K-STABLE FANO THREEFOLDS OF RANK 2 AND DEGREE 30

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ABSTRACT. We find all K-stable smooth Fano threefolds in the family №2.22.

Let X be a smooth Fano threefold. Then X belongs to one of the 105 families, which are labeled as №1.1, №1.2, ..., №9.1, №10.1. See [2], for the description of these families. If X is a general member of the family №$\mathscr{N}$, then [2, Main Theorem] gives

$$X \text{ is K-polystable} \iff \mathscr{N} \notin \left\{ \begin{array}{l} 2.23, 2.26, 2.28, 2.30, 2.31, 2.33, 2.35, 2.36, \\ 3.14, 3.16, 3.18, 3.21, 3.22, 3.23, \\ 3.24, 3.26, 3.28, 3.29, 3.30, 3.31, \\ 4.5, 4.8, 4.9, 4.10, 4.11, 4.12, \\ 5.2 \end{array} \right\}.$$

The goal of this note is to find all K-polystable smooth Fano threefolds in the family №2.22. This family contains both K-polystable and non-K-polystable smooth Fano threefolds, and a conjectural characterization of all K-polystable members has been given in [2, § 7.4]. We will confirm this conjecture — this will complete the description of all K-polystable smooth Fano threefolds of Picard rank 2 and degree 30 started in [2].

Starting from now, we suppose that X is a smooth Fano threefold in the family №2.22. Then X can be described both as the blow up of $\mathbb{P}^3$ along a smooth twisted quartic curve, and the blow up of V_5, the unique smooth threefold №1.15, along an irreducible conic. More precisely, there are a smooth twisted quartic curve $C_4 \subset \mathbb{P}^3$, a smooth conic $C \subset V_5$, and a commutative diagram

$$\begin{array}{ccc} & X & \\ {\scriptstyle\pi}\swarrow & & \searrow{\scriptstyle\phi} \\ \mathbb{P}^3 & \overset{\psi}{\dashrightarrow} & V_5, \end{array}$$

where π is the blow up of $\mathbb{P}^3$ along C_4, ϕ is the blow up of V_5 along C, and ψ is given by the linear system of cubic surfaces containing C_4. Here, V_5 is embedded in $\mathbb{P}^6$ as described in [2, § 5.10]. All smooth Fano threefolds in the family №2.22 can be obtained in this way.

The curve C_4 is contained in a unique smooth quadric surface $Q \subset \mathbb{P}^3$, and ϕ contracts the proper transform of this surface. Note that $\mathrm{Aut}(X) \cong \mathrm{Aut}(\mathbb{P}^3, C_4) \cong \mathrm{Aut}(Q, C_4)$. Choosing appropriate coordinates on $\mathbb{P}^3$, we may assume that Q is given by $x_0x_3 = x_1x_2$, where $[x_0 : x_1 : x_2 : x_3]$ are coordinates on $\mathbb{P}^3$. Fix the isomorphism $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ given by

$$\big([u : v], [x : y]\big) \mapsto \big[xu : xv : yu : yx\big],$$

where $([u : v], [x : y])$ are coordinates in $\mathbb{P}^1 \times \mathbb{P}^1$. Swapping $[u : v]$ and $[x : y]$ if necessary, we may assume that C_4 is a divisor of degree $(1, 3)$ in Q, so that C_4 is given in Q by

$$uf_3(x, y) = vg_3(x, y)$$

for some non-zero cubic homogeneous polynomials $f_3(x, y)$ and $g_3(x, y)$.

Let $\sigma\colon C_4\to\mathbb{P}^1$ be the map given by the projection $([u:v],[x:y])\mapsto[u:v]$. Then σ is a triple cover, which is ramified over at least two points. After an appropriate change of coordinates $[u:v]$, we may assume that σ is ramified over $[1:0]$ and $[0:1]$. Then both f_3 and g_3 have multiple zeros in $\mathbb{P}^1$. Changing coordinates $[x:y]$, we may assume that these zeros are $[0:1]$ and $[1:0]$, respectively. Keeping in mind that the curve C_4 is smooth, we see that C_4 is given by

$$u(x^3+ax^2y)=v(y^3+by^2x)$$

for some complex numbers a and b, after a suitable scaling of the coordinates. If $a=b=0$, then the curve C_4 is given by $ux^3=vy^3$, which gives $\mathrm{Aut}(X)\cong\mathrm{Aut}(Q,C_4)\cong\mathbb{G}_m\rtimes\boldsymbol{\mu}_2$. In this case, the threefold X is known to be K-polystable [2, § 4.4].

Example. Suppose that $ab=0$, but $a\neq 0$ or $b\neq 0$. We can scale the coordinates further and swap them if necessary, and assume that the curve C_4 is given by $ux^3=v(y^3+y^2x)$. In this case, the threefold X is not K-polystable [2, § 7.4].

A conjecture in [2, § 7.4] says that the non-K-polystable Fano threefold described in this example is the unique non-K-polystable smooth Fano threefold in the family №2.2. Let us show that this is indeed the case. To do this, we may assume that $a\neq 0$ and $b\neq 0$. Then, scaling the coordinates, we may assume that C_4 is given by

$$u\big(x^3+\lambda x^2y\big)=v\big(y^3+\lambda y^2x\big)\qquad(\bigstar)$$

for some non-zero complex number λ. Since the curve C_4 is smooth, we must have $\lambda\neq\pm 1$. Moreover, if $\lambda=\pm 3$, then we can change the coordinates on Q in such a way that C_4 would be given by $ux^3=v(y^3+y^2x)$, so that X is not K-polystable in this case.

We know from [2] that X is K-stable if C_4 is given by $(\bigstar)$ with λ sufficiently general. In particular, we know from [2, § 4.4] that the threefold X is K-stable when $\lambda=\pm\sqrt{3}$. Our main result is the following theorem.

Theorem. *Suppose that C_4 is given in $(\bigstar)$ with $\lambda\notin\{0,\pm1,\pm3\}$. Then X is K-stable.*

Let us prove this theorem. We suppose that C_4 is given by $(\bigstar)$ with $\lambda\notin\{0,\pm1,\pm3\}$. Then the triple cover $\sigma\colon C_4\to\mathbb{P}^1$ is ramified in four distinct points P_1, P_2, P_3, P_4, which implies that $\mathrm{Aut}(Q,C_4)$ is a finite group, since $\mathrm{Aut}(Q,C_4)\subset\mathrm{Aut}(C_4,P_1+P_2+P_3+P_4)$. Without loss of generality, we may assume that

$$P_1=\big([1:0],[0:1]\big)=[0:1:0:0]$$
$$P_2=\big([0:1],[1:0]\big)=[0:0:1:0],$$

where we use both the coordinates on $Q\cong\mathbb{P}^1\times\mathbb{P}^1$ and $\mathbb{P}^3$ simultaneously.

Observe that the group $\mathrm{Aut}(Q,C_4)$ contains an involution τ that is given by

$$\big([u:v],[x:y]\big)\mapsto\big([v:u],[y:x]\big).$$

Let us identify $\mathrm{Aut}(\mathbb{P}^3,C_4)=\mathrm{Aut}(Q,C_4)$ using the isomorphism $Q\cong\mathbb{P}^1\times\mathbb{P}^1$ fixed above. Then τ is given by $[x_0:x_1:x_2:x_3]\mapsto[x_3:x_2:x_1:x_0]$. Note that τ swaps P_1 and P_2, and the τ-fixed points in C_4 are $([1:1],[1:1])$ and $([1:-1],[1:-1])$, which are not ramification points of the triple cover σ. This shows that τ swaps the points P_3 and P_4. In fact, the group $\mathrm{Aut}(Q,C_4)$ is larger than its subgroup $\langle\tau\rangle\cong\boldsymbol{\mu}_2$. Indeed, one can change coordinates $([u:v],[x:y])$ on Q such that $P_1=([1:0],[0:1])$, $P_4=([0:1],[1:0])$, and the curve C_4 is given by

$$u\big(x^3+\lambda' x^2y\big)=v\big(y^3+\lambda' y^2x\big)$$

for some complex number $\lambda' \notin \{0, \pm 1, \pm 3\}$. This gives an involution $\iota \in \mathrm{Aut}(Q, C_4)$ such that $\iota(P_1) = P_4$ and $\iota(P_2) = P_3$. Let G be the subgroup $\langle \tau, \iota \rangle \subset \mathrm{Aut}(Q, C_4) = \mathrm{Aut}(\mathbb{P}^3, C_4)$. Then $G \cong \boldsymbol{\mu}_2^2$. Note that the group $\mathrm{Aut}(\mathbb{P}^3, C_4)$ can be larger for some $\lambda \in \mathbb{C} \setminus \{0, \pm 1, \pm 3\}$. For instance, if $\lambda = \pm\sqrt{3}$, then $\mathrm{Aut}(\mathbb{P}^3, C_4) \cong \mathfrak{A}_4$, c.f. [2, Example 4.4.6].

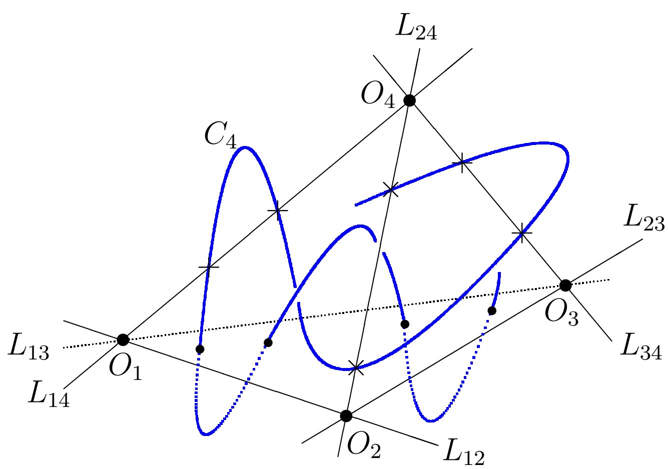
The G-action on C_4 is faithful, so that the curve C_4 does not contains G-fixed points. Hence, the quadric Q does not G-fixed points, since otherwise Q would contain a G-invariant curve of degree $(1, 0)$, which would intersect C_4 by a G-fixed point. This implies that the space $\mathbb{P}^3$ contains exactly four G-fixed points. Denote these points by O_1, O_2, O_3, O_4. These four points are not co-planar. For every $1 \leqslant i < j \leqslant 4$, let L_{ij} be the line in $\mathbb{P}^3$ that passes through O_i and O_j. Then the lines L_{12}, L_{13}, L_{14}, L_{23}, L_{24}, L_{34} are G-invariant, and they are the only G-invariant lines in $\mathbb{P}^3$. For each $1 \leqslant i \leqslant 4$, let Π_i be the plane in $\mathbb{P}^3$ determined by the three points $\{O_1, O_2, O_3, O_4\} \setminus \{O_i\}$. Then the four planes Π_1, Π_2, Π_3, Π_4 are the only G-invariant planes in $\mathbb{P}^3$.

Remark. Each plane Π_i intersects C_4 at four distinct points. Indeed, if $|\Pi_i \cap C_4| < 4$, then $\Pi_i \cap C_4$ is a G-orbit of length 2, and Π_i is tangent to C_4 at both the points of this orbit. Therefore, without loss of generality, we may assume that the intersection $\Pi_i \cap C_4$ is just the fixed locus of the involution τ. Then $\Pi_i \cap C_4 = ([1:1],[1:1]) \cup ([1:-1],[1:-1])$, so that $\Pi_i|_Q$ is a smooth conic that is given by $a(vx - uy) = b(ux - vy)$ for some $[a:b] \in \mathbb{P}^1$. But the conic $\Pi_i|_Q$ cannot tangent C_4 at the points $([1:1],[1:1])$ and $([1:-1],[1:-1])$, so that $|\Pi_i \cap C_4| = 4$.

The curve C_4 contains exactly three G-orbits of length 2, and these G-orbits are just the fixed loci of the involutions τ, ι, $\tau \circ \iota$ described earlier. Let L, L' and L'' be the three lines in $\mathbb{P}^3$ such that $L \cap C_4$, $L' \cap C_4$ and $L'' \cap C_4$ are the fixed loci of the involutions τ, ι and $\tau \circ \iota$, respectively. Then L, L' and L'' are G-invariant lines, so that they are three lines among L_{12}, L_{13}, L_{14}, L_{23}, L_{24}, L_{34}. In fact, one can show that the lines L, L', L'' meet at one point. Therefore, we may assume that $L \cap L' \cap L'' = O_4$ and $L = L_{14}$, $L' = L_{24}$, $L'' = L_{34}$. Then

$$\Pi_1 \cap C_4 = \big(L' \cap C_4\big) \cup \big(L'' \cap C_4\big), \Pi_2 \cap C_4 = \big(L \cap C_4\big) \cup \big(L'' \cap C_4\big), \Pi_3 \cap C_4 = \big(L \cap C_4\big) \cup \big(L' \cap C_4\big).$$

On the other hand, the intersection $\Pi_4 \cap C_4$ is a G-orbit of length 4.



Since C_4 is G-invariant, the action of the group G lifts to the threefold X, so that we also identify G with a subgroup of the group $\mathrm{Aut}(X)$. Let E be the π-exceptional surface, let $\widetilde{Q}$ be the proper transform of the quadric Q on the threefold X, let H_1, H_2, H_3 and H_4 be the proper transforms on X of the G-invariant planes Π_1, Π_2, Π_3 and Π_4, respectively, and let H be the proper transform on X of a general hyperplane in $\mathbb{P}^3$. Then

$$-K_X \sim 2\widetilde{Q}+E \sim \widetilde{Q}+2H_1 \sim \widetilde{Q}+2H_2 \sim \widetilde{Q}+2H_3 \sim \widetilde{Q}+2H_4 \sim 4H-E,$$

and the surfaces E, $\widetilde{Q}$, H_1, H_2, H_3, H_4 are G-invariant. Observe that $\widetilde{Q}\cong Q\cong \mathbb{P}^1\times\mathbb{P}^1$, and H_1, H_2, H_3, H_4 are smooth del Pezzo surfaces of degree 5.

Claim. *Let S be a possibly reducible G-invariant surface in X such that $-K_X\sim_{\mathbb{Q}}\mu S+\Delta$, where Δ is an effective $\mathbb{Q}$-divisor, and μ is a positive rational number such that $\mu>\frac{4}{3}$. Then S is one of the surfaces $\widetilde{Q}$, H_1, H_2, H_3, H_4.*

Proof. This follows from the fact that the cone $\mathrm{Eff}(X)$ is generated by E and $\widetilde{Q}$. □

Suppose X is not K-stable. Since $\mathrm{Aut}(X)$ is finite, the threefold X is not K-polystable. Then, by [3, Corollary 4.14], there is a G-invariant prime divisor F over X with $\beta(F)\leqslant 0$, see [2, § 1.2] for the precise definition of $\beta(F)$. Let us seek for a contradiction.

Let Z be the center of F on X. Then Z is not a surface by [2, Theorem 3.7.1], so that Z is either a G-invariant irreducible curve or a G-fixed point. In the latter case, the point $\pi(Z)$ must be one of the G-fixed points O_1, O_2, O_3, O_4, so that the point Z is not contained in $\widetilde{Q}\cup E$. Let us use Abban–Zhuang theory [1] to show that Z does not lie on $\widetilde{Q}\cup E$ in the former case.

Lemma. *The center Z cannot be contained in $\widetilde{Q}\cup E$.*

Proof. We suppose that $Z\subset\widetilde{Q}\cup E$. Then Z is an irreducible G-invariant curve, because neither $\widetilde{Q}$ nor E contains G-fixed points. Let us use notations introduced in [2, § 1.7]. Namely, we fix $u\in\mathbb{R}_{\geqslant 0}$. Then

$$-K_X-u\widetilde{Q}\sim_{\mathbb{R}}(4-2u)H+(u-1)E\sim_{\mathbb{R}}(1-u)\widetilde{Q}+2H,$$

so that $-K_X-u\widetilde{Q}$ is nef for $0\leqslant u\leqslant 1$, and not pseudo-effective for $u>2$. Thus, we have

$$P\big(-K_X-u\widetilde{Q}\big)=\begin{cases}-K_X-u\widetilde{Q} & \text{if } 0\leqslant u\leqslant 1,\\ (4-2u)H & \text{if } 1\leqslant u\leqslant 2,\end{cases}$$

and

$$N\big(-K_X-u\widetilde{Q}\big)=\begin{cases}0 & \text{if } 0\leqslant u\leqslant 1,\\ (u-1)E & \text{if } 1\leqslant u\leqslant 2.\end{cases}$$

If $Z\subset\widetilde{Q}$, then [2, Corollary 1.7.26] gives

$$1\geqslant\frac{A_X(F)}{S_X(F)}\geqslant\min\left\{\frac{1}{S_X(\widetilde{Q})},\frac{1}{S\big(W^{\widetilde{Q}}_{\bullet,\bullet};Z\big)}\right\},$$

where

$$S_X\big(\widetilde{Q}\big)=\frac{1}{(-K_X)^3}\int\limits_0^2\mathrm{vol}\big(-K_X-u\widetilde{Q}\big)du=\frac{1}{(-K_X)^3}\int\limits_0^2\Big(P\big(-K_X-u\widetilde{Q}\big)\Big)^3du$$

and

$$S\big(W^{\widetilde{Q}}_{\bullet,\bullet};Z\big)=\frac{3}{(-K_X)^3}\Bigg\{\int\limits_0^2\Big(P\big(-K_X-u\widetilde{Q}\big)^2\cdot\widetilde{Q}\Big)\cdot\mathrm{ord}_Z\Big(N\big(-K_X-u\widetilde{Q}\big)\big|_{\widetilde{Q}}\Big)du+\\+\int\limits_0^2\int\limits_0^\infty\mathrm{vol}\Big(P\big(-K_X-u\widetilde{Q}\big)\big|_{\widetilde{Q}}-vZ\Big)dvdu\Bigg\}.$$

Therefore, we conclude that $S(W^{\widetilde{Q}}_{\bullet,\bullet};Z)\geqslant 1$, because $S_X(\widetilde{Q})<1$, see [2, Theorem 3.7.1]. Similarly, if $Z\subset E$, then we get $S(W^{E}_{\bullet,\bullet};Z)\geqslant 1$.

Fix an isomorphism $\widetilde{Q}\cong Q\cong\mathbb{P}^1\times\mathbb{P}^1$ such that $E|_{\widetilde{Q}}$ is a divisor in $\widetilde{Q}$ of degree $(1,3)$. For $(a,b)\in\mathbb{R}^2$, let $\mathcal{O}_{\widetilde{Q}}(a,b)$ be the class of a divisor of degree (a,b) in $\mathrm{Pic}(\widetilde{Q})\otimes\mathbb{R}$. Then

$$P(-K_X-u\widetilde{Q})|_{\widetilde{Q}}\sim_{\mathbb{R}}\begin{cases}\mathcal{O}_{\widetilde{Q}}(3-u,u+1) & \text{if } 0\leqslant u\leqslant 1,\\ \mathcal{O}_{\widetilde{Q}}(4-2u,4-2u) & \text{if } 1\leqslant u\leqslant 2.\end{cases}$$

Therefore, if $Z=E\cap\widetilde{Q}$, then

$$\begin{aligned}S\big(W^{\widetilde{Q}}_{\bullet,\bullet};Z\big)&=\frac{1}{10}\Bigg\{\int\limits_1^2 2(4-2u)^2(u-1)du+\int\limits_0^1\int\limits_0^\infty\mathrm{vol}\Big(\mathcal{O}_{\widetilde{Q}}(3-u-v,u+1-3v)\Big)dvdu\\&\qquad+\int\limits_1^2\int\limits_0^\infty\mathrm{vol}\Big(\mathcal{O}_{\widetilde{Q}}(4-2u-v,4-2u-3v)\Big)dvdu\Bigg\}\\&=\frac{2}{30}+\frac{1}{10}\Bigg\{\int\limits_0^1\int\limits_0^{\frac{u+1}{3}}2(u+1-3v)(3-u-v)dvdu\\&\qquad+\int\limits_1^2\int\limits_0^{\frac{4-2u}{3}}2(4-2u-3v)(4-2u-v)dvdu\Bigg\}\\&=\frac{161}{540}.\end{aligned}$$

To estimate $S(W^{\widetilde{Q}}_{\bullet,\bullet};Z)$ in the case when $Z\subset\mathbb{Q}$ and $Z\neq E\cap\widetilde{Q}$, observe that $|Z-\Delta|\neq\varnothing$, where Δ is the diagonal curve in $\widetilde{Q}$. Indeed, this follows from the fact that $\widetilde{Q}$ contains neither G-invariant curves of degree $(0,1)$ nor G-invariant curves of degree $(1,0)$, which in turns easily follows from the fact that the curve $C_4\cong\mathbb{P}^1$ does not have G-fixed points.

Thus, if $Z \subset \widetilde{Q}$ and $Z \neq E \cap \widetilde{Q}$, then

$$
\begin{aligned}
S\big(W^{\widetilde{Q}}_{\bullet,\bullet}; Z\big) &\leqslant \frac{1}{10}\int_0^2\int_0^\infty \mathrm{vol}\Big(P\big(-K_X-u\widetilde{Q}\big)\big|_{\widetilde{Q}}-v\Delta\Big)dvdu\\
&= \frac{1}{10}\left\{\int_0^1\int_0^\infty \mathrm{vol}\Big(\mathcal{O}_{\widetilde{Q}}(3-u-v,u+1-v)\Big)dvdu+\right.\\
&\qquad\qquad\left.+\int_1^2\int_0^\infty \mathrm{vol}\Big(\mathcal{O}_{\widetilde{Q}}(4-2u-v,4-2u-v)\Big)dvdu\right\}\\
&= \frac{1}{10}\left\{\int_0^1\int_0^{u+1} 2(u+1-v)(3-u-v)dvdu+\int_1^2\int_0^{4-2u} 2(4-2u-v)^2dvdu\right\}\\
&= \frac{17}{30}.
\end{aligned}
$$

Therefore, $Z \not\subset \widetilde{Q}$, and hence $Z \subset E$ and $Z \neq \widetilde{Q} \cap E$.

One has $E \cong \mathbb{F}_n$ for some integer $n \geqslant 0$. It follows from the argument as in the proof of [2, Lemma 4.4.16] that n is either 0 or 2. Indeed, let $\mathbf{s}$ be the section of the projection $E \to C_4$ such that $\mathbf{s}^2=-n$, and let $\mathbf{l}$ be its fiber. Then $-E\big|_E \sim \mathbf{s}+k\mathbf{l}$ for some integer k. But

$$
-n+2k=E^3=-\mathrm{c}_1(\mathcal{N}_{C_4/\mathbb{P}^3})=-14,
$$

so that $k=\frac{n-14}{2}$. Then

$$
\widetilde{Q}\big|_E \sim \big(2H-E\big)\big|_E \sim \mathbf{s}+(k+8)\mathbf{l}=\mathbf{s}+\frac{n+2}{2}\mathbf{l},
$$

which implies that $\widetilde{Q}|_E \not\sim \mathbf{s}$. Moreover, we know that $\widetilde{Q}|_E$ is a smooth irreducible curve, since the quadric surface Q is smooth. Thus, since $\widetilde{Q}\big|_E \neq \mathbf{s}$, we have

$$
0 \leqslant \widetilde{Q}\big|_E\cdot\mathbf{s}=\Big(\mathbf{s}+\frac{n+2}{2}\mathbf{l}\Big)\cdot\mathbf{s}=-n+\frac{n+2}{2}=\frac{2-n}{2}
$$

so that $n=0$ or $n=2$. Now, let us show that $S(W^E_{\bullet,\bullet};Z)<1$ in both cases.

For $u \geqslant 0$,

$$
-K_X-uE \sim 2\widetilde{Q}+(1-u)E,
$$

so that $-K_X-uE$ is pseudo-effective if and only if $u \leqslant 1$, and it is nef if and only if $u \leqslant \frac{1}{3}$. Furthermore, if $\frac{1}{3} \leqslant u \leqslant 1$, then

$$
P(-K_X-uE)=(2-2u)(3H-E)
$$

and $N(-K_X-uE)=(3u-1)\widetilde{Q}$. Thus, if $n=0$, we have

$$
P(-K_X-uE)\big|_E=\begin{cases}(1+u)\mathbf{s}+(9-7u)\mathbf{l} & \text{if } 0 \leqslant u \leqslant \dfrac{1}{3},\\[2mm] (2-2u)\mathbf{s}+(10-10u)\mathbf{l} & \text{if } \dfrac{1}{3} \leqslant u \leqslant 1.\end{cases}
$$

Similarly, if $n=2$, then

$$P(-K_X-uE)\big|_E=\begin{cases}(1+u)\mathbf{s}+(10-6u)\mathbf{l} & \text{if } 0\leqslant u\leqslant\dfrac{1}{3},\\ (2-2u)\mathbf{s}+(12-12u)\mathbf{l} & \text{if } \dfrac{1}{3}\leqslant u\leqslant 1.\end{cases}$$

Recall that $Z\neq\widetilde{Q}\cap E$. Moreover, we have $Z\not\sim\mathbf{l}$, since $\pi(Z)$ is not one of the G-fixed points O_1, O_2, O_3, O_4. Thus, using [2, Corollary 1.7.26], we get

$$S(W^E_{\bullet,\bullet};Z)=\frac{1}{10}\int\limits_0^1\int\limits_0^\infty \mathrm{vol}\Big(P(u)\big|_E-vZ\Big)dvdu\leqslant\frac{1}{10}\int\limits_0^1\int\limits_0^\infty \mathrm{vol}\Big(P(u)\big|_E-v\mathbf{s}\Big)dvdu,$$

because the divisor $|Z-\mathbf{s}|\neq\varnothing$.

Consequently, if $n=0$, then

$$\begin{aligned}S(W^E_{\bullet,\bullet};Z)&\leqslant\\ &\frac{1}{10}\left\{\int\limits_0^{\frac{1}{3}}\int\limits_0^\infty \mathrm{vol}\Big((1+u)\mathbf{s}+(9-7u)\mathbf{l}-v\mathbf{s}\Big)dvdu+\right.\\ &\qquad\left.+\int\limits_{\frac{1}{3}}^1\int\limits_0^\infty \mathrm{vol}\Big((2-2u)\mathbf{s}+(10-10u)\mathbf{l}-v\mathbf{s}\Big)dvdu\right\}\\ &=\frac{1}{10}\left\{\int\limits_0^{\frac{1}{3}}\int\limits_0^{1+u}2(1+u-v)(9-7u)dvdu+\int\limits_{\frac{1}{3}}^1\int\limits_0^{2-2u}2(2-2u-v)(10-10u)dvdu\right\}\\ &=\frac{1783}{3240}.\end{aligned}$$

Similarly, if $n=2$, then

$$\begin{aligned}S(W^E_{\bullet,\bullet};Z)&\leqslant\\ &\frac{1}{10}\left\{\int\limits_0^{\frac{1}{3}}\int\limits_0^\infty \mathrm{vol}\Big((1+u)\mathbf{s}+(10-6u)\mathbf{l}-v\mathbf{s}\Big)dvdu+\right.\\ &\qquad\left.+\int\limits_{\frac{1}{3}}^1\int\limits_0^\infty \mathrm{vol}\Big((2-2u)\mathbf{s}+(12-12u)\mathbf{l}-v\mathbf{s}\Big)dvdu\right\}\\ &=\frac{1}{10}\left\{\int\limits_0^{\frac{1}{3}}\int\limits_0^{1+u}2(1+u-v)(10-6u)dvdu+\int\limits_{\frac{1}{3}}^1\int\limits_0^{2-2u}2(2-2u-v)(12-12u)dvdu\right\}\\ &=\frac{1043}{1620}.\end{aligned}$$

In both cases, we have $S(W^E_{\bullet,\bullet};Z)<1$, which is a contradiction. □

Now, we prove our main technical result using Abban–Zhuang theory, see also [2, § 1.7].

Proposition. *The center Z is not contained in $H_1\cup H_2\cup H_3\cup H_4$.*

Proof. Suppose that $Z \subset H_1 \cup H_2 \cup H_3 \cup H_4$. Without loss of generality, we may assume that either $Z \subset H_1$ or $Z \subset H_4$. We will only consider the case when $Z \subset H_1$, because the proof is very similar and simpler in the other case. Thus, we assume that $Z \subset H_1$. Then $\pi(Z) \subset \Pi_1$. Therefore, we see that one of the following two subcases are possible:

- either $\pi(Z)$ is one of the G-fixed points O_2, O_3, O_4,
- or Z is a G-invariant irreducible curve in H_1.

We will deal with these subcases separately. In both subcases, we let $S = H_1$ for simplicity. Recall that S is a smooth del Pezzo surface of degree 5, the surface S is G-invariant, and the action of the group G on the surface S is faithful. Note also that $Z \not\subset \widetilde{Q}$ by Lemma.

Let us use notations introduced in [2, § 1.7]. Take $u \in \mathbb{R}_{\geqslant 0}$. Then

$$-K_X - uS \sim_{\mathbb{R}} (4-u)H - E \sim_{\mathbb{R}} \widetilde{Q} + (2-u)H \sim_{\mathbb{R}} (u-1)\widetilde{Q} + (2-u)\big(3H - E\big).$$

Let $P(u) = P(-K_X - uS)$ and $N(u) = N(-K_X - uS)$. Then

$$P\big(u\big) = \begin{cases} -K_X - uS & \text{if } 0 \leqslant u \leqslant 1, \\ (2-u)\big(3H - E\big) & \text{if } 1 \leqslant u \leqslant 2, \end{cases}$$

and

$$N\big(u\big) = \begin{cases} 0 & \text{if } 0 \leqslant u \leqslant 1, \\ (u-1)\widetilde{Q} & \text{if } 1 \leqslant u \leqslant 2. \end{cases}$$

Note that $S_X(S) < 1$, see [2, Theorem 3.7.1]. In fact, one can compute $S_X(S) = \frac{17}{30}$.

Let $\varphi\colon S \to \Pi_1$ be birational morphism induced by π. Then φ is a G-equivariant blow up of the four intersection points $\Pi_1 \cap C_4$. Let ℓ be the proper transform on S of a general line in Π_1, and let e_1, e_2, e_3, e_4 be φ-exceptional curves, and let ℓ_{ij} be the proper transform on the surface S of the line in Π_1 that passes through $\varphi(e_i)$ and $\varphi(e_j)$, where $1 \leqslant i < j \leqslant 4$. Then the cone $\overline{\mathrm{NE}(S)}$ is generated by the curves e_1, e_2, e_3, e_4, ℓ_{12}, ℓ_{13}, ℓ_{14}, ℓ_{23}, ℓ_{24}, ℓ_{34}. Recall also that

$$\Pi_1 \cap C_4 = \big(L_{24} \cap C_4\big) \cup \big(L_{34} \cap C_4\big).$$

Therefore, we may assume that $L_{24} \cap C_4 = \varphi(e_1) \cup \varphi(e_2)$ and $L_{34} \cap C_4 = \varphi(e_3) \cup \varphi(e_4)$, so that we have $\varphi(\ell_{12}) = L_{24}$ and $\varphi(\ell_{34}) = L_{34}$.

Observe that, the group $\mathrm{Pic}^G(S)$ is generated by the divisor classes ℓ, $e_1 + e_2$, $e_3 + e_4$, because both $L_{24} \cap C_4$ and $L_{34} \cap C_4$ are G-orbits of length 2. Therefore, if Z is a curve, then $\varphi(Z)$ is a curve of degree $d \geqslant 1$, so that $Z \sim d\ell - m_{12}(e_1 + e_2) - m_{34}(e_3 + e_4)$ for some non-negative integers m_{12} and m_{34}, which gives

$$\begin{aligned} Z &\sim (d - 2m_{12})\ell + m_{12}(2\ell - e_1 - e_2 - e_3 - e_4) + (m_{12} - m_{34})(e_3 + e_4) \\ &\sim (d - 2m_{12})(\ell_{12} + e_1 + e_2) + m_{12}(\ell_{12} + \ell_{34}) + (m_{12} - m_{34})(e_3 + e_4) \end{aligned}$$

and

$$\begin{aligned} Z &\sim (d - 2m_{34})\ell + m_{34}(2\ell - e_1 - e_2 - e_3 - e_4) + (m_{34} - m_{12})(e_1 + e_2) \\ &\sim (d - 2m_{34})(\ell_{34} + e_3 + e_4) + m_{34}(\ell_{12} + \ell_{34}) + (m_{34} - m_{12})(e_1 + e_2). \end{aligned}$$

Moreover, if $Z \neq \ell_{12}$ and $Z \neq \ell_{34}$, then $d - 2m_{12} = Z \cdot \ell_{12} \geqslant 0$ and $d - 2m_{34} = Z \cdot \ell_{34} \geqslant 0$. Hence, if Z is a curve, then $|Z - \ell_{12}| \neq \varnothing$ or $|Z - \ell_{34}| \neq \varnothing$.

On the other hand, if Z is a curve, then [2, Corollary 1.7.26] gives

$$1 \geqslant \frac{A_X(F)}{S_X(F)} \geqslant \min\left\{\frac{1}{S_X(S)}, \frac{1}{S\big(W^S_{\bullet,\bullet}; Z\big)}\right\} = \min\left\{\frac{30}{17}, \frac{1}{S\big(W^S_{\bullet,\bullet}; Z\big)}\right\},$$

where

$$S\big(W^{S}_{\bullet,\bullet};Z\big)=\frac{3}{(-K_X)^3}\int_0^2\int_0^\infty \mathrm{vol}\big(P(u)\big|_S-vZ\big)dvdu,$$

because $Z\not\subset\widetilde{Q}$. Moreover, if $S(W^{S}_{\bullet,\bullet};Z)=1$, then [2, Corollary 1.7.26] gives

$$1\geqslant\frac{A_X(E)}{S_X(E)}=\frac{1}{S_X(S)}=\frac{30}{17},$$

which is absurd. Thus, if Z is a curve, then $S(W^{S}_{\bullet,\bullet};Z)>1$, which gives

$$\begin{aligned}1<S\big(W^{S}_{\bullet,\bullet};Z\big)&=\frac{1}{30}\int_0^2\int_0^\infty \mathrm{vol}\big(P(u)\big|_S-vZ\big)dvdu\\&\leqslant\max\left\{\frac{1}{10}\int_0^2\int_0^\infty \mathrm{vol}\big(P(u)\big|_S-v\ell_{12}\big)dvdu,\frac{1}{10}\int_0^2\int_0^\infty \mathrm{vol}\big(P(u)\big|_S-v\ell_{34}\big)dvdu\right\},\end{aligned}$$

because $|Z-\ell_{12}|\neq\varnothing$ or $|Z-\ell_{34}|\neq\varnothing$. Note also that

$$S\big(W^{S}_{\bullet,\bullet};\ell_{12}\big)=\frac{1}{10}\int_0^2\int_0^\infty \mathrm{vol}\big(P(u)\big|_S-v\ell_{12}\big)dvdu=\frac{1}{10}\int_0^2\int_0^\infty \mathrm{vol}\big(P(u)\big|_S-v\ell_{34}\big)dvdu.$$

Hence, if Z is a curve, then

$$1<S\big(W^{S}_{\bullet,\bullet};Z\big)\leqslant S\big(W^{S}_{\bullet,\bullet};\ell_{12}\big)=\frac{1}{10}\int_0^2\int_0^\infty \mathrm{vol}\big(P(u)\big|_S-v\ell_{12}\big)dvdu.$$

Let us compute $S(W^{S}_{\bullet,\bullet};\ell_{12})$. For $0\leqslant u\leqslant 1$ and $v\geqslant 0$, we have

$$P(u)\big|_S-v\ell_{12}=\big(-K_X-uS\big)\big|_S-v\ell_{12}\sim_{\mathbb{R}}(4-u-v)\ell-(1-v)\big(e_1+e_2\big)-e_3-e_4.$$

Therefore, if $0\leqslant v\leqslant 1$, then this divisor is nef, and its volume is $u^2+2uv-v^2-8u-4v+12$. Similarly, if $1\leqslant v\leqslant 2-u$, then its Zariski decomposition is

$$P(u)\big|_S-v\ell_{12}\sim_{\mathbb{R}}\underbrace{(4-u-v)\ell-e_3-e_4}_{\text{positive part}}+\underbrace{(v-1)\big(e_1+e_2\big)}_{\text{negative part}},$$

so that its volume is $u^2+2uv+v^2-8u-8v+14$. Likewise, if $2-u\leqslant v\leqslant 3-u$, then the Zariski decomposition of the divisor $P(u)|_S-v\ell_{12}$ is

$$P(u)\big|_S-v\ell_{12}\sim_{\mathbb{R}}\underbrace{(3-u-v)(2\ell-e_3-e_4)}_{\text{positive part}}+\underbrace{(v-1)\big(e_1+e_2\big)+(v-2+u)\ell_{34}}_{\text{negative part}},$$

so that its volume is $2(3-u-v)^2$. If $v>3-u$, then $P(u)|_S-v\ell_{12}$ is not pseudo-effective, so that the volume of this divisor is zero. Thus, we have

$$\begin{aligned}
&\frac{1}{10}\int_0^1\int_0^\infty \mathrm{vol}\Big(P(u)\big|_S-v\ell_{12}\Big)dvdu\\
&\qquad=\frac{1}{10}\int_0^1\int_0^{3-u} \mathrm{vol}\Big(P(u)\big|_S-v\ell_{12}\Big)dvdu\\
&\qquad=\frac{1}{10}\left\{\int_0^1\int_0^1\Big(u^2+2uv-v^2-8u-4v+12\Big)dvdu+\right.\\
&\qquad\qquad\left.+\int_0^1\int_1^{2-u}\Big(u^2+2uv+v^2-8u-8v+14\Big)dvdu+\int_0^1\int_{2-u}^{3-u}2(3-u-v)^2dvdu\right\}\\
&\qquad=\frac{107}{120}.
\end{aligned}$$

Similarly, if $1\leqslant u\leqslant 2$, then

$$P(u)\big|_S-v\ell_{12}\sim_{\mathbb{R}}(6-3u-v)\ell+(v+u-2)(e_1+e_2)+(u-2)(e_3+e_4).$$

If $0\leqslant v\leqslant 2-u$, this divisor is nef, and its volume is $5u^2+2uv-v^2-20u-4v+20$. Likewise, if $2-u\leqslant v\leqslant 4-2u$, then its Zariski decomposition is

$$P(u)\big|_S-v\ell_{12}\sim_{\mathbb{R}}\underbrace{(4-2u-)(2\ell-e_3-e_4)}_{\text{positive part}}+\underbrace{(v-2+u)\big(e_1+e_2+\ell_{34}\big)}_{\text{negative part}},$$

and its volume is $2(v-2+u)^2$. If $v>4-2u$, this divisor is not pseudo-effective, so that

$$\begin{aligned}
&\frac{1}{10}\int_1^2\int_0^\infty \mathrm{vol}\Big(P(u)\big|_S-v\ell_{12}\Big)dvdu\\
&\qquad=\frac{1}{10}\int_1^2\int_0^{4-2u} \mathrm{vol}\Big(P(u)\big|_S-v\ell_{12}\Big)dvdu\\
&\qquad=\frac{1}{10}\left\{\int_1^2\int_0^{2-u}\Big(5u^2+2uv-v^2-20u-4v+20\Big)dvdu+\int_1^2\int_{2-u}^{4-2u}2(v-2+u)^2dvdu\right\}\\
&\qquad=\frac{19}{24}.
\end{aligned}$$

Therefore, we see that

$$\begin{aligned}
S\big(W^S_{\bullet,\bullet};\ell_{12}\big)&=\frac{1}{10}\int_0^2\int_0^\infty \mathrm{vol}\big(P(u)\big|_S-v\ell_{12}\big)dvdu\\
&=\frac{1}{10}\int_0^1\int_0^\infty \mathrm{vol}\Big(P(u)\big|_S-v\ell_{12}\Big)dvdu+\frac{1}{10}\int_1^2\int_0^\infty \mathrm{vol}\Big(P(u)\big|_S-v\ell_{12}\Big)dvdu\\
&=\frac{107}{120}+\frac{19}{24}=1,
\end{aligned}$$

which implies, in particular, that Z is not a curve.

Hence, we see that $\pi(Z)$ is one of the points O_2, O_3, O_4. Without loss of generality, we may assume that either $\pi(Z)=O_2$ or $\pi(Z)=O_4$, so that $Z\in\ell_{12}$ in both subcases. Now, using [2, Theorem 1.7.30], we see that

$$1\geqslant\frac{A_X(F)}{S_X(F)}\geqslant\min\left\{\frac{1}{S(W^{S,\ell_{12}}_{\bullet,\bullet,\bullet};Z)},\frac{1}{S(W^{S}_{\bullet,\bullet};\ell_{12})},\frac{1}{S_X(S)}\right\}=\min\left\{\frac{1}{S(W^{S,\ell_{12}}_{\bullet,\bullet,\bullet};Z)},1\right\},$$

where $S(W^{S,\ell_{12}}_{\bullet,\bullet,\bullet};Z)$ is defined in [2, § 1.7]. In fact, [2, Theorem 1.7.30] implies the strict inequality $S(W^{S,\ell_{12}}_{\bullet,\bullet,\bullet};Z)<1$, because $S_X(S)<1$. Let us compute $S(W^{S,\ell_{12}}_{\bullet,\bullet,\bullet};Z)$.

For $0\leqslant u\leqslant 2$ and $v\geqslant 0$, let $P(u,v)$ be the positive part of the Zariski decomposition of the divisor $P(u)|_S-v\ell_{12}$, and let $N(u,v)$ be its negative part.

If $0\leqslant u\leqslant 1$, then

$$P(u,v)=\begin{cases}(4-u-v)\ell-(1-v)\big(e_1+e_2\big)-e_3-e_4 & \text{if } 0\leqslant v\leqslant 1,\\ (4-u-v)\ell-e_3-e_4 & \text{if } 1\leqslant v\leqslant 2-u,\\ (3-u-v)(2\ell-e_3-e_4) & \text{if } 2-u\leqslant v\leqslant 3-u,\end{cases}$$

and

$$N(u,v)=\begin{cases}0 & \text{if } 0\leqslant v\leqslant 1,\\ (v-1)\big(e_1+e_2\big) & \text{if } 1\leqslant v\leqslant 2-u,\\ (v-1)\big(e_1+e_2\big)+(v-2+u)\ell_{34} & \text{if } 2-u\leqslant v\leqslant 3-u.\end{cases}$$

Similarly, if $1\leqslant u\leqslant 2$, then

$$P(u,v)=\begin{cases}(6-3u-v)\ell+(v+u-2)(e_1+e_2)+(u-2)(e_3+e_4) & \text{if } 0\leqslant v\leqslant 2-u,\\ (v-2+u)\big(e_1+e_2+\ell_{34}\big) & \text{if } 2-u\leqslant v\leqslant 4-2u,\end{cases}$$

and

$$N(u,v)=\begin{cases}0 & \text{if } 0\leqslant v\leqslant 1,\\ (v-2+u)\big(e_1+e_2+\ell_{34}\big) & \text{if } 2-u\leqslant v\leqslant 4-2u.\end{cases}$$

Recall from [2, Theorem 1.7.30] that

$$S\big(W^{S,\ell_{12}}_{\bullet,\bullet};Z\big)=F_Z\big(W^{S,\ell_{12}}_{\bullet,\bullet}\big)+\frac{3}{(-K_X)^3}\int\limits_0^2\int\limits_0^\infty\big(P(u,v)\cdot\ell_{12}\big)^2dvdu$$

for

$$F_Z\big(W^{S,\ell_{12}}_{\bullet,\bullet}\big)=\frac{6}{(-K_X)^3}\int\limits_0^2\int\limits_0^\infty\big(P(u,v)\cdot\ell_{12}\big)\mathrm{ord}_Z\Big(N_S'(u)\Big|_{\ell_{12}}+N(u,v)\Big|_{\ell_{12}}\Big)dvdu,$$

where $N_S'(u)$ is the part of the divisor $N(u)|_S$ whose support does not contain ℓ_{12}, so that $N_S'(u)=N(u)|_S$ in our case, which implies that $\mathrm{ord}_Z(N_S'(u)|_{\ell_{12}})=0$ for $0\leqslant u\leqslant 2$, because $Z\notin\widetilde{Q}$. Thus, if $\pi(Z)=O_2$, then $Z\notin\ell_{34}\cup e_1\cup e_2$, which gives $F_Z(W^{S,\ell_{12}}_{\bullet,\bullet})=0$.

On the other hand, if $\pi(Z)=O_4$, then $Z=\ell_{12}\cap\ell_{34}$ and $Z\not\in e_1\cup e_2$, so that

$$
\begin{aligned}
F_Z\big(W^{S,\ell_{12}}_{\bullet,\bullet}\big)&=\frac{1}{5}\int\limits_0^2\int\limits_0^\infty\big(P(u,v)\cdot\ell_{12}\big)\mathrm{ord}_Z\Big(N(u,v)\big|_{\ell_{12}}\Big)dvdu\\
&=\frac{1}{5}\left\{\int\limits_0^1\int\limits_{2-u}^{3-u}(6-2u-2v+6)(v-2+u)dvdu+\right.\\
&\qquad\left.+\int\limits_1^2\int\limits_{2-u}^{4-2u}(8-4u-2v+8)(v-2+u)dvdu\right\}\\
&=\frac{1}{12}.
\end{aligned}
$$

Therefore, we see that

$$
\begin{aligned}
&S\big(W^{S,\ell_{12}}_{\bullet,\bullet};Z\big)\leqslant\frac{1}{12}+\frac{1}{10}\int\limits_0^2\int\limits_0^\infty\big(P(u,v)\cdot\ell_{12}\big)^2dvdu\\
&=\frac{1}{12}+\frac{1}{10}\left\{\int\limits_0^1\int\limits_0^1\big(2-u+v\big)^2dvdu+\int\limits_0^1\int\limits_1^{2-u}\big(4-u-v\big)^2dvdu+\right.\\
&\left.+\int\limits_0^1\int\limits_{2-u}^{3-u}\big(6-2u-2v\big)^2dvdu+\int\limits_1^2\int\limits_0^{2-u}\big(2-u+v\big)^2dvdu+\int\limits_1^2\int\limits_{2-u}^{4-2u}\big(8-4u-2v\big)^2dvdu\right\}\\
&=1.
\end{aligned}
$$

However, as we already mentioned, one has $S(W^{S,\ell_{12}}_{\bullet,\bullet};Z)<1$ by [2, Theorem 1.7.30]. The obtained contradiction completes the proof of Proposition. ☐

Corollary. *Both Z and $\pi(Z)$ are irreducible curves, and $\pi(Z)$ is not entirely contained in $\Pi_1\cup\Pi_2\cup\Pi_3\cup\Pi_4\cup Q$.*

Using [2, Lemma 1.4.4], we see that $\alpha_{G,Z}(X)<\frac{3}{4}$. Now, using [2, Lemma 1.4.1], we see that there are a G-invariant effective $\mathbb{Q}$-divisor D on the threefold X and a positive rational number $\mu<\frac{3}{4}$ such that $D\sim_{\mathbb{Q}}-K_X$ and Z is contained in the locus $\mathrm{Nklt}(X,\mu D)$. Moreover, it follows from Claim that $\mathrm{Nklt}(X,\mu D)$ does not contain G-irreducible surfaces except maybe for $\widetilde{Q}$, H_1, H_2, H_3, H_3. Now, applying [2, Corollary A.1.13] to $(\mathbb{P}^3,\mu\pi(D))$, we see that $\pi(Z)$ must be a G-invariant line in $\mathbb{P}^3$. But this is impossible by Corollary, since all G-invariant lines in $\mathbb{P}^3$ are contained in $\Pi_1\cup\Pi_2\cup\Pi_3\cup\Pi_4$.

The obtained contradiction completes the proof of our Theorem.

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References

[1] H. Abban, Z. Zhuang, *K-stability of Fano varieties via admissible flags*, arXiv:2003.13788 (2020).

[2] C. Araujo, A.-M. Castravet, I. Cheltsov, K. Fujita, A.-S. Kaloghiros, J. Martinez-Garcia, C. Shramov, H. Süß, N. Viswanathan, *The Calabi problem for Fano threefolds*, MPIM preprint 2021-31.

[3] Z. Zhuang, *Optimal destabilizing centers and equivariant K-stability*, Invent. Math. **226** (2021), 195–223.

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