K-STABLE DIVISORS IN $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ OF DEGREE $(1,1,2)$

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Abstract. We prove that every smooth divisor in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ of degree $(1,1,2)$ is K-stable.

1. Introduction

The goal of this paper is to prove the following result:

Main Theorem. *Let X be a smooth divisor in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ of degree $(1,1,2)$. Then X is K-stable.*

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2. Smooth Fano threefolds in the deformation family №3.3

Let X be a divisor in $\mathbb{P}^1_{s,t} \times \mathbb{P}^1_{u,v} \times \mathbb{P}^2_{x,y,z}$ of tridegree $(1,1,2)$, where $([s:t],[u:v],[x,y,z])$ are coordinates on $\mathbb{P}^1_{s,t} \times \mathbb{P}^1_{u,v} \times \mathbb{P}^2_{x,y,z}$. Then X is given by the following equation:

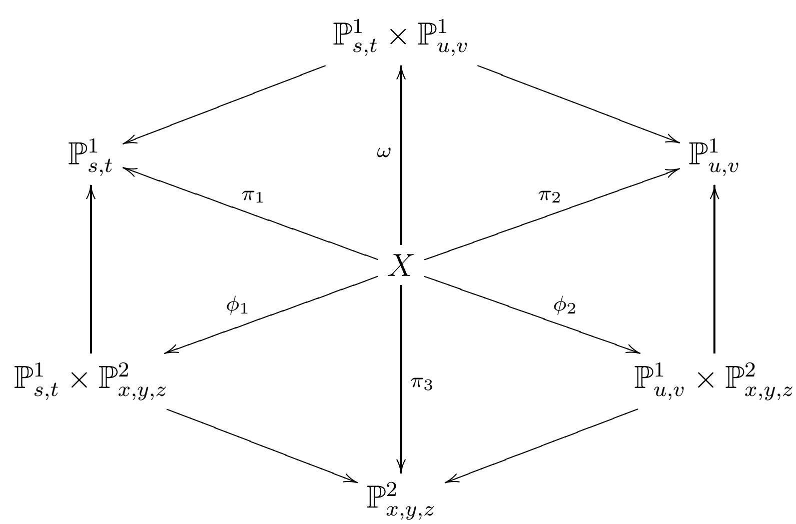
$$\begin{bmatrix} s & t \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 0,$$

where each $a_{ij} = a_{ij}(x,y,z)$ is a homogeneous polynomials of degree 2. We can also define X by

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0,$$

where each $b_{ij} = b_{ij}(s,t;u,v)$ is a bi-homogeneous polynomial of degree $(1,1)$.

Suppose that X is smooth. Then we have the following commutative diagram:



Throughout this paper, all varieties are assumed to be projective and defined over $\mathbb{C}$.

where all maps are induced by natural projections. Note that ω is a (standard) conic bundle whose discriminant curve $\Delta_{\mathbb{P}^1\times\mathbb{P}^1}\subset\mathbb{P}^1_{s,t}\times\mathbb{P}^1_{u,v}$ is a curve of degree $(3,3)$, which is given by

$$\det\begin{bmatrix} b_{11} & b_{12} & b_{13}\\ b_{21} & b_{22} & b_{23}\\ b_{31} & b_{32} & b_{33}\end{bmatrix}=0.$$

Similarly, the map π_3 is a conic bundle whose discriminant curve $\Delta_{\mathbb{P}^2}\subset\mathbb{P}^2_{x,y,z}$ is a smooth plane quartic curve, which is given by $a_{11}a_{22}=a_{12}a_{21}$. Both maps ϕ_1 and ϕ_2 are birational morphisms that blow up the following smooth genus 3 curves:

$$\big\{sa_{11}+ta_{21}=sa_{12}+ta_{22}=0\big\}\subset\mathbb{P}^1_{s,t}\times\mathbb{P}^2_{x,y,z},$$
$$\big\{ua_{11}+va_{12}=ua_{21}+va_{22}=0\big\}\subset\mathbb{P}^1_{u,v}\times\mathbb{P}^2_{x,y,z}.$$

Finally, both morphisms π_1 and π_2 are fibrations into quintic del Pezzo surfaces.

Let $H_1=\pi_1^*(\mathcal{O}_{\mathbb{P}^1}(1))$, let $H_2=\pi_2^*(\mathcal{O}_{\mathbb{P}^1}(1))$, let $H_3=\pi_3^*(\mathcal{O}_{\mathbb{P}^2}(1))$, let E_1 and E_2 be the exceptional divisors of the morphisms ϕ_1 and ϕ_2, respectively. Then

$$\begin{aligned}-K_X&\sim H_1+H_2+H_3,\\ E_1&\sim H_1+2H_3-H_2,\\ E_2&\sim H_2+2H_3-H_1.\end{aligned}$$

This gives $E_1+E_2\sim 4H_3$, which also follows from $E_1+E_2=\pi_3^*(\Delta_{\mathbb{P}^2})$. We have

$$-K_X\sim_{\mathbb{Q}}\frac{3}{2}H_1+\frac{1}{2}H_2+\frac{1}{2}E_2\sim_{\mathbb{Q}}\frac{1}{2}H_1+\frac{3}{2}H_2+\frac{1}{2}E_1.$$

In particular, we see that $\alpha(X)\leqslant\frac{2}{3}$. Note that $E_1\cong E_2\cong\Delta_{\mathbb{P}^2}\times\mathbb{P}^1$.

The Mori cone $\overline{\mathrm{NE}}(X)$ is simplicial and is generated by the curves contracted by ω, ϕ_1 and ϕ_2. The cone of effective divisors $\mathrm{Eff}(X)$ is generated by the classes of the divisors E_1, E_2, H_1, H_2.

Lemma 1. *Let S be a surface in the pencil $|H_1|$. Then S is a normal quintic del Pezzo surface that has at most Du Val singularities, the restriction $\pi_3|_S\colon S\to\mathbb{P}^2_{x,y,z}$ is a birational morphism, and the restriction $\pi_2|_S\colon S\to\mathbb{P}^1_{u,v}$ is a conic bundle. Moreover, one of the following cases hold:*

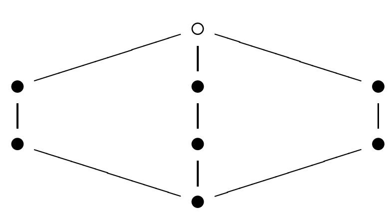
- *the surface S is smooth,*
- ($\mathbb{A}_1$) *the surface S has one singular point of type $\mathbb{A}_1$,*
- ($2\mathbb{A}_1$) *the surface S has two singular points of type $\mathbb{A}_1$,*
- ($\mathbb{A}_2$) *the surface S has one singular point of type $\mathbb{A}_2$,*
- ($\mathbb{A}_3$) *the surface S has one singular point of type $\mathbb{A}_3$.*

Furthermore, in each of these five case, the del Pezzo surface S is unique up to an isomorphism.

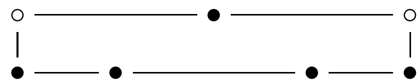
Proof. This is well-known [3, 4]. □

Remark 2. In the notations and assumptions of Lemma 1, suppose that the surface S is singular, and let $\varpi\colon\widetilde{S}\to S$ be its minimal resolution of singularities. Then the dual graph of the (-1)-curves and (-2)-curves on the surface $\widetilde{S}$ can be described as follows:

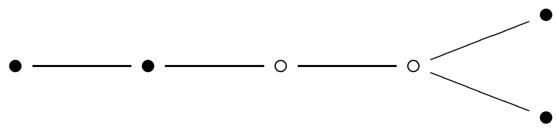
($\mathbb{A}_1$) if S has one singular point of type $\mathbb{A}_1$, then the dual graph is



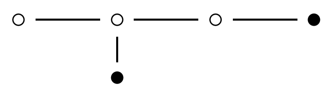
($2\mathbb{A}_1$) if S has two singular points of type $\mathbb{A}_1$, then the dual graph is



($\mathbb{A}_2$) if S has one singular point of type $\mathbb{A}_2$, then the dual graph is



($\mathbb{A}_3$) if S has one singular point of type $\mathbb{A}_3$, then the dual graph is



Here, as in the papers [4, 3], we denote a (-1)-curve by $\bullet$, and we denote a (-2)-curve by $\circ$.

Lemma 3. *Let S_1 be a surface in $|H_1|$, let S_2 be a surface in $|H_2|$, and let P be a point in $S_1 \cap S_2$. Then at least one of the surfaces S_1 or S_2 is smooth at P.*

Proof. Local computations. □

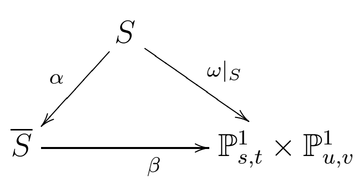
Corollary 4. *In the notations and assumptions of Lemma 3, suppose the conic $S_1 \cdot S_2$ is reduced. Then at least one of the surfaces S_1 or S_2 is smooth along $S_1 \cap S_2$.*

Lemma 5. *Let P be a point in X, let C be the scheme fiber of the conic bundle ω that contains P, and let Z be the scheme fiber of the conic bundle π_3 that contains P. Then C or Z is smooth at P.*

Proof. Local computations. □

Lemma 6. *Let C be a fiber of the morphism π_3, let S be a general surface in $|H_3|$ that contains C. Then S is smooth, $K_S^2 = 4$ and $-K_S \sim (H_1 + H_2)|_S$, which implies that $-K_S$ is nef and big. Moreover, one of the following three cases holds:*

1. *the conic C is smooth, $-K_S$ is ample, and the restriction $\omega|_S \colon S \to \mathbb{P}^1_{s,t} \times \mathbb{P}^1_{u,v}$ is a double cover branched over a smooth curve of degree $(2, 2)$,*
2. *the conic C is smooth, the divisor $-K_S$ is not ample, the conic $\omega(C)$ is an irreducible component of the discriminant curve $\Delta_{\mathbb{P}^1 \times \mathbb{P}^1}$, the conic C is contained in $\mathrm{Sing}(\omega^{-1}(\Delta_{\mathbb{P}^1 \times \mathbb{P}^1}))$, and the restriction map $\omega|_S \colon S \to \mathbb{P}^1_{s,t} \times \mathbb{P}^1_{u,v}$ fits the following commutative diagram:*



   where α is a birational morphism that contracts two disjoint (-2)-curves, and β is a double cover branched over a singular curve of degree $(2, 2)$, which is a union of the curve $\omega(C)$ and another smooth curve of degree $(1, 1)$, which intersect transversally at two distinct points,
3. *the conic C is singular, $-K_S$ is ample, and the restriction $\omega|_S \colon S \to \mathbb{P}^1_{s,t} \times \mathbb{P}^1_{u,v}$ is a double cover branched over a smooth curve of degree $(2, 2)$.*

Proof. The smoothness of the surface S easily follows from local computations. If $-K_S$ is ample, the remaining assertions are obvious. So, to complete the proof, we assume that $-K_S$ is not ample.

Then the restriction $\omega|_S\colon S \to \mathbb{P}^1_{s,t} \times \mathbb{P}^1_{u,v}$ fits the commutative diagram

$$\begin{array}{ccc} & S & \\ {}^{\alpha}\swarrow & & \searrow^{\omega|_S} \\ \overline{S} & \xrightarrow{\ \beta\ } & \mathbb{P}^1_{s,t} \times \mathbb{P}^1_{u,v} \end{array}$$

where α is a birational morphism that contracts all (-2)-curves in S, and β is a double cover branched over a singular curve of degree $(2,2)$. Let ℓ be a (-2)-curve in S. Then

$$(H_1 + H_2) \cdot \ell = -K_S \cdot \ell = 0,$$

so that $\omega(\ell)$ is a point in $\mathbb{P}^1_{s,t} \times \mathbb{P}^1_{u,v}$. But $\pi_3(\ell)$ is a line in $\mathbb{P}^2_{x,y,z}$ that contains the point $\pi_3(C)$. This shows that the curve ℓ is an irreducible component of a singular fiber of the conic bundle ω. Therefore, we see that $\omega(\ell) \in \Delta_{\mathbb{P}^1\times\mathbb{P}^1}$. This implies that the conic bundle ω maps an irreducible component of the conic C to an irreducible component of the curve $\Delta_{\mathbb{P}^1\times\mathbb{P}^1}$, because S is a general surface in the linear system $|H_3|$ that contains the curve C.

If C is singular, an irreducible component of the curve $\Delta_{\mathbb{P}^1\times\mathbb{P}^1}$ is a curve of degree $(1,0)$ or $(0,1)$, which is impossible [8, § 3.8]. Therefore, we see that the conic C is smooth and irreducible, and the curve $\omega(C) \cong C$ is an irreducible component of the discriminant curve $\Delta_{\mathbb{P}^1\times\mathbb{P}^1}$. Since the conic bundle ω is standard [8], the surface $\omega^{-1}(\omega(C))$ is irreducible and non-normal, which easily implies that the conic C is contained in its singular locus.

Choosing appropriate coordinates on $\mathbb{P}^1_{s,t} \times \mathbb{P}^1_{u,v} \times \mathbb{P}^2_{x,y,z}$, we may assume that $\pi_3(C) = [0:0:1]$, the conic C is given by $x = y = sv - tu = 0$, $([0:1],[0:1])$ is a smooth point of the curve $\Delta_{\mathbb{P}^1\times\mathbb{P}^1}$, and the fiber $\omega^{-1}([0:1],[0:1])$ is given by $s = u = xy = 0$. Then X is given by

$$(a_1su + b_1sv + c_1tu)x^2 + (a_2su + b_2sv + c_2tu + tv)xy + \\ + b_4(sv - tu)xz + (a_3su + b_3sv + c_3tu)y^2 + b_5(sv - tu)yz + (sv - tu)z^2 = 0$$

for some numbers a_1, a_2, a_3, b_1, b_2, b_3, b_4, b_5, c_1, c_2, c_3. One can check that $\Delta_{\mathbb{P}^1\times\mathbb{P}^1}$ indeed splits as a union of the curve $\omega(C)$ and the curve in $\mathbb{P}^1_{s,t} \times \mathbb{P}^1_{u,v}$ of degree $(2,2)$ that is given by

$$a_1b_5^2stu^2 - a_1b_5^2s^2uv + a_2b_4b_5s^2uv - a_2b_4b_5stu^2 - a_3b_4^2s^2uv + a_3b_4^2stu^2 - b_1b_5^2s^2v^2 + \\ + b_1b_5^2stuv + b_2b_4b_5s^2v^2 - b_2b_4b_5stuv - b_3b_4^2s^2v^2 + b_3b_4^2stuv - b_4^2c_3stuv + b_4^2c_3t^2u^2 + b_4b_5c_2stuv - \\ - b_4b_5c_2t^2u^2 - b_5^2c_1stuv + b_5^2c_1t^2u^2 + 4a_1a_3s^2u^2 + 4a_1b_3s^2uv + 4a_1c_3stu^2 - a_2^2s^2u^2 - 2a_2b_2s^2uv - \\ - 2a_2c_2stu^2 + 4a_3b_1s^2uv + 4a_3c_1stu^2 + 4b_1b_3s^2v^2 + 4b_1c_3stuv - b_2^2s^2v^2 - 2b_2c_2stuv + 4b_3c_1stuv + \\ + b_4b_5stv^2 - b_4b_5t^2uv + 4c_1c_3t^2u^2 - c_2^2t^2u^2 - 2a_2stuv - 2b_2stv^2 - 2c_2t^2uv - t^2v^2 = 0.$$

The surface S is cut out on X by the equation $y = \lambda x$, where λ is a general complex number. Then the double cover $\beta\colon \overline{S} \to \mathbb{P}^1_{s,t} \times \mathbb{P}^1_{u,v}$ is branched over a singular curve of degree $(2,2)$, which splits as a union of the curve $\omega(C)$ and the curve in $\mathbb{P}^1_{s,t} \times \mathbb{P}^1_{u,v}$ of degree $(1,1)$ that is given by

$$\lambda^2b_5^2tu - \lambda^2b_5^2sv + 4\lambda^2a_3su + 4\lambda^2b_3sv - 2b_4\lambda b_5sv + 2\lambda b_4b_5tu + \\ + 4\lambda^2c_3tu + 4\lambda a_2su + 4\lambda b_2sv - b_4^2sv + b_4^2tu + 4\lambda c_2tu + 4a_1su + 4b_1sv + 4c_1tu + 4\lambda tv = 0.$$

Since λ is general and X is smooth, these two curves intersect transversally by two points, which implies the remaining assertions of the lemma. $\square$

Note that the case (2) in Lemma 6 indeed can happen. For instance, if X is given by

$$(sv + tu)x^2 + (su - sv + tv)xy + (5sv - 5tu)zx + 3suy^2 + (sv - tu)zy + (sv - tu)z^2 = 0,$$

then X is smooth, and general surface in $|H_3|$ that contains the curve $\pi_3^{-1}([0:0:1])$ is a smooth weak del Pezzo surface, which is not a quartic del Pezzo surface.

Lemma 7. *Let C be a smooth fiber of the morphism ω, and let S be a general surface in $|H_1+H_2|$ that contains the curve C. Then S is a smooth del Pezzo surface of degree 2, and $-K_S\sim H_3|_S$.*

Proof. Left to the reader. □

Observe that $-K_X^3=18$, and X is a smooth Fano threefold in the deformation family №3.3. Moreover, every smooth Fano threefold in this deformation family can be obtained in this way.

3. Applications of Abban–Zhuang theory

Let us use notations and assumptions of Section 2. Let $f\colon\widetilde{X}\to X$ be a birational map such that $\widetilde{X}$ is a normal threefold, and let $\mathbf{F}$ be a prime divisor in $\widetilde{X}$. Then, to prove that X is K-stable, it is enough to show that $\beta(\mathbf{F})=A_X(\mathbf{F})-S_X(\mathbf{F})>0$, where $A_X(\mathbf{F})=1+\mathrm{ord}_{\mathbf{F}}(K_{\widetilde{X}}/K_X)$ and

$$S_X(\mathbf{F})=\frac{1}{-K_X^3}\int_0^\infty \mathrm{vol}\big(f^*(-K_X)-u\mathbf{F}\big)du.$$

This follows from the valuative criterion for K-stability [5, 7].

Let $\mathfrak{C}$ be the center of the divisor $\mathbf{F}$ on the threefold X. By [6, Theorem 10.1], we have

$$S_X(S)=\frac{1}{-K_X^3}\int_0^\infty \mathrm{vol}\big(-K_X-uS\big)du<1$$

for every surface $S\subset X$. Hence, if $\mathfrak{C}$ is a surface, then $\beta(\mathbf{F})>0$. Thus, to show that X is K-stable, we may assume that $\mathfrak{C}$ is either a curve or a point. If $\mathfrak{C}$ is a curve, then [2, Corollary 1.7.26] gives

Corollary 8. *Suppose that $\beta(\mathbf{F})\leqslant 0$ and $\mathfrak{C}$ is a curve. Let S be an irreducible normal surface in the threefold X that contains $\mathfrak{C}$. Set*

$$S\big(W^S_{\bullet,\bullet};\mathfrak{C}\big)=\frac{3}{(-K_X)^3}\int_0^\tau\big(P(u)^2\cdot S\big)\cdot\mathrm{ord}_{\mathfrak{C}}\big(N(u)\big|_S\big)du+ \\ +\frac{3}{(-K_X)^3}\int_0^\tau\int_0^\infty\mathrm{vol}\big(P(u)\big|_S-v\mathfrak{C}\big)dvdu,$$

where τ is the largest rational number u such that $-K_X-uS$ is pseudo-effective, $P(u)$ is the positive part of the Zariski decomposition of $-K_X-uS$, and $N(u)$ is its negative part. Then $S(W^S_{\bullet,\bullet};\mathfrak{C})>1$.

Let P be a point in $\mathfrak{C}$. Then

$$\frac{A_X(\mathbf{F})}{S_X(\mathbf{F})}\geqslant\delta_P(X)=\inf_{\substack{E/X\\ P\in C_X(E)}}\frac{A_X(E)}{S_X(E)},$$

where the infimum is taken over all prime divisors E over X whose centers on X that contain P. Therefore, to prove that the Fano threefold X is K-stable, it is enough to show that $\delta_P(X)>1$. On the other hand, we can estimate $\delta_P(X)$ by using [1, Theorem 3.3] and [2, Corollary 1.7.30]. Namely, let S be an irreducible surface in X with Du Val singularities such that $P\in S$. Set

$$\tau=\sup\Big\{u\in\mathbb{Q}_{\geqslant 0}\ \Big|\ \text{the divisor } -K_X-uS \text{ is pseudo-effective}\Big\}.$$

For $u\in[0,\tau]$, let $P(u)$ be the positive part of the Zariski decomposition of the divisor $-K_X-uS$, and let $N(u)$ be its negative part. Then [1, Theorem 3.3] and [2, Corollary 1.7.30] give

$$\delta_P(X)\geqslant\min\left\{\frac{1}{S_X(S)},\delta_P\big(S;W^S_{\bullet,\bullet}\big)\right\} \tag{3.1}$$

for

$$\delta_P\big(S;W^S_{\bullet,\bullet}\big)=\inf_{\substack{F/S,\\ P\subseteq C_S(F)}}\frac{A_S(F)}{S(W^S_{\bullet,\bullet};F)},$$

where

$$S\big(W^S_{\bullet,\bullet};F\big)=\frac{3}{-K_X^3}\int_0^\tau\big(P(u)^2\cdot S\big)\cdot\mathrm{ord}_F\big(N(u)\big|_S\big)du+\frac{3}{-K_X^3}\int_0^\tau\int_0^\infty\mathrm{vol}\big(P(u)\big|_S-vF\big)dvdu,$$

and now the infimum is taken over all prime divisors F over S whose centers on S that contain P. Let us show how to apply (3.1) in some cases. Recall that $S_X(S)<1$ by [6, Theorem 10.1].

Lemma 9. *Let C be the fiber of the conic bundle π_3 that contains P, and let S be a general surface in $|H_3|$ that contains C. Suppose S is a smooth del Pezzo of degree 4, and C is smooth. Then $\delta_P(X)>1$.*

Proof. One has $\tau=1$. Moreover, for $u\in[0,1]$, we have $N(u)=0$ and $P(u)|_S=-K_S+(1-u)C$. Let $L=-K_S+(1-u)C$. Using Lemma 23 and arguing as in the proof of Lemma 26, we get

$$\begin{aligned}S\big(W^S_{\bullet,\bullet};F\big)&=\frac{1}{6}\int_0^1 4(1+(1-u))S_L(F)du\leqslant\\ &\leqslant A_S(F)\int_0^1\frac{4}{6}(1+(1-u))\frac{19+8(1-u)+(1-u)^2}{24}du=\frac{143}{144}A_S(F)\end{aligned}$$

for any prime divisor F over S such that $P\in C_S(F)$. Then (3.1) gives $\delta_P(X)>1$. ☐

Similarly, we obtain the following result:

Lemma 10. *Let S be the surface in $|H_1|$ that contain P. Then*

$$\delta_P(X)\geqslant\min\left\{\frac{1}{S_X(S)},\frac{2592\delta_P(S)}{2560+63\delta_P(S)}\right\}$$

for $\delta_P(S)=\delta_P(S,-K_S)$, where $\delta_P(S,-K_S)$ is defined in Appendix A.

Proof. We have $\tau=\frac{3}{2}$. Moreover, we have

$$P(u)=\begin{cases}(1-u)H_1+H_2+H_3 \text{ if } 0\leqslant u\leqslant 1,\\ (2-u)H_2+(3-2u)H_3 \text{ if } 1\leqslant u\leqslant\dfrac{3}{2},\end{cases}$$

and

$$N(u)=\begin{cases}0 \text{ if } 0\leqslant u\leqslant 1,\\ (u-1)E_2 \text{ if } 1\leqslant u\leqslant\dfrac{3}{2}.\end{cases}$$

Note also that $E_2|_S$ is a smooth genus 3 curve contained in the smooth locus of the surface S.

Recall that S is a quintic del Pezzo surface with at most Du Val singularities, and the restriction morphism $\pi_2|_S\colon S\to\mathbb{P}^1_{u,v}$ is a conic bundle. Note that the morphism $\pi_3|_S\colon S\to\mathbb{P}^2_{x,y,z}$ is birational. Let C be a fiber of the conic bundle $\pi_2|_S$, and let L be the preimage in S of a general line in $\mathbb{P}^2_{x,y,z}$. Then $-K_S\sim C+L$ and

$$P(u)\big|_S\sim_{\mathbb{R}}\begin{cases}C+L \text{ if } 0\leqslant u\leqslant 1,\\ (2-u)C+(3-2u)L \text{ if } 1\leqslant u\leqslant\dfrac{3}{2},\end{cases}$$

Since $2L-C$ is pseudoeffective, the divisor $\frac{7-4u}{3}(-K_S)-(2-u)C-(3-2u)L$ is also pseudoeffective.

Let F be a divisor over S such that $P \in C_S(F)$. Then it follows from Lemma 26 that

$$S\big(W^S_{\bullet,\bullet};F\big) \leqslant \frac{1}{6}A_S(F)\int_1^{\frac{3}{2}}(u-1)\big(P(u)\big|_S\big)^2du+\frac{1}{6}\int_0^{\frac{3}{2}}\int_0^\infty \mathrm{vol}\big(P(u)\big|_S-vF\big)dvdu=$$

$$=\frac{7}{288}A_S(F)+\frac{1}{6}\int_0^1\int_0^\infty \mathrm{vol}\big(-K_S-vF\big)dvdu+\frac{1}{6}\int_1^{\frac{3}{2}}\int_0^\infty \mathrm{vol}\big((2-u)C+(3-2u)L-vF\big)dvdu\leqslant$$

$$\leqslant\frac{7}{288}A_S(F)+\frac{1}{6}\int_0^1 5\frac{A_S(F)}{\delta_P(S)}du+\frac{1}{6}\int_1^{\frac{3}{2}}\int_0^\infty \mathrm{vol}\left(\frac{7-4u}{3}\big(-K_S\big)-vF\right)dvdu=$$

$$=\frac{7}{288}A_S(F)+\frac{5}{6\delta_P(S)}A_S(F)+\frac{1}{6}\int_1^{\frac{3}{2}}\left(\frac{7-4u}{3}\right)^3\int_0^\infty \mathrm{vol}\big(-K_S-vF\big)dvdu\leqslant$$

$$=\frac{7}{288}A_S(F)+\frac{5}{6\delta_P(S)}A_S(F)+\frac{1}{6}\int_1^{\frac{3}{2}}\left(\frac{7-4u}{3}\right)^3 5\frac{A_S(F)}{\delta_P(S)}du=$$

$$=\frac{7}{288}A_S(F)+\frac{5}{6\delta_P(S)}A_S(F)+\frac{25}{162\delta_P(S)}A_S(F)=\left(\frac{80}{81\delta_P(S)}+\frac{7}{288}\right)A_S(F),$$

Then $\delta_P(S;W^S_{\bullet,\bullet}) \geqslant \frac{1}{\frac{80}{81\delta_P(S)}+\frac{7}{288}}=\frac{2592\delta_P(S)}{2560+63\delta_P(S)}$ and the required assertion follows from (3.1). ☐

Keeping in mind that $S_X(S)<1$ by [6, Theorem 10.1] and the δ-invariant of the smooth quintic del Pezzo surface is $\frac{15}{13}$ by [2, Lemma 2.11], we obtain

Corollary 11. *Let S be the surface in $|H_1|$ that contain P. If S is smooth, then $\delta_P(X)>1$.*

Similarly, using Lemmas 24 and 25 from Appendix A, we obtain

Corollary 12. *Let S be the surface in $|H_1|$ that contain P. Suppose that S has at most singular points of type $\mathbb{A}_1$, and P is not contained in any line in S that passes through a singular point. Then $\delta_P(X)>1$.*

Alternatively, we can estimate $\delta_P(X)$ using [2, Theorem 1.7.30]. Namely, let C be an irreducible smooth curve in S that contains P. Suppose S is smooth at P. Since $S\not\subset \mathrm{Supp}(N(u))$, we write

$$N(u)\big|_S=d(u)C+N'_S(u),$$

where $N'_S(u)$ is an effective $\mathbb{R}$-divisor on S such that $C\not\subset\mathrm{Supp}(N'_S(u))$, and $d(u)=\mathrm{ord}_C(N(u)|_S)$. Now, for every $u\in[0,\tau]$, we define the pseudo-effective threshold $t(u)\in\mathbb{R}_{\geqslant 0}$ as follows:

$$t(u)=\inf\Big\{v\in\mathbb{R}_{\geqslant 0}\ \Big|\ \text{the divisor } P(u)\big|_S-vC \text{ is pseudo-effective}\Big\}.$$

For $v\in[0,t(u)]$, we let $P(u,v)$ be the positive part of the Zariski decomposition of $P(u)|_S-vC$, and we let $N(u,v)$ be its negative part. As in Corollary 8, we let

$$S\big(W^S_{\bullet,\bullet};C\big)=\frac{3}{(-K_X)^3}\int_0^\tau\big(P(u)^2\cdot S\big)\cdot\mathrm{ord}_C\big(N(u)\big|_S\big)du+ \\ +\frac{3}{(-K_X)^3}\int_0^\tau\int_0^\infty\mathrm{vol}\big(P(u)\big|_S-vC\big)dvdu.$$

Note that $C\not\subset\mathrm{Supp}(N(u,v))$ for every $u\in[0,\tau)$ and $v\in(0,t(u))$. Thus, we can let

$$F_P\big(W^{S,C}_{\bullet,\bullet,\bullet}\big)=\frac{6}{(-K_X)^3}\int_0^\tau\int_0^{t(u)}\big(P(u,v)\cdot C\big)\cdot\mathrm{ord}_P\big(N'_S(u)\big|_C+N(u,v)\big|_C\big)dvdu.$$

Finally, we let

$$S\big(W^{S,C}_{\bullet,\bullet,\bullet};P\big)=\frac{3}{(-K_X)^3}\int_0^\tau\int_0^{t(u)}\big(P(u,v)\cdot C\big)^2dvdu+F_P\big(W^{S,C}_{\bullet,\bullet,\bullet}\big).$$

Then [2, Theorem 1.7.30] gives

Corollary 13. *One has*

$$(\bigstar)\qquad \frac{A_X(\mathbf{F})}{S_X(\mathbf{F})}\geqslant\delta_P(X)\geqslant\min\left\{\frac{1}{S(W^{S,C}_{\bullet,\bullet,\bullet};P)},\frac{1}{S(W^{S}_{\bullet,\bullet};C)},\frac{1}{S_X(S)}\right\}.$$

Moreover, if both inequalities in $(\bigstar)$ *are equalities and* $\mathfrak{C}=P$, *then* $\delta_P(X)=\frac{1}{S_X(S)}$.

Let us show how to compute $S(W^{S}_{\bullet,\bullet};C)$ and $S(W^{S,C}_{\bullet,\bullet,\bullet};P)$ in some cases.

Lemma 14. *Suppose that* $\omega(P)\notin\Delta_{\mathbb{P}^1\times\mathbb{P}^1}$. *Let* S *be a general surface in* $|H_1+H_2|$ *that contains* P, *and let* C *be the fiber of the morphism* ω *containing* P. *Then* $S(W^{S}_{\bullet,\bullet};C)=\frac{31}{36}$ *and* $S(W^{S,C}_{\bullet,\bullet,\bullet};P)=1$.

Proof. We have $\tau=1$. Moreover, for $u\in[0,1]$, we have $N(u)=0$ and $P(u)|_S=-K_S+2(1-u)C$. On the other hand, it follows from Lemma 7 that S is a smooth del Pezzo surface of degree 2, and the restriction map $\pi_3|_S\colon S\to\mathbb{P}^2_{x,y,z}$ is a double cover that is ramified over a smooth quartic curve. Therefore, applying the Galois involution of this double cover to C, we obtain another smooth irreducible curve $Z\subset S$ such that $C+Z\sim-2K_S$, $C^2=Z^2=0$ and $C\cdot Z=4$, which gives

$$P(u)|_S-vC\sim_{\mathbb{R}}\Big(\frac{5}{2}-2u-v\Big)C+\frac{1}{2}Z.$$

Then $P(u)|_S-vC$ is pseudoeffective $\iff$ $P(u)|_S-vC$ is nef $\iff$ $v\leqslant\frac{5}{2}-2u$. Thus, we have

$$\mathrm{vol}\big(P(u)|_S-vC\big)=\big(-K_S+2(1-u)C\big)^2=10-8u-4v$$

and $P(u,v)\cdot C=2$. Now, integrating, we get $S(W^{S}_{\bullet,\bullet};C)=\frac{31}{36}$ and $S(W^{S,C}_{\bullet,\bullet,\bullet};P)=1$. ☐

Lemma 15. *Suppose that* $P\notin E_1\cup E_2$. *Let* S *be a general surface in* $|H_3|$ *that contains* P, *and let* C *be the fiber of the morphism* π_3 *containing* P. *Suppose that* S *is a smooth del Pezzo surface. Then* $S(W^{S}_{\bullet,\bullet};C)=\frac{7}{9}$ *and* $S(W^{S,C}_{\bullet,\bullet,\bullet};P)=1$.

Proof. We have $\tau=1$. Moreover, for $u\in[0,1]$, we have $N(u)=0$ and $P(u)|_S=-K_S+(1-u)C$. Since S is a smooth del Pezzo surface, the restriction map $\omega|_S\colon S\to\mathbb{P}^1_{s,t}\times\mathbb{P}^1_{u,v}$ is a double cover ramified over a smooth elliptic curve. Therefore, using the Galois involution of this double cover, we get an irreducible curve $Z\subset S$ such that $C+Z\sim-K_S$, $C^2=Z^2=0$, $C\cdot Z=2$, which gives

$$P(u)|_S-vC\sim_{\mathbb{R}}(2-u-v)C+Z.$$

Then $P(u)|_S-vC$ is pseudoeffective $\iff$ $P(u)|_S-vC$ is nef $\iff$ $v\leqslant 2-u$. Thus, we have

$$\mathrm{vol}\big(P(u)|_S-vC\big)=\big(-K_S+(1-u)C\big)^2=8-4u-4v$$

and $P(u,v)\cdot C=2$. Now, integrating, we obtain $S(W^{S}_{\bullet,\bullet};C)=\frac{7}{9}$ and $S(W^{S,C}_{\bullet,\bullet,\bullet};P)=1$. ☐

Lemma 16. *Suppose that* $P\notin E_1\cup E_2$. *Let* S *be a general surface in* $|H_3|$ *that contains* P, *and let* C *be the fiber of the morphism* π_3 *containing* P. *Suppose* S *is not a smooth del Pezzo surface. Then* $S(W^{S}_{\bullet,\bullet};C)=\frac{8}{9}$ *and* $S(W^{S,C}_{\bullet,\bullet,\bullet};P)=\frac{7}{9}$.

Proof. We have $\tau=1$. Moreover, for $u\in[0,1]$, we have $N(u)=0$ and $P(u)|_S=-K_S+(1-u)C$. It follows from Lemma 6 that S contains two (-2)-curves $\mathbf{e}_1$ and $\mathbf{e}_2$ such that $-K_S\sim 2C+\mathbf{e}_1+\mathbf{e}_2$. On the surface S, we have $C^2=0$, $C\cdot\mathbf{e}_1=C\cdot\mathbf{e}_2=1$, $\mathbf{e}_1^2=\mathbf{e}_2^2=-2$, and

$$P(u)|_S-vC\sim_{\mathbb{R}}(3-u-v)C+\mathbf{e}_1+\mathbf{e}_2.$$

Then $P(u)|_S - vC$ is pseudoeffective $\iff$ $v \leqslant 3-u$. Moreover, we have

$$P(u,v) = \begin{cases} (3-u-v)C + \mathbf{e}_1 + \mathbf{e}_2 \text{ if } 0 \leqslant v \leqslant 1-u, \\ \dfrac{3-u-v}{2}\big(2C + \mathbf{e}_1 + \mathbf{e}_2\big) \text{ if } 1-u \leqslant v \leqslant 3-u, \end{cases}$$

$$N(u,v) = \begin{cases} 0 \text{ if } 0 \leqslant v \leqslant 1-u, \\ \dfrac{u+v-1}{2}(\mathbf{e}_1 + \mathbf{e}_2) \text{ if } 1-u \leqslant v \leqslant 3-u, \end{cases}$$

$$\mathrm{vol}\big(P(u)|_S - vC\big) = \begin{cases} 8-4u-4v \text{ if } 0 \leqslant v \leqslant 1-u, \\ (u+v-3)^2 \text{ if } 1-u \leqslant v \leqslant 3-u. \end{cases}$$

Now, integrating $\mathrm{vol}(P(u)|_S - vC)$, we obtain $S(W^S_{\bullet,\bullet}; C) = \frac{8}{9}$.

To compute $S(W^{S,C}_{\bullet,\bullet,\bullet}; P)$, observe that $F_P(W^{S,C}_{\bullet,\bullet,\bullet}) = 0$, because $P \not\in \mathbf{e}_1 \cup \mathbf{e}_2$, since S is a general surface in $|H_3|$ that contains C. On the other hand, we have

$$P(u,v) \cdot C = \begin{cases} 2 \text{ if } 0 \leqslant v \leqslant 1-u, \\ 3-u-v \text{ if } 1-u \leqslant v \leqslant 3-u. \end{cases}$$

Hence, integrating $(P(u,v)\cdot C)^2$, we get $S(W^{S,C}_{\bullet,\bullet,\bullet}; P) = \frac{7}{9}$ as required. □

Lemma 17. *Suppose $P \in (E_1 \cup E_2) \setminus (E_1 \cap E_2)$. Let S be a general surface in $|H_3|$ that contains P, let C be the irreducible component of the fiber of the conic bundle π_3 containing P such that $P \in C$. Then $S(W^S_{\bullet,\bullet}; C) = 1$ and $S(W^{S,C}_{\bullet,\bullet,\bullet}; P) \leqslant \frac{31}{36}$.*

Proof. We have $\tau = 1$. For $u \in [0,1]$, we have $N(u) = 0$ and $P(u)|_S \sim_{\mathbb{R}} -K_S + (1-u)(C + C')$, where C' is the irreducible curve in S such that $C + C'$ is the fiber of the conic bundle π_3 that passes through the point P. Since $P \not\in E_1 \cap E_2$, we see that $P \not\in C'$.

By Lemma 6, the surface S is a smooth del Pezzo surface of degree 4, so we can identify it with a complete intersection of two quadrics in $\mathbb{P}^4$. Then C and C' are lines in S, and S contains four additional lines that intersect C. Denote them by L_1, L_2, L_3, L_4, and let $Z = L_1 + L_2 + L_3 + L_4$. Then the intersections of the curves C, C' and Z on the surface S are given in the table below.

$\bullet$	C	C'	Z
C	-1	1	4
C'	1	-1	0
Z	4	0	-4

Observe that $-K_S \sim_{\mathbb{Q}} \frac{3}{2}C + \frac{1}{2}C' + \frac{1}{2}Z$. This gives $P(u)|_S - vC \sim_{\mathbb{R}} (\frac{5}{2} - u - v)C + (\frac{3}{2} - u)C' + \frac{1}{2}Z$, which implies that $P(u)|_S - vC$ is pseudoeffective $\iff$ $v \leqslant \frac{5}{2} - u$.

Moreover, we have

$$P(u,v) = \begin{cases} \Big(\dfrac{5}{2} - u - v\Big)C + \Big(\dfrac{3}{2} - u\Big)C' + \dfrac{1}{2}Z \text{ if } 0 \leqslant v \leqslant 1, \\ \Big(\dfrac{5}{2} - u - v\Big)(C + C') + \dfrac{1}{2}Z \text{ if } 1 \leqslant v \leqslant 2-u, \\ \Big(\dfrac{5}{2} - u - v\Big)(C + C' + Z) \text{ if } 2-u \leqslant v \leqslant \dfrac{5}{2} - u, \end{cases}$$

$$N(u,v) = \begin{cases} 0 \text{ if } 0 \leqslant v \leqslant 1, \\ (v-1)C' \text{ if } 1 \leqslant v \leqslant 2-u, \\ (v-1)C' + (v+u-2)Z \text{ if } 2-u \leqslant v \leqslant \dfrac{5}{2} - u, \end{cases}$$

$$P(u,v)\cdot C=\begin{cases}1+v \text{ if } 0\leqslant v\leqslant 1,\\ 2 \text{ if } 1\leqslant v\leqslant 2-u,\\ 10-4u-4v \text{ if } 2-u\leqslant v\leqslant \frac{5}{2}-u,\end{cases}$$

$$\mathrm{vol}\big(P(u)|_S-vC\big)=\begin{cases}8-v^2-4u-2v \text{ if } 0\leqslant v\leqslant 1,\\ 9-4u-4v \text{ if } 1\leqslant v\leqslant 2-u,\\ (5-2u-2v)^2 \text{ if } 2-u\leqslant v\leqslant \frac{5}{2}-u.\end{cases}$$

Now, integrating $\mathrm{vol}(P(u)|_S-vC)$ and $(P(u,v)\cdot C)^2$, we get $S(W^S_{\bullet,\bullet};C)=1$ and

$$S\big(W^{S,C}_{\bullet,\bullet,\bullet};P\big)=\frac{5}{6}+F_P\big(W^{S,C}_{\bullet,\bullet,\bullet}\big)=\frac{5}{6}+\frac{1}{3}\int_0^1\int_0^{\frac{5}{2}-u}\big(P(u,v)\cdot C\big)\cdot\mathrm{ord}_P\big(N(u,v)\big|_C\big)dvdu\leqslant$$
$$\leqslant\frac{5}{6}+\frac{1}{3}\int_0^1\int_2^{\frac{5}{2}-u}(10-4u-4v)(v+u-2)dvdu=\frac{31}{36},$$

because $P\not\in C'$, and the curves Z and C intersect each other transversally. □

4. The proof of Main Theorem

Let us use notations and assumptions of Sections 2 and 3. Recall that $\mathbf{F}$ is a prime divisor over the threefold X, and $\mathfrak{C}$ is its center in X. To prove Main Theorem, we must show that $\beta(\mathbf{F})>0$.

Lemma 18. *Suppose that $\mathfrak{C}$ is a curve. Then $\beta(\mathbf{F})>0$.*

Proof. Suppose $\beta(\mathbf{F})\leqslant 0$. Then $\delta_P(X)\leqslant 1$ for every point $P\in\mathfrak{C}$. Let us seek for a contradiction.

Let S_1 be a general surface in the linear system $|H_1|$. Then S_1 is smooth. Hence, if $S_1\cap\mathfrak{C}\neq\varnothing$, then $\delta_P(X)\leqslant 1$ for every point $P\in S_1\cap\mathfrak{C}$, which contradicts Corollary 11. We see that $S_1\cdot\mathfrak{C}=0$. Similarly, we see that $S_2\cdot\mathfrak{C}=0$. Therefore, we see that $\omega(\mathfrak{C})$ is a point.

Let C be the scheme fiber of the conic bundle ω over the point $\omega(\mathfrak{C})$. Then $\mathfrak{C}$ is an irreducible component of the curve C. If the fiber C is smooth, then we $\mathfrak{C}=C$.

Suppose that C is smooth. If S is a general surface in the linear system $|H_1+H_2|$ that contains $\mathfrak{C}$, then $S(W^S_{\bullet,\bullet};\mathfrak{C})=\frac{31}{36}<1$ by Lemma 14, which contradicts Corollary 8. So, the curve C is singular.

Note that $\pi_3(\mathfrak{C})$ is a line in $\mathbb{P}^2_{x,y,z}$. On the other hand, the discriminant curve $\Delta_{\mathbb{P}^2}$ is an irreducible smooth quartic curve in $\mathbb{P}^2_{x,y,z}$. Therefore, in particular, the line $\pi_3(\mathfrak{C})$ is not contained in $\Delta_{\mathbb{P}^2}$. Now, let P be a general point in $\mathfrak{C}$, let Z be the fiber of the conic bundle π_3 that passes through P, and let S be a general surface in $|H_3|$ that contains the curve Z. Then Z and S are both smooth, and it follows from Lemma 6 that S is a del Pezzo of degree 4, so that $\delta_P(X)>1$ by Lemma 9. □

Hence, to complete the proof of Main Theorem, we may assume that $\mathfrak{C}$ is a point. Set $P=\mathfrak{C}$. Let $\mathscr{C}$ be the fiber of the conic bundle ω that contains P.

Lemma 19. *Suppose that $P\not\in E_1\cap E_2$. Then $\beta(\mathbf{F})>0$.*

Proof. Apply Lemmas 15, 16, 17 and Corollary 13. □

Thus, to complete the proof of Main Theorem, we may assume, in addition, that $P\in E_1\cap E_2$. Then the conic $\mathscr{C}$ is smooth at P by Lemma 5. In particular, we see that $\mathscr{C}$ is reduced.

Lemma 20. *Suppose that $\mathscr{C}$ is smooth. Then $\beta(\mathbf{F})>0$.*

Proof. Apply Lemma 14 and Corollary 13. □

To complete the proof of Main Theorem, we may assume that $\mathscr{C}$ is singular. Write $\mathscr{C} = \ell_1 + \ell_2$, where ℓ_1 and ℓ_2 are irreducible components of the conic $\mathscr{C}$. Then $P \neq \ell_1 \cap \ell_2$, since $P \notin \mathrm{Sing}(\mathscr{C})$.

Let S_1 and S_2 be general surfaces in $|H_1|$ and $|H_2|$ that passes through the point P, respectively. Then $\mathscr{C} = S_1 \cap S_2$, and it follows from Corollary 4 that S_1 or S_2 is smooth along the conic $\mathscr{C}$. Without loss of generality, we may assume that S_1 is smooth along $\mathscr{C}$. We let $S = S_1$.

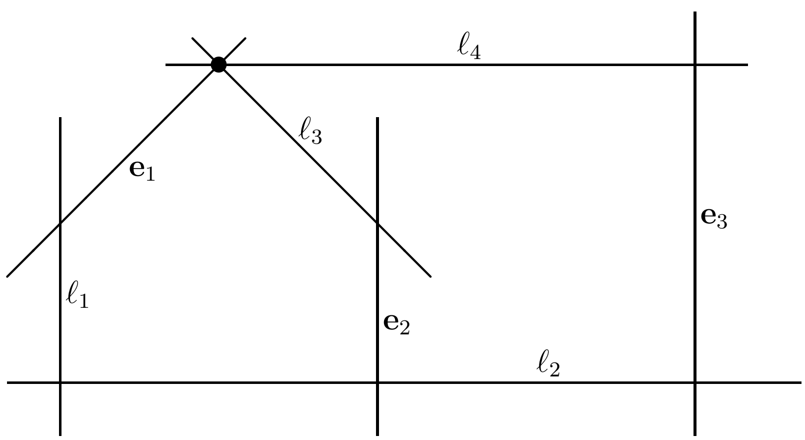
If S is smooth, then $\delta_P(X) > 1$ by Corollary 11. Thus, we may assume that S is singular.

Recall that S is a quintic del Pezzo surface, and ℓ_1 and ℓ_2 are lines in its anticanonical embedding. The preimages of the lines ℓ_1 and ℓ_2 on the minimal resolution of the surface S are (-1)-curves, which do not intersect (-2)-curves. By Lemma 1 and Remark 2, one of the following cases holds:

($\mathbb{A}_1$) the surface S has one singular point of type $\mathbb{A}_1$,
($2\mathbb{A}_1$) the surface S has two singular points of type $\mathbb{A}_1$.

In both cases, the restriction morphism $\pi_3|_S\colon S \to \mathbb{P}^2_{x,y,z}$ is birational. In ($\mathbb{A}_1$)-case, this morphism contracts three disjoint irreducible smooth rational curves $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ such that $E_1|_S = 2\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$, the curves $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are sections of the conic bundle $\pi_2|_S\colon S \to \mathbb{P}^1_{u,v}$, the curve $\mathbf{e}_1$ passes through the singular point of the surface S, but $\mathbf{e}_2$ and $\mathbf{e}_3$ are contained in the smooth locus of the surface S. In ($2\mathbb{A}_1$)-case, the morphism $\pi_3|_S$ contracts two disjoint curves $\mathbf{e}_1$ and $\mathbf{e}_2$ such that $E_1\big|_S = 2\mathbf{e}_1 + 2\mathbf{e}_2$, the curves $\mathbf{e}_1$ and $\mathbf{e}_2$ are sections of the conic bundle $\pi_2|_S$, and each curve among $\mathbf{e}_1$ and $\mathbf{e}_2$ contains one singular point of the surface S. In both cases, we may assume that $\ell_1 \cap \mathbf{e}_1 \neq \varnothing$.

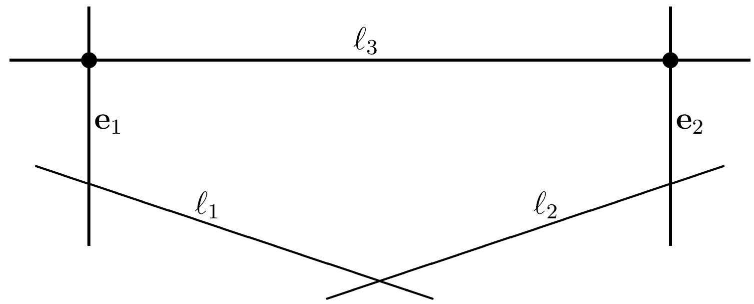
Let us identify the surface S with its image in $\mathbb{P}^5$ via the anticanonical embedding $S \hookrightarrow \mathbb{P}^5$. Then ℓ_1 and ℓ_2 and the curves contracted by $\pi_3|_S$ are lines. In ($\mathbb{A}_1$)-case, the surface S contains two additional lines ℓ_3 and ℓ_4 such that $\ell_3 + \ell_4 \sim \ell_1 + \ell_2$, the intersection $\ell_3 \cap \ell_4$ is the singular point of the surface S, and the intersection graph of the lines ℓ_1, ℓ_2, ℓ_3, ℓ_4, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ is shown here:



In this picture, we denoted by $\bullet$ the singular point of the surface S. Moreover, on the surface S, the intersections of the lines ℓ_1, ℓ_2, ℓ_3, ℓ_4, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are given in the table below.

$\bullet$	ℓ_1	ℓ_2	ℓ_3	ℓ_4	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$
ℓ_1	-1	1	0	0	1	0	0
ℓ_2	1	-1	0	0	0	1	1
ℓ_3	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	0
ℓ_4	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	0	1
$\mathbf{e}_1$	1	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	0
$\mathbf{e}_2$	0	1	1	0	0	-1	0
$\mathbf{e}_3$	0	1	0	1	0	0	-1

Likewise, in $(2\mathbb{A}_1)$-case, the surface S contains one additional lines ℓ_3 such that $2\ell_3 \sim \ell_1 + \ell_2$, the line ℓ_3 passes through both singular points of the del Pezzo surface S, and the intersection graph of the lines on the surface S is shown in the following picture:



As above, singular points of the surface S are denote by $\bullet$. The intersections of the lines ℓ_1, ℓ_2, ℓ_3, $\mathbf{e}_1$, $\mathbf{e}_2$ on the surface S are given in the table below.

$\bullet$	ℓ_1	ℓ_2	ℓ_3	$\mathbf{e}_1$	$\mathbf{e}_2$
ℓ_1	-1	1	0	1	0
ℓ_2	1	-1	0	0	1
ℓ_3	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$
$\mathbf{e}_1$	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
$\mathbf{e}_2$	0	1	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$

Remark 21. By [3, Lemma 2.9], the lines in S generate the group $\mathrm{Cl}(S)$ and the cone of effective divisors $\mathrm{Eff}(S)$, and every extremal ray of the Mori cone $\overline{\mathrm{NE}}(S)$ is generated by the class of a line.

In $(\mathbb{A}_1)$-case, the point P is one of the points $\mathbf{e}_1 \cap \ell_1$, $\mathbf{e}_2 \cap \ell_2$ or $\mathbf{e}_3 \cap \ell_2$, because $P \in E_1 \cap E_2$. On the other hand, if $P = \mathbf{e}_2 \cap \ell_2$ or $P = \mathbf{e}_3 \cap \ell_2$, it follows from Corollary 12 that $\delta_P(X) > 1$. In $(2\mathbb{A}_1)$-case, either $P = \mathbf{e}_1 \cap \ell_1$ or $P = \mathbf{e}_2 \cap \ell_2$. Therefore, to complete the proof of Main Theorem, we may assume that $P = \mathbf{e}_1 \cap \ell_1$ in both cases.

Now, we will apply Corollary 13 to the surface S with $C = \mathbf{e}_1$ at the point P. We have $\tau = \frac{3}{2}$. As in the proof of Corollary 10, we see that

$$P(u) = \begin{cases} (1-u)H_1 + H_2 + H_3 \text{ if } 0 \leqslant u \leqslant 1, \\ (2-u)H_2 + (3-2u)H_3 \text{ if } 1 \leqslant u \leqslant \dfrac{3}{2}, \end{cases}$$

and

$$N(u) = \begin{cases} 0 \text{ if } 0 \leqslant u \leqslant 1, \\ (u-1)E_2 \text{ if } 1 \leqslant u \leqslant \dfrac{3}{2}. \end{cases}$$

Since $H_1|_S \sim 0$, $H_2|_S \sim \ell_1 + \ell_2$, $H_3|_S \sim \ell_1 + 2\mathbf{e}_1$, we have

$$P(u)\big|_S - v\mathbf{e}_1 \sim_{\mathbb{R}} \begin{cases} (2-v)\mathbf{e}_1 + 2\ell_1 + \ell_2 \text{ if } 0 \leqslant u \leqslant 1, \\ (6-4u-v)\mathbf{e}_1 + (5-3u)\ell_1 + (2-u)\ell_2 \text{ if } 1 \leqslant u \leqslant \dfrac{3}{2}. \end{cases}$$

Thus, since the intersection form of the curves ℓ_1 and ℓ_2 is semi-negative definite, we get

$$t(u) = \begin{cases} 2 \text{ if } 0 \leqslant u \leqslant 1, \\ 6-4u \text{ if } 1 \leqslant u \leqslant \dfrac{3}{2}. \end{cases}$$

Similarly, if $0 \leqslant u \leqslant 1$, then

$$P(u,v) = \begin{cases} (2-v)\mathbf{e}_1 + 2\ell_1 + \ell_2 \text{ if } 0 \leqslant v \leqslant 1, \\ (2-v)\mathbf{e}_1 + (3-v)\ell_1 + \ell_2 \text{ if } 1 \leqslant v \leqslant 2, \end{cases}$$

$$N(u,v) = \begin{cases} 0 \text{ if } 0 \leqslant v \leqslant 1, \\ (v-1)\ell_1 \text{ if } 1 \leqslant v \leqslant 2, \end{cases}$$

$$P(u,v)\cdot\mathbf{e}_1 = \begin{cases} \dfrac{v+2}{2} \text{ if } 0 \leqslant v \leqslant 1, \\ \dfrac{4-v}{2} \text{ if } 1 \leqslant v \leqslant 2, \end{cases}$$

$$\mathrm{vol}\big(P(u)\big|_S - v\mathbf{e}_1\big) = \begin{cases} \dfrac{10-4v-v^2}{2} \text{ if } 0 \leqslant v \leqslant 1, \\ \dfrac{(2-v)(6-v)}{2} \text{ if } 1 \leqslant v \leqslant 2. \end{cases}$$

Likewise, if $1 \leqslant u \leqslant \frac{3}{2}$, then

$$P(u,v) = \begin{cases} (6-4u-v)\mathbf{e}_1 + (5-3u)\ell_1 + (2-u)\ell_2 \text{ if } 0 \leqslant v \leqslant 3-2u, \\ (6-4u-v)\mathbf{e}_1 + (8-5u-v)\ell_1 + (2-u)\ell_2 \text{ if } 3-2u \leqslant v \leqslant 6-4u, \end{cases}$$

$$N(u,v) = \begin{cases} 0 \text{ if } 0 \leqslant v \leqslant 3-2u, \\ (v+2u-3)\ell_1 \text{ if } 3-2u \leqslant v \leqslant 6-4u, \end{cases}$$

$$P(u,v)\cdot\mathbf{e}_1 = \begin{cases} \dfrac{4+v-2u}{2} \text{ if } 0 \leqslant v \leqslant 3-2u, \\ \dfrac{10-6u-v}{2} \text{ if } 3-2u \leqslant v \leqslant 6-4u, \end{cases}$$

$$\mathrm{vol}\big(P(u)\big|_S - v\mathbf{e}_1\big) = \begin{cases} \dfrac{66+24u^2+4uv-v^2-80u-8v}{2} \text{ if } 0 \leqslant v \leqslant 3-2u, \\ \dfrac{(6-4u-v)(14-8u-v)}{2} \text{ if } 3-2u \leqslant v \leqslant 6-4u. \end{cases}$$

Integrating, we get $S(W^S_{\bullet,\bullet};\mathbf{e}_1) = \frac{137}{144}$ and $S(W^{S,\mathbf{e}_1}_{\bullet,\bullet,\bullet};P) = \frac{59}{96} + F_P(W^{S,\mathbf{e}_1}_{\bullet,\bullet,\bullet})$. To compute $F_P(W^{S,\mathbf{e}_1}_{\bullet,\bullet,\bullet})$, we let $Z = E_2|_S$. Then Z is a smooth curve of genus 3 such that $\pi(Z)$ is a smooth quartic in $\mathbb{P}^2_{x,y,z}$. Moreover, the curve Z is contained in the smooth locus of the surface S, and

$$Z \sim \begin{cases} 4\mathbf{e}_1 + \ell_3 + \ell_4 + 2\ell_1 \text{ in } (\mathbb{A}_1)\text{-case}, \\ 2\ell_1 + 2\ell_2 + 2\mathbf{e}_1 + 2\mathbf{e}_2 \text{ in } (2\mathbb{A}_1)\text{-case}. \end{cases}$$

In particular, we have $Z\cdot\mathbf{e}_1 = 1$. Since $\mathbf{e}_1 \not\subset Z$, we have

$$N'_S(u) = \begin{cases} 0 \text{ if } 0 \leqslant u \leqslant 1, \\ (u-1)Z \text{ if } 1 \leqslant u \leqslant \dfrac{3}{2}. \end{cases}$$

Note that $P\in Z$, because $P\in E_1\cap E_2$. Thus, since $\mathbf{e}_1\cdot Z=1$ and $\mathbf{e}_1\cdot\ell_1=1$, we have

$$F_P\big(W^{S,\mathbf{e}_1}_{\bullet,\bullet,\bullet}\big)=\frac{1}{3}\int_1^{\frac{3}{2}}\int_0^{6-4u}\big(P(u,v)\cdot\mathbf{e}_1\big)(u-1)dvdu+\frac{1}{3}\int_0^{\frac{3}{2}}\int_0^{t(u)}\big(P(u,v)\cdot\mathbf{e}_1\big)\big(N(u,v)\cdot\mathbf{e}_1\big)dvdu=$$
$$+\frac{1}{3}\int_1^{\frac{3}{2}}\int_0^{3-2u}\frac{(4+v-2u)(u-1)}{2}dvdu+\frac{1}{3}\int_1^{\frac{3}{2}}\int_{3-2u}^{6-4u}\frac{(10-6u-v)(u-1)}{2}dvdu+$$
$$+\frac{1}{3}\int_0^{1}\int_1^{2}\frac{(4-v)(v-1)}{2}dvdu+\frac{1}{3}\int_1^{\frac{3}{2}}\int_{3-2u}^{6-4u}\frac{(10-6u-v)(v+2u-3)}{2}dvdu=\frac{71}{288},$$

so that $S(W^{S,\mathbf{e}_1}_{\bullet,\bullet,\bullet};P)=\frac{31}{36}$. Now, applying Corollary 13, we get $\delta_P(X)>1$, because $S_X(S)<1$. Therefore, we see that $\beta(\mathbf{F})>0$. By [5, 7], this completes the proof of Main Theorem.

Remark 22. Instead of using Corollary 13, we can finish the proof of Main Theorem as follows. Let F be a divisor over S such that $P\in C_S(F)$, and let $\mathcal{C}$ be a fiber of the conic bundle $\pi_2|_S$. Then, arguing as in the proof of Corollary 10, we get

$$S\big(W^S_{\bullet,\bullet};F\big)\leqslant\left(\frac{7}{288}+\frac{5}{6\delta_P(S)}\right)A_S(F)+\frac{1}{6}\int_1^{\frac{3}{2}}\int_0^{\infty}\mathrm{vol}\big((2-u)\mathcal{C}+(3-2u)H_3\big|_S-vF\big)dvdu.$$

But $\delta_P(S)=1$ by Lemmas 24 and 25, since $P=\mathbf{e}_1\cap\ell_1$. Thus, we have

$$(\heartsuit)\quad S\big(W^S_{\bullet,\bullet};F\big)\leqslant\frac{247}{288}A_S(F)+\frac{1}{6}\int_1^{\frac{3}{2}}\int_0^{\infty}\mathrm{vol}\big((2-u)\mathcal{C}+(3-2u)H_3\big|_S-vF\big)dvdu=$$
$$=\frac{247}{288}A_S(F)+\frac{1}{6}\int_1^{\frac{3}{2}}(3-2u)^3\int_0^{\infty}\mathrm{vol}\left(\frac{2-u}{3-2u}\mathcal{C}+H_3\big|_S-vF\right)dvdu=$$
$$=\frac{247}{288}A_S(F)+\frac{1}{6}\int_1^{\frac{3}{2}}(3-2u)^3\int_0^{\infty}\mathrm{vol}\left(-K_S+\frac{u-1}{3-2u}\mathcal{C}-vF\right)dvdu.$$

Set $L=-K_S+t\mathcal{C}$ for $t\in\mathbb{R}_{\geqslant0}$. Then L is ample and $L^2=5+4t$. Define $\delta_P(S,L)$ as in Appendix A. Then, applying [2, Corollary 1.7.24] to the flag $P\in\mathbf{e}_1\subset S$, we get

$$\delta_P(S,L)\geqslant\begin{cases}1\text{ if }0\leqslant t\leqslant\dfrac{-3+\sqrt{21}}{6},\\[2mm] \dfrac{15+12t}{6t^2+18t+13}\text{ if }\dfrac{-3+\sqrt{21}}{6}\leqslant t.\end{cases}$$

The proof of this inequality is very similar to our computations of $S(W^S_{\bullet,\bullet};\mathbf{e}_1)$ and $S(W^{S,\mathbf{e}_1}_{\bullet,\bullet,\bullet};P)$, so that we omit the details. Now, we let $t=\frac{u-1}{3-2u}$. Then $t\geqslant\frac{-3+\sqrt{21}}{6}\iff u\geqslant\frac{3}{2}(1-\frac{1}{\sqrt{21}})$, so

$$\frac{1}{6}\int_1^{\frac{3}{2}}(3-2u)^3\int_0^{\infty}\mathrm{vol}\big(-K_S+t\mathcal{C}-vF\big)dvdu=$$
$$=\frac{1}{6}\int_1^{\frac{3}{2}}(3-2u)^3(5+4t)S_L(F)du\leqslant\frac{1}{6}\int_1^{\frac{3}{2}(1-\frac{1}{\sqrt{21}})}(3-2u)^3(5+4t)A_S(F)du+$$
$$+\frac{1}{6}\int_{\frac{3}{2}(1-\frac{1}{\sqrt{21}})}^{\frac{3}{2}}(3-2u)^3(5+4t)\frac{15+12t}{6t^2+18t+13}A_S(F)du=\frac{247}{2016}A_S(F).$$

Now, using ($\heartsuit$), we get $S(W^S_{\bullet,\bullet};F)\leqslant\frac{247}{288}A_S(F)+\frac{247}{2016}A_S(F)=\frac{247}{252}A_S(F)$. Then $\delta_P(S;W^S_{\bullet,\bullet})\geqslant\frac{252}{247}$, so that $\delta_P(X)>1$ by (3.1), since $S_X(S)<1$ by [6, Theorem 10.1].

Appendix A. δ-invariants of del Pezzo surfaces

In this appendix, we present three rather sporadic results about δ-invariants of del Pezzo surfaces with at most du Val singularities, which are used in the proof of Main Theorem.

Let S be a del Pezzo surface that has at most du Val singularities, let L be an ample $\mathbb{R}$-divisor on the surface S, and let P be a point in S. Set

$$\delta_P(S,L)=\inf_{\substack{F/S\\ P\in C_S(F)}}\frac{A_S(F)}{S_L(F)},$$

where infimum is taken over all prime divisors F over S such that $P\in C_S(F)$, and

$$S_L(F)=\frac{1}{L^2}\int_0^\infty \mathrm{vol}\big(L-uF\big)du.$$

It would be nice to find an explicit formula for $\delta_P(S,L)$. But this problem seems to be very difficult. So, we will only estimate $\delta_P(S,L)$ in thee very special cases when $K_S^2\in\{4,5\}$.

Suppose that $4\leqslant K_S^2\leqslant 5$. Let us identify S with its image in the anticanionical embedding.

Lemma 23. *Suppose that $K_S^2=4$. Let C be a possibly reducible conic in S that passes through P, and let $L=-K_S+tC$ for $t\in\mathbb{R}_{\geqslant 0}$. If the conic C is smooth, then*

$$(\clubsuit)\qquad \delta_P(S,L)\geqslant\begin{cases}\dfrac{24}{19+8t+t^2} & \text{if } 0\leqslant t\leqslant 1,\\ \dfrac{6(1+t)}{5+6t+3t^2} & \text{if } t\geqslant 1.\end{cases}$$

Similarly, if C is a reducible conic, then

$$(\spadesuit)\qquad \delta_L(S,L)\geqslant\frac{24(1+t)}{19+30t+12t^2}.$$

Proof. The proof of this lemma is similar to the proof of [2, Lemma 2.12]. Namely, as in that proof, we will apply [2, Theorem 1.7.1], [2, Corollary 1.7.12], [2, Corollary 1.7.25] to get ($\clubsuit$) and ($\spadesuit$). Let us use notations introduced in [2, § 1] applied to S polarized by the ample divisor L.

First, we suppose that P is not contained in any line in S. In particular, the conic C is smooth. Let $\sigma\colon\widetilde{S}\to S$ be the blowup of the point P, let E be the exceptional curve of the blow up σ, and let $\widetilde{C}$ be the proper transform on $\widetilde{S}$ of the conic C. Then $\widetilde{S}$ is a smooth cubic surface in $\mathbb{P}^3$, and there exists a unique line $\mathbf{l}\subset\widetilde{S}$ such that $-K_{\widetilde{S}}\sim\widetilde{C}+E+\mathbf{l}$. Take $u\in\mathbb{R}_{\geqslant 0}$. Then

$$\sigma^*(L)-uE\sim_{\mathbb{R}}(1+t)\widetilde{C}+(2+t-u)E+\mathbf{l},$$

which implies that $\sigma^*(L)-uE$ is pseudoeffective $\iff$ $u\leqslant 2+t$. Similarly, we see that

$$\mathscr{P}(u)\sim_{\mathbb{R}}\begin{cases}(1+t)\widetilde{C}+(2+t-u)E+\mathbf{l}\text{ if }0\leqslant u\leqslant 2,\\ (3+t-u)\widetilde{C}+(2+t-u)E+\mathbf{l}\text{ if }2\leqslant u\leqslant 2+t,\end{cases}$$

$$\mathscr{N}(u)=\begin{cases}0\text{ if }0\leqslant u\leqslant 2,\\ (u-2)\widetilde{C}\text{ if }2\leqslant u\leqslant 2+t,\end{cases}$$

$$\mathscr{P}(u)\cdot E=\begin{cases}u\text{ if }0\leqslant u\leqslant 2,\\ 2\text{ if }2\leqslant u\leqslant 2+t,\end{cases}$$

$$\mathrm{vol}\big(\sigma^*(L)-uE\big)=\begin{cases}4+4t-u^2\text{ if }0\leqslant u\leqslant 2,\\ 4(2+t-u)\text{ if }2\leqslant u\leqslant 2+t,\end{cases}$$

where we denote by $\mathscr{P}(u)$ the positive part of the Zariski decomposition of the divisor $\sigma^*(L)-uE$, and we denote by $\mathscr{N}(u)$ its negative part. This gives

$$S_L(E)=\frac{8+12t+3t^2}{6(1+t)}.$$

Moreover, applying [2, Corollary 1.7.25], we obtain

$$S(W^E_{\bullet,\bullet};Q)\leqslant\frac{4+6t+3t^2}{6(1+t)}$$

for every point $Q\in E$. Note that $A_S(E)=2$. Thus, it follows from [2, Corollary 1.7.12] that

$$\delta_P(S,L)\geqslant\frac{6(1+t)}{4+6t+3t^2}>\frac{24}{19+8t+t^2}.$$

To complete the proof of the lemma, we may assume that S contains a line ℓ such that $P\in\ell$. Then $\ell\cdot C=0$ or $\ell\cdot C=1$. If $\ell\cdot C=0$, then ℓ must be an irreducible component of the conic C. Let us apply [2, Theorem 1.7.1] and [2, Corollary 1.7.25] to the flag $P\in\ell$ to estimate $\delta_P(S,L)$. Take $u\in\mathbb{R}_{\geqslant 0}$. Let $P(u)$ be the positive part of the Zariski decomposition of the divisor $L-u\ell$, and let $N(u)$ be its negative part. We must compute $P(u)$, $N(u)$, $P(u)\cdot\ell$ and $\mathrm{vol}(L-u\ell)$,

There exists a birational morphism $\pi\colon S\to\mathbb{P}^2$ that blows up five points $O_1,\dots,O_5\in\mathbb{P}^2$ such that no three of them are collinear. For every $i\in\{1,\dots,5\}$, let $\mathbf{e}_i$ be the π-exceptional curve such that $\pi(\mathbf{e}_i)=O_i$. Similarly, let $\mathbf{l}_{ij}$ be the strict transform of the line in $\mathbb{P}^2$ that contains O_i and O_j, where $1\leqslant i<j\leqslant 5$. Finally, let B be the strict transform of the conic on $\mathbb{P}^2$ that passes through the points $O_1,\dots,O_5$. Then $\mathbf{e}_1,\dots,\mathbf{e}_5,\mathbf{l}_{12},\dots,\mathbf{l}_{45},B$ are all lines in S, and each extremal ray of the Mori cone $\overline{\mathrm{NE}}(S)$ is generated by a class of one of these 16 lines.

Suppose that the conic C is irreducible. Then $C\cdot\ell=1$. In this case, without loss of generality, we may assume that $\ell=\mathbf{e}_1$ and $C\sim\mathbf{l}_{12}+\mathbf{e}_2$. If $0\leqslant t\leqslant 1$, then

$$P(u)=\begin{cases}L-u\ell\text{ if }0\leqslant u\leqslant 1,\\ L-u\ell-(u-1)(\mathbf{l}_{12}+\mathbf{l}_{13}+\mathbf{l}_{14}+\mathbf{l}_{15})\text{ if }1\leqslant u\leqslant 1+t,\\ L-u\ell-(u-1)(\mathbf{l}_{12}+\mathbf{l}_{13}+\mathbf{l}_{14}+\mathbf{l}_{15})-(u-t-1)B\text{ if }1+t\leqslant u\leqslant\dfrac{3+t}{2},\end{cases}$$

$$N(u)=\begin{cases}0\text{ if }0\leqslant u\leqslant 1,\\ (u-1)(\mathbf{l}_{12}+\mathbf{l}_{13}+\mathbf{l}_{14}+\mathbf{l}_{15})\text{ if }1\leqslant u\leqslant 1+t,\\ (u-1)(\mathbf{l}_{12}+\mathbf{l}_{13}+\mathbf{l}_{14}+\mathbf{l}_{15})+(u-t-1)B\text{ if }1+t\leqslant u\leqslant\dfrac{3+t}{2},\end{cases}$$

$$P(u)\cdot\ell=\begin{cases}1+t+u\text{ if }0\leqslant u\leqslant 1,\\ 5+t-3u\text{ if }1\leqslant u\leqslant 1+t,\\ 6+2t-4u\text{ if }1+t\leqslant u\leqslant\dfrac{3+t}{2},\end{cases}$$

$$\mathrm{vol}\big(L-u\ell\big)=\begin{cases}4(1+t)-2u(1+t)-u^2\text{ if }0\leqslant u\leqslant 1,\\ (2-u)(4+2t-3u)\text{ if }1\leqslant u\leqslant 1+t,\\ (3+t-2u)^2\text{ if }1+t\leqslant u\leqslant\dfrac{3+t}{2},\end{cases}$$

and $L-u\ell$ is not pseudoeffective for $u>\frac{3+t}{2}$. Similarly, if $t\geqslant 1$, then

$$P(u)=\begin{cases}L-u\ell\text{ if }0\leqslant u\leqslant 1,\\ L-u\ell-(u-1)(\mathbf{l}_{12}+\mathbf{l}_{13}+\mathbf{l}_{14}+\mathbf{l}_{15})\text{ if }1\leqslant u\leqslant 2,\end{cases}$$

$$N(u)=\begin{cases}0 \text{ if } 0\leqslant u\leqslant 1,\\ (u-1)(\mathbf{l}_{12}+\mathbf{l}_{13}+\mathbf{l}_{14}+\mathbf{l}_{15}) \text{ if } 1\leqslant u\leqslant 2,\end{cases}$$

$$P(u)\cdot\ell=\begin{cases}1+t+u \text{ if } 0\leqslant u\leqslant 1,\\ 5+t-3u \text{ if } 1\leqslant u\leqslant 2,\end{cases}$$

$$\mathrm{vol}\big(L-u\ell\big)=\begin{cases}4(1+t)-2u(1+t)-u^2 \text{ if } 0\leqslant u\leqslant 1,\\ (2-u)(4+2t-3u) \text{ if } 1\leqslant u\leqslant 2,\end{cases}$$

and $L-u\ell$ is not pseudoeffective for $u>2$. Then

$$S_L\big(\ell\big)=\begin{cases}\dfrac{17+4t-t^2}{24} \text{ if } 0\leqslant t\leqslant 1,\\ \dfrac{2+3t}{3(1+t)} \text{ if } t\geqslant 1.\end{cases}$$

Observe that $P\not\in\mathbf{l}_{ij}$ for every $1\leqslant i<j\leqslant 5$. Thus, if $t\leqslant 1$, then [2, Corollary 1.7.25] gives

$$S(W^{\ell}_{\bullet,\bullet};P)=\begin{cases}\dfrac{19+8t+t^2}{24} \text{ if } P\in B,\\ \dfrac{9+15t+3t^2+t^3}{12(1+t)} \text{ if } P\not\in B.\end{cases}$$

Similarly, if $t\geqslant 1$, then [2, Corollary 1.7.25] gives

$$S\big(W^{\ell}_{\bullet,\bullet};P\big)=\frac{5+6t+3t^2}{6(1+t)}.$$

Now, using [2, Theorem 1.7.1], we get (♣).

To complete the proof of the lemma, we may assume that the conic C is reducible. In this case, we let ℓ be an irreducible component of the conic C that contains P. Without loss of generality, we may assume that $\ell=\mathbf{e}_1$ and $C=\mathbf{e}_1+B$. Then

$$P(u)=\begin{cases}L-u\ell \text{ if } 0\leqslant u\leqslant 1,\\ L-u\ell-(u-1)B \text{ if } 1\leqslant u\leqslant 1+t,\\ L-u\ell-(u-t-1)(\mathbf{l}_{12}+\mathbf{l}_{13}+\mathbf{l}_{14}+\mathbf{l}_{15})-(u-1)B \text{ if } 1+t\leqslant u\leqslant \dfrac{3+2t}{2},\end{cases}$$

$$N(u)=\begin{cases}0 \text{ if } 0\leqslant u\leqslant 1,\\ (u-1)B \text{ if } 1\leqslant u\leqslant 1+t,\\ (u-t-1)(\mathbf{l}_{12}+\mathbf{l}_{13}+\mathbf{l}_{14}+\mathbf{l}_{15})+(u-1)B \text{ if } 1+t\leqslant u\leqslant \dfrac{3+2t}{2},\end{cases}$$

$$P(u)\cdot\ell=\begin{cases}1+u \text{ if } 0\leqslant u\leqslant 1,\\ 2 \text{ if } 1\leqslant u\leqslant 1+t,\\ 6+4t-4u \text{ if } 1+t\leqslant u\leqslant \dfrac{3+2t}{2},\end{cases}$$

$$\mathrm{vol}\big(L-u\ell\big)=\begin{cases}4(1+t)-2u-u^2 \text{ if } 0\leqslant u\leqslant 1,\\ 5+4t-4u \text{ if } 1\leqslant u\leqslant 1+t,\\ (3+2t-2u)^2 \text{ if } 1+t\leqslant u\leqslant \dfrac{3+2t}{2},\end{cases}$$

and the divisor $L - u\ell$ is not pseudoeffective for $u > \frac{3+2t}{2}$. This gives

$$S_L\big(\ell\big) = \frac{17 + 30t + 12t^2}{24(1+t)}.$$

Moreover, using [2, Corollary 1.7.25], we compute

$$S\big(W^{\ell}_{\bullet,\bullet}; P\big) = \begin{cases} \dfrac{19 + 30t + 12t^2}{24(1+t)} \text{ if } P \in B, \\ \dfrac{19 + 24t}{24(1+t)} \text{ if } P \in \mathbf{l}_{12} \cup \mathbf{l}_{13} \cup \mathbf{l}_{14} \cup \mathbf{l}_{15}, \\ \dfrac{3 + 4t}{4(1+t)} \text{ otherwise.} \end{cases}$$

Now, using [2, Theorem 1.7.1], we get (♠) as claimed. □

In the remaining part of this appendix, we suppose that $K_S^2 = 5$, $L = -K_S$, and S has isolated ordinary double points, i.e. singular points of type $\mathbb{A}_1$. As usual, we set $\delta_P(S) = \delta_P(S, -K_S)$ and

$$\delta(S) = \inf_{P \in S} \delta_P(S).$$

Let $\eta\colon \widetilde{S} \to S$ be the minimal resolution of the quintic del Pezzo surface S. Since $-K_{\widetilde{S}} \sim \eta^*(-K_S)$, we can estimate the number $\delta_P(S)$ as follows. Let O be a point in the surface $\widetilde{S}$ such that $\eta(O) = P$, and let C be a smooth irreducible rational curve in $\widetilde{S}$ such that

- if $P \in \mathrm{Sing}(S)$, then C is the η-exceptional curve such that $\eta(C) = P$,
- if $P \notin \mathrm{Sing}(S)$, then C is appropriately chosen curve that contains O.

As usual, we set

$$\tau = \sup\Big\{u \in \mathbb{Q}_{\geqslant 0} \;\Big|\; \text{the divisor } -K_{\widetilde{S}} - uC \text{ is pseudo-effective}\Big\}.$$

For $u \in [0, \tau]$, let $P(u)$ be the positive part of the Zariski decomposition of the divisor $-K_{\widetilde{S}} - uC$, and let $N(u)$ be its negative part. Let

$$S_S(C) = \frac{1}{K_S^2}\int_0^{\infty} \mathrm{vol}\big(-K_{\widetilde{S}} - uC\big)du = \frac{1}{K_S^2}\int_0^{\tau} P(u)^2 du$$

and let

$$S\big(W^C_{\bullet,\bullet}, O\big) = \frac{2}{K_S^2}\int_0^{\tau}\big(P(u)\cdot C\big)\mathrm{ord}_O\big(N(u)\big|_C\big)du + \frac{1}{K_S^2}\int_0^{\tau}(P(u)\cdot C)^2 du.$$

If $P \notin \mathrm{Sing}(S)$, then [2, Theorem 1.7.1] and [2, Corollary 1.7.25] give

(♦) $$\frac{1}{S_S(C)} \geqslant \delta_P(S) \geqslant \min\left\{\frac{1}{S_S(C)}, \frac{1}{S\big(W^C_{\bullet,\bullet}, O\big)}\right\}.$$

Similarly, if $P \in \mathrm{Sing}(S)$, then [2, Corollary 1.7.12] and [2, Corollary 1.7.25] give

(♢) $$\frac{1}{S_S(C)} \geqslant \delta_P(S) \geqslant \min\left\{\frac{1}{S_S(C)}, \inf_{O \in C}\frac{1}{S\big(W^C_{\bullet,\bullet}, O\big)}\right\}.$$

Lemma 24. *Suppose S has one singular point. Then $\delta(S) = \frac{15}{17}$, and the following assertions hold:*

- *If P is not contained in any line in S that contains the singular point of S, then $\delta_P(S) \geqslant \frac{15}{13}$.*
- *If P is not the singular point of the surface S, but P is contained in a line in S that passes through the singular point of the surface S, then $\delta_P(S) = 1$.*
- *If P is the singular point of the surface S, then $\delta_P(S) = \frac{15}{17}$.*

Proof. We let P_0 be the singular point of the surface S, and let ℓ_0 be the π-exceptional curve. Then it follows from [4] that there exists a birational morphism $\pi\colon \widetilde{S}\to\mathbb{P}^2$ such that $\pi(\ell_0)$ is a line, the map π blows up three points Q_1, Q_2, Q_3 contained in $\pi(\ell_0)$ and another point $Q_0\in\mathbb{P}^2\setminus\pi(\ell_0)$.

For $i\in\{0,1,2,3\}$, let $\mathbf{e}_i$ be the π-exceptional curve such that $\pi(\mathbf{e}_i)=Q_i$. For every $i\in\{1,2,3\}$, let ℓ_i be the strict transform of the line in $\mathbb{P}^2$ that passes through Q_0 and Q_i. Then ℓ_0, ℓ_1, ℓ_2, ℓ_3, $\mathbf{e}_0$, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are the only irreducible curves in the surface $\widetilde{S}$ that have negative self-intersections. Moreover, the intersections of these curves are given in the following table:

	ℓ_0	ℓ_1	ℓ_2	ℓ_3	$\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$
ℓ_0	-2					1	1	1
ℓ_1		-1			1	1		
ℓ_2			-1		1		1	
ℓ_3				-1	1			1
$\mathbf{e}_0$		1	1	1	-1			
$\mathbf{e}_1$	1	1				-1		
$\mathbf{e}_2$	1		1				-1	
$\mathbf{e}_3$	1			1				-1

Note that $\eta(\ell_1)$, $\eta(\ell_2)$, $\eta(\ell_3)$, $\eta(\mathbf{e}_0)$, $\eta(\mathbf{e}_1)$, $\eta(\mathbf{e}_2)$, $\eta(\mathbf{e}_3)$ are all lines contained in the surface S. Among them, only the lines $\eta(\mathbf{e}_1)$, $\eta(\mathbf{e}_2)$, $\eta(\mathbf{e}_3)$ pass through the singular point P_0.

For $(a_0,a_1,a_2,a_3,b_0,b_1,b_2,b_3)\in\mathbb{R}^8$, we write

$$[a_0,a_1,a_2,a_3,b_0,b_1,b_2,b_3]:=\sum_{i=0}^{3}a_i\ell_i+\sum_{i=0}^{3}b_ie_i\in\mathrm{Pic}(\widetilde{S})\otimes\mathbb{R}.$$

If $P=P_0$, then $C=\ell_0$, which implies that $\tau=2$ and

$$P(u)=\begin{cases}[-u,1,1,1,2,0,0,0] \text{ if } 0\leqslant u\leqslant 1,\\ [-u,1,1,1,2,1-u,1-u,1-u] \text{ if } 1\leqslant u\leqslant 2,\end{cases}$$

$$N(u)=\begin{cases}0 \text{ if } 0\leqslant u\leqslant 1,\\ (u-1)(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3) \text{ if } 1\leqslant u\leqslant 2,\end{cases}$$

$$P(u)\cdot C=\begin{cases}2 \text{ if } 0\leqslant u\leqslant 1,\\ 3-u \text{ if } 1\leqslant u\leqslant 2,\end{cases}\quad P(u)^2=\begin{cases}5-2u^2 \text{ if } 0\leqslant u\leqslant 1,\\ (4-u)(2-u) \text{ if } 1\leqslant u\leqslant 2,\end{cases}$$

which implies that $S_S(C)=\frac{17}{15}$ and $S(W^C_{\bullet,\bullet};O)=1$. Therefore, using ($\diamondsuit$), we obtain $\delta_{P_0}(S)=\frac{15}{17}$.

To proceed, we may assume that $P\neq P_0$. If $O\in\mathbf{e}_0$, we let $C=\mathbf{e}_0$. Then $\tau=2$, and

$$P(u)=\begin{cases}[0,1,1,1,2-u,0,0,0] \text{ if } 0\leqslant u\leqslant 1,\\ [0,2-u,2-u,2-u,2-u,0,0,0] \text{ if } 1\leqslant u\leqslant 2,\end{cases}$$

$$N(u)=\begin{cases}0 \text{ if } 0\leqslant u\leqslant 1,\\ (u-1)(\ell_1+\ell_2+\ell_3) \text{ if } 1\leqslant u\leqslant 2,\end{cases}$$

$$P(u)\cdot C=\begin{cases}1+u \text{ if } 0\leqslant u\leqslant 1,\\ 4-2u \text{ if } 1\leqslant u\leqslant 2,\end{cases}\quad P(u)^2=\begin{cases}5-2u-u^2 \text{ if } 0\leqslant u\leqslant 1,\\ 2(2-u)^2 \text{ if } 1\leqslant u\leqslant 2,\end{cases}$$

which implies that $S_S(C)=\frac{13}{15}$ and $S(W^C_{\bullet,\bullet};O)\leqslant\frac{13}{15}$, so that $\delta_P(S)=\frac{15}{13}$ by ($\blacklozenge$).

If $O \in \ell_1$, we let $C = \ell_1$. In this case, we have $\tau = 2$, and

$$P(u) = \begin{cases} [0, 1-u, 1, 1, 2, 0, 0, 0] \text{ if } 0 \leqslant u \leqslant 1, \\ [1-u, 1-u, 1, 1, 3-u, 2-2u, 0, 0] \text{ if } 1 \leqslant u \leqslant 2, \end{cases}$$

$$N(u) = \begin{cases} 0 \text{ if } 0 \leqslant u \leqslant 1, \\ (u-1)(\ell_0 + \mathbf{e}_0 + 2\mathbf{e}_1) \text{ if } 1 \leqslant u \leqslant 2, \end{cases}$$

$$P(u) \cdot C = \begin{cases} 1+u \text{ if } 0 \leqslant u \leqslant 1, \\ 4-2u \text{ if } 1 \leqslant u \leqslant 2, \end{cases} \quad P(u)^2 = \begin{cases} 5-2u-u^2 \text{ if } 0 \leqslant u \leqslant 1, \\ 2(2-u)^2 \text{ if } 1 \leqslant u \leqslant 2, \end{cases}$$

so that $S_S(C) = \frac{13}{15}$. If $O \in \ell_1 \setminus (\mathbf{e}_0 \cup \mathbf{e}_1)$, then $S(W^C_{\bullet,\bullet}; O) = \frac{11}{15}$. If $O = \ell_1 \cap \mathbf{e}_1$, then $S(W^C_{\bullet,\bullet}; O) = 1$. Thus, using (♦), we see that $\delta_P(S) = \frac{15}{13}$ if $O \in \ell_1 \setminus \mathbf{e}_1$, and $\delta_P(S) \geqslant 1$ if $O = \ell_1 \cap \mathbf{e}_1$.

Similarly, $\delta_P(S) = \frac{15}{13}$ if $O \in \ell_2 \setminus \mathbf{e}_2$ or $O \in \ell_3 \setminus \mathbf{e}_3$, and $\delta_P(S) \geqslant 1$ if $O = \ell_2 \cap \mathbf{e}_2$ or $O = \ell_3 \cap \mathbf{e}_3$.

If $O \in \mathbf{e}_1$, we let $C = \mathbf{e}_1$. In this case, we have $\tau = 2$, and

$$P(u) = \begin{cases} \Big[-\frac{u}{2}, 1, 1, 1, 2, -u, 0, 0\Big] \text{ if } 0 \leqslant u \leqslant 1, \\ \Big[-\frac{u}{2}, 2-u, 1, 1, 2, -u, 0, 0\Big] \text{ if } 1 \leqslant u \leqslant 2, \end{cases}$$

$$N(u) = \begin{cases} \frac{u}{2}\ell_0 \text{ if } 0 \leqslant u \leqslant 1, \\ \frac{u}{2}\ell_0 + (u-1)\ell_1 \text{ if } 1 \leqslant u \leqslant 2, \end{cases}$$

$$P(u) \cdot C = \begin{cases} \frac{2+u}{2} \text{ if } 0 \leqslant u \leqslant 1, \\ \frac{4-u}{2} \text{ if } 1 \leqslant u \leqslant 2, \end{cases} \quad P(u)^2 = \begin{cases} 5-2u-\frac{u^2}{2} \text{ if } 0 \leqslant u \leqslant 1, \\ \frac{(6-u)(2-u)}{2} \text{ if } 1 \leqslant u \leqslant 2, \end{cases}$$

which implies that $S_S(C) = 1$ and $S(W^C_{\bullet,\bullet}; O) \leqslant \frac{13}{15}$ if $O \in \mathbf{e}_1 \setminus \ell_0$, so that $\delta_P(S) = 1$ by (♦).

Likewise, we see that $\delta_P(S) = 1$ in the case when $O \in \mathbf{e}_2$ or $O \in \mathbf{e}_3$. Thus, to complete the proof, we may assume that P is not contained in any line in S.

Now, we let C be the unique curve in the pencil $|\ell_1 + \mathbf{e}_1|$ that contains P. By our assumption, the curve C is smooth and irreducible. Then $\tau = 2$, and

$$P(u) = \begin{cases} \Big[-\frac{u}{2}, 1-u, 1, 1, 2, -u, 0, 0\Big] \text{ if } 0 \leqslant u \leqslant 1, \\ \Big[-\frac{u}{2}, 1-u, 1, 1, 3-u, -u, 0, 0\Big] \text{ if } 1 \leqslant u \leqslant 2, \end{cases}$$

$$N(u) = \begin{cases} \frac{u}{2}\ell_0 \text{ if } 0 \leqslant u \leqslant 1, \\ \frac{1}{2}u\ell_0 + (u-1)\mathbf{e}_0 \text{ if } 1 \leqslant u \leqslant 2, \end{cases}$$

$$P(u) \cdot C = \begin{cases} \frac{4-u}{2} \text{ if } 0 \leqslant u \leqslant 1, \\ \frac{3(2-u)}{2} \text{ if } 1 \leqslant u \leqslant 2, \end{cases} \quad P(u)^2 = \begin{cases} 5-4u+\frac{u^2}{2} \text{ if } 0 \leqslant u \leqslant 1, \\ \frac{3(2-u)^2}{2} \text{ if } 1 \leqslant u \leqslant 2. \end{cases}$$

Then $S_S(C) = \frac{11}{15}$ and $S(W^C_{\bullet,\bullet}; O) = \frac{23}{30}$. Thus, it follows from (♦) that $\delta_P(S) \geqslant \frac{30}{23} > \frac{15}{13}$. □

Finally, let us estimate $\delta_P(S)$ in the case when the del Pezzo surface S has two singular points. In this case, the surface S contains a line that passes through both its singular points [4].

Lemma 25. *Suppose S has two singular points. Let ℓ be the line in S that passes through both singular points of the surface S. Then $\delta(S)=\frac{15}{19}$. Moreover, the following assertions hold:*

- *If P is not contained in any line in S that contains a singular point of S, then $\delta_P(S)\geqslant\frac{15}{13}$.*
- *If P is not contained in the line ℓ, but P is contained in a line in S that passes through a singular point of the surface S, then $\delta_P(S)=1$.*
- *If $P\in\ell$, then $\delta_P(S)=\frac{15}{19}$.*

Proof. Let $\mathbf{e}_1$ and $\mathbf{e}_2$ be η-exceptional curves. Then $\widetilde{S}$ contains (-1)-curves ℓ_1, ℓ_2, ℓ_3, ℓ_4, ℓ_5 such that the intersections of the curves ℓ_1, ℓ_2, ℓ_3, ℓ_4, ℓ_5, $\mathbf{e}_1$, $\mathbf{e}_2$ on $\widetilde{S}$ are given in the following table.

	ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	e_1	e_2
ℓ_1	-1					1	1
ℓ_2		-1	1			1	
ℓ_3		1	-1	1			
ℓ_4			1	-1	1		
ℓ_5				1	-1		1
e_1	1	1				-2	
e_2	1				1		-2

The curves $\eta(\ell_1)$, $\eta(\ell_2)$, $\eta(\ell_3)$, $\eta(\ell_4)$, $\eta(\ell_5)$ are the only lines in S. Moreover, we have $\ell=\eta(\ell_1)$, and $\eta(\ell_1)$, $\eta(\ell_2)$, $\eta(\ell_5)$ are the only lines in S that contain a singular point of the surface S.

As in the proof of Lemma 24, for $(a_1,a_2,a_3,a_4,a_5,b_1,b_2)\in\mathbb{R}^7$, we write

$$[a_1,a_2,a_3,a_4,a_5,b_1,b_2]:=\sum_{i=1}^{5}a_i\ell_i+\sum_{i=1}^{2}b_ie_i\in\mathrm{Pic}(\widetilde{S})\otimes\mathbb{R}.$$

If $O\in\ell_1\setminus(\mathbf{e}_1\cup\mathbf{e}_2)$, we let $C=\ell_1$. In this case, we have $\tau=3$, and

$$P(u)=\begin{cases}\Big[1-u,1,1,1,1,\dfrac{2-u}{2},\dfrac{2-u}{2}\Big]\text{ if }0\leqslant u\leqslant 2,\\ [1-u,3-u,3-u,0,0,0]\text{ if }2\leqslant u\leqslant 3,\end{cases}$$

$$N(u)=\begin{cases}\dfrac{u}{2}(\mathbf{e}_1+\mathbf{e}_2)\text{ if }0\leqslant u\leqslant 2,\\ (u-2)(\ell_2+\ell_5)+(u-1)(\mathbf{e}_1+\mathbf{e}_2)\text{ if }2\leqslant u\leqslant 3,\end{cases}$$

$$P(u)\cdot C=\begin{cases}1\text{ if }0\leqslant u\leqslant 2,\\ 3-u\text{ if }2\leqslant u\leqslant 3,\end{cases}\qquad P(u)^2=\begin{cases}5-2u\text{ if }0\leqslant u\leqslant 2,\\ (3-u)^2\text{ if }2\leqslant u\leqslant 3,\end{cases}$$

which implies that $S_S(C)=\frac{19}{15}$ and $S(W^C_{\bullet,\bullet};O)\leqslant\frac{17}{15}$, so that $\delta_P(S)=\frac{15}{19}$ by ($\blacklozenge$).

If $O\in\mathbf{e}_1$, then $C=\mathbf{e}_1$. In this case, we have $\tau=2$, and

$$P(u)=\begin{cases}[1,1,1,1,1,1-u,1]\text{ if }0\leqslant u\leqslant 1,\\ [3-2u,2-u,1,1,1,1-u,2-u]\text{ if }1\leqslant u\leqslant 2,\end{cases}$$

$$N(u)=\begin{cases}0\text{ if }0\leqslant u\leqslant 1,\\ 2(u-1)\ell_1+(u-1)\ell_2+(u-1)\mathbf{e}_2\text{ if }1\leqslant u\leqslant 2,\end{cases}$$

$$P(u)\cdot C=\begin{cases}2u\text{ if }0\leqslant u\leqslant 1,\\ 3-u\text{ if }1\leqslant u\leqslant 2,\end{cases}\qquad P(u)^2=\begin{cases}5-2u^2\text{ if }0\leqslant u\leqslant 1,\\ (2-u)(4-u)\text{ if }1\leqslant u\leqslant 2,\end{cases}$$

which implies that $S_S(C)=\frac{17}{15}$ and $S(W^C_{\bullet,\bullet};O)\leqslant\frac{19}{15}$, so that $\delta_P(S)\geqslant\frac{19}{15}$ by ($\lozenge$).

On the other hand, we already know that $S_S(\ell)=\frac{19}{15}$, which implies that $\delta_P(S)=\frac{19}{15}$ if $P=\eta(\mathbf{e}_1)$. Similarly, we see that $\delta_P(S)=\frac{19}{15}$ if $P=\eta(\mathbf{e}_2)$. Hence, we may assume that $O\not\in\mathbf{e}_1\cup\mathbf{e}_2\cup\ell_1$.

If $O\in\ell_2$, we let $C=\ell_2$. In this case, we have $\tau=2$, and

$$P(u)=\begin{cases}\Big[1,1-u,1,1,1,\dfrac{2-u}{2},1\Big] \text{ if } 0\leqslant u\leqslant 1,\\ \Big[1,1-u,2-u,1,1,\dfrac{2-u}{2},1\Big] \text{ if } 1\leqslant u\leqslant 2,\end{cases}$$

$$N(u)=\begin{cases}\dfrac{u}{2}\mathbf{e}_1 \text{ if } 0\leqslant u\leqslant 1,\\ \dfrac{u}{2}\mathbf{e}_1+(u-1)\ell_3 \text{ if } 1\leqslant u\leqslant 2,\end{cases}$$

$$P(u)\cdot C=\begin{cases}\dfrac{2+u}{2} \text{ if } 0\leqslant u\leqslant 1,\\ \dfrac{4-u}{2} \text{ if } 1\leqslant u\leqslant 2,\end{cases}\qquad P(u)^2=\begin{cases}5-2u-\dfrac{u^2}{2} \text{ if } 0\leqslant u\leqslant 1,\\ \dfrac{(6-u)(2-u)}{2} \text{ if } 1\leqslant u\leqslant 2,\end{cases}$$

which implies that $S_S(C)=1$ and $S(W^C_{\bullet,\bullet};O)\leqslant\frac{13}{15}$, so that $\delta_P(S)=1$ by ($\blacklozenge$).

Similarly, we see that $\delta_P(S)=1$ if $O\in\ell_5$. Hence, if P is contained in a line in S that passes through a singular point of the surface S, then $\delta_P(S)=1$. Thus, we may assume that $O\not\in\ell_2\cup\ell_2$.

If $P\in\ell_3$, we let $C=\ell_3$. In this case, we have $\tau=2$, and

$$P(u)=\begin{cases}[1,1,1-u,1,1,1,1] \text{ if } 0\leqslant u\leqslant 1,\\ [1,3-2u,1-u,2-u,1,2-u,1] \text{ if } 1\leqslant u\leqslant 2,\end{cases}$$

$$N(u)=\begin{cases}0 \text{ if } 0\leqslant u\leqslant 1,\\ (u-1)(\ell_4+2\ell_2+\mathbf{e}_1) \text{ if } 1\leqslant u\leqslant 2,\end{cases}$$

$$P(u)\cdot C=\begin{cases}1+u \text{ if } 0\leqslant u\leqslant 1,\\ 4-2u \text{ if } 1\leqslant u\leqslant 2,\end{cases}\qquad P(u)^2=\begin{cases}5-2u-u^2 \text{ if } 0\leqslant u\leqslant 1,\\ 2(2-u)^2 \text{ if } 1\leqslant u\leqslant 2,\end{cases}$$

which implies that $S_S(C)=\frac{13}{15}$ and $S(W^C_{\bullet,\bullet};O)\leqslant\frac{13}{15}$, so that $\delta_P(S)=\frac{15}{13}$ by ($\blacklozenge$).

Similarly, we see that $\delta_P(S)=\frac{15}{13}$ if $O\in\ell_4$. Therefore, we may also assume that $O\not\in\ell_3\cup\ell_4$.

Let C be the curve in the pencil $|\ell_2+\ell_3|$ that contains O. Then C is smooth and irreducible, since O is not contained in the curves ℓ_1, ℓ_2, ℓ_3, ℓ_4, ℓ_5, $\mathbf{e}_1$, $\mathbf{e}_2$ by assumption. Then $\tau=2$, and

$$P(u)=\begin{cases}\Big[1,1-u,1-u,1,1,\dfrac{2-u}{2},1\Big] \text{ if } 0\leqslant u\leqslant 1,\\ \Big[1,1-u,1-u,2-u,1,\dfrac{2-u}{2},1\Big] \text{ if } 1\leqslant u\leqslant 2,\end{cases}$$

$$N(u)=\begin{cases}\dfrac{u}{2}\mathbf{e}_1 \text{ if } 0\leqslant u\leqslant 1,\\ \dfrac{u}{2}\mathbf{e}_1+(u-1)\ell_4 \text{ if } 1\leqslant u\leqslant 2,\end{cases}$$

$$P(u)\cdot C=\begin{cases}\dfrac{4-u}{2} \text{ if } 0\leqslant u\leqslant 1,\\ \dfrac{3(2-u)}{2} \text{ if } 1\leqslant u\leqslant 2,\end{cases}\qquad P(u)^2=\begin{cases}5-4u+\dfrac{u^2}{2} \text{ if } 0\leqslant u\leqslant 1,\\ \dfrac{3(2-u)^2}{2} \text{ if } 1\leqslant u\leqslant 2.\end{cases}$$

This implies that $S_S(C)=\frac{11}{15}$ and $S(W^C_{\bullet,\bullet};O)=\frac{23}{30}$, so that $\delta_P(S)\geqslant\frac{30}{23}>\frac{15}{13}$ by ($\blacklozenge$). $\square$

Appendix B. Nemuro Lemma

Now, let X be any smooth Fano threefold, let $\pi\colon X\to\mathbb{P}^1$ be a fibration into del Pezzo surfaces, let S be a fiber of the morphism π such that S is an irreducible reduced normal del Pezzo surface that has at worst du Val singularities, and let P be a point in S. As in Section 3, set

$$\tau=\sup\Big\{u\in\mathbb{Q}_{\geqslant 0}\ \Big|\ \text{the divisor } -K_X-uS \text{ is pseudo-effective}\Big\}.$$

For $u\in[0,\tau]$, let $P(u)$ be the positive part of the Zariski decomposition of the divisor $-K_X-uS$, and let $N(u)$ be its negative part. Suppose, in addition, that

$$N(u)=\sum_{j=1}^{l}f_j(u)E_j$$

for some irreducible reduced surfaces $E_1,\ldots,E_l$ on the Fano threefold X that are different from S, where each $f_i\colon[0,\tau]\to\mathbb{R}_{\geqslant 0}$ is some function. For every $j\in\{1,\ldots,l\}$, we set $c_j=\mathrm{lct}_P(S;E_j|_S)$. As in Appendix A, we set $\delta_P(S)=\delta_P(S,-K_S)$. Define $S(W^S_{\bullet,\bullet};F)$ and $\delta_P(S;W^S_{\bullet,\bullet})$ as in [2, § 1], or define these numbers using the formulas used in (3.1).

Lemma 26. *Let F be any prime divisor over S such that $P\in C_S(F)$. Then*

$$(\diamondsuit)\qquad \begin{aligned} S\big(W^S_{\bullet,\bullet};F\big)&\leqslant A_S(F)\frac{3}{(-K_X)^3}\int_0^\tau\sum_{j=1}^{\tau}\frac{f_j(u)}{c_j}\big(P(u)\big|_S\big)^2du+\\ &+\frac{3}{(-K_X)^3}\int_0^\tau\int_0^\infty \mathrm{vol}\big(P(u)\big|_S-vF\big)dvdu\leqslant\\ &\leqslant A_S(F)\left(\frac{3}{(-K_X)^3}\sum_{j=1}^{l}\int_0^\tau\frac{f_j(u)}{c_j}\big(P(u)\big|_S\big)^2du+\frac{3}{(-K_X)^3}\frac{\tau(-K_S)^2}{\delta_P(S)}\right).\end{aligned}$$

In particular, we have

$$\delta_P\big(S;W^S_{\bullet,\bullet}\big)\geqslant\left(\frac{3}{(-K_X)^3}\sum_{j=1}^{l}\int_0^\tau\frac{f_j(u)}{c_j}\big(P(u)\big|_S\big)^2du+\frac{3}{(-K_X)^3}\frac{\tau(-K_S)^2}{\delta_P(S)}\right)^{-1}.$$

Proof. Since the log pair $(S,c_jE_j|_S)$ is log canonical at P, we conclude that $\mathrm{ord}_F(E_j|_S)\leqslant\frac{A_S(F)}{c_j}$. Thus, we get the first inequality in $(\diamondsuit)$. Moreover, since $P(u)|_S=-K_S-N(u)|_S$, we have

$$\int_0^\tau\int_0^\infty\mathrm{vol}\big(P(u)|_S-vF\big)dvdu\leqslant\int_0^\tau(-K_S)^2S_S(F)du=\tau(-K_S)^2S_S(F)\leqslant A_S(F)\frac{\tau(-K_S)^2}{\delta_P(S)}.$$

Hence, the assertion follows. ☐

Corollary 27. *Suppose that $N(u)=0$ for every $u\in[0,\tau]$, i.e. we have $l=0$. Then*

$$\delta_P(S,W^S_{\bullet,\bullet})\geqslant\frac{(-K_X)^3\delta_P(S)}{3\tau(-K_S)^2}.$$

Corollary 28. *Suppose that $l=1$, $E_1|_S$ is a smooth curve contained in $S\setminus\mathrm{Sing}(S)$, and*

$$f_1(u)=\begin{cases}0 \ \text{if } u\in[0,t],\\ c(u-t)\ \text{if } u\in[t,\tau],\end{cases}$$

for some $t\in(0,\tau)$ and some $c\in\mathbb{R}_{>0}$. Then

$$\delta_P\big(S;W^S_{\bullet,\bullet}\big)\geqslant\left(\frac{3}{(-K_X)^3}\int_t^\tau c(u-t)\big(P(u)\big|_S\big)^2du+\frac{3}{(-K_X)^3}\frac{\tau(-K_S)^2}{\delta_P(S)}\right)^{-1}.$$

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