K-POLYSTABILITY OF TWO SMOOTH FANO THREEFOLDS

IVAN CHELTSOV AND HENDRIK SÜSS

ABSTRACT. We give new proofs of the K-polystability of two smooth Fano threefolds. One of them is a smooth divisor in $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^2$ of degree $(1,1,1)$, which is unique up to isomorphism. Another one is the blow up of the complete intersection

$$\Big\{x_0x_3+x_1x_4+x_2x_5=x_0^2+\omega x_1^2+\omega^2x_2^2+\big(x_3^2+\omega x_4^2+\omega^2x_5^2\big)+\big(x_0x_3+\omega x_1x_4+\omega^2x_2x_5\big)\Big\}\subset\mathbb{P}^5$$

in the conic cut out by $x_0=x_1=x_2=0$, where ω is a primitive cube root of unity.

1. Introduction

Let X be a smooth Fano threefold. Then X is contained in one of 105 families, which are explicitly described in [4], These families are labeled as №1.1, №1.2, ..., №9.1, №10.1, and members of each family can be parametrized by an irreducible rational variety.

Theorem 1.1 ([1]). *Suppose that X is a general member of the family №$\mathscr{N}$. Then*

$$X \textit{ is K-polystable} \iff \mathscr{N}\not\in\left\{\begin{array}{c}2.23, 2.262.28, 2.30, 2.31, 2.33, 2.35, 2.36, 3.14,\\ 3.16, 3.18, 3.21, 3.22, 3.23, 3.24, 3.26, 3.28, 3.29,\\ 3.30, 3.31, 4.5, 4.8, 4.9, 4.10, 4.11, 4.12, 5.2\end{array}\right\}.$$

In the proof of this theorem, many explicitly given smooth Fano threefolds has been proven to be K-polystable. Among them are the two threefolds described in the abstract.

Let G be a reductive subgroup in $\mathrm{Aut}(X)$, and let $f\colon\widetilde{X}\to X$ be a G-equivariant birational morphism with smooth $\widetilde{X}$, and let E be any G-invariant prime divisor in $\widetilde{X}$. We say that E is a G-invariant prime divisor *over* X, and we let $C_X(E)=f(E)$. Then

$$K_{\widetilde{X}}\sim f^*(K_X)+\sum_{i=1}^n a_iE_i$$

where $E_1,\ldots,E_n$ are f-exceptional surfaces, and $a_1,\ldots,a_n$ are strictly positive integers. If $E=E_i$ for some $i\in\{1,\ldots,n\}$, we let $A_X(E)=a_i+1$. Otherwise, we let $A_X(E)=1$. The number $A_X(E)$ is known as the log discrepancy of the divisor E. Then we let

$$S_X(E)=\frac{1}{(-K_X)^n}\int_0^\infty\mathrm{vol}\big(f^*(-K_X)-xE\big)dx$$

and $\beta(E)=A_X(E)-S_X(E)$. We have the following result:

Theorem 1.2 ([3, 6, 9]). *The smooth Fano threefold X is K-polystable if $\beta(F)>0$ for every G-invariant prime divisor F over X.*

Now, we let

$$\alpha_G(X)=\sup\left\{\epsilon\in\mathbb{Q}\;\middle|\;\begin{array}{l}\text{the log pair }\Big(X,\dfrac{\epsilon}{m}\mathcal{D}\Big)\text{ is log canonical for any }m\in\mathbb{Z}_{>0}\\ \text{and every }G\text{-invariant linear subsystem }\mathcal{D}\subset\big|-mK_X\big|\end{array}\right\}.$$

This number, known as the global log canonical threshold [2], has been defined in [8] in a different way. But both definitions agree by [2, Theorem A.3]. If G is finite, then

$$\alpha_G(X) = \sup\left\{\epsilon \in \mathbb{Q} \;\middle|\; \begin{array}{l}\text{the log pair } (X, \epsilon D) \text{ is log canonical for every} \\ G\text{-invariant effective } \mathbb{Q}\text{-divisor } D \sim_{\mathbb{Q}} -K_X\end{array}\right\}.$$

by [1, Lemma 1.4.1]. We have the following result:

Theorem 1.3 ([8, 1])**.** *If $\alpha_G(X) \geqslant \frac{3}{4}$, then X is K-polystable.*

In this short note, we give a new proof of the K-polystability of the threefolds described in the abstract using Theorems 1.2 and 1.3. This is done in Sections 2 and 3.

2. Smooth divisor in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ of degree $(1, 1, 1)$

Let X be the unique smooth Fano threefold in the family №3.17. Then X is the divisor

$$\Big\{x_0y_0z_2 + x_1y_1z_0 = x_0y_1z_1 + x_1y_0z_1\Big\} \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2,$$

where $([x_0 : x_1], [y_0 : y_1], [z_0 : z_1 : z_2])$ are coordinates on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$.

Let $G = \mathrm{Aut}(X)$. Then $G \cong \mathrm{PGL}_2(\mathbb{C}) \rtimes \boldsymbol{\mu}_2$, where $\boldsymbol{\mu}_2$ is generated by an involution ι that acts as

$$\big([x_0 : x_1], [y_0 : y_1], [z_0 : z_1 : z_2]\big) \mapsto \big([y_0 : y_1], [x_0 : x_1], [z_0 : z_1 : z_2]\big).$$

and $\mathrm{PGL}_2(\mathbb{C})$ acts on each factor via an appropriate irreducible $\mathrm{SL}_2(\mathbb{C})$-representation. More precisely, an element $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{PGL}_2(\mathbb{C})$ acts as follows:

$$\begin{aligned}\big([x_0 : x_1], [y_0 : y_1], [z_0 : z_1 : z_2]\big) \mapsto \big(&[ax_0 + cx_1 : bx_0 + dx_1], [ay_0 + cy_1 : by_0 + dy_1], \\ &[a^2z_0 + 2acz_1 + c^2z_2 : abz_0 + (ad + bc)z_1 + cdz_2 : b^2z_0 + 2bdz_1 + d^2z_2]\big)\end{aligned}$$

There are birational contractions $\pi_1 \colon X \to \mathbb{P}^1 \times \mathbb{P}^2$ and $\pi_2 \colon X \to \mathbb{P}^1 \times \mathbb{P}^2$ that contracts smooth irreducible surfaces E_1 and E_1 to smooth curves C_1 and C_2 of bi-degrees $(1, 2)$. Moreover, there exists $\mathrm{PGL}_2(\mathbb{C})$-equivariant commutative diagram

$$\begin{array}{ccccc} & & X & & \\ & \swarrow^{\pi_1} & & \searrow^{\pi_2} & \\ \mathbb{P}^1 \times \mathbb{P}^2 & & & & \mathbb{P}^1 \times \mathbb{P}^2 \\ & \searrow_{\mathrm{pr}_2} & & \swarrow_{\mathrm{pr}_2} & \\ & & \mathbb{P}^2 & & \end{array}$$

where pr_2 is the projection to the second factor, the $\mathrm{PGL}_2(\mathbb{C})$-action on $\mathbb{P}^2$ is faithful, and $\mathrm{pr}_2(C_1) = \mathrm{pr}_2(C_2)$ is the unique $\mathrm{PGL}_2(\mathbb{C})$-invariant conic, which is given by $z_0z_2 - z_1^2 = 0$.

By [1, Lemma 4.2.6], the threefold X is K-polystable. Let us give an alternative proof of this assertion.

Let $\mathrm{pr}_1 \colon \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^1$ be the projection to the first factor. Using $\mathrm{pr}_1 \circ \pi_1$ and $\mathrm{pr}_1 \circ \pi_2$, we obtain a $\mathrm{PGL}_2(\mathbb{C})$-equivariant $\mathbb{P}^1$-bundle $\phi \colon X \to \mathbb{P}^1 \times \mathbb{P}^1$, where the $\mathrm{PGL}_2(\mathbb{C})$-action on the surface $\mathbb{P}^1 \times \mathbb{P}^1$ is diagonal. Let $C = E_1 \cap E_2$. Then $\phi(C)$ is a diagonal curve. Denote its preimage on X by R. Then $C = R \cap E_1 \cap E_2$ and

$$-K_X \sim E_1 + E_2 + R.$$

Let $H_1 = (\mathrm{pr}_1 \circ \pi_1)^*(\mathcal{O}_{\mathbb{P}^1}(1))$, let $H_2 = (\mathrm{pr}_1 \circ \pi_2)^*(\mathcal{O}_{\mathbb{P}^1}(1))$ and let $H_L = (\mathrm{pr}_2 \circ \pi_2)^*(\mathcal{O}_{\mathbb{P}^2}(1))$. Then $\mathrm{Pic}(X) = \langle H_1, H_L, E_1\rangle$, $E_2 \sim 2H_L - E_1$, $R \sim H_1 + H_2 \sim 2H_1 + H_L - E_1$ and

$$-K_X \sim 2H_1 + 3H_L - E_1.$$

Observe that the curve C and the surface R are the only proper G-invariant irreducible subvarieties in X. This easily implies that $\alpha_G(X) = \frac{2}{3}$, so that we cannot apply Theorem 1.3 to prove that X is K-polystable. Let us apply Theorem 1.2 instead.

Let $\eta\colon Y \to X$ be a G-equivariant birational morphism, let D be a prime G-invariant divisor in Y, let t be a non-negative real number, and let

$$S_X(D,t) = \frac{1}{-K_X^3}\int_0^t \mathrm{vol}\big(\eta^*(-K_X) - xD\big)dx.$$

Then we have $S_X(D) = S(D,\infty)$ and $\beta(D) = A_X(D) - S_X(D)$. By Theorem 1.2, to prove that X is K-polystable it is enough to show that $\beta(D) > 0$. Let us first show this in the case when η is an identify map:

Lemma 2.1. *One has $S_X(R) = \frac{4}{9}$ and $\beta(R) = \frac{5}{9}$.*

Proof. Since $-K_X = E_1 + E_2 + R$, the pseudoeffective threshold $\tau(E)$ is 1, so that

$$S_X(R) = \frac{1}{-K_X^3}\int_0^1 (-K_X - xR)^3 dx = \\ = \int_0^1 -R^3x^3 + R^2(-K_X)x^2 - R(-K_X)^2 + (-K_X)^3\, dx = \\ = \frac{1}{36}\int_0^1 12x^2 - 48x + 36\, dx = \frac{4}{9}.$$

Since $A_X(R) = 1$, we have $\beta(R) = \frac{5}{9}$. ∎

Let $f\colon \widetilde{X} \to X$ be the blow-up of the curve C, let E be the exceptional surface of f, let $\widetilde{R}$, $\widetilde{E}_1$, $\widetilde{E}_2$ be the proper transforms on $\widetilde{X}$ of the surfaces R, E_1, E_2, respectively. Then

$$\begin{cases} \widetilde{E}_1 \sim f^*(E_1) - E, \\ \widetilde{E}_2 \sim 2f^*(H_L) - f^*(E_2) - E, \\ \widetilde{R} \sim f^*(2H_1 + H_L - E_1) - E. \end{cases}$$

Lemma 2.2. *One has $S_X(E) = \frac{11}{9}$ and $\beta(E) = \frac{7}{9}$. Moreover, if $0 \leqslant t \leqslant 1$, then*

$$S_X(E,t) = \frac{1}{36}\int_0^t (36 - 18t + 4x^3)dx = \frac{1}{36}t^4 - \frac{1}{4}t^3 + t.$$

Proof. We have

$$f^*(-K_X) - xE \sim f^*(R + E_1 + E_2) - xE \sim \widetilde{R} + \widetilde{E}_1 + \widetilde{E}_2 + (3-x)E.$$

so that $\tau(E) = 3$. If $0 \leqslant x \leqslant 1$, then $f^*(-K_X) - xE$ is nef. Thus, if $x \in [0,1]$, then

$$\mathrm{vol}(f^*(-K_X) - xE) = \Big(f^*(-K_X) - xE\Big)^3 = \\ = f^*(-K_X)^3 + 3x^2 f^*(-K_X)E^2 - x^3E^3 = 36 - 18x^2 + 4x^3.$$

If $3>x>1$, then both surfaces $\widetilde{E}_1$ and $\widetilde{E}_2$ lies in the asymptotic base locus of the big divisor $f^*(-K_X)-xE$. Moreover, if $x\in[1,2]$, then its Zariski decomposition is

$$f^*(-K_X)-xE\sim_{\mathbb{R}}\frac{1}{2}(x-1)\big(\widetilde{E}_1+\widetilde{E}_2\big)+\underbrace{\Big(f^*(-K_X)-xE-\frac{1}{2}(x-1)\big(\widetilde{E}_1+\widetilde{E}_2\big)\Big)}_{\text{nef part}}.$$

Thus, if $x\in[1,2]$, then we have

$$\mathrm{vol}(f^*(-K_X)+xE)=\Big(f^*(-K_X)-xE-\frac{1}{2}(x-1)\big(\widetilde{E}_1+\widetilde{E}_2\big)\Big)^3=6x^2-36x+52.$$

If $x\in(2,3)$, then the nef part of the Zariski decomposition of $f^*(-K_X)-xE$ is

$$f^*(-K_X)-xE-\frac{1}{2}(x-1)\big(\widetilde{E}_1+\widetilde{E}_2\big)+(x-2)\widetilde{R}.$$

Thus, if $x\in[2,3]$, then

$$\mathrm{vol}(f^*(-K_X)-xE)=\Big(f^*(-K_X)-xE-\frac{1}{2}(x-1)\big(\widetilde{E}_1+\widetilde{E}_2\big)+(x-2)\widetilde{R}\Big)^3=4(3-x)^3.$$

Summarizing and integrating, we see that

$$S_X(E)=\frac{1}{36}\int_0^1(36-18x^2+4x^3)dx+\frac{1}{36}\int_1^2(6x^2-36x+52)dx+\frac{1}{36}\int_2^3 4(3-x)^3dx=\frac{11}{9},$$

which gives $\beta(E)=\frac{7}{9}$, because $A_X(E)=2$. Similarly, we compute $S_X(E,t)$. □

The action of the group G lift to the threefold $\widetilde{X}$, and $E\cap\widetilde{R}$ is a G-invariant irreducible curve, which is contained in the pencil $|\widetilde{R}|_E|$. Therefore, using [7, Theorem 5.1], we see that the group $\mathrm{PGL}_2(\mathbb{C})$ must act trivially on the fibers of the natural projection $E\to C$. Since the curves $\widetilde{E}_1|_E$ and $\widetilde{E}_1|_E$ are swapped by G, we see conclude that $|\widetilde{R}|_E|$ contains exactly two G-invariant curves: $E\cap\widetilde{R}$ and another curve, which we denote by C'.

Now, let $g\colon\widehat{X}\to\widetilde{X}$ be the blow up of the curve C', let R' be the f-exceptional surface, let $\widehat{E}_1$, $\widehat{E}_2$, $\widehat{E}$, $\widehat{R}$ be the proper transforms on $\widehat{X}$ of the surfaces E_1, E_2, E, $\widetilde{R}$, respectively. Then we have

$$(f\circ g)^*(-K_X)\sim_{\mathbb{R}}\widehat{E}_1+\widehat{E}_2+\widehat{R}+3\widehat{E}+3R',$$

which implies that the pseudoeffective threshold $\tau(R')=3$. On the other hand, we have

Lemma 2.3. *One has $\beta(R')\geqslant\frac{5}{9}$.*

Proof. Let x be a non-negative real number such that $x<3$. Then $\widehat{E}$ lies in the stable base locus of the divisor $(f\circ g)^*(-K_X)-xF$, and the positive part of the Zariski decomposition of this divisor has the following form:

$$(f\circ g)^*(-K_X)-\frac{t}{2}\widehat{E}-xR'-D$$

for an effective $\mathbb{R}$-divisor D. Indeed, if ℓ is a general fiber of the projection $\widehat{E}\to C$, then

$$\Big((f\circ g)^*(-K_X)-xR'\Big)\cdot\ell=-x$$

and $\widehat{E}\cdot\ell=-2$, which implies the required assertion. Thus, we have

$$S_X(F)\leqslant 2S_X(E)=\frac{22}{9},$$

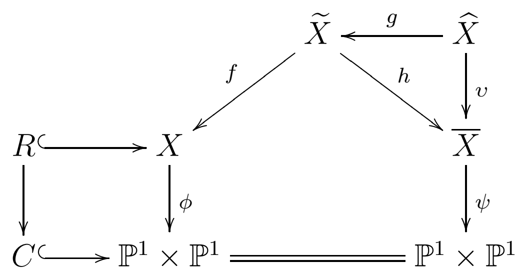
because $S_X(E) = \frac{11}{9}$ by Lemma 2.2. Then

$$\beta(F) = A_X(F) - S_X(F) = 3 - S_X(F) \geqslant 3 - \frac{22}{9} = \frac{5}{9}$$

as required. $\square$

The action of the group G lifts to $\widehat{X}$, and the surfaces R', $\widehat{E}$ and $\widehat{R}$ are G-invariant.

Remark 2.4. There exists the following G-equivariant commutative diagram:



where h is the contraction of the surface $\widetilde{R}$, υ is the contraction of the surfaces R' and $\widehat{R}$, and ψ is a $\mathbb{P}^1$-bundle. Moreover, one can show that $\overline{X} \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2,0) \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,2))$, so that there is an involution $\sigma \in \mathrm{Aut}(\overline{X})$ such that σ swaps the curves $\upsilon(R')$ and $\upsilon(\widehat{R})$. Then σ lifts to $\widehat{V}$ and swaps the divisors R' and $\widehat{R}$.

The threefold $\widetilde{X}$ contains two G-invariant irreducible curves: the curves $E \cap \widetilde{R}$ and C'. The threefold $\widehat{X}$ also contains just two G-invariant irreducible curves: $\widehat{E} \cap \widehat{R}$ and $\widehat{E} \cap R'$, which are swapped by the involution σ from Remark 2.4. Blowing up one of the curves, we obtain a new threefold that contains exactly three G-invariant irreducible curves that can be described in a very similar manner. Now, iterating this process, we obtain infinitely many G-invariant prime divisors over X, which can be described using weighted blow ups.

Definition 2.5. Let V be a smooth threefold that contains two smooth irreducible distinct surfaces A and B that intersect transversally along a smooth irreducible curve Z, and let $\theta\colon U \to V$ be the weighted blow up with weights (a, b) of the curve Z with respect to the local coordinates along Z that are given by the equations of the surfaces A and B, and let F be the exceptional surface of the weighted blow up θ. Then

- the morphism θ is said to be an (a, b)-blowup between A and B,
- the surface F is said to be an (a, b)-divisor between A and B.

Observe that $(1, 1)$-blow up in this construction is the usual blow up of the intersection curve. To proceed, we need the following well-known result:

Lemma 2.6. *In the assumptions of Definition 2.5 and notations introduced in this definition, suppose that $(a, b) = (1, 1)$ and $Z \cong \mathbb{P}^1$. Let $n = |\alpha - \beta|$, where α and β be integers such that*

$$Z^2 = \begin{cases} \alpha \text{ on the surface } A, \\ \beta \text{ on the surface } B. \end{cases}$$

Denote by $\widetilde{A}$ and $\widetilde{B}$ the proper transforms on U of the surfaces A and B, respectively. Then $F \cong \mathbb{F}_n$, the surfaces $\widetilde{A}$ and $\widetilde{B}$ are disjoint, $\widetilde{A}|_E$ and $\widetilde{B}|_E$ are sections of the natural projection $F \to Z$ such that $(\widetilde{A}|_E)^2 = (\beta - \alpha)$ and $(\widetilde{B}|_E)^2 = (\alpha - \beta)$.

Proof. Left to the reader. $\square$

Now, we are ready to prove

Lemma 2.7. *All G-invariant prime divisors over X can be described as follows:*

(1) *the surfaces R, E or R',*
(2) *an (a,b)-divisor between E and $\widetilde{R}$,*
(3) *an (a,b)-divisor between $\widehat{E}$ and R'.*

Proof. Let F be a G-invariant prime divisor over X such that F is different from R, E, R'. Then its center on $\widetilde{X}$ is one of the curves $E\cap\widetilde{R}$ or C'. Keeping in mind Remark 2.4, we may assume that its center on $\widetilde{X}$ is $E\cap\widetilde{R}$. Let us show that F is an exceptional divisor of a weighted blow up between the surfaces E and $\widetilde{R}$,

Let $V_0=X$ and $Z_0=E\cap\widetilde{R}$. Then there exists a sequence of G-equivariant blow ups

$$V_m \xrightarrow{\theta_m} V_{m-1} \xrightarrow{\theta_{m-1}} \cdots \xrightarrow{\theta_2} V_1 \xrightarrow{\theta_1} V_0$$

such that θ_1 is the blow up of the curve Z_0, the surface F is the θ_m-exceptional surface, the morphism θ_k is a blow up of a G-invariant irreducible smooth curve $Z_{k-1}\subset V_{k-1}$ such that the curve Z_{k-1} is contained in the θ_{k-1}-exceptional surface provided that $k\geqslant 2$.

For every $k\in\{1,\ldots,m\}$, let F_k be the θ_k-exceptional surface, so that we have $F=F_m$. To prove that $F=F_m$ is an exceptional divisor of a weighted blow up between E and $\widetilde{R}$, it sufficient to prove the following assertion for every k:

- the surface F_k contains exactly two $\mathrm{PGL}_2(\mathbb{C})$-invariant irreducible curves,
- the two $\mathrm{PGL}_2(\mathbb{C})$-invariant irreducible curves in F_k are disjoint,
- if $\mathscr{C}$ is a $\mathrm{PGL}_2(\mathbb{C})$-invariant irreducible curve in F_k, then $\mathscr{C}$ is cut out by the strict transform of one of the following surfaces:
  - the surface F_r for some $r\in\{1,\ldots,m\}$ such that $r\neq k$,
  - the surface E,
  - the surface $\widetilde{R}$.

Clearly, it is enough to prove this assertion only for $k=m$. Let us do this.

Let $F_0=E$ and $F_{-1}=\widetilde{R}$. For every $k\in\{-1,0,1,\ldots,m-1\}$, let $\overline{F}_k$ be the proper transform of the surface F_k on the threefold V_m. We claim that

(i) $F_m\cong\mathbb{F}_n$ for some $n>0$;
(ii) the surface F_m contains exactly two $\mathrm{PGL}_2(\mathbb{C})$-invariant irreducible curves,
(iii) the two $\mathrm{PGL}_2(\mathbb{C})$-invariant irreducible curves in F_m are disjoint,
(iv) if $\mathscr{C}$ is a $\mathrm{PGL}_2(\mathbb{C})$-invariant irreducible curve in F_m, then $\mathscr{C}^2\in\{-n,n\}$,
(v) if $\mathscr{C}$ is a $\mathrm{PGL}_2(\mathbb{C})$-invariant irreducible curve in F_m, then

$$\mathscr{C}=F_m\cap\overline{F}_r$$

for some $r\in\{-1,0,1,\ldots,m-1\}$ and the following assertions hold:
  - if $\mathscr{C}^2=n$ on the surface F_m, then $\mathscr{C}^2\leqslant 0$ on the surface $\overline{F}_r$,
  - if $\mathscr{C}^2=-n$ on the surface F_m, then $\mathscr{C}^2>0$ on the surface $\overline{F}_r$.

Let us prove this (stronger than we need) statement by induction on m.

Suppose that $m=1$. We already know that $F_0=E\cong\mathbb{P}^1\times\mathbb{P}^1$ and $F_{-1}=\widetilde{R}\cong\mathbb{P}^1\times\mathbb{P}^1$. Moreover, we have $Z_0^2=0$ on the surface F_0, and we have $Z_0^2=2$ on the surface F_{-1}. Then $F_1\cong\mathbb{F}_2$ by Lemma 2.6. Moreover, since $\mathrm{PGL}_2(\mathbb{C})$ acts faithfully on the curve Z_0, it acts faithfully on the surface F_1. Furthermore, if $\mathscr{C}$ is a $\mathrm{PGL}_2(\mathbb{C})$-invariant irreducible curve in F_1, then it follows from [7, Theorem 5.1] that either $\mathscr{C}=\overline{F}_0\cap F_1$ or $\mathscr{C}=\overline{F}_{-1}\cap F_1$. Using Lemma 2.6 again, we see that

- if $\mathscr{C}=\overline{F}_0\cap F_1$, then $\mathscr{C}^2=2$ on the surface F_1, and $\mathscr{C}^2=0$ on the surface $\overline{F}_0$,
- if $\mathscr{C}=\overline{F}_{-1}\cap F_1$, then $\mathscr{C}^2=-2$ on the surface F_1, while $\mathscr{C}^2=2$ on the surface $\overline{F}_0$.

Thus, we conclude that our claim holds for $m=1$. This is the base of induction.

Suppose that our claim holds for $m\geqslant 1$. Let us show that it holds for $m+1$ blow ups. Let $\mathscr{C}$ be a $\mathrm{PGL}_2(\mathbb{C})$-invariant irreducible curve in F_m, let $\Theta\colon\mathcal{V}\to V_m$ be its blow up, and let $\mathcal{F}$ be the Θ-exceptional surface. By induction, we know that $F_m\cong\mathbb{F}_n$ for $n>0$. Moreover, we also know that

$$\mathscr{C}=F_m\cap\overline{F}_r$$

for some $r\in\{-1,0,1,\ldots,m-1\}$. Furthermore, one of the following two assertions holds:

- either $\mathscr{C}^2=n>0$ on the surface F_m, and $\mathscr{C}^2\leqslant 0$ on the surface $\overline{F}_r$,
- or $\mathscr{C}^2=-n<0$ on the surface F_m, and $\mathscr{C}^2>0$ on the surface $\overline{F}_r$.

Let $\mathcal{F}_m$ and $\mathcal{F}_r$ be the strict transforms on $\mathcal{V}$ of the surfaces F_m and $\overline{F}_r$, respectively. Then $\mathcal{F}\cap\mathcal{F}_m$ and $\mathcal{F}\cap\mathcal{F}_r$ are disjoint $\mathrm{PGL}_2(\mathbb{C})$-invariant irreducible curves that are sections of the projection $\mathcal{F}\to\mathscr{C}$. Let γ be the self-intersection $\mathscr{C}^2$ on the surface $\overline{F}_r$. Then it follows from Lemma 2.6 that $F_{m+1}\cong\mathbb{F}_s$ for

$$s=n+|\gamma|>0.$$

Thus, by [7, Theorem 5.1], the curves $\mathcal{F}\cap\mathcal{F}_m$ and $\mathcal{F}\cap\mathcal{F}_r$ are the only $\mathrm{PGL}_2(\mathbb{C})$-invariant irreducible curves in the surface $\mathcal{F}$. Let $\mathcal{C}_1=\mathcal{F}\cap\mathcal{F}_m$ and $\mathcal{C}_2=\mathcal{F}\cap\mathcal{F}_r$.

Suppose that $\mathscr{C}^2=n$ on the surface F_m. In this case, we have $\gamma\leqslant 0$ and $s=n-\gamma>0$. By Lemma 2.6, we have $\mathcal{C}_1^2=n>0$ on the surface $\mathcal{F}_m$, and $\mathcal{C}_1^2=-s$ on the surface $\mathcal{F}$. Similarly, we see that $\mathcal{C}_2^2=\gamma\leqslant 0$ on the surface $\mathcal{F}_r$, and $\mathcal{C}_2^2=s>0$ on the surface $\mathcal{F}$. Thus, we see that the required claim holds for $m+1$ blow ups in this case.

Finally, we suppose that $\mathscr{C}^2=-n$ on the surface F_m. Then $\gamma>0$ and $s=n+\gamma>0$. By Lemma 2.6, we have $\mathcal{C}_1^2=-n<0$ on the surface $\mathcal{F}_m$, and $\mathcal{C}_1^2=s$ on the surface $\mathcal{F}$. Similarly, we have $\mathcal{C}_2^2=\gamma>0$ on the surface $\mathcal{F}_r$, and $\mathcal{C}_2^2=-s<0$ on the surface $\mathcal{F}$. Therefore, we proved that the required claim holds for $m+1$ blow up also in this case. Hence, it holds for any number of blow ups (by induction). □

By Lemmas 2.1, 2.2, 2.3, we have $\beta(R)>0$, $\beta(E)>0$, $\beta(R')>0$, respectively. Thus, to prove that X is K-polystable, it is enough to check that $\beta(F)>0$ in the following cases:

(1) when F is the (a,b)-divisor between E and $\widetilde{R}$,
(2) when F is the (a,b)-divisor between $\widehat{E}$ and R'.

We start with the first case.

Proposition 2.8. *Let $\nu\colon Y\to\widetilde{X}$ be the (a,b)-blow up between the surfaces E and $\widetilde{R}$, and let F be the ν-exceptional surface. Then $\beta(F)>0$.*

Proof. Let $\overline{E}_1$, $\overline{E}_2$, $\overline{E}$, $\overline{R}$ be the proper transforms on Y of the surfaces E_1, E_2, E, $\widetilde{R}$, respectively. Take a non-negative real number x. Put $\eta=f\circ\nu$. Then

$$\eta^*(-K_X)-xF\sim_{\mathbb{R}}\overline{E}_1+\overline{E}_2+\overline{R}+3\overline{E}+(a+3b-x)F,$$

so that the pseudoeffective threshold $\tau=\tau(F)$ is at least $a+3b$.

Suppose that $x < \tau$. Then $\overline{E}$ lies in the stable base locus of the divisor $\eta^*(-K_X) - xF$. Moreover, we claim that the positive part of the Zariski decomposition of this divisor has the following form:

$$\eta^*(-K_X) - \frac{t}{a+b}\overline{E} - xF - D$$

for an effective $\mathbb{R}$-divisor D. Indeed, if ℓ is a general fiber of the projection $\overline{E} \to C$, then

$$\Big(\eta^*(-K_X) - xF\Big) \cdot \ell = -\frac{x}{a},$$

because $\eta^*(-K_X) \cdot \ell = 0$ and $F \cdot \ell = \frac{1}{a}$. On the other hand, we have $\overline{E} \cdot \ell = -\frac{a+b}{a}$, which implies the required claim. Thus, if $7b > 2a$, then

$$S_X(F) \leqslant (a+b) S_X(E) = \frac{11}{9}(a+b),$$

because $S_X(E) = \frac{11}{9}$ by Lemma 2.2. Thus, if $\frac{b}{a} > \frac{2}{7}$, then

$$\beta(F) = A_X(F) - S_X(F) = a + 2b - S_X(F) \geqslant a + 2b - \frac{11}{9}(a+b) = \frac{7b - 2a}{9} > 0$$

as required. Hence, we may assume that $\frac{b}{a} \leqslant \frac{2}{7}$.

If $x > 2b$, then the surface $\overline{R}$ lies in the stable base locus of the divisor $\eta^*(-K_X) - xF$. Moreover, in this case, the Zariski decomposition of this divisor has the following the form:

$$\eta^*(-K_X) - \frac{t}{a+b}\overline{E} - \frac{t-2b}{a+b}\overline{R} - xF - D$$

for some effective $\mathbb{R}$-divisor D (supported in $\overline{E}_1$, $\overline{E}_2$, $\overline{E}$, $\overline{R}$, F). Indeed, if ℓ is a general fiber of the natural projection $\overline{R} \to \phi(C)$. Then $\overline{R} \cdot \ell = -\frac{a+b}{b}$ and

$$\Big(\eta^*(-K_X) - xF\Big) \cdot \ell = 2 - \frac{x}{b},$$

which implies that the Zariski decomposition has the required form for $x > 2b$. Then

$$S_X(F) \leqslant \frac{1}{36}\int_0^{2b} \mathrm{vol}\left(\varphi^*(-K_X) - \frac{t}{a+b}E\right) dt + \frac{1}{36}\int_{2b}^{\infty} \mathrm{vol}\left(\varphi^*(-K_X) - \frac{t-2b}{a+b}R\right) dt =$$
$$= (a+b) \cdot S\left(E, \frac{2b}{a+b}\right) + (a+b) \cdot S(R) < \frac{5}{9}(a+b) + \frac{4}{9}(a+b) = a+b.$$

because we have $S(R) = \frac{4}{9}$ by Lemma 2.1, and we have $S\big(E, \frac{2b}{a+b}\big) < \frac{5}{9}$ by Lemma 2.2. This gives $\beta(F) > 0$, since $A_X(F) = a + 2b$. □

Finally, we deal with (a,b)-divisors between $\widehat{E}$ and R'.

Proposition 2.9. *Let $\nu\colon Y \to \widehat{X}$ be the (a,b)-blow up between the surfaces $\widehat{E}$ and R', and let F be the ν-exceptional surface. Then $\beta(F) > 0$.*

Proof. Let $\overline{E}_1$, $\overline{E}_2$, $\overline{E}$, $\overline{R}$, $\overline{R}'$ be the proper transforms on Y of E_1, E_2, E, $\widetilde{R}$, R', respectively. Take a non-negative real number x. Put $\eta = f \circ g \circ \nu$. Then

$$\eta^*(-K_X) - xF \sim_{\mathbb{R}} \overline{E}_1 + \overline{E}_2 + \overline{R} + 3\overline{E} + 3\overline{R}' + (3a + 3b - x)F,$$

so that the pseudoeffective threshold $\tau = \tau(F)$ is at least $3a + 3b$.

Suppose that $x < \tau$. Then $\overline{E}$ lies in the stable base locus of the divisor $\eta^*(-K_X) - xF$. Moreover, we claim that the positive part of the Zariski decomposition of this divisor has the following form:

$$\eta^*(-K_X) - \frac{t}{2a+b}\overline{E} - xF - D$$

for an effective $\mathbb{R}$-divisor D. Indeed, if ℓ is a general fiber of the projection $\overline{E} \to C$, then

$$\Big(\eta^*(-K_X) - xF\Big) \cdot \ell = -\frac{x}{a},$$

because $\eta^*(-K_X) \cdot \ell = 0$ and $F \cdot \ell = \frac{1}{a}$. On the other hand, we have $\overline{E} \cdot \ell = -\frac{2a+b}{a}$, which implies the required claim. Thus, we have

$$S_X(F) \leqslant (2a+b) S_X(E) = \frac{11}{9}(2a+b),$$

because $S_X(E) = \frac{11}{9}$ by Lemma 2.2. Then

$$\beta(F) = A_X(F) - S_X(F) = 3a + 2b - S_X(F) \geqslant 3a + 2b - \frac{11}{9}(a+b) = \frac{5a+7b}{9} > 0$$

as required. □

Thus, we see that $\beta(F) > 0$ for every G-invariant prime divisor F over the threefold X. Then X is K-polystable by Theorem 1.2.

3. Blow up of a complete intersection of two quadrics in a conic

Let $Q_1 = \{f = 0\} \subset \mathbb{P}^5$, where $f = x_0x_3 + x_1x_4 + x_2x_5$, and let $Q_2 = \{g = 0\} \subset \mathbb{P}^5$, where $g = x_0^2 + \omega x_1^2 + \omega^2 x_2^2 + (x_3^2 + \omega x_4^2 + \omega^2 x_5^2) + (x_0x_3 + \omega x_1x_4 + \omega^2 x_2x_5)$, and ω is a primitive cubic root of unity. Let $V_4 = Q_1 \cap Q_2$. Then V_4 is smooth. Let G be a subgroup in $\mathrm{Aut}(\mathbb{P}^5)$ such that $G \cong \boldsymbol{\mu}_2^2 \rtimes \boldsymbol{\mu}_3$, where the generator of $\boldsymbol{\mu}_3$ acts by

$$\big[x_0 : x_1 : x_2 : x_3 : x_4 : x_5\big] \mapsto \big[x_1 : x_2 : x_0 : x_4 : x_5 : x_3\big],$$

the generator of the first factor of $\boldsymbol{\mu}_2^2$ acts by

$$\big[x_0 : x_1 : x_2 : x_3 : x_4 : x_5\big] \mapsto \big[-x_0 : x_1 : -x_2 : -x_3 : x_4 : -x_5\big],$$

and the generator of the second factor of $\boldsymbol{\mu}_2^2$ acts by

$$\big[x_0 : x_1 : x_2 : x_3 : x_4 : x_5\big] \mapsto \big[-x_0 : -x_1 : x_2 : -x_3 : -x_4 : x_5\big].$$

Then $G \cong \mathfrak{A}_4$, and $\mathbb{P}^5 = \mathbb{P}(\mathbb{U}_3 \oplus \mathbb{U}_3)$, where $\mathbb{U}_3$ is the unique (unimodular) irreducible three-dimensional representation of the group G. Note that Q_1 and Q_2 are G-invariant, so that V_4 is also G-invariant. Thus, we may identify G with a subgroup in $\mathrm{Aut}(V_4)$.

Let τ be an involution in $\mathrm{Aut}(\mathbb{P}^5)$ that is given by

$$\big[x_0 : x_1 : x_2 : x_3 : x_4 : x_5\big] \mapsto \big[x_3 : x_4 : x_5 : x_0 : x_1 : x_2\big].$$

Then Q_1 and Q_2 are τ-invariant, so that V_4 is also τ-invariant.

The group G does not have fixed points in $\mathbb{P}^5$, and there are no G-invariant lines in $\mathbb{P}^5$. Moreover, every G-invariant plane in $\mathbb{P}^5$ is given by

$$\begin{cases} \lambda x_0 + \mu x_3 = 0, \\ \lambda x_1 + \mu x_4 = 0, \\ \lambda x_2 + \mu x_5 = 0, \end{cases}$$

where $[\lambda : \mu] \in \mathbb{P}^1$. Using this, we see that V_4 contains exactly four G-invariant conics. These conics are cut out on V_4 by the following G-invariant planes: the plane Π_1 given by $x_0 = x_1 = x_2 = 0$, the plane $\Pi_2 = \tau(\Pi_1)$, the plane Π_3 given by

$$\begin{cases} x_0 = \omega x_3, \\ x_1 = \omega x_4, \\ x_2 = \omega x_5, \end{cases}$$

and the plane $\Pi_4 = \tau(\Pi_3)$. We let $C_1 = V_4 \cap \Pi_1$, $C_2 = V_4 \cap \Pi_2$, $C_3 = V_4 \cap \Pi_3$, $C_4 = V_4 \cap \Pi_4$. Then the conics C_1, C_2, C_3, C_4 are pairwise disjoint, $C_2 = \tau(C_1)$ and $C_4 = \tau(C_3)$,

For every $i \in \{1, 2, 3, 4\}$, we let $\pi_i \colon X_i \to V_4$ be the blow up of the conic C_i, and we denote by E_i the exceptional surface of the blow up π_i. Then $X_1 \cong X_2$ and $X_3 \cong X_4$ are smooth Fano threefolds №2.16, and the action of the group G lifts to its action on them.

For every $i \in \{1, 2, 3, 4\}$, we have the following G-equivariant diagram:

$$\begin{array}{ccc} & X_i & \\ {}^{\pi_i}\swarrow & & \searrow^{\eta_i} \\ V_4 & \dashrightarrow & \mathbb{P}^2 \end{array}$$

where the dashed arrow is a linear projection from the plane Π_i, and η_i is a conic bundle that is given by the linear system $|\pi_i^*(H) - E_i|$, where H is a hyperplane section of the threefold V_4. In each case, we have $\mathbb{P}^2 = \mathbb{P}(\mathbb{U}_3)$.

Lemma 3.1. *One has $E_1 \cong E_2 \cong E_3 \cong E_4 \cong \mathbb{P}^1 \times \mathbb{P}^1$.*

Proof. One has $E_i \cong \mathbb{F}_n$ for some integer $n \geqslant 0$. We have $-E_i|_{E_i} \sim s_{E_i} + a f_{E_i}$ where s_{E_i} is a section of the projection $E_i \to C_i$ such that $s_{E_i}^2 = -n$, and f_{E_i} is a fiber of this projection. Since $E_i^3 = 2 + K_{V_4} \cdot C_i = -2$, we have $-2 = (s_{E_i} + a f_{E_i})^2 = -n + 2a$, so that $a = \frac{n-2}{2}$. On the other hand, we have $(\pi_i^*(H) - E_i)|_{E_i} \sim s_{E_i} + \frac{n+2}{2} f_{E_i}$. Since $|\pi_i^*(H) - E_i|$ is base point free, we have $\frac{n+2}{2} \geqslant n$, so that either $n = 0$ or $n = 2$. If $n = 2$, then s_{E_i} is contracted by η_i to a point, which is impossible, since G does not have fixed points in $\mathbb{P}^2$. Hence, we see that $n = 0$, so that $E_i \cong \mathbb{P}^1 \times \mathbb{P}^1$. □

For each $i \in \{1, 2, 3, 4\}$, let Δ_i be the discriminant curve in $\mathbb{P}^2$ of the conic bundle η_i. Then Δ_i is a (possibly reducible) quartic curve with at most ordinary double points.

Lemma 3.2. *The curves Δ_1, Δ_2, Δ_3, Δ_4 are smooth.*

Proof. If $i = 1$, then the linear projection $V_4 \dashrightarrow \mathbb{P}^2$ from the plane Π_1 is given by

$$\big[x_0 : x_1 : x_2 : x_3 : x_4 : x_5\big] \mapsto \big[x_0 : x_1 : x_2\big].$$

Using this, one can deduce that Δ_1 is given by $4x_0^4 - x_0^2 x_1^2 - x_0^2 x_2^2 + 4x_1^4 - x_1^2 x_2^2 + 4x_2^4 = 0$. This curve is smooth. Thus, the curve $\Delta_2 \cong \Delta_1$ is also smooth.

Let $y_0 = x_0 - \omega x_3$, $y_1 = x_1 - \omega x_4$, $y_2 = x_2 - \omega x_5$, $y_3 = x_3$, $y_4 = x_4$, $y_5 = x_5$. In new coordinates, the linear projection $V_4 \dashrightarrow \mathbb{P}^2$ from the plane Π_3 is given by

$$\big[y_0 : y_1 : y_2 : y_3 : y_4 : y_5\big] \mapsto \big[y_0 : y_1 : y_2\big].$$

Then Δ_3 is given by $4x_0^4 - \omega x_0^2 x_1^2 + (\omega + 1) x_2^2 x_0^2 - 4(\omega + 1) x_1^4 - x_1^2 x_2^2 + 4\omega x_2^4$. This curve is smooth, so that $\Delta_4 \cong \Delta_3$ is also smooth. □

Observe that $\mathbb{P}^2 = \mathbb{P}(\mathbb{U}_3)$ has three G-invariant conics. Denote them by $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$, and denote by $F_{1,i}$, $F_{2,i}$ and $F_{3,i}$ their preimages on X_i via η_i, respectively. Then

$$F_{1,i} \sim F_{2,i} \sim F_{3,i} \sim \pi_i^*(2H) - 2E_i.$$

For every $i \in \{1, 2, 3, 4\}$ and $j \in \{1, 2, 3\}$, let $\overline{F}_{j,i} = \pi_i(F_{j,i})$. Then $\overline{F}_{j,i}$ is an irreducible surface in $|2H|$ that is singular along the conic C_i. Without loss of generality, we may assume that $\overline{F}_{1,1}$ is cut out on V_4 by the equation $f_{1,1} = 0$ for $f_{1,1} = x_0^2 + x_1^2 + x_3^2$, and the surface $\overline{F}_{2,1}$ is cut out on V_4 by the equation $f_{2,1} = 0$ for $f_{2,1} = x_0^2 + \omega x_1^2 + \omega^2 x_3^2$. Then the surface $\overline{F}_{3,1}$ is cut out on V_4 by the equation $f_{3,1} = 0$, where $f_{3,1} = x_0^2 + \omega^2 x_1^2 + \omega x_3^2$. Using the involution τ, we also see that $\overline{F}_{1,2} = \tau(\overline{F}_{1,1})$, $\overline{F}_{2,2} = \tau(\overline{F}_{2,1})$ and $\overline{F}_{3,2} = \tau(\overline{F}_{3,1})$, so that we let $f_{1,2} = \tau^*(f_{1,1})$, $f_{2,2} = \tau^*(f_{2,1})$ and $f_{3,2} = \tau^*(f_{3,1})$. Then $\overline{F}_{1,3}$ is cut out by $f_{1,3} = 0$, where $f_{1,3} = (x_0 - \omega x_3)^2 + (x_1 - \omega x_4)^2 + (x_2 - \omega x_5)^2$. Likewise, the surface $\overline{F}_{2,3}$ is cut out on V_4 by the equation $f_{2,3} = 0$, where $f_{2,3} = (x_0 - \omega x_3)^2 + \omega(x_1 - \omega x_4)^2 + \omega^2(x_2 - \omega x_5)^2$, Similarly, $\overline{F}_{3,3}$ is cut out by $f_{3,3} = 0$, where $f_{3,3} = (x_0 - \omega x_3)^2 + \omega^2(x_1 - \omega x_4)^2 + \omega(x_2 - \omega x_5)^2$. Finally, we conclude that $\overline{F}_{1,4} = \tau(\overline{F}_{1,3})$, $\overline{F}_{2,4} = \tau(\overline{F}_{2,3})$ and $\overline{F}_{3,4} = \tau(\overline{F}_{3,3})$, so that we let $f_{1,4} = \tau^*(f_{1,3})$, $f_{2,4} = \tau^*(f_{2,3})$ and $f_{3,4} = \tau^*(f_{3,3})$.

Remark 3.3. The incidence relation between the surfaces $\overline{F}_{1,1}$, $\overline{F}_{2,1}$, $\overline{F}_{3,1}$, $\overline{F}_{1,2}$, $\overline{F}_{2,2}$, $\overline{F}_{3,2}$, $\overline{F}_{1,3}$, $\overline{F}_{2,3}$, $\overline{F}_{3,3}$, $\overline{F}_{1,4}$, $\overline{F}_{2,4}$, $\overline{F}_{3,4}$ and the conics C_1, C_2, C_3, C_4 is described in the following table:

	$\overline{F}_{1,1}$	$\overline{F}_{2,1}$	$\overline{F}_{3,1}$	$\overline{F}_{1,2}$	$\overline{F}_{2,2}$	$\overline{F}_{3,2}$	$\overline{F}_{1,3}$	$\overline{F}_{2,3}$	$\overline{F}_{3,3}$	$\overline{F}_{1,4}$	$\overline{F}_{2,4}$	$\overline{F}_{3,4}$
C_1	Node	Node	Cusp	No	Yes	No	No	Yes	No	No	Yes	No
C_2	No	Yes	No	Node	Node	Cusp	No	Yes	No	No	Yes	No
C_3	Yes	No	No	Yes	No	No	Node	Node	Cusp	Yes	No	No
C_4	Yes	No	No	Yes	No	No	Yes	No	No	Node	Node	Cusp

Here, No means that the surface does not contains the conic, and in all other cases the surface contains the conic. Likewise, Node means the the surface has an ordinary double point in general point of the conic, and Cusp means that the surface has an ordinary cusp in general point of the conic. In all remaining cases the surface is smooth at general point of the conic (we will see later that it is smooth along this conic).

Corollary 3.4. *For every* $i \in \{1, 2, 3, 4\}$*, one has* $\alpha_G(X_i) \leqslant \frac{3}{4}$.

Proof. Observe that $F_{3,i} + E_i \sim -K_{X_i}$. Moreover, it follows from Remark 3.3 that the surface $F_{3,i}$ is tangent to E_i along a section of the projection $E_i \to C_i$. Thus, we conclude that $\alpha_G(X_i) \leqslant \mathrm{lct}(X_i, F_{3,i} + E_i) \leqslant \frac{3}{4}$ as required. $\square$

Recall that the group $G \cong \boldsymbol{\mu}_2^2 \rtimes \boldsymbol{\mu}_3$ has three different one-dimensional representations: the trivial representation with the character χ_0, the non-trivial representation with the character χ_1 that sends the generator of $\boldsymbol{\mu}_3$ to ω, and the non-trivial representation with the character χ_2 that sends the generator of $\boldsymbol{\mu}_3$ to ω^2. On the other hand, the polynomials f, g, $f_{1,1}$, $f_{2,1}$, $f_{3,1}$, $f_{1,2}$, $f_{2,2}$, $f_{3,2}$, $f_{1,3}$, $f_{2,3}$, $f_{3,3}$, $f_{1,4}$, $f_{2,4}$, $f_{3,4}$ are semi-invariants of the group G considered as a subgroup in $\mathrm{SL}_6(\mathbb{C})$. These polynomials split into three groups with respect to the characters χ_0, χ_1 and χ_2 as follows:

(χ_0) f, $f_{1,1}$, $f_{1,2}$, $f_{1,3}$, $f_{1,4}$ are G-invariants;
(χ_1) $f_{3,1}$, $f_{3,2}$, $f_{3,3}$, $f_{3,4}$ are G-semi-invariants with character χ_1;
(χ_2) g, $f_{2,1}$, $f_{2,2}$, $f_{2,3}$, $f_{2,4}$ are G-semi-invariants with character χ_2.

Note that $f_{1,4} = -(\omega+2)f_{1,1}+(\omega+2)f_{1,2}+f_{1,3}$ and $(\omega+1)f_{1,1}-\omega f_{1,2}-(\omega+1)f_{1,3}+2f = 0$, which implies that $\overline{F}_{1,1}$, $\overline{F}_{1,2}$, $\overline{F}_{1,3}$, $\overline{F}_{1,4}$ generate a pencil on V_4, which we denote by $\mathcal{P}_0$. Similarly, we have $f_{3,4} = -(\omega+2)f_{3,1} + (\omega+2)f_{3,2} + f_{3,3}$, and the surfaces $\overline{F}_{3,1}$, $\overline{F}_{3,2}$, $\overline{F}_{3,3}$, $\overline{F}_{3,4}$ generate two-dimensional linear system (net), which we denote by $\mathcal{M}_1$. This linear system $\mathcal{M}_1$ contains four pencils, which we denote by $\mathcal{P}_{1,1}$, $\mathcal{P}_{1,2}$, $\mathcal{P}_{1,3}$ and $\mathcal{P}_{1,4}$, that consist of surfaces containing the conics C_1, C_2, C_3 and C_4, respectively. Likewise, we have $f_{2,4} = -(\omega+2)f_{2,1}+(\omega+2)f_{2,2}+f_{2,3}$ and $(\omega-1)f_{2,1}-(\omega+2)f_{2,2}-(\omega+1)f_{2,3}+2g = 0$, so that $\overline{F}_{2,1}$, $\overline{F}_{2,2}$, $\overline{F}_{2,3}$, $\overline{F}_{2,4}$ generates a pencil on V_4, which we denote by $\mathcal{P}_2$.

For every $i \in \{1,2,3,4\}$, denote by $\mathcal{P}_0^i$, $\mathcal{P}_{1,1}^i$, $\mathcal{P}_{1,2}^i$, $\mathcal{P}_{1,3}^i$, $\mathcal{P}_{1,4}^i$ and $\mathcal{P}_2^i$ the strict transforms on X_i of the pencils $\mathcal{P}_0$, $\mathcal{P}_{1,1}$, $\mathcal{P}_{1,2}$, $\mathcal{P}_{1,3}$, $\mathcal{P}_{1,4}$ and $\mathcal{P}_2$. Then

$$\begin{aligned}
\mathcal{P}_{1,1}^1 &\sim \mathcal{P}_2^1 \sim -K_{X_1},\\
\mathcal{P}_{1,2}^2 &\sim \mathcal{P}_2^2 \sim -K_{X_2},\\
\mathcal{P}_{1,3}^3 &\sim \mathcal{P}_0^3 \sim -K_{X_3},\\
\mathcal{P}_{1,4}^4 &\sim \mathcal{P}_0^4 \sim -K_{X_4}.
\end{aligned}$$

Moreover, we have $F_{3,1}+E_1 \in \mathcal{P}_{1,1}^1$, $F_{2,1}+E_1 \in \mathcal{P}_2^1$ $F_{3,2}+E_2 \in \mathcal{P}_{1,2}^2$, $F_{2,2}+E_2 \in \mathcal{P}_2^2$, $F_{3,3}+E_3 \in \mathcal{P}_{1,3}^3$, $F_{1,3}+E_3 \in \mathcal{P}_0^3$, $F_{3,4}+E_4 \in \mathcal{P}_{1,4}^4$, $F_{1,4}+E_3 \in \mathcal{P}_0^4$. Thus, we see that the restrictions $\mathcal{P}_{1,1}^1|_{X_1}$, $\mathcal{P}_2^1|_{X_1}$, $\mathcal{P}_{1,2}^2|_{X_2}$, $\mathcal{P}_2^2|_{X_2}$, $\mathcal{P}_{1,3}^3|_{X_3}$, $\mathcal{P}_0^3|_{X_3}$, $\mathcal{P}_{1,4}^4|_{X_4}$, $\mathcal{P}_0^4|_{X_4}$ are G-invariant curves in E_1, E_2, E_3, E_4, respectively. Denote them by Z_1, Z_1', Z_2, Z_2', Z_3, Z_3', Z_4, Z_4', respectively. Observe that $Z_1 \neq Z_1'$, $Z_2 \neq Z_2'$, $Z_3 \neq Z_3'$ and $Z_4 \neq Z_4'$. This follows from the exact sequence of G-representations

$$0 \to H^0\Big(\mathcal{O}_{X_i}\big(-K_{X_i}-E_i\big)\Big) \to H^0\Big(\mathcal{O}_{X_i}\big(-K_{X_i}\big)\Big) \twoheadrightarrow H^0\Big(\mathcal{O}_{E_i}\big(-K_{X_i}\big|_{E_i}\big)\Big),$$

where the surjectivity of the last map follows from Kodaira vanishing. Alternatively, one can show this using the explicit equations of the pencils $\mathcal{P}_0$, $\mathcal{P}_{1,1}$, $\mathcal{P}_{1,2}$, $\mathcal{P}_{1,3}$, $\mathcal{P}_{1,4}$ and $\mathcal{P}_2$.

Recall that $E_1 \cong E_2 \cong E_3 \cong E_4 \cong \mathbb{P}^1 \times \mathbb{P}^1$ by Lemma 3.1. For every $i \in \{1,2,3,4\}$, let s_{E_i} be a section of the projection $E_i \to C_i$ such that $s_{E_i}^2 = 0$, and let f_{E_i} be a fiber of this projection. Then $-E_i|_{E_i} = s_{E_i} - f_{E_i}$, so that $-K_{X_i} \sim s_{E_i} + 3f_{E_i}$. Hence, we see that $Z_i \sim Z_i' \sim s_{E_i} + 3f_{E_i}$, which immediately implies that both curve Z_i and Z_i' are irreducible, because C_i does not have G-orbits of lengths 1, 2 and 3.

For each $i \in \{1,2,3,4\}$, the conic bundle η_i gives a double cover $E_i \to \mathbb{P}^2$, whose branching curve is $\mathcal{C}_3$. Indeed, one has $F_{3,i} \sim \pi_i^*(2H) - 2E_i$, and $\overline{F}_{3,i}$ has a cusp at general point of the conic C_i. Since $F_{3,i}|_{E_i} \sim 2s_{E_i} + 2f_{E_i}$, we have $\overline{F}_{3,i}|_{E_i} = 2C_i^i$ for some irreducible curve $C_i^i \in |s_{E_i} + f_{E_i}|$. Since the double cover $E_i \to \mathbb{P}^2$ is given by a linear subsystem in $|s_{E_i} + f_{E_i}|$, we conclude that $\eta_i(C_i^i)$ is the branching curve of this double cover. But $\eta_i(C_i^i) = \mathcal{C}_3$, since $F_{3,i}$ is the preimage of the curve $\mathcal{C}_3$ via η_i.

For every i and j in $\{1,2,3,4\}$ such that $j \neq i$, denote by C_j^i the strict transform of the conic C_j on the threefold X_i. Then $-K_{X_i}\cdot C_1^i = -K_{X_i}\cdot C_2^i = -K_{X_i}\cdot C_3^i = -K_{X_i}\cdot C_4^i = 4$ and $-K_{X_i}\cdot Z_i = -K_{X_i}\cdot Z_i' = 6$. Observe also that C_1^i, C_2^i, C_3^i, C_4^i, Z_i, Z_i' are smooth rational curves. Moreover, we have the following result:

Lemma 3.5. *Let C be an irreducible G-invariant curve in X_i such that $C \cong \mathbb{P}^1$ and $-K_{X_i} \cdot C < 8$. Then C is one of the curves C_1^i, C_2^i, C_3^i, C_4^i, Z_i, Z_i'.*

Proof. The proof is the same for every $i \in \{1, 2, 3, 4\}$. Thus, for simplicity of notations, we assume that $i = 1$. Suppose that C is not one of the curves C_1^1, C_2^1, C_3^1, C_4^1, Z_1, Z_1'. Let us seek for a contradiction.

First, we suppose that $C \subset E_1$. Then $C \sim a s_{E_1} + b f_{E_1}$ for some non-negative integers a and b. Since $-K_{X_1}|_{E_1} \sim s_{E_1} + 3 f_{E_1}$, we see that $3a + b = -K_{X_i} \cdot C < 8$. Moreover, since $C_1^1 \cdot C = a + b$, we conclude that $a + b \geqslant 4$ and $a + b \neq 5$, because C_1^1 does not have G-orbits of lengths 1, 2, 3 and 5. Thus, since C is irreducible, we conclude that $a = 1$ and $b = 3$.

Let us describe the action of G on the surface $E_1 \cong \mathbb{P}^1 \times \mathbb{P}^1$. Since G acts faithfully on $C_1 \cong \mathbb{P}^1$, this action is given by the unique (unimodular) irreducible two-dimensional representation of the central extension $2.G \cong \mathrm{SL}_2(\mathbb{F}_3)$ of the group G, which we denote by $\mathbb{W}_3$. Since $|s_{E_1} + f_{E_1}|$ contains a G-invariant curve, and the projection $E_1 \to C_1$ is G-equivariant, and we deduce that the action of G on the surface E_1 is given by the identification $E_1 = \mathbb{P}(\mathbb{W}_2) \times \mathbb{P}(\mathbb{W}_2)$. Thus, the G-invariant curves in $|s_{E_1} + 3 f_{E_1}|$ corresponds to one-dimensional subrepresentations of the group $2.G$ in $\mathbb{W}_2 \otimes \mathrm{Sym}^3(\mathbb{W}_2)$. Using the following GAP script, we conclude that there are two such subrepresentations:

```
G:=Group("SL(2,3)");
R:=IrreducibleModules(G,CyclotomicField(3));
M:=TensorProduct(R[4],SymmetricPower(R[4],3));
IndecomposableSummands(M);
```

These subrepresentations corresponds to the curves Z_1 and Z_1', so that C must be one of them, which is impossible by assumption.

Thus, we see that C is not contained in E_1. Let $\overline{C} = \pi_1(C)$. Then $\pi_1^*(H) \cdot C = H \cdot \overline{C} \geqslant 2$. Moreover, if $H \cdot \overline{C} = 2$, then $\overline{C}$ is one of the conics C_1, C_2, C_3 or C_4, because these are the only G-invariant conics in V_4. Since $C \not\subset E_1$ and C is not one of the curves C_2^1, C_3^1, C_4^1, we see that $H \cdot \overline{C} \neq 2$, so that $\pi_1^*(H) \cdot C \geqslant 3$.

Note also that $\eta_1(C)$ is a curve, because G does not have fixed points in $\mathbb{P}^2$. Similarly, we see that $\eta_1(C)$ is not a line. Hence, we conclude that $(\pi_1^*(H) - E_1) \cdot C \geqslant \deg(\eta_1(C)) \geqslant 2$. One the other hand, we have $E_1 \cdot C$ must be even since C does not have G-orbits of odd length. Moreover, we have

$$7 \geqslant -K_{X_1} \cdot C = \big(\pi_1^*(2H) - E_1\big) \cdot C = \pi_1^*(H) \cdot C + \big(\pi_1^*(H) - E_1\big) \cdot C \geqslant 5,$$

so that $-K_{X_1} \cdot C = 6$, $\pi_1^*(H) \cdot C = 3$ and $(\pi_1^*(H) - E_1) \cdot C = 3$, which gives $E_1 \cdot C = 0$. Hence, we see that $\overline{C}$ is a smooth rational cubic curve, and $\eta_1(C)$ is a singular cubic curve. This is impossible, since G does not have fixed points in $\mathbb{P}^2$. □

Lemma 3.6. *Let S be a G-invariant surface such that $-K_{X_i} \sim_{\mathbb{Q}} aS + \Delta$ for a rational number a and an effective G-invariant $\mathbb{Q}$-divisor Δ on X_i. Then $a \leqslant 1$.*

Proof. If $S = E_i$, then $2 = -K_{X_i} \cdot \mathscr{C} = aS \cdot \mathscr{C} + \Delta \cdot \mathscr{C} \geqslant aE_i \cdot \mathscr{C} = 2a$ for a general fiber $\mathscr{C}$ of the conic bundle ν_i. Thus, we may assume that $S \neq E_i$. Then $\pi_i(S)$ is a surface in V_4, and $2H \sim_{\mathbb{Q}} a\pi_i(S) + \pi_i(\Delta)$. Hence, if $a > 1$, then $\pi_i(Z) \sim H$, which is impossible, because $\mathbb{P}^5$ does not contain G-invariant hyperplanes. □

Now we are ready to state the main technical result of this section:

Lemma 3.7. *Let a and λ be positive rational numbers such that $a \geqslant 1$ and $\lambda < \frac{3}{4}$, and let D be an effective G-invariant $\mathbb{Q}$-divisor on X_i such that $D \sim_{\mathbb{Q}} \pi_i^*(2H) - aE_i$. Then E_i, C_i^i, Z_i and Z_i' are not log canonical centers of the log pair $(X_i, \lambda D)$.*

Let us use this result to prove

Proposition 3.8. *One has $\alpha_G(X_1) = \alpha_G(X_2) = \alpha_G(X_3) = \alpha_G(X_4) = \frac{3}{4}$.*

Proof. Suppose that $\alpha_G(X_i) < \frac{3}{4}$. Let us seek for a contradiction. Since X_i does not have G-fixed points, it follows from [1, Lemma A.4.8] and Lemma 3.6 that there exists a G-invariant $\mathbb{Q}$-divisor D on the threefold X_i such that $D \sim_{\mathbb{Q}} -K_{X_i}$, the log pair $(X_i, \lambda D)$ is strictly log canonical for some positive rational number $\lambda < \frac{3}{4}$, and the only center of log canonical singularities of this log pair is an irreducible G-invariant smooth irreducible rational curve $Z \subset X_i$ such that $-K_{X_i} \cdot Z < 8$. Then it must be one of the curves C_1^i, C_2^i, C_3^i, C_4^i, Z_i, Z_i' by Lemma 3.5. On the other hand, it follows from Lemma 3.7 that Z is not one of the curves C_i^i, Z_i, Z_i', so that $Z = C_j^i$ for some $j \in \{1, 2, 3, 4\}$ such that $j \neq i$.

Let $\nu\colon V \to X_i$ be the blow up of the curve Z, let F be the ν-exceptional surface, let $\widetilde{D}$ be strict transform of the divisor D via ν, and let $m = \mathrm{mult}_Z(D)$. Then $m \geqslant \frac{1}{\lambda}$ and

$$K_V + \lambda\widetilde{D} + \big(\lambda m - 1\big)F \sim_{\mathbb{Q}} \nu^*\big(K_{X_i} + \lambda D\big).$$

Thus, either $\lambda m - 1 \geqslant 1$ or the surface F contains an irreducible G-invariant smooth rational curve $\widetilde{Z}$ such that $\nu(\widetilde{Z}) = Z$, the curve $\widetilde{Z}$ is a section of the projection $F \to Z$, and $\widetilde{Z}$ is a center of log canonical singularities the log pair $(V, \lambda\widetilde{D} + (\lambda m - 1)F)$.

Let $\upsilon\colon V \to X_j$ be the birational contraction of the strict transform of the surface E_i, and let $\overline{D} = \upsilon(\widetilde{D})$. Then $\upsilon(F) = E_j$ and $\overline{D} \sim_{\mathbb{Q}} \pi_j(2H) - mE_j$, so that

$$\overline{D} + \Big(m - \frac{1}{\lambda}\Big)E_j \sim_{\mathbb{Q}} \pi_j(2H) - \frac{1}{\lambda}E_j.$$

Then the surface E_j and the curves C_j^j, Z_j and Z_j' are not log canonical centers of the log pair $(X_j, \lambda\overline{D} + (\lambda m - 1)E_j)$ by Lemma 3.7. In particular, we see that $\lambda m - 1 < 1$, so that the surface E_j contains an irreducible G-invariant smooth rational curve $\overline{Z}$ such that $\pi_j(\overline{Z}) = Z$, the curve $\overline{Z}$ is a section of the projection $E_j \to C_j$, and $\overline{Z}$ is a center of log canonical singularities of the log pair $(X_j, \lambda\overline{D} + (\lambda m - 1)E_j)$. Let us repeat that the curve $\overline{Z}$ is not one of the curves C_j^j, Z_j and Z_j' by Lemma 3.7.

Recall that $E_j \cong \mathbb{P}^1 \times \mathbb{P}^1$. Write $\overline{D}|_{E_j} = \delta Z + \Upsilon$. where δ is a non-negative rational number, and Υ is an effective $\mathbb{Q}$-divisor on E_j such that its support does not contain the curve Z. Then $\delta \geqslant \frac{1}{\lambda} > \frac{4}{3}$ by [5, Theorem 5.50]. But

$$\overline{D}\big|_{E_j} \sim_{\mathbb{Q}} \Big(\pi_j(2H) - mE_j\Big)\big|_{E_j} \sim_{\mathbb{Q}} 4f_{E_j} + m(s_{E_j} - f_{E_j}) = ms_{E_j} + (4 - m)f_{E_j},$$

and $Z \sim s_{E_j} + kf_{E_j}$ for some non-negative integer k. This gives

$$\Upsilon \sim_{\mathbb{Q}} ms_{E_j} + (4 - m)f_{E_j} - \delta\big(s_{E_j} + kf_{E_j}\big) = (m - \delta)s_{E_j} + (4 - m - \delta k)f_{E_j}.$$

Since $m \geqslant \frac{1}{\lambda} > \frac{4}{3}$ and $\delta > \frac{4}{3}$, we get $k = 0$ or $k = 1$, so that $Z = C_j^j$ by Lemma 3.5, which is impossible by Lemma 3.7. □

By Proposition 3.8 and Theorem 1.3, the smooth Fano threefolds $X_1 \cong X_2$ and $X_3 \cong X_4$ are K-polystable. However, to complete the proof of Proposition 3.8, we have to prove technical Lemma 3.7. Note that it is enough to prove this lemma for X_1 and X_3, so that we will assume in the following that either $i=1$ or $i=3$.

Fix rational numbers a and λ such that $a\geqslant 1$ and $0<\lambda<\frac{3}{4}$. Let D be a G-invariant effective $\mathbb{Q}$-divisor on the threefold X_i such that $D\sim_{\mathbb{Q}}\pi_i^*(2H)-aE_i$. Then we must show that E_i, C_i^i, Z_i and Z_i' are also not log canonical centers of the pair $(X_i,\lambda D)$. Replacing D by $D+(a-1)E_i$, we may assume that $a=1$, so that $D\sim_{\mathbb{Q}}-K_{X_i}$. Write $D=\varepsilon E_i+\Delta$, where $\varepsilon\in\mathbb{Q}_{\geqslant 0}$, and Δ is effective $\mathbb{Q}$-divisor on X_i whose support does not contain E_i. Then $\varepsilon\leqslant 1$ by Lemma 3.6, so that E_i is not a log canonical center of the log pair $(X_i,\lambda D)$.

Lemma 3.9. *Neither Z_i nor Z_i' is a log canonical center of the pair $(X_i,\lambda D)$.*

Proof. Denote by Z one of the curves Z_i or Z_i'. Let $m_\Delta=\mathrm{mult}_Z(\Delta)$ and $m=\mathrm{mult}_Z(D)$. Then $m=m_\Delta+\varepsilon$. Let us bound m. To do this, write $\Delta\big|_{E_i}=\delta Z+\Upsilon$, where δ is a rational number such that $\delta\geqslant m_\Delta$, and Υ is an effective $\mathbb{Q}$-divisor on the surface $E_i\cong\mathbb{P}^1\times\mathbb{P}^1$ such that its support does not contain Z. Observe that

$$\Delta\big|_{E_i}\sim_{\mathbb{Q}}\Big(\pi_i(2H)-(1+\varepsilon)E_i\Big)\Big|_{E_i}\sim_{\mathbb{Q}}4f_{E_i}+(1+\varepsilon)(s_{E_i}-f_{E_i})=(1+\varepsilon)s_{E_i}+(3-\varepsilon)f_{E_i}$$

and $Z\sim s_{E_i}+3f_{E_i}$. This gives $\Upsilon\sim_{\mathbb{Q}}(1+\varepsilon-\delta)s_{E_i}+(3-\varepsilon-3\delta)f_{E_i}$, which gives $\delta\leqslant 1-\frac{\varepsilon}{3}$. In particular, we get $m=m_\Delta+\varepsilon\leqslant\delta+\varepsilon\leqslant 1+\frac{2\varepsilon}{3}\leqslant\frac{5}{3}$.

Let $\nu\colon V\to X_i$ be the blow up of the curve Z, and let F be the ν-exceptional surface. Then the action of the group G lifts to the threefold V, since Z is G-invariant.

Recall that Z is cut out on E_i by a G-invariant surface in $|-K_{X_i}|$. Since $Z\cong\mathbb{P}^1$, this gives $\mathcal{N}_{Z/X_i}\cong\mathcal{O}_{\mathbb{P}^1}(6)\oplus\mathcal{O}_{\mathbb{P}^1}(-2)$, because $-K_{X_i}\cdot Z=6$, and $Z^2=6$ on the surface E_i. Thus, we have $F\cong\mathbb{F}_8$. Moreover, since $F^3=-4$, we deduce that $-F\big|_F\sim s_F+2f_F$, where s_F is a section of the projection $F\to Z$ such that $s_F^2=-8$, and f_F is a fiber of this projection. Let $\widetilde{E}_i$ and $\widetilde{D}$ be the proper transforms of the divisors E_i and D on the threefold V, respectively. Then $\widetilde{E}_i|_F\sim(\nu^*(E_i)-F)|_F\sim s_F$, since $E\cdot Z=-2$. Thus, we see that $\widetilde{E}_i|_F=s_F$. Similarly, we get $\widetilde{D}\big|_F\sim_{\mathbb{Q}}ms_F+(2m+6)f_F$.

Now we suppose that Z is a log canonical center of the pair $(X_i,\lambda D)$. Let us seek for a contradiction. Since $\lambda m-1<1$ and $K_V+\lambda\widetilde{D}+(\lambda m-1)F\sim_{\mathbb{Q}}\nu^*(K_{X_i}+\lambda D)$, the surface F contains an irreducible G-invariant smooth rational curve $\widetilde{Z}$ such that $\nu(\widetilde{Z})=Z$, the curve $\widetilde{Z}$ is a section of the projection $F\to Z$, and $\widetilde{Z}$ is a center of log canonical singularities the log pair $(V,\lambda\widetilde{D}+(\lambda m-1)F)$. Write $\widetilde{D}|_F=\theta\widetilde{Z}+\Omega$, where θ is a non-negative rational number, and Ω is an effective $\mathbb{Q}$-divisor on F such that its support does not contain the curve $\widetilde{Z}$. Then using [5, Theorem 5.50], we get $\theta\geqslant\frac{1}{\lambda}>\frac{4}{3}$. On the other hand, we have $\widetilde{Z}\sim s_F+kf_F$ for some non-negative integer k such that either $k=0$ or $k\geqslant 8$. Thus, we have $\Omega\sim_{\mathbb{Q}}(m-\theta)s_F+(2m+6-\theta k)f_F$. Hence, if $k\neq 0$, then $0\leqslant 2m+6-\theta k\leqslant 2m+6-8\theta<2m+6-\frac{32}{3}=\frac{6m-14}{3}$, so that $m>\frac{7}{3}$, which is impossible, since $m\leqslant\frac{5}{3}$. Then $k=0$, so that $\widetilde{Z}=s_F=\widetilde{E}_i\cap F$.

Recall that $D=\varepsilon E_i+\Delta$, where ε is a non-negative rational number such that $\varepsilon\leqslant 1$, and Δ is an effective $\mathbb{Q}$-divisor on the threefold X_i whose support does not contain E_i. Denote by $\widetilde{\Delta}$ the proper transform of this divisor on the threefold V. Then $\widetilde{Z}$ is a center of log canonical singularities the log pair $(V,\lambda\varepsilon\widetilde{E}_i+\lambda\widetilde{\Delta}+(\lambda m_\Delta+\lambda\varepsilon-1)F)$, where $m_\Delta=\mathrm{mult}_Z(\Delta)$. Using [5, Theorem 5.50] again, we see that $\widetilde{Z}$ is a center of log canonical singularities the log pair $(\widetilde{E}_i,\lambda\widetilde{\Delta}|_{\widetilde{E}_i}+(\lambda m_\Delta+\lambda\varepsilon-1)F|_{\widetilde{E}_i})$, where $F|_{\widetilde{E}_i}=\widetilde{Z}$. This simply means that $\lambda\widetilde{\Delta}|_{\widetilde{E}_i}+(\lambda m_\Delta+\lambda\varepsilon-1)F|_{\widetilde{E}_i}=c\widetilde{Z}+\Xi$ for some rational number $c\geqslant 1$, where Ξ is an effective $\mathbb{Q}$-divisor on $\widetilde{E}_i$ whose support does not contain the curve $\widetilde{Z}$.

Now, let us compute the numerical class of the restriction $\widetilde{\Delta}\big|_{\widetilde{E}_i}$. Observe that $\widetilde{E}_i\cong E_i$. Denote by $s_{\widetilde{E}_i}$ and $f_{\widetilde{E}_i}$ the strict transforms on $\widetilde{E}_i$ of the curves s_{E_i} and f_{E_i}, respectively. Then $\widetilde{\Delta}|_{\widetilde{E}_i}\sim_{\mathbb{Q}}(1+\varepsilon)s_{\widetilde{E}_i}+(3-\varepsilon)f_{\widetilde{E}_i}-m_\Delta\widetilde{Z}=(1+\varepsilon-m_\Delta)s_{\widetilde{E}_i}+(3-\varepsilon-3m_\Delta)f_{\widetilde{E}_i}$. Thus, we see that

$$
\begin{aligned}
c\big(s_{\widetilde{E}_i}+3f_{\widetilde{E}_i}\big)+\Xi\sim_{\mathbb{Q}}\lambda\widetilde{\Delta}\big|_{\widetilde{E}_i}+\big(\lambda m_\Delta+\lambda\varepsilon-1\big)F\big|_{\widetilde{E}_i}\sim_{\mathbb{Q}}&\\
\sim_{\mathbb{Q}}\lambda(1+\varepsilon-m_\Delta)s_{\widetilde{E}_i}+\lambda(3-\varepsilon-3m_\Delta)f_{\widetilde{E}_i}+\big(\lambda m_\Delta+\lambda\varepsilon-1\big)\widetilde{Z}\sim_{\mathbb{Q}}&\\
\sim_{\mathbb{Q}}(\lambda+2\lambda\varepsilon-1)s_{\widetilde{E}_i}+(3\lambda+2\lambda\varepsilon-3)f_{\widetilde{E}_i}&,
\end{aligned}
$$

so that $\Xi\sim_{\mathbb{Q}}(\lambda+2\lambda\varepsilon-1-c)s_{\widetilde{E}_i}+(3\lambda+2\lambda\varepsilon-3-3c)f_{\widetilde{E}_i}$, which gives $3\lambda+2\lambda\varepsilon-3-3c\geqslant 0$. Since $c\geqslant 1$ and $\lambda<\frac{3}{4}$, we deduce that $\varepsilon\geqslant\frac{3}{\lambda}-\frac{3}{2}>4-\frac{3}{2}=\frac{5}{2}$. But $\varepsilon\leqslant 1$. The obtained contradiction completes the proof of the lemma. ☐

To complete the proof of Lemma 3.7, we must show that C_i^i is not a log canonical center of the log pair $(X_i,\lambda D)$. Let $Z=C_i^i$. Suppose that Z is a log canonical center of the pair $(X_i,\lambda D)$. Let us seek for a contradiction. Observe that $\mathrm{mult}_Z\big(D\big)\geqslant\frac{1}{\lambda}>\frac{4}{3}$. Observe also that Z is not a log canonical center of the log pair $(X_i,\lambda(F_{3,i}+E_i))$ and $D\sim_{\mathbb{Q}}F_{3,i}+E_i$. Thus, replacing D by a divisor $(1+\mu)D-\mu(F_{3,i}+E_i)$ for an appropriate non-negative rational number μ, we may assume that either the surface $F_{3,i}$ or the surface E_i is not contained in the support of the $\mathbb{Q}$-divisor D. Then we conclude that $F_{3,i}$ is not contained in the support of the $\mathbb{Q}$-divisor D, because

Lemma 3.10. *The surface E_i is contained in the support of the $\mathbb{Q}$-divisor D.*

Proof. Let $\mathscr{C}$ be a general fiber of the projection $E_i\to Z$. If the surface E_i is contained in the support of the $\mathbb{Q}$-divisor D, then $1=-K_{X_i}\cdot\mathscr{C}=D\cdot\mathscr{C}\geqslant\mathrm{mult}_Z(D)\geqslant\frac{1}{\lambda}>\frac{4}{3}$, which is absurd. ☐

Let $\nu\colon V\to X_i$ be the blow up of the curve Z, let F be the ν-exceptional surface, and let $\widetilde{E}_i$ be the strict transform of the surface F via ν. Then $F\cong\mathbb{F}_n$ for some integer $n\geqslant 0$, and $F|_F\sim -s_F+af_F$ for some integer a, where s_F is a section of the projection $F\to Z$ such that $s_F^2=-n$, and f_F is a fiber of this projection. Since $-K_{X_i}\cdot Z=4$, we conclude that $F^3=-2$. Thus, we have $-2=F^3=(-s_F+af_F)^2=-n-2a$, so that $a=\frac{2-n}{2}$. On the other hand, we have $\widetilde{E}_i\big|_F\sim s_F+\frac{n-2}{2}f_F$, since $E_i\cdot Z=(-s_{E_i}+f_{E_i})\cdot(s_{E_i}+f_{E_i})=0$. But $\widetilde{E}_i\big|_F$ is an irreducible curve, which implies that $n=2$, since $\frac{n-2}{2}<n$. Thus, we see that $F\cong\mathbb{F}_2$ and $-F|_F\sim\widetilde{E}_i|_F=s_F$. Observe also that the action of the group G lifts to the threefold V, since Z is G-invariant.

Remark 3.11. The divisor $-K_V$ is nef and big. Indeed, the linear system $|\pi_i^*(2H)-2E_i|$ is base point free. Let $\mathcal{M}$ be its strict transform on V. Then $\mathcal{M}+\widetilde{E}_i$ is a linear subsystem of the linear system $|-K_V|$, so that the base locus of the linear system $|-K_V|$ is contained in $\widetilde{E}_i$. But $\widetilde{E}_i\cong E_i$ and $-K_V|_{\widetilde{E}_i}\sim 2f_{\widetilde{E}_i}$, where $f_{\widetilde{E}_i}$ is a strict transform of the curve f_{E_i} on the surface $\widetilde{E}_i$. Then $-K_V|_{\widetilde{E}_i}$ is nef, so that $-K_V$ is also nef. Since $-K_V^3=12$, we see that $-K_V$ is big.

Let $m = \mathrm{mult}_Z(D)$, and let $\widetilde{D}$ be the proper transform of the divisor D via ν. Then

$$\widetilde{D}\big|_F \sim_{\mathbb{Q}} \big(\nu^*(-K_{X_i}) - mF\big)\big|_F \sim_{\mathbb{Q}} ms_F + 4f_F.$$

Let $\mathscr{C}$ be a sufficiently general fiber of the conic bundle ν_i that is contained in $F_{3,i}$, and let $\widetilde{\mathscr{C}}$ be its strict transform on the threefold V. Then $\mathscr{C}$ is an irreducible curve that is not contained in the support of the divisor D, because we assumed that $F_{3,i} \not\subset \mathrm{Supp}(D)$. Moreover, the curve $\mathscr{C}$ intersects the curve Z, because $F_{3,i}|_{E_i} = 2Z$. Thus, we have

$$2 - m = 2 - mF \cdot \widetilde{\mathscr{C}} = \big(\nu^*(-K_{X_i}) - mF\big) \cdot \widetilde{\mathscr{C}} = \widetilde{D} \cdot \widetilde{\mathscr{C}} \geqslant 0,$$

so that $m \leqslant 2$. Since $\lambda m - 1 < 1$ and $K_V + \lambda\widetilde{D} + (\lambda m - 1)F \sim_{\mathbb{Q}} \nu^*(K_{X_i} + \lambda D)$, the surface F contains an irreducible G-invariant smooth curve $\widetilde{Z}$ such that $\nu(\widetilde{Z}) = Z$, the curve $\widetilde{Z}$ is a section of the projection $F \to Z$, and $\widetilde{Z}$ is a center of log canonical singularities the log pair $(V, \lambda\widetilde{D} + (\lambda m - 1)F)$. Let $\widetilde{m} = \mathrm{mult}_{\widetilde{Z}}(\widetilde{D})$. Then

$$m + \widetilde{m} \geqslant \frac{2}{\lambda} > \frac{8}{3}, \tag{3.12}$$

because the multiplicity of the divisor $\lambda\widetilde{D} + (\lambda m - 1)F$ at the curve $\widetilde{Z}$ must be at least 1.

Lemma 3.13. *Either $\widetilde{Z} = s_F$ or $\widetilde{Z} \sim s_F + 2f_F$.*

Proof. Write $\widetilde{D}|_F = \theta\widetilde{Z} + \Omega$, where θ is a non-negative rational number, and Ω is an effective $\mathbb{Q}$-divisor on F such that its support does not contain $\widetilde{Z}$. Using [5, Theorem 5.50], we get $\theta \geqslant \frac{1}{\lambda} > \frac{4}{3}$. But $\widetilde{Z} \sim s_F + kf_F$ for $k \in \mathbb{Z}$ such that $k = 0$ or $k \geqslant 2$. Thus, we have

$$\Omega \sim_{\mathbb{Q}} ms_F + 4f_F - \theta\widetilde{Z} \sim_{\mathbb{Q}} (m - \theta)s_F + (4 - \theta k)f_F.$$

Hence, if $k \neq 0$, then $0 \leqslant 4 - \theta k < 4 - \frac{4}{3}k$, so that $k = 2$. Then $\widetilde{Z} = s_F$ or $\widetilde{Z} \sim s_F + 2f_F$. □

Let $\widetilde{F}_{3,i}$ be the proper transform on V of the surface $F_{3,i}$. If $\widetilde{Z} = s_F$, then $\widetilde{Z} = \widetilde{E}_i \cap \widetilde{F}_{3,i}$, because $F_{3,i}$ is tangent to E_i along the curve Z and $\widetilde{E}_i \cap |_F = \widetilde{Z}$. Using this, we get

Lemma 3.14. *One has $\widetilde{Z} \neq s_F$.*

Proof. If $\widetilde{Z} = s_F$, then $\widetilde{\mathscr{C}}$ intersects the curve $\widetilde{Z}$, so hat $2 - m \geqslant 2 - mF \cdot \widetilde{\mathscr{C}} = \widetilde{D} \cdot \widetilde{\mathscr{C}} \geqslant \widetilde{m}$, which contradicts (3.12). □

Thus, we see that $\widetilde{Z} \sim s_F + 2f_F$.

Remark 3.15. The curve $\widetilde{Z}$ is unique G-invariant curve in the linear system $|s_F + 2f_F|$, because $(s_F + 2f_F) \cdot \widetilde{Z} = 2$, and $\widetilde{Z}$ does not have G-orbits of length less than 4.

Let $\rho\colon Y \to V$ be the blow up of the curve $\widetilde{Z}$, and let R be the ρ-exceptional surface. Then $-K_Y^3 = 2$.

Lemma 3.16. *The divisor $-K_Y$ is nef.*

Proof. Let $\widehat{F}_{3,i}$, $\widehat{E}_i$, $\widehat{F}$ be the strict transforms of the surfaces $F_{3,i}$, E_i, F, respectively. Then $|-K_Y|$ contains the divisor $\widehat{F}_{3,i} + \widehat{E}_i + \widehat{F}$. Therefore, to prove the required assertion, it is enough to prove that the restrictions $-K_Y|_{\widehat{F}_{3,i}}$, $-K_Y|_{\widehat{E}_i}$ and $-K_Y|_{\widehat{F}}$ are nef.

The nefness of the restriction $-K_Y|_{\widehat{E}_i}$ follows from the nefness of the restriction $-K_V|_{\widetilde{E}_i}$, because $\widetilde{Z}$ is disjoint from the surface $\widetilde{E}_i$. To check the nefness of the restriction $-K_Y|_{\widehat{F}}$, note that $\widetilde{Z} \sim s_F + 2f_F$ and $-K_V|_F \sim s_F + 4f_F$, so that $-K_Y|_{\widehat{F}}$ is rationally equivalent to the sum of two fibers of the projection $\widehat{F} \to \mathbb{P}^1$. Hence, the restriction $-K_Y|_{\widehat{F}}$ is nef.

Thus, we must prove that $-K_Y|_{\widehat{F}_{3,i}}$ is nef. To do this, recall that $F_{3,i}$ is a preimage via the conic bundle η_i of a G-invariant conic in $\mathbb{P}^2$, which we denoted earlier by $\mathcal{C}_3$. Using explicit equation of the surface $F_{3,i}$, one can check that this conic intersects the discriminant curve Δ_i by four points that form a G-orbit of length 4, so that $\mathcal{C}_3$ has simple tangency with Δ_i at every intersection point. Denote the points in $\mathcal{C}_3 \cap \Delta_i$ by P_1, P_2, P_3 and P_4. For each $k \in \{1,2,3,4\}$, we have $\eta_i^{-1}(P_k) = \ell_k + \ell_k'$, where ℓ_k and ℓ_k' are smooth rational curve that intersect transversally at one point. Thus, in total we obtain eight smooth rational curves ℓ_1, ℓ_1', ℓ_2, ℓ_2', ℓ_3, ℓ_3', ℓ_4, ℓ_4'. Denote their images in V_4 by $\overline{\ell}_1$, $\overline{\ell}_1'$, $\overline{\ell}_2$, $\overline{\ell}_2'$, $\overline{\ell}_3$, $\overline{\ell}_3'$, $\overline{\ell}_4$, $\overline{\ell}_4'$, respectively. Then these eight curves are lines, which we will describe later. Similarly, denote their strict transforms on V by $\widetilde{\ell}_1$, $\widetilde{\ell}_1'$, $\widetilde{\ell}_2$, $\widetilde{\ell}_2'$, $\widetilde{\ell}_3$, $\widetilde{\ell}_3'$, $\widetilde{\ell}_4$, $\widetilde{\ell}_4'$, respectively. Then, by construction, we have

$$-K_V\cdot\widetilde{\ell}_1 = -K_V\cdot\widetilde{\ell}_1' = -K_V\cdot\widetilde{\ell}_2 = -K_V\cdot\widetilde{\ell}_2' = -K_V\cdot\widetilde{\ell}_3 = -K_V\cdot\widetilde{\ell}_3' = -K_V\cdot\widetilde{\ell}_4 = -K_V\cdot\widetilde{\ell}_4' = 0.$$

Finally, let us denote the strict transforms on Y of these eight curves by $\widehat{\ell}_1$, $\widehat{\ell}_1'$, $\widehat{\ell}_2$, $\widehat{\ell}_2'$, $\widehat{\ell}_3$, $\widehat{\ell}_3'$, $\widehat{\ell}_4$, $\widehat{\ell}_4'$, respectively. For every $k \in \{1,2,3,4\}$, we have $-K_Y \cdot \widehat{\ell}_k = -R \cdot \widehat{\ell}_k$ and $-K_Y \cdot \widehat{\ell}_k' = -R \cdot \widehat{\ell}_k'$. Therefore, if $\widetilde{Z}$ intersects a curve $\widehat{\ell}_k$ or $\widehat{\ell}_k'$, then $-K_Y$ is not nef, because in these case we have $-K_Y \cdot \widehat{\ell}_k < 0$ or $-K_Y \cdot \widehat{\ell}_k' < 0$, respectively.

First, let us show that the curves $\widehat{\ell}_1$, $\widehat{\ell}_1'$, $\widehat{\ell}_2$, $\widehat{\ell}_2'$, $\widehat{\ell}_3$, $\widehat{\ell}_3'$, $\widehat{\ell}_4$, $\widehat{\ell}_4'$ are the only curves in $\widehat{F}_{3,i}$ that a priori may have negative intersections with the divisor $-K_Y$. After thus, we will explicitly check that $\widetilde{Z}$ does not intersects any of the curves $\widetilde{\ell}_1$, $\widetilde{\ell}_1'$, $\widetilde{\ell}_2$, $\widetilde{\ell}_2'$, $\widetilde{\ell}_3$, $\widetilde{\ell}_3'$, $\widetilde{\ell}_4$, $\widetilde{\ell}_4'$, which would imply that $-K_Y$ is indeed nef.

By construction, the curves ℓ_1, ℓ_1', ℓ_2, ℓ_2', ℓ_3, ℓ_3', ℓ_4, ℓ_4' form two G-irreducible curves each consisting of four irreducible components. Without loss of generality, we may assume that $\ell_1 + \ell_2 + \ell_3 + \ell_4$ is one of these curves, and $\ell_1' + \ell_2' + \ell_3' + \ell_4'$ is another curve.

Observe that $\widetilde{F}_{3,i}|_F \sim s_F + 4f_F$ and the intersection $\widetilde{F}_{3,i} \cap F$ contains the curve s_F. This implies that $\widetilde{F}_{3,i}|_F = s_F + e_1 + e_2 + e_3 + e_4$, where e_k is a fiber of the projection $F \to Z$ such that $\nu(e_k) = \ell_k \cap \ell_k'$. Since $\widetilde{F}_{3,i}|_{\widetilde{E}_i} = s_F$, we see that $\widetilde{F}_{3,i}$ is smooth. Moreover, we have $(s_F \cdot s_F)_{\widetilde{F}_{3,i}} = -2$, because $\widetilde{E}_i^2 \cdot \widetilde{F}_{3,i} = -2$. Now, using this and $F^2 \cdot \widetilde{F}_{3,i} = -2$, we conclude that $(e_1 \cdot e_1)_{\widetilde{F}_{3,i}} = (e_2 \cdot e_2)_{\widetilde{F}_{3,i}} = (e_3 \cdot e_3)_{\widetilde{F}_{3,i}} = (e_4 \cdot e_5)_{\widetilde{F}_{3,i}} = -2$. Thus, we conclude that $F_{3,i}$ has an ordinary double point at each point $\ell_k \cap \ell_k'$, and the birational morphism ν induces the minimal resolution of singularities $\widetilde{F}_{3,i} \to F_{3,i}$, which contracts the curve e_k to the point $\ell_k \cap \ell_k'$.

The composition $\eta_i \circ \nu$ induces a conic bundle $\widetilde{F}_{3,i} \to \mathcal{C}_3$. The curve s_F is its section, and its (scheme) fibers over the points P_1, P_2, P_3, P_4 are $e_1 + \widetilde{\ell}_1 + \widetilde{\ell}_1'$, $e_2 + \widetilde{\ell}_1 + \widetilde{\ell}_2'$, $e_3 + \widetilde{\ell}_1 + \widetilde{\ell}_3'$, $e_4 + \widetilde{\ell}_1 + \widetilde{\ell}_4'$, respectively. Thus, for every $k \in \{1,2,3,4\}$, the curves $\widetilde{\ell}_k$ and $\widetilde{\ell}_k'$ are disjoint (-1)-curves on the surface $\widetilde{F}_{3,i}$, which both do not intersect the section s_F, because s_F intersects the (-2)-curve e_k. Moreover, we have

$$-K_V\big|_{\widetilde{F}_{3,i}} \sim s_F + \sum_{k=1}^{4} \big(e_k + \widetilde{\ell}_k + \widetilde{\ell}_k'\big),$$

because $-K_V \sim \nu^*(F_{3,i}) + \widetilde{E}_i$ and $\widetilde{E}_i|_{\widetilde{F}_{3,i}} = s_F$.

The curve $\widetilde{Z}$ intersects the surface $\widetilde{F}_{3,i}$ transversally by a G-orbit of length 4, because it intersects the (reducible) curve $s_F+e_1+e_2+e_3+e_4$ transversally by the points $\widetilde{Z}\cap e_1$, $\widetilde{Z}\cap e_2$, $\widetilde{Z}\cap e_3$, $\widetilde{Z}\cap e_4$, which form one G-orbit. Thus, the morphism ρ induces a birational morphism $\varrho\colon \widehat{F}_{3,i}\to \widetilde{F}_{3,i}$ that is a a blow up of this G-orbit. Using this, we see that

$$-K_Y\big|_{\widehat{F}_{3,i}}\sim \varrho^*\Big(s_F+\sum_{k=1}^{4}\big(e_k+\widetilde{\ell}_k+\widetilde{\ell}'_k\big)\Big)-r_1-r_2-r_3-r_4$$

where r_k is the exceptional curve of ϱ that is contracted to the point $\widetilde{Z}\cap e_k$. Observe that these four points $\widetilde{Z}\cap e_1$, $\widetilde{Z}\cap e_2$, $\widetilde{Z}\cap e_3$, $\widetilde{Z}\cap e_4$ are not contained in the curve s_F, because the curves $\widetilde{Z}$ and s_F are disjoint. Moreover, we have three mutually excluding options:

(1) the G-orbit $\widetilde{Z}\cap\widetilde{F}_{3,i}$ is contained in the curve $\widetilde{\ell}_1+\widetilde{\ell}_2+\widetilde{\ell}_3+\widetilde{\ell}_4$;
(2) the G-orbit $\widetilde{Z}\cap\widetilde{F}_{3,i}$ is contained in the curve $\widetilde{\ell}'_1+\widetilde{\ell}'_2+\widetilde{\ell}'_3+\widetilde{\ell}'_4$;
(3) the G-orbit $\widetilde{Z}\cap\widetilde{F}_{3,i}$ is contained in the curves $\widetilde{\ell}_1+\widetilde{\ell}_2+\widetilde{\ell}_3+\widetilde{\ell}_4$ and $\widetilde{\ell}'_1+\widetilde{\ell}'_2+\widetilde{\ell}'_3+\widetilde{\ell}'_4$.

As we already mentioned, the divisor $-K_Y$ is not nef in the first two cases. In the third case, we have

$$-K_Y\big|_{\widehat{F}_{3,i}}\sim \widehat{s}_F+\sum_{k=1}^{4}\big(\widehat{e}_k+\widehat{\ell}_k+\widehat{\ell}'_k\big),$$

where $\widehat{s}_F$ and $\widehat{e}_k$ are strict transforms of the curves s_F and e_k on the surface $\widehat{F}_{3,i}$. Moreover, in this case, we have $\widehat{s}_F\cdot\widehat{s}_F=-2$, $\widehat{s}_F\cdot\widehat{e}_k=1$, $\widehat{e}_k=1\cdot\widehat{e}_k=-1$, $\widehat{e}_k\cdot\widehat{e}_k=-3$, $\widehat{e}_k\cdot\widehat{\ell}_k=1$, $\widehat{e}_k\cdot\widehat{\ell}'_k=1$ on the surface $\widehat{F}_{3,i}$, and all other intersections are zero. This immediately implies that the divisor $-K_Y\big|_{\widehat{F}_{3,i}}$ is nef in the third case, so that $-K_Y$ is also nef.

Therefore, we proved that the divisor $-K_Y$ is nef if and only if the curve $\widetilde{Z}$ does not intersect the curves $\widetilde{\ell}_1+\widetilde{\ell}_2+\widetilde{\ell}_3+\widetilde{\ell}_4$, and $\widetilde{\ell}_1+\widetilde{\ell}_2+\widetilde{\ell}_3+\widetilde{\ell}_4$. Observe that these curves intersects the ν-exceptional surface F by two (distinct) G-orbits of length 4, respectively. Denote these G-orbits by Θ and Θ', respectively. Hence, to complete the proof, it is enough to check that neither Θ nor Θ' is contained in the curve $\widetilde{Z}$.

We have $h^0(\mathcal{O}_V(-K_V))=9$ by the Riemann–Roch formula and the Kawamata–Viehweg vanishing, since $-K_V$ is big and nef by Remark 3.11. Moreover, we have $-K_V|_F\sim s_F+4f_F$ and $h^0(\mathcal{O}_F(s_F+4f_F))=8$. Furthermore, we have $h^0(\mathcal{O}_V(-K_V-F))=1$, since the linear system $|-K_V-F|$ contains unique effective divisor: $\widetilde{F}_{3,i}+\widetilde{E}_i$. This gives the following exact sequence of G-representations:

$$(3.17)\quad 0\longrightarrow H^0\Big(\mathcal{O}_V\big(\widetilde{F}_{3,i}+\widetilde{E}_i\big)\Big)\longrightarrow H^0\Big(\mathcal{O}_V\big(-K_V\big)\Big)\longrightarrow H^0\Big(\mathcal{O}_F\big(s_F+4f_F\big)\Big)\longrightarrow 0.$$

Here, the kernel of the third map is the one-dimensional G-representation generated by the section vanishing on the divisor $\widetilde{F}_{3,i}+\widetilde{E}_i+F$.

Note that $s_F\cong\mathbb{P}^1$ and $(s_F+4f_F)\cdot s_F=2$. Thus, the Riemann–Roch formula and the Kawamata–Viehweg vanishing give the following exact sequence of G-representations:

$$0\longrightarrow H^0\big(\mathcal{O}_F(4f_F)\big)\longrightarrow H^0\big(\mathcal{O}_F(s_F+4f_F)\big)\longrightarrow H^0\big(\mathcal{O}_{\mathbb{P}^1}(2)\big)\longrightarrow 0.$$

Since s_F does not have G-orbits of length 2, we have $H^0(\mathcal{O}_{\mathbb{P}^1}(2))\cong\mathbb{U}_3$, where $\mathbb{U}_3$ is the unique irreducible three-dimensional representation of the group G. Similarly, since

Z has exactly two G-orbits of length 4, we have $H^0(\mathcal{O}_F(4f_F)) \cong \mathbb{U}_1 \oplus \mathbb{U}_1' \oplus \mathbb{U}_3$, where $\mathbb{U}_1$ and $\mathbb{U}_1'$ are different one-dimensional representations of the group G. Thus, one has

$$H^0\Big(\mathcal{O}_F\big(s_F + 4f_F\big)\Big) \cong \mathbb{U}_1 \oplus \mathbb{U}_1' \oplus \mathbb{U}_3 \oplus \mathbb{U}_3.$$

We may assume that $\mathbb{U}_1$ is generated by a section that vanishes at $s_F + e_1 + e_2 + e_3 + e_4$.

Let $\mathbb{V}$ and $\mathbb{V}'$ be sub-representations in $H^0(\mathcal{O}_F(s_F + 4f_F))$ that consist of all sections vanishing at the G-orbits Θ and Θ', respectively. Then $\dim(\mathbb{V}) = \dim(\mathbb{V}') = 4$, so that

$$\mathbb{V} \cong \mathbb{V}' \cong \mathbb{U}_1 \oplus \mathbb{U}_3,$$

since both G-orbits Θ and Θ' are contained in $s_F + e_1 + e_2 + e_3 + e_4$ by construction. Let $\widetilde{\mathbb{V}}$ and $\widetilde{\mathbb{V}}'$ be the the preimages in $H^0(\mathcal{O}_V(-K_V))$ via the restriction map in (3.17) of the sub-representations $\mathbb{V}$ and $\mathbb{V}'$, respectively. Then, as G-representations, we have

$$\widetilde{\mathbb{V}} \cong \widetilde{\mathbb{V}}' \cong \mathbb{U}_1 \oplus \mathbb{U}_1'' \oplus \mathbb{U}_3,$$

where $\mathbb{U}_1''$ is a one-dimensional representation of the group G. Since $\widetilde{\mathbb{V}}$ and $\widetilde{\mathbb{V}}'$ contain unique three-dimensional subrepresentation of the group G, these (two) three-dimensional subrepresentations define two G-invariant linear subsystems $\mathcal{M}_V$ and $\mathcal{M}_V'$ of the linear system $|-K_V|$, respectively. They can be characterized as (unique) three-dimensional G-invariant linear subsystems in $|-K_V|$ that contains G-orbits Θ and Θ', respectively. Then $\mathcal{M}_V|_F$ and $\mathcal{M}_V'|_F$ are (unique) three-dimensional G-invariant linear subsystems of the linear system $|s_F + 4f_F|$ that contain Θ and Θ', respectively. Thus, if $\Theta \subset \widetilde{Z}$, then

$$\mathcal{M}_V\big|_F = \widetilde{Z} + |2f_F|,$$

so that $\widetilde{Z} \subseteq \mathrm{Bs}(\mathcal{M}_V)$. Similarly, if $\Theta' \subset \widetilde{Z}$, then $\mathcal{M}_V'|_F = \widetilde{Z} + |2f_F|$, so that $\widetilde{Z} \subseteq \mathrm{Bs}(\mathcal{M}_V')$.

Let $\mathcal{M}$ and $\mathcal{M}'$ be strict transforms on V_4 of the linear systems $\mathcal{M}_V$ and $\mathcal{M}_V'$, respectively. Then $\mathcal{M}$ and $\mathcal{M}'$ are linear subsystems in $|2H|$, so that they do not have fixed components, because $|H|$ does not have G-invariant divisors. Let M_1 and M_2 be two distinct surfaces in $\mathcal{M}$. If $\Theta \subset \widetilde{Z}$, then

$$\Big(M_1 \cdot M_2\Big)_{C_i} \geqslant 3. \tag{3.18}$$

Similarly, if $\Theta' \subset \widetilde{Z}$, then

$$\Big(M_1' \cdot M_2'\Big)_{C_i} \geqslant 3, \tag{3.19}$$

where M_1' and M_2' are two surfaces in $\mathcal{M}'$. Both conditions (3.18) and (3.19) are easy to check provided that we know generators of the linear system $\mathcal{M}$ and $\mathcal{M}'$.

Observe that the curve $\widetilde{\ell}_1 + \widetilde{\ell}_2 + \widetilde{\ell}_3 + \widetilde{\ell}_4$ is contained in the base locus of the linear system $\mathcal{M}_V$. Indeed, one has $\mathcal{M}_V \subset |-K_V|$ and $-K_V \cdot \widetilde{\ell}_i = 0$ for every $i \in \{1, 2, 3, 4\}$, while $\Theta \subseteq \mathrm{Bs}(\mathcal{M}_V)$ by construction, and Θ is contained in $\widetilde{\ell}_1 + \widetilde{\ell}_2 + \widetilde{\ell}_3 + \widetilde{\ell}_4$ by definition. Likewise, we see that $\widetilde{\ell}_1' + \widetilde{\ell}_2' + \widetilde{\ell}_3' + \widetilde{\ell}_4'$ is contained in the base locus of the linear system $\mathcal{M}_V'$. Hence, the G-irreducible curves $\overline{\ell}_1 + \overline{\ell}_2 + \overline{\ell}_3 + \overline{\ell}_4$ and $\overline{\ell}_1' + \overline{\ell}_2' + \overline{\ell}_3' + \overline{\ell}_4'$ are contained in the base loci of the linear systems $\mathcal{M}$ and $\mathcal{M}'$, respectively. Moreover, the base loci of these linear systems also contain the conic C_i. Using these linear conditions, we can find the generators of these linear systems, and check the conditions (3.18) and (3.19).

Since $X_1 \cong X_2$ and $X_3 \cong X_4$, it is enough to consider only the cases $i = 1$ and $i = 3$. First, we deal with the case $i = 1$. In this case, the curves $\overline{\ell}_1+\overline{\ell}_2+\overline{\ell}_3+\overline{\ell}_4$ and $\overline{\ell}_1'+\overline{\ell}_2'+\overline{\ell}_3'+\overline{\ell}_4'$ can be described as follows: up to a swap and a reshuffle, we may assume that

- $\overline{\ell}_1$ is the line $[\lambda : \omega\lambda : -(\omega+1)\lambda : \mu-(\omega+2)\lambda : \mu : \mu+(\omega-1)\lambda]$,
- $\overline{\ell}_2$ is the line $[\lambda : -\omega\lambda : -(\omega+1)\lambda : -\mu-(\omega+2)\lambda : \mu : -\mu+(\omega-1)\lambda]$,
- $\overline{\ell}_3$ is the line $[\lambda : \omega\lambda : (\omega+1)\lambda : \mu-(\omega+2)\lambda : \mu : -\mu+(-\omega+1)\lambda]$,
- $\overline{\ell}_4$ is the line $[\lambda : -\omega\lambda : (\omega+1)\lambda : -\mu-(\omega+2)\lambda : \mu : \mu+(-\omega+1)\lambda]$,

and

- $\overline{\ell}_1'$ is the line $[\lambda : \omega\lambda : -(\omega+1)\lambda : \mu+(2\omega+1)\lambda : \mu : \mu+(\omega+2)\lambda]$,
- $\overline{\ell}_2'$ is the line $[\lambda : -\omega\lambda : -(\omega+1)\lambda : -\mu+(2\omega+1)\lambda : \mu : -\mu+(\omega+2)\lambda]$,
- $\overline{\ell}_3'$ is the line $[\lambda : \omega\lambda : (\omega+1)\lambda : \mu+(2\omega+1)\lambda : \mu : -\mu-(\omega+2)\lambda]$,
- $\overline{\ell}_4'$ is the line $[\lambda : -\omega\lambda : (\omega+1)\lambda : -\mu+(2\omega+1)\lambda : \mu : \mu-(\omega+2)\lambda]$,

where $[\lambda : \mu] \in \mathbb{P}^1$. Therefore, the linear subsystem in $|2H|$ that consists of all surfaces containing the conic C_1 and the curve $\overline{\ell}_1+\overline{\ell}_2+\overline{\ell}_3+\overline{\ell}_4$ is five-dimensional. Moreover, it is generated by the G-invariant surfaces $\overline{F}_{1,1}$, $\overline{F}_{3,1}$, $\overline{F}_{2,3}$, and the G-invariant two-dimensional linear subsystem (net) that is cut out on V_4 by

$$
\begin{aligned}
(3.20)\quad \lambda\Big((1-\omega)x_0x_5-(2\omega+1)x_2x_3+3x_0x_2\Big)+ \\
+\mu\Big((\omega+1)x_1x_3+(2\omega+1)x_0x_1+x_4x_0\Big)+ \\
+\gamma\Big((\omega+2)x_1x_2-\omega x_1x_5+x_2x_4\Big)=0,
\end{aligned}
$$

where $[\lambda : \mu : \gamma] \in \mathbb{P}^2$. Therefore, we conclude that (3.20) defines the linear system $\mathcal{M}$. It follows from (3.20) that the base locus of this linear system consists of the conic C_1, the curve $\overline{\ell}_1+\overline{\ell}_2+\overline{\ell}_3+\overline{\ell}_4$, and the conic C_3. Similarly, we see that $\mathcal{M}'$ is given by

$$
\begin{aligned}
\lambda\Big((2\omega+1)x_0x_5+(\omega+2)x_2x_3+3x_0x_2\Big)+ \\
+\mu\Big((\omega+1)x_1x_3+(1-\omega)x_0x_1+x_4x_0\Big)+ \\
+\gamma\Big((2\omega+1)x_1x_2+\omega x_1x_5-x_2x_4\Big)=0,
\end{aligned}
$$

where $[\lambda : \mu : \gamma] \in \mathbb{P}^2$. We also see that the base locus of the linear system $\mathcal{M}'$ consists of the conic C_1, the curve $\overline{\ell}_1'+\overline{\ell}_2'+\overline{\ell}_3'+\overline{\ell}_4'$, and the conic C_4. Now one can check that neither (3.18) nor (3.19) holds. Thus, if $i = 1$ or $i = 2$, then $-K_Y$ is nef.

Finally, we consider the case $i = 3$. Now, up to a swap, the linear system $\mathcal{M}$ is again given by (3.20), and the linear system $\mathcal{M}'$ is given by

$$
\begin{aligned}
\lambda\Big((\omega+1)x_1x_5+x_4x_2-(\omega-1)x_4x_5\Big)+ \\
+\mu\Big(\omega x_0x_5-x_3x_2+(2\omega+1)x_3x_5\Big)+ \\
+\gamma\Big(\omega x_0x_5-x_3x_2+(2\omega+1)x_3x_5\Big)=0,
\end{aligned}
$$

where $[\lambda:\mu:\gamma]\in\mathbb{P}^2$. Note that the base locus of the net $\mathcal{M}'$ consists of the conic C_3, the curve $\overline{\ell}_1'+\overline{\ell}_2'+\overline{\ell}_3'+\overline{\ell}_4'$, and the conic C_2. As above, one can check that neither (3.18) nor (3.19) holds. Thus, the divisor $-K_Y$ is nef. □

Let $\widehat{D}$ be the proper transform of the divisor D on the threefold Y. Then

$$\widehat{D}\sim_{\mathbb{Q}}\sim(\pi_i\circ\nu\circ\rho)^*(2H)-(\nu\circ\rho)^*(E_i)-m\rho^*(F)-\widetilde{m}R.$$

Since $-K_Y$ is nef, we see that $-K_Y^2\cdot\widehat{D}\geqslant 0$. To compute $-K_Y^2\cdot\widehat{D}$, observe that

$$H^3=4,\pi_i^*(H)\cdot E^2=-2,(\pi_i\circ\nu)^*(H)\cdot F^2=-2,$$
$$(\pi_i\circ\nu\circ\rho)^*(H)\cdot R^2=-2,E^3=-2,F^3=-2,R^3=-2,$$

and other intersections involved in the computation $-K_Y^2\cdot\widehat{D}$ are all zero. This gives

$$0\leqslant -K_Y^2\cdot\widehat{D}=\Big((\pi_i\circ\nu\circ\rho)^*(2H)-(\nu\circ\rho)^*(E_i)-\rho^*(F)-R\Big)^2\cdot\widehat{D}=14-6(m+\widetilde{m}),$$

so that $m+\widetilde{m}\leqslant\frac{7}{3}$, which is impossible by (3.12). The obtained contradiction completes the proof of Lemma 3.7, which completes the proof of Proposition 3.8. Thus, we see that the threefolds X_1, X_2, X_3 and X_4 are K-polystable.

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Ivan Cheltsov
University of Edinburgh, Edinburgh, Scotland
I.Cheltsov@ed.ac.uk

Hendrik Süß
University of Manchester, Manchester, England
suess@sdf-eu.org