2H) - \tilde{Z})\right) &= \\ &= 2 + h^1\left(\mathcal{O}_{\tilde{S}}(\tilde{C}_{12} + f^*(2H) - \tilde{Z})\right) - h^2\left(\mathcal{O}_{\tilde{S}}(\tilde{C}_{12} + f^*(2H) - \tilde{Z})\right) = \\ &= 2 + h^1\left(\mathcal{O}_{\tilde{S}}(\tilde{C}_{12} + f^*(2H) - \tilde{Z})\right) - h^0\left(\mathcal{O}_{\tilde{S}}(\tilde{Z} - \tilde{C}_{12} - f^*(2H))\right) = \\ &= 2 + h^1\left(\mathcal{O}_{\tilde{S}}(\tilde{C}_{12} + f^*(2H) - \tilde{Z})\right) \geq 2. \end{aligned}$$

Hence, the linear system  $|\tilde{C}_{12} + f^*(2H) - \tilde{Z}|$  is at least a pencil. Moreover, it does not contain base curves. Indeed, suppose it does. Then

$$\tilde{C}_{12} + f^*(2H) - \tilde{Z} \sim F + M,$$

where  $F$  is the fixed part of the linear system  $|\tilde{C}_{12} + f^*(2H) - \tilde{Z}|$ , and  $M$  is its mobile part. Then

$$0 \leq f(F) \cdot H = F \cdot f^*(H) = 8 - M \cdot f^*(H) < 8.$$

By Theorem 4.16, this implies that  $F$  is contracted by  $f$ , so that  $F = mE$  for some  $m \geq 1$ . Then

$$M^2 = \left(\tilde{C}_{12} + f^*(2H) - \tilde{Z} - mE\right)^2 = -32m^2,$$

which is absurd. This shows that the linear system  $|\tilde{C}_{12} + f^*(2H) - \tilde{Z}|$  is free from base curves. Since

$$(\tilde{C}_{12} + f^*(2H) - \tilde{Z})^2 = 0,$$

we see that it has no base points, so that it is composed of a base point free pencil. Thus, we see that there exists a  $\overline{G}_{80}$ -equivariant morphism  $v: \tilde{S} \rightarrow \mathbb{P}^1$ , whose general fibers are smooth elliptic curves (by the adjunction formula). This is impossible, because  $\overline{H}$  is the only non-trivial proper normal subgroup of the group  $\overline{G}_{80}$  by Lemma 4.1, and the group  $\overline{H}$  does not act faithfully on rational and elliptic curves by Lemma 2.3. This completes the proof of Lemma 4.26.  $\square$

Recall from Remark 4.18 that there are two  $\overline{G}_{80}$ -invariant curves of degree 8 in  $\mathbb{P}^3$ . These curves are just the intersections  $S_1 \cap S_4$  or  $S_2 \cap S_3$  taken with the reduced structure. Furthermore, these curves are smooth curves of genus 5 by Theorem 4.16.

**Proposition 4.27** (cf. [2, 15, 16]). *Let  $C$  be a  $\overline{G}_{80}$ -invariant curve of degree 8 in  $\mathbb{P}^3$ , and let  $\pi: X \rightarrow \mathbb{P}^3$  be a blow up of the curve  $C$ . Then there exists a  $\overline{G}_{80}$ -commutative diagram*

$$(4.28) \quad \begin{array}{ccccc} X & \xrightarrow{\quad \iota \quad} & X & & \\ \pi \downarrow & \searrow \alpha & \swarrow \alpha & & \downarrow \pi \\ & V_8 & \xrightarrow{v} & V_8 & \\ & \pi \downarrow & & & \downarrow \pi \\ \mathbb{P}^3 & \xrightarrow{\quad \tau \quad} & \mathbb{P}^3 & & \end{array}$$

where  $V_8$  is the Fano threefold with terminal Gorenstein singularities such that  $-K_{V_8}^3 = 8$ , the map  $\alpha$  is a flopping contraction, the map  $\tau$  is a birational map,  $v$  is an isomorphism, and  $\iota$  is a composition of flops that acts on  $\text{Pic}(X)$  in the following way:

$$\begin{cases} \iota^*(E) \sim 24\pi^*(H) - 7E, \\ \iota^*(\pi^*(H)) \sim 7\pi^*(H) - 2E. \end{cases}$$

Moreover, the diagram (4.28) is also  $\overline{G}_{160}$ -commutative.

*Proof.* Without loss of generality, we may assume that  $C = S_1 \cap S_4$ . Denote by  $\tilde{S}_1$  and  $\tilde{S}_4$  the proper transforms of the surfaces  $S_1$  and  $S_4$  on the threefold  $X$ , respectively. Then  $\tilde{S}_1$  and  $\tilde{S}_4$  are smooth, and

$$\tilde{S}_1 \sim \tilde{S}_4 \sim -K_X \sim \pi^*(4H) - E.$$

Here  $H$  is a plane in  $\mathbb{P}^3$ , and  $E$  is the  $\pi$ -exceptional divisor. Since  $-K_X^3 = 8$ , we see that the intersection  $\tilde{S}_1 \cap \tilde{S}_4$  is a  $\overline{G}_{80}$ -invariant smooth curve isomorphic to  $C$ . Let us denote this curve by  $\tilde{C}$ . Note that  $\tilde{C}$  is the scheme-theoretic intersection of the surfaces  $\tilde{S}_1$  and  $\tilde{S}_4$ , and

$$\tilde{S}_1 \cdot \tilde{C} = \tilde{S}_4 \cdot \tilde{C} = -K_X^3 = 8,$$

so that  $-K_X$  is nef and big. By the Base Point Free Theorem (see [24, Theorem 1.3.6]), the linear system  $| -nK_X |$  gives a birational morphism  $\alpha: X \rightarrow V_8$  for some  $n \gg 0$ , where  $V_8$  is the Fano threefold with canonical Gorenstein singularities such that  $-K_{V_8}^3 = 8$ .

Let us show that the birational morphism  $\alpha$  is small (cf. [25, Theorems 4.9 and 4.11]). Suppose that  $\alpha$  is not small. Then there exists a prime divisor  $G$  that is contracted by  $\alpha$ . One has  $-K_X^2 \cdot G = 0$ . On the other hand, we have  $G \sim \pi^*(aH) - bE$  for some positive integers  $a$  and  $b$ . Then  $a = 3b$ , since

$$0 = -K_X^2 \cdot G = \left( \pi^*(4H) - E \right) \cdot \left( \pi^*(aH) - bE \right)^2 = 8(a - 3b).$$

If  $\alpha(G)$  is a point, then

$$0 = -K_X \cdot G^2 = 4(a^2 - 4ab + 8b^2) = 20b^2,$$

which is absurd. Thus, we see that  $\alpha(G)$  is a curve, so that the intersection  $G \cap \tilde{S}_1$  contains a curve. Then

$$0 = -K_X^2 \cdot G = G|_{\tilde{S}_1} \cdot \tilde{S}_2|_{\tilde{S}_1} = G|_{\tilde{S}_1} \cdot \tilde{C},$$



which implies that  $\tilde{C}$  is not an ample divisor on the surface  $\tilde{S}_1$ . On the other hand, arguing as in the proof of Corollary 4.22, one can show that the linear system  $|2\tilde{C} - \pi^*(H)|_{\tilde{S}_1}|$  is free from base points, which implies the ampleness of the divisor

$$\pi^*(H)|_{\tilde{S}_1} + (2\tilde{C} - \pi^*(H)|_{\tilde{S}_1}) \sim 2\tilde{C}.$$

The obtained contradiction implies that the birational morphism  $\alpha$  is small.

Since  $\alpha$  is a small birational morphism, the singularities of the threefold  $V_8$  are terminal. Moreover, there exists a  $\overline{G}_{80}$ -commutative diagram

$$(4.29) \quad \begin{array}{ccccc} & X & \overset{\chi}{\dashrightarrow} & Y & \\ \pi \swarrow & & \searrow \alpha & \swarrow \beta & \searrow \phi \\ \mathbb{P}^3 & & V_8 & & Z \end{array}$$

such that  $\chi$  is a composition of flops,  $\beta$  is a small birational morphism, and  $\phi$  is an extremal contraction such that  $-K_Y$  is  $\phi$ -ample. Observe that the threefold  $Y$  is smooth, because the threefold  $X$  is smooth and  $\chi$  is a composition of flops. Now arguing as in the proof of [15, Proposition 4.4], we see that the morphism  $\phi$  is birational, one has  $Z \cong \mathbb{P}^3$ , and  $\phi$  is a blow up of a smooth curve of degree 8 and genus 5. Moreover, this proof also implies that

$$(4.30) \quad \tilde{D} \sim 24\pi^*(H) - 7E,$$

where  $\tilde{D}$  is the proper transform on  $X$  of the  $\phi$ -exceptional surface.

By Corollary 4.2, we may assume that the diagram (4.29) is  $\overline{G}_{80}$ -commutative. Then  $\phi$  is a blow up of one of the two  $\overline{G}_{80}$ -invariant smooth curves of degree 8 and genus 5. By Remark 4.18, the group  $\overline{G}_{320}$  swaps these curves. Thus, composing the map  $\phi$  with an appropriate element in  $\overline{G}_{320}$ , we may assume that  $\phi$  is a blow up of the curve  $C$ . Therefore, we can identify  $X$  with  $Y$  and  $\phi$  with  $\pi$ . This gives the  $\overline{G}_{80}$ -commutative diagram (4.28) with

$$\iota = \chi, \quad \tau = \pi \circ \iota \circ \pi^{-1}, \quad v = \alpha \circ \iota \circ \alpha^{-1}.$$

Since  $V_8$  is an anticanonical model of the threefold  $X$ , the map  $v$  must be biregular. The action of  $\iota$  on  $\text{Pic}(X)$  follows from (4.30), since  $\iota^*(K_X) \sim K_X$ . The diagram (4.28) also  $\overline{G}_{160}$ -commutative, because  $\text{Aut}(C) \cong \overline{G}_{160}$  by Remark 4.18.  $\square$

Using Remark 4.20, one can show that the morphism  $\alpha$  in Proposition 4.27 contracts the proper transforms of the curve  $\mathcal{L}_{10}$ . Using this, one can prove that  $V_8$  in Proposition 4.27 is a complete intersection of three quadrics in  $\mathbb{P}^6$  that has 10 isolated ordinary double points.

## 5. BIRATIONAL RIGIDITY

Let  $G$  be a finite subgroup in  $\text{PGL}_4(\mathbb{C})$ . In this section we prove Theorem 1.1, which is the main result of our paper. To prove Theorem 1.1, we have to find out when  $\mathbb{P}^3$  is  $G$ -birationally rigid. Let us first deal with two cases when  $\mathbb{P}^3$  is not  $G$ -birationally rigid.

**Proposition 5.1.** *Suppose that  $G$  is a transitive subgroup in  $\text{PGL}_4(\mathbb{C})$  such that there exists a  $G$ -orbit in  $\mathbb{P}^3$  of length 4. Then there exists a  $G$ -birational map  $\mathbb{P}^3 \dashrightarrow V_{24}$  such that  $V_{24}$  is a toric Fano threefold, one has  $-K_{V_{24}}^3 = 24$ , the singular locus of  $V_{24}$  consists of 8 quotient singularities of type  $\frac{1}{2}(1, 1, 1)$ . In particular, the singularities of  $V_{24}$  are*

terminal and  $\mathbb{Q}$ -factorial. Moreover, one has  $\text{Pic}(V_{24}) \cong \mathbb{Z}^3$ . Furthermore, if there is no  $G$ -invariant pair of lines in  $\mathbb{P}^3$ , then  $\text{Pic}(V_{24})^G \cong \mathbb{Z}$ . In particular, in the latter case the projective space  $\mathbb{P}^3$  is not  $G$ -birationally rigid.

*Proof.* By assumption, there exists a  $G$ -orbit in  $\mathbb{P}^3$  of length 4. Denote the points of this  $G$ -orbit by  $P_1, P_2, P_3$  and  $P_4$ . Without loss of generality, we may assume that

$$P_1 = [1 : 0 : 0 : 0], \quad P_2 = [0 : 1 : 0 : 0], \quad P_3 = [0 : 0 : 1 : 0], \quad P_4 = [0 : 0 : 0 : 1].$$

For every  $1 \leq i < j \leq 4$ , let  $L_{ij}$  be the line in  $\mathbb{P}^3$  that passes through  $P_i$  and  $P_j$ . Let  $\mathcal{M}$  be the linear system consisting of all sextic surfaces in  $\mathbb{P}^3$  that are singular along every line  $L_{ij}$ . Then the surfaces in  $\mathcal{M}$  are given by

$$(5.2) \quad a_0x^2y^2z^2 + a_1x^2y^2w^2 + a_2x^2z^2w^2 + a_3y^2z^2w^2 + \\ + a_4x^3yzw + a_5x^2y^2zw + a_6x^2yz^2w + a_7x^2yzw^2 + a_8xy^3zw + a_9xy^2z^2w + \\ + a_{10}xy^2zw^2 + a_{11}xyz^3w + a_{12}xyz^2w^2 + a_{13}xyzw^3 = 0$$

for  $[a_0 : \dots : a_{13}] \in \mathbb{P}^{13}$ . The linear system  $\mathcal{M}$  gives a rational map  $\psi: \mathbb{P}^3 \dashrightarrow \mathbb{P}^{13}$ . Note that  $\psi$  is well-defined away from the lines  $L_{12}, L_{13}, L_{14}, L_{23}, L_{24}$  and  $L_{34}$ . Furthermore, the image of the map  $\psi$  is three-dimensional. Denote it by  $V_{24}$ . By construction, the threefold  $V_{24}$  is toric.

Let  $\Pi_1, \Pi_2, \Pi_3$  and  $\Pi_4$  be the planes in  $\mathbb{P}^3$  that are given by  $x = 0, y = 0, z = 0$  and  $w = 0$ , respectively. Then  $\psi$  contracts these planes to 4 different points.

To resolve the indeterminacy of the rational map  $\psi$ , let  $\alpha: X \rightarrow \mathbb{P}^3$  be the blow up of the points  $P_1, P_2, P_3$  and  $P_4$ . Denote by  $\tilde{L}_{ij}$  the proper transform of the line  $L_{ij}$  on the threefold  $X$ . Let  $\beta: W \rightarrow X$  be the blow up of the curves  $\tilde{L}_{12}, \tilde{L}_{13}, \tilde{L}_{14}, \tilde{L}_{23}, \tilde{L}_{24}$  and  $\tilde{L}_{34}$ . Then there exists a  $G$ -commutative diagram:

$$(5.3) \quad \begin{array}{ccccc} & W & \xrightarrow{\quad \rho \quad} & U & \\ \beta \swarrow & & & & \searrow \gamma \\ X & & & & Y \\ \alpha \downarrow & \phi \nearrow & & & \downarrow \delta \\ \mathbb{P}^3 & \xrightarrow{\quad \psi \quad} & & & V_{24} \end{array}$$

such that  $\rho$  is a composition of 12 Atiyah flops, and  $\gamma, \delta$  and  $\phi$  are birational morphisms that we are about to describe.

To describe  $\rho$ , denote by  $\tilde{\Pi}_i$ ,  $1 \leq i \leq 4$ , the proper transform on  $X$  of the plane  $\Pi_i$ . Then each  $\tilde{\Pi}_i$  is a smooth del Pezzo surface of degree 6. Denote by  $E_i$  the exceptional divisor of the birational morphism  $\alpha$  that is mapped to the point  $P_i$ . Let  $\ell_{ij}$  be the intersection curve  $E_i \cap \tilde{\Pi}_j$ , and let  $\bar{\ell}_{ij}$  be its proper transform on the threefold  $W$ . This gives us 12 disjoint curves  $\bar{\ell}_{ij}$ ,  $1 \leq i \neq j \leq 4$ . The map  $\rho$  is a composition of Atiyah flops in these curves.

To describe  $\gamma$  and  $\delta$ , denote by  $\bar{E}_i$  and  $\bar{\Pi}_i$ ,  $1 \leq i \leq 4$ , the proper transforms on  $W$  of the surfaces  $E_i$  and  $\Pi_i$ , respectively. Then all of them are smooth del Pezzo surfaces of degree 6. Moreover, the curves flopped by  $\rho$  are pairwise disjoint  $(-1)$ -curves on these surfaces. Denote by  $\hat{E}_i$  and  $\hat{\Pi}_i$  the proper transforms on  $U$  of the surfaces  $\bar{E}_i$  and  $\bar{\Pi}_i$ , respectively. Then the surfaces  $\hat{E}_1, \hat{E}_2, \hat{E}_3, \hat{E}_4, \hat{\Pi}_1, \hat{\Pi}_2, \hat{\Pi}_3$  and  $\hat{\Pi}_4$  are pairwise disjoint,

all of them are isomorphic to  $\mathbb{P}^2$ , and their normal bundles in  $U$  are  $\mathcal{O}_{\mathbb{P}^2}(-2)$ . The map  $\gamma$  is the contraction of the surfaces  $\widehat{E}_1, \widehat{E}_2, \widehat{E}_3$  and  $\widehat{E}_4$  to 4 quotient singular points of type  $\frac{1}{2}(1, 1, 1)$ . Similarly, the map  $\delta$  is the contraction of the surfaces  $\gamma(\widehat{\Pi}_1), \gamma(\widehat{\Pi}_2), \gamma(\widehat{\Pi}_3)$  and  $\gamma(\widehat{\Pi}_4)$ . Thus, the threefold  $V_{24}$  has exactly 8 singular points, and each of them is a quotient singularity of type  $\frac{1}{2}(1, 1, 1)$ . In particular, the singularities of  $V_{24}$  are terminal and  $\mathbb{Q}$ -factorial.

To describe  $\phi$ , denote by  $\widehat{F}_{12}, \widehat{F}_{13}, \widehat{F}_{14}, \widehat{F}_{23}, \widehat{F}_{24}$  and  $\widehat{F}_{34}$  the proper transforms on  $Y$  of the  $\beta$ -exceptional surfaces that are mapped to the curves  $\widetilde{L}_{12}, \widetilde{L}_{13}, \widetilde{L}_{14}, \widetilde{L}_{23}, \widetilde{L}_{24}$  and  $\widetilde{L}_{34}$ , respectively. Then  $\phi$  is the contraction of these six surfaces to the lines  $L_{12}, L_{13}, L_{14}, L_{23}, L_{24}$  and  $L_{34}$ , respectively. By [27, Theorem 4.9], the morphism  $\phi$  is the *symbolic blow up* of the reducible curve  $L_{12} + L_{13} + L_{14} + L_{23} + L_{24} + L_{34}$ , see also [37, §6.1].

By construction, the threefold  $V_{24}$  is toric Fano threefold, one has  $-K_{V_{24}}^3 = 24$  and  $\text{Pic}(V_{24}) \cong \mathbb{Z}^3$ . If there is no  $G$ -invariant pair of lines in  $\mathbb{P}^3$ , then the group  $G$  permutes the six lines  $L_{ij}$  transitively. This gives  $\text{Pic}(V_{24})^G \cong \mathbb{Z}$ .  $\square$

In the notation of [5], the threefold  $V_{24}$  in Proposition 5.1 is the terminal toric Fano threefold  $\mathbb{N}^{\circ}47$ . One can show that a general hyperplane section of the threefold  $V_{24} \subset \mathbb{P}^{13}$  is a smooth Enriques surface. Of course, this is well-known: the equation (5.2) was found by Enriques to construct what is now known as an Enriques surface. The variety  $V_{24}$  is one of the Fano–Enriques threefolds classified by Bayle in [1]. To be precise, the threefold  $V_{24}$  is a quotient of  $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$  by an involution  $\tau$  that acts as

$$\left([x_1 : y_1], [x_2 : y_2], [x_3 : y_3]\right) \mapsto \left([y_1 : x_1], [y_2 : x_2], [y_3 : x_3]\right).$$

**Proposition 5.4.** *Suppose that  $G$  is a primitive subgroup in  $\text{PGL}_4(\mathbb{C})$  such that  $G \cong \mathfrak{S}_5$ , and  $G$  leaves invariant a pair of disjoint twisted cubic curves. Denote them by  $C_1$  and  $C_2$ . Let  $\pi: X \rightarrow \mathbb{P}^3$  be the blow up of the curves  $C_1$  and  $C_2$ . Then there is a  $G$ -commutative diagram*

$$\begin{array}{ccccc} & X & \overset{\chi}{\dashrightarrow} & Y & \\ \pi \swarrow & & \searrow \alpha & \swarrow \beta & \searrow \phi \\ \mathbb{P}^3 & & V_{12} & & \mathbb{P}^1 \times \mathbb{P}^1 \end{array}$$

Here  $V_{12}$  is a divisor of bi-degree  $(2, 2)$  in  $\mathbb{P}^2 \times \mathbb{P}^2$  with ten isolated terminal Gorenstein singularities,  $\alpha$  is a small birational morphism that contracts proper transforms of ten common secants of the curves  $C_1$  and  $C_2$ , the map  $\chi$  is a composition of ten flops, the morphism  $\beta$  is a flopping contraction, the morphism  $\phi$  is a  $\mathbb{P}^1$ -bundle, and the action of the group  $G$  on  $\mathbb{P}^1 \times \mathbb{P}^1$  is faithful. In particular, the projective space  $\mathbb{P}^3$  is not  $G$ -birationally rigid.

*Proof.* Denote by  $\Gamma$  the subgroup in  $G$  isomorphic to  $\mathfrak{A}_5$ . Let  $\pi_1: X_1 \rightarrow \mathbb{P}^3$  be the blow up of the curve  $C_1$ , and let  $\pi_2: X_2 \rightarrow \mathbb{P}^3$  be the blow up of the curve  $C_2$ . Then  $X_1$  and  $X_2$  are isomorphic Fano threefolds. Moreover, it follows from Example 1.4 that there is a

$\Gamma$ -commutative diagram

$$\begin{array}{ccccc}
 & X_1 & \xleftarrow{\omega_2} & X & \xrightarrow{\omega_1} & X_2 \\
 \phi_1 \swarrow & & \searrow \pi_1 & \downarrow \pi & \swarrow \pi_2 & \searrow \phi_2 \\
 \mathbb{P}^2 & \xleftarrow{\theta_1} & \mathbb{P}^3 & \xrightarrow{\theta_2} & \mathbb{P}^2
 \end{array}$$

Here  $\phi_1$  and  $\phi_2$  are  $\mathbb{P}^1$ -bundles,  $\theta_1$  and  $\theta_2$  are rational maps that are given by the linear systems of quadrics passing through  $C_1$  and  $C_2$ , respectively, and  $\omega_1$  and  $\omega_2$  are blow ups of proper transforms of the curves  $C_1$  and  $C_2$ , respectively. Note that the action of  $\Gamma$  on  $\mathbb{P}^2$  is faithful.

We claim that there are exactly 10 lines in  $\mathbb{P}^3$  that are common secants of both twisted cubic curves  $C_1$  and  $C_2$  (cf. [45, §2] or [18, Remark 2]). Indeed, denote by  $E_1^1$  the  $\pi_1$ -exceptional surface. Then the morphism  $\phi_1$  is given by the linear system  $|\pi_1^*(\mathcal{O}_{\mathbb{P}^2}(2)) - E_1^1|$ , and its fibers are proper transforms of the secant lines of the curve  $C_1$ . Denote by  $\overline{C}_2$  the proper transform of the curve  $C_2$  on the threefold  $X_1$ . Since  $\overline{C}_2$  is  $\Gamma$ -invariant, the curve  $\phi_1(\overline{C}_2)$  is also  $\Gamma$ -invariant. By [10, Lemma 5.3.1], there are no  $\Gamma$ -invariant irreducible curves in  $\mathbb{P}^2$  of degree 1 and 3. Thus we see that either  $\phi_1(\overline{C}_2)$  is an irreducible conic, or  $\phi_1(\overline{C}_2)$  is an irreducible curve of degree 6. The former case is clearly impossible. Indeed, if  $\phi_1(\overline{C}_2)$  is a conic, then the induced morphism  $\overline{C}_2 \rightarrow \phi_1(\overline{C}_2)$  is a triple cover, so that the curve  $C_2$  has infinitely many 3-secant lines, which are also secants of the curve  $C_1$ . However, twisted cubic curves in  $\mathbb{P}^2$  do not have 3-secant lines at all, because they are cut out by quadrics. Therefore, we see that  $\phi_1(\overline{C}_2)$  is an irreducible curve of degree 6, so that the induced morphism  $\overline{C}_2 \rightarrow \phi_1(\overline{C}_2)$  is birational. By [10, Lemma 5.3.1], the sextic curve  $\phi_1(\overline{C}_2)$  is contained in the pencil that consists of all  $\Gamma$ -invariant curves of degree 6 in  $\mathbb{P}^2$ . The curve  $\phi_1(\overline{C}_2)$  must be singular, since the curve  $C_2$  is rational. All singular curves in this pencil are described in [10, Remark 6.1.5]. Using this description, we deduce that  $\phi_1(\overline{C}_2)$  has 10 double points (cf. [18, Remark 5]), and these singular points form one  $\Gamma$ -orbit in  $\mathbb{P}^2$ . Note that the singular points of  $\phi_1(\overline{C}_2)$  are ordinary double points. Indeed, since the arithmetic genus of the curve  $\phi_1(\overline{C}_2)$  equals 10, its singular points are either ordinary double points or ordinary cusps. In the latter case, the curve  $\overline{C}_2$  would have a  $\Gamma$ -orbit of length 10, which is impossible by [10, Lemma 5.1.5]. Thus, every singular point of the curve  $\phi_1(\overline{C}_2)$  is an ordinary double point. The fibers of  $\phi_1$  over the singular points of the curve  $\phi_1(\overline{C}_2)$  are exactly the proper transforms of the common secants of both curves  $C_1$  and  $C_2$ . Vice versa, if  $\ell$  is a common secant of the curves  $C_1$  and  $C_2$ , then  $\theta_1(\ell)$  must be a singular point of the curve  $\phi_1(\overline{C}_2)$ . Hence, we conclude that there are exactly 10 lines in  $\mathbb{P}^3$  that are secants of both  $C_1$  and  $C_2$ , and the group  $\Gamma$  acts transitively on them.

Denote by  $E_1^1$  the  $\pi_1$ -exceptional surface, and denote by  $E_2^2$  the  $\pi_2$ -exceptional surface. Then

$$-K_X \sim \omega_2^*(\pi_1^*(\mathcal{O}_{\mathbb{P}^2}(2)) - E_1^1) + \omega_1^*(\pi_2^*(\mathcal{O}_{\mathbb{P}^2}(2)) - E_2^2).$$

Moreover, both divisors  $\pi_1^*(\mathcal{O}_{\mathbb{P}^2}(2)) - E_1^1$  and  $\pi_2^*(\mathcal{O}_{\mathbb{P}^2}(2)) - E_2^2$  are nef. In particular, the divisor  $-K_X$  is nef and big, since  $-K_X^3 = 12$ .

Denote the 10 common secants of both  $C_1$  and  $C_2$  by  $L_1, \dots, L_{10}$ , and denote their proper transforms on  $X$  by  $\tilde{L}_1, \dots, \tilde{L}_{10}$ , respectively. Then  $-K_X$  intersects these curves trivially. Moreover, the curves  $\tilde{L}_1, \dots, \tilde{L}_{10}$  are the only irreducible curves in  $X$  that have

trivial intersection with  $-K_X$ . Indeed, let  $C$  be an irreducible curve in the threefold  $X$  such that  $-K_X \cdot C = 0$ . Then

$$\omega_2(C) \cdot (\pi_1^*(\mathcal{O}_{\mathbb{P}^2}(2)) - E_1^1) = \omega_1(C) \cdot (\pi_2^*(\mathcal{O}_{\mathbb{P}^2}(2)) - E_2^2) = 0,$$

and  $\pi(C)$  is an irreducible curve. Since  $\phi_1$  is given by the linear system  $|\pi_1^*(\mathcal{O}_{\mathbb{P}^2}(2)) - E_1^1|$ , we see that  $\omega_2(C)$  is contracted by  $\phi_1$  to a point. Similarly, we see that  $\omega_1(C)$  is contracted by  $\phi_2$  to a point. Hence, the curve  $\pi(C)$  is a common secant of both curves  $C_1$  and  $C_2$ , which implies that  $C$  is one of the curves  $\tilde{L}_1, \dots, \tilde{L}_{10}$ .

Consider the morphism

$$\alpha = \pi_1 \circ \omega_2 = \pi_2 \circ \omega_1.$$

We already know that the image of  $\alpha$  is three-dimensional. Now we claim that  $\alpha$  is birational. Indeed, choose a general fiber of  $\alpha$ , and suppose that it contains at least two different points, say  $A$  and  $A'$ . Then  $\omega_1(A) = \omega_1(A')$ , so that  $\pi(A)$  and  $\pi(A')$  lie on a secant of the twisted cubic  $C_2$ . Similarly, we have  $\omega_2(A) = \omega_2(A')$ , so that  $\pi(A)$  and  $\pi(A')$  lie on a secant of  $C_1$ . On the other hand, since the fiber of  $\alpha$  was chosen to be general, we can assume that the points  $\pi(A)$  and  $\pi(A')$  are different points of  $\mathbb{P}^3$ . Hence they lie on one of the lines  $L_1, \dots, L_{10}$ , which gives a contradiction.

We have constructed the morphism  $\alpha: X \rightarrow V_{12}$ , where  $V_{12}$  is a divisor of bi-degree  $(2, 2)$  in  $\mathbb{P}^2 \times \mathbb{P}^2$ . The variety  $V_{12}$  is an anticanonical model of the threefold  $X$ , and the morphism  $\alpha$  is given by the linear system  $|-K_X|$ . The threefold  $V_{12}$  has 10 isolated terminal Gorenstein singular points, which are the images of the curves  $\tilde{L}_1, \dots, \tilde{L}_{10}$ . Thus, there exists a  $G$ -commutative diagram

$$\begin{array}{ccccc} & X & \overset{\chi}{\dashrightarrow} & Y & \\ \pi \swarrow & & \searrow \alpha & \swarrow \beta & \searrow \phi \\ \mathbb{P}^3 & \dashrightarrow_{\psi} & V_{12} & & Z \end{array}$$

Here  $\psi$  is a birational map that is given by the linear system of quartic surfaces in  $\mathbb{P}^3$  that contain both curves  $C_1$  and  $C_2$ , and  $\chi$  is a composition of flops in the curves  $\tilde{L}_1, \dots, \tilde{L}_{10}$ . The map  $\beta$  is a flopping contraction, and the morphism  $\phi$  is a  $G$ -extremal contraction such that the divisor  $-K_Y$  is  $\phi$ -ample. Now arguing as in [43, 16] and using [32, Theorem 3.3] together with [14, Theorem 4], we see that  $\phi$  is a  $\mathbb{P}^1$ -bundle, and  $Z \cong \mathbb{P}^1 \times \mathbb{P}^1$ . Finally, using the fact that our  $\mathbb{P}^3$  contains exactly two  $\Gamma$ -orbits of length 12 by [9, Lemma 3.2], we conclude that the group  $G$  acts faithfully on  $Z$ .  $\square$

In fact, one can show that the variety  $V_{12}$  in Proposition 5.4 has isolated ordinary double points, and  $\chi$  is a composition of Atiyah flops. Moreover, one can show that a general fiber of the rational map  $\phi \circ \chi \circ \pi^{-1}$  is a twisted cubic curve  $Z$  such that both intersections  $Z \cap C_1$  and  $Z \cap C_2$  consist of 5 points.

To prove Theorem 1.1, we have to show that  $\mathbb{P}^3$  is  $G$ -birationally rigid if and only if  $G$  is a primitive subgroup in  $\mathrm{PGL}_4(\mathbb{C})$  that is not isomorphic to  $\mathfrak{A}_5$  or  $\mathfrak{S}_5$ . Actually, we will also prove one more closely related result. Recall from [10, Definition 3.1.1] that  $\mathbb{P}^3$  is said to be  $G$ -birationally super-rigid if it is  $G$ -birationally rigid and

$$\mathrm{Bir}^G(\mathbb{P}^3) = \mathrm{Aut}^G(\mathbb{P}^3),$$

where  $\text{Bir}^G(\mathbb{P}^3)$  and  $\text{Aut}^G(\mathbb{P}^3)$  are the normalizers of the group  $G$  in  $\text{Bir}(\mathbb{P}^3)$  and  $\text{Aut}(\mathbb{P}^3)$ , respectively. In addition to Theorem 1.1, we will prove the following

**Theorem 5.5.** *The projective space  $\mathbb{P}^3$  is  $G$ -birationally super-rigid if and only if  $G$  is a primitive subgroup that is isomorphic neither to  $\mathfrak{A}_5$ , nor to  $\mathfrak{S}_5$ , nor to  $\text{PSL}_2(\mathbf{F}_7)$ , nor to  $\mathfrak{A}_6$ , nor to  $\mu_2^4 \rtimes \mu_5$ , nor to  $\mu_2^4 \rtimes D_{10}$ .*

To prove Theorems 1.1 and 5.5, we will use the following well-known result, which goes back to the classical works of Noether, Fano, and Iskovskikh.

**Theorem 5.6** ([10, Theorem 3.3.1]). *Let  $\Gamma$  be a subset in  $\text{Bir}^G(\mathbb{P}^3)$ . Suppose that for every non-empty  $G$ -invariant mobile linear system  $\mathcal{D}$  on  $\mathbb{P}^3$ , there exists  $\gamma \in \Gamma$  such that the log pair  $(\mathbb{P}^3, \frac{4}{n}\gamma(\mathcal{D}))$  has canonical singularities, where  $n$  be a positive integer that is defined by  $\gamma(\mathcal{D}) \sim \mathcal{O}_{\mathbb{P}^3}(n)$ . Then  $\mathbb{P}^3$  is  $G$ -birationally rigid, and the group  $\text{Bir}^G(\mathbb{P}^3)$  is generated by  $\Gamma$  and  $\text{Aut}^G(\mathbb{P}^3)$ .*

**Corollary 5.7.** *Suppose that for every non-empty  $G$ -invariant mobile linear system  $\mathcal{D}$  on  $\mathbb{P}^3$ , the log pair  $(\mathbb{P}^3, \frac{4}{n}\mathcal{D})$  has canonical singularities, where  $n$  be a positive integer that is defined by  $\mathcal{D} \sim \mathcal{O}_{\mathbb{P}^3}(n)$ . Then  $\mathbb{P}^3$  is  $G$ -birationally super-rigid.*

The following simple fact is useful for the proof of  $G$ -birational rigidity of  $\mathbb{P}^3$ .

**Lemma 5.8.** *Let  $\mathcal{D}$  be a non-empty  $G$ -invariant mobile linear system on  $\mathbb{P}^3$ , and let  $Z$  be a  $G$ -irreducible curve in  $\mathbb{P}^3$ . Then*

$$\text{mult}_Z(\mathcal{D}) \leq \frac{n}{\sqrt{\deg(Z)}},$$

where  $n$  is a positive integer such that  $\mathcal{D} \sim \mathcal{O}_{\mathbb{P}^3}(n)$ .

*Proof.* Let  $D_1$  and  $D_2$  be two general surfaces in  $\mathcal{D}$ . Then

$$D_1 \cdot D_2 = mZ + \Delta,$$

where  $m$  is a positive integer such that  $m \geq \text{mult}_Z^2(\mathcal{D})$ , and  $\Delta$  is an effective one-cycle on  $\mathbb{P}^3$  such that  $\text{Supp}(\Delta)$  does not contain irreducible components of the curve  $Z$ . Let  $H$  be a general plane in  $\mathbb{P}^3$ . Then

$$n^2 = H \cdot D_1 \cdot D_2 = H \cdot (mZ + \Delta) \geq m \cdot \deg(Z) \geq \text{mult}_Z^2(\mathcal{D}) \cdot \deg(Z),$$

which implies the required inequality.  $\square$

To illustrate Lemma 5.8 and Corollary 5.7, let us prove

**Lemma 5.9.** *Suppose that  $G$  contains a proper subgroup isomorphic to  $\mathfrak{A}_6$ . Then  $\mathbb{P}^3$  is  $G$ -birationally super-rigid.*

*Proof.* Let  $\mathcal{D}$  be a non-empty  $G$ -invariant mobile linear system on  $\mathbb{P}^3$ . Then  $\mathcal{D} \sim \mathcal{O}_{\mathbb{P}^3}(n)$  for some positive integer  $n$ . By Corollary 5.7, to complete the proof, we have to show that the singularities of the log pair  $(\mathbb{P}^3, \frac{4}{n}\mathcal{D})$  are canonical. By assumption, the group  $G$  contains a subgroup isomorphic to  $\mathfrak{A}_6$ . Denote it by  $\Theta$ . It follows from [8, §4] that there are two  $\Theta$ -irreducible curves  $\mathcal{L}_6$  and  $\mathcal{L}'_6$  in  $\mathbb{P}^3$  such that  $\mathcal{L}_6 \cap \mathcal{L}'_6 = \emptyset$ , and each of them is a disjoint union of 6 lines. Moreover, it follows from [8, Theorem 4.3] that the log pair  $(\mathbb{P}^3, \frac{4}{n}\mathcal{D})$  has canonical singularities provided that

$$\max\left\{\text{mult}_{\mathcal{L}_6}(\mathcal{D}), \text{mult}_{\mathcal{L}'_6}(\mathcal{D})\right\} \leq \frac{n}{4}.$$

On the other hand, it follows from the proof of [8, Theorem 4.9] that

$$\min \left\{ \text{mult}_{\mathcal{L}_6}(\mathcal{D}), \text{mult}_{\mathcal{L}'_6}(\mathcal{D}) \right\} \leq \frac{n}{4}.$$

Note that  $\Theta$  is a primitive subgroup in  $\text{PGL}_4(\mathbb{C})$ , so that  $G$  is primitive as well. Thus, it follows from Blichfeldt's classification [4, Chapter VII] that  $G$  is one of the following groups:  $\mathfrak{S}_6$ ,  $\mathfrak{A}_7$ , or a simple group of order 25920. If  $G \cong \mathfrak{S}_6$ , then the log pair  $(\mathbb{P}^3, \frac{4}{n}\mathcal{D})$  has canonical singularities, because  $G$  permutes the curves  $\mathcal{L}_6$  and  $\mathcal{L}'_6$  in this case, so that

$$\text{mult}_{\mathcal{L}_6}(\mathcal{D}) = \text{mult}_{\mathcal{L}'_6}(\mathcal{D}).$$

Similarly, if  $G$  is a simple group of order 25920, then the log pair  $(\mathbb{P}^3, \frac{4}{n}\mathcal{D})$  also has canonical singularities, because this group contains  $\mathfrak{S}_6$  as a subgroup (see [11, p. 26]).

To complete the proof, we may assume that  $G \cong \mathfrak{A}_7$ . Let  $Z$  be a  $G$ -orbit of the curve  $\mathcal{L}_6$ . Then  $\deg(Z) = 42$ , so that  $\text{mult}_{\mathcal{L}_6}(\mathcal{D}) \leq \frac{n}{4}$  by Lemma 5.8. Similarly, we see that  $\text{mult}_{\mathcal{L}'_6}(\mathcal{D}) \leq \frac{n}{4}$ . Thus, it follows from [8, Theorem 4.3] that the log pair  $(\mathbb{P}^3, \frac{4}{n}\mathcal{D})$  has canonical singularities.  $\square$

*Remark 5.10.* Let  $\Theta$  be a subgroup of the group  $G$ . If  $\mathbb{P}^3$  is  $\Theta$ -birationally super-rigid, then it is also  $G$ -birationally super-rigid. This follows from Corollary 5.7 and equivariant Minimal Model Program (see [6, Corollary 1.4.3]). However, if  $\mathbb{P}^3$  is  $\Theta$ -birationally rigid, then we cannot immediately conclude that  $\mathbb{P}^3$  is  $G$ -birationally rigid. A priori, the projective space  $\mathbb{P}^3$  can be  $G$ -birational to a Fano threefold  $X$  with terminal singularities such that the  $G$ -invariant class group of the threefold  $X$  is of rank 1, but the  $\Theta$ -invariant class group is not of rank 1. We refer the reader to [26] for a discussion of a similar problem for the action of Galois groups.

Now we will state two technical propositions and use them to prove Theorems 1.1 and 5.5. After this, we will present the proofs of these propositions. Let  $\overline{G}_{144}$  be the primitive subgroup in  $\text{PGL}_4(\mathbb{C})$  constructed in §2 and dealt with in §3.

**Proposition 5.11.** *Let  $\mathcal{D}$  be a non-empty  $\overline{G}_{144}$ -invariant mobile linear system on  $\mathbb{P}^3$ , and let  $n$  be a positive integer such that  $\mathcal{D} \sim \mathcal{O}_{\mathbb{P}^3}(n)$ . Then the log pair  $(\mathbb{P}^3, \frac{4}{n}\mathcal{D})$  has canonical singularities.*

Proposition 5.11 and Corollary 5.7 imply the following:

**Corollary 5.12.** *Suppose that  $G$  contains  $\overline{G}_{144}$ . Then  $\mathbb{P}^3$  is  $G$ -birationally super-rigid.*

Similarly, let  $\overline{G}_{80}$  and  $\overline{G}_{160}$  be primitive subgroups in  $\text{PGL}_4(\mathbb{C})$  constructed in §2 and dealt with in §4. Let  $\mathcal{C}_1$  and  $\mathcal{C}_2$  be two distinct  $\overline{G}_{80}$ -invariant irreducible smooth curves of degree 8 and genus 5 in  $\mathbb{P}^3$  constructed in §4.

**Proposition 5.13.** *Let  $\mathcal{D}$  be a non-empty  $\overline{G}_{80}$ -invariant mobile linear system on  $\mathbb{P}^3$ , and let  $n$  be a positive integer such that  $\mathcal{D} \sim \mathcal{O}_{\mathbb{P}^3}(n)$ . Suppose that*

$$\max \left\{ \text{mult}_{\mathcal{C}_1}(\mathcal{D}), \text{mult}_{\mathcal{C}_2}(\mathcal{D}) \right\} \leq \frac{n}{4}.$$

*Then the log pair  $(\mathbb{P}^3, \frac{4}{n}\mathcal{D})$  has canonical singularities.*

Proposition 5.13 and Theorem 5.6 imply the following:

**Corollary 5.14.** *Suppose that  $G = \overline{G}_{80}$  or  $G = \overline{G}_{160}$ . Then  $\mathbb{P}^3$  is  $G$ -birationally rigid.*

*Proof.* Let  $\Gamma$  be a subgroup in  $\text{Bir}^G(\mathbb{P}^3)$  that is generated by the birational maps constructed in Proposition 4.27. Let  $\mathcal{D}$  be a non-empty  $G$ -invariant mobile linear system on  $\mathbb{P}^3$ . For every  $\sigma \in \Gamma$ , let  $n_\sigma$  be a positive integer such that  $\sigma(\mathcal{D}) \sim \mathcal{O}_{\mathbb{P}^3}(n_\sigma)$ . Then there is  $\gamma \in \Gamma$  such that

$$n_\gamma = \min \left\{ n_\sigma \mid \sigma \in \Gamma \right\}.$$

Let  $n = n_\gamma$  and  $\mathcal{M} = \gamma(\mathcal{D})$ . By Theorem 5.6, to prove the required assertion, it is enough to show that the log pair  $(\mathbb{P}^3, \frac{4}{n}\mathcal{M})$  has canonical singularities. Suppose that this is not true. By Proposition 5.13, either  $\text{mult}_{\mathcal{C}_1}(\mathcal{M}) > \frac{n}{4}$  or  $\text{mult}_{\mathcal{C}_2}(\mathcal{M}) > \frac{n}{4}$ . In the former case, we let  $C = \mathcal{C}_1$ . In the latter case, we let  $C = \mathcal{C}_2$ . Let  $\tau$  be the birational map constructed in Proposition 4.27 starting from the curve  $C$ . Then  $\tau \in \Gamma$  and

$$\tau(\mathcal{M}) \sim \mathcal{O}_{\mathbb{P}^3}(7n - 24\text{mult}_C(\mathcal{M}))$$

by Proposition 4.27. This contradicts the minimality of  $n$ , since  $7n - 24\text{mult}_C(\mathcal{D}) < n$ .  $\square$

To prove Theorems 1.1 and 5.5, we need the last auxiliary result.

**Lemma 5.15.** *Suppose that the group  $G$  contains  $\overline{G}_{80}$ , and that  $G$  does not coincide with  $\overline{G}_{80}$  and  $\overline{G}_{160}$ . Then  $\mathbb{P}^3$  is  $G$ -birationally super-rigid.*

*Proof.* Let  $\mathcal{D}$  be a non-empty  $G$ -invariant mobile linear system on  $\mathbb{P}^3$ . Then  $\mathcal{D} \sim \mathcal{O}_{\mathbb{P}^3}(n)$  for some positive integer  $n$ . By Corollary 5.7, to complete the proof, we have to show that the singularities of the log pair  $(\mathbb{P}^3, \frac{4}{n}\mathcal{D})$  are canonical. This follows from Proposition 5.13 provided that

$$\max \left\{ \text{mult}_{\mathcal{C}_1}(\mathcal{D}), \text{mult}_{\mathcal{C}_2}(\mathcal{D}) \right\} \leq \frac{n}{4},$$

because  $\mathcal{D}$  is also  $\overline{G}_{80}$ -invariant. Let us prove the latter inequality.

Let  $Z$  be a  $G$ -orbit of the curve  $\mathcal{C}_1$ . Recall from §4 that  $\mathcal{C}_1$  is  $\overline{G}_{160}$ -invariant. Moreover, it follows from Remark 4.18 that  $\overline{G}_{160}$  is the full automorphism group of the curve  $\mathcal{C}_1$ . Thus, we have  $\deg(Z) \geq 2\deg(\mathcal{C}_1) = 16$ , so that  $\text{mult}_{\mathcal{C}_1}(\mathcal{D}) \leq \frac{n}{4}$  by Lemma 5.8. Similarly, we see that  $\text{mult}_{\mathcal{C}_2}(\mathcal{D}) \leq \frac{n}{4}$ . Therefore, the log pair  $(\mathbb{P}^3, \frac{4}{n}\mathcal{D})$  has canonical singularities by Proposition 5.13.  $\square$

Now we are ready to prove the main results of our paper.

*Proof of Theorems 1.1 and 5.5.* If  $G$  is intransitive, then  $\mathbb{P}^3$  is  $G$ -birational to a  $\mathbb{P}^1$ -bundle over  $\mathbb{P}^2$  or to a  $\mathbb{P}^2$ -bundle over  $\mathbb{P}^1$ , so that  $\mathbb{P}^3$  is not  $G$ -birationally rigid. If there is a  $G$ -invariant union of two skew lines in  $\mathbb{P}^3$ , then  $\mathbb{P}^3$  is  $G$ -birational to a  $\mathbb{P}^1$ -bundle over  $\mathbb{P}^1 \times \mathbb{P}^1$ . Thus, if  $G$  is imprimitive, then  $\mathbb{P}^3$  is not  $G$ -birationally rigid by Proposition 5.1. Therefore, we may assume that  $G$  is a primitive subgroup in  $\text{PGL}_4(\mathbb{C})$ . Now we use the classification provided by Blichfeldt in [4, Chapter VII].

If  $G \cong \mathfrak{A}_5$  and there exists a  $G$ -invariant quadric in  $\mathbb{P}^3$ , then  $\mathbb{P}^3$  is not  $G$ -birationally rigid by Examples 1.3, 1.5, or 1.6. If  $G \cong \mathfrak{A}_5$  and there exists a  $G$ -invariant twisted cubic curve in  $\mathbb{P}^3$ , then  $\mathbb{P}^3$  is not  $G$ -birationally rigid by Example 1.4. If  $G \cong \mathfrak{S}_5$  and there exists a  $G$ -invariant quadric in  $\mathbb{P}^3$ , then  $\mathbb{P}^3$  is not  $G$ -birationally rigid by Examples 1.5 or 1.6. If  $G \cong \mathfrak{S}_5$  and there exists a  $G$ -invariant pair of twisted cubic curves in  $\mathbb{P}^3$ , then  $\mathbb{P}^3$  is not  $G$ -birationally rigid by Proposition 5.4.

If  $G \cong \text{PSL}_2(\mathbf{F}_7)$ , then  $\mathbb{P}^3$  is  $G$ -birationally rigid and is not  $G$ -birationally super-rigid by [7, Theorem 1.9]. Similarly, if  $G \cong \mathfrak{A}_6$ , then  $\mathbb{P}^3$  is  $G$ -birationally rigid and is not



$G$ -birationally super-rigid by [8, Theorem 1.24]. Moreover, if  $G$  contains  $\mathfrak{A}_6$  as a proper subgroup, then  $\mathbb{P}^3$  is  $G$ -birationally super-rigid by Lemma 5.9.

Recall from [4, Chapter VII] that up to conjugation all remaining finite primitive subgroups of  $\mathrm{PGL}_4(\mathbb{C})$  belongs to one of the following two classes. The first class consists of primitive groups that leave a smooth quadric surface in  $\mathbb{P}^3$  invariant and contain  $\overline{G}_{144}$  (see [4, §121]). The second class consists of primitive groups containing an imprimitive normal subgroup  $\mu_2^4$  (see [4, §124]). These classes overlap. For instance, the group  $\overline{G}_{144} \cong \mu_2^4 \rtimes (\mu_3 \times \mu_3)$  itself is contained in both of them (see [4, §125]). However, if  $G$  is in the second class and is not in the first class, then  $G$  must contain the subgroup  $\overline{G}_{80}$  (see [4, §124]).

Now we proceed with  $G$ -birational rigidity. If the group  $G$  contains  $\overline{G}_{144}$  as a subgroup, then the projective space  $\mathbb{P}^3$  is  $G$ -birationally super-rigid by Corollary 5.12. If  $G = \overline{G}_{80}$  or  $G = \overline{G}_{160}$ , then  $\mathbb{P}^3$  is  $G$ -birationally rigid and is not  $G$ -birationally super-rigid by Corollary 5.14. Finally, if  $G$  contains  $\overline{G}_{80}$  and neither  $G = \overline{G}_{80}$  nor  $G = \overline{G}_{160}$ , then  $\mathbb{P}^3$  is  $G$ -birationally super-rigid by Lemma 5.15.  $\square$

Thus, to complete the proofs of our main results, it is enough to prove Propositions 5.11 and 5.13. This is what we are going to do in the remaining part of this section. Namely, suppose that either  $G = \overline{G}_{144}$  or  $G = \overline{G}_{80}$ . Let  $\mathcal{D}$  be a  $G$ -invariant mobile linear system on  $\mathbb{P}^3$ . Then  $\mathcal{D} \sim \mathcal{O}_{\mathbb{P}^3}(n)$  for a positive integer  $n$ . If  $G = \overline{G}_{80}$ , we also suppose that

$$(5.16) \quad \max\{\mathrm{mult}_{C_1}(\mathcal{D}), \mathrm{mult}_{C_2}(\mathcal{D})\} \leq \frac{n}{4}.$$

To prove Propositions 5.11 and 5.13, we have to show that the singularities of the log pair  $(\mathbb{P}^3, \frac{4}{n}\mathcal{D})$  are canonical. Suppose that this is not true. Let us show that this assumption leads to a contradiction.

**Lemma 5.17.** *Let  $C$  be a  $G$ -irreducible curve in  $\mathbb{P}^3$ . Then  $\mathrm{mult}_C(\mathcal{D}) \leq \frac{n}{4}$ .*

*Proof.* By Lemma 5.8, we may assume that  $\deg(Z) \leq 15$ . If  $G = \overline{G}_{144}$ , the assertion follows from Theorem 3.22 together with Corollaries 3.8 and 3.21. If  $G = \overline{G}_{80}$ , the assertion follows from (5.16), Theorem 4.16, Lemma 4.11 and Corollary 4.22.  $\square$

Thus, the log pair  $(\mathbb{P}^3, \frac{4}{n}\mathcal{D})$  is canonical away from finitely many points in  $\mathbb{P}^3$  (see, for instance, [13, Proposition 5.14]). On the other hand, this log pair is not canonical at some point  $P \in \mathbb{P}^3$  by assumption. Denote by  $\Sigma$  the  $G$ -orbit of the point  $P$ . Let  $D_1$  and  $D_2$  be two sufficiently general surfaces in  $\mathcal{D}$ . Then

$$(5.18) \quad \mathrm{mult}_O(D_1 \cdot D_2) > \frac{n^2}{4}$$

for every point  $O \in \Sigma$  by [12, Corollary 3.4].

For every  $r \geq 2$ , denote by  $\mathcal{M}_r$  the linear system of surfaces of degree  $r$  in  $\mathbb{P}^3$  that contain  $\Sigma$ . If  $\mathcal{M}_r$  is not empty, let  $M_r$  be a general surface in  $\mathcal{M}_r$ . In this case (5.18) implies that

$$rn^2 = M_r \cdot D_1 \cdot D_2 \geq \sum_{O \in \Sigma} \mathrm{mult}_O(D_1 \cdot D_2) > \frac{n^2}{4}|\Sigma|$$

provided that  $M_r$  does not contain any irreducible component of the one-cycle  $D_1 \cdot D_2$ . Thus, if the base locus of the linear system  $\mathcal{M}_r$  does not contain curves, then (5.18)

implies the inequality

$$(5.19) \quad |\Sigma| < 4r.$$

We expect that the base locus of the linear system  $\mathcal{M}_r$  contains no curves for  $r = \lfloor \frac{|\Sigma|}{4} \rfloor$ . Unfortunately, we failed to prove this, so instead we use an alternative approach that utilizes results obtained in §3 and §4. Let us start with

**Corollary 5.20.** *If  $\overline{G} = \overline{G}_{144}$ , then  $|\Sigma| \geq 16$ . Similarly, if  $G = \overline{G}_{80}$ , then  $|\Sigma| \geq 20$ .*

*Proof.* If  $G = \overline{G}_{144}$  and  $|\Sigma| < 16$ , then  $|\Sigma| = 12$  by Lemma 3.10. If  $G = \overline{G}_{80}$  and  $|\Sigma| < 20$ , then  $|\Sigma| = 16$  by Lemma 4.7. Thus, the base locus of the linear system  $\mathcal{M}_3$  contains no curves by Corollaries 3.13, 4.7 and 4.14. This gives a contradiction with (5.19).  $\square$

Recall from Lemma 2.10 that there is unique  $\overline{G}_{144}$ -invariant quadric surface in  $\mathbb{P}^3$ .

**Lemma 5.21.** *Suppose that  $G = \overline{G}_{144}$ . Let  $\mathcal{Q}$  be the unique  $G$ -invariant quadric surface in  $\mathbb{P}^3$ . Then  $\Sigma$  is not contained in  $\mathcal{Q}$ .*

*Proof.* Suppose that  $\Sigma$  is contained in  $\mathcal{Q}$ . Then the log pair  $(\mathbb{P}^3, \mathcal{Q} + \frac{4}{n}\mathcal{D})$  is not log canonical at every point of  $\Sigma$ . Thus, the log pair  $(\mathcal{Q}, \frac{4}{n}\mathcal{D}|_{\mathcal{Q}})$  is not log canonical at every point of  $\Sigma$  by Inversion of Adjunction (see [28, Theorem 5.50]). Hence, there is  $\mu < \frac{4}{n}$  such that the log pair  $(\mathcal{Q}, \mu\mathcal{D}|_{\mathcal{Q}})$  is not Kawamata log terminal. By Corollary 3.7, this log pair is Kawamata log terminal away from finitely many points in  $\mathcal{Q}$ .

Let  $\mathcal{I}$  be the multiplier ideal sheaf of the log pair  $(\mathcal{Q}, \mu\mathcal{D}|_{\mathcal{Q}})$ . Then the ideal  $\mathcal{I}$  defines a (non-empty) zero-dimensional subscheme  $\mathcal{L}$  of the quadric  $\mathcal{Q}$  such that the support of the subscheme  $\mathcal{L}$  contains  $\Sigma$ . Moreover, this subscheme is  $G$ -invariant, because  $\mathcal{D}$  is  $G$ -invariant. Let  $H$  be a plane section of the quadric  $\mathcal{Q}$ . Then

$$h^1(\mathcal{I} \otimes \mathcal{O}_{\mathcal{Q}}(2H)) = 0$$

by Nadel vanishing theorem (see [30, Theorem 9.4.8]). Now using the exact sequence of sheaves

$$0 \longrightarrow \mathcal{I} \otimes \mathcal{O}_{\mathcal{Q}}(2H) \longrightarrow \mathcal{O}_{\mathcal{Q}}(2H) \longrightarrow \mathcal{O}_{\mathcal{L}} \otimes \mathcal{O}_{\mathcal{Q}}(2H) \longrightarrow 0,$$

we see that

$$|\Sigma| \leq h^0(\mathcal{O}_{\mathcal{L}}) = h^0(\mathcal{O}_{\mathcal{L}} \otimes \mathcal{O}_{\mathcal{Q}}(2H)) \leq h^0(\mathcal{O}_{\mathcal{Q}}(2H)) = 9,$$

which is impossible by Lemma 3.4.  $\square$

**Lemma 5.22.** *Let  $C$  be a  $G$ -irreducible curve in  $\mathbb{P}^3$  such that  $\deg(C) \leq 15$ . Then  $\Sigma \not\subset C$ .*

*Proof.* Suppose that  $\Sigma \subset C$ . Let us seek for a contradiction. We have

$$D_1 \cdot D_2 = mC + \Delta,$$

where  $m$  is a non-negative integer, and  $\Delta$  is an effective one-cycle on  $\mathbb{P}^3$  such that  $\text{Supp}(\Delta)$  does not contain irreducible components of the curve  $C$ . Hence, it follows from (5.18) that

$$(5.23) \quad \text{mult}_O(\Delta) > \frac{n^2}{4} - m \cdot \text{mult}_O(C)$$

for every point  $O \in \Sigma$ . Let  $H$  be a plane in  $\mathbb{P}^3$ . Thus

$$(5.24) \quad n^2 = H \cdot D_1 \cdot D_2 = H \cdot (mC + \Delta) = m \cdot \deg(C) + H \cdot \Delta \geq m \cdot \deg(C).$$

For every  $r \geq 2$ , denote by  $\mathcal{B}_r$  the linear system that consists of surfaces of degree  $r$  in  $\mathbb{P}^3$  containing  $C$ . If the base locus of  $\mathcal{B}_r$  does not contain  $G$ -irreducible curves different from  $C$ , then it follows from (5.23) that

$$(5.25) \quad r(n^2 - m \cdot \deg(C)) = B_r \cdot (D_1 \cdot D_2 - mC) = \\ = B_r \cdot \Delta \geq \sum_{O \in \Sigma} \text{mult}_O(\Delta) > |\Sigma| \left( \frac{n^2}{4} - m \cdot \text{mult}_O(C) \right),$$

where  $B_r$  is a sufficiently general surface in the linear system  $\mathcal{B}_r$ .

Suppose that  $G = \overline{G}_{144}$ . By Lemma 5.21 and Theorem 3.22, the curve  $C$  is a smooth irreducible curve of degree 12 and genus 13. In particular, we have  $|\Sigma| \geq 24$  by Lemma 3.2. Moreover, it follows from Corollary 3.20 that the base locus of the linear system  $\mathcal{B}_6$  does not contain curves different from  $C$ . Thus, it follows from (5.25) that

$$6(n^2 - 12m) > 24 \left( \frac{n^2}{4} - m \right),$$

so that  $0 > 48m$ , which is absurd.

We see that  $G = \overline{G}_{80}$ . Let us use the notation of §4. By Theorem 4.16, one of the following cases holds:

- $\deg(C) = 8$ , and  $C$  is an irreducible smooth curve of genus 5;
- $\deg(C) = 12$ , and  $C$  is an irreducible smooth curve of genus 5;
- $\deg(C) = 10$ , and  $C$  is one of the curves  $\mathcal{L}_{10}$ ,  $\mathcal{L}'_{10}$ ,  $\mathcal{L}''_{10}$ .

Note that in the former two cases one has  $|\Sigma| \geq 40$  by Lemma 4.3 and Corollary 5.20. Also, we know from (5.24) that  $n^2 \geq m \cdot \deg(C) > 4m$ . Let us consider the above possibilities case by case.

If  $\deg(C) = 8$ , then the base locus of  $\mathcal{B}_4$  does not contain  $G$ -irreducible curves different from  $C$  by Theorem 4.16. Hence (5.25) gives

$$4n^2 - 32m = 4(n^2 - 8m) > |\Sigma| \left( \frac{n^2}{4} - m \right) \geq 40 \left( \frac{n^2}{4} - m \right) = 10n^2 - 40m.$$

This implies that  $6n^2 < 8m$ . On the other hand, one has  $8m \leq n^2$  by (5.24), which is absurd.

If  $\deg(C) = 12$ , then the base locus of  $\mathcal{B}_6$  does not contain curves different from  $C$  by Corollary 4.23. Hence (5.25) gives

$$6n^2 - 72m = 6(n^2 - 12m) > |\Sigma| \left( \frac{n^2}{4} - m \right) \geq 40 \left( \frac{n^2}{4} - m \right) = 10n^2 - 40m,$$

which is absurd.

Thus, we see that  $C$  is one of the curves  $\mathcal{L}_{10}$ ,  $\mathcal{L}'_{10}$  or  $\mathcal{L}''_{10}$ . In particular, by (5.24) we have  $n^2 \geq 10m > 8m$ .

If  $C = \mathcal{L}'_{10}$  or  $C = \mathcal{L}''_{10}$ , then the base locus of  $\mathcal{B}_4$  does not contain  $G$ -irreducible curves different from  $C$  by Lemma 4.10, so that Lemma 4.6 and (5.25) give

$$4n^2 - 40m = 4(n^2 - 10m) > |\Sigma| \left( \frac{n^2}{4} - 2m \right) \geq 20 \left( \frac{n^2}{4} - 2m \right) = 5n^2 - 40m,$$

which is absurd. Thus, neither  $C = \mathcal{L}'_{10}$  nor  $C = \mathcal{L}''_{10}$  (that is,  $\Sigma$  is not contained in  $\mathcal{L}'_{10}$  and in  $\mathcal{L}''_{10}$ ). We see that  $C = \mathcal{L}_{10}$ .

By Lemma 4.10, the base locus of  $\mathcal{B}_6$  does not contain  $G$ -irreducible curves different from  $C$ . If  $|\Sigma| \neq 20$ , then Lemma 4.6 implies that  $|\Sigma| \geq 40$ , so that (5.25) gives

$$6n^2 - 60m = 6(n^2 - 10m) > |\Sigma| \left( \frac{n^2}{4} - m \right) \geq 40 \left( \frac{n^2}{4} - m \right) = 10n^2 - 40m,$$

which is absurd. We see that  $C = \mathcal{L}_{10}$  and  $|\Sigma| = 20$ . Then  $\Sigma \subset \mathcal{L}'_{10} \cup \mathcal{L}''_{10}$  by Lemma 4.6. However, we just proved that  $\Sigma$  is not contained in  $\mathcal{L}'_{10}$  and  $\mathcal{L}''_{10}$ .  $\square$

**Corollary 5.26.** *One has  $|\Sigma| > 35$ .*

*Proof.* If  $G = \overline{G}_{144}$ , then  $|\Sigma| > 35$  by Lemma 5.21 and Theorem 3.10. If  $G = \overline{G}_{80}$ , one has  $|\Sigma| \geq 80$  by Corollary 4.7, Theorem 4.16, Corollary 5.20 and Lemma 5.22.  $\square$

Now we are going to use the *multiplication by two trick* that was introduced in [7, 8]. Recall that  $P \in \Sigma$  is a center of non-canonical singularities of the log pair  $(\mathbb{P}^3, \frac{4}{n}\mathcal{D})$ , so that  $\text{mult}_P(\mathcal{D}) > \frac{n}{4}$  (see [13, Proposition 5.14]). This immediately implies that  $P$  is a center of non-log canonical singularities of the log pair  $(\mathbb{P}^3, \frac{8}{n}\mathcal{D})$ . In particular, the log pair  $(\mathbb{P}^3, \frac{8}{n}\mathcal{D})$  is not log canonical at  $P$ . Thus, there is a positive rational number  $\mu < \frac{8}{n}$  such that  $(\mathbb{P}^3, \mu\mathcal{D})$  is log canonical at the point  $P$ , and it is not Kawamata log terminal at the point  $P$ . Of course, the same holds for every point in  $\Sigma$ , because  $\Sigma$  is the  $G$ -orbit of the point  $P$ , and the linear system  $\mathcal{D}$  is  $G$ -invariant.

**Lemma 5.27.** *The log pair  $(\mathbb{P}^3, \mu\mathcal{D})$  is Kawamata log terminal in a punctured neighborhood of every point in  $\Sigma$ .*

*Proof.* Suppose that the required assertion is not true. Then there exists a  $G$ -irreducible curve  $C$  in  $\mathbb{P}^3$  such that  $\Sigma \subset C$  and  $(\mathbb{P}^3, \mu\mathcal{D})$  is not Kawamata log terminal in every point of the curve  $C$ . Let  $D_1$  and  $D_2$  be two general surfaces in  $\mathcal{D}$ . Then

$$D_1 \cdot D_1 = mC + \Delta,$$

where  $m$  is a positive integer, and  $\Delta$  is an effective one-cycle on  $\mathbb{P}^3$  such that  $\text{Supp}(\Delta)$  does not contain irreducible components of the curve  $C$ . By [12, Theorem 3.1], one has

$$m \geq \frac{4}{\mu^2} > \frac{n^2}{16}.$$

Let  $H$  be a general plane in  $\mathbb{P}^3$ . Then

$$n^2 = H \cdot D_1 \cdot D_2 = H \cdot (mC + \Delta) = m \cdot \deg(C) + H \cdot \Delta \geq m \cdot \deg(C) > \frac{n^2}{16} \cdot \deg(C),$$

so that  $\deg(C) \leq 15$ , which is impossible by Lemma 5.22.  $\square$

Let  $\mathcal{I}$  be the multiplier ideal sheaf of the log pair  $(\mathbb{P}^3, \mu\mathcal{D})$ . Then the ideal  $\mathcal{I}$  is not trivial, since  $(\mathbb{P}^3, \mu\mathcal{D})$  is not Kawamata log terminal. Thus, it defines a (non-empty) subscheme  $\mathcal{L}$  of  $\mathbb{P}^3$  such that the support of the subscheme  $\mathcal{L}$  contains  $\Sigma$ . Since  $\mathcal{D}$  is mobile, the subscheme  $\mathcal{L}$  has no two-dimensional components. Moreover, the subscheme  $\mathcal{L}$  is reduced in a neighborhood of every point in  $\Sigma$ , because the log pair  $(\mathbb{P}^3, \mu\mathcal{D})$  is log canonical at every point of  $\Sigma$ . Furthermore, every point of  $\Sigma$  is an isolated irreducible

component of the subscheme  $\mathcal{L}$  by Lemma 5.27. On the other hand, it follows from Nadel vanishing theorem (see [30, Theorem 9.4.8]) that

$$h^1(\mathcal{I} \otimes \mathcal{O}_{\mathbb{P}^3}(4)) = 0.$$

Now using the exact sequence of sheaves

$$0 \longrightarrow \mathcal{I} \otimes \mathcal{O}_{\mathbb{P}^3}(4) \longrightarrow \mathcal{O}_{\mathbb{P}^3}(4) \longrightarrow \mathcal{O}_{\mathcal{L}} \otimes \mathcal{O}_{\mathbb{P}^3}(4) \longrightarrow 0,$$

we see that

$$|\Sigma| \leq h^0(\mathcal{O}_{\mathcal{L}} \otimes \mathcal{O}_{\mathbb{P}^3}(4)) \leq h^0(\mathcal{O}_{\mathbb{P}^3}(4)) = 35,$$

which is impossible by Corollary 5.26. The obtained contradiction completes the proof of Propositions 5.11 and 5.13. This in turns completes the proof of Theorems 1.1 and 5.5.

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