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BIRATIONAL SELF-MAPS OF THREEFOLDS OF (UN)-BOUNDED GENUS OR GONALITY

JÉRÉMY BLANC, IVAN CHELTSOV, ALEXANDER DUNCAN, AND YURI PROKHOROV

Abstract. We study the complexity of birational self-maps of a projective threefold X by looking at the birational type of surfaces contracted. These surfaces are birational to the product of the projective line with a smooth projective curve. We prove that the genus of the curves occuring is unbounded if and only if X is birational to a conic bundle or a fibration into cubic surfaces. Similarly, we prove that the gonality of the curves is unbounded if and only if X is birational to a conic bundle.

Contents

1. Introduction

Let X be a smooth projective complex algebraic variety. One way of studying the complexity of the geometry of elements of the group $\mathrm{Bir}(X)$ of birational self-maps of X consists of studying the complexity of the irreducible hypersurfaces contracted by elements of $\mathrm{Bir}(X)$. If X is a curve, then $\mathrm{Bir}(X) = \mathrm{Aut}(X)$, so there is nothing to be said. If X is a surface, every irreducible curve contracted by an element of $\mathrm{Bir}(X)$ is rational. The case of threefolds is then the first interesting to study in this context.

If $\dim(X) = 3$, then every irreducible surface contracted by a birational transformation $\varphi \in \mathrm{Bir}(X)$ is birational to $\mathbb{P}^1 \times C$ for some smooth projective curve C. There are then two natural integers that one can associate to C in this case, namely its genus $g(C)$ and its gonality $\mathrm{gon}(C)$ (the minimal degree of a dominant morphism $C \longrightarrow \mathbb{P}^1$). We then define the *genus* $g(\varphi)$ (respectively the *gonality*

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gon(φ)) of φ to be the maximum of the genera $g(C)$ (respectively of the gonalities gon(C)) of the smooth projective curves C such that a hypersurface of X contracted by φ is birational to $\mathbb{P}^1 \times C$.

This notion of genus of elements of $\mathrm{Bir}(X)$ was already defined in [Fru73] with another definition, which is in fact equivalent to ours by [Lam14]. Moreover, for each g, the set of elements of $\mathrm{Bir}(X)$ of genus $\leq g$ form a subgroup, so we get a natural filtration on $\mathrm{Bir}(X)$, studied in [Fru73, Lam14]. This naturally raises the question of finding the threefolds X for which this filtration is infinite, namely the threefolds X for which the genus of $\mathrm{Bir}(X)$ is unbounded (see [Lam14, Question 11]). Analogously, we get a filtration given by the gonality. Of course, the gonality being bounded if the genus is bounded, the unboundedness of the gonality is stronger than the unboundedness of the genus.

Note that the boundedness of the genus (respectively of the gonality) of elements of $\mathrm{Bir}(X)$ is a birational invariant. Our main result (Theorem 1.1) describes the threefolds having this property.

Recall that a variety Y is a *conic bundle* (respectively a *del Pezzo fibration of degree* d) if Y admits a morphism $Y \longrightarrow S$ such that the generic fibre is a conic (respectively a del Pezzo surface of degree d) over the field of rational functions of S. If the conic has a rational point (or equivalently the conic bundle has a rational section), then it is isomorphic to $\mathbb{P}^1$ and in this case we say that the conic bundle is *trivial*.

1.1. **Theorem.** *Let X be a smooth projective complex algebraic threefold.*

(i) *If X is birational to a conic bundle, the gonality and the genus of the elements* $\mathrm{Bir}(X)$ *are both unbounded.*
(ii) *If X is birational to a del Pezzo fibration of degree* 3*, the genus of the elements of* $\mathrm{Bir}(X)$ *is unbounded.*
(iii) *If X is not birational to a conic bundle, the gonality of the elements of* $\mathrm{Bir}(X)$ *is bounded.*
(iv) *If X is not birational to a conic bundle and to a del Pezzo fibration of degree* 3*, then both the genus and the gonality of elements of* $\mathrm{Bir}(X)$ *are bounded.*

We can generalise the above notions to higher dimensions. If X is a smooth projective variety of dimension $d \geq 3$, every irreducible hypersurface contracted by an element of $\mathrm{Bir}(X)$ is birational to $\mathbb{P}^1 \times S$ for some variety S of dimension $d-2$. When $d \geq 4$, $\dim(S) \geq 2$ and there are then many ways to study the complexity of this variety. One possibility is the covering gonality cov.gon(S) of S, namely the smallest integer c such that through a general point of S there is an irreducible curve $\Gamma \subseteq S$ birational to a smooth curve of gonality $\leq c$. As before, we say that *the covering gonality of* $\mathrm{Bir}(X)$ *is bounded* if the covering gonality of the irreducible varieties S such that a hypersurface contracted by an element $\mathrm{Bir}(X)$ is birational to $S \times \mathbb{P}^1$ is bounded. Since the covering gonality of a smooth curve is its gonality, this notion is the same as the gonality defined before, in the case of threefolds. As in dimension 3, this is again a birational invariant.

In Corollary 4.4, we prove that that if X is a solid Fano variety (see [AO18, Definition 1.4]), then the covering gonality of elements of $\mathrm{Bir}(X)$ are bounded by a constant that depends only on $\dim(X)$. In particular, the covering gonality of birational selfmaps of birationally rigid Fano varieties of dimension n (see [CP17, Definition 1.1.2]) are bounded by a constant that depends only on n.

In Proposition 2.4, we prove that if $\pi\colon X \longrightarrow B$ is a trivial conic bundle of any dimension ≥ 3, then the covering gonality of the elements of

$$\mathrm{Bir}(X/B) = \{\varphi \in \mathrm{Bir}(X) \mid \pi \circ \varphi = \pi\} \subseteq \mathrm{Bir}(X)$$

is unbounded. This raises the following two questions:

1.2. **Question.** Let B be a projective variety of dimension ≥ 3 and let $X \longrightarrow B$ be a non-trivial conic bundle. Is the covering gonality of elements of $\mathrm{Bir}(X/B)$ unbounded?

1.3. **Question.** Let X be a projective variety of dimension ≥ 4 that is not birational to a conic bundle. Is the covering gonality of elements of $\mathrm{Bir}(X)$ bounded?

A rough idea of the proof of Theorem 1.1 is as follows. Since the boundedness of the genus and gonality is a birational invariant, we can run the MMP and replace X with a birational model (with terminal singularities) such that either K_X is nef or X has a Mori fibre space structure X/B. In the former case any birational self-map is a pseudo-automorphism [Han87, Lemma 3.4] and so the genus and gonality are bounded in this case. If X/B is a Mori fibre space, then any birational map $X \dashrightarrow X$ is a composition of Sarkisov links (see Sect. 3). If a link involves a Mori fibre space X_i/B_i which is (generically) a conic bundle, then we apply an explicit construction of Sect. 2 to get unboundedness (and thus obtain Theorem 1.1(i)). If a link involves Mori fibre spaces X_i/B_i and X_{i+1}/B_{i+1}, then we use the boundedness result for Fano threefolds [KMMT00] (see also [Bir16]). This result is also used to prove the assertions 1.1(iii)-(iv) (see Lemma 4.5). The unboundeness of the genus for del Pezzo fibrations of degree 3 (Theorem 1.1(ii)) is obtained by finding 2-sections of large genus and applying Bertini involutions associated to these curves, see Section 5 for the detailed construction.

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2. The case of conic bundles

Every conic bundle is square birational equivalent to a conic bundle that can be seen as a conic in a (Zariski locally trivial) $\mathbb{P}^2$-bundle. Using a rational section of the $\mathbb{P}^2$-bundle, one can do the following construction:

2.1. **Construction.** *Let $\pi : Q \longrightarrow B$ be a conic bundle over an irreducible normal variety B and let Q_η be its generic fibre. Then Q_η is a conic over the function field $\mathbb{C}(B)$. The anicanonical linear system $|-K_{Q_\eta}|$ defines an embedding $Q_\eta \hookrightarrow \mathbb{P}^2_{\mathbb{C}(B)}$. Fix a $\mathbb{C}(B)$-point $s_\eta \in \mathbb{P}^2_{\mathbb{C}(B)} \setminus Q_\eta$. The projection $\mathrm{pr} : Q_\eta \longrightarrow \mathbb{P}^1_{\mathbb{C}(B)}$ from s_η is a double cover. Let $\iota_\eta : Q_\eta \longrightarrow Q_\eta$ be the corresponding Galois involution. It induces a fibrewise birational involution $\iota : Q \dashrightarrow Q$.*

Suppose now that our conic bundle $\pi : Q \longrightarrow B$ is embedded into a $\mathbb{P}^2$-bundle $\hat{\pi} : P \longrightarrow B$ and suppose that we are given a section $s : B \longrightarrow P$ whose image is not contained in Q. This section defines a point $s_\eta \in \mathbb{P}^2_{\mathbb{C}(B)} \setminus Q_\eta$ and therefore defines an involution $\iota : Q \dashrightarrow Q$ as above.

2.2. **Lemma.** [BL15, Lemma 15] *If, in the above notation, $\Gamma \subseteq B$ is an irreducible hypersurface that is not contained in the discriminant locus of π and such that $s(\Gamma) \subseteq Q$, the hypersurface $V = \pi^{-1}(\Gamma)$ of Q is contracted by ι onto the codimension 2 subset $s(\Gamma)$.*

2.3. **Corollary.** *Let $\pi\colon Q \longrightarrow B$ be a conic bundle over an irreducible normal variety B, given by the restriction of a $\mathbb{P}^2$-bundle $\hat{\pi}\colon P \longrightarrow B$. Let $\Gamma \subseteq B$ be an irreducible hypersurface such that the restriction of π gives a trivial conic bundle $V = \pi^{-1}(\Gamma) \longrightarrow \Gamma$. Then, there exists an involution*

$$\iota \in \mathrm{Bir}(Q/B) = \{\varphi \in \mathrm{Bir}(Q) \mid \pi\varphi = \pi\}$$

that contracts the hypersurface V onto the image of a rational section of $\Gamma \dashrightarrow V$.

Proof. Since the restriction of π gives a trivial conic bundle $V = \pi^{-1}(\Gamma) \longrightarrow \Gamma$, there is a rational section $s_\Gamma\colon \Gamma \dashrightarrow V \subseteq Q \subseteq P$. We then extend this section to a rational section $s\colon B \dashrightarrow P$ whose image is not contained in Q. This can be done locally, on a open subset where $P \longrightarrow B$ is a trivial $\mathbb{P}^2$-bundle. Lemma 2.2 provides an involution $\iota \in \mathrm{Bir}(Q/B)$ that contracts V onto the image of s_Γ. □

We can now give the proof of Theorem 1.1(i) (and of the small generalisation to higher dimensions mentioned in the introduction):

2.4. **Proposition.** *Let B be a projective variety of dimension ≥ 2, let $\pi\colon Q \longrightarrow B$ be a conic bundle and let us assume that either π is trivial (admits a rational section) or that $\dim(B) = 2$. Then, the covering gonality (and the genus if $\dim(Q) = 3$) of elements of $\mathrm{Bir}(Q/B)$ is unbounded.*

Proof. We can assume that π is the restriction of a $\mathbb{P}^2$-bundle $\hat{\pi}\colon P \longrightarrow B$.

Let $\Gamma \subseteq B$ be an irreducible hypersurface which is not contained in the discriminant locus of π. Then the restriction of π gives a conic bundle $\pi_\Gamma\colon V =$

$\pi^{-1}(\Gamma) \longrightarrow \Gamma$. If π is a trivial conic bundle, then so is π_Γ. If $\dim(B) = 2$, then Γ is a curve, and π is again a trivial conic bundle by Tsen's Theorem [Kol96, Corollary 6.6.2 p. 232]. In both cases, we can apply Corollary 2.3 to find an element of $\mathrm{Bir}(Q/B)$ that contracts the hypersurface $V \subseteq Q$, birational to $\mathbb{P}^1 \times \Gamma$.

This can be done for any irreducible hypersurface of B not contained in the discriminant locus. Thus, the covering gonality of elements of $\mathrm{Bir}(Q/B)$ are unbounded (to see this, simply embed B in a projective space and take a general hypersurface of large degree). The same argument applies to the genus when $\dim(Q) = 3$. □

We recall the following classical result:

2.5. **Lemma.** *Let B be a projective curve. A del Pezzo fibration X/B of degree ≥ 4 is birational to a conic bundle X'/B'.*

Proof. The generic fibre of X/B is a del Pezzo surface F of degree $d \geq 4$ over the function field $\mathbb{C}(B)$. Applying a MMP over B, we can assume that the generic fibre satisfies $\mathrm{rk}\,\mathrm{Pic}(F) = 1$ or has a structure of conic bundle. In the latter case, the proof is over, so we assume that $\mathrm{rk}\,\mathrm{Pic}(F) = 1$. It is sufficient to show that F birationally has a conic bundle structure $F \longrightarrow C$, where C is a curve defined over $\mathbb{C}(B)$; this is for instance the case if F is rational. As B is a curve, the field $\mathbb{C}(B)$ has the C_1 property, so F has a rational $\mathbb{C}(B)$-point $x \in F$ (see [Kol96, Theorem IV.6.8, page 233]).

If $d = 9$, the existence of x implies that F is isomorphic to $\mathbb{P}^2$. If $d = 8$, the fact that $\mathrm{rk}\,\mathrm{Pic}(F) = 1$ implies that F is isomorphic to a smooth quadric in $\mathbb{P}^3$, and the projection from x gives a birational map to $\mathbb{P}^2$. We cannot have $d = 7$, as the unique (-1)-curve of $F_{\overline{\mathbb{C}(B)}}$ would be defined over $\mathbb{C}(B)$, contradicting the assumption $\mathrm{rk}\,\mathrm{Pic}(F) = 1$.

It remains to study the cases where $d \in \{4, 5, 6\}$. Since $\mathrm{rk}\,\mathrm{Pic}(F) = 1$, the $\mathbb{C}(B)$-rational point $x \in F$ does not lie on a (-1)-curve. Therefore, by blowing-up $x \in F$ we obtain a del Pezzo surface Y over $\mathbb{C}(B)$ of degree $d - 1$ with $\mathrm{rk}\,\mathrm{Pic}(Y) = 2$. Thus on Y there exists a Mori contraction $Y \longrightarrow Y'$ which is different from $Y \longrightarrow F$. The type of $Y \longrightarrow F$ can be computed explicitely (see [Isk96, Theorem 2.6]): If $d = 5$ (resp. $d = 6$), then $Y \longrightarrow Y'$ is a birational contraction to $\mathbb{P}^2_{\mathbb{C}(B)}$, (resp. to a quadric in $\mathbb{P}^3_{\mathbb{C}(B)}$ having a rational point), so F is again rational. If $d = 4$, then $Y' \simeq \mathbb{P}^1_{\mathbb{C}(B)}$ and $Y \longrightarrow Y'$ is a conic bundle. This proves our lemma. □

3. Reminders on the Sarkisov program

3.1. **Definition.** A variety X with a surjective morphism $\eta\colon X \longrightarrow B$ is a *Mori fibre space* if the following conditions hold:

(i) η has connected fibres, B is normal, $\dim X > \dim B \geq 0$ and the relative Picard rank $\rho(X/B) = \rho(X) - \rho(B)$ is equal to 1;

(ii) X is $\mathbb{Q}$-factorial with at most terminal singularities;
(iii) The anticanonical divisor $-K_X$ is η-ample.

The Mori fibre space is denoted X/B.

An isomorphism of fibre spaces $X/B \xrightarrow{\simeq} X'/B'$ is an isomorphism $\varphi\colon X \longrightarrow X'$ that sits in a commutative diagram

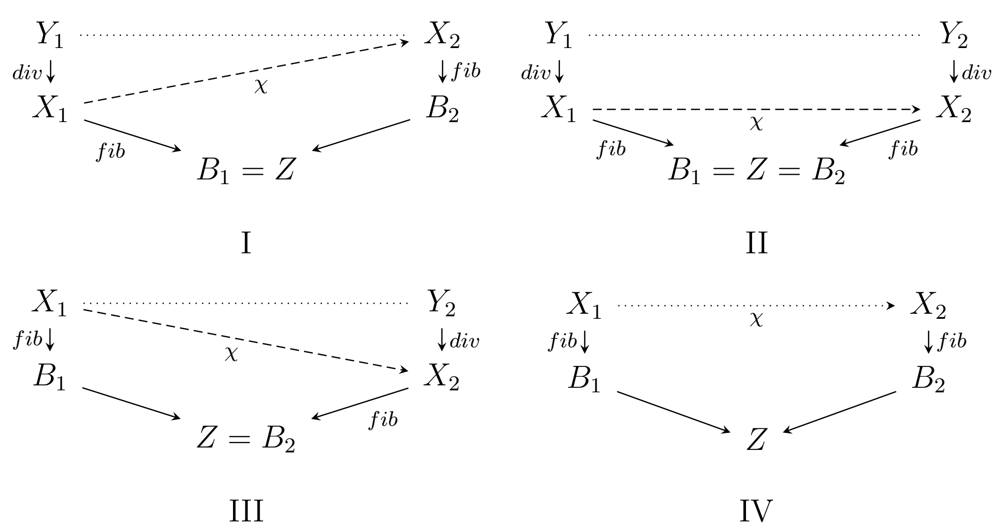
$$\begin{array}{ccc} X & \xrightarrow{\varphi} & X' \\ \downarrow & & \downarrow \\ B & \xrightarrow{\psi} & B' \end{array}$$

where $\psi\colon B \xrightarrow{\simeq} B'$ is an isomorphism.

3.2. **Remark.** In the case that we study, namely when $\dim(X) = 3$, we obtain three possible cases for a Mori fibre space X/B:

(i) If $\dim(B) = 0$, then X is a Fano variety of Picard rank 1;
(ii) If $\dim(B) = 1$, then X is a del Pezzo fibration over the curve B;
(iii) If $\dim(B) = 2$, then X is a conic bundle over the surface B.

3.3. **Definition.** A Sarkisov link $\chi\colon X_1 \dashrightarrow X_2$ between two Mori fibre spaces X_1/B_1 and X_2/B_2 is a birational map which fits into one of the following commutative diagrams.



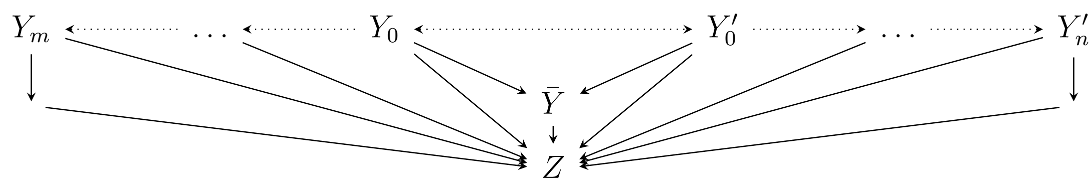
Here the dotted arrows are pseudo-isomorphisms (isomorphisms outside of codimension ≥ 2 subsets) given by a sequence of log-flips, the plain arrows are surjective morphisms of relative Picard rank 1, the arrows written *"div"* are divisorial contractions, and the variety Z is normal with at most Kawamata log terminal singularities. We say that the *base of the Sarkisov link* is the variety Z (which is dominated by, but not necessarily equal to, the bases B_1 and B_2 of the two Mori fibre spaces), and that the above diagram is the *Sarkisov diagram associated to* χ.

The notion of Sarkisov links is important, because of the following result.

3.4. **Theorem.** *Every birational map between Mori fibre spaces decomposes into a composition of Sarkisov links and isomorphisms of Mori fibre spaces.*

Proof. In dimension 2, this is essentially due to Castelunovo [Cas01], although not stated directly in these terms. The case of dimension 3 was done for the first time in [Cor95, Theorem 3.7]. The proof in any dimension is available in [HM13, Theorem 1.1]. □

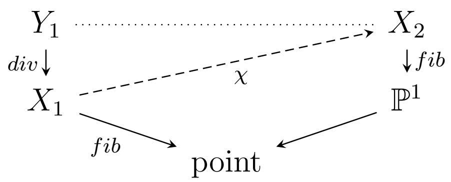
3.5. **Remark.** In fact, it follows from the definition that there are strong constraints on the sequence of anti-flips, flops and flips (that is, about the sign of the intersection of the exceptional curves against the canonical divisor). Precisely, as explained in [BLZ19, Remark 3.10], the top (dotted) row of a Sarkisov diagram has the following form:



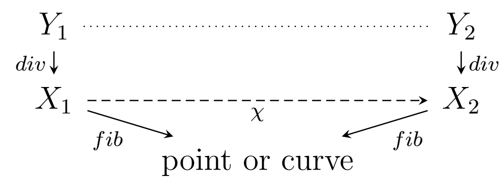
where $Y_0 \dashrightarrow Y'_0$ is a flop over Z (or an isomorphism), $m, n \geq 0$, and each $Y_i \dashrightarrow Y_{i+1}$, $Y'_i \dashrightarrow Y'_{i+1}$ is a flip over Z (or an isomorphism). Indeed, one can decompose the pseudo-isomorphism into a sequence of log-flips and for $Y = Y_i$ or $Y = Y'_i$, a general contracted curve C of the fibration Y/Z satisfies $K_Y \cdot C < 0$, hence at least one of the two extremal rays of the cone $\mathrm{NE}(Y/Z)$ is strictly negative against K_Y. In particular, both Y_0/Z and Y'_0/Z are relatively weak Fano (or Fano if the flop is an isomorphism) over Z.

If $Y_0 \dashrightarrow Y'_0$ is an isomorphism, we choose $\bar{Y}$ to be isomorphic to both; if $Y_0 \dashrightarrow Y'_0$ is a flop and not an isomorphism, the map is naturally associated to a variety $\bar{Y}$, that is a Fano with terminal (but not $\mathbb{Q}$-factorial) singularities such that $\operatorname{rk}\operatorname{Cl}(\bar{Y}/Z) = 2$. It is called *the central model* in [CS11]. Two contractions $Y_0 \longrightarrow \bar{Y}$ and $Y'_0 \longrightarrow \bar{Y}$ are small $\mathbb{Q}$-factorialisations of $\bar{Y}$. Hence the whole diagram is uniquely determined by $\bar{Y}/Z$.

3.6. **Remark.** In the sequel, we will mostly work with varieties of dimension 3 not birational to conic bundles, as the case of conic bundles as already treated in Section 2. The Mori fibre spaces will be then either Fano of rank 1 or del Pezzo fibrations. As a Fano is rationally connected, a del Pezzo fibration over a base not equal to $\mathbb{P}^1$ is not birational to a Fano variety. All the Sarkisov links that we can have between Mori fibre spaces not birational to conic bundles are then as follows:



I



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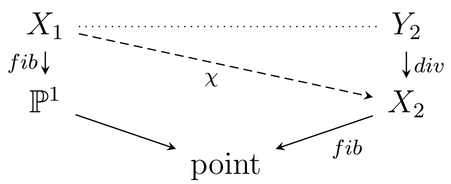
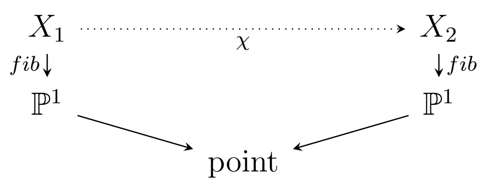
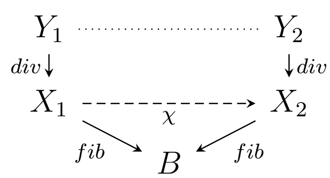


III



IV

3.7. **Lemma.** *Let us consider a Sarkisov link of type* II *between three-dimensional Mori fibre spaces X_1/B and X_2/B over a base B of dimension* 1*.*



Denoting by $E_i \subset Y_i$ the exceptional divisor of Y_i/X_i and by $e_i \subset X_i$ its image, one of the following case holds:

(i) *χ induces an isomorphism between the generic fibres of X_1/B and X_2/B, and e_i is contained in a fibre of X_i/B for $i = 1, 2$.*

(ii) *χ induces a birational map between the generic fibres of X_1/B and X_2/B which is not an isomorphism and e_i is a curve of X_i such that e_i/B is a finite morphism of degree $r_i \in \{1, \ldots, 8\}$, for $i = 1, 2$.*

Moreover, in case (ii)*, if one of the degree d_i of the del Pezzo fibration X_i/B is ≤ 3, then $d_1 = d_2$ and $r_1 = r_2$, and $(d_i, r_i) \in \{(3, 2), (3, 1), (2, 1)\}$, and the generic fibres of X_1/B and X_2/B are isomorphic. In particular, e_i is birational to B if $d_i \leq 2$.*

Proof. The image e_i is a curve or a point, so is either (a) contained in a fibre of X_i/Z, or (b) maps surjectively to Z via a finite morphism of degree $d_i \geq 1$. Case (a) happens if and only if the generic fibres of X_i/Z and Y_i/Z are isomorphic. As the generic fibres of X_i/Z are del Pezzo surfaces of rank 1, for $i = 1, 2$, case (a) happens for $i = 1$ if and only if it happens for $i = 2$. This provides the dichotomy (i)-(ii) above.

In case (ii), we look at the birational map between the generic fibres X_i/Z which are del Pezzo surfaces of degree 1. The classification of such maps, given in [Isk96, Theorem 2.6], implies that $r_i \le 8$ for $i = 1, 2$ and that if $d_i \le 3$ then $d_1 = d_2$, $r_1 = r_2$, and $(d_i, r_i) \in \{(3, 2), (3, 1), (2, 1)\}$, and the generic fibres of X_1/B and X_2/B are isomorphic. □

In Case (ii) in Lemma 3.7, the case of del Pezzo fibrations of degree 3 is the most interesting one, as the degree ≤ 2 only gives curves e_i of bounded genus, and the case of degree ≥ 4 is covered by Lemma 2.5.

3.8. **Remark.** Let X/B be a Mori fibre space such that $\dim(B) \in \{0, 1\}$. It may happen that no Sarkisov link starts from X. If $\dim(B) = 0$, this means (almost by definition) that X is a birationally super-rigid Fano threefold. Many examples of such Fano threefolds can be found in [CP17]. Similarly, if $\dim(B) = 1$ and no Sarkisov link starts from X, then X/B is a del Pezzo fibration of degree 1 that is birationally rigid over B (see [Cor00, Definition 1.3]). Vice versa, if X/B is a del Pezzo fibration of degree 1 that is birationally rigid over B, then the only Sarkisov links that can start at X are described in Case (i) of Lemma 3.7. For some (birationally rigid over the base) del Pezzo fibrations such links do not exist (see [Kry18]). However, in general they may exist and are not well understood (see [Par01, Par03]).

4. Bounding the gonality and genus of curves

We first state a consequence of the boundedness of weak-Fano terminal varieties. The next lemma applies to Sarkisov links involving a Fano threefold of rank 1 (when one of the B_i has dimension 0) and to Sarkisov links of type IV between del Pezzo fibrations (when $\dim(B_1) = \dim(B_2) = 1$ and $\dim(Z) = 0$).

4.1. **Lemma.** *There are integers $g_0, c_0 \ge 1$ such that, for each Sarkisov link $\chi\colon X_1 \dashrightarrow X_2$ between two Mori fibre spaces X_1/B_1 and X_2/B_2 such that $\dim(X_1) = \dim(X_2) = 3$ over a base Z of dimension 0, the following hold:*

- (i) *Each divisorial contraction involved in the Sarkisov diagram of χ contracts a divisor birational to $\mathbb{P}^1 \times \Gamma$ where $\operatorname{gon}(\Gamma) \le c_0$, and where the genus of Γ is smaller than or equal to g_0.*
- (ii) *For each $i \in \{1, 2\}$, if $\dim(B_i) = 1$, then each fibre of X_i/B_i is birational to $\mathbb{P}^1 \times \Gamma$ with $\operatorname{gon}(\Gamma) \le c_0$, and where the genus of Γ is smaller than or equal to g_0.*

Proof. As in Remark 3.5, we consider the variety Y_0 which is pseudo-isomorphic to the top varieties in the Sarkisov diagram and which is a weak-Fano variety of rank 2, since the base Z of the Sarkisov link is of dimension 0.

By [KMMT00] (see also [Bir16, Theorem 1.1]), the set of Fano threefolds with terminal singularities is bounded. As explained in Remark 3.5, the Sarkisov link is determined by the Fano variety $\overline{Y}$. The two rational maps $Y_0 \dashrightarrow X_i \longrightarrow B_i$ are then determined by the two extremal rays of the cone $\mathrm{NE}(Y_0)$, so their fibres and contracted divisors are also in bounded families. This achieves the result. □

The following result is a direct consequence of Lemma 4.1.

4.2. **Corollary.** *There are integers $g_0, c_0 \geq 1$ such that for each Mori fibre space X/B where $\dim(X) = 3$, $\dim(B) = 0$, not birational to a conic bundle or a del Pezzo fibration, the genus and the gonality of elements of $\mathrm{Bir}(X)$ are bounded by g_0 and c_0 respectively.*

Proof. Every element of $\mathrm{Bir}(X)$ decomposes into a product of Sarkisov links and isomorphisms of Mori fibre spaces (Theorem 3.4). The base of each of these Sarkisov links has dimension 0, as X is not birational to a conic bundle or a del Pezzo fibration. Hence the genus and the gonality of each Sarkisov link are bounded by g_0 and c_0 respectively by Lemma 4.1(i). This provides the result. □

In fact, Corollary 4.2 can be easily extended to higher dimension, and concerns then the *solid Fano varieties*, defined as below (see [AO18, Definition 1.4]):

4.3. **Definition.** A Fano variety X being a Mori fibre space over a point is said to be (birationally) *solid* if it is not birational to any Mori fibre space over a positive dimensional base.

4.4. **Corollary.** *For every integer $n \geq 3$, there is an integer $c_n \geq 1$ (that only depends on n) such that for each solid Fano variety X of dimension n, the covering gonality of elements of $\mathrm{Bir}(X)$ are bounded by c_n.*

Proof. The proof is similar as the one of Lemma 4.1 and Corollary 4.2:

By [Bir16, Theorem 1.1], the set of Fano varieties of dimension n with terminal singularities is bounded. As explained in Remark 3.5, any Sarkisov link over a base Z of dimension 0 is determined by the Fano variety $\overline{Y}$. The covering gonality of the divisors contracted by such Sarkisov links are thus bounded. The result then follows, as every element of $\mathrm{Bir}(X)$ is a composition of Sarkisov links over a base of dimension 0. □

We can now extend Corollary 4.2 to a more general situation:

4.5. **Lemma.** *There are integers $g_0, c_0 \geq 1$ such that the following holds:*

Suppose X/B is a Mori fibre space with $\dim(X) = 3$ and there exists a Sarkisov link over a base of dimension 0 involving X/B. Then, for every element $\varphi \in Bir(X)$ having a decomposition into Sarkisov links that involves no conic bundles, the gonality of φ is bounded by c_0. Moreover, if no del Pezzo fibration of degree ≥ 3 arises in the decomposition of φ, the genus of φ is bounded by g_0.

Proof. We choose integers g_0 and c_0 from Lemma 4.1, and assume that $c_0 \geq 8$.

If $\dim(B) = 0$, then X a Fano variety, and is thus rationally connected [Zha06]. If $\dim(B) \geq 1$, then $\dim(B) = 1$ as X is not birational to a conic bundle. By assumption, X/B is involved in a Sarkisov link over a base of dimension 0. The variety X is then birational to a Fano variety (the variety $\bar{Y}$ of Remark 3.5), so is again rationally connected. Moreover, Lemma 4.1(ii) implies that each fibre of X/B is birational to $C \times \mathbb{P}^1$ for some curve of genus and gonality bounded by g_0 and c_0.

We take an element $\varphi \in \mathrm{Bir}(X)$ that we decompose, using Theorem 3.4), as

$$\varphi = \theta_r \circ \chi_r \circ \cdots \circ \theta_1 \circ \chi_1 \circ \theta_0,$$

where each χ_i is a Sarkisov link between X'_{i-1}/B'_{i-1} and X_i/B_i, each θ_i is an isomorphism of Mori fibre spaces

$$\begin{array}{ccc} X_i & \xrightarrow{\theta_i} & X'_i \\ \downarrow & & \downarrow \\ B_i & \xrightarrow[\simeq]{} & B'_i \end{array}$$

where $X_0/B_0 = X'_r/B'_r = X/B$. By assumption, none of the X_i/B_i (or X'_i/B'_i) is a conic bundle, which means that $\dim(B_i) \in \{0, 1\}$ for each $i \in \{0, \dots, r\}$.

By Remark 3.6 and Lemma 3.7, we obtain three different types of Sarkisov links χ_i:

a) Sarkisov links χ_i with a base of dimension 0.

b) Sarkisov links χ_i of type II *over a curve $B_{i-1} = B_i$ inducing no isomorphism between the generic fibres of X'_{i-1}/B_{i-1} and X_i/B_i.*

c) Sarkisov links χ_i of type II *over a curve $B_{i-1} = B_i$ inducing an isomorphism between the generic fibres of X'_{i-1}/B_{i-1} and X_i/B_i.*

The genus and the gonality of the Sarkisov link in case a) are bounded by g_0 and c_0 respectively (Lemma 4.1(i)).

In Case $b) - c)$, the Sarkisov link is between two del Pezzo fibrations X'_{i-1}/B'_{i-1} and X_i/B_i over a curve $B'_{i-1} = B_i$. As X_{i-1} and X_i are rationally connected (because they are birational to X), we obtain $B'_{i-1} = B_i \simeq \mathbb{P}^1$.

Case b) corresponds to Case (ii) of Lemma 3.7). The two divisorial contractions contract divisors onto curves of X'_{i-1}/B'_{i-1} and X_i/B_i of gonality at most 8, which are moreover rational (and thus of genus bounded by g_0) if the degree of the del Pezzo fibrations is ≤ 2 (Lemma 3.7).

The remaining "bad" case is Case c) (Case (i) of Lemma 3.7). In this case, the surfaces $F_{i-1} \subseteq X'_{i-1}$ and $F_i \subseteq X_i$ contracted by χ_i and χ_i^{-1} correspond to fibres of the del Pezzo fibrations X'_{i-1}/B'_{i-1} and X_i/B_i. If the fibre is general, it is rational and we are done, but it could be that it is birational to $\mathbb{P}^1 \times C$ for some non-rational curve C, a priori of large genus / gonality (see Question 4.6). To solve the problem, we denote by $1 \leq j \leq i \leq k \leq r$ the smallest integer $j \in \{1, \dots, i\}$ and the biggest integer $k \in \{i, \dots, r\}$ such that $\chi_j, \dots, \chi_k$ are links of type II, and obtain that

$$\nu = \theta_k \circ \chi_k \circ \theta_{k-1} \circ \chi_{k-1} \circ \cdots \circ \chi_{j+1} \circ \theta_j \circ \chi_j \circ \theta_j$$

is a birational map between del Pezzo fibrations X_{j-1}/B_i and X'_k/B'_k which fits in a commutative diagram

$$\begin{array}{ccc} X_{j-1} & \overset{\nu}{\dashrightarrow} & X'_k \\ \downarrow & & \downarrow \\ B_i & \xrightarrow[\simeq]{} & B'_k \simeq \mathbb{P}^1. \end{array}$$

Every surface contracted by ν is either a fibre, or birational to a surface contracted by a Sarkisov link in $\chi_j, \dots, \chi_k$ of type b). We now prove that the fibres contracted by ν are birational to $C \times \mathbb{P}^1$ for some curve C of genus and gonality bounded by g_0 and c_0 respectively. If $j = 1$, this is because $X_0/B_0 = X/B$ and we already observed at the beginning that each fibre of X/B had this property. If $j > 1$, then the Sarkisov link χ_{j-1} is not of type II, so is over a base of dimension 0, i.e. is in Case a). Hence, each fibre of X_{j-1}/B_{j-1} is birational to $C \times \mathbb{P}^1$ for some curve C of genus $\leq g_0$ and gonality $\leq c_0$ (Lemma 4.1(ii)).

Hence, even if χ_i can a priori be of arbitrary large genus or gonality, the gonality of ν is bounded by c_0, and the genus is bounded by g_0 if no del Pezzo fibration of degree 3 appears in the decomposition.

As φ decomposes into links of type a) and maps having the same form as ν (compositions of Sarkisov links of type II), the gonality of ν is bounded by c_0, and the genus is bounded by g_0 if no del Pezzo fibration of degree 3 appears in the decomposition. □

The following question is naturally raised by the proof of Lemma 4.5.

4.6. **Question.** Is there an integer $g \geq 1$ such that for each Mori fibre space $X \longrightarrow B$ which is a del Pezzo fibration, each fibre is birational to $C \times \mathbb{P}^1$ for some curve C of genus (respectively gonality) $\leq g$?

In the case of birational maps between del Pezzo fibrations, we do not have an absolute bound (which would follow from a positive answer to Question 4.6), but we can easily obtain the following result on birational maps involving links over a base of dimension 1. This is for instance the case for all elements of $\mathrm{Bir}(X)$ if X/B is a Mori fibre space not birational to a conic bundle with X not rationally connected.

4.7. **Lemma.** *Let X/B be a Mori fibre space such that $\dim(X) = 3$ and $\dim(B) = 1$ (a del Pezzo fibration over a curve). There are integers $c, g \geq 0$ (depending on X/B) such that the following holds:*

For each birational map $\varphi \in \mathrm{Bir}(X)$ that decomposes into Sarkisov links of type II, each over a base of dimension 1, the gonality of φ is bounded by c. Moreover, the genus of φ is bounded by g if no del Pezzo fibration of degree ≥ 3 occurs in the decomposition.

Proof. As each Sarkisov link occuring in the decomposition of φ is of type II, the base of the Sarkisov link is isomorphic to B, and so are all bases of the Mori fibre spaces involved.

The map φ is a square birational map, i.e. sends a general fibre of X/B onto a general fibre. There are finitely many fibres of X/B that are not rational, so we only need to bound the gonality and the genus of the curves C such that $C \times \mathbb{P}^1$ is birational to a surface contracted by φ that is not a fibre. Such a surface is contracted by a Sarkisov link χ between del Pezzo fibrations X_{i-1}/B_{i-1} and X_i/B_i, which is not an isomorphism between the generic fibres. We proceed as in the proof of Lemma 4.5: The two divisorial contractions contract divisors onto multisections of X_{i-1}/B_{i-1} and X_i/B_i. As the variety Y_0 in the middle (see Remark 3.5) is a weak del Pezzo fibration over the base, we can only blow-up curves with gonality at most 8. Moreover, the curves are rational if X_{i-1}/B_{i-1} and X_i/B_i are del Pezzo fibrations of degree ≤ 2. This achieves the result. □

We can now apply Lemma 4.5 and obtain the following result, which gives the proof of parts (iii) and (iv) of Theorem 1.1. Note that here the bound depends on X and is not an absolute bound as in Lemma 4.5. This is of course needed, as one can consider the blow-up of any threefold along a curve of arbitrary large genus.

4.8. **Proposition.** *Let X be a projective threefold not birational to a conic bundle. Then, then the following hold:*

(i) *The gonality of the elements of* $\mathrm{Bir}(X)$ *is bounded.*
(ii) *If X is not birational to a del Pezzo fibration of degree* 3*, the genus of the elements of* $\mathrm{Bir}(X)$ *is also bounded.*

Proof. Using the minimal model program (MMP), we see that X is birational to a $\mathbb{Q}$-factorial variety X' with at most terminal singularities, which is either a Mori fibre space X'/B' or where $K_{X'}$ is nef. We may replace X with X', as this does not change the boundedness (but can a priori change the bound).

If K_X is nef, then the genus and the gonality of elements of $\mathrm{Bir}(X)$ are bounded, as no element of $\mathrm{Bir}(X)$ contracts any hypersurface.

We can then assume that X/B is a Mori fibre space. We have $\dim(B) \in \{0, 1\}$ as X is not birational to a conic bundle. If no Sarkisov link starts from X, the result is trivially true, since in this case Theorem 3.4 gives

$$\mathrm{Bir}(X) = \mathrm{Aut}(X).$$

Note that such threefolds do exists (see Remark 3.8). To complete the proof, we may assume that $\mathrm{Bir}(X) \neq \mathrm{Aut}(X)$, so that, in particular, there is a Sarkisov link that starts from X. If X/B is involved in a Sarkisov link over a base of dimension 0, the result follows from Lemma 4.5. So we may assume that the base of every Sarkisov link between Mori fibre spaces birational to X has dimension 1. Then $\dim(B) = 1$ and every Sarkisov link involved in any decomposition of any element $\varphi \in \mathrm{Bir}(X)$ is of type II between two del Pezzo fibrations. In particular, the del Pezzo fibration X/B is birationally rigid over B (see [Cor00, Definition 1.3] and Remark 3.8). Now the result then follows from Lemma 4.7. □

5. Del Pezzo fibrations of degree 3

We recall the following result, proven in [Cor96] (see also [Kol97] for generalisations):

5.1. **Proposition** ([Cor96, Theorem 1.10])**.** *Let B be a smooth curve and let $X \longrightarrow B$ be a del Pezzo fibration of degree* 3 (*cubic surface fibration*)*. Then, there exists a birational map*

$$\begin{array}{ccc} X & \overset{\varphi}{\dashrightarrow} & X' \\ & \searrow \quad \swarrow & \\ & B & \end{array}$$

such that X'/B is another del Pezzo fibration of degree 3 *having the following properties:*

(i) *X' is a projective threefold with terminal singularities of index* 1*;*
(ii) *every fibre of X'/B is reduced and irreducible, and is a Gorenstein del Pezzo surface;*
(iii) *the anti-canonical system $-K_{X'}$ is relatively very ample and defines an embedding in a $\mathbb{P}^3$-bundle over B.*

We will also need the following lemma:

5.2. **Lemma.** *Let $0 \in V \subset \mathbb{C}^4$ be an isolated cDV singularity and let $C \subset V$ be a smooth curve. Let $\sigma : \hat{V} \longrightarrow V$ be the blowup of C. Then $\hat{V}$ is normal.*

Proof. Let $x_1, \ldots, x_4$ be coordinates in $\mathbb{C}^4$. We may assume that C is the x_1-axis, given by $x_2 = x_3 = x_4 = 0$. As V contains C, it is given by the equation

$$\phi = x_2\phi_2 + x_3\phi_3 + x_4\phi_4 = 0,$$

where the functions $\phi_i = \phi_i(x_1, \ldots, x_4)$ vanish at the origin. The origin being a cDV singularity, at least one of ϕ_i's contains a linear term and at least one of the ϕ_i's contains a term x_1^k for some $k > 0$ (because V is smooth at a general point of C). By changing coordinates x_2, x_3, x_4 linearly, we may assume that ϕ_4 contains a non-zero linear term $\ell(x_1, x_4)$ (which depends on x_1, x_4). Note that the fibre $\sigma^{-1}(0) \subset \hat{V}$ is a plane $\mathbb{P}^2$ and $\hat{V}$ is a hypersurface in a nonsingular fourfold. By Serre's criterion it is sufficient to show that $\hat{V}$ is smooth at some point of $\sigma^{-1}(0)$. Consider the affine chart $U_4 := \{x_4' \neq 0\}$. Then $\sigma^{-1}(U_4)$ is given by

$$\{x_2'\phi_2' + x_3'\phi_3' + \phi_4' = 0\} \subset \mathbb{C}^4_{x_1', \ldots, x_4'},$$

where $\phi_i' = \phi_i(x_1', x_2'x_4', x_3'x_4', x_4')$. The fibre $\sigma^{-1}(0) \cap U_4$ in this chart is given by $x_1' = x_4' = 0$. Then

$$\mathrm{mult}_0(x_2'\phi_2') \geq 2, \quad \mathrm{mult}_0(x_3'\phi_3') \geq 2,$$

and the linear part $\ell(x_1, x_4)$ of ϕ_4' is nontrivial. Therefore, $\hat{V}$ is smooth at the origin of $\sigma^{-1}(U_4)$. $\square$

5.3. **Proposition.** *Let B be a smooth curve and let $X \longrightarrow B$ be a del Pezzo fibration of degree 3. Then, the genus of elements of* $\mathrm{Bir}(X/B)$ *is unbounded (even if the gonality can be bounded).*

Proof. We can assume that B is projective, and apply Proposition 5.1 to reduce to the case where X, B satisfy the conditions (i)-(iii) of this proposition.

We then take a curve $C \subset X$ which is a section of X/B (this exists as the field $\mathbb{C}(B)$ has the C_1 property, as before by [Kol96, Theorem IV.6.8, page 233]). We can moreover assume that a general point of C is not contained in any of the 27 lines of the corresponding fibre ([Kol96, Theorem IV.6.10, page 234]).

We denote by $\eta\colon \hat{X} \longrightarrow X$ the blow-up of C. Note that $\hat{X}$ is embedded in the blow-up $\hat{P}$ of P along C as a hypersurface. In particular, the canonical class $K_{\hat{X}}$ is a well-defined Cartier divisor. By Lemma 5.2, $\hat{X}$ is normal. Consider, for some large integer n, the divisor

$$D = -K_{\hat{X}} + n\hat{F},$$

where $\hat{F}$ is a general fibre of $\hat{X} \longrightarrow B$.

We first observe that D is base-point-free (for n big enough). To see this, we view X as a closed hypersurface of a $\mathbb{P}^3$-bundle $P \longrightarrow B$ (Condition (iii) of Proposition 5.1), and then view $\hat{X}$ as a closed hypersurface of the blow-up $\hat{P}$ of P along C. By the adjunction formula, the divisor D is the restriction of a divisor D_P on $\hat{P}$ which is equal, on each fibre of $\hat{P} \longrightarrow B$, to a strict transform of a hyperplane of $\mathbb{P}^3$ through the point blown-up. Taking n big enough, we obtain that D_P is without base-points, and this implies that D is base-point free.

Take two general elements D_1, D_2 of the linear system of D and consider the curve $\hat{Q} = D_1 \cap D_2$. Observe that $\hat{Q}$ intersects a general fibre $\hat{F}$ of $\hat{X} \longrightarrow B$ at two points. Indeed, for $i = 1, 2$, the intersection of D_i with a general fibre $\hat{F}$ is the strict transform of a hyperplane section of the cubic surface F, fibre of $X \longrightarrow B$, passing through the point $C \cap F$ blown-up by η. The two hyperplane sections intersect F into 3 points, so 2 outside of $C \cap F$.

We can then associate to $Q = \eta(\hat{Q})$ the birational involution $\varphi \in \mathrm{Bir}(X/B)$ which performs a Bertini involution on a general fibre F of X/B, associated to the two points $Q \cap F$: the fibre F is a smooth cubic surface in $\mathbb{P}^3$ and the blow-up of the two points $Q \cap F$ is a del Pezzo surface of degree 1 on which there is the unique Bertini involution associated to the double covering (see for instance [Isk96, Page 613]). Hence, the involution φ lifts to a birational involution of the blow-up $Y \longrightarrow X$ of Q, which is an isomorphism on a general fibre. There is then a surface birational to $Q \times \mathbb{P}^1$ contracted by φ, which corresponds to the union of two curves contracted onto the two points in each fibre.

It remains to see that $\hat{Q}$ is smooth and irreducible, with unbounded genus when n is taken big enough. This will give the result, as Q is then birational to $\hat{Q}$. Since the linear system $|D|$ is base point free and ample, it is not composed with a pencil. Since $\hat{X}$ is normal, it is smooth in codimension one and by Bertini's theorem we see that $\hat{Q} = D_1 \cap D_2$, where D_1, $D_2 \in |D|$ are general members, is a smooth curve contained in the smooth locus of $\hat{X}$.

By adjunction formula we get, for $i = 1, 2$,

$$\begin{array}{rcccl} K_{D_1} & = & (K_{\hat{X}} + D_1)|_{D_1} & = & n \cdot (\hat{F}|_{D_1}), \\ K_Q & = & (K_{D_1} + Q)|_{D_1} & = & (n\hat{F}|_{D_1} + D_2|_{D_1})|_Q, \end{array}$$

which gives

$$\begin{array}{rcl} \deg(-K_Q) & = & (n\hat{F} + D_2) \cdot D_1 \cdot D_2 \\ & = & (n\hat{F} + D) \cdot D^2 \\ & = & (-K_{\hat{X}} + 2n\hat{F}) \cdot (-K_{\hat{X}} + n\hat{F})^2 \\ & = & 4n\hat{F} \cdot (K_{\hat{X}})^2 - (K_{\hat{X}})^3 \\ & = & 4n - (K_{\hat{X}})^3 \end{array}$$

For n large enough, the curve $\hat{Q}$ cannot be reducible as its genus would then be bounded (because it would be the union of two disjoint curves isomorphic to B). The curve Q, birational to $\hat{Q}$, is again of unbounded genus, when n is large enough. □

The proof of Theorem 1.1 is now finished:

Proof of Theorem 1.1. Part (i) is given by Proposition 2.4.

Part (ii) is given by Proposition 5.3.

Parts (iii) and (iv) are respectively parts (i) and (ii) of Proposition 4.8. □

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Jérémy Blanc, Departement Mathematik und Informatik, Universität Basel, Spiegelgasse 1, 4051 Basel, Switzerland
E-mail address: jeremy.blanc@unibas.ch

Ivan Cheltsov, School of Mathematics, The University of Edinburgh, Edinburgh EH9 3JZ, UK.
E-mail address: I.Cheltsov@ed.ac.uk

Alexander Duncan, Department of Mathematics, University of South Carolina, Columbia, SC 29208, USA.
E-mail address: duncan@math.sc.edu

Yuri Prokhorov, Steklov Mathematical Institute of Russian Academy of Sciences, Moscow, Russia
E-mail address: prokhoro@mi-ras.ru