KUMMER QUARTIC DOUBLE SOLIDS

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Abstract. We study equivariant birational geometry of (rational) quartic double solids ramified over (singular) Kummer surfaces.

A Kummer quartic surface is an irreducible normal surface in $\mathbb{P}^3$ of degree 4 that has the maximal possible number of 16 singular points, which are ordinary double singularities. Any such surface is the Kummer variety of the Jacobian surface of a smooth genus 2 curve. Vice versa, the Jacobian surface of a smooth genus 2 curve admits a natural involution such that the quotient surface is a Kummer quartic surface in $\mathbb{P}^3$.

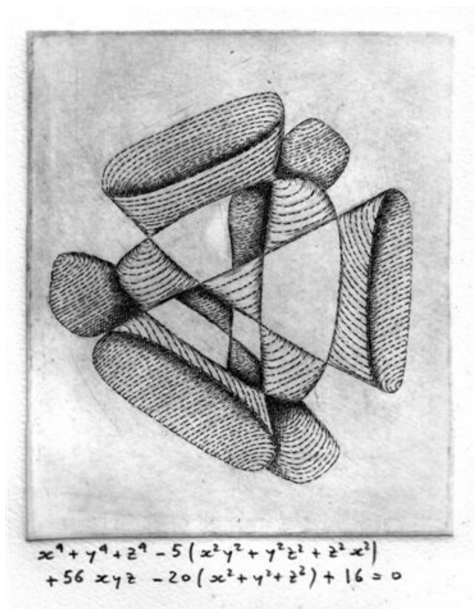

Figure 1. A Kummer surface by Patrice Jeener.

Let $\mathscr{S}$ be a Kummer surface in $\mathbb{P}^3$, and let $\mathscr{C}$ be the smooth genus 2 curve such that

$$\mathscr{S} \cong \mathrm{J}\big(\mathscr{C}\big)/\langle\tau\rangle, \tag{1}$$

where τ is the involution of the Jacobian $\mathrm{J}(\mathscr{C})$ that sends a point P to the point $-P$. Recall from [22, 28, 20, 17] that the quartic surface $\mathscr{S}$ can be given by the equation

$$a(x_0^4+x_1^4+x_2^4+x_3^4)+2b(x_0^2x_1^2+x_2^2x_3^2)+ \\ +2c(x_0^2x_2^2+x_1^2x_3^2)+2d(x_0^2x_3^2+x_1^2x_2^2)+4ex_0x_1x_2x_3=0 \tag{2}$$

for some $[a:b:c:d:e]\in\mathbb{P}^4$ such that

$$a(a^2+e^2-b^2-c^2-d^2)+2bcd=0. \tag{3}$$

Throughout this paper, all varieties are assumed to be projective and defined over $\mathbb{C}$.

Note that the curve $\mathscr{C}$ is hyperelliptic, and equation (3) defines a cubic threefold in $\mathbb{P}^4$, which is projectively equivalent to *the Segre cubic threefold* [28, 17].

Using a formula from the book [7] implemented in Magma [27], we can easily extract an equation of the surface $\mathscr{S}$ from the curve $\mathscr{C}$. However, the resulting equation may differ from (2). For instance, if $\mathscr{C}$ is the unique genus 2 curve such that $\mathrm{Aut}(\mathscr{C})\cong\boldsymbol{\mu}_2.\mathfrak{S}_4$, then $\mathscr{C}$ is isomorphic to the curve

$$\big\{z^2=xy(x^4-y^4)\big\}\subset\mathbb{P}(1,1,3)$$

where x, y, z are homogeneous coordinates on $\mathbb{P}(1,1,3)$ of weights 1, 1, 2, respectively. In this case, Magma produces the following Kummer quartic surface:

$$\big\{x_0^4+2x_0^2x_2x_3-2x_0^2x_2^2+4x_0x_2^2x_2-4x_0x_2x_3^2+x_2^2x_3^2-2x_2x_2^2x_3+x_2^4=0\big\}\subset\mathbb{P}^3,$$

which is projectively equivalent to the surface given by (2) with parameters $a=b=1$, $c=d=-1$, $e=-4$ that do not satisfy (3). But this surface is projectively equivalent to

$$\big\{x_0^4+x_1^4+x_2^4+x_3^4-4ix_0x_1x_2x_3=0\big\}\subset\mathbb{P}^3, \tag{4}$$

which is given by (2) with parameters $a=1$, $b=c=d=0$, $e=-i$ that do satisfy (3). Here, we use the following Magma code provided to us by Michela Artebani:

```
R<x>:=PolynomialRing(Rationals());
C:=HyperellipticCurve(x^5-x);
GroupName(GeometricAutomorphismGroup(C));
KummerSurfaceScheme(C);
```

It is not very difficult to recover the hyperelliptic curve $\mathscr{C}$ from the quartic surface $\mathscr{S}$. Indeed, $\mathbb{P}^3$ contains exactly 16 planes $\Pi_1,\ldots,\Pi_{16}$ such that $\mathscr{S}|_{\Pi_i}=2\mathcal{C}_i$ for each of them, where $\mathcal{C}_i$ is a smooth conic, called *trope*. One can show that

- each plane Π_i contains exactly six singular points of the surface $\mathscr{S}$,
- each singular point of the surface $\mathscr{S}$ is contained in six planes among $\Pi_1,\ldots,\Pi_{16}$.

Moreover, for every trope $\mathcal{C}_i$, there exists a double cover $\mathscr{C}\to\mathcal{C}_i$ which is ramified over the six points $\mathcal{C}_i\cap\mathrm{Sing}(\mathscr{S})$. This gives us an algorithm how to recover $\mathscr{C}$ from $\mathscr{S}$.

Example 5. Suppose that the surface $\mathscr{S}$ is given by the equation (2) with

$$\begin{cases} a=2,\\ b=-t^2-1,\\ c=-t^2-1,\\ d=-t^2-1,\\ e=t^3+3t,\end{cases}$$

where $t\in\mathbb{C}\setminus\{\pm1,\pm\sqrt{3}i\}$. Then the surface $\mathscr{S}$ is given by the following equation:

$$x_0^4+x_1^4+x_2^4+x_3^4+2(t^3+3t)x_0x_1x_2x_3=(t^2+1)(x_0^2x_1^2+x_2^2x_3^2+x_0^2x_2^2+x_1^2x_3^2+x_0^2x_3^2+x_1^2x_2^2). \tag{6}$$

Its singular locus $\mathrm{Sing}(\mathscr{S})$ consists of the following 16 points:

$$\begin{gathered}[1:1:1:t],[-1:1:-1:t],[-1:-1:1:t],[1:-1:-1:t],\\ [1:1:t:1],[1:-1:t:-1],[-1:-1:t:1],[-1:1:t:-1],\\ [t:1:1:1],[t:-1:1:-1],[t:1:-1:-1],[t:-1:-1:1],\\ [1:t:1:1],[-1:t:-1:1],[1:t:-1:-1],[-1:t:1:-1].\end{gathered}$$

Moreover, the planes $\Pi_1, \ldots, \Pi_{16}$ are listed in the following table:

$\Pi_1 = \{x_0 + x_1 + x_2 + tx_3 = 0\}$	$\Pi_2 = \{x_0 - x_1 + x_2 - tx_3 = 0\}$
$\Pi_3 = \{x_0 + x_1 - x_2 - tx_3 = 0\}$	$\Pi_4 = \{x_0 - x_1 - x_2 + tx_3 = 0\}$
$\Pi_5 = \{x_0 + x_1 + tx_2 + x_3 = 0\}$	$\Pi_6 = \{x_0 - x_1 + tx_2 - x_3 = 0\}$
$\Pi_7 = \{x_0 - tx_2 + x_1 - x_3 = 0\}$	$\Pi_8 = \{x_0 - x_1 - tx_2 + x_3 = 0\}$
$\Pi_9 = \{x_0 + tx_1 + x_2 + x_3 = 0\}$	$\Pi_{10} = \{x_0 - tx_1 - x_2 + x_3 = 0\}$
$\Pi_{11} = \{x_0 - tx_1 + x_2 - x_3 = 0\}$	$\Pi_{12} = \{x_0 + tx_1 - x_2 - x_3 = 0\}$
$\Pi_{13} = \{tx_0 + x_1 + x_2 + x_3 = 0\}$	$\Pi_{14} = \{tx_0 - x_1 - x_2 + x_3 = 0\}$
$\Pi_{15} = \{tx_0 - x_1 + x_2 - x_3 = 0\}$	$\Pi_{16} = \{tx_0 + x_1 - x_2 - x_3 = 0\}$

Then the trope $\mathcal{C}_1$ is the smooth conic

$$\big\{x_0 + x_1 + x_2 + tx_3 = tx_1x_3 + tx_2x_3 + x_1^2 + x_1x_2 + x_2^2 - x_3^2 = 0\big\} \subset \mathbb{P}^3.$$

This conic contains the following six singular points of our surface:

$$[1 : -1 : t : -1],\ [-1 : 1 : t : -1],\ [t : -1 : 1 : -1],$$
$$[t : 1 : -1 : -1],\ [1 : t : -1 : -1],\ [-1 : t : 1 : -1].$$

Projecting from $[t : 1 : -1 : -1]$, we get an isomorphism $\mathcal{C}_1 \cong \mathbb{P}^1$ that maps these points to

$$[t + 1 : -2],\ [1 : 0],\ [-1 : 1],\ [1 - t : 1 + t],\ [0 : 1],\ [t - 1 : 2].$$

Therefore, the hyperelliptic curve $\mathscr{C}$ is isomorphic to the curve

$$\Big\{z^2 = xy\big(x - y\big)\big((t - 1)x + 2y\big)\big(2x - (t + 1)y\big)\big((t + 1)x - (t - 1)y\big)\Big\} \subset \mathbb{P}(1, 1, 3).$$

In particular, it follows from [6] or Magma computations that

$$\mathrm{Aut}\big(\mathscr{C}\big) \cong \begin{cases} \boldsymbol{\mu}_2.\mathfrak{S}_4 \text{ if } t \in \{0, \pm i, 1 \pm 2i, -1 \pm 2i\}, \\ \boldsymbol{\mu}_2.\mathrm{D}_{12} \text{ if } t \in \{0, \pm 3\}, \\ \boldsymbol{\mu}_2 \times \mathfrak{S}_3 \text{ if } t \text{ is general.} \end{cases}$$

For instance, to identify $\mathrm{Aut}(\mathscr{C})$ in the case when $t = i$, one can use the following script:

```
K:=CyclotomicField(4);
R<x>:=PolynomialRing(K);
i:=Roots(x^2+1,K)[1,1];
t:=i;
f:=x*(x-1)*((t-1)*x+2)*(2*x-(t+1))*((t+1)*x-(t-1));
C:=HyperellipticCurve(f);
GroupName(GeometricAutomorphismGroup(C));
```

In this example, we assume that $t \notin \{\pm 1, \pm\sqrt{3}i\}$, because

- if $t = \pm 1$ or $t = \infty$, then the equation (6) defines a union of 4 planes,
- if $t = \pm\sqrt{3}i$, the equation (6) defines a double quadric.

These are semistable degenerations with minimal $\mathrm{PGL}_4(\mathbb{C})$-orbits [32, Theorem 2.4].

Let $\mathrm{Aut}(\mathbb{P}^3,\mathscr{S})$ be the subgroup in $\mathrm{PGL}_4(\mathbb{C})$ consisting of projective transformations that leave $\mathscr{S}$ invariant. Then $\mathrm{Aut}(\mathbb{P}^3,\mathscr{S})$ contains a subgroup $\mathbb{H}\cong\boldsymbol{\mu}_2^4$ generated by

$$A_1=\begin{pmatrix}-1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}, A_2=\begin{pmatrix}-1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, A_3=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}, A_4=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}.$$

The action of this subgroup on $\mathscr{S}$ is induced by the translations of $\mathrm{J}(\mathscr{C})$ by two-torsion points, so $\mathrm{Sing}(\mathscr{S})$ is an $\mathbb{H}$-orbit. Similarly, we see that $\mathbb{H}$ acts transitively on the set

$$\Big\{\Pi_1,\Pi_2,\Pi_3,\Pi_4,\Pi_5,\Pi_6,\Pi_7,\Pi_8,\Pi_9,\Pi_{10},\Pi_{11},\Pi_{12},\Pi_{13},\Pi_{14},\Pi_{15},\Pi_{16}\Big\}. \tag{7}$$

If $\mathscr{S}$ is general, then $\mathrm{Aut}(\mathbb{P}^3,\mathscr{S})=\mathbb{H}$, and $\mathrm{Aut}(\mathscr{C})$ is generated by the hyperelliptic involution [23]. However, if $\mathscr{S}$ is special, then $\mathrm{Aut}(\mathbb{P}^3,\mathscr{S})$ can be larger than $\mathbb{H}$.

Example 8. Let us use assumptions and notations of Example 5. For $t\in\mathbb{C}\backslash\{\pm1,\pm\sqrt{3}i\}$, the group $\mathrm{Aut}(\mathbb{P}^3,\mathscr{S})$ contains the subgroup isomorphic to $\boldsymbol{\mu}_2^4\rtimes\mathfrak{S}_3$ generated by

$$A_1,A_2,A_3,A_4,\begin{pmatrix}0&0&0&1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix},\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}.$$

In fact, this is the whole group $\mathrm{Aut}(\mathbb{P}^3,\mathscr{S})$ if t is general. On the other hand, if $t=0$, then it follows from [8] that $\mathrm{Aut}(\mathbb{P}^3,\mathscr{S})\cong\boldsymbol{\mu}_2^4\rtimes\mathrm{D}_{12}$, and this group is generated by

$$A_1,A_2,A_3,A_4,\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}0&0&0&1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix},\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}.$$

If $t=\pm i$, then $\mathscr{S}$ is the surface (4), and $\mathrm{Aut}(\mathbb{P}^3,\mathscr{S})\cong\boldsymbol{\mu}_2^4\rtimes\mathfrak{S}_4$ is generated by

$$A_1,A_2,A_3,A_4,\begin{pmatrix}0&0&0&1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix},\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}i&0&0&0\\0&i&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}.$$

Example 9. Suppose that $\mathscr{S}$ is given by the equation (2) with

$$\begin{cases}a=2\zeta_5^3+2\zeta_5^2+6\zeta_5-1,\\b=4\zeta_5^3+4\zeta_5^2-10\zeta_5+9,\\c=-6\zeta_5^3-6\zeta_5^2+4\zeta_5+3,\\d=11,\\e=-20\zeta_5^3+24\zeta_5^2-16\zeta_5+10.\end{cases}$$

Then $\mathrm{Aut}(\mathscr{C})\cong\boldsymbol{\mu}_2\times\boldsymbol{\mu}_5$ and $\mathrm{Aut}(\mathbb{P}^3,\mathscr{S})\cong\boldsymbol{\mu}_2^4\rtimes\boldsymbol{\mu}_5$, which is generated by

$$A_1,A_2,A_3,A_4,\begin{pmatrix}-i&0&0&i\\0&1&1&0\\1&0&0&1\\0&-i&i&0\end{pmatrix}.$$

Looking at Examples 5, 8 and 9, one can spot a relation between $\mathrm{Aut}(\mathscr{S})$ and $\mathrm{Aut}(\mathscr{C})$. In fact, this relation holds for all Kummer surfaces in $\mathbb{P}^3$ by the following well-known result, about which we learned from Igor Dolgachev.

Lemma 10. *Let $\iota \in \mathrm{Aut}(\mathscr{C})$ be the hyperelliptic involution of the curve $\mathscr{C}$. Then*

$$\mathrm{Aut}\big(\mathbb{P}^3, \mathscr{S}\big) \cong \boldsymbol{\mu}_2^4 \rtimes \big(\mathrm{Aut}(\mathscr{C})/\langle\iota\rangle\big).$$

Proof. Let us identify $\mathscr{C}$ with the theta divisor in $\mathrm{J}(\mathscr{C})$ via the Abel–Jacobi map whose base point is one of the fixed points of the involution ι (one of the six Weierstrass points). Then the linear system $|2\mathscr{C}|$ gives a morphism $\mathrm{J}(\mathscr{C}) \to \mathbb{P}^3$ whose image is the surface $\mathscr{S}$. Taking the Stein factorization of the morphism $\mathrm{J}(\mathscr{C}) \to \mathscr{S}$, we get the isomorphism (1).

On the other hand, elements in $\mathrm{Aut}(\mathscr{C})$ give automorphisms in $\mathrm{Aut}(\mathrm{J}(\mathscr{C}))$ that leave the linear system $|2\mathscr{C}|$ invariant. This gives us a homomorphism $\mathrm{Aut}(\mathscr{C}) \to \mathrm{Aut}(\mathbb{P}^3, \mathscr{S})$, whose kernel is the hyperelliptic involution ι, since ι induces the involution $\tau \in \mathrm{Aut}(\mathrm{J}(\mathscr{C}))$.

The image of the group $\mathrm{Aut}(\mathscr{C})$ in $\mathrm{Aut}(\mathbb{P}^3, \mathscr{S})$ normalizes the subgroup $\mathbb{H}$, because elements in $\mathbb{H}$ are induced by the translations of the Jacobian $\mathrm{J}(\mathscr{C})$ by two-torsion points. This gives a monomorphism $\vartheta\colon \boldsymbol{\mu}_2^4 \rtimes (\mathrm{Aut}(\mathscr{C})/\langle\iota\rangle) \to \mathrm{Aut}(\mathbb{P}^3, \mathscr{S})$.

We claim that ϑ is an epimorphism. Indeed, the action of an element $g \in \mathrm{Aut}(\mathbb{P}^3, \mathscr{S})$ on the surface $\mathscr{S}$ lifts to its its action on the Jacobian $\mathrm{J}(\mathscr{C})$ that leaves $[2\mathscr{C}]$ invariant, so composing g with some $h \in \mathbb{H}$, we obtain an element $g \circ h$ that preserves the class $[\mathscr{C}]$. Thus, since $[\mathscr{C}]$ is a principal polarization, the composition $g \circ h$ preserves $\mathscr{C}$, and it acts faithfully on $\mathscr{C}$, since $\mathscr{C}$ generates $\mathrm{J}(\mathscr{C})$. This gives $g \circ h \in \mathrm{im}(\vartheta)$, so ϑ is surjective. □

Since $\mathrm{Aut}(\mathscr{C})$ is isomorphic to a group among $\boldsymbol{\mu}_2$, $\boldsymbol{\mu}_2^2$, D_8, D_{12}, $\boldsymbol{\mu}_2.\mathrm{D}_{12}$, $\boldsymbol{\mu}_2.\mathfrak{S}_4$, $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_5$, we conclude that $\mathrm{Aut}(\mathbb{P}^3, \mathscr{S})$ is isomorphic to one of the following groups:

$$\boldsymbol{\mu}_2^4,\ \boldsymbol{\mu}_2^4 \rtimes \boldsymbol{\mu}_2,\ \boldsymbol{\mu}_2^4 \rtimes \boldsymbol{\mu}_2^2,\ \boldsymbol{\mu}_2^4 \rtimes \mathfrak{S}_3,\ \boldsymbol{\mu}_2^4 \rtimes \mathrm{D}_{12},\ \boldsymbol{\mu}_2^4 \rtimes \mathfrak{S}_4,\ \boldsymbol{\mu}_2^4 \rtimes \boldsymbol{\mu}_5.$$

Note that the group $\mathrm{Aut}(\mathscr{S})$ is always larger that $\mathrm{Aut}(\mathbb{P}^3, \mathscr{S})$ [23].

Remark 11 ([4, 28, 20]). Let $\mathfrak{N}$ be the normalizer of the subgroup $\mathbb{H}$ in the group $\mathrm{PGL}_4(\mathbb{C})$. Then $\mathrm{Aut}(\mathscr{C}) \subset \mathfrak{N}$, and there is a non-split exact sequence $1 \longrightarrow \mathbb{H} \longrightarrow \mathfrak{N} \longrightarrow \mathfrak{S}_6 \longrightarrow 1$, which can be described as follows. Let

$$B_1 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } B_2 = \begin{pmatrix} -i & 0 & 0 & i \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -i & i & 0 \end{pmatrix}.$$

Then $\langle B_1, B_2\rangle \in \mathfrak{N}$. Since $B_1^2 \in \mathbb{H}$, $B_2^5 = (B_1B_2)^6 = [B_1, B_2]^3 = \mathrm{Id}_{\mathbb{P}^3}$, $[B_1, B_2B_1B_2]^2 \in \mathbb{H}$, the images of B_1 and B_2 in the quotient $\mathfrak{N}/\mathbb{H}$ generate the whole group $\mathfrak{N}/\mathbb{H} \cong \mathfrak{S}_6$. Set

$$S_1 = \big\{x_0^4 + x_1^4 + x_2^4 + x_3^4 - 6\big(x_0^2x_1^2 + x_2^2x_3^2\big) - 6\big(x_0^2x_2^2 + x_1^2x_3^2\big) - 6\big(x_0^2x_3^2 + x_1^2x_2^2\big) = 0\big\},$$
$$S_2 = \big\{x_0^4 + x_1^4 + x_2^4 + x_3^4 - 6\big(x_0^2x_1^2 + x_2^2x_3^2\big) + 6\big(x_0^2x_2^2 + x_1^2x_3^2\big) + 6\big(x_0^2x_3^2 + x_1^2x_2^2\big) = 0\big\},$$
$$S_3 = \big\{x_0^4 + x_1^4 + x_2^4 + x_3^4 + 6\big(x_0^2x_1^2 + x_2^2x_3^2\big) - 6\big(x_0^2x_2^2 + x_1^2x_3^2\big) + 6\big(x_0^2x_3^2 + x_1^2x_2^2\big) = 0\big\},$$
$$S_4 = \big\{x_0^4 + x_1^4 + x_2^4 + x_3^4 + 6\big(x_0^2x_1^2 + x_2^2x_3^2\big) + 6\big(x_0^2x_2^2 + x_1^2x_3^2\big) - 6\big(x_0^2x_3^2 + x_1^2x_2^2\big) = 0\big\},$$
$$S_5 = \big\{x_0^4 + x_1^4 + x_2^4 + x_3^4 - 12x_0x_1x_2x_3 = 0\big\},$$
$$S_6 = \big\{x_0^4 + x_1^4 + x_2^4 + x_3^4 + 12x_0x_1x_2x_3 = 0\big\}.$$

Then S_1, S_2, S_3, S_4, S_5, S_6 are $\mathbb{H}$-invariant surfaces, and the quotient $\mathfrak{N}/\mathbb{H}$ permutes them. For instance, the transformation B_1 acts on the set $\{S_1, S_2, S_3, S_4, S_5, S_6\}$ as $(1\,2)(3\,4)(5\,6)$, and B_2 acts as the permutation $(1\,2\,6\,3\,5)$. This gives an explicit isomorphism $\mathfrak{N}/\mathbb{H} \cong \mathfrak{S}_6$.

Remark 12. The quotient $\mathrm{Aut}(\mathscr{S})/\mathbb{H}$ naturally linearly acts on the threefold (3) fixing the point $[a : b : c : d : e]$ that corresponds to $\mathscr{S}$. Projecting the threefold from this point, we obtain a (rational) double cover of $\mathbb{P}^3$ that is branched along the surface $\mathscr{S}$.

Let $\pi\colon X \to \mathbb{P}^3$ be the double cover branched along the surface $\mathscr{S}$. Set $H = \pi^*(\mathcal{O}_{\mathbb{P}^3}(1))$. Then $\mathrm{Pic}(X) = \mathbb{Z}[H]$, $H^3 = 2$ and $-K_X \sim 2H$, so X is a del Pezzo threefold of degree 2, which has 16 ordinary double points. We say that X is *a Kummer quartic double solid* [33].

The threefold X is a hypersurface in $\mathbb{P}(1, 1, 1, 1, 2)$ given by

$$(13)\quad w^2 = a(x_0^4 + x_1^4 + x_2^4 + x_3^4) + 2b(x_0^2x_1^2 + x_2^2x_3^2)+ \\ + 2c(x_0^2x_2^2 + x_1^2x_3^2) + 2d(x_0^2x_3^2 + x_1^2x_2^2) + 4ex_0x_1x_2x_3 = 0,$$

where we consider x_0, x_1, x_2, x_3 as homogeneous coordinates on $\mathbb{P}(1, 1, 1, 1, 2)$ of weight 1, and w is a homogeneous coordinate on $\mathbb{P}(1, 1, 1, 1, 2)$ of weight 2.

It is well-known that the threefold X is rational [31, 33, 29, 11], see also Remark 12. Moreover, it follows from [29] that there exists the following commutative diagram:

$$(14)\qquad \begin{array}{ccc} \widehat{X} & \xrightarrow{\ \varphi\ } & X \\ {\scriptstyle\eta}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ \mathbb{P}^3 & \overset{\chi}{\dashrightarrow} & \mathbb{P}^3 \end{array}$$

where η is a blow up of six distinct points that are contained in a twisted cubic $C_3 \subset \mathbb{P}^3$, the morphism φ is a contraction of the proper transform of the curve C_3 and proper transforms of 15 lines in $\mathbb{P}^3$ that pass through two blown up points, and χ is a rational map given by the linear system of quadric surfaces that pass through six blown up points.

Corollary 15 ([15, 19])**.** *One has* $\mathrm{Cl}(X) \cong \mathbb{Z}^7$.

Remark 16. The vertices of the quadric cones in $\mathbb{P}^3$ that pass through six blown up points in the diagram (14) span a quartic surface $\mathfrak{S}$ which is known as *the Weddle surface* [22, 33]. This surface has nodes at the six blown points, and χ induces a birational map $\mathfrak{S} \dashrightarrow \mathscr{S}$. On the other hand, the double cover of $\mathbb{P}^3$ branched along $\mathfrak{S}$ is irrational [33, 11].

Let $\sigma \in \mathrm{Aut}(X)$ be the Galois involution of the double cover π. Then σ is contained in the center of the group $\mathrm{Aut}(X)$. Moreover, since π is $\mathrm{Aut}(X)$-equivariant, it induces a homomorphism $\upsilon\colon \mathrm{Aut}(X) \to \mathrm{Aut}(\mathbb{P}^3, \mathscr{S})$ with $\ker(\upsilon) = \langle\sigma\rangle$, so we have exact sequence

$$1 \longrightarrow \langle\sigma\rangle \longrightarrow \mathrm{Aut}(X) \xrightarrow{\ \upsilon\ } \mathrm{Aut}\big(\mathbb{P}^3, \mathscr{S}\big) \longrightarrow 1.$$

The main result of this paper is the following theorem (cf. [2, 3, 10]).

Theorem 17. *Let G be any subgroup in* $\mathrm{Aut}(X)$ *such that* $\mathrm{Cl}^G(X) \cong \mathbb{Z}$ *and* $\mathbb{H} \subseteq \upsilon(G)$. *Then the Fano threefold X is G-birationally super-rigid.*

Corollary 18. *Let G be any subgroup in* $\mathrm{Aut}(X)$ *such that G contains σ and* $\mathbb{H} \subseteq \upsilon(G)$. *Then X is G-birationally super-rigid.*

The condition $\mathrm{Cl}^G(X) \cong \mathbb{Z}$ in Theorem 17 simply means that X is a G-Mori fibre space, which is required by the definition of G-birational super-rigidity (see [13, Definition 3.1.1]). The condition $\mathbb{H} \subseteq \upsilon(G)$ does not imply that $\mathrm{Cl}^G(X) \cong \mathbb{Z}$, see Examples 28 and 29 below. The following example shows that we cannot remove the condition $\mathbb{H} \subseteq \upsilon(G)$.

Example 19. Observe that $\mathrm{Cl}^{\langle\sigma\rangle}(X) \cong \mathbb{Z}$. Let S_1 and S_2 be two general surfaces in $|H|$, and let $C = S_1 \cap S_2$. Then C is a smooth irreducible $\langle\sigma\rangle$-invariant curve, $\pi(C)$ is a line, and there exists $\langle\sigma\rangle$-commutative diagram

$$\begin{array}{ccc} & V & \\ {\scriptstyle\alpha}\swarrow & & \searrow{\scriptstyle\beta} \\ X & \dashrightarrow & \mathbb{P}^1 \end{array}$$

where α is the blow up of the curve C, the dashed arrow $\dashrightarrow$ is given by the pencil generated by the surfaces S_1 and S_2, and β is a fibration into del Pezzo surfaces of degree 2. Therefore, the threefold X is not $\langle\sigma\rangle$-birationally rigid.

Let G be a subgroup in $\mathrm{Aut}(X)$ such that $\upsilon(G)$ contains $\mathbb{H}$. Before proving Theorem 17, let us explain how to check the condition $\mathrm{Cl}^G(X) \cong \mathbb{Z}$. For a homomorphism $\rho\colon \mathbb{H} \to \boldsymbol{\mu}_2$, consider the action of the group $\mathbb{H}$ on the threefold X given by

$$\begin{aligned} A_1\colon [x_0 : x_1 : x_2 : x_3 : w] &\mapsto [-x_0 : x_1 : -x_2 : x_3 : \rho(A_1)w], \\ A_2\colon [x_0 : x_1 : x_2 : x_3 : w] &\mapsto [-x_0 : x_1 : -x_2 : x_3 : \rho(A_2)w], \\ A_3\colon [x_0 : x_1 : x_2 : x_3 : w] &\mapsto [x_1 : x_2 : x_3 : x_2 : \rho(A_3)w], \\ A_4\colon [x_0 : x_1 : x_2 : x_3 : w] &\mapsto [x_3 : x_2 : x_1 : x_0 : \rho(A_4)w]. \end{aligned}$$

This gives a lift of the subgroup $\mathbb{H}$ to $\mathrm{Aut}(X)$. Let $\mathbb{H}^\rho$ be the resulting subgroup in $\mathrm{Aut}(X)$. Since $\mathbb{H} \subset \upsilon(G)$, we may assume that $\mathbb{H}^\rho \subset G$. If ρ is trivial, we let $\mathbb{H} = \mathbb{H}^\rho$ for simplicity.

For every plane Π_i, one has $\pi^*(\Pi_i) = \Pi_i^+ + \Pi_i^-$, where Π_i^+ and Π_i^- are two irreducible surfaces such that $\Pi_i^+ \neq \Pi_i^-$ and $\sigma(\Pi_i^+) = \Pi_i^-$. Note that we do not have a canonical way to distinguish between the surfaces Π_i^+ and Π_i^-. Namely, if $\pi^*(\Pi_i)$ is given by

$$\begin{cases} h_i(x_0, x_1, x_2, x_3) = 0, \\ w^2 = g_i^2(x_0, x_1, x_2, x_3), \end{cases}$$

where h_i is a linear polynomial such that $\Pi_i = \{h_i = 0\}$, and g_i is a quadratic polynomial such that the trope $\mathcal{C}_i$ is given by $h_i = g_i = 0$, then

$$\Pi_i^\pm = \big\{w \pm g_i(x_0, x_1, x_2, x_3) = h_i(x_0, x_1, x_2, x_3)\big\} \subset \mathbb{P}(1, 1, 1, 1, 2).$$

But the choice of $\pm$ here is not uniquely defined, because we can always swap g_i with $-g_i$.

On the other hand, since $\mathbb{H}$ acts transitively on the set (7), the set

$$\Big\{\Pi_1^+, \Pi_1^-, \Pi_2^+, \Pi_2^-, \Pi_3^+, \Pi_3^- \ldots, \Pi_{14}^+, \Pi_{14}^-, \Pi_{15}^+, \Pi_{15}^-, \Pi_{16}^+, \Pi_{16}^-\Big\}$$

splits into two $\mathbb{H}^\rho$-orbits consisting of 16 surfaces such that each of them contains exactly one surface among Π_i^+ and Π_i^- for every i. Hence, we may assume that these $\mathbb{H}^\rho$-orbits are

$$\Big\{\Pi_1^+, \Pi_2^+, \Pi_3^+, \Pi_4^+, \Pi_5^+, \Pi_6^+, \Pi_7^+, \Pi_8^+, \Pi_9^+, \Pi_{10}^+, \Pi_{11}^+, \Pi_{12}^+, \Pi_{13}^+, \Pi_{14}^+, \Pi_{15}^+, \Pi_{16}^+\Big\}$$

and

$$\Big\{\Pi_1^-, \Pi_2^-, \Pi_3^-, \Pi_4^-, \Pi_5^-, \Pi_6^-, \Pi_7^-, \Pi_8^-, \Pi_9^-, \Pi_{10}^-, \Pi_{11}^-, \Pi_{12}^-, \Pi_{13}^-, \Pi_{14}^-, \Pi_{15}^-, \Pi_{16}^-\Big\}.$$

Note that the surfaces $\Pi_1^+, \Pi_1^-, \ldots, \Pi_{16}^+, \Pi_{16}^-$ are not $\mathbb{Q}$-Cartier divisors on X, and their strict transforms on the threefold $\widehat{X}$ in (14) can be described as follows:

(a) six of them are η-exceptional surfaces;
(b) another six of them are strict transforms of quadric cones in $\mathbb{P}^3$ that contain all blown up points and are singular at one of them;
(c) the remaining twenty of them are proper transforms of the planes in $\mathbb{P}^3$ that pass through three blown up points.

Note also that σ acts birationally on $\widehat{X}$ as a composition of flops of φ-contracted curves. Moreover, it is not difficult to see that σ swaps six surfaces in (a) with six surfaces in (b), and σ maps the strict transform of the plane in $\mathbb{P}^3$ that passes through three blown up points to the strict transform of the plane that passes through other blown up points.

Corollary 20. *The surfaces* $\Pi_1^+, \Pi_1^-, \ldots, \Pi_{16}^+, \Pi_{16}^-$ *generate the group* $\mathrm{Cl}(X)$*.*

Corollary 21. *Either* $\mathrm{Cl}^{\mathbb{H}^\rho}(X) \cong \mathbb{Z}$ *or* $\mathrm{Cl}^{\mathbb{H}^\rho}(X) \cong \mathbb{Z}^2$*.*

Now, we are ready to state a criterion for $\mathrm{Cl}^G(X) \cong \mathbb{Z}$. To do this, we set

$$\Pi^{\pm} = \sum_{i=1}^{16} \Pi_i^{\pm}.$$

Then Π^+ and Π^- are $\mathbb{H}^\rho$-invariant divisors, $\sigma(\Pi^+) = \Pi^-$ and $\Pi^+ + \Pi^- \sim 16H$.

Lemma 22. *One has* $\mathrm{Cl}^G(X) \cong \mathbb{Z}$ *is at least one of the following conditions is satisfied:*

(i) *the group* G *swaps* Π^+ *and* Π^-*;*
(ii) *the divisor* Π^+ *is Cartier;*
(iii) *the divisor* Π^- *is Cartier;*
(iv) *the surfaces* $\Pi_1^+, \ldots, \Pi_{16}^+$ *generate the group* $\mathrm{Cl}(X)$*;*
(v) *the surfaces* $\Pi_1^-, \ldots, \Pi_{16}^-$ *generate the group* $\mathrm{Cl}(X)$*.*

Proof. The assertion follows from Corollary 21, since we assume that $\mathbb{H}^\rho \subset G$. □

This lemma is easy to apply if we fix $\mathscr{S}$ and the group $G \subset \mathrm{Aut}(X)$ such that $\mathbb{H} \subset \upsilon(G)$. For instance, to check whether the surfaces $\Pi_1^+, \ldots, \Pi_{16}^+$ generate the group $\mathrm{Cl}(X)$ or not, we can use the fact that $\mathrm{Cl}(X) \cong \mathbb{Z}^7$ is naturally equipped with an intersection form [29]. Namely, fix a smooth del Pezzo surface $S \in |H|$, and let

$$D_1 \bullet D_2 = D_1\big|_S \cdot D_2\big|_S \in \mathbb{Z}$$

for any two Weil divisors D_1 and D_2 in $\mathrm{Cl}(X)$. Then

$$\Pi_i^{\pm} \bullet \Pi_j^{\pm} = \begin{cases} 0 \text{ if } i \neq j \text{ and } \Pi_i^{\pm} \cap \Pi_j^{\pm} \text{ does not contain curves,} \\ 1 \text{ if } i \neq j \text{ and } \Pi_i^{\pm} \cap \Pi_j^{\pm} \text{ contains a curve,} \\ -1 \text{ if } i = j \text{ and } \Pi_i^{\pm} = \Pi_j^{\pm}, \\ 2 \text{ if } i = j \text{ and } \Pi_i^{\pm} \neq \Pi_j^{\pm}, \end{cases}$$

where two $\pm$ in $\Pi_i^{\pm}$ and $\Pi_j^{\pm}$ are independent.

Remark 23. Let Λ be the sublattice in $\mathrm{Cl}(X)$ consisting of divisors D such that $D \bullet H = 0$. Then Λ is isomorphic to a root lattice of type D_6 by [29, Theorem 1.7], and the natural homomorphism $\mathrm{Aut}(X) \to \mathrm{Aut}(\Lambda)$ is injective by [29], where $\mathrm{Aut}\big(\Lambda\big) \cong (\boldsymbol{\mu}_2^5 \rtimes \mathfrak{S}_6) \rtimes \boldsymbol{\mu}_2$.

Applying Lemma 22, we get

Corollary 24. *If* $\mathrm{rank}(\Pi_i^+ \bullet \Pi_j^+) = 7$ *or* $\mathrm{rank}(\Pi_i^- \bullet \Pi_j^-) = 7$*, then* $\mathrm{Cl}^{\mathbb{H}^\rho}(X) \cong \mathbb{Z}$*.*

Let us show how to apply Corollary 24 in the case when $\mathscr{S}$ is the surface (6).

Example 25. Let us use assumptions and notations of Example 5. Suppose, in addition, that $\rho\colon \mathbb{H} \to \boldsymbol{\mu}_2$ is the trivial homomorphism. Therefore, we have $\mathbb{H}^\rho = \mathbb{H}$. Set $t = \frac{2s}{s^2+1}$. Observe that $\pi^*(\Pi_1)$ is given in $\mathbb{P}(1,1,1,1,2)$ by the following equations:

$$\begin{cases} x_0 + x_1 + x_2 + \dfrac{2s}{s^2+1} x_3 = 0, \\ w^2 = \dfrac{(s^2-1)^2}{(s^2+1)^4}\big((s^2+1)x_1^2 + (s^2+1)x_1x_2 + 2sx_1x_3 + (s^2+1)x_2^2 + 2sx_2x_3 - (s^2+1)x_3^2\big)^2. \end{cases}$$

Thus, without loss of generality, we may assume that the surface Π_1^+ is given by

$$\begin{cases} x_0 + x_1 + x_2 + \dfrac{2s}{s^2+1} x_3 = 0, \\ w = \dfrac{s^2-1}{(s^2+1)^2}\big((s^2+1)x_1^2 + (s^2+1)x_1x_2 + 2sx_1x_3 + (s^2+1)x_2^2 + 2sx_2x_3 - (s^2+1)x_3^2\big). \end{cases}$$

Then the defining equations of the remaining surfaces $\Pi_2^+, \ldots, \Pi_{16}^+$ are listed in Figure 2. Now, the intersection matrix $(\Pi_i^+ \bullet \Pi_j^+)$ can be computed as follows:

$$\begin{pmatrix}
-1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
1 & -1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & -1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & -1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & -1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 & -1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 1 & -1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & -1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & -1 & 1 & 1 & 0 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & -1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & -1 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & -1 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & -1 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & -1 & 1 \\
0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & -1
\end{pmatrix}$$

The rank of this matrix is 7. Therefore, we conclude that $\mathrm{Cl}^{\mathbb{H}}(X) \cong \mathbb{Z}$ by Corollary 24. Note that we can also prove this using Lemma 22(ii). To do this, it is enough to show that the divisor Π^+ is a Cartier divisor, which can be done locally at any point in $\mathrm{Sing}(X)$. For instance, let $P = [t:1:1:1:0] \in \mathrm{Sing}(X)$. Among $\Pi_1^+, \ldots, \Pi_{16}^+$, only

$$\Pi_2^+, \Pi_3^+, \Pi_7^+, \Pi_8^+, \Pi_{10}^+, \Pi_{11}^+$$

pass through P. Choosing a generator of the local class group $\mathrm{Cl}_P(X) \cong \mathbb{Z}$, we see that the classes of the surfaces Π_2^+, Π_3^+, Π_7^+, Π_8^+, Π_{10}^+, Π_{11}^+ are 1, -1, 1, -1, 1, -1, respectively. Hence, we see that Π^+ is locally Cartier at P, which implies that Π^+ is globally Cartier, because the group $\mathbb{H}$ acts transitively on the set $\mathrm{Sing}(X)$.

Example 26. Let us use assumptions and notations of Example 5. Then $\mathrm{Aut}(X)$ contains a unique subgroup G such that $G \cong \upsilon(G) \cong \boldsymbol{\mu}_2^4 \rtimes \boldsymbol{\mu}_3$, and $\upsilon(G)$ is generated by

$$A_1, A_2, A_3, A_4, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

One can check that G contains the subgroup $\mathbb{H} = \mathbb{H}^\rho$, where ρ is a trivial homomorphism. Therefore, it follows from Example 25 that $\mathrm{Cl}^G(X) \cong \mathbb{Z}$.

If $\mathrm{Cl}^G(X) \cong \mathbb{Z}^2$, then exists a uniquely determined G-Sarkisov link

(27)
$$\begin{array}{ccccccc} & & V & \overset{\varsigma}{\dashrightarrow} & V & & \\ & \swarrow^{\varphi} & \downarrow{\scriptstyle\varpi} & & {\scriptstyle\varpi}\downarrow & \searrow^{\varphi} & \\ Z & & X & \xrightarrow{\ \sigma\ } & X & & Z \end{array}$$

where ϖ is a G-equivariant small resolution, ς flops ϖ-contracted curves, and

- either φ is a G-extremal birational contraction, and Z is a Fano threefold,
- or φ is a conic bundle, and Z is a surface,
- or φ is a del Pezzo fibration, and $Z \cong \mathbb{P}^1$.

Note that $\mathrm{Cl}^G(X) \cong \mathbb{Z}^2$ is indeed possible. Let us give two (related) examples.

Example 28. Let us use all assumptions and notations of Example 25, and let $G = \mathbb{H}^\rho$, where the homomorphism ρ is defined by $\rho(A_1) = -1$, $\rho(A_2) = 1$, $\rho(A_3) = -1$, $\rho(A_4) = 1$. Then, arguing as in Example 25, we compute $\mathrm{Cl}^G(X) \cong \mathbb{Z}^2$. What is (27) in this case?

Example 29. Let us use all assumptions and notations of Example 9. Then

$$\mathrm{Aut}(X) \cong \boldsymbol{\mu}_2 \times \mathrm{Aut}\big(\mathbb{P}^3, \mathscr{S}\big) \cong \boldsymbol{\mu}_2 \times \big(\boldsymbol{\mu}_2^4 \rtimes \boldsymbol{\mu}_5\big),$$

and the group $\mathrm{Aut}(X)$ contains a unique subgroup isomorphic to $\mathrm{Aut}(\mathbb{P}^3, \mathscr{S}) \cong \boldsymbol{\mu}_2^4 \rtimes \boldsymbol{\mu}_5$. Suppose that G is this subgroup. It follows from Remark 23 that $\mathrm{Cl}(X) \otimes \mathbb{Q}$ is a faithful seven-dimensional G-representation. Using this, it is easy to see that $\mathrm{Cl}(X) \otimes \mathbb{Q}$ splits as a sum of an irreducible five-dimensional representation and two trivial one-dimensional representations. Hence, we conclude that $\mathrm{Cl}^G(X) \cong \mathbb{Z}^2$. What is (27) in this case?

Before proving Theorem 17, let us prove its two baby cases, which follow from [14, 12].

Proposition 30. *Suppose $G = \mathrm{Aut}(X)$, and $\mathscr{S}$ is the quartic surface from Example 9. Then $\mathrm{Cl}^G(X) \cong \mathbb{Z}$ and X is G-birationally super-rigid.*

Proof. Since $\sigma \in G$, we get $\mathrm{Cl}^G(X) \cong \mathbb{Z}$. Let us show that X is G-birationally super-rigid.

Note that the $\upsilon(G)$-equivariant birational geometry of the projective space $\mathbb{P}^3$ has been studied in [14]. In particular, we know from [14, Corollary 4.7] and [14, Theorem 4.16] that

- $\mathbb{P}^3$ does not contain $\upsilon(G)$-orbits of length less that 16,
- $\mathbb{P}^3$ does not contain $\upsilon(G)$-invariant curves of degree less than 8.

Let $\mathcal{M}$ be a G-invariant linear system on X such that $\mathcal{M}$ has no fixed components. Choose a positive integer n such that $\mathcal{M} \subset |nH|$. Then, by [13, Corollary 3.3.3], to prove that the threefold X is G-birationally super-rigid it is enough to show that $(X, \frac{2}{n}\mathcal{M})$ has canonical singularities. Suppose that the singularities of this log pair are not canonical.

Let Z be a center of non-canonical singularities of the pair $(X, \frac{2}{n}\mathcal{M})$ that has the largest dimension. Since the linear system $\mathcal{M}$ does not have fixed components, we conclude that either Z is an irreducible curve, or Z is a point. In both cases, we have

$$\mathrm{mult}_Z\big(\mathcal{M}\big) > \frac{n}{2}$$

by [24, Theorem 4.5].

Let M_1 and M_2 be general surfaces in $\mathcal{M}$. If Z is a curve, then

$$M_1 \cdot M_2 = (M_1 \cdot M_2)_Z \mathscr{Z} + \Delta$$

where $\mathscr{Z}$ is the G-irreducible curve in X whose irreducible component is the curve Z, and Δ is an effective one-cycle whose support does not contain $\mathscr{Z}$, which gives

$$\begin{aligned} 2n^2 = n^2H^2 = H \cdot M_1 \cdot M_2 = \big(M_1 \cdot M_2\big)_Z \mathscr{Z} + \Delta = \\ = \big(M_1 \cdot M_2\big)_Z \big(H \cdot \mathscr{Z}\big) + H \cdot \Delta \geqslant \big(M_1 \cdot M_2\big)_Z (H \cdot \mathscr{Z}) \geqslant \\ \geqslant \mathrm{mult}^2_Z\big(\mathcal{M}\big)\big(H \cdot \mathscr{Z}\big) > \frac{n^2}{4}\big(H \cdot \mathscr{Z}\big) \geqslant \frac{n^2}{4}\deg\big(\pi(\mathscr{Z})\big), \end{aligned}$$

so $\pi(\mathscr{Z})$ is a $\upsilon(G)$-invariant curve of degree $\leqslant 7$, which contradicts [14, Theorem 4.16].

We see that Z is a point, and $(X, \frac{2}{n}\mathcal{M})$ is canonical away from finitely many points.

We claim that $Z \notin \mathrm{Sing}(X)$. Indeed, suppose Z is a singular point of the threefold X. Let $h\colon \overline{X} \to X$ be the blow up of the locus $\mathrm{Sing}(X)$, let $E_1, \ldots, E_{16}$ be the h-exceptional surfaces, let $\overline{M}_1$ and $\overline{M}_1$ be the proper transforms on $\overline{X}$ of the surfaces M_1 and M_2, respectively. Write $E = E_1 + \cdots + E_{16}$. Since $\mathrm{Sing}(X)$ is a G-orbit, we have

$$\overline{M}_1 \sim \overline{M}_2 \sim h^*(H) - \epsilon E$$

for some integer $\epsilon \geqslant 0$. Using [16, Theorem 3.10] or [9, Theorem 1.7.20], we get $\epsilon > \frac{n}{2}$. On the other hand, the linear system $|h^*(3H) - E|$ is not empty and does not have base curves away from the locus $E_1 \cup E_2 \cup \cdots \cup E_{16}$, because $\mathrm{Sing}(\mathscr{S})$ is cut out by cubic surfaces in $\mathbb{P}^3$. In particular, the divisor $h^*(3H) - E$ is nef, so

$$0 \leqslant \big(h^*(3H) - E\big) \cdot \overline{M}_1 \cdot \overline{M}_2 = \big(h^*(3H) - E\big) \cdot \big(h^*(3nH) - \epsilon E\big)^2 = 6n^2 - 32\epsilon,$$

which is impossible, since $\epsilon > \frac{n}{2}$. So, we see that Z is a smooth point of the threefold X.

Then the pair $(X, \frac{3}{n}\mathcal{M})$ is not log canonical at Z. Let μ be the largest rational number such that the log pair $(X, \mu\mathcal{M})$ is log canonical. Then $\mu < \frac{3}{n}$ and

$$\mathrm{Orb}_G(Z) \subseteq \mathrm{Nklt}\big(X, \mu\mathcal{M}\big).$$

Observe that $\mathrm{Nklt}(X, \mu\mathcal{M})$ is at most one-dimensional, since $\mathcal{M}$ has no fixed components. Moreover, this locus is G-invariant, because $\mathcal{M}$ is G-invariant.

We claim that $\mathrm{Nklt}(X, \mu\mathcal{M})$ does not contain curves. Indeed, suppose this is not true. Then $\mathrm{Nklt}(X, \mu\mathcal{M})$ contains a G-irreducible curve C. We write $M_1 \cdot M_2 = mC + \Omega$, where m is a non-negative integer, and Ω is an effective one-cycle whose support does not contain the curve C. Then it follows from [16, Theorem 3.1] that

$$m \geqslant \frac{4}{\mu^2} > \frac{4n^2}{9}.$$

Therefore, we have

$$2n^2 = n^2H^3 = H \cdot M_1 \cdot M_2 = m\big(H \cdot C\big) + H \cdot \Omega \geqslant m\big(H \cdot C\big) > \frac{4n^2}{9}\big(H \cdot C\big),$$

which implies that $H\cdot C\leqslant 4$. Then $\pi(C)$ is a $\upsilon(G)$-invariant curve in $\mathbb{P}^3$ of degree $\leqslant 4$, which contradicts [14, Theorem 4.16]. Thus, the locus $\mathrm{Nklt}(X,\mu\mathcal{M})$ contains no curves.

Let $\mathcal{I}$ be the multiplier ideal sheaf of the pair $(X,\mu\mathcal{M})$, and let $\mathcal{L}$ be the corresponding subscheme in X. Then $\mathcal{L}$ is a zero-dimensional (reduced) subscheme such that

$$\mathrm{Orb}_G\big(Z\big)\subseteq\mathrm{Supp}\big(\mathcal{L}\big)=\mathrm{Nklt}\big(X,\mu\mathcal{M}\big).$$

Applying Nadel's vanishing [26, Theorem 9.4.8], we get

$$h^1\big(X,\mathcal{I}\otimes\mathcal{O}_X(H)\big)=0.$$

This gives

$$4=h^0\big(X,\mathcal{O}_X(H)\big)\geqslant h^0\big(\mathcal{O}_{\mathcal{L}}\otimes\mathcal{O}_X(H)\big)=h^0\big(\mathcal{O}_{\mathcal{L}}\big)\geqslant|\mathrm{Orb}_G(Z)|.$$

In particular, we conclude that the length of the $\upsilon(G)$-orbit of the point $\pi(Z)$ is at most 4, which is impossible by [14, Corollary 4.7]. $\square$

Proposition 31. *Suppose that $\mathscr{S}$ is the surface from Example 5, and G is the subgroup described in Example 26. Then $\mathrm{Cl}^G(X)\cong\mathbb{Z}$ and X is G-birationally super-rigid.*

Proof. Recall from Example 26 that $G\cong\boldsymbol{\mu}_2^4\rtimes\boldsymbol{\mu}_3$ and $\mathrm{Cl}^G(X)\cong\mathbb{Z}$.

The $\upsilon(G)$-equivariant geometry of the projective space $\mathbb{P}^3$ has been studied in [12]. In particular, we know from [12] that $\mathbb{P}^3$ does not contain $\upsilon(G)$-orbits of length 1, 2 or 3, and the only $\upsilon(G)$-orbits in $\mathbb{P}^3$ of length 4 are

$$\begin{aligned}\Sigma_4&=\big\{[1:0:0:0],[0:1:0:0],[0:0:1:0],[0:0:0:1]\big\},\\ \Sigma_4'&=\big\{[1:1:1:-1],[1:1:-1:1],[1:-1:1:1],[-1:1:1:1]\big\},\\ \Sigma_4''&=\big\{[1:1:1:1],[1:1:-1:-1],[1:-1:-1:1],[-1:-1:1:1]\big\}\end{aligned}$$

We also know from [12] the classification of $\upsilon(G)$-invariant curves in $\mathbb{P}^3$ of degree at most 7. To present it, let $\mathcal{L}_4$, $\mathcal{L}_4'$, $\mathcal{L}_4''$, $\mathcal{L}_4'''$, $\mathcal{L}_6$, $\mathcal{L}_6'$, $\mathcal{L}_6''$, $\mathcal{L}_6'''$, $\mathcal{L}_6''''$ be $\upsilon(G)$-irreducible curves in $\mathbb{P}^3$ whose irreducible components are the lines

$$\begin{aligned}&\big\{2x_0+(1+\sqrt{3}i)x_2-(1-\sqrt{3}i)x_3=2x_1+(1-\sqrt{3}i)x_2+(1+\sqrt{3}i)x_3=0\big\},\\ &\big\{2x_0+(1-\sqrt{3}i)x_2-(1+\sqrt{3}i)x_3=2x_1+(1+\sqrt{3}i)x_2+(1-\sqrt{3}i)x_3=0\big\},\\ &\big\{2x_0-(1-\sqrt{3}i)x_2+(1+\sqrt{3}i)x_3=2x_1+(1+\sqrt{3}i)x_2+(1-\sqrt{3}i)x_3=0\big\},\\ &\big\{2x_0-(1+\sqrt{3}i)x_2+(1-\sqrt{3}i)x_3=2x_1+(1-\sqrt{3}i)x_2+(1+\sqrt{3}i)x_3=0\big\},\\ &\big\{x_0=x_1=0\big\},\\ &\big\{x_0+x_1=x_2-x_3=0\big\},\\ &\big\{x_0+x_1=x_2+x_3=0\big\},\\ &\big\{x_0+ix_2=x_1+ix_3=0\big\},\\ &\big\{x_0+ix_3=x_1+ix_2=0\big\},\end{aligned}$$

respectively. Then the curves $\mathcal{L}_4$, $\mathcal{L}_4'$, $\mathcal{L}_4''$, $\mathcal{L}_4'''$, $\mathcal{L}_6$, $\mathcal{L}_6'$, $\mathcal{L}_6''$, $\mathcal{L}_6'''$, $\mathcal{L}_6''''$ are unions of 4, 4, 4, 4, 6, 6, 6, 6, 6 lines, respectively. Moreover, it follows from [12] that

$$\mathcal{L}_4,\ \mathcal{L}_4',\ \mathcal{L}_4'',\ \mathcal{L}_4''',\ \mathcal{L}_6,\ \mathcal{L}_6',\ \mathcal{L}_6'',\ \mathcal{L}_6''',\ \mathcal{L}_6''''$$

are the only $\upsilon(G)$-invariant curves in $\mathbb{P}^3$ of degree at most 7.

Now, using the defining equation of the surface $\mathscr{S}$, one can check that any irreducible component of any curve among $\mathcal{L}_4$, $\mathcal{L}_4'$, $\mathcal{L}_4''$, $\mathcal{L}_4'''$, $\mathcal{L}_6$, $\mathcal{L}_6'$, $\mathcal{L}_6''$, $\mathcal{L}_6'''$, $\mathcal{L}_6''''$ intersects the quartic surface $\mathscr{S}$ transversally by 4 distinct points, so that its preimage in X via the double cover π is a smooth elliptic curve. Thus, if C is a G-invariant curve in X, then $H\cdot C\geqslant 8$.

Suppose that X is not G-birationally super-rigid. It follows from [13, Corollary 3.3.3] that there are a positive integer n and a G-invariant linear subsystem $\mathcal{M}\subset|nH|$ such that $\mathcal{M}$ does not have fixed components, but the log pair $(X,\frac{2}{n}\mathcal{M})$ is not canonical.

Arguing as in the proof of Proposition 30, we see that the log pair $(X,\frac{2}{n}\mathcal{M})$ is canonical away from finitely many points. Let P be a point in X that is a center of non-canonical singularities of the log pair $(X,\frac{2}{n}\mathcal{M})$. Now, arguing as in the proof of Proposition 30 again, we see that P is a smooth point of the threefold X.

Then the log pair $(X,\frac{3}{n}\mathcal{M})$ is not log canonical at P. Let μ be the largest rational number such that $(X,\mu\mathcal{M})$ is log canonical. Then $\mu<\frac{3}{n}$ and

$$\mathrm{Orb}_G(P)\subseteq\mathrm{Nklt}\big(X,\mu\mathcal{M}\big).$$

Observe that $\mathrm{Nklt}(X,\mu\mathcal{M})$ is at most one-dimensional, since $\mathcal{M}$ has no fixed components. Moreover, this locus is G-invariant, because $\mathcal{M}$ is G-invariant. Furthermore, arguing as in the proof of Proposition 30, we see that

$$\dim\Big(\mathrm{Nklt}\big(X,\mu\mathcal{M}\big)\Big)=0.$$

Let $\mathcal{I}$ be the multiplier ideal sheaf of the pair $(X,\mu\mathcal{M})$, and let $\mathcal{L}$ be the corresponding subscheme in X. Then $\mathcal{L}$ is a zero-dimensional (reduced) subscheme such that

$$\mathrm{Orb}_G\big(P\big)\subseteq\mathrm{Supp}\big(\mathcal{L}\big)=\mathrm{Nklt}\big(X,\mu\mathcal{M}\big).$$

On the other hand, applying Nadel's vanishing theorem [26, Theorem 9.4.8], we get

$$h^1\big(X,\mathcal{I}\otimes\mathcal{O}_X(H)\big)=0.$$

This gives

$$4=h^0\big(X,\mathcal{O}_X(H)\big)\geqslant h^0\big(\mathcal{O}_{\mathcal{L}}\otimes\mathcal{O}_X(H)\big)=h^0\big(\mathcal{O}_{\mathcal{L}}\big)\geqslant|\mathrm{Orb}_G(P)|.$$

Thus, we conclude that $|\mathrm{Orb}_G(P)|=4$ and

$$\pi(P)\in\Sigma_4\cup\Sigma_4'\cup\Sigma_4''.$$

Let M_1 and M_2 be two general surfaces in $\mathcal{M}$. Using [30] or [16, Corollary 3.4], we get

$$\big(M_1\cdot M_2\big)_P>n^2. \tag{32}$$

Let $\mathcal{S}$ be a linear subsystem in $|3H|$ that consists of all surfaces that are singular at every point of the G-orbit $\mathrm{Orb}_G(P)$. Then its base locus does not contain curves, which implies that there is a surface $S\in\mathcal{S}$ that does not contain components of the cycle $M_1\cdot M_2$. Thus, using (32) and $\mathrm{mult}_P(S)\geqslant 2$, we get

$$6n^2=S\cdot M_1\cdot M_2\geqslant\sum_{O\in\mathrm{Orb}_G(P)}2\big(M_1\cdot M_2\big)_O=2|\mathrm{Orb}_G(P)|\big(M_1\cdot M_2\big)_P=8\big(M_1\cdot M_2\big)_P>8n^2,$$

which is absurd. This completes the proof of Proposition 31. □

In the remaining part of the paper, we prove Theorem 17, and consider one application. Let us recall from [22, 28, 20, 18, 1] basic facts about the $\mathbb{H}$-equivariant geometry of $\mathbb{P}^3$. Set

$$
\begin{aligned}
\mathcal{Q}_1 &= \big\{x_0^2+x_1^2+x_2^2+x_3^2=0\big\},\\
\mathcal{Q}_2 &= \big\{x_0^2+x_1^2=x_2^2+x_3^2\big\},\\
\mathcal{Q}_3 &= \big\{x_0^2-x_1^2=x_2^2-x_3^2\big\},\\
\mathcal{Q}_4 &= \big\{x_0^2-x_1^2=x_3^2-x_2^2\big\},\\
\mathcal{Q}_5 &= \big\{x_0x_2+x_1x_3=0\big\},\\
\mathcal{Q}_6 &= \big\{x_0x_3+x_1x_2=0\big\},\\
\mathcal{Q}_7 &= \big\{x_0x_1+x_2x_3=0\big\},\\
\mathcal{Q}_8 &= \big\{x_0x_2=x_1x_3\big\},\\
\mathcal{Q}_9 &= \big\{x_0x_3=x_1x_2\big\},\\
\mathcal{Q}_{10} &= \big\{x_0x_1=x_2x_3\big\}.
\end{aligned}
$$

Then $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$, $\mathcal{Q}_4$, $\mathcal{Q}_5$, $\mathcal{Q}_6$, $\mathcal{Q}_7$, $\mathcal{Q}_8$, $\mathcal{Q}_9$, $\mathcal{Q}_{10}$ are all $\mathbb{H}$-invariant quadric surfaces in $\mathbb{P}^3$. These quadrics are smooth, and $\mathbb{H}\cong\boldsymbol{\mu}_2^4$ acts *naturally* on each quadric $\mathcal{Q}_i\cong\mathbb{P}^1\times\mathbb{P}^1$.

For a non-trivial element $g\in\mathbb{H}$, the locus of its fixed points in $\mathbb{P}^3$ consists of two skew lines, which we will denote by L_g and $L_g^\prime$. For two non-trivial elements $g\neq h$ in $\mathbb{H}$, one has

$$\big\{L_g,L_g^\prime\big\}\cap\big\{L_h,L_h^\prime\big\}=\varnothing.$$

In total, this gives 30 lines $\ell_1,\ldots,\ell_{30}$, whose equations are listed in the following table:

$\ell_1=\big\{x_0=x_1=0\big\}$	$\ell_2=\big\{x_2=x_3=0\big\}$
$\ell_3=\big\{x_0=x_2=0\big\}$	$\ell_4=\big\{x_1=x_3=0\big\}$
$\ell_5=\big\{x_0=x_3=0\big\}$	$\ell_6=\big\{x_1=x_2=0\big\}$
$\ell_7=\big\{x_0+x_1=x_2+x_3=0\big\}$	$\ell_8=\big\{x_0-x_1=x_2-x_3=0\big\}$
$\ell_9=\big\{x_0+x_2=x_1+x_3=0\big\}$	$\ell_{10}=\big\{x_0-x_2=x_1-x_3=0\big\}$
$\ell_{11}=\big\{x_0+x_3=x_1+x_2=0\big\}$	$\ell_{12}=\big\{x_0-x_3=x_1-x_2=0\big\}$
$\ell_{13}=\big\{x_0+x_1=x_2-x_3=0\big\}$	$\ell_{14}=\big\{x_0-x_1=x_2+x_3=0\big\}$
$\ell_{15}=\big\{x_0+x_2=x_1-x_3=0\big\}$	$\ell_{16}=\big\{x_0-x_2=x_1+x_3=0\big\}$
$\ell_{17}=\big\{x_0+x_3=x_1-x_2=0\big\}$	$\ell_{18}=\big\{x_0-x_3=x_1+x_2=0\big\}$
$\ell_{19}=\big\{x_0+ix_1=x_2+ix_3=0\big\}$	$\ell_{20}=\big\{x_0-ix_1=x_2-ix_3=0\big\}$
$\ell_{21}=\big\{x_0+ix_2=x_1+ix_3=0\big\}$	$\ell_{22}=\big\{x_0-ix_2=x_1-ix_3=0\big\}$
$\ell_{23}=\big\{x_0+ix_3=x_1+ix_2=0\big\}$	$\ell_{24}=\big\{x_0-ix_3=x_1-ix_2=0\big\}$
$\ell_{25}=\big\{x_0-ix_1=x_2+ix_3=0\big\}$	$\ell_{26}=\big\{x_0+ix_1=x_2-ix_3=0\big\}$
$\ell_{27}=\big\{x_0+ix_2=x_1-ix_3=0\big\}$	$\ell_{28}=\big\{x_0-ix_2=x_1+ix_3=0\big\}$
$\ell_{29}=\big\{x_0+ix_3=x_1-ix_2=0\big\}$	$\ell_{30}=\big\{x_0-ix_3=x_1+ix_2=0\big\}$

Note that $\ell_1, \ldots, \ell_{30}$ are irreducible components of the curves $\mathcal{L}_6$, $\mathcal{L}_6'$, $\mathcal{L}_6''$, $\mathcal{L}_6'''$, $\mathcal{L}_6''''$ which have been introduced in the proof of Proposition 31. One can check that

- for every $k \in \{1, \ldots, 15\}$, the curve $\ell_{2k-1} + \ell_{2k}$ is $\mathbb{H}$-irreducible,
- each line among $\ell_1, \ldots, \ell_{30}$ is contained in 4 quadrics among $\mathcal{Q}_1, \ldots, \mathcal{Q}_{10}$,
- each quadric among $\mathcal{Q}_1, \ldots, \mathcal{Q}_{10}$ contains 12 lines among $\ell_1, \ldots, \ell_{30}$,
- every two quadrics among $\mathcal{Q}_1, \ldots, \mathcal{Q}_{10}$ intersect by 4 lines among $\ell_1, \ldots, \ell_{30}$.

The incidence relation between $\ell_1, \ldots, \ell_{30}$ and $\mathcal{Q}_1, \ldots, \mathcal{Q}_{10}$ is presented in Figure 3.

Now, let us describe the intersection points of the lines $\ell_1, \ldots, \ell_{30}$. To do this, we set

$$\begin{aligned}
\Sigma_4^1 &= \big\{[1:0:0:0],[0:1:0:0],[0:0:1:0],[0:0:0:1]\big\},\\
\Sigma_4^2 &= \big\{[1:1:1:-1],[1:1:-1:1],[1:-1:1:1],[-1:1:1:1]\big\},\\
\Sigma_4^3 &= \big\{[1:1:1:1],[-1:-1:1:1],[1:-1:-1:1],[-1:1:-1:1]\big\},\\
\Sigma_4^4 &= \big\{[0:0:1:1],[1:1:0:0],[0:0:-1:1],[1:-1:0:0]\big\},\\
\Sigma_4^5 &= \big\{[1:0:1:0],[0:1:0:1],[-1:0:1:0],[0:-1:0:1]\big\},\\
\Sigma_4^6 &= \big\{[0:1:1:0],[1:0:0:1],[0:-1:1:0],[-1:0:0:1]\big\},\\
\Sigma_4^7 &= \big\{[i:0:0:1],[0:i:1:0],[-i:0:0:1],[0:-i:1:0]\big\},\\
\Sigma_4^8 &= \big\{[i:0:1:0],[0:i:0:1],[0:-i:0:1],[-i:0:1:0]\big\},\\
\Sigma_4^9 &= \big\{[i:1:0:0],[0:0:i:1],[-i:1:0:0],[0:0:-i:1]\big\},\\
\Sigma_4^{10} &= \big\{[i:i:1:1],[-i:-i:1:1],[i:-i:-1:1],[-i:i:-1:1]\big\},\\
\Sigma_4^{11} &= \big\{[1:i:i:1],[1:-i:-i:1],[-1:-i:i:1],[-1:i:-i:1]\big\},\\
\Sigma_4^{12} &= \big\{[1:i:-i:1],[-1:i:i:1],[-1:-i:-i:1],[1:-i:i:1]\big\},\\
\Sigma_4^{13} &= \big\{[i:1:i:1],[-i:1:-i:1],[-i:-1:i:1],[i:-1:-i:1]\big\},\\
\Sigma_4^{14} &= \big\{[i:1:-i:1],[i:-1:i:1],[-i:-1:-i:1],[-i:1:i:1]\big\},\\
\Sigma_4^{15} &= \big\{[i:i:-1:1],[-i:-i:-1:1],[i:-i:1:1],[-i:i:1:1]\big\}.
\end{aligned}$$

Then the subsets $\Sigma_4^1, \ldots, \Sigma_4^{15}$ are $\mathbb{H}$-orbits of length 4. Moreover, one has

$$\Sigma_4^1 \cup \Sigma_4^2 \cup \cdots \cup \Sigma_4^{15} = \mathrm{Sing}\big(\ell_1 + \ell_2 + \cdots + \ell_{30}\big).$$

So, for every ℓ_i and ℓ_j such that $\ell_i \neq \ell_j$ and $\ell_i \cap \ell_j \neq \varnothing$, one has $\ell_i \cap \ell_j \in \Sigma_4^1 \cup \Sigma_4^2 \cup \cdots \cup \Sigma_4^{15}$. Furthermore, one can also check that

- every line among $\ell_1, \ldots, \ell_{30}$ contains 6 points in $\Sigma_4^1 \cup \Sigma_4^2 \cup \cdots \cup \Sigma_4^{15}$,
- every point in $\Sigma_4^1 \cup \Sigma_4^2 \cup \cdots \cup \Sigma_4^{15}$ is contained in 3 lines among $\ell_1, \ldots, \ell_{30}$.

As in Remark 11, let $\mathfrak{N}$ be the normalizer of the subgroup $\mathbb{H}$ in the group $\mathrm{PGL}_4(\mathbb{C})$. Then $\mathrm{Aut}(\mathscr{C}) \subset \mathfrak{N}$ and $\mathfrak{N} \cong \mathbb{H}.\mathfrak{S}_6$, see Remark 11. Moreover, one can show that

- the group $\mathfrak{N}$ acts transitively on the set $\{\mathcal{Q}_1, \ldots, \mathcal{Q}_{10}\}$,
- the group $\mathfrak{N}$ acts transitively on the set $\{\ell_1, \ldots, \ell_{30}\}$,
- the group $\mathfrak{N}$ acts transitively on the set $\{\Sigma_4^1, \ldots, \Sigma_4^{15}\}$.

Now, we are ready to describe $\mathbb{H}$-orbits in $\mathbb{P}^3$. They can be described as follows:

(1) $\Sigma_4^1, \ldots, \Sigma_4^{15}$ are $\mathbb{H}$-orbits of length 4;
(2) $\mathbb{H}$-orbit of every point in $(\ell_1 \cup \ell_2 \cup \cdots \cup \ell_{30}) \setminus (\Sigma_4^1 \cup \Sigma_4^2 \cup \cdots \cup \Sigma_4^{15})$ has length 8;
(3) $\mathbb{H}$-orbit of every point in $\mathbb{P}^3 \setminus (\ell_1 \cup \ell_2 \cup \cdots \cup \ell_{30})$ has length 16.

Lemma 33. *The surface $\mathscr{S}$ does not contain $\mathbb{H}$-orbits of length* 4.

Proof. The assertion follows from [34, Theorem 3], since the $\mathbb{H}$-action on the minimal resolution of the quartic surface $\mathscr{S}$ is symplectic. Alternatively, we can check this explicitly. Indeed, it is enough to check that $\mathscr{S}$ does not contain Σ_4^1, since the group $\mathfrak{N}$ transitively permutes the orbits $\Sigma_4^1,\ldots,\Sigma_4^{15}$. If $\Sigma_4^1\subset\mathscr{S}$, then $\mathscr{S}$ is given by (2) with $a=bcd=0$, which implies that $\mathscr{S}$ has non-isolated singularities. □

Corollary 34. *Every line among $\ell_1,\ldots,\ell_{30}$ intersects $\mathscr{S}$ transversally by* 4 *points.*

Proof. Fix $k\in\{1,\ldots,15\}$. If $|\ell_{2k-1}\cap\mathscr{S}|<4$, then the subset $(\ell_{2k-1}\cup\ell_{2k})\cap\mathscr{S}$ contains an $\mathbb{H}$-orbit of length 4, which contradicts Lemma 33. Therefore, we have $|\ell_{2k-1}\cap\mathscr{S}|=4$. Similarly, we see that $|\ell_{2k}\cap\mathscr{S}|=4$. □

Now, let us prove one result that plays a crucial role in the proof of Theorem 17.

Lemma 35. *Let C be a possibly reducible $\mathbb{H}$-irreducible curve in $\mathbb{P}^3$ such that* $\deg(C)<8$. *Then one of the following two possibilities hold:*

(a) *either $C=\ell_{2k-1}+\ell_{2k}$ for some $k\in\{1,\ldots,15\}$;*
(b) *or C is a union of* 4 *disjoint lines and $C\subset\mathcal{Q}_i$ for some $i\in\{1,\ldots,10\}$.*

Proof. Intersecting C with quadric surfaces $\mathcal{Q}_1,\ldots,\mathcal{Q}_{10}$, we conclude that $\deg(C)$ is even. This gives $\deg(C)\in\{2,4,6\}$.

Suppose that C is reducible. Since $|\mathbb{H}|=16$, we have the following possibilities:

(i) C is a union of 2 lines,
(ii) C is a union of 4 lines,
(iii) C is a union of 2 irreducible conics,
(iv) C is a union of 3 irreducible conics.
(v) C is a union of 2 irreducible plane cubics,
(vi) C is a union of 2 twisted cubics,

Since $\mathbb{P}^3$ does not have $\mathbb{H}$-orbits of length 2 and 3, cases (iii), (iv) and (v) are impossible. Similarly, case (vi) is also impossible, because $\boldsymbol{\mu}_2^3$ cannot faithfully act on a rational curve. Thus, either C is a union of 2 lines, or C is a union of 4 lines.

Suppose that $C=L_1+L_2$, where L_1 and L_2 are lines. Then $\mathrm{Stab}_{\mathbb{H}}(L_1)\cong\boldsymbol{\mu}_2^3$, and this group cannot act faithfully on $L_1\cong\mathbb{P}^1$. Therefore, there exists a non-trivial $g\in\mathrm{Stab}_{\mathbb{H}}(L_1)$ such that g pointwise fixes the line L_1. But this means that L_1 is one of the lines $\ell_1,\ldots,\ell_{30}$, so we have $C=\ell_{2k-1}+\ell_{2k}$ for some $k\in\{1,\ldots,15\}$ as required.

Suppose $C=L_1+L_2+L_3+L_4$, where L_1, L_2, L_3, L_4 are lines. Then $\mathrm{Stab}_{\mathbb{H}}(L_1)\cong\boldsymbol{\mu}_2^2$. Note that $\mathrm{Stab}_{\mathbb{H}}(L_1)$ must act faithfully on L_1, because L_1 is not one of the lines $\ell_1,\ldots,\ell_{30}$. This implies that L_1 does not have $\mathrm{Stab}_{\mathbb{H}}(L_1)$-fixed points, which implies that $\mathbb{P}^3$ also does not have $\mathrm{Stab}_{\mathbb{H}}(L_1)$-fixed points. All subgroups in $\mathbb{H}$ isomorphic to $\boldsymbol{\mu}^2$ with these property are conjugated by the action of the group $\mathfrak{N}$. Thus, we may assume that

$$\mathrm{Stab}_{\mathbb{H}}\big(L_1\big)=\langle A_1A_2,A_3\rangle.$$

This subgroup leaves invariant rulings of the quadric surface $\mathcal{Q}_8\cong\mathbb{P}^1\times\mathbb{P}^1$. To be precise, for every $[\lambda:\mu]\in\mathbb{P}^1$, the group $\langle A_1A_2,A_3\rangle$ leaves invariant the line

$$\big\{\lambda x_0+\mu x_3=\lambda x_1+\mu x_2=0\big\}\subset\mathcal{Q}_8,$$

and these are all $\langle A_1A_2,A_3\rangle$-invariant lines in $\mathbb{P}^3$. So, the lines L_1, L_2, L_3, L_4 are disjoint, and all of them are contained in the quadric $\mathcal{Q}_8$. Thus, we are done in this case.

Therefore, to complete the proof of the lemma, we may assume that C is irreducible. Observe that the curve C is not planar, because $\mathbb{P}^3$ does not contain $\mathbb{H}$-invariant planes. Moreover, the curve C is singular: otherwise its genus is $\leqslant 4$ by the Castelnuovo bound, but $\mathbb{H}$ cannot faithfully act on a smooth curve of genus less than 5 by [12, Lemma 3.2]. Therefore, we conclude that $\deg(C)=6$, since otherwise the curve C would be planar.

We claim that the curve C does not contain $\mathbb{H}$-orbits of length 4. Suppose that it does. Since $\mathfrak{N}$ transitively permutes the orbits $\Sigma_4^1,\ldots,\Sigma_4^{15}$, we may assume that $\Sigma_4^1\subset C$. Then

$$\Sigma_4^1\subset\mathrm{Sing}(C),$$

because the stabilizer in $\mathbb{H}$ of a smooth point in C must be a cyclic group [35, Lemma 2.7]. Let $\iota\colon\mathbb{P}^3\dashrightarrow\mathbb{P}^3$ be the standard Cremona involution, which is given by

$$[x_0:x_1:x_2:x_3]\mapsto[x_1x_2x_3:x_0x_2x_3:x_0x_1x_3:x_0x_1x_2].$$

Then ι centralizes $\mathbb{H}$. On the other hand, the curve $\iota(C)$ is a conic, because $\deg(C)=6$, and C is singular at every point of the $\mathbb{H}$-orbit Σ_4^1. But $\mathbb{P}^3$ contains no $\mathbb{H}$-invariant conics, because it contains no $\mathbb{H}$-invariant planes. Thus, C contains no $\mathbb{H}$-orbits of length 4.

Note that $\mathcal{Q}_1\cap\mathcal{Q}_2\cap\cdots\cap\mathcal{Q}_{10}=\varnothing$. So, at least one quadric among $\mathcal{Q}_1,\ldots,\mathcal{Q}_{10}$ does not contain the curve C. Without loss of generality, we may assume that $C\not\subset\mathcal{Q}_1$. Then

$$12=\mathcal{Q}_1\cdot C\geqslant|\mathcal{Q}_1\cap C|,$$

which implies that the intersection $\mathcal{Q}_1\cap C$ is an $\mathbb{H}$-orbit of length 8, because we already proved that C does not contain $\mathbb{H}$-orbits of length 4. For a point $P\in\mathcal{Q}_1\cap C$, we have

$$12=\mathcal{Q}_1\cdot C=|\mathrm{Orb}_{\mathbb{H}}(P)|\big(\mathcal{Q}_1\cdot C\big)_P=8\big(\mathcal{Q}_1\cdot C\big)_P,$$

which is impossible, since 12 is not divisible by 8. □

Corollary 36. *Let $\mathcal{Q}$ be any quadric among $\mathcal{Q}_1,\ldots,\mathcal{Q}_{10}$, and let $C=\mathscr{S}|_{\mathcal{Q}}$. Then*

(i) *either C is a smooth curve of degree 8 and genus 9,*
(ii) *or $C=\mathcal{L}_4+\mathcal{L}_4'$ for $\mathbb{H}$-irreducible curves $\mathcal{L}_4$ and $\mathcal{L}_4'$ consisting of 4 disjoint lines such that the intersection $\mathcal{L}_4\cap\mathcal{L}_4'$ is an $\mathbb{H}$-orbit of length 16.*

Proof. If C is reducible or non-reduced, Lemma 35 and Corollary 33 imply the assertion. Thus, we may assume that C is irreducible and reduced. Then its arithmetic genus is 9. If C is smooth, we are done. If C is singular, then the genus of it normalization is $\leqslant 1$, because C does not contain $\mathbb{H}$-orbits of length 4 by Corollary 33. But $\mathbb{H}$ cannot faithfully act on a smooth curve of genus less than 5 by [12, Lemma 3.2]. □

Now, we are ready to prove Theorem 17.

Proof of Theorem 17. Let G be a subgroup in $\mathrm{Aut}(X)$ such that $\mathrm{Cl}^G(X)\cong\mathbb{Z}$ and

$$\mathbb{H}\subseteq\upsilon(G),$$

so G contains a subgroup $\mathbb{H}^\rho$ for some homomorphism $\rho\colon\mathbb{H}\to\boldsymbol{\mu}_2$. We must prove that the threefold X is G-birationally super-rigid. Suppose it is not G-birationally super-rigid. Then there are a positive integer n and a G-invariant linear subsystem $\mathcal{M}\subset|nH|$ such that the linear system $\mathcal{M}$ does not have fixed components, but $(X,\frac{2}{n}\mathcal{M})$ is not canonical.

Starting from this moment, we are going to forget about the group G. In the following, we will work only with its subgroup $\mathbb{H}^\rho$. Note that $\upsilon(\mathbb{H}^\rho)=\mathbb{H}$.

Let Z be the center of non-canonical singularities of the log pair $(X, \frac{2}{n}\mathcal{M})$ that has maximal dimension. We claim that Z must be a point. Indeed, suppose that Z is a curve. Let M be sufficiently general surface in the linear system $\mathcal{M}$. Then

$$\mathrm{mult}_Z\big(M\big) > \frac{n}{2} \tag{37}$$

by [24, Theorem 4.5]. Let us seek for a contradiction.

Let $\mathscr{Z}$ be an $\mathbb{H}^\rho$-irreducible curve in X whose irreducible components is the curve Z, Then, arguing as in the proof of Proposition 30, we see that

$$H \cdot \mathscr{Z} \leqslant 7.$$

In particular, we conclude that $\pi(\mathscr{Z})$ is a $\mathbb{H}$-invariant curve of degree $\leqslant 7$. By Lemma 35, the curve $\pi(Z)$ is a line, and one of the following two possibilities hold:

(a) either $\pi(\mathscr{Z}) = \ell_{2k-1} + \ell_{2k}$ for some $k \in \{1, \ldots, 15\}$;
(b) or $\pi(\mathscr{Z})$ is a union of 4 disjoint lines and $\pi(\mathscr{Z}) \subset \mathcal{Q}_i$ for some $i \in \{1, \ldots, 10\}$.

Let us deal with these two cases separately.

Suppose we are in case (a). Without loss of generality, we may assume $\pi(\mathscr{Z}) = \ell_1 + \ell_2$. Let C_1 and C_2 be the preimages on the threefold X of the lines ℓ_1 and ℓ_2, respectively. Then it follows from Corollary 34 that C_1 and C_2 are smooth irreducible elliptic curves. In particular, the curves C_1 and C_2 are disjoint and

$$\mathscr{Z} = C_1 + C_2.$$

Let $f\colon \widetilde{X} \to X$ be the blow up of the curves C_1 and C_2, let E_1 and E_2 be the f-exceptional surfaces such that $f(E_1) = C_1$ and $f(E_2) = C_2$, and let $\widetilde{M}$ be the proper transform on the threefold $\widetilde{X}$ of the surface M. Then $|f^*(2H) - E_1 - E_2|$ is base point free, so

$$0 \leqslant \big(f^*(2H) - E_1 - E_2\big)^2 \cdot \widetilde{M} = \\ = \big(f^*(2H) - E_1 - E_2\big)^2 \cdot \big(f^*(nH) - \mathrm{mult}_Z\big(M\big)(E_1 + E_2)\big) = 4n - 8\mathrm{mult}_Z\big(M\big),$$

which contradicts (37). This shows that case (a) is impossible.

Suppose we are in case (b). Without loss of generality, we may assume that $\pi(\mathscr{Z}) \subset \mathcal{Q}_1$. Let S be the preimage of the quadric surface $\mathcal{Q}_1$ via the double cover π. Then it follows from Corollary 36 that S is an irreducible normal surface such that

(i) either S is a smooth K3 surface,
(ii) or S is a singular K3 surface that has 16 ordinary double points.

Note that $\mathscr{Z} \subset S$ by construction. Let $\mathcal{C}$ be the preimage in X of a sufficiently general line in the quadric $\mathcal{Q}_1$ that intersect the line $\pi(Z)$. Then $\mathcal{C}$ is a smooth irreducible elliptic curve, which is contained in the smooth locus of the K3 surface S. Observe that $H \cdot \mathcal{C} = 2$. Moreover, we also have $|\mathcal{C} \cap \mathscr{Z}| \geqslant 4$. Thus, since $\mathcal{C} \not\subset \mathrm{Supp}(M)$, we get

$$2n = nH \cdot \mathcal{C} = M \cdot \mathcal{C} \geqslant \sum_{O \in C \cap \mathscr{Z}} \mathrm{mult}_O(M) \geqslant \mathrm{mult}_Z\big(M\big)|\mathcal{C} \cap \mathscr{Z}| \geqslant 4\mathrm{mult}_Z\big(M\big),$$

which contradicts (37). This shows that case (b) is also impossible.

Hence, we see that Z is a point. In particular, the pair $(X, \frac{2}{n}\mathcal{M})$ is canonical away from finitely many points. Now, arguing as in the proof of Proposition 30, we get $Z \notin \mathrm{Sing}(X)$.

Let M_1 and M_2 be two general surfaces in $\mathcal{M}$. Using [30] or [16, Corollary 3.4], we get

$$\big(M_1 \cdot M_2\big)_Z > n^2. \tag{38}$$

Let $P=\pi(Z)$. Then, arguing as in the proof of Proposition 31, we get $|\mathrm{Orb}_{\mathbb{H}}(P)|\neq 4$.

We claim that $|\mathrm{Orb}_{\mathbb{H}}(P)|\neq 8$. Indeed, suppose $|\mathrm{Orb}_{\mathbb{H}}(P)|=8$. Then

$$P\in \ell_1\cup\ell_2\cup\cdots\cup\ell_{30}.$$

Without loss of generality, we may assume that $P\in\ell_1$. Let C_1 and C_2 be the preimages on the threefold X of the lines ℓ_1 and ℓ_2, respectively. Recall that C_1 and C_2 are smooth irreducible elliptic curves, and the curve C_1+C_2 is $\mathbb{H}^\rho$-irreducible. Write

$$M_1\cdot M_2=m(C_1+C_2)+\Delta,$$

where m is a non-negative integer, and Δ is an effective one-cycle whose support does not contain the curves C_1 and C_2. Then $m\leqslant\frac{n^2}{2}$, because

$$2n^2=H\cdot M_1\cdot M_2=mH\cdot(C_1+C_2)+H\cdot\Delta\leqslant mH\cdot(C_1+C_2)=4m.$$

On the other hand, since C_1 and C_2 are smooth curves, it follows from (38) that

$$\mathrm{mult}_O\big(\Delta\big)>n^2-m \tag{39}$$

for every point $O\in\mathrm{Orb}_{\mathbb{H}^\rho}(Z)$. Note also that $Z\in C_1$ and $|\mathrm{Orb}_{\mathbb{H}^\rho}(Z)|\geqslant 8$.

Let $\mathcal{D}$ be the linear subsystem in $|2H|$ that consists of surfaces passing through $C_1\cup C_2$. Then, as we already implicitly mentioned, the linear system $\mathcal{D}$ does not have base curves except for C_1 and C_2. Therefore, if D is a general surface in $\mathcal{D}$, then D does not contain irreducible components of the one-cycle Δ, so (39) gives

$$\begin{aligned}4n^2-8m=D\cdot\Delta\geqslant\sum_{O\in\mathrm{Orb}_{\mathbb{H}^\rho}(Z)}\mathrm{mult}_O\big(\Delta\big)=\\ =|\mathrm{Orb}_{\mathbb{H}^\rho}(Z)|\mathrm{mult}_Z\big(\Delta\big)>|\mathrm{Orb}_{\mathbb{H}^\rho}(Z)|(n^2-m)\geqslant 8(n^2-m),\end{aligned}$$

which is absurd. This shows that $|\mathrm{Orb}_{\mathbb{H}}(P)|\neq 8$.

In particular, we see that $|\mathrm{Orb}_{\mathbb{H}^\rho}(Z)|=|\mathrm{Orb}_{\mathbb{H}}(P)|=16$ and $P\notin\ell_1\cup\ell_2\cup\cdots\cup\ell_{30}$.

We claim that $P\notin\mathcal{Q}_1\cup\mathcal{Q}_2\cup\cdots\cup\mathcal{Q}_{10}$. Indeed, suppose that $P\in\mathcal{Q}_1\cup\mathcal{Q}_2\cup\cdots\cup\mathcal{Q}_{10}$. Without loss of generality, we may assume that

$$\pi(Z)=P\in\mathcal{Q}_1.$$

As above, denote by S the preimage of the quadric surface $\mathcal{Q}_1$ via the double cover π. Then S is a K3 surface with at most ordinary double singularities, and it follows from the inversion of adjunction [24, Theorem 5.50] that $(S,\frac{2}{n}\mathcal{M}|_S)$ is not log canonical at Z. Let λ be the largest rational number such that $(S,\lambda\mathcal{M}|_S)$ is log canonical at Z. Then

$$\mathrm{Orb}_{\mathbb{H}^\rho}(Z)\subseteq\mathrm{Nklt}\big(S,\lambda\mathcal{M}|_S\big).$$

Note that the locus $\mathrm{Nklt}(S,\lambda\mathcal{M}|_S)$ is $\mathbb{H}^\rho$-invariant, because $\mathcal{M}$ and S are $\mathbb{H}^\rho$-invariant.

Suppose $\mathrm{Nklt}(S,\lambda\mathcal{M}|_S)$ contains an $\mathbb{H}^\rho$-irreducible curve C that passes through Z. This means that $\lambda\mathcal{M}|_S=C+\Omega$, where Ω is an effective $\mathbb{Q}$-linear system on S. Then

$$H\cdot C\leqslant H\cdot(C+\Omega)=4n\lambda<8,$$

hence $\pi(C)$ is a union of 4 disjoint lines in $\mathcal{Q}_1$ by Lemma 35, since $P\notin\ell_1\cup\ell_2\cup\cdots\cup\ell_{30}$. Let $\mathcal{C}$ be the preimage in X of a general line in $\mathcal{Q}_1$ that intersect $\pi(C)$. Then

$$4\leqslant\mathcal{C}\cdot C\leqslant\mathcal{C}\cdot(C+\Omega)=\lambda n\big(H\cdot\mathcal{C}\big)=2\lambda n<4,$$

which is absurd. So, the locus $\mathrm{Nklt}(S,\lambda\mathcal{M}|_S)$ contains no curves that pass through Z.

Let $\mathcal{I}_S$ be the multiplier ideal sheaf of the pair $(S,\lambda\mathcal{M}|_S)$, let $\mathcal{L}_S$ be the corresponding subscheme in S. Then

$$\mathrm{Supp}\big(\mathcal{L}_S\big)=\mathrm{Nklt}\big(S,\lambda\mathcal{M}|_S\big).$$

Now, applying Nadel's vanishing theorem [26, Theorem 9.4.8], we get

$$h^1\big(S,\mathcal{I}_S\otimes\mathcal{O}_S(2H|_S)\big)=0.$$

Now, using the Riemann–Roch theorem and Serre's vanishing, we obtain

$$10=h^0\big(S,\mathcal{O}_S(2H|_S)\big)\geqslant h^0\big(\mathcal{O}_{\mathcal{L}_S}\otimes\mathcal{O}_S(2H|_S)\big)\geqslant|\mathrm{Orb}_{\mathbb{H}^\rho}(Z)|.$$

because $\mathcal{L}_S$ has at least $|\mathrm{Orb}_{\mathbb{H}^\rho}(Z)|$ disjoint zero-dimensional components, whose supports are points in $\mathrm{Orb}_{\mathbb{H}^\rho}(Z)$, because $\mathrm{Orb}_{\mathbb{H}^\rho}(Z)\subseteq\mathrm{Nklt}(S,\lambda\mathcal{M}|_S)$, and $\mathrm{Nklt}(S,\lambda\mathcal{M}|_S)$ does not contain curves that are not disjoint from $\mathrm{Orb}_{\mathbb{H}^\rho}(Z)$. Hence, we see that $|\mathrm{Orb}_{\mathbb{H}^\rho}(Z)|\leqslant 10$, which is impossible, since $|\mathrm{Orb}_{\mathbb{H}^\rho}(Z)|=16$. This shows that

$$\pi(Z)=P\notin\mathcal{Q}_1\cup\mathcal{Q}_2\cup\cdots\cup\mathcal{Q}_{10}.$$

Let us summarize what we proved so far. Recall that $\mathcal{M}$ is a mobile $\mathbb{H}^\rho$-invariant linear subsystem in $|nH|$, the log pair $(X,\frac{2}{n}\mathcal{M})$ is canonical away from finitely many points, but the singularities of the pair $(X,\frac{2}{n}\mathcal{M})$ are not canonical at the point $Z\in X$ such that

- $Z\notin\mathrm{Sing}(X)$,
- $\pi(Z)\notin\ell_1\cup\ell_2\cup\cdots\cup\ell_{30}$,
- $\pi(Z)\notin\mathcal{Q}_1\cup\mathcal{Q}_2\cup\cdots\cup\mathcal{Q}_{10}$,
- $|\mathrm{Orb}_{\mathbb{H}^\rho}(Z)|=|\mathrm{Orb}_{\mathbb{H}}(\pi(Z))|=16$.

By Lemma 35, $\pi(Z)$ is not contained in any $\mathbb{H}$-invariant curve whose degree is at most 7. Let us use this and Nadel's vanishing [26, Theorem 9.4.8] to derive a contradiction.

As in the proofs of Propositions 30 and 31, we observe that $(X,\frac{3}{n}\mathcal{M})$ is not log canonical at the point Z, because X is smooth at Z. Let μ be the largest rational number such that the log pair $(X,\mu\mathcal{M})$ is log canonical at Z. Then $\mu<\frac{3}{n}$ and

$$\mathrm{Orb}_{\mathbb{H}^\rho}(Z)\subseteq\mathrm{Nklt}\big(X,\mu\mathcal{M}\big).$$

Moreover, if the locus $\mathrm{Nklt}(X,\mu\mathcal{M})$ contains an $\mathbb{H}^\rho$-irreducible curve C, then arguing as in the proof of Proposition 30, we see that

$$\deg\big(\pi(C)\big)\leqslant H\cdot C\leqslant 4,$$

which implies that the curve C does not pass through Z. Hence, we conclude that every point of the orbit $\mathrm{Orb}_{\mathbb{H}^\rho}(Z)$ is an isolated irreducible component of the locus $\mathrm{Nklt}(X,\mu\mathcal{M})$.

Let $\mathcal{I}$ be the multiplier ideal sheaf of the pair $(X,\mu\mathcal{M})$, and let $\mathcal{L}$ be the corresponding subscheme in X. Then

$$\mathrm{Supp}\big(\mathcal{L}\big)=\mathrm{Nklt}\big(X,\mu\mathcal{M}\big),$$

so the subscheme $\mathcal{L}$ contains at least $|\mathrm{Orb}_{\mathbb{H}^\rho}(Z)|=16$ zero-dimensional components whose supports are points in the orbit $\mathrm{Orb}_{\mathbb{H}^\rho}(Z)$. On the other hand, we have

$$h^1\big(X,\mathcal{I}\otimes\mathcal{O}_X(H)\big)=0$$

by Nadel's vanishing theorem [26, Theorem 9.4.8]. This gives

$$4=h^0\big(X,\mathcal{O}_X(H)\big)\geqslant h^0\big(\mathcal{O}_{\mathcal{L}}\otimes\mathcal{O}_X(H)\big)\geqslant|\mathrm{Orb}_{\mathbb{H}^\rho}(Z)|=16,$$

which is absurd. The obtained contradiction completes the proof of Theorem 17. □

Let us conclude this paper with one application of Theorem 17, which was the initial motivation for this paper — we were looking for various embeddings $\boldsymbol{\mu}_2^4 \rtimes \boldsymbol{\mu}_3 \hookrightarrow \mathrm{Bir}(\mathbb{P}^3)$.

Example 40 (cf. Examples 5, 8, 26)**.** Let $G_{48,50}$ be the subgroup in $\mathrm{PGL}_4(\mathbb{C})$ generated by

$$A_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, A_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, A_5 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then one can check that $G_{48,50} \cong \boldsymbol{\mu}_2^4 \rtimes \boldsymbol{\mu}_3$ and the GAP ID of the group $G_{48,50}$ is [48,50]. For every $t \in \mathbb{C} \setminus \{\pm 1, \pm\sqrt{3}i\}$, let S_t be the quartic surface in $\mathbb{P}^3$ given by the equation (6), i.e. the surface S_t is the quartic surface in $\mathbb{P}^3$ given by the following equation:

$$x_0^4+x_1^4+x_2^4+x_3^4-(t^2+1)(x_0^2x_1^2+x_2^2x_3^2+x_0^2x_2^2+x_1^2x_3^2+x_0^2x_3^2+x_1^2x_2^2)+2(t^3+3t)x_0x_1x_2x_3 = 0.$$

Then S_t is $G_{48,50}$-invariant, and S_t has 16 ordinary double singularities (see Example 5). Now, let X_t be the hypersurface in $\mathbb{P}(1,1,1,1,2)$ that is given by

$$w^2 = x_0^4+x_1^4+x_2^4+x_3^4-(t^2+1)(x_0^2x_1^2+x_2^2x_3^2+x_0^2x_2^2+x_1^2x_3^2+x_0^2x_3^2+x_1^2x_2^2)+2(t^3+3t)x_0x_1x_2x_3,$$

where we consider x_0, x_1, x_2, x_3 as homogeneous coordinates on $\mathbb{P}(1,1,1,1,2)$ of weight 1, and w is a coordinate of weight 2. Consider the faithful action $G_{48,50} \curvearrowright X_t$ given by

$$\begin{aligned} A_1 &\colon [x_0 : x_1 : x_2 : x_3 : w] \mapsto [-x_0 : x_1 : -x_2 : x_3 : w], \\ A_2 &\colon [x_0 : x_1 : x_2 : x_3 : w] \mapsto [-x_0 : x_1 : -x_2 : x_3 : w], \\ A_3 &\colon [x_0 : x_1 : x_2 : x_3 : w] \mapsto [x_1 : x_2 : x_3 : x_2 : w], \\ A_4 &\colon [x_0 : x_1 : x_2 : x_3 : w] \mapsto [x_3 : x_2 : x_1 : x_0 : w], \\ A_5 &\colon [x_0 : x_1 : x_2 : x_3 : w] \mapsto [x_1 : x_2 : x_0 : x_3 : w]. \end{aligned}$$

Since the threefold X_t is $G_{48,50}$-invariant, this gives an embedding $G_{48,50} \hookrightarrow \mathrm{Aut}(X_t)$. Then it follows from Theorem 17 that the threefold X_t is $G_{48,50}$-birationally super-rigid. In particular, for any $t_1 \neq t_2$ in $\mathbb{C} \setminus \{\pm 1, \pm\sqrt{3}i\}$, the following conditions are equivalent:

- the threefolds X_{t_1} and X_{t_2} are $G_{48,50}$-birational;
- the surfaces S_{t_1} and S_{t_2} are projectively equivalent.

Recall that X_t is rational. For $t \in \mathbb{C} \setminus \{\pm 1, \pm\sqrt{3}i\}$, fix a birational map $\chi_t \colon \mathbb{P}^3 \dashrightarrow X_t$, and consider the monomorphism $\eta_t \colon G_{48,50} \hookrightarrow \mathrm{Bir}(\mathbb{P}^3)$ that is given by $g \mapsto \chi_t^{-1} \circ g \circ \chi_t$. Then, for any $t_1 \neq t_2$ in $\mathbb{C} \setminus \{\pm 1, \pm\sqrt{3}i\}$, we have the following assertion:

$$\eta_{t_1}(G_{48,50}) \text{ and } \eta_{t_1}(G_{48,50}) \text{ are conjugate in } \mathrm{Bir}(\mathbb{P}^3) \iff X_{t_1} \text{ and } X_{t_2} \text{ are } G_{48,50}\text{-birational.}$$

Thus, if $t_1 \neq t_2$ are general, then $\eta_{t_1}(G_{48,50})$ and $\eta_{t_1}(G_{48,50})$ are not conjugate in $\mathrm{Bir}(\mathbb{P}^3)$. Similarly, we see that $\eta_t(G_{48,50})$ is not conjugate in $\mathrm{Bir}(\mathbb{P}^3)$ to the group $G_{48,50} \subset \mathrm{PGL}_4(\mathbb{C})$, which also follows from [12]. Can we show this using other obstructions [5, 25, 21]?

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FIGURE 2. Defining equations of the surfaces $\Pi_1^+, \dots, \Pi_{16}^+$ in Example 25.

Π_1^+	$x_0 + x_1 + x_2 + \frac{2s}{s^2+1}x_3 = 0$ $w = \frac{s^2-1}{(s^2+1)^2}\big((s^2+1)x_1^2 + (s^2+1)x_1x_2 + 2sx_1x_3 + (s^2+1)x_2^2 + 2sx_2x_3 - (s^2+1)x_3^2\big)$
Π_2^+	$x_0 - x_1 + x_2 + \frac{2s}{s^2+1}x_3 = 0$ $w = \frac{s^2-1}{(s^2+1)^2}\big((s^2+1)x_1^2 - (s^2+1)x_1x_2 + 2sx_1x_3 + (s^2+1)x_2^2 - 2sx_2x_3 - (s^2+1)x_3^2\big) = 0$
Π_3^+	$x_0 + x_1 - x_2 - \frac{2s}{s^2+1}x_3 = 0$ $w = \frac{s^2-1}{(s^2+1)^2}\big((s^2+1)x_1^2 - (s^2+1)x_1x_2 - 2sx_1x_3 + (s^2+1)x_2^2 + 2sx_2x_3 - (s^2+1)x_3^2\big) = 0$
Π_4^+	$x_0 - x_1 - x_2 + \frac{2s}{s^2+1}x_3 = 0$ $w = \frac{s^2-1}{(s^2+1)^2}\big((s^2+1)x_1^2 + (s^2+1)x_1x_2 - 2sx_1x_3 + (s^2+1)x_2^2 - 2sx_2x_3 - (s^2+1)x_3^2\big) = 0$
Π_5^+	$x_0 + x_1 + \frac{2s}{s^2+1}x_2 + x_3 = 0$ $w = \frac{s^2-1}{(s^2+1)^2}\big((s^2+1)x_0^2 + 2sx_2x_0 + (s^2+1)x_3x_0 - (s^2+1)x_2^2 + 2sx_2x_3 + (s^2+1)x_3^2\big) = 0$
Π_6^+	$x_0 - x_1 + \frac{2s}{s^2+1}x_2 - x_3 = 0$ $w = \frac{s^2-1}{(s^2+1)^2}\big((s^2+1)x_0^2 + 2sx_2x_0 - (s^2+1)x_3x_0 - (s^2+1)x_2^2 - 2sx_2x_3 + (s^2+1)x_3^2\big) = 0$
Π_7^+	$\frac{2s}{s^2+1}x_2 - x_1 - x_0 + x_3 = 0$ $w = \frac{s^2-1}{(s^2+1)^2}\big((s^2+1)x_0^2 - 2sx_2x_0 - (s^2+1)x_3x_0 - (s^2+1)x_2^2 + 2sx_2x_3 + (s^2+1)x_3^2\big) = 0$
Π_8^+	$x_0 - x_1 - \frac{2s}{s^2+1}x_2 + x_3 = 0$ $w = \frac{s^2-1}{(s^2+1)^2}\big((s^2+1)x_0^2 - 2sx_2x_0 + (s^2+1)x_3x_0 - (s^2+1)x_2^2 - 2sx_2x_3 + (s^2+1)x_3^2\big) = 0$
Π_9^+	$x_0 + \frac{2s}{s^2+1}x_1 + x_2 + x_3 = 0$ $w = \frac{s^2-1}{(s^2+1)^2}\big((s^2+1)x_0^2 + 2sx_1x_0 + (s^2+1)x_3x_0 - (s^2+1)x_1^2 + 2sx_1x_3 + (s^2+1)x_3^2\big) = 0$
Π_{10}^+	$x_0 - \frac{2s}{s^2+1}x_1 - x_2 + x_3 = 0$ $w = \frac{s^2-1}{(s^2+1)^2}\big((s^2+1)x_0^2 - 2sx_1x_0 + (s^2+1)x_3x_0 - (s^2+1)x_1^2 - 2sx_1x_3 + (s^2+1)x_3^2\big) = 0$
Π_{11}^+	$x_0 - \frac{2s}{s^2+1}x_1 + x_2 - x_3 = 0$ $w = \frac{s^2-1}{(s^2+1)^2}\big((s^2+1)x_0^2 - 2sx_1x_0 - (s^2+1)x_3x_0 - (s^2+1)x_1^2 + 2sx_1x_3 + (s^2+1)x_3^2\big) = 0$
Π_{12}^+	$x_0 + \frac{2s}{s^2+1}x_1 - x_2 - x_3 = 0$ $w = \frac{s^2-1}{(s^2+1)^2}\big((s^2+1)x_0^2 + 2sx_1x_0 - (s^2+1)x_3x_0 - (s^2+1)x_1^2 - 2sx_1x_3 + (s^2+1)x_3^2\big) = 0$
Π_{13}^+	$\frac{2s}{s^2+1}x_0 + x_1 + x_2 + x_3 = 0$ $w = \frac{s^2-1}{(s^2+1)^2}\big((s^2+1)x_0^2 - 2sx_1x_0 - 2sx_2x_0 - (s^2+1)x_1^2 - (s^2+1)x_1x_2 - (s^2+1)x_2^2\big) = 0$
Π_{14}^+	$\frac{2s}{s^2+1}x_0 - x_1 - x_2 + x_3 = 0$ $w = \frac{s^2-1}{(s^2+1)^2}\big((s^2+1)x_0^2 + 2sx_1x_0 + 2sx_2x_0 - (s^2+1)x_1^2 - (s^2+1)x_1x_2 - (s^2+1)x_2^2\big) = 0$
Π_{15}^+	$\frac{2s}{s^2+1}x_0 - x_1 + x_2 - x_3 = 0$ $w = \frac{s^2-1}{(s^2+1)^2}\big((s^2+1)x_0^2 + 2sx_1x_0 - 2sx_2x_0 - (s^2+1)x_1^2 + (s^2+1)x_1x_2 - (s^2+1)x_2^2\big) = 0$
Π_{16}^+	$\frac{2s}{s^2+1}x_0 + x_1 - x_2 - x_3 = 0$ $w = \frac{s^2-1}{(s^2+1)^2}\big((s^2+1)x_0^2 - 2sx_1x_0 + 2sx_2x_0 - (s^2+1)x_1^2 + (s^2+1)x_1x_2 - (s^2+1)x_2^2\big) = 0$

FIGURE 3. Ten $\mathbb{H}$-invariant quadrics in $\mathbb{P}^3$ and thirty lines in them.

	$\mathcal{Q}_1$	$\mathcal{Q}_2$	$\mathcal{Q}_3$	$\mathcal{Q}_4$	$\mathcal{Q}_5$	$\mathcal{Q}_6$	$\mathcal{Q}_7$	$\mathcal{Q}_8$	$\mathcal{Q}_9$	$\mathcal{Q}_{10}$
ℓ_1	$-$	$-$	$-$	$-$	$+$	$+$	$-$	$+$	$+$	$-$
ℓ_2	$-$	$-$	$-$	$-$	$+$	$+$	$-$	$+$	$+$	$-$
ℓ_3	$-$	$-$	$-$	$-$	$-$	$+$	$+$	$-$	$+$	$+$
ℓ_4	$-$	$-$	$-$	$-$	$-$	$+$	$+$	$-$	$+$	$+$
ℓ_5	$-$	$-$	$-$	$-$	$+$	$-$	$+$	$+$	$-$	$+$
ℓ_6	$-$	$-$	$-$	$-$	$+$	$-$	$+$	$+$	$-$	$+$
ℓ_7	$-$	$-$	$+$	$+$	$-$	$-$	$-$	$+$	$+$	$-$
ℓ_8	$-$	$-$	$+$	$+$	$-$	$-$	$-$	$+$	$+$	$-$
ℓ_9	$-$	$+$	$+$	$-$	$-$	$-$	$-$	$-$	$+$	$+$
ℓ_{10}	$-$	$+$	$+$	$-$	$-$	$-$	$-$	$-$	$+$	$+$
ℓ_{11}	$-$	$+$	$-$	$+$	$-$	$-$	$-$	$+$	$-$	$+$
ℓ_{12}	$-$	$+$	$-$	$+$	$-$	$-$	$-$	$+$	$-$	$+$
ℓ_{13}	$-$	$-$	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$-$
ℓ_{14}	$-$	$-$	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$-$
ℓ_{15}	$-$	$+$	$+$	$-$	$-$	$+$	$+$	$-$	$-$	$-$
ℓ_{16}	$-$	$+$	$+$	$-$	$-$	$+$	$+$	$-$	$-$	$-$
ℓ_{17}	$-$	$+$	$-$	$+$	$+$	$-$	$+$	$-$	$-$	$-$
ℓ_{18}	$-$	$+$	$-$	$+$	$+$	$-$	$+$	$-$	$-$	$-$
ℓ_{19}	$+$	$+$	$-$	$-$	$+$	$-$	$-$	$-$	$+$	$-$
ℓ_{20}	$+$	$+$	$-$	$-$	$+$	$-$	$-$	$-$	$+$	$-$
ℓ_{21}	$+$	$-$	$-$	$+$	$-$	$-$	$+$	$-$	$+$	$-$
ℓ_{22}	$+$	$-$	$-$	$+$	$-$	$-$	$+$	$-$	$+$	$-$
ℓ_{23}	$+$	$-$	$+$	$-$	$+$	$-$	$+$	$+$	$-$	$-$
ℓ_{24}	$+$	$-$	$+$	$-$	$-$	$-$	$+$	$+$	$-$	$-$
ℓ_{25}	$+$	$+$	$-$	$-$	$-$	$+$	$-$	$+$	$-$	$-$
ℓ_{26}	$+$	$+$	$-$	$-$	$-$	$+$	$-$	$+$	$-$	$-$
ℓ_{27}	$+$	$-$	$-$	$+$	$-$	$+$	$-$	$-$	$-$	$+$
ℓ_{28}	$+$	$-$	$-$	$+$	$-$	$+$	$-$	$-$	$-$	$+$
ℓ_{29}	$+$	$-$	$+$	$-$	$+$	$-$	$-$	$-$	$-$	$+$
ℓ_{30}	$+$	$-$	$+$	$-$	$+$	$-$	$-$	$-$	$-$	$+$