K-STABLE SMOOTH FANO THREEFOLDS OF PICARD RANK TWO

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Abstract. We prove that all smooth Fano threefolds in the families №2.1, №2.2, №2.3, №2.4, №2.6 and №2.7 are K-stable, and we also prove that smooth Fano threefolds in the family №2.5 that satisfy one very explicit generality condition are K-stable.

1. Introduction

Let X be a smooth Fano threefold. Then X belongs to one of the 105 families, which are labeled as №1.1, №1.2, …, №9.1, №10.1. See [3], for the description of these families. In 76 families, K-polystable smooth Fano threefolds are classified [2, 3, 4, 7, 8, 12, 15, 17]. The remaining 29 deformation families are

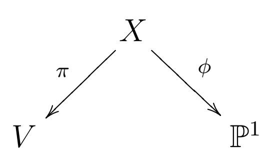
№1.9, №1.10, №2.1, …, №2.7, №2.9, …, №2.21, №3.2, №3.4, …, №3.8, №3.11.

General members of these 29 families are K-polystable [3]. In this paper, we prove

Main Theorem. *Let X be a smooth Fano threefold contained in one of the following deformation families: №2.1, №2.2, №2.3, №2.4, №2.6, №2.7. Then X is K-stable.*

and

Auxiliary Theorem. *Let X be a smooth Fano threefold in the deformation family №2.5. Recall that there exists the following Sarkisov link:*



where V is a smooth cubic threefold in $\mathbb{P}^4$, the morphism π is a blow up of a smooth plane cubic curve, and ϕ is a morphism whose fibers are normal cubic surfaces. Suppose that

(★) *no fiber of the morphism ϕ has a Du Val singular point of type $\mathbb{D}_5$ or $\mathbb{E}_6$.*

Then X is K-stable.

Let us describe the structure of this paper. In Section 2, we prove Auxiliary Theorem, and we prove that all smooth Fano threefolds in the families №2.1 and №2.3 are K-stable. In Sections 3, 4, 5, 6, we prove that all smooth Fano threefolds in the families №2.2, №2.4, №2.6, №2.7 are K-stable, respectively. Note that Section 6 is very technical and long.

In this paper, we use two applications of the Abban–Zhuang theory [1] which have been discovered in [3, 15]. For the background material, we refer the reader to [3, 15, 19].

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Throughout this paper, all varieties are assumed to be projective and defined over $\mathbb{C}$.

2. Families №2.1, №2.3, №2.5

Fix $d \in \{1, 2, 3\}$. Let V be one of the following smooth Fano threefolds:

$\boxed{d = 1}$ a smooth sextic hypersurface in $\mathbb{P}(1, 1, 1, 2, 3)$;
$\boxed{d = 2}$ a smooth quartic hypersurface in $\mathbb{P}(1, 1, 1, 1, 2)$;
$\boxed{d = 3}$ a smooth cubic threefold in $\mathbb{P}^4$.

Then $-K_V \sim 2H$ for an ample divisor $H \in \mathrm{Pic}(V)$ such that $H^3 = d$ and $\mathrm{Pic}(V) = \mathbb{Z}[H]$. Let S_1 and S_2 be two distinct surfaces in the linear system $|H|$, and let $\mathcal{C} = S_1 \cap S_2$. Suppose that the curve $\mathcal{C}$ is smooth. Then $\mathcal{C}$ is an elliptic curve by the adjunction formula. Let $\pi\colon X \to V$ be the blow up of the curve $\mathcal{C}$, and let E be the π-exceptional surface.

- If $d = 1$, then X is a smooth Fano threefold in the deformation family №2.1.
- If $d = 2$, then X is a smooth Fano threefold in the deformation family №2.3.
- If $d = 3$, then X is a smooth Fano threefold in the deformation family №2.5.

Moreover, all smooth Fano threefolds in these families can be obtained in this way.

Note that $(-K_X)^3 = 4d$. Moreover, we have the following commutative diagram:

$$\begin{array}{ccc} & X & \\ {}^{\pi}\swarrow & & \searrow^{\phi} \\ V & \dashrightarrow & \mathbb{P}^1 \end{array}$$

where $V \dashrightarrow \mathbb{P}^1$ is the rational map given by the pencil that is generated by S_1 and S_2, and ϕ is a morphism whose general fiber is a smooth del Pezzo surface of degree d.

The goal of this section is to show that X is K-stable in the case when $d = 1$ or $d = 2$, and to show that X is K-stable in the case when $d = 3$ and X satisfies the condition ($\bigstar$). To show that X is K-stable, it is enough to show that $\delta_O(X) > 1$ for every point $O \in X$. This follows from the valuative criterion for K-stability [14, 16].

Lemma 2.1. *Let O be a point in X, let A be the fiber of the morphism ϕ such that $O \in A$. Suppose that A has at most Du Val singularities at the point O. Then*

$$\delta_O(X) \geqslant \begin{cases} \min\Big\{\dfrac{16}{11}, \dfrac{16}{15}\delta_O(A)\Big\} \text{ if } O \notin E, \\ \min\Big\{\dfrac{16}{11}, \dfrac{16\delta_O(A)}{\delta_O(A)+15}\Big\} \text{ if } O \in E. \end{cases}$$

Proof. Let u be a non-negative real number. Then $-K_X - uA \sim_{\mathbb{R}} (2-u)A + E$, which implies that divisor $-K_X - uA$ is pseudoeffective if and only if $u \leqslant 2$. For every $u \in [0, 2]$, let us denote by $P(u)$ the positive part of Zariski decomposition of the divisor $-K_X - uA$, and let us denote by $N(u)$ its negative part. Then

$$P(u) = \begin{cases} (2-u)A + E \text{ if } 0 \leqslant u \leqslant 1, \\ (2-u)H \text{ if } 1 \leqslant u \leqslant 2, \end{cases}$$

and

$$N(u) = \begin{cases} 0 \text{ if } 0 \leqslant u \leqslant 1, \\ (u-1)E \text{ if } 1 \leqslant u \leqslant 2. \end{cases}$$

Integrating, we get $S_X(A)=\frac{11}{16}$. Using [1, Theorem 3.3] and [3, Corollary 1.7.30], we get

$$
\delta_O(X)\geqslant\min\left\{\frac{1}{S_X(A)},\inf_{\substack{F/A\\ O\in C_A(F)}}\frac{A_A(F)}{S(W^A_{\bullet,\bullet};F)}\right\}=\min\left\{\frac{16}{11},\inf_{\substack{F/A\\ O\in C_A(F)}}\frac{A_A(F)}{S(W^A_{\bullet,\bullet};F)}\right\}\tag{2.1}
$$

where the infimum is taken by all prime divisors F over the surface A with $O\in C_A(F)$, and $S(W^A_{\bullet,\bullet};F)$ can be computed using [3, Corollary 1.7.24] as follows:

$$
S\big(W^A_{\bullet,\bullet};F\big)=\frac{3}{4d}\int_1^2\big(P(u)\big|_A\big)^2(u-1)\mathrm{ord}_F\big(E\big|_A\big)du+\frac{3}{4d}\int_0^2\int_0^\infty\mathrm{vol}\big(P(u)\big|_A-vF\big)dvdu.
$$

Now, let F be any prime divisor over the surface A such that $O\in C_A(F)$. Since

$$
P(u)\big|_A=\begin{cases}-K_A \text{ if } 0\leqslant u\leqslant 1,\\ (2-u)(-K_A) \text{ if } 1\leqslant u\leqslant 2,\end{cases}
$$

we have

$$
\begin{aligned}
S\big(W^A_{\bullet,\bullet};F\big)&=\frac{3}{4d}\int_1^2 d(2-u)^2(u-1)\mathrm{ord}_F\big(E\big|_A\big)du+\\
&+\frac{3}{4d}\int_0^1\int_0^\infty\mathrm{vol}\big(-K_A-vF\big)dvdu+\frac{3}{4d}\int_1^2\int_0^\infty\mathrm{vol}\big((2-u)(-K_A)-vF\big)dvdu=\\
&=\frac{\mathrm{ord}_F\big(E\big|_A\big)}{16}+\frac{3}{4d}\int_0^\infty\mathrm{vol}\big(-K_A-vF\big)dv+\frac{3}{4d}\int_1^2(2-u)^3\int_0^\infty\mathrm{vol}\big(-K_A-vF\big)dvdu=\\
&=\frac{\mathrm{ord}_F\big(E\big|_A\big)}{16}+\frac{3}{4d}\int_0^\infty\mathrm{vol}\big(-K_A-vF\big)dv+\frac{3}{16d}\int_0^\infty\mathrm{vol}\big(-K_A-vF\big)dv=\\
&=\frac{\mathrm{ord}_F\big(E\big|_A\big)}{16}+\frac{15}{16d}\int_0^\infty\mathrm{vol}\big(-K_A-vF\big)dv=\\
&=\frac{\mathrm{ord}_F\big(E\big|_A\big)}{16}+\frac{15}{16}S_A(F)\leqslant\frac{\mathrm{ord}_F\big(E\big|_A\big)}{16}+\frac{15A_A(F)}{16\delta_O(A)}.
\end{aligned}
$$

Therefore, if $O\not\in E$, then $\mathrm{ord}_F(E|_A)=0$, which implies that

$$
S\big(W^A_{\bullet,\bullet};F\big)\leqslant\frac{15A_A(F)}{16\delta_P(A)}.
$$

Similarly, if $O\in E$, then $\mathrm{ord}_F(E|_A)\leqslant A_A(F)$, because $(A,E|_A)$ is log canonical, so that

$$
S\big(W^A_{\bullet,\bullet};F\big)=\frac{\mathrm{ord}_F\big(E\big|_A\big)}{16}+\frac{15}{16}S_A(F)\leqslant\frac{A_A(F)}{16}+\frac{15A_A(F)}{16\delta_P(A)}=\frac{\delta_P(A)+15}{16\delta_P(A)}A_A(F).
$$

Now, using (2.1), we obtain the required inequality. □

Suppose X is not K-stable. Let us seek for a contradiction. Using the valuative criterion for K-stability [14, 16], we see that there exists a prime divisor $\mathbf{F}$ over X such that

$$
\beta(\mathbf{F})=A_X(\mathbf{F})-S_X(\mathbf{F})\leqslant 0,
$$

where $A_X(\mathbf{F})$ is a log discrepancy of the divisor $\mathbf{F}$, and $S_X(\mathbf{F})$ is defined in [14] or [3, § 1.2]. Let Z be the center of the divisor $\mathbf{F}$ on X. Then Z is not a surface [3, Theorem 3.7.1]. We see that Z is an irreducible curve or a point. Let P be a point in Z. Then $\delta_P(X)\leqslant 1$.

Lemma 2.2. *One has $P \notin E$.*

Proof. Let us compute $S_X(E)$. Note that $S_X(E) < 1$ by [3, Theorem 3.7.1]. Fix $u \in \mathbb{R}_{\geqslant 0}$. Then the divisor $-K_X - uE$ is pseudoeffective $\iff$ it is nef $\iff$ $u \leqslant 1$. Thus, we have

$$S_X(E) = \frac{1}{4d}\int_0^1 \big(-K_X - uE\big)^3 du = \frac{1}{4d}\int_0^1 d(2u^3 - 6u + 4)du = \frac{3}{8}.$$

Suppose that $P \in E$. Let us seek for a contradiction.

Note that $E \cong \mathcal{C} \times \mathbb{P}^1$. Let $\mathbf{s}$ be a fiber of the projection $\phi|_E \colon E \to \mathbb{P}^1$ that contains P, and let $\mathbf{f}$ be a fiber of the projection $\pi|_E \colon E \to \mathcal{C}$. Fix $u \in [0,1]$ and take $v \in \mathbb{R}_{\geqslant 0}$. Then

$$-K_X - uE\big|_E - v\mathbf{s} \equiv (1+u-v)\mathbf{s} + d(1-u)\mathbf{f}.$$

Thus, the divisor $-K_X - uE\big|_E - v\mathbf{s}$ is pseudoeffective $\iff$ it is nef $\iff$ $v \leqslant 1+v$. Therefore, using [3, Corollary 1.7.26], we get

$$S\left(W^E_{\bullet,\bullet};\mathbf{s}\right) = \frac{3}{4d}\int_0^1\int_0^{1+u} 2d(1-u)(1+u-v)dvdu = \frac{11}{16}.$$

Similarly, using [3, Theorem 1.7.30], we get

$$S\left(W^{E,\mathbf{s}}_{\bullet,\bullet,\bullet};P\right) = \frac{3}{4d}\int_0^1\int_0^{1+u} \big(d(1-u)\big)^2 dvdu = \frac{5d}{16}.$$

Therefore, it follows from [3, Theorem 1.7.30] that

$$1 \geqslant \frac{A_X(\mathbf{F})}{S_X(\mathbf{F})} \geqslant \min\left\{\frac{1}{S_X(E)}, \frac{1}{S\left(W^E_{\bullet,\bullet},\mathbf{s}\right)}, \frac{1}{S\left(W^{E,\mathbf{s}}_{\bullet,\bullet,\bullet};P\right)}\right\} = \min\left\{\frac{8}{3}, \frac{16}{11}, \frac{16}{5d}\right\} \geqslant \frac{16}{15} > 1,$$

which is a contradiction. □

Let A be the fiber of the del Pezzo fibration ϕ such that A passes through the point P. Then A is a del Pezzo surface of degree $d \in \{1,2,3\}$ that has at most isolated singularities. In particular, we see that A is normal. Applying Lemmas 2.1 and 2.2, we obtain

Corollary 2.3. *One has $\delta_P(A) \leqslant \frac{15}{16}$.*

Proof. Since $1 \geqslant \frac{A_X(\mathbf{F})}{S_X(\mathbf{F})} \geqslant \delta_P(X)$, we get $\delta_P(A) \leqslant \frac{15}{16}$ by Lemmas 2.1 and 2.2. □

Corollary 2.4. *The surface A is singular.*

Proof. If A is smooth, then $\delta_P(A) \geqslant \delta(A) \geqslant \frac{3}{2}$ [3, § 2], which contradicts Corollary 2.3. □

Let $\overline{S}$ be a general surface in $|H|$ that passes through $\pi(P)$, and let S be the proper transform on X of the surface $\overline{S}$. Then

- the surface $\overline{S}$ is a smooth del Pezzo surface of degree d,
- the surface $\overline{S}$ intersects the curve $\mathcal{C}$ transversally at d points,
- the induced morphism $\pi|_S \colon S \to \overline{S}$ is a blow up of the points $\overline{S} \cap \mathcal{C}$.

Observe that $\phi|_S \colon S \to \mathbb{P}^1$ is an elliptic fibration given by the pencil $|-K_S|$. Set $C = A\big|_S$. Then C is a reduced curve of arithmetic genus 1 in $|-K_S|$ that has at most d components. In particular, if $d = 1$, then C is irreducible. Therefore, the following cases may happen:

(1) the curve C is irreducible, and C is smooth at P,
(2) the curve C is irreducible, and C has an ordinary node at P,

(3) the curve C is irreducible, and C has an ordinary cusp at P,
(4) the curve C is reducible.

Fix $u \in \mathbb{R}_{\geqslant 0}$. Then $-K_X - uS$ is nef $\iff u \leqslant 1 \iff -K_X - uS$ is pseudoeffective. Using this, we see that

$$S_X(S) = \frac{1}{4d}\int_0^1 (-K_X - uS)^3 du = \frac{1}{4d}\int_0^1 d(4-u)(1-u)^2 du = \frac{5}{16} < 1,$$

which also follows from [3, Theorem 3.7.1]. Moreover, if $u \in [0,1]$, then

$$(-K_X - uS)|_S \sim_{\mathbb{R}} (1-u)(\pi|_S)^*(-K_{\overline{S}}) - K_S \sim_{\mathbb{R}} (1-u)\sum_{i=1}^{d} \mathbf{e}_i + (2-u)(-K_S),$$

where $\mathbf{e}_1, \ldots, \mathbf{e}_d$ are exceptional curves of the blow up $\pi|_S\colon S \to \overline{S}$.

Lemma 2.5. *Suppose that C is irreducible. Then C is singular at the point P.*

Proof. As in the proof of Lemma 2.2, it follows from [3, Theorem 1.7.30] that

$$1 \geqslant \frac{A_X(\mathbf{F})}{S_X(\mathbf{F})} \geqslant \min\left\{\frac{1}{S_X(S)}, \frac{1}{S(W^S_{\bullet,\bullet}; C)}, \frac{1}{S(W^{S,C}_{\bullet,\bullet,\bullet}; P)}\right\},$$

where $S(W^S_{\bullet,\bullet}; C)$ and $S(W^{S,C}_{\bullet,\bullet,\bullet}; P)$ are defined in [3, § 1.7]. Since we know that $S_X(S) < 1$, we see that $S(W^S_{\bullet,\bullet}; C) \geqslant 1$ or $S(W^{S,C}_{\bullet,\bullet,\bullet}; P) \geqslant 1$. Let us compute these numbers.

Let $P(u,v)$ be the positive part of the Zariski decomposition of $(-K_X - uS)|_S - vC$, and let $N(u,v)$ be its negative part, where $u \in [0,1]$ and $v \in \mathbb{R}_{\geqslant 0}$. Since

$$(-K_X - uS)|_S - vC \sim_{\mathbb{R}} (1-u)\sum_{i=1}^{d} \mathbf{e}_i + (2-u-v)C,$$

we see that $(-K_X - uS)|_S - vC$ is pseudoeffective $\iff v \leqslant 2-u$. Moreover, we have

$$P(u,v) = \begin{cases} (1-u)\displaystyle\sum_{i=1}^{d} \mathbf{e}_i + (2-u-v)C \text{ if } 0 \leqslant v \leqslant 1, \\ (2-u-v)\Big(C + \displaystyle\sum_{i=1}^{d} \mathbf{e}_i\Big) \text{ if } 1 \leqslant v \leqslant 2-u, \end{cases}$$

and

$$N(u,v) = \begin{cases} 0 \text{ if } 0 \leqslant v \leqslant 1, \\ (v-1)\displaystyle\sum_{i=1}^{d} \mathbf{e}_i \text{ if } 1 \leqslant v \leqslant 2-u. \end{cases}$$

Thus, it follows from [3, Corollary 1.7.26] that

$$S\big(W^S_{\bullet,\bullet}; C\big) = \frac{3}{4d}\int_0^1\int_0^{2-u} P(u,v)^2 dv du =$$

$$= \frac{3}{4d}\int_0^1\int_0^1 d(1-u)(3-u-2v) dv du + \frac{3}{4d}\int_0^1\int_1^{2-u} d(2-u-v)^2 dv du = \frac{11}{16} < 1.$$

Thus, we conclude that $S(W^{S,C}_{\bullet,\bullet,\bullet}; P) \geqslant 1$.

Since $P\not\in \mathbf{e}_1\cup\cdots\cup\mathbf{e}_d$ by Lemma 2.2, it follows from [3, Theorem 1.7.30] that

$$S\big(W^{S,C}_{\bullet,\bullet,\bullet};P\big)=\frac{3}{4d}\int_0^1\int_0^{2-u}\big(P(u,v)\cdot C\big)^2dvdu=$$
$$=\frac{3}{4d}\int_0^1\int_0^1 d^2(u-1)^2dvdu+\frac{3}{4d}\int_0^1\int_1^{2-u} d^2(2-u-v)^2dvdu=\frac{5d}{16}<1,$$

which is a contradiction. □

Now, let us show that C is reducible for $d\in\{1,2\}$.

Lemma 2.6. *Suppose that C is irreducible. Then $d=3$ and C has a cusp at P.*

Proof. By Lemma 2.5, the curve C is singular at the point P.

Now, let $\sigma\colon\widetilde{S}\to S$ be the blow up of the point P, let $\mathbf{f}$ be the σ-exceptional curve, and let $\widetilde{\mathbf{e}}_1,\ldots,\widetilde{\mathbf{e}}_d,\widetilde{C}$ be the proper transforms on $\widetilde{S}$ of the curves $\mathbf{e}_1,\ldots,\mathbf{e}_d,C$, respectively. Then the curve $\widetilde{C}$ is smooth, and it follows from [3, Remark 1.7.32] that

$$1\geqslant\frac{A_X(\mathbf{F})}{S_X(\mathbf{F})}\geqslant\min\left\{\inf_{O\in\mathbf{f}}\frac{1}{S(W^{\widetilde{S},\mathbf{f}}_{\bullet,\bullet,\bullet};O)},\frac{2}{S(V^S_{\bullet,\bullet};\mathbf{f})},\frac{1}{S_X(S)}\right\},$$

where $S(W^{\widetilde{S},\mathbf{f}}_{\bullet,\bullet,\bullet};O)$ and $S(V^S_{\bullet,\bullet};\mathbf{f})$ are defined in [3, § 1.7]. Since we know that $S_X(S)<1$, we see that $S(V^S_{\bullet,\bullet};\mathbf{f})\geqslant 2$ or there exists a point $O\in\mathbf{f}$ such that $S(W^{\widetilde{S},\mathbf{f}}_{\bullet,\bullet,\bullet};O)\geqslant 1$.

Let us compute $S(V^S_{\bullet,\bullet};\mathbf{f})$. Fix $u\in[0,1]$ and $v\in\mathbb{R}_{\geqslant 0}$. Since $\sigma^*(C)\sim\widetilde{C}+2\mathbf{f}$, we get

$$\sigma^*\big((-K_X-uS)|_S\big)-v\mathbf{f}\sim_{\mathbb{R}}(2-u)\widetilde{C}+(4-2u-v)\mathbf{f}+(1-u)\sum_{i=1}^d\widetilde{\mathbf{e}}_i.$$

Then $\sigma^*((-K_X-uS)|_S)-v\mathbf{f}$ is pseudoeffective $\iff$ $v\leqslant 4-2u$.

Let $P(u,v)$ be the positive part of the Zariski decomposition of $\sigma^*((-K_X-uS)|_S)-v\mathbf{f}$, and let $N(u,v)$ be its negative part. Then

$$P(u,v)=\begin{cases}(2-u)\widetilde{C}+(4-2u-v)\mathbf{f}+(1-u)\sum\limits_{i=1}^d\widetilde{\mathbf{e}}_i\text{ if }0\leqslant v\leqslant\dfrac{d-du}{2},\\[2ex] \dfrac{8+d-4u-du-2v}{4}\widetilde{C}+(4-2u-v)\mathbf{f}+(1-u)\sum\limits_{i=1}^d\widetilde{\mathbf{e}}_i\text{ if }\dfrac{1-u}{2}\leqslant v\leqslant\dfrac{4+d-du}{2},\\[2ex] \dfrac{4-2u-v}{4-d}\Big(2\widetilde{C}+(4-d)\mathbf{f}+2\sum\limits_{i=1}^d\widetilde{\mathbf{e}}_i\Big)\text{ if }\dfrac{4+d-du}{2}\leqslant v\leqslant 4-2u,\end{cases}$$

and

$$N(u,v)=\begin{cases}0\text{ if }0\leqslant v\leqslant\dfrac{d-du}{2},\\[2ex] \dfrac{2v+du-d}{4}\widetilde{C}\text{ if }\dfrac{d-du}{2}\leqslant v\leqslant\dfrac{4+d-du}{2},\\[2ex] \dfrac{2v+du-2d}{4-d}\widetilde{C}+\dfrac{2v+du-4-d}{4-d}\sum\limits_{i=1}^d\widetilde{\mathbf{e}}_i\text{ if }\dfrac{4+d-du}{2}\leqslant v\leqslant 4-2u.\end{cases}$$

Thus, using [3, Corollary 1.7.26], we get

$$S\big(W^{S}_{\bullet,\bullet};\mathbf{f}\big)=\frac{3}{4d}\int_0^1\int_0^{\frac{d-du}{2}}\big(du^2-4du-v^2+3d\big)dvdu+ \\ +\frac{3}{4d}\int_0^1\int_{\frac{d-du}{2}}^{\frac{4+d-du}{2}}\frac{d(1-u)(12-du+d-4u-4v)}{4}dvdu+ \\ +\frac{3}{4d}\int_0^1\int_{\frac{4+d-du}{2}}^{4-2u}\frac{d(4-2u-v)^2}{4-d}dvdu=\frac{44+5d}{32}<2.$$

Therefore, there exists a point $O\in\mathbf{f}$ such that $S(W^{\widetilde{S},\mathbf{f}}_{\bullet,\bullet,\bullet};O)\geqslant 1$.

Let us compute $S(W^{\widetilde{S},\mathbf{f}}_{\bullet,\bullet,\bullet};O)$. Observe that

$$P(u,v)\cdot\mathbf{f}=\begin{cases} v \text{ if } 0\leqslant v\leqslant\dfrac{d-du}{2},\\ \dfrac{d-du}{2} \text{ if } \dfrac{d-du}{2}\leqslant v\leqslant\dfrac{4+d-du}{2},\\ \dfrac{d(4-2u-v)}{4-d} \text{ if } \dfrac{4+d-du}{2}\leqslant v\leqslant 4-2u.\end{cases}$$

Thus, it follows from [3, Remark 1.7.32] that

$$S(W^{\widetilde{S},\mathbf{f}}_{\bullet,\bullet,\bullet};O)=\frac{3}{4d}\int_0^1\int_0^{4-2u}\Big(\big(P(u,v)\cdot\mathbf{f}\big)\Big)^2dvdu+ \\ +\frac{6}{4d}\int_0^1\int_0^{4-2u}\big(P(u,v)\cdot\mathbf{f}\big)\mathrm{ord}_O\big(N(u,v)\big|_{\mathbf{f}}\big)dvdu= \\ =\frac{3}{4d}\int_0^1\int_0^{\frac{d-du}{2}}v^2dvdu+\frac{3}{4d}\int_{\frac{d-du}{2}}^{\frac{4+d-du}{2}}\Big(\frac{d-du}{2}\Big)^2dvdu+\frac{3}{4d}\int_0^1\int_{\frac{4+d-du}{2}}^{4-2u}\Big(\frac{d(4-2u-v)}{4-d}\Big)^2dvdu+ \\ +\frac{6}{4d}\int_0^1\int_0^{4-2u}\big(P(u,v)\cdot\mathbf{f}\big)\mathrm{ord}_O\big(N(u,v)\big|_{\mathbf{f}}\big)dvdu= \\ =\frac{5d}{32}+\frac{3}{2}\int_0^1\int_0^{4-2u}\big(P(u,v)\cdot\mathbf{f}\big)\mathrm{ord}_O\big(N(u,v)\big|_{\mathbf{f}}\big)dvdu.$$

Therefore, if $O\not\in\widetilde{C}$, we obtain $S(W^{\widetilde{S},\mathbf{f}}_{\bullet,\bullet,\bullet};O)=\frac{5d}{32}$, which contradicts to $S(W^{\widetilde{S},\mathbf{f}}_{\bullet,\bullet,\bullet};O)\geqslant 1$. Similarly, if $O\in\widetilde{C}$ and $\widetilde{C}$ intersects the curve $\mathbf{f}$ transversally at the point O, then

$$S(W^{\widetilde{S},\mathbf{f}}_{\bullet,\bullet,\bullet};O)=\frac{5d}{32}+\frac{6}{4d}\int_0^1\int_0^{4-2u}\big(P(u,v)\cdot\mathbf{f}\big)\mathrm{ord}_O\big(N(u,v)\big|_{\mathbf{f}}\big)dvdu=\frac{44+5d}{64}<1.$$

which again contradicts $S(W^{\widetilde{S},\mathbf{f}}_{\bullet,\bullet,\bullet};O)\geqslant 1$. Therefore, the curves $\widetilde{C}$ and $\mathbf{f}$ are tangent at O, which implies that C has a cusp at the point P.

Thus, to proceed, we may assume that $d=1$ or $d=2$.

Now, let us consider the following commutative diagram:

$$\begin{array}{ccccc} \widetilde{S} & \xleftarrow{\ \rho\ } & \widehat{S} & \xleftarrow{\ \eta\ } & \overline{S} \\ \big\downarrow{\scriptstyle \sigma} & & & & \big\downarrow{\scriptstyle \psi} \\ S & & \xleftarrow{\qquad \upsilon \qquad} & & \mathscr{S} \end{array}$$

where ρ is the blow up of the point $\widetilde{C}\cap\mathbf{f}$, the morphism η is the blow up of the point in the ρ-exceptional curve that is contained in the the proper transform of the curve $\widetilde{C}$, the map ψ is the contraction of the proper transforms of both $(\sigma\circ\rho)$-exceptional curves, and υ is the birational contraction of the proper transform of the η-exceptional curve. Let $\mathscr{F}$ be the υ-exceptional curve, let $\mathscr{C}$ be the proper transform on $\mathscr{S}$ of the curve C. Then $\mathscr{F}$ and $\mathscr{C}$ are smooth, $\mathscr{C}^2=-6$, $\mathscr{F}^2=-\frac{1}{6}$, $\mathscr{C}\cdot\mathscr{F}=1$, and $\upsilon^*(C)=\mathscr{C}+6\mathscr{F}$.

Observe that $\mathscr{F}$ contains two singular points of the surfaces $\mathscr{S}$, which are quotient singular points of type $\frac{1}{2}(1,1)$ and $\frac{1}{3}(1,1)$. Denote these points by Q_2 and Q_3, respectively. Note that $\mathscr{C}$ does not contain Q_2 and Q_3. Write $\Delta_{\mathscr{F}}=\frac{1}{2}Q_2+\frac{2}{3}Q_3$. Then, since $A_S(\mathscr{F})=5$, it follows from [3, Remark 1.7.32] that

$$1\geqslant\frac{A_X(\mathbf{F})}{S_X(\mathbf{F})}\geqslant\min\left\{\inf_{Q\in\mathscr{F}}\frac{1-\mathrm{ord}_Q(\Delta_{\mathscr{F}})}{S(W^{\mathscr{S},\mathscr{F}}_{\bullet,\bullet,\bullet};Q)},\frac{5}{S(V^S_{\bullet,\bullet};\mathscr{F})},\frac{1}{S_X(S)}\right\}.$$

But we already proved that $S_X(S)<1$. Hence, we conclude that $S(V^S_{\bullet,\bullet};\mathscr{F})\geqslant 5$ or there exists a point $Q\in\mathscr{F}$ such that $S(W^{\mathscr{S},\mathscr{F}}_{\bullet,\bullet,\bullet};Q)\geqslant 1-\mathrm{ord}_Q(\Delta_{\mathscr{F}})$.

Let us compute $S(V^S_{\bullet,\bullet};\mathscr{F})$. Take $v\in\mathbb{R}_{\geqslant 0}$. Then

$$\upsilon^*\big(P(u)|_S\big)-v\mathscr{F}\sim_{\mathbb{R}}(2-u)\mathscr{C}+(1-u)\sum_{i=1}^{d}\mathscr{E}_i+(12-6u-v)\mathscr{F},$$

where $\mathscr{E}_i$ is the proper transform on $\mathscr{S}$ of the (-1)-curve $\mathbf{e}_i$. Using this, we conclude that the divisor $\upsilon^*(P(u)|_S)-v\mathscr{F}$ is pseudoeffective $\iff$ $v\leqslant 12-6u$.

Let $\mathscr{P}(u,v)$ be the positive part of the Zariski decomposition of $\upsilon^*(P(u)|_S)-v\mathscr{F}$, and let $\mathscr{N}(u,v)$ be its negative part. Then

$$\mathscr{P}(u,v)=\begin{cases}(2-u)\mathscr{C}+(1-u)\sum_{i=1}^{d}\mathscr{E}_i+(12-6u-v)\mathscr{F}\ \text{if}\ 0\leqslant v\leqslant d(1-u),\\ \frac{12+d-(6+d)u-v}{6}\mathscr{C}+(1-u)\sum_{i=1}^{d}\mathscr{E}_i+(12-6u-v)\mathscr{F}\ \text{if}\ d(1-u)\leqslant v\leqslant 6+d-du,\\ \frac{12-6u-v}{6-d}\Big(\mathscr{C}+\sum_{i=1}^{d}\mathscr{E}_i+(6-d)\mathscr{F}\Big)\ \text{if}\ 6+d-du\leqslant v\leqslant 12-6u,\end{cases}$$

and

$$\mathscr{N}(u,v)=\begin{cases}0\ \text{if}\ 0\leqslant v\leqslant d(1-u),\\ \frac{v+d(u-1)}{6}\mathscr{C}\ \text{if}\ d(1-u)\leqslant v\leqslant 6+d-du,\\ \frac{v+du-2d}{6-d}\mathscr{C}+\frac{v+du-6-d}{6-d}\sum_{i=1}^{d}\mathscr{E}_i\ \text{if}\ 6+d-du\leqslant v\leqslant 12-6u.\end{cases}$$

This gives

$$\mathscr{P}(u,v)^2=\begin{cases}\dfrac{6du^2-24du-v^2+18d}{6} \text{ if } 0\leqslant v\leqslant d(1-u),\\ \dfrac{d(1-u)(18-(d+6)u+d-2v)}{6} \text{ if } d(1-u)\leqslant v\leqslant 6+d-du,\\ \dfrac{d(12-6u-v)^2}{6(6-d)} \text{ if } 6+d-du\leqslant v\leqslant 12-6u.\end{cases}$$

Thus, using [3, Remark 1.7.30] and integrating, we get

$$S(W^{\mathscr{S}}_{\bullet,\bullet};\mathscr{F})=\frac{3}{4d}\int_0^1\int_0^{12-6u}\mathscr{P}(u,v)^2dvdu=\frac{66+5d}{16}\leqslant\frac{19}{4}<5=A_S(\mathscr{F}),$$

since $d=1$ or $d=2$. Thus, there is a point $Q\in\mathscr{F}$ such that $S(W^{\mathscr{S},\mathscr{F}}_{\bullet,\bullet,\bullet};Q)\geqslant 1-\mathrm{ord}_Q(\Delta_{\mathscr{F}})$.

Now, using [3, Remark 1.7.32] again, we see that

$$S\big(W^{\mathscr{S},\mathscr{F}}_{\bullet,\bullet,\bullet};Q\big)=\frac{3}{4d}\int_0^1\int_0^{12-6u}\Big(\big(\mathscr{P}(u,v)\cdot\mathscr{F}\big)\Big)^2dvdu+ \\ +\frac{6}{4d}\int_0^1\int_0^{12-6u}\big(P(u,v)\cdot\mathscr{F}\big)\mathrm{ord}_Q\big(\mathscr{N}(u,v)\big|_{\mathscr{F}}\big)dvdu.$$

On the other hand, we have

$$\mathscr{P}(u,v)\cdot\mathscr{F}=\begin{cases}\dfrac{v}{6} \text{ if } 0\leqslant v\leqslant d(1-u),\\ \dfrac{d(1-u)}{6} \text{ if } d(1-u)\leqslant v\leqslant 6+d-du,\\ \dfrac{d(12-6u-v)}{6(6-d)} \text{ if } 6+d-du\leqslant v\leqslant 12-6u,\end{cases}$$

and

$$\mathscr{N}(u,v)\cdot\mathscr{F}=\begin{cases}0 \text{ if } 0\leqslant v\leqslant d(1-u),\\ \dfrac{v-d(1-u)}{6} \text{ if } d(1-u)\leqslant v\leqslant 6+d-du,\\ \dfrac{v+du-2d}{6-d} \text{ if } 6+d-du\leqslant v\leqslant 12-6u.\end{cases}$$

In particular, we have

$$S(W^{\mathscr{S},\mathscr{F}}_{\bullet,\bullet,\bullet};Q)=\frac{5d}{96}+\frac{6}{4d}\int_0^1\int_0^{12-6u}\big(P(u,v)\cdot\mathscr{F}\big)\mathrm{ord}_Q\big(\mathscr{N}(u,v)\big|_{\mathscr{F}}\big)dvdu.$$

Hence, if $Q\not\in\mathscr{C}$, then $\frac{1}{3}\leqslant 1-\mathrm{ord}_Q(\Delta_{\mathscr{F}})\leqslant S(W^{\mathscr{S},\mathscr{F}}_{\bullet,\bullet,\bullet};Q)=\frac{5d}{96}<\frac{1}{3}$, which is absurd. Thus, we conclude that $Q=\mathscr{C}\cap\mathscr{F}$. Then

$$S(W^{\mathscr{S},\mathscr{F}}_{\bullet,\bullet,\bullet};Q)=\frac{5d}{96}+\frac{6}{4d}\int_0^1\int_0^{12-6u}\big(P(u,v)\cdot\mathscr{F}\big)\mathrm{ord}_Q\big(\mathscr{N}(u,v)\big|_{\mathscr{F}}\big)dvdu\leqslant \\ \leqslant\frac{5d}{96}+\frac{6}{4d}\int_0^1\int_0^{12-6u}\big(P(u,v)\cdot\mathscr{F}\big)\big(\mathscr{N}(u,v)\cdot\mathscr{F}\big)dvdu=\frac{11}{16}<1,$$

which is a contradiction, since $S(W^{\mathscr{S},\mathscr{F}}_{\bullet,\bullet,\bullet};Q)\geqslant 1-\mathrm{ord}_Q(\Delta_{\mathscr{F}})=1$. □

In particular, we conclude that either $d=2$ or $d=3$.

Corollary 2.7. *All smooth Fano threefolds in the family №2.1 are K-stable.*

Recall that A is the fiber of the del Pezzo fibration $\phi\colon X \to \mathbb{P}^1$ that passes through P. Note also that we have the following possibilities:

- $d = 2$, and A is a double cover of $\mathbb{P}^2$ branched over a reduced quartic curve;
- $d = 3$, and A is a normal cubic surface in $\mathbb{P}^3$.

Observe that $C = S \cap A$, where S is a general surface in $|\pi^*(H)|$ that contains the point P. Since C is singular at P, the surface A must be singular at P, which confirms Corollary 2.4. Now, using classifications of reduced singular plane quartic curves and singular normal cubic surfaces [6], we see that $P = \mathrm{Sing}(A)$, and one of the following three cases holds:

- $d = 2$, and A is a double cover of $\mathbb{P}^2$ branched over 4 lines intersecting in a point;
- $d = 3$, and A is a cone in $\mathbb{P}^3$ over a smooth plane cubic curve;
- $d = 3$, and A has Du Val singular point of type $\mathbb{D}_4$, $\mathbb{D}_5$, or $\mathbb{E}_6$.

Let us show that the first case is impossible.

Lemma 2.8. *One has $d = 3$.*

Proof. Suppose that $d = 2$. Then $P = \mathrm{Sing}(A)$, and A is a double cover of $\mathbb{P}^2$ branched over a reduced reducible plane quartic curve that is a union of 4 distinct lines passing through one point. Let us seek for a contradiction.

Let $\alpha\colon \widetilde{X} \to X$ be the blow up of the point P, let E_P be the α-exceptional divisor, and let $\widetilde{A}$ be the proper transform on $\widetilde{X}$ of the surface A. Then $\widetilde{A} \cap E_P$ is a line $L \subset E_P \cong \mathbb{P}^2$, and the surface $\widetilde{A}$ is singular along this line. Let $\beta\colon \overline{X} \to \widetilde{X}$ be the blow up of the line L, let E_L be the β-exceptional divisor, let $\overline{A}$ be the proper transform on $\overline{X}$ of the surface $\widetilde{A}$, and let $\overline{E}_P$ be the proper transforms on $\overline{X}$ of the surface E_P. Then

- $E_L \cong \mathbb{F}_2$,
- the intersection $\overline{E}_P \cap E_L$ is the (-2)-curve in E_L,
- the surface $\overline{A}$ is smooth, and the exists a $\mathbb{P}^1$-bundle $\overline{A} \to \mathcal{C}$,
- $\overline{A} \cap E_L$ is a smooth elliptic curve that is a section of the $\mathbb{P}^1$-bundle $\overline{A} \to \mathcal{C}$,
- the surfaces $\overline{A}$ and $\overline{E}_P$ are disjoint,
- $\overline{E}_P \cong \mathbb{P}^2$ and $\overline{E}_P|_{\overline{E}_P} \cong \mathcal{O}_{\mathbb{P}^2}(-2)$.

There is a birational contraction $\gamma\colon \overline{X} \to \widehat{X}$ of the surface $\overline{E}_P$ such that $\widehat{X}$ is a projective threefold that has one singular point $O = \gamma(\overline{E}_P)$, which is a terminal cyclic quotient singularities of type $\frac{1}{2}(1,1,1)$. Thus, there exists the following commutative diagram

$$\begin{array}{ccc} \overline{X} & \xrightarrow{\ \beta\ } & \widetilde{X} \\ {\scriptstyle\gamma}\big\downarrow & & \big\downarrow{\scriptstyle\alpha} \\ \widehat{X} & \xrightarrow[\ \sigma\ ]{} & X \end{array}$$

where σ is a birational morphism that contracts the surface $\gamma(E_L)$ to the point P.

Let $G = \gamma(E_L)$, let $\widehat{A} = \gamma(\overline{A})$, and let $\widehat{E}$ be the proper transform on $\widehat{X}$ of the surface E. Then $A_X(G) = 4$, $\sigma^*(-K_X) \sim 2\widehat{A} + \widehat{E} + 8G$ and $\sigma^*(A) \sim \widehat{A} + 4G$.

Note that $\widehat{A} \cong \overline{A}$ and $G \cong \mathbb{P}(1,1,2)$, so we can identify G with a quadric cone in $\mathbb{P}^3$. Note also that O is the vertex of the cone G. Moreover, by construction, we have $O \notin \widehat{A}$. Furthermore, the exists a $\mathbb{P}^1$-bundle $\widehat{A} \to \mathcal{C}$ such that $G|_{\widehat{A}}$ is its section.

Let $\mathbf{g}$ be a ruling of the quadric cone G, let $\mathbf{l}$ be a fiber of the $\mathbb{P}^1$-bundle $\widehat{A} \to \mathcal{C}$, and let $\mathbf{f}$ be a fiber of the $\mathbb{P}^1$-bundle $\pi \circ \sigma|_{\widehat{E}}\colon \widehat{E} \to \mathcal{C}$. Then $G|_G \sim_{\mathbb{Q}} -\mathbf{g}$ and $\widehat{A}|_G \sim_{\mathbb{Q}} 4\mathbf{g}$. Moreover, the intersections of the surfaces G, $\widehat{A}$, $\widehat{E}$ with the curves $\mathbf{g}$, $\mathbf{l}$, $\mathbf{f}$ are given here:

	G	$\widehat{A}$	$\widehat{E}$
$\mathbf{g}$	$-\frac{1}{2}$	2	0
$\mathbf{l}$	1	-4	1
$\mathbf{f}$	0	1	-1

Fix a non-negative real number u. We have $\sigma^*(-K_X)-uG \sim_{\mathbb{R}} 2\widehat{A}+\widehat{E}+(8-u)G$, which implies that $\sigma^*(-K_X)-uG$ is pseudo-effective $\iff$ $u \in [0,8]$. Furthermore, if $u \in [0,8]$, then the Zariski decomposition of the divisor $\sigma^*(-K_X)-uG$ can be described as follows:

$$P(u) = \begin{cases} 2\widehat{A}+\widehat{E}+(8-u)G \text{ if } 0 \leqslant u \leqslant 1, \\ \dfrac{9-u}{4}\widehat{A}+\widehat{E}+(8-u)G \text{ if } 1 \leqslant u \leqslant 5, \\ \dfrac{8-u}{3}\big(\widehat{A}+\widehat{E}+3G\big) \text{ if } 5 \leqslant u \leqslant 8, \end{cases}$$

and

$$N(u) = \begin{cases} 0 \text{ if } 0 \leqslant u \leqslant 1, \\ \dfrac{u-1}{4}\widehat{A} \text{ if } 1 \leqslant u \leqslant 5, \\ \dfrac{u-2}{3}\widehat{A}+\dfrac{u-5}{3}\widehat{E} \text{ if } 5 \leqslant u \leqslant 8, \end{cases}$$

where $P(u)$ and $N(u)$ are the positive and the negative parts of the decomposition. Then

$$\mathrm{vol}\big(\sigma^*(-K_X)-uG\big) = P(u)^3 = \begin{cases} 8-\dfrac{u^3}{8} \text{ if } 0 \leqslant u \leqslant 1, \\ \dfrac{18-3u}{2} \text{ if } 1 \leqslant u \leqslant 5, \\ \dfrac{(8-u)^3}{18} \text{ if } 5 \leqslant u \leqslant 8. \end{cases}$$

Integrating, we get $S_X(G)=\frac{27}{8}<4=A_X(G)$. But [15, Corollary 4.18] gives

$$1 \geqslant \frac{A_X(\mathbf{F})}{S_X(\mathbf{F})} \geqslant \min\left\{\frac{A_X(G)}{S_X(G)}, \inf_{Q\in G}\delta_Q\big(G,V^G_{\bullet,\bullet}\big)\right\} = \min\left\{\frac{32}{27}, \inf_{Q\in G}\delta_Q\big(G,V^G_{\bullet,\bullet}\big)\right\},$$

where $\delta_Q(G,V^G_{\bullet,\bullet})$ is defined in [15]. Moreover, if Q is a point in G and Z is a smooth curve in G that passes through Q, then it follows from [15, Corollary 4.18] that

$$\delta_Q\big(G,V^G_{\bullet,\bullet}\big) \geqslant \min\left\{\frac{1}{S\big(V^G_{\bullet,\bullet};Z\big)}, \frac{1-\mathrm{ord}_Q(\Delta_Z)}{S\big(W^{G,Z}_{\bullet,\bullet,\bullet};Q\big)}\right\},$$

where $S(V^G_{\bullet,\bullet};Z)$ and $S(W^{G,Z}_{\bullet,\bullet,\bullet};Q)$ are defined in [15], and

$$\Delta_Z = \begin{cases} 0 \text{ if } O \notin Z, \\ \dfrac{1}{2}O \text{ if } O \in Z. \end{cases}$$

Let us show that $\delta_Q\big(G, V^G_{\bullet,\bullet}\big) > 1$ for every $Q \in G$, which would imply a contradiction.

Let $\mathscr{C} = \widehat{A}|_G$, and let ℓ be a curve in $|\mathbf{f}|$ that passes through Q. Then $O \notin \ell$ and $O \in \ell$, so that $\Delta_\ell = 0$ and $\Delta_\ell = \frac{1}{2}O$. Take $v \in \mathbb{R}_{\geqslant 0}$. Then

$$P(u)\big|_G - v\ell \sim_{\mathbb{R}} \begin{cases} (u-v)\mathbf{g} \text{ if } 0 \leqslant u \leqslant 1, \\ (1-v)\mathbf{g} \text{ if } 1 \leqslant u \leqslant 5, \\ \dfrac{8-u-3v}{3}\mathbf{g} \text{ if } 5 \leqslant u \leqslant 8. \end{cases}$$

Now, using [15, Theorem 4.8], we get

$$S\big(W^G_{\bullet,\bullet};\ell\big) = \frac{3}{8}\int_0^8\int_0^\infty \mathrm{vol}\big(P(u)\big|_G - v\ell\big)dvdu = \frac{3}{8}\int_0^1\int_0^u \frac{(u-v)^2}{2}dvdu + \\ + \frac{3}{12}\int_1^5\int_0^1 \frac{(1-v)^2}{4}dvdu + \frac{3}{12}\int_5^8\int_0^{\frac{8-u}{3}} \frac{(8-u-3v)^2}{18}dvdu = \frac{5}{16}.$$

Similarly, if $Q \notin \mathscr{C}$, then it follows from [15, Theorem 4.17] that

$$S\big(W^{G,\ell}_{\bullet,\bullet,\bullet};Q\big) = \frac{3}{8}\int_0^1\int_0^u \Big(\big(P(u)\big|_G - v\ell\big)\cdot\ell\Big)^2 dvdu + \frac{3}{8}\int_1^5\int_0^1 \Big(\big(P(u)\big|_G - v\ell\big)\cdot\ell\Big)^2 dvdu + \\ + \frac{3}{8}\int_5^8\int_0^{\frac{8-u}{3}} \Big(\big(P(u)\big|_G - v\ell\big)\cdot\ell\Big)^2 dvdu + F_Q = \frac{3}{8}\int_0^1\int_0^u \frac{(u-v)^2}{4}dvdu + \\ + \frac{3}{8}\int_1^5\int_0^1 \frac{(1-v)^2}{4}dvdu + \frac{3}{8}\int_5^8\int_0^{\frac{8-u}{3}} \frac{(8-u-3v)^2}{36}dvdu = \frac{5}{32}.$$

so that $S(W^{G,\ell}_{\bullet,\bullet,\bullet};Q) = \frac{5}{32}$, which implies that $\delta_Q(G, V^G_{\bullet,\bullet}) \geqslant \frac{16}{5}$. Likewise, if $Q \in \mathscr{C}$, then

$$S\big(W^G_{\bullet,\bullet};\mathscr{C}\big) = \frac{3}{8}\int_0^8 \big(P(u)^2\cdot G\big)\cdot \mathrm{ord}_{\mathscr{C}}\Big(N(u)\big|_G\Big)du + \frac{3}{8}\int_0^8\int_0^\infty \mathrm{vol}\big(P(u)\big|_G - v\mathscr{C}\big)dvdu = \\ = \frac{3}{8}\int_1^5 \frac{u-1}{8}du + \frac{3}{8}\int_5^8 \frac{(u-2)(8-u)^2}{54}du + \frac{3}{8}\int_0^1\int_0^{\frac{u}{4}} \frac{(u-4v)^2}{2}dvdu + \\ + \frac{3}{8}\int_1^5\int_0^{\frac{1}{4}} \frac{(1-4v)^2}{2}dvdu + \frac{3}{8}\int_5^8\int_0^{\frac{8-u}{12}} \frac{(8-u-12v)^2}{18}dvdu = \frac{11}{16}$$

and

$$S\big(W^{G,\mathscr{C}}_{\bullet,\bullet,\bullet};Q\big) = \frac{3}{8}\int_0^1\int_0^{\frac{u}{4}} \Big(\big(P(u)\big|_G - v\mathscr{C}\big)\cdot\mathscr{C}\Big)^2 dvdu + \\ + \frac{3}{8}\int_1^5\int_0^{\frac{1}{4}} \Big(\big(P(u)\big|_G - v\mathscr{C}\big)\cdot\mathscr{C}\Big)^2 dvdu + \frac{3}{8}\int_5^8\int_0^{\frac{8-u}{12}} \Big(\big(P(u)\big|_G - v\mathscr{C}\big)\cdot\mathscr{C}\Big)^2 dvdu = \\ = \frac{3}{8}\int_0^1\int_0^{\frac{u}{4}} (2u-8v)^2 dvdu + \frac{3}{8}\int_1^5\int_0^{\frac{1}{4}} (2-8v)^2 dvdu + \\ + \frac{3}{8}\int_5^8\int_0^{\frac{8-u}{12}} \Big(\frac{(16-2u-24v)^2}{9}dvdu = \frac{5}{8}.$$

This implies that $\delta_Q(G, V^G_{\bullet,\bullet}) \geqslant \min\big\{\frac{16}{11}, \frac{8}{5}\big\} = \frac{16}{11}$, which is a contradiction. □

Corollary 2.9. *All smooth Fano threefolds in the family №2.3 are K-stable.*

We see that $d=3$, so that A is a singular cubic surface in $\mathbb{P}^3$ such that $P=\mathrm{Sing}(A)$. Let $\sigma\colon \widehat{X}\to X$ be the blow up of the point P, and let G be the σ-exceptional surface. Denote by $\widehat{A}$ and $\widehat{E}$ the proper transforms on $\widehat{X}$ of the surfaces A and E, respectively.

Lemma 2.10. *The surface A has Du Val singularities.*

Proof. Suppose that A is a cone in $\mathbb{P}^3$ with vertex P. Take $u\in\mathbb{R}_{\geqslant 0}$. Then

$$\sigma^*(-K_X)-vG\sim_{\mathbb{R}} 2\widehat{A}+\widehat{E}+(6-u)G.$$

Thus, the divisor $\sigma^*(-K_X)-vG$ is pseudo-effective $\iff$ $u\in[0,6]$. Moreover, if $u\in[0,6]$, then the Zariski decomposition of the divisor $\sigma^*(-K_X)-vG$ can be described as follows:

$$P(u)=\begin{cases} 2\widehat{A}+E+(6-u)G \text{ if } 0\leqslant u\leqslant 1,\\ \dfrac{7-u}{3}\widehat{A}+\widehat{E}+(6-u)G \text{ if } 1\leqslant u\leqslant 4,\\ \dfrac{6-u}{2}(\widehat{A}+\widehat{E}+2G) \text{ if } 4\leqslant u\leqslant 6,\end{cases}$$

and

$$N(u)=\begin{cases} 0 \text{ if } 0\leqslant u\leqslant 1,\\ \dfrac{u-1}{3}\widehat{A} \text{ if } 1\leqslant u\leqslant 4,\\ \dfrac{u-2}{2}\widehat{A}+\dfrac{u-4}{2}\widehat{E} \text{ if } 4\leqslant u\leqslant 6,\end{cases}$$

where $P(u)$ and $N(u)$ are the positive and the negative parts of the Zariski decomposition, respectively. Using this, we compute

$$S_X(G)=\frac{3}{12}\int_0^1 u^3du+\frac{3}{12}\int_1^4 udu+\frac{3}{12}\int_4^6\frac{u(6-u)^2}{4}du=\frac{43}{16}<3=A_X(G).$$

Let us apply [15, Theorem 4.8], [15, Corollary 4.17], [15, Corollary 4.18] using notations introduced in [15, § 4]. To start with, we apply [15, Corollary 4.18] to get

$$(2.2)\qquad 1\geqslant\frac{A_X(\mathbf{F})}{S_X(\mathbf{F})}\geqslant\min\left\{\frac{A_X(G)}{S_X(G)},\inf_{Q\in G}\delta_Q\big(G,V^G_{\bullet,\bullet}\big)\right\}=\min\left\{\frac{48}{43},\inf_{Q\in G}\delta_Q\big(G,V^G_{\bullet,\bullet}\big)\right\},$$

where $\delta_Q(G,V^G_{\bullet,\bullet})$ is defined in [15, § 4]. Let Q be an arbitrary point in the surface G, and let ℓ is a general line in $G\cong\mathbb{P}^2$ that contains Q. Then [15, Corollary 4.18] gives

$$\delta_Q\big(G,V^G_{\bullet,\bullet}\big)\geqslant\min\left\{\frac{1}{S\big(V^G_{\bullet,\bullet};\ell\big)},\frac{1}{S\big(W^{G,\ell}_{\bullet,\bullet,\bullet};Q\big)}\right\},$$

where $S(V^F_{\bullet,\bullet};\ell)$ and $S(W^{G,\ell}_{\bullet,\bullet,\bullet};Q)$ are defined in [15, § 4]. Take $v\in\mathbb{R}_{\geqslant 0}$. Then

$$P(u)\big|_G-v\ell\sim_{\mathbb{R}}\begin{cases}(u-v)\ell \text{ if } 0\leqslant u\leqslant 1,\\ (1-v)\ell \text{ if } 1\leqslant u\leqslant 4,\\ \dfrac{6-u-2v}{2}\ell \text{ if } 4\leqslant u\leqslant 6.\end{cases}$$

Let $\mathscr{C} = \widehat{A}|_G$. Then $\mathscr{C}$ is a smooth cubic curve in $G \cong \mathbb{P}^2$. Let

$$N'(u) = N(u)\big|_G = \begin{cases} 0 \text{ if } 0 \leqslant u \leqslant 1, \\ \dfrac{u-1}{3}\mathscr{C} \text{ if } 1 \leqslant u \leqslant 4, \\ \dfrac{u-2}{2}\mathscr{C} \text{ if } 4 \leqslant u \leqslant 6. \end{cases}$$

Now, using [15, Theorem 4.8], we get

$$S\big(W^G_{\bullet,\bullet};\ell\big) = \frac{3}{12}\int_0^6\int_0^\infty \mathrm{vol}\big(P(u)\big|_G - v\ell\big)dvdu = \frac{3}{12}\int_0^1\int_0^u (u-v)^2dvdu + \\ + \frac{3}{12}\int_1^4\int_0^1 (1-v)^2dvdu + \frac{3}{12}\int_4^6\int_0^{\frac{6-u}{2}}\left(\frac{6-u-2v}{2}\right)^2dvdu = \frac{5}{16}.$$

Similarly, it follows from [15, Theorem 4.17] that

$$S\big(W^{G,\ell}_{\bullet,\bullet,\bullet};Q\big) = \frac{3}{12}\int_0^1\int_0^u \Big(\big(P(u)\big|_G - v\ell\big)\cdot\ell\Big)^2dvdu + \frac{3}{12}\int_1^4\int_0^1 \Big(\big(P(u)\big|_G - v\ell\big)\cdot\ell\Big)^2dvdu + \\ + \frac{3}{12}\int_4^6\int_0^{\frac{6-u}{2}} \Big(\big(P(u)\big|_G - v\ell\big)\cdot\ell\Big)^2dvdu + F_Q = \frac{3}{12}\int_0^1\int_0^u (u-v)^2dvdu + \\ + \frac{3}{12}\int_1^4\int_0^1 (1-v)^2dvdu + \frac{3}{12}\int_4^6\int_0^{\frac{6-u}{2}}\left(\frac{6-u-2v}{2}\right)^2dvdu + F_Q = \frac{5}{16} + F_Q,$$

where

$$F_Q = \frac{6}{12}\int_1^4\int_0^1 \Big(\big(P(u)\big|_G - v\ell\big)\cdot\ell\Big)\mathrm{ord}_Q\big(N'(u)\big|_\ell\big)dvdu + \\ + \frac{6}{12}\int_4^6\int_0^{\frac{6-u}{2}} \Big(\big(P(u)\big|_G - v\ell\big)\cdot\ell\Big)\mathrm{ord}_Q\big(N'(u)\big|_\ell\big)dvdu \leqslant \\ \leqslant \frac{6}{12}\int_1^4\int_0^1 \frac{(1-v)(u-1)}{3}dvdu + \frac{6}{12}\int_4^6\int_0^{\frac{6-u}{2}} \frac{(6-u-2v)(u-2)}{4}dvdu = \frac{7}{12}.$$

So, we have $S(W^{G,\ell}_{\bullet,\bullet,\bullet};Q) \leqslant \frac{43}{48}$. Then $\delta_Q(G, V^G_{\bullet,\bullet}) > 1$, which contradicts (2.2). □

Thus, we see that P is a Du Val singular point of the surface A of type $\mathbb{D}_4$, $\mathbb{D}_5$, $\mathbb{E}_6$. Now, arguing as in the proof of [19, Lemma 9.11], we see that $\beta(\mathbf{F}) > 0$ if

(1) the inequality $\beta(G) > 0$ holds,
(2) and for every prime divisor $\mathbf{E}$ over X such that $C_X(\mathbf{E})$ is a curve containing P, the following inequality holds:

$$\frac{A_X(\mathbf{E})}{S_X(\mathbf{E})} \geqslant \frac{4}{3}.$$

Since $\beta(\mathbf{F}) \leqslant 0$ by our assumption, we see that at least one of these conditions must fail.

Lemma 2.11. *One has $\beta(G) \geqslant \frac{465}{2048}$.*

Proof. Let $\widehat{A}$ and $\widehat{E}$ be the proper transforms on $\widehat{X}$ of the surfaces A and E, respectively. Take $u \in \mathbb{R}_{\geqslant u}$. Then

$$\sigma^*(-K_X) - uG \sim \sigma^*(2H - E) - uG \sim \sigma^*(2A + E) - uG \sim 2\widehat{A} + \widehat{E} + (4 - u)G,$$

which easily implies that the divisor $-K_{\widehat{X}} - G$ is pseudoeffective $\iff u \leqslant 4$, because we can contract the surfaces $\widehat{A}$ and $\widehat{E}$ simultaneously after flops. Then

$$\beta(G) = A_X(G) - S_X(G) = 3 - \frac{1}{12}\int_0^4 \mathrm{vol}\big(\sigma^*(-K_X) - uG\big)du.$$

Note that $\sigma^*(-K_X) - uG$ is nef for $u \in [0, 1]$, because the divisor $-K_X$ is very ample. Thus, if $u \in [0, 1]$, then

$$\mathrm{vol}\big(\sigma^*(-K_X) - uG\big) = \big(\sigma^*(-K_X) - uG\big)^3 = 12 - u^3.$$

Similarly, if $1 \leqslant u \leqslant \frac{3}{2}$, then

$$\mathrm{vol}\big(\sigma^*(-K_X) - uG\big) \leqslant\leqslant \mathrm{vol}\big(\sigma^*(-K_X) - G\big) = \big(\sigma^*(-K_X) - G\big)^3 = 11.$$

Finally, let us estimate $\mathrm{vol}(\sigma^*(-K_X) - uG)$ in the case when $4 \geqslant u > \frac{3}{2}$.

Let Z be a general hyperplane section of the cubic surface A that passes through P, and let $\widehat{Z}$ be its proper transform on the threefold $\widehat{X}$. Then Z is an irreducible cuspidal cubic curve, and $\widehat{Z} \subset \widehat{A}$. Observe that $(\sigma^*(-K_X) - uG) \cdot \widehat{Z} = 3 - 2u$ and $\widehat{A} \cdot \widehat{Z} = -4$, so $\widehat{A}$ is contained in the asymptotic base locus of the divisor $\sigma^*(-K_X) - uG$ for $u > \frac{3}{2}$. Moreover, if $\sigma^*(-K_X) - uG \sim_{\mathbb{R}} \widehat{D} + \lambda\widehat{A}$ for $\lambda \in \mathbb{R}_{\geqslant 0}$ and an effective $\mathbb{R}$-divisor $\widehat{D}$ whose support does not contain $\widehat{A}$, then $\widehat{Z} \not\subset \widehat{D}$, which implies that

$$0 \leqslant \widehat{D} \cdot \widehat{Z} = \Big(\sigma^*(-K_X) - uG - \lambda\widehat{A}\Big) \cdot \widehat{Z} = 3 - 2u - 4\lambda,$$

so that $\lambda \geqslant \frac{3-2u}{4}$. Thus, if $4 \geqslant u > \frac{3}{2}$, then

$$\mathrm{vol}\big(\sigma^*(-K_X) - uG\big) \leqslant \mathrm{vol}\Big(\sigma^*(2H - E) - uG - \frac{2u - 3}{4}\widehat{A}\Big).$$

Moreover, if $4 \geqslant u > \frac{3}{2}$, then

$$\sigma^*(2H - E) - uG - \frac{2u - 3}{4}\widehat{A} \sim_{\mathbb{R}} \frac{11 - 2u}{4}\sigma^*(H) - \frac{7 - 2u}{2}\sigma^*(E) - \frac{3}{2}G.$$

Therefore, if $4 \geqslant u > \frac{3}{2}$, then

$$\mathrm{vol}\big(\sigma^*(-K_X) - uG\big) \leqslant \mathrm{vol}\Big(\frac{11 - 2u}{4}\sigma^*(H) - \frac{7 - 2u}{2}\sigma^*(E)\Big) = \mathrm{vol}\Big(\frac{11 - 2u}{4}H - \frac{7 - 2u}{2}E\Big).$$

Furthermore, if $\frac{7}{2} \geqslant u > \frac{3}{2}$, then $\frac{11-2u}{4}H - \frac{7-2u}{2}E$ is nef, so that

$$\mathrm{vol}\Big(\frac{11 - 2u}{4}H - \frac{7 - 2u}{2}E\Big) = \Big(\frac{11 - 2u}{4}H - \frac{7 - 2u}{2}E\Big)^3 = \frac{25 - 6u}{16}.$$

Similarly, if $4 \geqslant u > \frac{7}{2}$, then

$$\mathrm{vol}\Big(\frac{11 - 2u}{4}H - \frac{7 - 2u}{2}E\Big) = \Big(\frac{11 - 2u}{4}H\Big)^3 = \frac{(11 - 2u)^3}{256}.$$

Now, we can estimate $\beta(G)$ as follows

$$\beta(G) = 3 - \frac{1}{12}\int_0^4 \mathrm{vol}\big(\sigma^*(-K_X) - uG\big)du \geqslant 3 - \frac{1}{12}\int_0^1 (12-u^3)du - \frac{1}{12}\int_1^{\frac{3}{2}} 11du-$$
$$- \frac{1}{12}\int_{\frac{3}{2}}^{\frac{7}{2}} \frac{25-6u}{16}du - \frac{1}{12}\int_{\frac{7}{2}}^4 \frac{(11-2u)^3}{256}du = 3 - \frac{5679}{2048} = \frac{465}{2048}$$

as claimed. □

Therefore, there exists a prime divisor $\mathbf{E}$ over X such that $C_X(\mathbf{E})$ is a curve, $P \in C_X(\mathbf{E})$, and $A_X(\mathbf{E}) < \frac{4}{3}S_X(\mathbf{E})$. Set $Z = C_X(\mathbf{E})$. Then $\delta_O(X) < \frac{4}{3}$ for every point $O \in Z$.

Lemma 2.12. *One has $Z \subset A$, and Z is a line in the cubic surface A.*

Proof. Let O be a general point in Z, and let A_O be the fiber of ϕ that passes through O. If $Z \not\subset A$, then A_O is smooth, so that $\delta_O(A_O) \geqslant \frac{3}{2}$ by [3, Lemma 2.13], which gives

$$\frac{4}{3} > \frac{A_X(\mathbf{E})}{S_X(\mathbf{E})} \geqslant \delta_O(X) \geqslant \min\left\{\frac{16}{11}, \frac{16\delta_O(A_O)}{16\delta_O(A_O)+15}\right\} \geqslant \min\left\{\frac{16}{11}, \frac{16\times\frac{3}{2}}{\frac{3}{2}+15}\right\} = \frac{16}{11} > \frac{4}{3},$$

by Lemma 2.1. This shows that $Z \subset A$ and $A_O = A$.

To complete the proof of the lemma, we have to show that Z is a line in the surface A. Suppose that Z is not a line. Then the point O is not contained in a line in the surface A, because A contains finitely many lines [6]. Now, arguing as in the proof of [3, Lemma 2.13], we get $\delta_O(A) \geqslant \frac{3}{2}$. So, applying Lemma 2.1 again, we get a contradiction as above. □

Now, our Auxiliary Theorem follows from the following lemma:

Lemma 2.13. *The surface A does not have a singular point of type $\mathbb{D}_4$.*

Proof. Suppose A has singularity of type $\mathbb{D}_4$. Then, it follows from [6] that, for a suitable choice of coordinates x, y, z, t on the projective space $\mathbb{P}^3$, one of the following cases hold:

(A) $A = \{tx^2 = y^3 - z^3\} \subset \mathbb{P}^3$,
(B) $A = \{tx^2 = y^3 - z^3 + xyz\} \subset \mathbb{P}^3$.

Note that $P = [0:0:0:1]$, and A contains 6 lines [6]. In case (A), these lines are

$$L_1 = \{x = y - z = 0\},$$
$$L_2 = \{x = y - \omega_3 z = 0\},$$
$$L_3 = \{x = y + \omega_3^2 z = 0\},$$
$$L_4 = \{t = y - z = 0\},$$
$$L_5 = \{t = y + \omega_3 z = 0\},$$
$$L_6 = \{t = y + \omega_3^2 z = 0\},$$

where ω_3 is a primitive cube root of unity. In case (B), these lines are

$$
\begin{aligned}
L_1 &= \{x = y - z = 0\},\\
L_2 &= \{x = y - \omega_3 z = 0\},\\
L_3 &= \{x = y - \omega_3^2 z = 0\},\\
L_4 &= \{x + 3(y - z) = y - z - 9t = 0\},\\
L_5 &= \{x + 3\omega_3(y - \omega_3 z) = \omega_3 y - \omega_3^2 z - 9t = 0\},\\
L_6 &= \{x + 3\omega_3^2(y - \omega_3^2 z) = \omega_3^2 y - \omega_3 z - 9t = 0\}.
\end{aligned}
$$

Note that $P = L_1 \cap L_3 \cap L_3$, $P \notin L_4 \cup L_5 \cup L_6$ and $-K_A \sim 2L_1 + L_4 \sim 2L_2 + L_5 \sim 2L_3 + L_6$.

By Lemma 2.12, we may assume that $Z = L_1$.

Recall that $S_X(A) = \frac{11}{16}$, see the proof of Lemma 2.1. Using [3, Theorem 1.7.30], we get

$$
\frac{4}{3} > \frac{A_X(\mathbf{E})}{S_X(\mathbf{E})} \geqslant \min\left\{\frac{1}{S_X(A)}, \frac{1}{S(W^A_{\bullet,\bullet}; L_1)}\right\} = \min\left\{\frac{16}{11}, \frac{1}{S(W^A_{\bullet,\bullet}; L_1)}\right\},
$$

where $S(W^A_{\bullet,\bullet}; L_1)$ is defined in [3, § 1.7]. Therefore, we conclude that $S(W^A_{\bullet,\bullet}; L_1) < \frac{4}{3}$. Let us compute $S(W^A_{\bullet,\bullet}; L_1)$ using [3, Corollary 1.7.26].

To do this, we use notations introduced in the proof of Lemma 2.1 applied to $O = P$. Then, using [3, Corollary 1.7.26] and computations from the proof of Lemma 2.1, we get

$$
S\big(W^A_{\bullet,\bullet}; L_1\big) = \frac{1}{4}\int_0^1\int_0^\infty \mathrm{vol}\big(-K_A - vL_1\big)dvdu + \frac{1}{4}\int_1^2\int_0^\infty \mathrm{vol}\big((2-u)(-K_A) - vL_1\big)dvdu,
$$

since $L_1 \not\subset \mathrm{Supp}(N(u))$, since $L_1 \not\subset E$. Let us compute $S(W^A_{\bullet,\bullet}; L_1)$. Take $v \in \mathbb{R}_{\geqslant 0}$. Then

$$
-K_A - vL_1 \sim_{\mathbb{R}} (2 - v)L_1 + L_4.
$$

Thus, the divisor $-K_A - vL_1$ is pseudoeffective $\iff$ $v \leqslant 2$, since $L_4^2 = -1$. Fix $v \in [0, 2]$. Let $P(u, v)$ be the positive part of the Zariski decomposition of the divisor $-K_A - vL_1$, and let $N(u, v)$ be its negative part. Then

$$
P(u, v) = \begin{cases} (2 - v)L_1 + L_4 \text{ if } 0 \leqslant v \leqslant 1,\\ (2 - v)(L_1 + L_4) \text{ if } 1 \leqslant v \leqslant 2,\end{cases}
$$

and

$$
N(u, v) = \begin{cases} 0 \text{ if } 0 \leqslant v \leqslant 1,\\ (v - 1)L_4 \text{ if } 1 \leqslant v \leqslant 2.\end{cases}
$$

Thus, if $0 \leqslant v \leqslant 1$, then $\mathrm{vol}(-K_A - vL_1) = 3 - 2v$, because $L_1^2 = 0$ and $L_1 \cdot L_4 = 0$. Similarly, if $1 \leqslant v \leqslant 2$, then $\mathrm{vol}(-K_A - vL_1) = (v - 2)^2$. This gives

$$
\frac{1}{4}\int_0^1\int_0^\infty \mathrm{vol}\big(-K_A - vL_1\big)dvdu = \frac{1}{4}\int_0^1\int_0^1(3 - 2v)dvdu + \frac{1}{4}\int_0^1\int_1^2(v - 2)^2dvdu = \frac{7}{12}
$$

and

$$
\frac{1}{4}\int_1^2\int_0^\infty \mathrm{vol}\big((2 - u)(-K_A) - vL_1\big)dvdu = \frac{1}{4}\int_1^2\int_0^\infty(2 - u)^3\mathrm{vol}\big((-K_A) - vL_1\big)dvdu =
$$

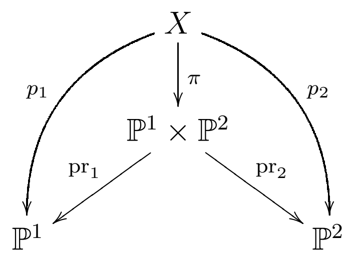
$$
= \frac{1}{4}\int_1^2\int_0^1(2 - u)^3(3 - 2v)dvdu + \frac{1}{4}\int_1^2\int_1^2(2 - u)^3(v - 2)^2dvdu = \frac{7}{48}.
$$

Combining, we get $S(W^A_{\bullet,\bullet}; L_1) = \frac{35}{48} < \frac{3}{4}$. This is a contradiction. □

3. Family №2.2.

Let R be a smooth surface of degree $(2,4)$ in $\mathbb{P}^1 \times \mathbb{P}^2$, let $\pi\colon X \to \mathbb{P}^1 \times \mathbb{P}^2$ be a double cover ramified over surface R. Then X is a Fano threefold in the deformation family № 2.2. Moreover, all smooth Fano threefolds in this family can be obtained this way.

Let $\mathrm{pr}_1\colon \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^1$ and $\mathrm{pr}_2\colon \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^2$ be the projections to the first and the second factors, respectively. Set $p_1 = \mathrm{pr}_1 \circ \pi$ and $p_2 = \mathrm{pr}_2 \circ \pi$. We have the following commutative diagram:



where p_1 is a fibration into del Pezzo surfaces of degree 2, and p_2 is a conic bundle.

Lemma 3.1. *Let S be a fiber of the morphism p_1. Then S is irreducible and normal.*

Proof. Left to the reader. □

Lemma 3.2. *Let S be a fiber of the morphism p_1, let C be a fiber of the morphism p_2, and let P be a point in $S \cap C$. Then S or C is smooth at P.*

Proof. Local computations. □

Now, we are ready to prove that X is K-stable. Recall from [9] that $\mathrm{Aut}(X)$ is finite. Thus, the threefold X is K-stable if and only if it is K-polystable [19].

Let τ be the Galois involution of the double cover $\pi\colon X \to \mathbb{P}^1 \times \mathbb{P}^2$, and let $G = \langle\tau\rangle$. Suppose that X is not K-polystable. Then it follows from [20, Corollary 4.14] that there exists a G-invariant prime divisor $\mathbf{F}$ over X such that

$$\beta(\mathbf{F}) = A_X(\mathbf{F}) - S_X(\mathbf{F}) \leqslant 0.$$

Let Z be the center of this divisor on X. Then Z is not a surface by [3, Theorem 3.7.1]. Hence, we see that either Z is a G-invariant irreducible curve, or Z is a G-fixed point. Let us seek for a contradiction.

Let P be a general point in Z, and let S be the fiber of p_1 that passes through P.

Lemma 3.3. *The surface S is singular at P.*

Proof. Suppose that S is smooth at P. Let C be a general curve in $|-K_S|$ that contains P. Then C is smooth. Applying [3, Theorem 1.7.30] to the flag $P \in C \subset S$, we get

$$1 \geqslant \frac{A_X(\mathbf{F})}{S_X(\mathbf{F})} \geqslant \min\left\{\frac{1}{S_X(S)}, \frac{1}{S(W^S_{\bullet,\bullet};C)}, \frac{1}{S(W^{S,C}_{\bullet,\bullet,\bullet};P)}\right\}.$$

Since $S_X(S) < 1$ by [3, Theorem 3.7.1], we see that $S(W^S_{\bullet,\bullet};C) \geqslant 1$ or $S(W^{S,C}_{\bullet,\bullet,\bullet};P) \geqslant 1$. We refer the reader to [3, § 1.7] for definitions of $S_X(S)$, $S(W^S_{\bullet,\bullet};C)$, $S(W^{S,C}_{\bullet,\bullet,\bullet};P)$.

Note that [3, Theorem 1.7.30] requires S to have Du Val singularities, but S may have non-Du Val singularities. Nevertheless, we still can apply [3, Theorem 1.7.30] here, since the proof of [3, Theorem 1.7.30] remains valid in our case, because S is smooth along C.

Let us compute $S(W^S_{\bullet,\bullet};C)$ and $S(W^{S,C}_{\bullet,\bullet,\bullet};P)$. Take $u \in \mathbb{R}_{\geqslant 0}$ and $v \in \mathbb{R}_{\geqslant 0}$. Then

$$-K_X-uS \text{ is nef } \iff -K_X-uS \text{ is pseudoeffective } \iff u\leqslant 1,$$

Similarly, if $u\in[0,1]$, then $(-K_X-uS)|_S-vC\sim_{\mathbb{R}}(1-v)(-K_S)$, so

$$(-K_X-uS)|_S-vC \text{ is nef } \iff (-K_X-uS)|_S-vC \text{ is pseudoeffective } \iff v\leqslant 1.$$

Now, applying [3, Corollary 1.7.26], we get

$$S(W^S_{\bullet,\bullet};C)=\frac{3}{6}\int_0^1\int_0^1\big((1-v)(-K_S)\big)^2dvdu=\frac{1}{2}\int_0^1\int_0^1 2(1-v)^2dv=\frac{1}{3}<1.$$

Similarly, using [3, Theorem 1.7.30], we get

$$S(W^{S,C}_{\bullet,\bullet,\bullet},P)=\frac{3}{6}\int_0^1\int_0^1\big((1-v)(-K_S)\cdot C\big)^2dvdu=\frac{2}{3}<1.$$

But we already know that $S(W^S_{\bullet,\bullet};C)\geqslant 1$ or $S(W^{S,C}_{\bullet,\bullet,\bullet};P)\geqslant 1$. This is a contradiction. $\square$

If Z is a curve, then S is smooth at P by Lemma 3.1, because P is a general point in Z. Hence, we conclude that $Z=P$, because S is singular at the point P by Lemma 3.3. Recall that Z is G-invariant. This implies that $\tau(P)\in R$.

Let C be the fiber of p_2 that passes through P. Then C is smooth at P by Lemma 3.2, because S is singular at P. Since $\tau(P)\in R$, we see that C is irreducible and smooth.

Let T be a sufficiently general surface in linear system $|\pi_2^*(\mathcal{O}_{\mathbb{P}^2}(1))|$ that contains C. Since C is smooth, it follows from Bertini's theorem that the surface T is smooth.

As in the proof of Lemma 3.3, it follows from [3, Theorem 1.7.30] that

$$1\geqslant\frac{A_X(\mathbf{F})}{S_X(\mathbf{F})}\geqslant\min\left\{\frac{1}{S_X(T)},\frac{1}{S(W^T_{\bullet,\bullet};C)},\frac{1}{S(W^{T,C}_{\bullet,\bullet,\bullet};P)}\right\}.$$

Moreover, it follows from [3, Theorem 3.7.1] that $S_X(T)<1$. Thus, we conclude that

$$\max\big\{S(W^T_{\bullet,\bullet};C),S(W^{T,C}_{\bullet,\bullet,\bullet};P)\big\}\geqslant 1.$$

In fact, since P is the center of the divisor F on X, [3, Theorem 3.7.1] gives

$$\max\big\{S(W^T_{\bullet,\bullet};C),S(W^{T,C}_{\bullet,\bullet,\bullet};P)\big\}>1. \tag{3.1}$$

Now, let us compute $S(W^T_{\bullet,\bullet};C)$ and $S(W^{T,C}_{\bullet,\bullet,\bullet};P)$ using the results obtained in [3, § 1.7].

Take $u\in\mathbb{R}_{\geqslant 0}$ and $v\in\mathbb{R}_{\geqslant 0}$. Then

$$-K_X-uT \text{ is nef } \iff -K_X-uT \text{ is pseudoeffective } \iff u\leqslant 1,$$

Similarly, if $u\in[0,1]$, then

$(-K_X-uT)|_T-vC$ is nef $\iff$ $(-K_X-uT)|_T-vC$ is pseudoeffective $\iff$ $v\leqslant 1-u$, because $(-K_X-uT)|_T-vC\sim_{\mathbb{R}}S|_T+(1-u-v)C$. So, using [3, Corollary 1.7.26], we get

$$S(W^T_{\bullet,\bullet};C)=\frac{3}{6}\int_0^1\int_0^{1-u}\big(S|_T+(1-u-v)C\big)^2dvdu=\frac{1}{2}\int_0^1\int_0^{1-u}4(1-u-v)dvdu=\frac{1}{3}<1.$$

Hence, it follows from (3.1) that $S(W^{T,C}_{\bullet,\bullet,\bullet},P)>1$. Now, using [3, Theorem 1.7.30], we get

$$S(W^{T,C}_{\bullet,\bullet,\bullet},P)=\frac{3}{6}\int_0^1\int_0^{1-u}\Big(\big(S|_T+(1-u-v)C\big)\cdot C\Big)^2dvdu=\frac{3}{6}\int_0^1\int_0^{1-u}4\,dvdu=1,$$

which is a contradiction. This shows that X is K-stable.

Corollary 3.4. *All smooth Fano threefolds in the family №2.2 are K-stable.*

4. Family №2.4.

Let $\mathscr{S}$ and $\mathscr{S}'$ be smooth cubic surfaces in $\mathbb{P}^3$ such that their intersection is a smooth curve of genus 10. Set $\mathscr{C} = \mathscr{S} \cap \mathscr{S}'$, and let $\pi\colon X \to \mathbb{P}^3$ be the blow up of the curve $\mathscr{C}$. Then X is a smooth Fano threefold in the family №2.4, and every smooth Fano threefold in this family can be obtained in this way. Moreover, there exists a commutative diagram

$$\begin{array}{ccc} & X & \\ {\scriptstyle\pi}\swarrow & & \searrow{\scriptstyle\phi} \\ \mathbb{P}^3 & \dashrightarrow & \mathbb{P}^1 \end{array}$$

where $\mathbb{P}^3 \dashrightarrow \mathbb{P}^1$ is a map that is given by the pencil generated by the surfaces $\mathscr{S}$ and $\mathscr{S}'$, and ϕ is a fibration into cubic surfaces. Note that $-K_X^3 = 10$ and $\mathrm{Aut}(X)$ is finite [9].

Let $H = \pi^*(\mathcal{O}_{\mathbb{P}^3}(1))$, and let E be the π-exceptional surface. Then $-K_X \sim 4H - E$, the morphism ϕ is given by the linear system $|3H - E|$, and $E \cong \mathscr{S} \times \mathbb{P}^1$.

The goal of this section is to prove that X is K-stable. Suppose that X is not K-stable. Let us seek for a contradiction. First, using the valuative criterion for K-stability [14, 16], we see that there exists a prime divisor $\mathbf{F}$ over X such that

$$\beta(\mathbf{F}) = A_X(\mathbf{F}) - S_X(\mathbf{F}) \leqslant 0.$$

Let Z be the center of the divisor $\mathbf{F}$ on X. Then Z is not a surface by [3, Theorem 3.7.1]. Therefore, either Z is an irreducible curve or Z is a point. Fix a point $P \in Z$.

Let A be the surface in $|3H - E|$ that contains P. Fix $u \in \mathbb{R}_{\geqslant 0}$. Let $\mathscr{P}(u)$ be the positive part of the Zariski decomposition of $-K_X - uA$, and let $\mathscr{N}(u)$ be its negative part. Then

$$-K_X - uA \sim_{\mathbb{R}} (4 - 3u)H - (1 - u)E \sim_{\mathbb{R}} \Big(\frac{4}{3} - u\Big)A + \frac{1}{3}E.$$

This implies that $-K_X - uA$ is pseudoeffective $\iff u \leqslant \frac{4}{3}$. Moreover, we have

$$\mathscr{P}(u) = \begin{cases} (4 - 3u)H - (1 - u)E \text{ if } 0 \leqslant u \leqslant 1, \\ (4 - 3u)H \text{ if } 1 \leqslant v \leqslant \dfrac{4}{3}, \end{cases}$$

and

$$\mathscr{N}(u) = \begin{cases} 0 \text{ if } 0 \leqslant u \leqslant 1, \\ (u - 1)E \text{ if } 1 \leqslant u \leqslant \dfrac{4}{3}. \end{cases}$$

Integrating, we obtain $S_X(A) = \frac{67}{120} < 1$, which also follows from [3, Theorem 3.7.1].

Note that $\pi(A)$ is a normal cubic surface in $\mathbb{P}^3$, and $\pi(A)$ is smooth along the curve $\mathscr{C}$. In particular, we see that $A \cong \pi(A)$, and A is smooth along the intersection $E \cap A$.

Lemma 4.1. *The surface A is singular at the point P.*

Proof. Suppose that A is smooth at P. Let C be a general curve in $|-K_A|$ that passes through the point P. Then C is a smooth irreducible elliptic curve. Take $v \in \mathbb{R}_{\geqslant 0}$. Then

$$\mathscr{P}(u)\big|_S - vC \sim_{\mathbb{R}} \begin{cases} (1 - v)C \text{ if } 0 \leqslant u \leqslant 1, \\ (4 - 3u - v)C \text{ if } 1 \leqslant u \leqslant \dfrac{4}{3}. \end{cases}$$

Therefore, using [3, Corollary 1.7.26], we obtain

$$S\big(W^{A}_{\bullet,\bullet};C\big)=\frac{3}{10}\int_0^1\int_0^1 3(1-v)^2dvdu+\frac{3}{10}\int_1^{\frac{4}{3}}\int_0^{4-3u}3(4-3u-v)^2dvdu=\frac{13}{40}.$$

Similarly, using [3, Theorem 1.7.30], we obtain

$$S\big(W^{A,C}_{\bullet,\bullet,\bullet};P\big)\leqslant\frac{3}{10}\int_0^1\int_0^1\big(3(1-v)\big)^2dvdu+\frac{3}{10}\int_1^{\frac{4}{3}}\int_0^{4-3u}\big(3(4-3u-v)\big)^2dvdu+ \\ +\underbrace{\frac{6}{10}\int_1^{\frac{4}{3}}\int_0^{4-3u}3(4-3u-v)(u-1)dvdu}_{\text{if } P\in E}=\frac{39}{40}+\frac{1}{120}=\frac{59}{60}.$$

Therefore, it follows from [3, Theorem 1.7.30] that

$$1\geqslant\frac{A_X(\mathbf{F})}{S_X(\mathbf{F})}\geqslant\min\left\{\frac{1}{S_X(A)},\frac{1}{S(W^{A}_{\bullet,\bullet};C)},\frac{1}{S(W^{A,C}_{\bullet,\bullet,\bullet};P)}\right\}\geqslant\min\left\{\frac{120}{67},\frac{40}{13},\frac{60}{59}\right\}>1,$$

which is absurd. ☐

Corollary 4.2. *The point P is not contained in the surface E.*

Since $A\cong\pi(A)$, we may consider A as a cubic surface in $\mathbb{P}^3$. Then

- either $\mathrm{mult}_P(A)=2$ and A has Du Val singularities.
- or $\mathrm{mult}_P(A)=3$ and A is a cone over a plane smooth cubic curve with vertex P.

Lemma 4.3. *One has* $\mathrm{mult}_P(A)\neq 3$.

Proof. Let $\sigma\colon\widehat{X}\to X$ be a blow up of the point P, and let G be the σ-exceptional surface. Denote by $\widehat{A}$ and $\widehat{E}$ the proper transforms on $\widehat{X}$ of the surfaces A and E, respectively. Suppose that $\mathrm{mult}_P(A)=3$. Take $u\in\mathbb{R}_{\geqslant 0}$. Then

$$\sigma^*(-K_X)-vG\sim_{\mathbb{R}}\frac{1}{3}\widehat{E}+\frac{4}{3}\widehat{A}+(4-u)G.$$

Thus, the divisor $\sigma^*(-K_X)-vG$ is pseudo-effective $\iff$ $u\in[0,4]$. Moreover, if $u\in[0,4]$, then the Zariski decomposition of the divisor $\sigma^*(-K_X)-vG$ can be described as follows:

$$P(u)=\begin{cases}\frac{1}{3}\widehat{E}+\frac{4}{3}\widehat{A}+(4-u)G \text{ if } 0\leqslant u\leqslant 1,\\ \frac{1}{3}\widehat{E}+\frac{5-u}{3}\widehat{A}+(4-u)G\text{if } 1\leqslant u\leqslant 4,\end{cases}$$

and

$$N(u)=\begin{cases}0 \text{ if } 0\leqslant u\leqslant 1,\\ \frac{u-1}{3}\widehat{A} \text{ if } 1\leqslant u\leqslant 4,\end{cases}$$

where $P(u)$ and $N(u)$ are the positive and the negative parts of the Zariski decomposition, respectively. Using this, we compute

$$S_X(G)=\frac{3}{10}\int_0^1 u^3,du+\frac{3}{10}\int_1^4 udu=\frac{93}{40}<3=A_X(G).$$

As in the proof of Lemma 2.10, let us use results obtained in [15, § 4] to get a contradiction. Namely, applying [15, Corollary 4.18], we get

$$1 \geqslant \frac{A_X(\mathbf{F})}{S_X(\mathbf{F})} \geqslant \min\left\{\frac{A_X(G)}{S_X(G)}, \inf_{Q\in G}\delta_Q\big(G, V^G_{\bullet,\bullet}\big)\right\} = \min\left\{\frac{40}{31}, \inf_{Q\in G}\delta_Q\big(G, V^G_{\bullet,\bullet}\big)\right\},$$

where $\delta_Q(G, V^G_{\bullet,\bullet})$ is defined in [15, § 4]. So, there is $Q \in G$ such that $\delta_Q(G, V^G_{\bullet,\bullet}) < \frac{40}{31}$.

Let ℓ is a general line in $G \cong \mathbb{P}^2$ that contains Q. Then [15, Corollary 4.18] gives

$$\delta_Q\big(G, V^G_{\bullet,\bullet}\big) \geqslant \min\left\{\frac{1}{S\big(V^F_{\bullet,\bullet};\ell\big)}, \frac{1}{S\big(W^{G,\ell}_{\bullet,\bullet,\bullet};Q\big)}\right\}.$$

Let us compute $S(V^G_{\bullet,\bullet};\ell)$ and $S(W^{G,\ell}_{\bullet,\bullet,\bullet};Q)$. Take $v \in \mathbb{R}_{\geqslant 0}$. Then

$$P(u)\big|_G - v\ell \sim_{\mathbb{R}} \begin{cases}(u-v)\ell \text{ if } 0 \leqslant u \leqslant 1,\\ (1-v)\ell \text{ if } 1 \leqslant u \leqslant 4.\end{cases}$$

Let $\mathscr{C} = \widehat{A}|_G$. Then $\mathscr{C}$ is a smooth cubic curve in $G \cong \mathbb{P}^2$. Let

$$N'(u) = N(u)\big|_G = \begin{cases}0 \text{ if } 0 \leqslant u \leqslant 1,\\ \dfrac{u-1}{3}\mathscr{C} \text{ if } 1 \leqslant u \leqslant 4.\end{cases}$$

Now, using [15, Theorem 4.8], we get

$$S\big(W^G_{\bullet,\bullet};\ell\big) = \frac{3}{10}\int_0^4\int_0^\infty \mathrm{vol}\big(P(u)\big|_G - v\ell\big)dvdu =$$
$$= \frac{3}{10}\int_0^1\int_0^u (u-v)^2 dvdu + \frac{3}{10}\int_1^4\int_0^1 (1-v)^2 dvdu = \frac{13}{40}.$$

Similarly, it follows from [15, Theorem 4.17] that $S(W^{G,\ell}_{\bullet,\bullet,\bullet};Q)$ can be computes as follows:

$$\frac{3}{10}\int_0^1\int_0^u \Big(\big(P(u)\big|_G - v\ell\big)\cdot\ell\Big)^2 dvdu + \frac{3}{10}\int_1^4\int_0^1 \Big(\big(P(u)\big|_G - v\ell\big)\cdot\ell\Big)^2 dvdu + F_Q =$$
$$= \frac{3}{12}\int_0^1\int_0^u (u-v)^2 dvdu + \frac{3}{10}\int_1^4\int_0^1 (1-v)^2 dvdu + F_Q = \frac{13}{40} + F_Q,$$

where $F_Q = 0$ if $Q \notin \widehat{A}|_G$, and

$$F_Q = \frac{6}{10}\int_1^4\int_0^1 \frac{(1-v)(u-1)}{3} dvdu = \frac{9}{20}$$

otherwise. This gives $S(W^{G,\ell}_{\bullet,\bullet,\bullet};Q) \leqslant \frac{31}{40}$. Combining the estimates, we get $\delta_Q(G, V^G_{\bullet,\bullet}) \geqslant \frac{40}{31}$, which is a contradiction. This completes the proof of the lemma. □

Hence, we see that the surface A has Du Val singularities. Let S be a general surface in the linear system $|H|$ that contains P. Then S is smooth, and $-K_X - uS \sim_{\mathbb{R}} (4-u)H - E$. Thus, the divisor $-K_X - uS$ is pseudoeffective $\iff$ it is nef $\iff$ $u \leqslant 1$. Then

$$S_X(S) = \frac{3}{10}\int_0^1 (-K_X - uS)^3 du = \frac{3}{10}\int_0^1 u(1-u)(7-u)du = \frac{13}{40} < 1.$$

Let $C = A|_S$. Then C is a reduced curve in $|-K_S|$ that is singular at P, and $C \cong \pi(C)$. Moreover, the curve $\pi(C)$ is a general hyperplane section of the cubic surface $\pi(A) \subset \mathbb{P}^3$

that passes through the point $\pi(P)$. Therefore, since $\pi(A)$ is not a cone by Lemma 4.3, we conclude that the curve C is irreducible. Hence, one of the following two cases holds:

(1) the curve C has an ordinary node at P,
(2) the curve C has an ordinary cusp at P.

Let $\Pi = \pi(S)$. Then Π is a plane in $\mathbb{P}^3$ such that $\pi(P) \in \Pi$ and $\Pi \cap \pi(A) = \pi(C)$, and the morphism $\pi|_S \colon S \to \Pi$ is a composition of blow ups of 9 intersection points $\Pi \cap \mathscr{C}$, which we denote by $O_1, \ldots, O_9$. Note that $\pi(C)$ is a reduced plane cubic curve that passes through these nine points, and $\pi(C)$ is smooth away from $\pi(P)$.

Lemma 4.4. *The curve C cannot have an ordinary double point at the point P.*

Proof. For each $i \in \{1, \ldots, 9\}$, let L_i be the proper transform on S of the line in Π that passes through P and O_i. Then $L_i \neq L_j$ for $i \neq j$, since $\pi(C)$ is irreducible. We have

$$(-K_X - uS)\big|_S \sim_{\mathbb{R}} \frac{1-u}{6} \sum_{i=1}^{9} L_i$$

Let $\sigma \colon \widehat{S} \to S$ be the blow up of S at the point P, let $\mathbf{f}$ be the σ-exceptional curve, let $\widehat{C}, \widehat{L}_1, \ldots, \widehat{L}_9$ be the proper transforms on $\widehat{S}$ of the curves $C, L_1, \ldots, L_9$, respectively. Then $\widehat{L}_1, \ldots, \widehat{L}_9$ are disjoint. On the surface $\widehat{S}$, we have $\mathbf{f}^2 = -1$ and

$$\widehat{C}^2 = -4, \widehat{C} \cdot \widehat{L}_1 = \cdots = \widehat{C} \cdot \widehat{L}_9 = 0, \widehat{C} \cdot \mathbf{f} = 2, \widehat{L}_1^2 = \cdots = \widehat{L}_9^2 = -1, \widehat{L}_1 \cdot \mathbf{f} = \cdots = \widehat{L}_9 \cdot \mathbf{f} = 1.$$

Fix $u \in [0, 1]$. Let v be a non-negative real number. Then

$$\sigma^*\big((-K_X - uS)\big|_S\big) - v\mathbf{f} \sim_{\mathbb{R}} \frac{5+u}{6}\widehat{C} + \frac{1-u}{6} \sum_{i}^{9} \widehat{L}_i + \frac{19-7u-6v}{6}\mathbf{f}. \tag{4.1}$$

Thus, the divisor $\sigma^*((-K_X - uS)|_S) - v\mathbf{f}$ is pseudoeffective $\iff$ it is nef $\iff$ $v \leqslant \frac{19-7u}{6}$. Let $P(u, v)$ and $N(u, v)$ be the positive and the negative parts of its Zariski decomposition. Then, using (4.1), we compute

$$P(u,v) = \begin{cases} \frac{5+u}{6}\widehat{C} + \frac{1-u}{6} \sum_{i}^{9} \widehat{L}_i + \frac{19-7u-6v}{6}\mathbf{f} \text{ if } 0 \leqslant v \leqslant \frac{3-3u}{2}, \\ \frac{19-7u-6v}{12}\widehat{C} + \frac{1-u}{6} \sum_{i}^{9} \widehat{L}_i + \frac{19-7u-6v}{6}\mathbf{f} \text{ if } \frac{3-3u}{2} \leqslant v \leqslant 3-u, \\ \frac{19-7u-6v}{12}\big(\widehat{C} + 2\sum_{i}^{9} \widehat{L}_i + 2\mathbf{f}\big) \text{ if } 3-u \leqslant v \leqslant \frac{19-7u}{6}, \end{cases}$$

$$N(u,v) = \begin{cases} 0 \text{ if } 0 \leqslant v \leqslant \frac{3-3u}{2}, \\ \frac{-3+3u+2v}{4}\widehat{C} \text{ if } \frac{3-3u}{2} \leqslant v \leqslant 3-u, \\ \frac{2v+3u-3}{4}\widehat{C} + (v+u-3) \sum_{i}^{9} \widehat{L}_i \text{ if } 3-u \leqslant v \leqslant \frac{19-7u}{6}, \end{cases}$$

$$\mathrm{vol}\Big(\sigma^*\big((-K_X-uS)\big|_S\big)-v\mathbf{f}\Big)=\begin{cases}(1-u)(7-u)-v^2 \text{ if } 0\leqslant v\leqslant \dfrac{3-3u}{2},\\ \dfrac{(1-u)(37-13u-12v)}{4} \text{ if } \dfrac{3-3u}{2}\leqslant v\leqslant 3-u,\\ \dfrac{(19-7u-6v)^2}{4} \text{ if } 3-u\leqslant v\leqslant \dfrac{19-7u}{6},\end{cases}$$

$$P(u,v)\cdot\mathbf{f}=\begin{cases}v \text{ if } 0\leqslant v\leqslant \dfrac{3-3u}{2},\\ \dfrac{3(1-u)}{2} \text{ if } \dfrac{3-3u}{2}\leqslant v\leqslant 3-u,\\ \dfrac{3(19-7u-6v)}{2} \text{ if } 3-u\leqslant v\leqslant \dfrac{19-7u}{6}.\end{cases}$$

Now, using [3, Corollary 1.7.26], we get

$$S\big(W^{\widehat{S}}_{\bullet,\bullet};\mathbf{f}\big)=\frac{3}{10}\int_0^1\int_0^{\frac{19-7u}{6}}\mathrm{vol}\Big(\sigma^*\big((-K_X-uS)\big|_S\big)-v\mathbf{f}\Big)dudv=$$
$$=\frac{3}{10}\int_0^1\int_0^{\frac{3-3u}{2}}\big((1-u)(7-u)-v^2\big)dv+\frac{3}{10}\int_0^1\int_{\frac{3-3u}{2}}^{3-u}\frac{(1-u)(37-13u-12v)}{4}dv+$$
$$+\frac{3}{10}\int_0^1\int_{3-u}^{\frac{19-7u}{6}}\frac{(19-7u-6v)^2}{4}dvdu=\frac{767}{480}<2=A_S(\mathbf{f}).$$

Moreover, if Q is a point in $\mathbf{f}$, then [3, Remark 1.7.32] gives

$$S\big(W^{\widehat{S},\mathbf{f}}_{\bullet,\bullet,\bullet};Q\big)=F_Q+\frac{3}{10}\int_0^1\int_0^{\frac{19-7u}{6}}\big(P(u,v)\cdot\mathbf{f}\big)^2dvdu=F_Q+\frac{3}{10}\int_0^1\int_0^{\frac{3-3u}{2}}v^2dv+$$
$$+\frac{3}{10}\int_0^1\int_{\frac{3-3u}{2}}^{3-u}\left(\frac{3(1-u)}{2}\right)^2dv+\frac{3}{10}\int_0^1\int_{3-u}^{\frac{19-7u}{6}}\left(\frac{3(19-7u-6v)}{2}\right)^2dvdu=F_Q+\frac{147}{320},$$

where

$$F_Q=\frac{6}{10}\int_0^1\int_0^{\frac{19-7u}{6}}\big(P(u,v)\cdot\mathbf{f}\big)\mathrm{ord}_Q\Big(N(u,v)\big|_{\mathbf{f}}\Big)dvdu,$$

which implies the following assertions:

- if $Q\not\in\widehat{C}\cup\widehat{L}_1\cup\cdots\cup\widehat{L}_9$, then $F_Q=0$;
- if $Q\in\widehat{L}_1\cup\cdots\cup\widehat{L}_9$, then
$$F_Q=\frac{6}{10}\int_0^1\int_{3-u}^{\frac{19-7u}{6}}\frac{3(19-7u-6v)(v+u-3)}{2}dvdu=\frac{1}{960};$$
- if $Q\in\widehat{C}$ and $\widehat{C}$ intersects $\mathbf{f}$ transversally at P, then
$$F_Q=\frac{6}{10}\int_0^1\int_{\frac{3-3u}{2}}^{3-u}\frac{3(1-u)(2v+3u-3)}{8}dvdu+$$
$$\frac{6}{10}\int_0^1\int_{3-u}^{\frac{19-7u}{6}}\frac{3(19-7u-6v)(2v+3u-3)}{8}dvdu=\frac{643}{1920};$$
- if $Q\in\widehat{C}$ and $\widehat{C}$ is tangent to $\mathbf{f}$ at the point P, then $F_Q=\frac{643}{960}$.

Thus, if C has a node at P, then $S(W^{\widehat{S},\mathbf{f}}_{\bullet,\bullet,\bullet};Q)\leqslant\frac{305}{384}$, so [3, Remark 1.7.32] gives

$$1\geqslant\frac{A_X(\mathbf{F})}{S_X(\mathbf{F})}\geqslant\min\left\{\inf_{Q\in\mathbf{f}}\frac{1}{S(W^{\widehat{S},\mathbf{f}}_{\bullet,\bullet,\bullet};Q)},\frac{2}{S(V^{S}_{\bullet,\bullet};\mathbf{f})},\frac{1}{S_X(S)}\right\}\geqslant\min\left\{\frac{384}{305},\frac{960}{767},\frac{40}{13}\right\}=\frac{960}{767}>1,$$

which is a contradiction. This shows that C has a cusp at P. □

Lemma 4.5. *The curve C cannot have an ordinary cusp at the point P.*

Proof. Suppose C has a cusp. Let L be an irreducible curve in S such that $\pi(L)$ is a line and $\pi(L)\cap\pi(C)=\pi(P)$. Then $(-K_X-uS)|_S\sim_{\mathbb{R}}(1-u)L+C$.

Now, we consider the following commutative diagram:

$$\begin{array}{ccccc}
S_1 & \xleftarrow{\sigma_2} & S_2 & \xleftarrow{\sigma_3} & S_3\\
\downarrow{\scriptstyle\sigma_1} & & & & \downarrow{\scriptstyle\upsilon}\\
S & & \xleftarrow{\quad\sigma\quad} & & \widehat{S}
\end{array}$$

where σ_1 is the blow up of P, σ_2 is the blow up of the point in the σ_1-exceptional curve contained in the proper transform of C, σ_3 is the blow up of the point in the σ_2-exceptional curve contained in the proper transform of C, υ is the birational contraction of the proper transforms of $\sigma_1\circ\sigma_2$-exceptional curves, and σ is the birational contraction of the proper transform of the σ_3-exceptional curve. Then $\widehat{S}$ has two singular points:

(1) a cyclic quotient singularity of type $\frac{1}{2}(1,1)$, which we denote by Q_2;
(2) a cyclic quotient singularity of type $\frac{1}{3}(1,1)$, which we denote by Q_3.

Let $\mathbf{f}$ be the σ-exceptional curve, let $\widehat{C}$ be the proper transform on $\widehat{S}$ of the curve C, and let $\widehat{L}$ be the proper transform of the curve L. Then the curves $\mathbf{f}$, $\widehat{C}$, $\widehat{L}$ are smooth. Moreover, it is not very difficult to check that $Q_2\in\mathbf{f}\ni Q_3$, $Q_2\notin\widehat{C}\not\ni Q_3$, $Q_2\in\widehat{L}\not\ni Q_2$. Further, we have $A_S(\mathbf{f})=5$, $\sigma^*(C)\sim\widehat{C}+6\mathbf{f}$, $\sigma^*(L)\sim\widehat{L}+3\mathbf{f}$. On the surface $\widehat{S}$, we have

$$\widehat{L}^2=-\frac{1}{2},\widehat{L}\cdot\widehat{C}=0,\widehat{C}\cdot\mathbf{f}=\frac{1}{2},\widehat{C}^2=-6,\widehat{C}\cdot\mathbf{f}=1,\mathbf{f}^2=-\frac{1}{6}.$$

Note that $Q_2=\mathbf{f}\cap\widehat{L}$, and $\widehat{C}$ intersects $\mathbf{f}$ transversally by one point.

Fix $u\in[0,1]$. Let v be a non-negative real number. Then

$$\sigma^*\big((-K_X-uS)\big|_S\big)-v\mathbf{f}\sim_{\mathbb{R}}(1-u)\widehat{L}+\widehat{C}+(9-3u-v)\mathbf{f}. \tag{4.2}$$

Thus, the divisor $\sigma^*((-K_X-uS)|_S)-v\mathbf{f}$ is pseudoeffective $\iff$ it is nef $\iff$ $v\leqslant 9-3u$. Let $P(u,v)$ and $N(u,v)$ be the positive and the negative parts of its Zariski decomposition. Then using (4.2), we compute

$$P(u,v)=\begin{cases}(1-u)\widehat{L}+\widehat{C}+(9-3u-v)\mathbf{f}\text{ if }0\leqslant v\leqslant 3-3u,\\ (1-u)\widehat{L}+\dfrac{9-3u-v}{6}\widehat{C}+(9-3u-v)\mathbf{f}\text{ if }3-3u\leqslant v\leqslant 8-2u,\\ \dfrac{9-3u-v}{6}\big(6\widehat{L}+\widehat{C}+6\mathbf{f}\big)\text{ if }8-2u\leqslant v\leqslant 9-3u,\end{cases}$$

and

$$N(u,v)=\begin{cases}0 \text{ if } 0\leqslant v\leqslant 3-3u,\\ \dfrac{v+3u-3}{6}\widehat{C} \text{ if } 3-3u\leqslant v\leqslant 8-2u,\\ \dfrac{v+3u-3}{6}\widehat{C}+(v+2u-8)\widehat{L} \text{ if } 8-2u\leqslant v\leqslant 9-3u.\end{cases}$$

This gives

$$\mathrm{vol}\Big(\sigma^*\big((-K_X-uS)\big|_S\big)-v\mathbf{f}\Big)=\begin{cases}(1-u)(7-u)-\dfrac{v^2}{6} \text{ if } 0\leqslant v\leqslant 3-3u,\\ \dfrac{(1-u)(17-5u-2v)}{2} \text{ if } 3-3u\leqslant v\leqslant 8-2u,\\ \dfrac{(9-3u-v)^2}{2} \text{ if } 8-2u\leqslant v\leqslant 9-3u,\end{cases}$$

and

$$P(u,v)\cdot\mathbf{f}=\begin{cases}\dfrac{v}{6} \text{ if } 0\leqslant v\leqslant 3-3u,\\ \dfrac{1-u}{2} \text{ if } 3-3u\leqslant v\leqslant 8-2u,\\ \dfrac{9-3u-v}{2} \text{ if } 8-2u\leqslant v\leqslant 9-3u.\end{cases}$$

Now, we use [3, Corollary 1.7.26] to get

$$S\big(W^{\widehat{S}}_{\bullet,\bullet};\mathbf{f}\big)=\frac{3}{10}\int_0^1\int_0^{3-3u}\Big((1-u)(7-u)-\frac{v^2}{6}\Big)dv+$$
$$+\frac{3}{10}\int_0^1\int_{3-3u}^{8-2u}\frac{(1-u)(17-5u-2v)}{2}dv+\frac{3}{10}\int_0^1\int_{8-2u}^{9-3u}\frac{(9-3u-v)^2}{2}dvdu=\frac{173}{40}.$$

Similarly, if Q is a point in $\mathbf{f}$, then [3, Remark 1.7.32] gives

$$S\big(W^{\widehat{S},\mathbf{f}}_{\bullet,\bullet,\bullet};Q\big)=F_Q+\frac{3}{10}\int_0^1\int_0^{9-3u}\big(P(u,v)\cdot\mathbf{f}\big)^2dvdu=$$
$$=F_Q+\frac{3}{10}\int_0^1\int_0^{3-3u}\Big(\frac{v}{6}\Big)^2dvdu+\frac{3}{10}\int_0^1\int_{3-3u}^{8-2u}\Big(\frac{1-u}{2}\Big)^2dvdu+$$
$$+\frac{3}{10}\int_0^1\int_{8-2u}^{9-3u}\Big(\frac{9-3u-v}{2}\Big)^2dvdu=F_Q+\frac{5}{32},$$

where F_Q can be computes as follows:

- if $Q\neq\widehat{C}\cap\mathbf{f}$ and $Q\neq\widehat{L}\cap\mathbf{f}$, then $F_Q=0$;
- if $Q=\widehat{L}\cap\mathbf{f}$, then

$$F_Q=\frac{6}{10}\int_0^1\int_{8-2u}^{9-3u}\frac{(9-3u-v)(v+2u-8)}{2}dvdu=\frac{1}{80};$$

- if $Q = \widehat{C} \cap \mathbf{f}$, then

$$F_Q = \frac{6}{10}\int_0^1\int_{3-3u}^{8-2u}\frac{(1-u)(v+3u-3)}{12}dv+ \\ +\frac{6}{10}\int_0^1\int_{8-2u}^{9-3u}\frac{(9-3u-v)(v+3u-3)}{12}dudv=\frac{193}{480}.$$

Therefore, we conclude that

$$S\big(W^{\widehat{S},\mathbf{f}}_{\bullet,\bullet,\bullet};Q\big)=\begin{cases}\frac{5}{32} \text{ if } Q\not\in \widehat{C}\cup\widehat{L},\\ \frac{27}{80} \text{ if } Q=\widehat{L}\cap\mathbf{f},\\ \frac{67}{120} \text{ if } Q=\widehat{C}\cap\mathbf{f}.\end{cases}$$

Let $\Delta_{\mathbf{f}} = \frac{1}{2}Q_2 + \frac{2}{3}Q_3$. Then, using [3, Remark 1.7.32] and our computations, we get

$$1\geqslant\frac{A_X(\mathbf{F})}{S_X(\mathbf{F})}\geqslant\min\left\{\inf_{Q\in\mathbf{f}}\frac{1-\mathrm{ord}_Q\big(\Delta_{\mathbf{f}}\big)}{S(W^{\widehat{S},\mathbf{f}}_{\bullet,\bullet,\bullet};Q)},\frac{5}{S(V^S_{\bullet,\bullet};\mathbf{f})},\frac{1}{S_X(S)}\right\}\geqslant\min\left\{\frac{40}{13},\frac{200}{173},\frac{120}{67}\right\}=\frac{200}{173}>1,$$

which is a contradiction. □

Combining Lemmas 4.4 and 4.5, we obtain a contradiction.

Corollary 4.6. *All smooth Fano threefolds in the family №2.4 are K-stable.*

5. Family №2.6 (Verra threefolds)

Smooth Fano threefolds in the family №2.6 can be described as follows:

(a) smooth divisors of degree $(2,2)$ in $\mathbb{P}^2\times\mathbb{P}^2$, which are known as Verra threefolds,
(b) double covers of the (unique) smooth divisor in $\mathbb{P}^2\times\mathbb{P}^2$ of degree $(1,1)$ branched over smooth anticanonical K3 surfaces.

Note that every double cover of the smooth divisor in $\mathbb{P}^2\times\mathbb{P}^2$ of degree $(1,1)$ branched over a smooth anticanonical surface is K-stable [13, Example 4.4]. In fact, this also implies that general Verra threefold is K-stable [3, Example 3.5.8].

The goal of this section is to prove that all smooth Verra threefolds are K-stable.

Let X be a smooth divisor of degree $(2,2)$ in $\mathbb{P}^2\times\mathbb{P}^2$, let $\pi_1\colon X\to\mathbb{P}^2$ be the projection to the first factor of $\mathbb{P}^2\times\mathbb{P}^2$, and let $\pi_2\colon X\to\mathbb{P}^2$ be the projection to the second factor. Then π_1 and π_2 are conic bundles [18]. Set $H_1=\pi_1^*(\mathcal{O}_{\mathbb{P}^2}(1))$ and $H_2=\pi_2^*(\mathcal{O}_{\mathbb{P}^2}(1))$. Then

$$-K_X\sim H_1+H_2,$$

and the group $\mathrm{Pic}(X)$ is generated by H_1 and H_2. Note that the group $\mathrm{Aut}(X)$ is finite [9]. Thus, the threefold X is K-stable $\iff$ it is K-polystable. See [19] for details.

Lemma 5.1. *Fix a point $P\in X$. Let C_1 be the fiber of the conic bundle π_1 that contains P, and let C_2 be the fiber of the conic bundle π_2 contains P. Then C_1 or C_2 is smooth at P.*

Proof. Local computations. □

Let Δ_1 and Δ_2 be the discriminant curves of the conic bundles π_1 and π_2, respectively. Then Δ_1 and Δ_2 are reduced curves of degree 6 that have at most ordinary double points as singularities. For basic properties of the discriminant curves Δ_1 and Δ_2, see [18, § 3.8]. In particular, we know that no line in $\mathbb{P}^2$ can be an irreducible component of these curves.

Lemma 5.2. *Fix a point $P \in X$. Let C_2 be the fiber of the conic bundle π_2 that contains P, let S be a general surface in $|H_2|$ that contains C_2. Then one of the following cases holds:*

(1) *C_2 is smooth, S is smooth, the divisor $-K_S$ is ample;*
(2) *C_2 is smooth, S is smooth, the divisor $-K_S$ is nef, the surface S has exactly four irreducible curves that have trivial intersection with the divisor $-K_S$, these curves are disjoint and none of them passes through P, and $C_2 \subset \mathrm{Sing}(\pi_1^{-1}(\Delta_1))$;*
(3) *C_2 is singular and reduced, S is smooth, the divisor $-K_S$ is ample;*
(4) *C_2 is not reduced, $P \notin \mathrm{Sing}(S) \subset \mathrm{Supp}(C_2)$, and $\mathrm{Sing}(S)$ consists of two points, which are ordinary double points of the surface S, and $-K_S$ is ample.*

Proof. The assertion about the singularities of the surface S are local and well-known.

We have $-K_S \sim H_1|_S$ and $K_S^2 = 2$, so S is a weak del Pezzo surface of degree 2, and the restriction $\pi_1|_S\colon S \to \mathbb{P}^2$ is the anticanonical morphism. Let ℓ be an irreducible curve in the surface S such that

$$-K_S \cdot \ell = 0.$$

Then ℓ is an irreducible component of the fiber $\pi_1^{-1}(\pi_1(\ell))$. But $\pi_2(\ell)$ is the line $\pi_2(S)$, which implies that the scheme fiber $\pi_1^{-1}(\pi_1(\ell))$ is singular, which implies that $\pi_1(\ell) \in \Delta_1$. Since $\ell_1 \cap C_2 \neq \varnothing$ and $\pi_2(S)$ is a general line in $\mathbb{P}^2$ that passes through the point $\pi_2(P)$, we see that $\pi_1(C_2) \subset \Delta_1$. This implies that C_2 is irreducible and reduced.

Let $R = \pi_1^{-1}(\pi_1(C_2))$. Then the surface R is singular along a curve that is isomorphic to the conic $\pi_1(C_2) \cong C_2$. Let $f\colon \widetilde{R} \to R$ be the blow up of this curve. Then $\widetilde{R}$ is smooth, and the composition morphism $\pi_1 \circ f$ induces a $\mathbb{P}^1$-bundle $\widetilde{R} \to \widetilde{C}_2$, where $\widetilde{C}_2$ is double cover of the conic $\pi_1(C_2)$ that is branched over the eight points $\pi_1(C_2) \cap (\Delta_1 - \pi_1(C_2))$. In particular, we see that $\widetilde{C}_2$ is an irrational curve, which implies that $C_2 = \mathrm{Sing}(\widetilde{R})$.

Vice versa, if the fiber C_2 is a smooth conic, and the conic $\pi_1(C_2)$ is an irreducible component of the discriminant curve Δ_1, then it follows from the Bertini theorem that

$$S \cdot \pi_1^{-1}(\pi_1(C_2)) = 2C_2 + \ell_1 + \ell_2 + \cdots + \ell_k$$

where $\ell_1, \ell_2, \ldots, \ell_k$ are k distinct irreducible reduced curves in X that are irreducible components of the fibers of the natural projection $\pi_1^{-1}(\pi_1(C_2)) \to \pi_1(C_2)$. Since

$$4 = 2H_2^2 \cdot H_1 = H_2 \cdot S \cdot \pi_1^{-1}(\pi_1(C_2)) = H_2 \cdot \Big(2C_2 + \sum_{i=1}^{k} \ell_i\Big) = k,$$

we see that S contains exactly 4 irreducible curves that intersects trivially with $-K_S$. Now, the generality in the choice of S implies that none of these curves contains P. $\square$

Example 5.3. Actually, the case (2) in Lemma 5.2 can happen. Indeed, in the assumption and notations of Lemma 5.2, let $P = ([0:0:1],[0:0:1])$, and suppose that X is given by

$$(u^2 + 2uw + v^2 + 2w^2)x^2 + (uv - w^2)xy + (uw - 2uv + 3v^2 + w^2)y^2 + (uw + v^2)z^2 = 0,$$

where $([u:v:w],[x:y:z])$ are coordinates on $\mathbb{P}^2 \times \mathbb{P}^2$. Then the threefold X is smooth. For instance, this can be checked using the following Magma script:

```
Q:=RationalField();
PxP<x,y,z,u,v,w>:=ProductProjectiveSpace(Q,[2,2]);
X:=Scheme(PxP,[(u^2+2*u*w+v^2+2*w^2)*x^2+(u*v-w^2)*x*y+
                ((-2*v+w)*u+3*v^2+w^2)*y^2+(u*w+v^2)*z^2]);
IsNonsingular(X);
```

Observe that the fiber C_1 is a singular reduced curve given by $u = v = 2x^2 - xy + y^2 = 0$, the fiber C_2 is a smooth curve that is given by $x = y = uw + v^2 = 0$, and the discriminant curve Δ_1 is a union of the conic $\pi_1(C_2)$ and the irreducible quartic plane curve given by

$$8u^3v - 4u^3w - 11u^2v^2 + 16u^2vw - 12u^2w^2 + 8uv^3 - \\ - 28uv^2w + 14uvw^2 - 16uw^3 - 12v^4 - 28v^2w^2 - 7w^4 = 0.$$

As in case (2) in Lemma 5.2, we have $C_2 = \mathrm{Sing}(\pi_1^{-1}(\pi_1(C_2)))$.

Let us prove that X is K-stable. Suppose it is not. Using the valuative criterion [14, 16], we see that there exists a prime divisor $\mathbf{F}$ over X such that

$$\beta(\mathbf{F}) = A_X(\mathbf{F}) - S_X(\mathbf{F}) \leqslant 0.$$

Let Z be the center of the divisor $\mathbf{F}$ on X. Then Z is not a surface by [3, Theorem 3.7.1].

Let P be any point in Z, let C_1 be the fiber of the conic bundle π_1 that contains P, and let C_2 be the fiber of the conic bundle π_2 that contains P. By Lemma 5.1, at least one curve among C_1 or C_2 is smooth at P. We may assume that C_2 is smooth at P.

Let S be a general surface in $|H_2|$ that contains C_2. Then S is smooth by Lemma 5.2. Moreover, one of the following three cases holds:

(1) C_2 is smooth, the divisor $-K_S$ is ample;
(2) C_2 is smooth, $\pi_1(C_2) \subset \Delta_1$, the divisor $-K_S$ is nef, the surface S has exactly four irreducible curves that have trivial intersection with the divisor $-K_S$, these curves are disjoint and none of them passes through P;
(3) C_2 is singular and reduced, the divisor $-K_S$ is ample.

Let C be the curve in X that is defined as follows:

- if C_2 is smooth and irreducible, we let $C = C_2$.
- if C_2 is reducible, we let C be its irreducible component that contains P.

Note that $-K_S \sim H_1|_S$ and $K_S^2 = 2$. Let $\eta\colon S \to \mathbb{P}^2$ be the restriction morphism $\pi_1|_S$. Then η is an anticanonical morphism of the surface S, which is generically two-to-one. Hence, the morphism η induces an involution $\tau \in \mathrm{Aut}(S)$. We let $C' = \tau(C)$.

Now, let u be a non-negative real number. Then we have $-K_X - uS \sim_{\mathbb{R}} H_1 + (1-u)H_2$, so the divisor $-K_X - uS$ is nef $\iff$ it is pseudoeffective $\iff$ $u \leqslant 1$. Then $S_X(S) = \frac{5}{12}$. Now, let us use notations introduced in [3, § 1.7]. Using [3, Theorem 1.7.30], we get

$$1 \geqslant \frac{A_X(\mathbf{F})}{S_X(\mathbf{F})} \geqslant \min\left\{\frac{1}{S_X(S)}, \frac{1}{S(W^S_{\bullet,\bullet}; C)}, \frac{1}{S(W^{S,C}_{\bullet,\bullet,\bullet}; P)}\right\},$$

where $S(W^S_{\bullet,\bullet}; C)$ and $S(W^{S,C}_{\bullet,\bullet,\bullet}; P)$ are defined in [3, § 1.7]. Hence, since $S_X(S) < 1$, we get

(5.1) $$\max\left\{S(W^S_{\bullet,\bullet}; C), S(W^{S,C}_{\bullet,\bullet,\bullet}; P)\right\} \geqslant 1.$$

Moreover, if $Z = P$, then it also follows from [3, Theorem 1.7.30] that

(5.2) $$\max\left\{S(W^S_{\bullet,\bullet}; C), S(W^{S,C}_{\bullet,\bullet,\bullet}; P)\right\} > 1.$$

Let us estimate $S(W^S_{\bullet,\bullet};C)$ and $S(W^{S,C}_{\bullet,\bullet,\bullet};P)$ using results obtained in [3, § 1.7].

Let $P(u)=-K_X-uS$. Then [3, Corollary 1.7.26] gives

$$S\big(W^S_{\bullet,\bullet};C\big)=\frac{1}{4}\int_0^1\int_0^\infty \mathrm{vol}\big(P(u)\big|_S-vC\big)dvdu,$$

Let $P(u,v)$ be the positive part of the Zariski decomposition of the divisor $P(u)|_S-vC$, and let $N(u,v)$ be its negative part, where $u\in[0,1]$ and $v\in\mathbb{R}_{\geqslant 0}$. Then

$$S\big(W^{S,C}_{\bullet,\bullet,\bullet};P\big)=F_P+\frac{1}{4}\int_0^1\int_0^\infty \big(P(u,v)\cdot C\big)^2dvdu$$

by [3, Theorem 1.7.30], where

$$F_P=\frac{1}{2}\int_0^1\int_0^\infty \big(P(u,v)\cdot C\big)\mathrm{ord}_P\Big(N(u,v)\big|_C\Big)dvdu.$$

Lemma 5.4. *Suppose that C_2 is smooth, and $-K_S$ is ample. Then*

$$S(W^S_{\bullet,\bullet};C)=\frac{13}{24},$$
$$S(W^{S,C}_{\bullet,\bullet,\bullet};P)=1.$$

Proof. We have $C\cdot C'=4$ and $(C')^2=C^2=0$. Fix $u\in[0,1]$ and $v\in\mathbb{R}_{\geqslant 0}$. Then

$$P(u)|_S-vC\sim_{\mathbb{Q}}\Big(\frac{3}{2}-u-v\Big)C+\frac{1}{2}C'.$$

which implies that $P(u)|_S-vC$ is pseudoeffective $\iff$ $P(u)|_S-vC$ is nef $\iff$ $v\leqslant\frac{3}{2}-u$. If $0\leqslant u\leqslant 1$ and $0\leqslant v\leqslant\frac{3}{2}-u$, then $P(u,v)=(\frac{3}{2}-u-v)C+\frac{1}{2}C'$ and $N(u,v)=0$, so

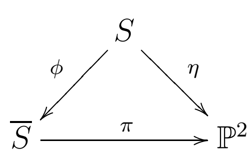
$$S\big(W^S_{\bullet,\bullet};C\big)=\frac{1}{4}\int_0^1\int_0^{\frac{3}{2}-u}\Bigg(\Big(\frac{3}{2}-u-v\Big)C+\frac{1}{2}C'\Bigg)^2dvdu=\frac{1}{4}\int_0^1\int_0^{\frac{3}{2}-u}6-4u-4v\,dvdu=\frac{13}{24}.$$

Similarly, we see that $F_P=0$ and $P(u,v)\cdot C=2$, which gives $S(W^{S,C}_{\bullet,\bullet,\bullet};P)=1$. □

Lemma 5.5. *Suppose that C_2 is smooth, $-K_S$ is not ample. Then*

$$S(W^S_{\bullet,\bullet};C)=\frac{7}{12},$$
$$S(W^{S,C}_{\bullet,\bullet,\bullet};P)=\frac{5}{6}.$$

Proof. In this case, we have the following commutative diagram:



where ϕ is a birational map that contracts four disjoint (-2)-curves, and $\overline{S}$ is a del Pezzo surface of degree 2 that has 4 isolated ordinary double points, and π is a double cover that is ramified in a reducible quartic curve that is a union of two irreducible conics such that $\eta(C)$ is one of these two conics. In particular, we have $C=\tau(C)$.

Let E_1, E_2, E_3, E_4 be the ϕ-exceptional curves. Fix $u\in[0,1]$ and $v\in\mathbb{R}_{\geqslant 0}$. Then

$$P(u)|_S-vC\sim_{\mathbb{Q}}(2-u-v)C+\frac{1}{2}(E_1+E_2+E_3+E_4)$$

so the divisor $P(u)|_S - vC$ is pseudoeffective $\iff v \leqslant 2-u$. Moreover, we have

$$P(u,v) = \begin{cases} (2-u-v)C + \dfrac{1}{2}(E_1+E_2+E_3+E_4) \text{ if } 0 \leqslant v \leqslant 1-u, \\ (2-u-v)\Big(C + \dfrac{1}{2}(E_1+E_2+E_3+E_4)\Big) \text{ if } 1-u \leqslant v \leqslant 2-u, \end{cases}$$

$$N(u,v) = \begin{cases} 0 \text{ if } 0 \leqslant v \leqslant 1-u, \\ \dfrac{u+v-1}{2}(E_1+E_2+E_3+E_4) \text{ if } 1-u \leqslant v \leqslant 2-u, \end{cases}$$

$$\mathrm{vol}\big(P(u)|_S - vC\big) = \begin{cases} (6-4u)-4 \text{ if } 0 \leqslant v \leqslant 1-u, \\ 2(2-u-v)^2 \text{ if } 1-u \leqslant v \leqslant 2-u. \end{cases}$$

Now, integrating $\mathrm{vol}(P(u)|_S - vC)$, we obtain $S(W^S_{\bullet,\bullet};C) = \frac{7}{12}$.

To compute $S(W^{S,C}_{\bullet,\bullet,\bullet};P)$, we first observe that $F_P = 0$, because $P \notin E_1 \cup E_2 \cup E_3 \cup E_4$, since S is a general surface in $|H_2|$ that contains C_2. On the other hand, we have

$$P(u,v)\cdot C = \begin{cases} 2 \text{ if } 0 \leqslant v \leqslant 1-u, \\ 4-2u-2v \text{ if } 1-u \leqslant v \leqslant 2-u. \end{cases}$$

Hence, integrating $(P(u,v)\cdot C)^2$, we get $S(W^{S,C}_{\bullet,\bullet,\bullet};P) = \frac{5}{6}$ as required. □

Lemma 5.6. *Suppose that C_2 is singular. Then*

$$S(W^S_{\bullet,\bullet};C) = \frac{3}{4},$$
$$S(W^{S,C}_{\bullet,\bullet,\bullet};P) \leqslant \frac{11}{12}.$$

Proof. The curve C_2 consists of two irreducible components: the curve C and another curve, which we denote by L. The curves C and L are smooth and intersects transversally at one point. Note that $P \neq C \cap L$, since C_2 is smooth at the point P by assumption.

The intersections of the curves C, L and $C' = \tau(C)$ on S are given in the table below.

$\bullet$	C	L	C'
C	-1	1	2
L	1	-1	0
C'	2	0	-1

Fix $u \in [0,1]$ and $v \in \mathbb{R}_{\geqslant 0}$. Since $C + C' \sim -K_S$, we have

$$P(u)|_S - vC \sim_{\mathbb{Q}} (2-u-v)C + (1-u)L + C',$$

so $P(u)|_S - vC$ is pseudoeffective $\iff v \leqslant 2-u$. Moreover, if $0 \leqslant u \leqslant \frac{1}{2}$, then

$$P(u,v) = \begin{cases} (2-u-v)C + (1-u)L + C' \text{ if } 0 \leqslant v \leqslant 1, \\ (2-u-v)(C+L) + C' \text{ if } 1 \leqslant v \leqslant \dfrac{3}{2}-u, \\ (2-u-v)(C+L+C') \text{ if } \dfrac{3}{2}-u \leqslant v \leqslant 2-u, \end{cases}$$

$$N(u,v)=\begin{cases}0 \text{ if } 0\leqslant v\leqslant 1,\\ (v-1)L \text{ if } 1\leqslant v\leqslant \dfrac{3}{2}-u,\\ (v-1)L+(2v+2u-3)C' \text{ if } \dfrac{3}{2}-u\leqslant v\leqslant 2-u,\end{cases}$$

$$\mathrm{vol}\big(P(u)|_S-vC\big)=\begin{cases}6-v^2-4u-2v \text{ if } 0\leqslant v\leqslant 1,\\ 7-4u-4v \text{ if } 1\leqslant v\leqslant \dfrac{3}{2}-u,\\ 4(u+v-2)^2 \text{ if } \dfrac{3}{2}-u\leqslant v\leqslant 2-u.\end{cases}$$

Similarly, if $\frac{1}{2}\leqslant u\leqslant 1$, then

$$P(u,v)=\begin{cases}(2-u-v)C+(1-u)L+C' \text{ if } 0\leqslant v\leqslant \dfrac{3}{2}-u,\\ (2-u-v)(C+2C')+(1-u)L \text{ if } \dfrac{3}{2}-u\leqslant v\leqslant 1,\\ (2-u-v)(C+L+C') \text{ if } 1\leqslant v\leqslant 2-u,\end{cases}$$

$$N(u,v)=\begin{cases}0 \text{ if } 0\leqslant v\leqslant \dfrac{3}{2}-u,\\ (2v+2u-3)C' \text{ if } \dfrac{3}{2}-u\leqslant v\leqslant 1,\\ (v-1)L+(2v+2u-3)C' \text{ if } 1\leqslant v\leqslant 2-u,\end{cases}$$

$$\mathrm{vol}\big(P(u)|_S-vC\big)=\begin{cases}6-v^2-4u-2v \text{ if } 0\leqslant v\leqslant \dfrac{3}{2}-u,\\ (5-2u-3v)(3-2u-v) \text{ if } \dfrac{3}{2}-u\leqslant v\leqslant 1,\\ 4(u+v-2)^2 \text{ if } 1\leqslant v\leqslant 2-u.\end{cases}$$

Hence, integrating $\mathrm{vol}(P(u)|_S-vC)$, we get $S(W^S_{\bullet,\bullet};C)=\frac{3}{4}$.

Now, let us compute $S(W^{S,C}_{\bullet,\bullet,\bullet};P)$. If $0\leqslant u\leqslant\frac{1}{2}$, then

$$P(u,v)\cdot C=\begin{cases}1+v \text{ if } 0\leqslant v\leqslant 1,\\ 2 \text{ if } 1\leqslant v\leqslant \dfrac{3}{2}-u,\\ 8-4u-4v \text{ if } \dfrac{3}{2}-u\leqslant v\leqslant 2-u.\end{cases}$$

Similarly, if $\frac{1}{2}\leqslant u\leqslant 1$, then

$$P(u,v)\cdot C=\begin{cases}1+v \text{ if } 0\leqslant v\leqslant \dfrac{3}{2}-u,\\ 7-4u-3v \text{ if } \dfrac{3}{2}-u\leqslant v\leqslant 1,\\ 8-4u-4v \text{ if } 1\leqslant v\leqslant 2-u.\end{cases}$$

Then, integrating, we get $S(W^{S,C}_{\bullet,\bullet,\bullet};P)=\frac{145}{192}+F_P$. To compute F_P, let us recall that $P\notin L$. Hence, if $P\notin C'$, then $F_P=0$, which implies that $S(W^{S,C}_{\bullet,\bullet,\bullet};P)=\frac{145}{192}<\frac{11}{12}$ as required.

We may assume that $P \in C \cap C'$. If C' intersects C transversally at P, then

$$\mathrm{ord}_P\Big(N(u,v)\big|_C\Big) = \begin{cases} 0 \text{ if } 0 \leqslant v\dfrac{3}{2} - u, \\ 2u+2v-3 \text{ if } \dfrac{3}{2} - u \leqslant v \leqslant 2-u, \end{cases}$$

which gives

$$F_P = \frac{1}{2}\int_0^{\frac{1}{2}}\int_{\frac{3}{2}-u}^{2-u}(8-4u-4v)(2u+2v-3)dvdu+$$
$$+\frac{1}{2}\int_{\frac{1}{2}}^{1}\int_{1}^{\frac{3}{2}-u}(7-4u-3v)(2u+2v-3)dvdu+\frac{1}{2}\int_{\frac{1}{2}}^{1}\int_{\frac{3}{2}-u}^{2-u}(8-4u-4v)(2u+2v-3)dvdu = \frac{31}{384},$$

so $S(W^{S,C}_{\bullet,\bullet,\bullet};P) = \frac{145}{192} + \frac{31}{384} = \frac{107}{128} < \frac{11}{12}$. If C' is tangent to C at the point P, then

$$\mathrm{ord}_P\Big(N(u,v)\big|_C\Big) = \begin{cases} 0 \text{ if } 0 \leqslant v\dfrac{3}{2} - u, \\ 2(2u+2v-3) \text{ if } \dfrac{3}{2} - u \leqslant v \leqslant 2-u, \end{cases}$$

which gives $F_P = \frac{31}{192}$, so $S(W^{S,C}_{\bullet,\bullet};P) = \frac{145}{192} + \frac{31}{192} = \frac{11}{12}$. ◻

Now, using (5.2) and Lemmas 5.4, 5.5, 5.6, we see that Z is a curve.

Lemma 5.7. *One has $H_1 \cdot Z \geqslant 1$ and $H_2 \cdot Z \geqslant 1$.*

Proof. If $H_2 \cdot Z = 0$, then $Z = C$, which is impossible by (5.1) and Lemmas 5.4, 5.5, 5.6. Hence, we see that $H_2 \cdot Z \geqslant 1$ and $\pi_2(Z)$ is a curve. Let us show that $H_1 \cdot Z \geqslant 1$.

Suppose that $H_1 \cdot Z = 0$. Then Z must be an irreducible component of the curve C_1. If C_1 is reduced, then arguing exactly as in the proofs of Lemmas 5.4, 5.5, 5.6, we obtain a contradiction with [3, Corollary 1.7.26]. Thus, we see that C_1 is not reduced.

So far, the point P was a point in Z. Let us choose $P \in Z$ such that $\pi_2(P) \in \pi_2(Z) \cap \Delta_2$. Then C_2 is singular. But it is smooth at P by Lemma 5.1, which fits our assumption above. Then $S(W^S_{\bullet,\bullet};C) = \frac{3}{4}$ and $S(W^{S,C}_{\bullet,\bullet,\bullet};P) \leqslant \frac{11}{12}$ by Lemma 5.6, which contradicts (5.1). ◻

Both $\pi_1(Z)$ and $\pi_2(Z)$ are curves. Similar to what we did in the proof of Lemma 5.7, let us choose the point $P \in Z$ such that $\pi_1(P) \in \Delta_1$. Then C_1 is singular at P, which implies that C_2 is smooth at P by Lemma 5.1. Now, using (5.1) and Lemmas 5.5 and 5.6, we see that $C = C_2$, the curve C_2 is smooth, the divisor $-K_S$ is ample.

We see that S is a del Pezzo surface, and $\eta\colon S \to \mathbb{P}^2$ is a double cover ramified in a smooth quartic curve, so we are almost in the same position as in the proof of Lemma 5.4. But now, we have one small advantage: the point $\eta(P)$ is contained in the ramification divisor of the double cover η, because C_1 is singular at P. Then $|-K_S|$ contains a unique curve that is singular at P. Denote this curve by R. We have the following possibilities:

(a) R is an irreducible curve that has a nodal singularity at P;
(b) R is an irreducible curve that has a cuspidal singularity at P;
(c) $R = R_1 + R_2$ for two (-1)-curves in S that intersect transversally at P;
(d) $R = R_1 + R_2$ for two (-1)-curves in S that are tangent at P.

Let $f\colon \widetilde{S}\to S$ be the blow up of the point P. Denote by $\widetilde{R}$ and $\widetilde{C}$ the proper transforms on the surface $\widetilde{S}$ of the curves R and C, respectively. Fix $u\in[0,1]$ and $v\in\mathbb{R}_{\geqslant 0}$. Then

$$f^*\big(P(u)|_S\big)-vE\sim_{\mathbb{Q}}\widetilde{R}+(1-u)\widetilde{C}+(3-u-v)E.$$

Let $\widetilde{P}(u,v)$ be the positive part of the Zariski decomposition of $\widetilde{R}+(1-u)\widetilde{C}+(3-u-v)E$, and let $\widetilde{N}(u,v)$ its negative part. Then it follows from [3, Remark 1.7.32] that

$$1\geqslant\frac{A_X(\mathbf{F})}{S_X(\mathbf{F})}\geqslant\min\left\{\frac{1}{S_X(S)},\frac{2}{S(W^{\widetilde{S}}_{\bullet,\bullet};E)},\inf_{O\in E}\frac{1}{S(W^{\widetilde{S},E}_{\bullet,\bullet,\bullet};O)}\right\},\tag{5.3}$$

where $S(W^{\widetilde{S}}_{\bullet,\bullet};E)$ and $S(W^{\widetilde{S},E}_{\bullet,\bullet,\bullet};O)$ are defined in [3] similar to $S(W^{S}_{\bullet,\bullet};C)$ and $S(W^{S,C}_{\bullet,\bullet,\bullet};P)$. These two numbers can be computed using [3, Remark 1.7.32]. Namely, we have

$$S\big(W^{\widetilde{S}}_{\bullet,\bullet};E\big)=\frac{1}{4}\int_0^1\int_0^\infty\big(\widetilde{P}(u,v)\big)^2dvdu$$

and

$$S\big(W^{\widetilde{S},E}_{\bullet,\bullet,\bullet};O\big)=\frac{1}{4}\int_0^1\int_0^\infty\Big(\big(\widetilde{P}(u,v)\cdot E\big)\Big)^2dvdu+F_O,$$

where O is a point in E and

$$F_O=\frac{1}{2}\int_0^1\int_0^\infty\big(\widetilde{P}(u,v)\cdot E\big)\mathrm{ord}_O\Big(\widetilde{N}(u,v)\big|_E\Big)dvdu.$$

Let us use these formulas to estimate $S(W^{\widetilde{S}}_{\bullet,\bullet};E)$ and $S(W^{\widetilde{S},E}_{\bullet,\bullet,\bullet};O)$.

If the curve R is irreducible, the intersections of the curves $\widetilde{C}$, $\widetilde{R}$, E can be computed as follows: $\widetilde{C}^2=-1$, $\widetilde{C}\cdot\widetilde{R}=0$, $\widetilde{C}\cdot E=1$, $\widetilde{R}^2=-2$, $\widetilde{R}\cdot E=2$, $E^2=-1$. If R is reducible, then $\widetilde{R}=\widetilde{R}_1+\widetilde{R}_2$ for two smooth irreducible curves $\widetilde{R}_1+\widetilde{R}_2$ such that the intersection form of the curves $\widetilde{C}$, $\widetilde{R}_1$, $\widetilde{R}_1$ and E is given in the following table:

$\bullet$	$\widetilde{C}$	$\widetilde{R}_1$	$\widetilde{R}_1$	E
$\widetilde{C}$	-1	0	0	1
$\widetilde{R}_1$	0	-2	1	1
$\widetilde{R}_1$	0	1	-2	1
E	1	1	1	-1

Then $\widetilde{R}+(1-u)\widetilde{C}+(3-u-v)E$ is pseudoeffective $\iff$ $v\leqslant 3-u$. Moreover, we have

$$\widetilde{P}(u,v)=\begin{cases}\widetilde{R}+(1-u)\widetilde{C}+(3-u-v)E\text{ if }0\leqslant v\leqslant 2-u,\\(1-u)\widetilde{C}+(3-u-v)(E+\widetilde{R})\text{ if }2-u\leqslant v\leqslant 2,\\(3-u-v)(E+\widetilde{R}+\widetilde{C})\text{ if }2\leqslant v\leqslant 3-u,\end{cases}$$

$$\widetilde{N}(u,v)=\begin{cases}0\text{ if }0\leqslant v\leqslant 2-u,\\(v+u-2)\widetilde{R}\text{ if }2-u\leqslant v\leqslant 2,\\(v+u-2)\widetilde{R}+(v-2)\widetilde{C}\text{ if }2\leqslant v\leqslant 3-u,\end{cases}$$

$$\big(\widetilde{P}(u,v)\big)^2 = \begin{cases} 6 - v^2 - 4u \text{ if } 0 \leqslant v \leqslant 2-u, \\ 14 + 2u^2 + 4uv + v^2 - 12u - 8v \text{ if } 2-u \leqslant v \leqslant 2, \\ 2(3-u-v)^2 \text{ if } 2 \leqslant v \leqslant 3-u, \end{cases}$$

Now, integrating, we obtain $S(W^{\widetilde{S}}_{\bullet,\bullet}; E) = \frac{17}{12}$.

Fix a point $O \in E$. To estimate $S(W^{\widetilde{S},E}_{\bullet,\bullet,\bullet}; O)$, first we observe that

$$\widetilde{P}(u,v) \cdot E = \begin{cases} v \text{ if } 0 \leqslant v \leqslant 2-u, \\ 4 - 2u - v \text{ if } 2-u \leqslant v \leqslant 2, \\ 6 - 2u - 2v \text{ if } 2 \leqslant v \leqslant 3-u. \end{cases}$$

Therefore, integrating $(\widetilde{P}(u,v) \cdot E)^2$, we obtain $S(W^{Y,\widetilde{S}}_{\bullet,\bullet,\bullet}; O) = \frac{13}{24} + F_O$.

If $O \notin \widetilde{C} \cup \widetilde{R}$, then $F_O = 0$. Similarly, if $O \in \widetilde{C}$, then $O \notin \widetilde{R}$, which gives

$$F_O = \frac{1}{2}\int_0^1 \int_2^{3-u} (6-2u-2v)(v-2)dvdu = \frac{1}{24}.$$

Finally, if $O \in \widetilde{R}$, then $O \notin \widetilde{C}$, which gives

$$F_O \leqslant \frac{1}{2}\int_0^1 \int_{2-u}^{2} 2(4-2u-v)(v+u-2)dvdu + \frac{1}{2}\int_0^1 \int_2^{3-u} 2(6-2u-2v)(v+u-2)dvdu = \frac{7}{24}.$$

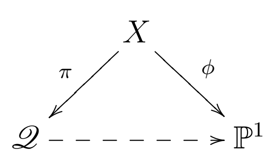
Summarizing, we get $S(W^{\widetilde{S},E}_{\bullet,\bullet,\bullet}; O) \leqslant \frac{5}{6}$ for every point $O \in E$, which contradicts (5.3), because $S_X(S) = \frac{5}{12}$ and $S(W^{\widetilde{S}}_{\bullet,\bullet}; E) = \frac{17}{12} < 2$. This shows that X is K-stable.

Corollary 5.8. *All smooth Fano threefolds in the family №2.6 are K-stable.*

6. Family №2.7

Now, let us fix three smooth quadric hypersurfaces $\mathscr{Q}$, $\mathscr{Q}'$ and $\mathscr{Q}''$ in $\mathbb{P}^4$ such that their intersection is a smooth curve of degree 8 and genus 5. We set $\mathscr{C} = \mathscr{Q} \cap \mathscr{Q}' \cap \mathscr{Q}''$. Let $\pi\colon X \to \mathscr{Q}$ be the blow up of the smooth curve $\mathscr{C}$. Then X is a smooth Fano threefold, which is contained in the family №2.7. Moreover, all smooth Fano threefolds in this family can be obtained in this way. Note that $-K_X^3 = 14$ and $\mathrm{Aut}(X)$ is finite [9].

The pencil generated by the surfaces $\mathscr{Q}'|_{\mathscr{Q}}$ and $\mathscr{Q}''|_{\mathscr{Q}}$ gives a rational map $\mathscr{Q} \dashrightarrow \mathbb{P}^1$, which fits the following commutative diagram:



where ϕ is a fibration into quartic del Pezzo surfaces. Let E be the π-exceptional surface, and let $H = \pi^*(\mathcal{O}_{\mathbb{P}^4}(1)|_{\mathscr{Q}})$. Then $-K_X \sim 3H - E$, the morphism ϕ is given by the linear system $|2H - E|$, and $E \cong \mathscr{C} \times \mathbb{P}^1$.

Lemma 6.1. *Let S be a fiber of the morphism ϕ. Then S has at most Du Val singularities.*

Proof. Since $E|_S$ is a smooth curve, the surface S is smooth along $E|_S$, which implies that it has at most isolated singularities. Hence, we conclude that S is normal and irreducible.

Note that $S \cong \pi(S)$. Suppose that the singularities of the surface $\pi(S)$ are not Du Val. Then it follows from [5, Theorem 1] that $\pi(S)$ is a cone in $\mathbb{P}^4$ over a quartic elliptic curve.

Let P be the vertex of the cone $\pi(S)$, and let T_P be the hyperplane section of the quadric hypersurface $\mathscr{Q}$ that is singular at P. Then T_P contains all lines in $\mathscr{Q}$ that pass through P, which implies that T_P contains $\pi(S)$. This is impossible, since T_P is a quadric cone. $\square$

The goal of this section is to prove that X is K-stable. To do this, we fix a point $P \in X$. By [14, 16], to prove that X is K-stable, it is enough to show that $\delta_P(X) > 1$.

Let S be the fiber of the morphism ϕ containing P. Then S is a quartic del Pezzo surface, and S has at most Du Val singularity at P by Lemma 6.1. Moreover, if S is singular at P, then P is a singular point of the surface S of type $\mathbb{A}_1$, $\mathbb{A}_2$, $\mathbb{A}_3$, $\mathbb{A}_4$, $\mathbb{D}_4$ or $\mathbb{D}_5$, see [11].

Lemma 6.2. *If $\delta_P(S) > \frac{54}{55}$ or $P \in \mathrm{Sing}(S)$ and $\delta_P(S) > \frac{27}{28}$, then $\delta_P(X) > 1$.*

Proof. Take $u \in \mathbb{R}_{\geqslant 0}$. Since $S \sim 2H - E$, we have

$$-K_X - uS \sim_{\mathbb{R}} (3-2u)H - (u-1)E \sim_{\mathbb{R}} \Big(\frac{3}{2} - u\Big)S + \frac{1}{2}E.$$

Using this, we conclude that the divisor $-K_X - uS$ is pseudoeffective if and only if $u \leqslant \frac{3}{2}$. For $u \leqslant \frac{3}{2}$, let $P(u)$ be the positive part of Zariski decomposition of the divisor $-K_X - uS$, and let $N(u)$ be the negative part of Zariski decomposition of the divisor $-K_X - uS$. Then

$$P(u) = \begin{cases} -K_X - uS \text{ if } 0 \leqslant u \leqslant 1, \\ (3-2u)H \text{ if } 1 \leqslant u \leqslant \dfrac{3}{2}, \end{cases}$$

and

$$N(u) = \begin{cases} 0 \text{ if } 0 \leqslant u \leqslant 1, \\ (u-1)E \text{ if } 1 \leqslant u \leqslant \dfrac{3}{2}. \end{cases}$$

This gives

$$S_X(S) = \frac{1}{14}\int_0^{\frac{3}{2}} \big(P(u)\big)^3 du = \frac{1}{14}\int_0^1 (14-12u)du + \frac{1}{14}\int_1^{\frac{3}{2}} 2(3-2u)^3 du = \frac{33}{56} < 1.$$

Now, using [1, Theorem 3.3] and [3, Corollary 1.7.30], we get

$$\delta_P(X) \geqslant \min\left\{\frac{1}{S_X(S)}, \inf_{\substack{F/S \\ P \in C_S(F)}} \frac{A_S(F)}{S(W^S_{\bullet,\bullet}; F)}\right\}, \tag{6.1}$$

where the infimum is taken by all prime divisors F over the surface S such that $P \in C_S(F)$, and $S(W^S_{\bullet,\bullet}; F)$ can be computed using [3, Corollary 1.7.24] as follows:

$$S\big(W^S_{\bullet,\bullet}; F\big) = \frac{3}{14}\int_0^{\frac{3}{2}} \big(P(u)\big|_S\big)^2 \mathrm{ord}_F\big(N(u)\big|_S\big) du + \frac{3}{14}\int_0^{\frac{3}{2}}\int_0^{\infty} \mathrm{vol}\big(P(u)\big|_S - vF\big) dv du.$$

Let $E_S = E|_S$. Then $E_S \in |-2K_S|$ and $E_S \cong \mathscr{C}$. Moreover, the surface S is smooth in a neighborhood of the curve E_S. Furthermore, we have

$$P(u)\big|_S = \begin{cases} -K_S \text{ if } 0 \leqslant u \leqslant 1, \\ (3-2u)(-K_S) \text{ if } 1 \leqslant u \leqslant \dfrac{3}{2}, \end{cases}$$

and

$$N(u)\big|_S=\begin{cases}0 \text{ if } 0\leqslant u\leqslant 1,\\ (u-1)E_S \text{ if } 1\leqslant u\leqslant \dfrac{3}{2}.\end{cases}$$

Let F be any prime divisor over S such that $P\in C_S(F)$. Then

$$S\big(W^S_{\bullet,\bullet};F\big)=\frac{3}{14}\int_1^{\frac{3}{2}}4(u-1)(3-2u)^2\mathrm{ord}_F\big(E_S\big)du+$$
$$+\frac{3}{14}\int_0^1\int_0^\infty \mathrm{vol}\big(-K_S-vF\big)dvdu+$$
$$+\frac{3}{14}\int_1^{\frac{3}{2}}\int_0^\infty \mathrm{vol}\big((3-2u)(-K_S)-vF\big)dvdu=$$
$$=\frac{\mathrm{ord}_F\big(E_S\big)}{56}+\frac{3}{14}\int_0^\infty \mathrm{vol}\big(-K_S-vF\big)dv+\frac{3}{14}\int_1^{\frac{3}{2}}(3-2u)^3\int_0^\infty \mathrm{vol}\big(-K_S-vF\big)dvdu=$$
$$=\frac{\mathrm{ord}_F\big(E_S\big)}{56}+\frac{3}{14}\int_0^\infty \mathrm{vol}\big(-K_S-vF\big)dv+\frac{3}{112}\int_0^\infty \mathrm{vol}\big(-K_S-vF\big)dv=$$
$$=\frac{\mathrm{ord}_F\big(E_S\big)}{56}+\frac{27}{112}\int_0^\infty \mathrm{vol}\big(-K_S-vF\big)dv=\frac{\mathrm{ord}_F\big(E_S\big)}{56}+\frac{27}{28}S_S(F)\leqslant\frac{A_S(F)}{56}+\frac{27A_S(F)}{28\delta_P(S)},$$

because log pair (S,E_S) is log canonical. Thus, if $\delta_P(S)>\frac{54}{55}$, then (6.1) gives $\delta_P(X)>1$. Similarly, if $P\in\mathrm{Sing}(S)$, then $P\not\in E_S$, so that $\mathrm{ord}_F(E_S)=0$, which implies that

$$S\big(W^S_{\bullet,\bullet};F\big)=\frac{27}{28}S_S(F)\leqslant\frac{27A_S(F)}{28\delta_P(S)}.$$

Hence, in this case, it follows from (6.1) that $\delta_P(X)>1$ provided that $\delta_P(S)>\frac{27}{28}$. □

Corollary 6.3. *If S is smooth, then $\delta_P(X)>1$.*

Proof. If S is smooth, then $\delta(S)=\frac{4}{3}$ by [3, Lemma 2.12]. Now apply Lemma 6.2. □

Corollary 6.4. *If P is not contained in a line in S, then $\delta_P(X)>1$.*

Proof. If P is not contained in a line in S, then P is a smooth point of the surface S, and the blow up of the surface S at this point is a (possibly singular) cubic surface in $\mathbb{P}^3$. Thus, arguing exactly as in the end of the proof of [3, Lemma 2.12], we obtain $\delta_P(S)\geqslant\frac{3}{2}$, which implies that $\delta_P(X)>1$ by Lemma 6.2. □

Now, let T be a surface in the linear system $|H|$ such that $P\in T$, and let $\mathcal{Q}=\pi(T)$. Then $\mathcal{Q}$ is a hyperplane section of the hypersurface $\mathscr{Q}$, so both $\mathcal{Q}$ and T are irreducible. In the following, we will choose T such that the surface $\mathcal{Q}$ is smooth, so that $\mathcal{Q}\cong\mathbb{P}^1\times\mathbb{P}^1$.

Lemma 6.5. *Suppose that T is a general surface in the linear system $|H|$ such that $P\in T$. Then the (scheme) intersection $S\cap T$ is an irreducible reduced curve.*

Proof. Let $\rho\colon\widetilde{S}\to S$ be a blow up of the quartic del Pezzo surface S at the point P, and let Z be the proper transform of the curve $T|_S$ on the surface $\widetilde{S}$. Then $|Z|$ has no base points and gives the morphism $\eta\colon \widetilde{S} \to \mathbb{P}^3$ that fits the following commutative diagram:

$$\begin{array}{ccc} & \widetilde{S} & \\ {}^{\rho}\swarrow & & \searrow^{\eta} \\ S & \dashrightarrow & \mathbb{P}^3 \end{array}$$

where $S \dashrightarrow \mathbb{P}^3$ is a projection from P. Moreover, if P is a smooth point of the surface S, then $Z^2 = 3$, and the image of the morphism η is an irreducible cubic surface in $\mathbb{P}^3$. Similarly, if P is a singular point of the surface S, then we have $Z^2 = 4 - \mathrm{mult}_P(S) = 2$, and the image of the morphism η is an irreducible quadric surface. Therefore, we conclude that the curve Z must be irreducible and reduced (by Bertini theorem), which implies that the intersection $S \cap T$ is also irreducible and reduced. □

Remark 6.6. Suppose that S is singular at P, and T is a general surface in $|H|$ that passes through the point P. Then, choosing appropriate coordinates $[x : y : z : t : w]$ on $\mathbb{P}^4$, we may assume that $\pi(P) = [0 : 0 : 0 : 0 : 1]$, and the surface $\pi(S)$ is given in $\mathbb{P}^4$ by

$$\begin{cases} at^2 + btx + f_2(x, y, z) = 0, \\ wt = g_2(x, y, z), \end{cases}$$

where a and b are complex numbers, $f_2(x, y, z)$ and $g_2(x, y, z)$ are non-zero quadratic homogeneous polynomials. In the chart $w \neq 0$, the surface $\pi(S)$ is given by

$$\begin{cases} at^2 + btx + f_2(x, y, z) = 0, \\ t = g_2(x, y, z), \end{cases}$$

where now we consider x, y, z, t as affine coordinates on $\mathbb{C}^4$. Then $\pi(S) \cap \mathcal{Q}$ is cut out on the surface $\pi(S)$ by $c_1x + c_2y + c_3z + c_4t = 0$, where c_1, c_2, c_3, c_4 are general numbers. The affine part of the surface $\pi(S)$ is isomorphic to the hypersurface in $\mathbb{C}^3$ given by

$$ag_2^2(x, y, z) + bxg_2(x, y, z) + f_2(x, y, z) = 0,$$

and the affine part of the curve $\pi(S) \cap \mathcal{Q}$ is cut out by $c_1x + c_2y + c_3z + c_4g_2(x, y, z) = 0$. If P is a singular point of the surface S of type $\mathbb{D}_4$ or $\mathbb{D}_5$, then $S \cap T$ has an ordinary cusp at the point P, which easily implies that the intersection $S \cap T$ is reduced and irreducible. Similarly, if P is a Du Val singular point of the surface S of type $\mathbb{A}_1$, $\mathbb{A}_2$, $\mathbb{A}_3$ or $\mathbb{A}_4$, then the intersection $S \cap T$ has an isolated ordinary double singularity at P.

Observe that the morphism $\pi\colon X \to \mathscr{Q}$ induces a birational morphism $\varpi\colon T \to \mathcal{Q}$, and the morphism $\phi\colon X \to \mathbb{P}^1$ induces a fibration $\varphi\colon T \to \mathbb{P}^1$ that both fit the following commutative diagram:

$$\begin{array}{ccc} & T & \\ {}^{\varpi}\swarrow & & \searrow^{\varphi} \\ \mathcal{Q} & \dashrightarrow & \mathbb{P}^1 \end{array}$$

where $\mathcal{Q} \dashrightarrow \mathbb{P}^1$ is a map given by the pencil generated by the curves $\mathscr{Q}'|_{\mathcal{Q}}$ and $\mathscr{Q}''|_{\mathcal{Q}}$. In the following, we will always choose $T \in |H|$ such that the quadric surface $\mathcal{Q}$ is smooth, and T is either smooth or has one isolated ordinary singularity, which would imply that a general fiber of the induced fibration $\varphi\colon T \to \mathbb{P}^1$ is a smooth elliptic curve. Let $\mathcal{C} = S|_T$. Then $\mathcal{C}$ is the fiber of the (elliptic) fibration φ that contains the point P.

Let u be a non-negative real number. Then $-K_X-uT\sim_{\mathbb{R}}(3-u)H-E\sim_{\mathbb{R}}(1-u)H+S$, which implies that $-K_X-uT$ is nef $\iff$ $-K_X-uT$ is pseudoeffective $\iff$ $u\in[0,1]$. Integrating, we get $S_X(T)=\frac{9}{28}<1$. For simplicity, we let $P(u)=-K_X-uT$.

Lemma 6.7. *Suppose that S is singular at P. Then $\delta_P(X)>1$.*

Proof. Now, let us choose $T\in|H|$ such that T is a general surface in $|H|$ that contains P. Then T and $\mathcal{Q}$ are smooth, and ϖ is a blow up of the eight intersection points $\mathscr{C}\cap\mathcal{Q}$. Moreover, by Lemma 6.5, the curve $\mathcal{C}$ is an irreducible singular curve of arithmetic genus 1. Thus, we have $P=\mathrm{Sing}(\mathcal{C})$. Furthermore, using Remark 6.6, we see that

- either $\mathcal{C}$ has an isolated ordinary double singularity at P,
- or the curve $\mathcal{C}$ has an ordinary cusp at the point P.

Recall that $\mathcal{Q}$ is a smooth quadric surface, so that it contains exactly two lines that pass though $\pi(P)$. Since T is chosen to be general, these lines are disjoint from $\mathscr{C}\cap\mathcal{Q}$. Denote by L_1 and L_2 the proper transforms of these lines on T. Then

$$P(u)\big|_T\sim_{\mathbb{R}}\big((1-u)H+S\big)\big|_T\sim_{\mathbb{R}}(1-u)(L_1+L_2)+\mathcal{C}.$$

Now, we let $\sigma\colon\widetilde{T}\to T$ be the blow up of the point P, we let $\mathbf{e}$ be the σ-exceptional curve, and we denote by $\widetilde{\mathcal{C}}$, $\widetilde{L}_1$, $\widetilde{L}_2$ the proper transforms on $\widetilde{T}$ of the curves $\mathcal{C}$, L_1, L_2, respectively. Take a non-negative real number v. Then

$$\sigma^*\big(P(u)\big|_T\big)-v\mathbf{e}\sim_{\mathbb{R}}\widetilde{\mathcal{C}}+(1-u)(\widetilde{L}_1+\widetilde{L}_2)+(4-2u-v)\mathbf{e}. \tag{6.2}$$

Note that the curves $\widetilde{\mathcal{C}}$, $\widetilde{L}_1$, $\widetilde{L}_2$ are disjoint. Moreover, we have $\widetilde{L}_1^2=\widetilde{L}_2^2=-1$ and $\widetilde{\mathcal{C}}^2=-4$. Thus, using (6.2), we see that $\sigma^*(P(u)|_T)-v\mathbf{e}$ is pseudoeffective $\iff$ $v\leqslant 4-2u$.

Let $\widetilde{P}(u,v)$ be the positive part of Zariski decomposition of the divisor $\sigma^*(P(u)|_T)-v\mathbf{e}$, and let $\widetilde{N}(u,v)$ be its negative part. Then it follows from [3, Remark 1.7.32] that

$$\delta_P(X)\geqslant\min\left\{\frac{1}{S_X(T)},\frac{2}{S(W^{\widetilde{T}}_{\bullet,\bullet};\mathbf{e})},\inf_{O\in\mathbf{e}}\frac{1}{S(W^{\widetilde{T},\mathbf{e}}_{\bullet,\bullet,\bullet};O)}\right\}, \tag{6.3}$$

where $S(W^{\widetilde{T}}_{\bullet,\bullet};\mathbf{e})$ and $S(W^{\widetilde{T},\mathbf{e}}_{\bullet,\bullet,\bullet};O)$ are defined in [3, § 1.7], and these two numbers can be computed using formulas described in [3, Remark 1.7.32]. Namely, we have

$$S\big(W^{\widetilde{T}}_{\bullet,\bullet};\mathbf{e}\big)=\frac{3}{14}\int_0^1\int_0^{4-2u}\big(\widetilde{P}(u,v)\big)^2dvdu$$

and

$$S\big(W^{\widetilde{T},\mathbf{e}}_{\bullet,\bullet,\bullet};O\big)=\frac{3}{14}\int_0^1\int_0^{4-2u}\Big(\big(\widetilde{P}(u,v)\cdot\mathbf{e}\big)\Big)^2dvdu+\frac{3}{7}\int_0^1\int_0^{4-2u}\big(\widetilde{P}(u,v)\cdot\mathbf{e}\big)\mathrm{ord}_O\Big(\widetilde{N}(u,v)\big|_{\mathbf{e}}\Big)dvdu,$$

where O is a point in $\mathbf{e}$. Moreover, using (6.2), we compute $\widetilde{P}(u,v)$ and $\widetilde{N}(u,v)$ as follows:

$$\widetilde{P}(u,v)=\begin{cases}\widetilde{\mathcal{C}}+(1-u)(\widetilde{L}_1+\widetilde{L}_2)+(4-2u-v)\mathbf{e}\text{ if }0\leqslant v\leqslant 2-2u,\\ \dfrac{4-2u-v}{2}\widetilde{\mathcal{C}}+(1-u)(\widetilde{L}_1+\widetilde{L}_2)+(4-2u-v)\mathbf{e}\text{ if }2-2u\leqslant v\leqslant 3-u,\\ \dfrac{4-2u-v}{2}\big(\widetilde{\mathcal{C}}+2(\widetilde{L}_1+\widetilde{L}_2)+2\mathbf{e}\big)\text{ if }3-u\leqslant v\leqslant 4-2u,\end{cases}$$

and

$$\widetilde{N}(u,v)=\begin{cases} 0 \text{ if } 0\leqslant v\leqslant 2-2u,\\ \dfrac{v-2+2u}{2}\widetilde{\mathcal{C}} \text{ if } 2-2u\leqslant v\leqslant 3-u,\\ \dfrac{v-2+2u}{2}\widetilde{\mathcal{C}}+(v-3+u)(\widetilde{L}_1+\widetilde{L}_2) \text{ if } 3-u\leqslant v\leqslant 4-2u.\end{cases}$$

Thus, we have

$$\big(\widetilde{P}(u,v)\big)^2=\begin{cases} 2(1-u)(5-u)-v^2 \text{ if } 0\leqslant v\leqslant 2-2u,\\ 2(1-u)(7-3u-2v) \text{ if } 2-2u\leqslant v\leqslant 3-u,\\ 2(4-2u-v)^2 \text{ if } 3-u\leqslant v\leqslant 4-2u,\end{cases}$$

and

$$\widetilde{P}(u,v)\cdot\mathbf{e}=\begin{cases} v \text{ if } 0\leqslant v\leqslant 2-2u,\\ 2(1-u) \text{ if } 2-2u\leqslant v\leqslant 3-u,\\ 2(4-2u-v) \text{ if } 3-u\leqslant v\leqslant 4-2u.\end{cases}$$

Now, integrating, we get $S(W^{\widetilde{T}}_{\bullet,\bullet};\mathbf{e})=\frac{51}{28}<2$.

Let O be an arbitrary point in $\mathbf{e}$. If $O\not\in\widetilde{L}_1\cup\widetilde{L}_2\cup\widetilde{\mathcal{C}}$, then we compute $S(W^{\widetilde{T},\mathbf{e}}_{\bullet,\bullet,\bullet};O)=\frac{4}{7}$. Similarly, if $O\in\widetilde{L}_1\cup\widetilde{L}_2$, then $S(W^{\widetilde{T},\mathbf{e}}_{\bullet,\bullet,\bullet};O)=\frac{17}{28}$. Finally, if $O\in\widetilde{\mathcal{C}}$, then

$$S\big(W^{\widetilde{T},\mathbf{e}}_{\bullet,\bullet,\bullet};O\big)=\frac{4}{7}+\frac{3}{7}\int_0^1\int_{2-2u}^{3-u}2(1-u)\frac{v-2+2u}{2}\mathrm{ord}_O\big(\widetilde{\mathcal{C}}\big|_{\mathbf{e}}\big)dvdu+$$
$$+\frac{3}{7}\int_0^1\int_{3-u}^{4-2u}2(4-2u-v)\frac{v-2+2u}{2}\mathrm{ord}_O\big(\widetilde{\mathcal{C}}\big|_{\mathbf{e}}\big)dvdu=\frac{4}{7}+\frac{17}{56}\mathrm{ord}_O\big(\widetilde{\mathcal{C}}\big|_{\mathbf{e}}\big).$$

Hence, if $\widetilde{\mathcal{C}}$ intersects $\mathbf{e}$ transversally, then $S(W^{\widetilde{T},\mathbf{e}}_{\bullet,\bullet,\bullet};O)<1$, so that $\delta_P(X)>1$ by (6.3).

Therefore, to complete the proof of the lemma, we may assume that $\widetilde{\mathcal{C}}$ is tangent to $\mathbf{e}$. This means that $\mathcal{C}$ has a cusp at P, and the intersection $\widetilde{\mathcal{C}}\cap\mathbf{e}$ consists of a single point.

Now, as in the proof of Lemma 4.5, we consider the following commutative diagram:

$$\begin{array}{ccc} \widetilde{T} & \xleftarrow{\ \gamma\ } & \overline{T}\\ {\scriptstyle\sigma}\big\downarrow & & \big\downarrow{\scriptstyle\upsilon}\\ T & \xleftarrow{\ \varsigma\ } & \widehat{T}\end{array}$$

where γ is a composition of the blow up of the point $\widetilde{\mathcal{C}}\cap\mathbf{e}$ with the blow up of the unique intersection point of the proper transforms of the curves $\widetilde{\mathcal{C}}$ and $\mathbf{e}$, υ is the birational contraction of all $(\sigma\circ\gamma)$-exceptional curves that are not (-1)-curve, and ς is the birational contraction of the proper transform of the unique γ-exceptional curve that is (-1)-curve. Then $\widehat{T}$ has two singular points:

(1) a cyclic quotient singularity of type $\frac{1}{2}(1,1)$, which we denote by O_2;
(2) a cyclic quotient singularity of type $\frac{1}{3}(1,1)$, which we denote by O_3.

Let $\mathbf{f}$ be the ς-exceptional curve, let $\widehat{\mathcal{C}}$ be the proper transform on $\widehat{T}$ of the curve $\mathcal{C}$, and let $\widehat{L}_1$ and $\widehat{L}_2$ be the proper transforms on $\widehat{T}$ of the curves L_1 and L_2, respectively.

Then the curves $\mathbf{f}$, $\widehat{\mathcal{C}}$, $\widehat{L}_1$, $\widehat{L}_2$ are smooth, and the curve $\mathbf{f}$ contains both points O_2 and O_3, which are not contained in $\widehat{\mathcal{C}}$. Moreover, we have

$$\widehat{L}_1 \cap \widehat{L}_2 = \widehat{L}_1 \cap \mathbf{f} = \widehat{L}_2 \cap \mathbf{f} = O_3.$$

Furthermore, we have $A_T(\mathbf{f}) = 5$, $\varsigma^*(\mathcal{C}) \sim \widehat{\mathcal{C}} + 6\mathbf{f}$, $\varsigma^*(L_1) \sim \widehat{L}_1 + 2\mathbf{f}$, $\varsigma^*(L_2) \sim \widehat{L}_2 + 2\mathbf{f}$, and the intersection form of the curves $\mathbf{f}$, $\widehat{\mathcal{C}}$, $\widehat{L}_1$ and $\widehat{L}_2$ is given in the following table:

	$\mathbf{f}$	$\widehat{\mathcal{C}}$	$\widehat{L}_1$	$\widehat{L}_2$
$\mathbf{f}$	$-\frac{1}{6}$	1	$\frac{1}{3}$	$\frac{1}{3}$
$\widehat{\mathcal{C}}$	1	-6	0	0
$\widehat{L}_1$	$\frac{1}{3}$	0	$-\frac{2}{3}$	$\frac{1}{3}$
$\widehat{L}_2$	$\frac{1}{3}$	0	$\frac{1}{3}$	$-\frac{2}{3}$

For a non-negative real number v, we have

$$\varsigma^*\big(P(u)\big|_T\big) - v\mathbf{f} \sim_{\mathbb{R}} \widetilde{\mathcal{C}} + (1-u)(\widetilde{L}_1 + \widetilde{L}_2) + (10 - 4u - v)\mathbf{f},$$

which implies that the divisor $\varsigma^*(P(u)|_T) - v\mathbf{f}$ is pseudoeffective if and only if $v \leqslant 10 - 4u$, because the intersection form of the curves $\widehat{\mathcal{C}}$, $\widehat{L}_1$, $\widehat{L}_2$ is negative definite.

Let $\widehat{P}(u,v)$ be the positive part of Zariski decomposition of the divisor $\varsigma^*(P(u)|_T) - v\mathbf{f}$, and let $\widehat{N}(u,v)$ be its negative part. Set $\Delta_{\mathbf{f}} = \frac{1}{2}O_2 + \frac{2}{3}O_3$. By [3, Remark 1.7.32], we get

$$\delta_P(X) \geqslant \min\left\{\frac{1}{S_X(T)}, \frac{5}{S(W^{\widehat{T}}_{\bullet,\bullet}; \mathbf{f})}, \inf_{O \in \mathbf{f}} \frac{1 - \mathrm{ord}_O\big(\Delta_{\mathbf{f}}\big)}{S(W^{\widehat{T},\mathbf{f}}_{\bullet,\bullet,\bullet}; O)}\right\}, \tag{6.4}$$

where $S(W^{\widehat{T}}_{\bullet,\bullet}; \mathbf{f})$ and $S(W^{\widehat{T},\mathbf{f}}_{\bullet,\bullet,\bullet}; O)$ are defined as $S(W^{\widetilde{T}}_{\bullet,\bullet}; \mathbf{e})$ and $S(W^{\widetilde{T},\mathbf{e}}_{\bullet,\bullet,\bullet}; O)$ used earlier. Moreover, it follows from [3, Remark 1.7.32] that

$$S\big(W^{\widehat{T}}_{\bullet,\bullet}; \mathbf{f}\big) = \frac{3}{14}\int_0^1 \int_0^{10-4u} \big(\widehat{P}(u,v)\big)^2 dv du$$

and

$$S\big(W^{\widehat{T},\mathbf{f}}_{\bullet,\bullet,\bullet}; O\big) = \frac{3}{14}\int_0^1 \int_0^{10-4u} \Big(\big(\widehat{P}(u,v)\cdot\mathbf{f}\big)\Big)^2 dv du + \frac{3}{7}\int_0^1 \int_0^{10-4u} \big(\widehat{P}(u,v)\cdot\mathbf{f}\big)\mathrm{ord}_O\Big(\widehat{N}(u,v)\big|_{\mathbf{f}}\Big) dv du.$$

Moreover, we compute $\widehat{P}(u,v)$ and $\widehat{N}(u,v)$ as follows:

$$\widehat{P}(u,v) = \begin{cases} \widetilde{\mathcal{C}} + (1-u)(\widetilde{L}_1 + \widetilde{L}_2) + (10-4u-v)\mathbf{f} \text{ if } 0 \leqslant v \leqslant 4 - 4u, \\ \dfrac{10-4u-v}{6}\widetilde{\mathcal{C}} + (1-u)(\widetilde{L}_1 + \widetilde{L}_2) + (10-4u-v)\mathbf{f} \text{ if } 4 - 4u \leqslant v \leqslant 9 - 3u, \\ \dfrac{10-4u-v}{6}\big(\widehat{\mathcal{C}} + 6(\widehat{L}_1 + \widehat{L}_2) + 6\mathbf{f}\big) \text{ if } 9 - 3u \leqslant v \leqslant 10 - 4u, \end{cases}$$

and

$$\widehat{N}(u,v) = \begin{cases} 0 \text{ if } 0 \leqslant v \leqslant 4 - 4u, \\ \dfrac{v-4+4u}{6}\widehat{\mathcal{C}} \text{ if } 4 - 4u \leqslant v \leqslant 9 - 3u, \\ \dfrac{v-4+4u}{6}\widehat{\mathcal{C}} + (v - 9 + 3u)(\widetilde{L}_1 + \widetilde{L}_2) \text{ if } 9 - 3u \leqslant v \leqslant 10 - 4u. \end{cases}$$

This gives

$$\big(\widehat{P}(u,v)\big)^2=\begin{cases}2(1-u)(5-u)-\dfrac{v^2}{6}\ \text{if}\ 0\leqslant v\leqslant 4-4u,\\ \dfrac{2(1-u)(19-7u-2v)}{3}\ \text{if}\ 4-4u\leqslant v\leqslant 9-3u,\\ \dfrac{2(10-4u-v)^2}{3}\ \text{if}\ 9-3u\leqslant v\leqslant 10-4u,\end{cases}$$

and

$$\widehat{P}(u,v)\cdot\mathbf{f}=\begin{cases}\dfrac{v}{6}\ \text{if}\ 0\leqslant v\leqslant 4-4u,\\ \dfrac{2(1-u)}{3}\ \text{if}\ 4-4u\leqslant v\leqslant 9-3u,\\ \dfrac{2(10-4u-v)}{3}\ \text{if}\ 9-3u\leqslant v\leqslant 10-4u.\end{cases}$$

Now, integrating, we get $S(W^{\widehat{T}}_{\bullet,\bullet};\mathbf{f})=\frac{135}{28}<A_T(\mathbf{f})=5$.

Let O be a point in $\mathbf{f}$. If $O\not\in\widehat{L}_1\cup\widehat{L}_2\cup\widehat{\mathcal{C}}$, then

$$S\big(W^{\widehat{T},\mathbf{f}}_{\bullet,\bullet,\bullet};O\big)=\frac{3}{14}\int_0^1\int_0^{4-4u}\Big(\frac{v}{6}\Big)^2dvdu+\frac{3}{14}\int_0^1\int_{4-4u}^{9-3u}\Big(\frac{2(1-u)}{3}\Big)^2dvdu+$$
$$+\frac{3}{14}\int_0^1\int_{9-3u}^{10-4u}\Big(\frac{2(10-4u-v)}{3}\Big)^2dvdu=\frac{13}{63}.$$

Similarly, if $O=\mathbf{f}\cap\widetilde{\mathcal{C}}$, then

$$S\big(W^{\widehat{T},\mathbf{f}}_{\bullet,\bullet,\bullet};O\big)=\frac{13}{63}+\frac{3}{7}\int_0^1\int_{4-4u}^{9-3u}\frac{2(1-u)}{3}\times\frac{v-4+4u}{6}dvdu+$$
$$+\frac{3}{7}\int_0^1\int_{9-3u}^{10-4u}\frac{2(10-4u-v)}{3}\times\frac{v-4+4u}{6}dvdu=\frac{13}{63}+\frac{193}{504}=\frac{33}{56}.$$

Likewise, if $O=O_3$, we compute $S(W^{\widehat{T},\mathbf{f}}_{\bullet,\bullet,\bullet};O)=\frac{3}{14}$. So, using (6.4), we get $\delta_P(X)>1$. $\square$

Thus, to prove that $\delta_P(X)>1$, we may assume that S is singular, but $P\not\in\mathrm{Sing}(S)$.

Lemma 6.8. *Suppose that $P\not\in E$. Then $\delta_P(X)>1$.*

Proof. By Corollary 6.4, we may assume that S contains a line L that passes through P. Then $\pi(L)$ is a line in $\mathscr{Q}$. Note that $\pi(L)\cap\mathscr{C}\neq\varnothing$, and one of the following cases holds:

Case 1: the line $\pi(L)$ intersects the curve $\mathscr{C}$ transversally at 2 points,

Case 2: the line $\pi(L)$ is tangent to the curve $\mathscr{C}$ at their single intersection point.

Now, let us choose T to be a sufficiently general surface in $|H|$ that passes through L. If the intersection $\pi(L)\cap\mathscr{C}$ consists of two points, then $\varpi\colon T\to\mathcal{Q}$ is a blow up of eight distinct points of the transversal intersection $\mathcal{Q}\cap\mathscr{C}$, which implies that T is smooth. On the other hand, if $L\cap\mathscr{C}$ consists of one point, then T has one ordinary double point, which is not contained in the curve L. We have $\mathcal{C}=S|_T=L+Z$, where Z is a smooth rational irreducible curve such that $\pi(Z)$ is a smooth twisted cubic in $\mathscr{Q}$ that intersects the curve $\mathscr{C}$ transversally by six distinct points, which we denote by Q_3,Q_4,Q_5,Q_6,Q_7,Q_8.

Moreover, if $\pi(L)\cap\mathscr{C}$ consists of two distinct points, we denote these points by Q_1 and Q_2. Likewise, if $\pi(L)\cap\mathscr{C}$ consists of one point, we let $Q_1=Q_2=\pi(Z)\cap\mathscr{C}$. Then

Case 1: the morphism $\varpi\colon T\to\mathcal{Q}$ is the blow up of the points $Q_1,Q_2,\ldots,Q_8$,

Case 2: the morphism $\varpi\colon T\to\mathcal{Q}$ is a composition of the blow up of the points $Q_3,\ldots,Q_8$ with a weighted blow up with weights $(1,2)$ of the point $Q_1=Q_2$.

Since T is a general surface in $|H|$ that contains the line L, we may assume that $P\notin Z$. Likewise, we may assume further that Z is contained in the smooth locus of the surface T. Moreover, we may also assume that the quadric surface $\mathcal{Q}$ does not contain lines that pass through one point in the set $\{Q_1,Q_2,\pi(P)\}$ and one point in $\{Q_3,Q_4,Q_5,Q_6,Q_7,Q_8\}$. Indeed, let $\mathscr{Q}'$, $\mathscr{Q}''$, $\mathscr{Q}'''$ be the hyperplane sections of the quadric $\mathscr{Q}$ that are singular at the points Q_1, Q_2, $\pi(P)$, respectively. Then $\mathscr{Q}'$, $\mathscr{Q}''$, $\mathscr{Q}'''$ are cones, $\pi(L)\subset\mathscr{Q}'\cap\mathscr{Q}''\cap\mathscr{Q}'''$, and every line in $\mathscr{Q}$ containing a point in $\{Q_1,Q_2,\pi(P)\}$ is a ruling of one of these cones. On the other hand, we have $\mathscr{C}\not\subset\mathscr{Q}'\cup\mathscr{Q}''\cup\mathscr{Q}'''$, because $\mathscr{C}$ is not contained in a hyperplane. This implies that the quadric threefold $\mathscr{Q}$ contains at most finitely many lines that pass through a point in $\{Q_1,Q_2,\pi(P)\}$ and a point in $\mathscr{C}\setminus\{Q_1,Q_2\}$. Therefore, we can choose the surface $T\in|H|$ such that $L\subset T$, but $\mathcal{Q}=\pi(T)$ does not contain any of these lines.

Let us identify $\mathcal{Q}=\mathbb{P}^1\times\mathbb{P}^1$ such that the line $\pi(L)$ is a curve in $\mathcal{Q}$ of degree $(0,1)$. Denote by $\mathbf{e}_1,\ldots,\mathbf{e}_8$ the ϖ-exceptional curves such that $\varpi(\mathbf{e}_1)=Q_1,\ldots,\varpi(\mathbf{e}_8)=Q_8$. Let $\mathbf{g}_3,\ldots,\mathbf{g}_8$ be the strict transforms on T of the curves in $\mathcal{Q}$ of degree $(1,0)$ that pass through the points $Q_3,\ldots,Q_8$, respectively. Then $L,Z,\mathbf{e}_1,\ldots,\mathbf{e}_8,\mathbf{g}_3,\ldots,\mathbf{g}_8$ are smooth irreducible rational curves. In Case 1, their intersections are given in the following table:

	L	Z	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_5$	$\mathbf{e}_6$	$\mathbf{e}_7$	$\mathbf{e}_8$	$\mathbf{g}_3$	$\mathbf{g}_4$	$\mathbf{g}_5$	$\mathbf{g}_6$	$\mathbf{g}_7$	$\mathbf{g}_8$
L	-2	2	1	1	0	0	0	0	0	0	1	1	1	1	1	1
Z	2	-2	0	0	1	1	1	1	1	1	0	0	0	0	0	0
$\mathbf{e}_1$	1	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_2$	1	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_3$	0	1	0	0	-1	0	0	0	0	0	1	0	0	0	0	0
$\mathbf{e}_4$	0	1	0	0	0	-1	0	0	0	0	0	1	0	0	0	0
$\mathbf{e}_5$	0	1	0	0	0	0	-1	0	0	0	0	0	1	0	0	0
$\mathbf{e}_6$	0	1	0	0	0	0	0	-1	0	0	0	0	0	1	0	0
$\mathbf{e}_7$	0	1	0	0	0	0	0	0	-1	0	0	0	0	0	1	0
$\mathbf{e}_8$	0	1	0	0	0	0	0	0	0	-1	0	0	0	0	0	1
$\mathbf{g}_3$	1	0	0	0	1	0	0	0	0	0	-1	0	0	0	0	0
$\mathbf{g}_4$	1	0	0	0	0	1	0	0	0	0	0	-1	0	0	0	0
$\mathbf{g}_5$	1	0	0	0	0	0	1	0	0	0	0	0	-1	0	0	0
$\mathbf{g}_6$	1	0	0	0	0	0	0	1	0	0	0	0	0	-1	0	0
$\mathbf{g}_7$	1	0	0	0	0	0	0	0	1	0	0	0	0	0	-1	0
$\mathbf{g}_8$	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	-1

In Case 2, we have $\mathbf{e}_1=\mathbf{e}_2$, and $\mathbf{e}_1$ contains the singular point of T, so that $\mathbf{e}_1^2=-\frac{1}{2}$. The remaining intersection numbers are exactly the same as in Case 1.

Observe that $P\not\in Z\cup\mathbf{g}_3\cup\mathbf{g}_4\cup\mathbf{g}_5\cup\mathbf{g}_6\cup\mathbf{g}_7\cup\mathbf{g}_8\cup\mathbf{e}_1\cup\mathbf{e}_2$, since $P\not\in E$ by assumption.

Recall that $P(u)=-K_X-uT$ is nef $\iff$ $P(u)$ is pseudoeffective $\iff$ $u\in[0,1]$. Let v be a non-negative real number. Then, in both Cases 1 and 2, we have

$$P(u)\big|_T-vL\sim_{\mathbb{R}}\frac{9-5u-4v}{4}L+\frac{3+u}{4}Z+\frac{5-5u}{4}\big(\mathbf{e}_1+\mathbf{e}_2\big)+\frac{1-u}{4}\sum_{i=3}^{8}\mathbf{g}_i, \tag{6.5}$$

which implies that the divisor $P(u)|_T-vL$ is pseudoeffective $\iff$ $v\leqslant\frac{9-5u}{4}$.

Let $P(u,v)$ be the positive part of Zariski decomposition of the divisor $P(u)|_T-vL$, and let $N(u,v)$ be its negative part. Then it follows from [3, Theorem 1.7.30] that

$$\delta_P(X)\geqslant\min\left\{\frac{1}{S_X(T)},\frac{1}{S(W^T_{\bullet,\bullet};L)},\frac{1}{S(W^{T,L}_{\bullet,\bullet,\bullet};P)}\right\}, \tag{6.6}$$

where

$$S\big(W^T_{\bullet,\bullet};L\big)=\frac{3}{14}\int_0^1\int_0^{\frac{9-5u}{4}}\big(P(u,v)\big)^2dvdu$$

and

$$S\big(W^{T,L}_{\bullet,\bullet,\bullet};P\big)=\frac{3}{14}\int_0^1\int_0^{\frac{9-5u}{4}}\Big(\big(P(u,v)\cdot L\big)\Big)^2dvdu+\frac{3}{7}\int_0^1\int_0^{\frac{9-5u}{4}}\big(P(u,v)\cdot L\big)\mathrm{ord}_P\Big(N(u,v)\big|_L\Big)dvdu.$$

Let us compute $S(W^T_{\bullet,\bullet};L)$ and $S(W^{T,L}_{\bullet,\bullet,\bullet};P)$. If $0\leqslant u\leqslant\frac{1}{3}$, then, using (6.5), we get

$$P(u,v)=\begin{cases}\frac{9-5u-4v}{4}L+\frac{3+u}{4}Z+\frac{5-5u}{4}\big(\mathbf{e}_1+\mathbf{e}_2\big)+\frac{1-u}{4}\sum\limits_{i=3}^{8}\mathbf{g}_i \text{ if } 0\leqslant v\leqslant 1,\\ \frac{9-5u-4v}{4}\big(L+\mathbf{e}_1+\mathbf{e}_2\big)+\frac{3+u}{4}Z+\frac{1-u}{4}\sum\limits_{i=3}^{8}\mathbf{g}_i \text{ if } 1\leqslant v\leqslant\frac{3-3u}{2},\\ \frac{9-5u-4v}{4}\big(L+Z+\mathbf{e}_1+\mathbf{e}_2\big)+\frac{1-u}{4}\sum\limits_{i=3}^{8}\mathbf{g}_i \text{ if } \frac{3-3u}{2}\leqslant v\leqslant 2-u,\\ \frac{9-5u-4v}{4}\Big(L+Z+\mathbf{e}_1+\mathbf{e}_2+\sum\limits_{i=3}^{8}\mathbf{g}_i\Big) \text{ if } 2-u\leqslant v\leqslant\frac{9-5u}{4},\end{cases}$$

$$N(u,v)=\begin{cases}0 \text{ if } 0\leqslant v\leqslant 1,\\ (v-1)\big(\mathbf{e}_1+\mathbf{e}_2\big) \text{ if } 1\leqslant v\leqslant\frac{3-3u}{2},\\ (v-1)\big(\mathbf{e}_1+\mathbf{e}_2\big)+\frac{2v+3u-3}{2}Z \text{ if } \frac{3-3u}{2}\leqslant v\leqslant 2-u,\\ (v-1)\big(\mathbf{e}_1+\mathbf{e}_2\big)+\frac{2v+3u-3}{2}Z+(v-2+u)\sum\limits_{i=3}^{8}\mathbf{g}_i \text{ if } 2-u\leqslant v\leqslant\frac{9-5u}{4},\end{cases}$$

$$
\big(P(u,v)\big)^2=\begin{cases}
2u^2+(2v-12)u-2v^2-2v+10 \text{ if } 0\leqslant v\leqslant 1,\\
2u^2+(2v-12)u-6v+12 \text{ if } 1\leqslant v\leqslant \dfrac{3-3u}{2},\\
\dfrac{13u^2+16uv+4v^2-42u-24v+33}{2} \text{ if } \dfrac{3-3u}{2}\leqslant v\leqslant 2-u,\\
\dfrac{(9-5u-4v)^2}{2} \text{ if } 2-u\leqslant v\leqslant \dfrac{9-5u}{4},
\end{cases}
$$

and

$$
P(u,v)\cdot L=\begin{cases}
1-u+2v \text{ if } 0\leqslant v\leqslant 1,\\
3-u \text{ if } 1\leqslant v\leqslant \dfrac{3-3u}{2},\\
6-4u-2v \text{ if } \dfrac{3-3u}{2}\leqslant v\leqslant 2-u,\\
2(9-5u-4v) \text{ if } 2-u\leqslant v\leqslant \dfrac{9-5u}{4},
\end{cases}
$$

Similarly, if $\frac{1}{3}\leqslant u\leqslant 1$, then, using (6.5), we get

$$
P(u,v)=\begin{cases}
\dfrac{9-5u-4v}{4}L+\dfrac{3+u}{4}Z+\dfrac{5-5u}{4}\big(\mathbf{e}_1+\mathbf{e}_2\big)+\dfrac{1-u}{4}\sum_{i=3}^{8}\mathbf{g}_i \text{ if } 0\leqslant v\leqslant \dfrac{3-3u}{2},\\
\dfrac{9-5u-4v}{4}(L+Z)+\dfrac{5-5u}{4}\big(\mathbf{e}_1+\mathbf{e}_2\big)+\dfrac{1-u}{4}\sum_{i=3}^{8}\mathbf{g}_i \text{ if } \dfrac{3-3u}{2}\leqslant v\leqslant 1,\\
\dfrac{9-5u-4v}{4}\big(L+Z+\mathbf{e}_1+\mathbf{e}_2\big)+\dfrac{1-u}{4}\sum_{i=3}^{8}\mathbf{g}_i \text{ if } 1\leqslant v\leqslant 2-u,\\
\dfrac{9-5u-4v}{4}\Big(L+Z+\mathbf{e}_1+\mathbf{e}_2+\sum_{i=3}^{8}\mathbf{g}_i\Big) \text{ if } 2-u\leqslant v\leqslant \dfrac{9-5u}{4},
\end{cases}
$$

$$
N(u,v)=\begin{cases}
0 \text{ if } 0\leqslant v\leqslant \dfrac{3-3u}{2},\\
\dfrac{2v+3u-3}{2}Z \text{ if } \dfrac{3-3u}{2}\leqslant v\leqslant 1,\\
(v-1)\big(\mathbf{e}_1+\mathbf{e}_2\big)+\dfrac{2v+3u-3}{2}Z \text{ if } 1\leqslant v\leqslant 2-u,\\
(v-1)\big(\mathbf{e}_1+\mathbf{e}_2\big)+\dfrac{2v+3u-3}{2}Z+(v-2+u)\sum_{i=3}^{8}\mathbf{g}_i \text{ if } 2-u\leqslant v\leqslant \dfrac{9-5u}{4},
\end{cases}
$$

$$
\big(P(u,v)\big)^2=\begin{cases}
2u^2+(2v-12)u-2v^2-2v+10 \text{ if } 0\leqslant v\leqslant \dfrac{3-3u}{2},\\
\dfrac{(1-u)(29-13u-16v)}{2} \text{ if } \dfrac{3-3u}{2}\leqslant v\leqslant 1,\\
\dfrac{13u^2+16uv+4v^2-42u-24v+33}{2} \text{ if } 1\leqslant v\leqslant 2-u,\\
\dfrac{(9-5u-4v)^2}{2} \text{ if } 2-u\leqslant v\leqslant \dfrac{9-5u}{4},
\end{cases}
$$

and

$$P(u,v)\cdot L=\begin{cases}1-u+2v \text{ if } 0\leqslant v\leqslant \dfrac{3-3u}{2},\\ 4-4u \text{ if } \dfrac{3-3u}{2}\leqslant v\leqslant 1,\\ 6-4u-2v \text{ if } 1\leqslant v\leqslant 2-u,\\ 2(9-5u-4v) \text{ if } 2-u\leqslant v\leqslant \dfrac{9-5u}{4},\end{cases}$$

Therefore, we have $P\not\in \mathrm{Supp}(N(u,v))$, because $P\not\in Z\cup\mathbf{g}_3\cup\mathbf{g}_4\cup\mathbf{g}_5\cup\mathbf{g}_6\cup\mathbf{g}_7\cup\mathbf{g}_8\cup\mathbf{e}_1\cup\mathbf{e}_2$. So, integrating $(P(u,v))^2$ and $(P(u,v)\cdot L)^2$, we get $S(W^T_{\bullet,\bullet};L)=\frac{423}{448}$ and $S(W^{T,L}_{\bullet,\bullet,\bullet};P)=\frac{79}{84}$, which implies that $\delta_P(X)>1$ by (6.6). □

By Lemma 6.8, to show that $\delta_P(X)>1$, we may assume that $P\in E$. Then $\pi(P)\in\mathscr{C}$. Now, let us choose T to be a sufficiently general surface in $|H|$ that contains the point P, so that $\mathcal{Q}$ is a sufficiently general hyperplane section of the quadric $\mathscr{Q}$ that contains $\pi(P)$. Then T is smooth, and $\varpi\colon T\to\mathcal{Q}$ is a blow up of eight points of the intersection $\mathcal{Q}\cap\mathscr{C}$.

Let $Q_1=\pi(P)$, let $Q_2,\ldots,Q_8$ be the remaining seven points of the intersection $\mathcal{Q}\cap\mathscr{C}$, and let $\mathbf{e}_1,\ldots,\mathbf{e}_8$ be the ϖ-exceptional curves such that $\varpi(\mathbf{e}_1)=Q_1,\ldots,\varpi(\mathbf{e}_8)=Q_8$. For every $u\in[0,1]$, we set

$$t(u)=\inf\Big\{v\in\mathbb{R}_{\geqslant 0}\ \Big|\ \text{the divisor } P(u)\big|_T-v\mathbf{e}_1 \text{ is pseudo-effective}\Big\}.$$

Then $t(u)$ is not very easy to compute explicitly in terms of $u\in[0,1]$. Fix $v\in[0,t(u)]$. Let $P(u,v)$ be the positive part of the Zariski decomposition of the divisor $P(u)|_T-v\mathbf{e}_1$, and let $N(u,v)$ be its negative part. Then it follows from [3, Theorem 1.7.30] that

$$\delta_P(X)\geqslant\min\left\{\frac{1}{S_X(T)},\frac{1}{S(W^T_{\bullet,\bullet};\mathbf{e}_1)},\frac{1}{S(W^{T,\mathbf{e}_1}_{\bullet,\bullet,\bullet};P)}\right\},\tag{6.7}$$

where

$$S\big(W^T_{\bullet,\bullet};\mathbf{e}_1\big)=\frac{3}{14}\int_0^1\int_0^{t(u)}\big(P(u,v)\big)^2dvdu$$

and

$$S\big(W^{T,\mathbf{e}_1}_{\bullet,\bullet,\bullet};P\big)=\frac{3}{14}\int_0^1\int_0^{t(u)}\Big(\big(P(u,v)\cdot\mathbf{e}_1\big)\Big)^2dvdu+\\ +\frac{3}{7}\int_0^1\int_0^{t(u)}\big(P(u,v)\cdot\mathbf{e}_1\big)\mathrm{ord}_P\Big(N(u,v)\big|_{\mathbf{e}_1}\Big)dvdu.$$

Let us compute $S(W^T_{\bullet,\bullet};\mathbf{e}_1)$ and $S(W^{T,\mathbf{e}_1}_{\bullet,\bullet,\bullet};P)$. This will take a while.

Recall that $\varphi\colon T\to\mathbb{P}^1$ is an elliptic fibration, which is given by the linear system $|-K_T|$. As in the proof of Lemma 6.8, let us identify $\mathcal{Q}=\mathbb{P}^1\times\mathbb{P}^1$.

Lemma 6.9. *Let F be a curve in $|-K_T|$. Then F is irreducible and reduced.*

Proof. Suppose that F is reducible or non-reduced. Then the curve $\pi(F)$ is also reducible or non-reduced, and every irreducible component of the curve F is a smooth (-2)-curve. But $\pi(F)$ is a curve in $\mathcal{Q}$ of degree $(2,2)$ that passes through the points $Q_1,Q_2,\ldots,Q_8$. Therefore, we have one of the following possibilities:

(1) $\mathcal{Q}$ contains a line that passes through Q_1 and one point among $Q_2,\ldots,Q_8$,

(2) $\mathcal{Q}$ contains a line that passes through two points among $Q_2, \ldots, Q_8$,
(3) $\mathcal{Q}$ contains a conic that passes through Q_1 and three points among $Q_2, \ldots, Q_8$.

Recall that $\mathcal{Q}$ is a general hyperplane section of the quadric $\mathscr{Q}$ that contains $Q_1 = \pi(P)$. As we already mentioned in the proof of Lemma 6.8, the quadric $\mathscr{Q}$ contains finitely many lines that pass through Q_1 and a point in $\mathscr{C} \setminus Q_1$. Thus, since $\mathcal{Q}$ is assumed to be general, the quadric $\mathcal{Q}$ does not contain any of these lines, so that $\mathcal{Q}$ does not contain a line that passes through Q_1 and a point among $Q_2, \ldots, Q_8$.

Similarly, a parameter count implies that $\mathcal{Q}$ does not contain secant lines of the curve $\mathscr{C}$, so that $\mathcal{Q}$ does not contain a line that passes through two points among $Q_2, \ldots, Q_8$,

Finally, we suppose that $\mathcal{Q}$ contains an irreducible conic C that passes through Q_1 and three points among $Q_2, \ldots, Q_8$. Let $\rho\colon \mathscr{Q} \dashrightarrow \mathbb{P}^3$ be a linear projection from the point Q_1. Then $\rho(\mathscr{C})$ is a curve of degree 7, and the induced map $\mathscr{C} \dashrightarrow \rho(\mathscr{C})$ is an isomorphism, because $\mathscr{C}$ is an intersection of quadrics. Similarly, all points $\rho(Q_2), \ldots, \rho(Q_8)$ are distinct. Then $\rho(C)$ is a three-secant line of the curve $\rho(\mathscr{C})$. Note that $\rho(\mathscr{C})$ contains one-parameter family of three-secants [10, Appendix A]. But $\rho(\mathcal{Q})$ is a general plane in $\mathbb{P}^3$, which implies that $\rho(\mathcal{Q})$ does not contain three-secant lines of the curve $\rho(\mathscr{C})$ — a contradiction. □

Corollary 6.10. *Let γ be a class in the group $\mathrm{Pic}(T)$ such that $-K_T \cdot \gamma = 1$ and $\gamma^2 = -1$. Then the linear system $|\gamma|$ consists of a unique (-1)-curve.*

Proof. Apply the Riemann–Roch theorem, Serre duality and Lemma 6.9. □

Let us use Corollary 6.10, to describe infinitely many (-1)-curves in the surface T. Namely, let ℓ_1 and ℓ_2 be any curves in $\mathcal{Q} = \mathbb{P}^1 \times \mathbb{P}^1$ of degrees $(1,0)$ and $(0,1)$, respectively. For $n \in \mathbb{Z}_{\geqslant 0}$ and $i \in \{2,3,4,5,6,7,8\}$, let $B_{n,1,1}$, $B_{n,1,2}$, $B_{n,2,i}$, $B_{n,3}$, $B_{n,4,i}$ be the classes

$$\varpi^*\big(a_1\ell_1 + a_2\ell_2\big) - \sum_{i=1}^{8} b_i \mathbf{e}_i \in \mathrm{Pic}(T),$$

where $a_1, a_2, b_1, \ldots, b_8$ are non-negative integers given in the following table:

	a_1	a_2	b_1	$b_2, b_3, b_4, b_5, b_6, b_7, b_8$
$B_{n,1,1}$	$14n^2 + 7n + 1$	$14n^2 + 7n$	$7n^2 + 7n + 1$	$\forall j \ b_j = 7n^2 + 3n$
$B_{n,1,2}$	$14n^2 + 7n$	$14n^2 + 7n + 1$	$7n^2 + 7n + 1$	$\forall j \ b_j = 7n^2 + 3n$
$B_{n,2,i}$	$14n^2 + 13n + 3$	$14n^2 + 13n + 3$	$7n^2 + 10n + 3$	$b_i = 7n^2 + 6n + 2$ $\forall j \neq i \ b_j = 7n^2 + 6n + 1$
$B_{n,3}$	$14n^2 + 21n + 7$	$14n^2 + 21n + 7$	$7n^2 + 14n + 6$	$\forall j \ b_j = 7n^2 + 10n + 3$
$B_{n,4,i}$	$14n^2 + 29n + 15$	$14n^2 + 29n + 15$	$7n^2 + 18n + 11$	$b_i = 7n^2 + 14n + 6$ $\forall j \neq i \ b_j = 7n^2 + 14n + 7$

By Corollary 6.10, each linear system $|B_{n,1,1}|$, $|B_{n,1,2}|$, $|B_{n,2,i}|$, $|B_{n,3}|$, $|B_{n,4,i}|$ contains a unique (-1)-curve. Hence, we can identify the classes $B_{n,1,1}$, $B_{n,1,2}$, $B_{n,2,i}$, $B_{n,3}$, $B_{n,4,i}$ with (-1)-curves in $|B_{n,1,1}|$, $|B_{n,1,2}|$, $|B_{n,2,i}|$, $|B_{n,3}|$, $|B_{n,4,i}|$, respectively. Set

$$\begin{aligned}
B_{n,1} &= B_{n,1,1} + B_{n,1,2},\\
B_{n,2} &= B_{n,2,2} + B_{n,2,3} + B_{n,2,4} + B_{n,2,5} + B_{n,2,6} + B_{n,2,7} + B_{n,2,8},\\
B_{n,4} &= B_{n,4,2} + B_{n,4,3} + B_{n,4,4} + B_{n,4,5} + B_{n,4,6} + B_{n,4,7} + B_{n,4,8}.
\end{aligned}$$

Note that irreducible components of each curve $B_{n,1}$, $B_{n,2}$, $B_{n,4}$ are disjoint (-1)-curves, and $B_{n,1}\cap B_{n,2}=\varnothing$, $B_{n,2}\cap B_{n,3}=\varnothing$, $B_{n,3}\cap B_{n,4}=\varnothing$, $B_{n,4}\cap B_{n+1,1}=\varnothing$ for each $n\geqslant 0$.

Now, we let $I'_{0,1}=[0,\frac{1}{3}]$ and $I''_{0,1}=[\frac{1}{3},\frac{3}{8}]$. For every $n\in\mathbb{Z}_{>0}$, we also let

$$
\begin{aligned}
I'_{n,1}&=\left[\frac{-1+4n+14n^2}{6n+14n^2},\frac{1+13n+21n^2}{3+16n+21n^2}\right],\\
I''_{n,1}&=\left[\frac{1+13n+21n^2}{3+16n+21n^2},\frac{3+35n+49n^2}{8+42n+49n^2}\right].
\end{aligned}
$$

For every $n\in\mathbb{Z}_{\geqslant 0}$, we let

$$
\begin{aligned}
I'_{n,2}&=\left[\frac{3+35n+49n^2}{8+42n+49n^2},\frac{3+22n+28n^2}{6+26n+28n^2}\right],\\
I''_{n,2}&=\left[\frac{3+22n+28n^2}{6+26n+28n^2},\frac{2+7n}{3+7n}\right],\\
I'_{n,3}&=\left[\frac{2+7n}{3+7n},\frac{21+50n+28n^2}{26+54n+28n^2}\right],\\
I''_{n,3}&=\left[\frac{21+50n+28n^2}{26+54n+28n^2},\frac{39+91n+49n^2}{48+98n+49n^2}\right],\\
I'_{n,4}&=\left[\frac{39+91n+49n^2}{48+98n+49n^2},\frac{19+41n+21n^2}{23+44n+21n^2}\right],\\
I''_{n,4}&=\left[\frac{19+41n+21n^2}{23+44n+21n^2},\frac{17+32n+14n^2}{20+34n+14n^2}\right].
\end{aligned}
$$

Set $I_{n,1}=I'_{n,1}\cup I''_{n,1}$, $I_{n,2}=I'_{n,2}\cup I''_{n,2}$, $I_{n,3}=I'_{n,3}\cup I''_{n,3}$, $I_{n,4}=I'_{n,4}\cup I''_{n,4}$. Then

$$
[0,1)=\bigcup_{n\in\mathbb{Z}_{\geqslant 0}}\Big(I_{n,1}\cup I_{n,2}\cup I_{n,3}\cup I_{n,4}\Big),
$$

the intervals $I_{n,1}$, $I_{n,2}$, $I_{n,3}$, $I_{n,4}$ have positive volumes, and all their interiors are disjoint. Let us analyze $P(u,v)$ and $N(u,v)$ when u is contained in one of these intervals.

First, we deal with $u\in I_{n,1}$. If $u\in I'_{n,1}$ and $v\in\Big[0,\frac{2+14n+28n^2-u(1+14n+28n^2)}{1+7n+7n^2}\Big]$, then

$$
\begin{aligned}
P(u,v)=&\frac{19+70n+84n^2-u\left(16+70n+84n^2\right)-v\left(8+28n+21n^2\right)}{8+28n+21n^2}\mathbf{e}_1+\\
&+\frac{3+35n+49n^2-u\left(8+42n+49n^2\right)}{8+28n+21n^2}B_{n,1}+\frac{1-4n-14n^2+u\left(6n+14n^2\right)}{8+28n+21n^2}B_{n,2}
\end{aligned}
$$

and $N(u,v)=0$. The same holds if $u\in I''_{n,1}$ and $v\in\Big[0,\frac{7+26n+28n^2-u(6+26n+28n^2)}{3+10n+7n^2}\Big]$. Then

$$
\begin{aligned}
\big(P(u,v)\big)^2&=10-12u+2u^2-2v-v^2,\\
P(u,v)\cdot\mathbf{e}_1&=1+v.
\end{aligned}
$$

Similarly, if $u \in I'_{n,1}$ and $v \in \left[\frac{2+14n+28n^2-u\left(1+14n+28n^2\right)}{1+7n+7n^2}, \frac{7+26n+28n^2-u\left(6+26n+28n^2\right)}{3+10n+7n^2}\right]$, then

$$P(u,v) = \frac{19+70n+84n^2-u(16+70n+84n^2)-v(8+28n+21n^2)}{8+28n+21n^2}\mathbf{e}_1 + \\ + \frac{\left(19+70n+84n^2-u(16+70n+84n^2)-v(8+28n+21n^2)\right)(1+7n+7n^2)}{8+28n+21n^2}B_{n,1} + \\ + \frac{1-4n-14n^2+u(6n+14n^2)}{8+28n+21n^2}B_{n,2}$$

and

$$N(u,v) = \left(u\left(1+14n+28n^2\right)+v\left(1+7n+7n^2\right)-2-14n-28n^2\right)B_{n,1}.$$

Then

$$\left(P(u,v)\right)^2 = 10-12u+2u^2-2v-v^2 + \\ + 2\left(u\left(1+14n+28n^2\right)+v\left(1+7n+7n^2\right)-2-14n-28n^2\right)^2$$

and

$$P(u,v)\cdot\mathbf{e}_1 = 5+56n+280n^2+588n^3+392n^4 - \\ - 2u(1+7n+7n^2)(1+14n+28n^2)-v(1+28n+126n^2+196n^3+98n^4).$$

Likewise, if $u \in I''_{n,1}$ and $v \in \left[\frac{7+26n+28n^2-u\left(6+26n+28n^2\right)}{3+10n+7n^2}, \frac{2+14n+28n^2-u\left(1+14n+28n^2\right)}{1+7n+7n^2}\right]$, then

$$P(u,v) = \frac{(19+70n+84n^2-u(16+70n+84n^2)-v(8+28n+21n^2)}{8+28n+21n^2}\mathbf{e}_1 + \\ + \frac{\left(19+70n+84n^2-u(16+70n+84n^2)-v(8+28n+21n^2)\right)(1+n)(3+7n)}{8+28n+21n^2}B_{n,2} + \\ + \frac{3+35n+49n^2-u\left(8+42n+49n^2\right)}{8+28n+21n^2}B_{n,1}$$

and

$$N(u,v) = \left(u\left(6+26n+28n^2\right)+v\left(3+10n+7n^2\right)-7-26n-28n^2\right)B_{n,2}.$$

Then

$$\left(P(u,v)\right)^2 = 10-12u+2u^2-2v-v^2 + \\ + 7\left(u\left(6+26n+28n^2\right)+v\left(3+10n+7n^2\right)-7-26-28n^2\right)^2$$

and

$$P(u,v)\cdot\mathbf{e}_1 = 148+1036n+2751n^2+3234n^3+1372n^4 - \\ - 14u(1+n)(1+2n)(3+7n)^2-v\left(62+420n+994n^2+980n^3+343n^4\right).$$

If $u \in I'_{n,1}$ and $v \in \left[\frac{7+26n+28n^2-u\left(6+26n+28n^2\right)}{3+10n+7n^2}, \frac{19+70n+84n^2-u\left(16+70n+84n^2\right)}{8+28n+21n^2}\right]$, then

$$P(u,v) = \frac{19+70n+84n^2-u\left(16+70n+84n^2\right)-v\left(8+28n+21n^2\right)}{8+28n+21n^2}\mathbf{e}_1+$$
$$+\frac{\left(19+70n+84n^2-u\left(16+70n+84n^2\right)-v\left(8+28n+21n^2\right)\right)\left(1+7n+7n^2\right)}{8+28n+21n^2}B_{n,1}+$$
$$+\frac{\left(19+70n+84n^2-u\left(16+70n+84n^2\right)-v\left(8+28n+21n^2\right)\right)(1+n)(3+7n)}{8+28n+21n^2}B_{n,2}$$

and

$$N(u,v) = \left(u\left(1+14n+28n^2\right)+v\left(1+7n+7n^2\right)-2-14n-28n^2\right)B_{n,1}+$$
$$+\left(u\left(6+26n+28n^2\right)+v\left(3+10n+7n^2\right)-7-26n-28n^2\right)B_{n,2}.$$

The same holds if $u \in I''_{n,1}$ and $v \in \left[\frac{2+14n+28n^2-u\left(1+14n+28n^2\right)}{1+7n+7n^2}, \frac{19+70n+84n^2-u\left(16+70n+84n^2\right)}{8+28n+21n^2}\right]$. In both cases, we have

$$\left(P(u,v)\right)^2 = \left(19+70n+84n^2-u\left(16+70n+84n^2\right)-v\left(8+28n+21n^2\right)\right)^2,$$
$$P(u,v)\cdot\mathbf{e}_1 = \left(8+28n+21n^2\right)\left(19+70n+84n^2-u\left(16+70n+84n^2\right)-v\left(8+28n+21n^2\right)\right).$$

Hence, if $u \in I_{n,1}$, then

$$t(u) = \frac{19+70n+84n^2-u\left(16+70n+84n^2\right)}{8+28n+21n^2}.$$

Now, we deal with $u \in I_{n,2}$. If $u \in I'_{n,2}$ and $v \in \left[0, \frac{7+26n+28n^2-u\left(6+26n+28n^2\right)}{3+10n+7n^2}\right]$, then

$$P(u,v) = \frac{17+56n+56n^2-u\left(15+56n+56n^2\right)-7v(1+n)(1+2n)}{7(1+n)(1+2n)}\mathbf{e}_1+$$
$$+\frac{(1+n)(2+7n)-u(1+n)(3+7n)}{7(1+n)(1+2n)}B_{n,2}+\frac{u\left(8+42n+49n^2\right)-3-35n-49n^2}{7(1+n)(1+2n)}B_{n,3}$$

and $N(u,v)=0$. The same holds if $u \in I''_{n,2}$ and $v \in \left[0, \frac{15+42n+28n^2-u\left(14+42n+28n^2\right)}{6+14n+7n^2}\right]$. Then

$$\left(P(u,v)\right)^2 = 10-12u+2u^2-2v-v^2,$$
$$P(u,v)\cdot\mathbf{e}_1 = v+1.$$

If $u \in I'_{n,2}$ and $v \in \left[\frac{7+26n+28n^2-u\left(6+26n+28n^2\right)}{3+10n+7n^2}, \frac{15+42n+28n^2-u\left(14+42n+28n^2\right)}{6+14n+7n^2}\right]$, then

$$P(u,v) = \frac{17+56n+56n^2-u(15+56n+56n^2)-7v(1+n)(1+2n)}{7(1+n)(1+2n)}\mathbf{e}_1+$$
$$+\frac{(1+n)(3+7n)\left(17+56n+56n^2-u(15+56n+56n^2)-7v(1+n)(1+2n)\right)}{7(1+n)(1+2n)}B_{n,2}+$$
$$+\frac{u\left(8+42n+49n^2\right)-3-35n-49n^2}{7(1+n)(1+2n)}B_{n,3},$$

$$N(u,v) = \big(u\left(6+26n+28n^2\right) + v\left(3+10n+7n^2\right) - 7 - 26n - 28n^2\big)B_{n,2},$$

$$\big(P(u,v)\big)^2 = 10 - 12u + 2u^2 - 2v - v^2 + \\ + 7\left(u\left(6+26n+28n^2\right) + v\left(3+10n+7n^2\right) - 7 - 26n - 28n^2\right)^2,$$

$$P(u,v)\cdot \mathbf{e}_1 = 148 + 1036n + 2751n^2 + 3234n^3 + 1372n^4 - \\ - 14u(1+n)(1+2n)(3+7n)^2 - \\ - v\left(62 + 420n + 994n^2 + 980n^3 + 343n^4\right).$$

Similarly, if $u \in I''_{n,2}$ and $v \in \left[\frac{15+42n+28n^2-u\left(14+42n+28n^2\right)}{6+14n+7n^2}, \frac{7+26n+28n^3-u\left(6+26n+28n^2\right)}{3+10n+7n^2}\right]$, then

$$P(u,v) = \frac{17+56+56n^2 - u\left(15+56n+56n^2\right) - 7v(1+n)(1+2n)}{7(1+n)(1+2n)}\mathbf{e}_1 + \\ + \frac{(6+14n+7n^2)\big(17+56+56n^2 - u\left(15+56n+56n^2\right) - 7v(1+n)(1+2n)\big)}{7(1+n)(1+2n)}B_{n,3} + \\ + \frac{(1+n)(2+7n) - u(1+n)(3+7n)}{7(1+n)(1+2n)}B_{n,2},$$

$$N(u,v) = \big(u\left(14+42n+28n^2\right) + v\left(6+14n+7n^2\right) - 15 - 42n - 28n^2\big)B_{n,3},$$

$$\big(P(u,v)\big)^2 = 10 - 12u + 2u^2 - 2v - v^2 + \\ + \left(u\left(14+42n+28n^2\right) + v\left(6+14n+7n^2\right) - 15 - 42n - 28n^2\right)^2,$$

$$P(u,v)\cdot \mathbf{e}_1 = 7(1+n)(13+53n+70n^2+28n^3) - \\ - 14u(1+n)(1+2n)\left(6+14n+7n^2\right) - 7v(1+n)^2\left(5+14n+7n^2\right).$$

Likewise, if $u \in I'_{n,2}$ and $v \in \left[\frac{15+42n+28n^2-u\left(14+42n+28n^2\right)}{6+14n+7n^2}, \frac{17+56n+56n^2-u\left(15+56n+56n^2\right)}{7(1+n)(1+2n)}\right]$, then

$$P(u,v) = \frac{17+56n+56n^2 - u\left(15+56n+56n^2\right) - v(7(1+n)(1+2n))}{7(1+n)(1+2n)}\mathbf{e}_1 + \\ + \frac{(1+n)(3+7n)\big(17+56n+56n^2 - u\left(15+56n+56n^2\right) - 7v(1+n)(1+2n)\big)}{7(1+n)(1+2n)}B_{n,2} + \\ + \frac{(6+14n+7n^2)\left(17+56n+56n^2 - u\left(15+56n+56n^2\right) - 7v(1+n)(1+2n)\right)}{7(1+n)(1+2n)}B_{n,3}$$

and

$$N(u,v) = \big(u(6+26n+28n^2) + v(3+10n+7n^2) - 7 - 26n - 28n^2\big)B_{n,2} + \\ + \big(u\left(14+42n+28n^2\right) + v\left(6+14n+7n^2\right) - 15 - 42n - 28n^2\big)B_{n,3}.$$

The same holds if $u \in I''_{n,2}$ and $v \in \left[\frac{7+26n+28n^2-u\left(6+26n+28n^2\right)}{3+10n+7n^2}, \frac{17+56n+56n^2-u\left(15+56n+56n^2\right)}{7(1+n)(1+2n)}\right]$. Moreover, in both cases, we have

$$P(u,v)\cdot\mathbf{e}_1 = 14(1+n)(1+2n)\left(17+56n+56n^2-u\left(15+56n+56n^2\right)-7v(1+n)(1+2n)\right)$$

and

$$\big(P(u,v)\big)^2 = 2\left(17+56n+56n^2-u\left(15+56n+56n^2\right)-7v(1+n)(1+2n)\right)^2.$$

Thus, if $u \in I_{n,2}$, then

$$t(u) = \frac{17+56n+56n^2-u\left(15+56n+56n^2\right)}{7(1+n)(1+2n)}.$$

Now, we deal with $u \in I_{n,3}$. If $u \in I'_{n,3}$ and $v \in \left[0, \frac{15+42n+28n^2-u\left(14+42n+28n^2\right)}{6+14n+7n^2}\right]$, then

$$P(u,v) = \frac{59+112n+56n^2-u\left(57+112n+56n^2\right)-7v(1+n)(3+2n)}{7(1+n)(3+2n)}\mathbf{e}_1 + \\ + \frac{39+91n+49n^2-u\left(48+98n+49n^2\right)}{7(1+n)(3+2n)}B_{n,3} + \frac{u(1+n)(3+7n)-(1+n)(2+7n)}{7(1+n)(3+2n)}B_{n,4}$$

and $N(u,v)=0$. The same holds if $u \in I''_{n,3}$ and $v \in \left[0, \frac{31+58n+28n^2-u\left(30+58n+28n^2\right)}{11+18n+7n^2}\right]$. Then

$$\begin{aligned} \big(P(u,v)\big)^2 &= 10-12u+2u^2-2v-v^2, \\ P(u,v)\cdot\mathbf{e}_1 &= 1+v. \end{aligned}$$

If $u \in I'_{n,3}$ and $v \in \left[\frac{15+42n+28n^2-u\left(14+42n+28n^2\right)}{6+14n+7n^2}, \frac{31+58n+28n^2-u\left(30+58n+28n^2\right)}{11+18n+7n^2}\right]$, then

$$P(u,v) = \frac{59+112n+56n^2-u\left(57+112n+56n^2\right)-7v(1+n)(3+2n)}{7(1+n)(3+2n)}\mathbf{e}_1 + \\ + \frac{\left(6+14n+7n^2\right)\left(59+112n+56n^2-u\left(57+112n+56n^2\right)-7v(1+n)(3+2n)\right)}{7(1+n)(3+2n)}B_{n,3} + \\ + \frac{u(1+n)(3+7n)-(1+n)(2+7n)}{7(1+n)(3+2n)}B_{n,4}$$

$$N(u,v) = \left(u\left(14+42n+28n^2\right)+v\left(6+14n+7n^2\right)-15-42n-28n^2\right)B_{n,3},$$

$$\big(P(u,v)\big)^2 = 10-12u+2u^2-2v-v^2 + \\ + \left(u\left(14+42n+28n^2\right)+v\left(6+14n+7n^2\right)-15-42n-28n^2\right)^2,$$

$$P(u,v)\cdot\mathbf{e}_1 = 7(1+n)\left(13+53n+70n^2+28n^3\right) - \\ - 14u(1+n)(1+2n)\left(6+14n+7n^2\right) - 7v(1+n)^2\left(5+14n+7n^2\right).$$

Similarly, if $u \in I''_{n,3}$ and $v \in \left[\frac{31+58n+28n^2-u\left(30+58n+28n^2\right)}{11+18n+7n^2}, \frac{15+42n+28n^2-u\left(14+42n+28n^2\right)}{6+14n+7n^2}\right]$, then

$$P(u,v) = \frac{59+112n+56n^2-u(57+112n+56n^2)-7v(1+n)(3+2n)}{7(1+n)(3+2n)}\mathbf{e}_1+$$
$$+\frac{(1+n)(11+7n)\big(59+112n+56n^2-u(57+112n+56n^2)-7v(1+n)(3+2n)\big)}{7(1+n)(3+2n)}B_{n,4}+$$
$$+\frac{39+91n+49n^2-u\left(48+98n+49n^2\right)}{7(1+n)(3+2n)}B_{n,3},$$

$$N(u,v) = \left(u\left(30+58n+28n^2\right)+v\left(11+18n+7n^2\right)-31-58n-28n^2\right)B_{n,4},$$

$$\big(P(u,v)\big)^2 = 10-12u+2u^2-2v-v^2+$$
$$+7\left(u\left(30+58n+28n^2\right)+v\left(11+18n+7n^2\right)-31-58n-28n^2\right)^2,$$

$$P(u,v)\cdot\mathbf{e}_1 = 2388+8372n+10983n^2+6370n^3+1372n^4-$$
$$-14u(1+n)^2(11+7n)(15+14n)-v\left(846+2772n+3346n^2+1764n^3+343n^4\right).$$

If $u \in I'_{n,3}$ and $v \in \left[\frac{31+58n+28n^2-u\left(30+58n+28n^2\right)}{11+18n+7n^2}, \frac{59+112n+56n^2-u\left(57+112n+56n^2\right)}{7(1+n)(3+2n)}\right]$, then

$$P(u,v) = \frac{59+112n+56n^2-u\left(57+112n+56n^2\right)-7v(1+n)(3+2n)}{7(1+n)(3+2n)}\mathbf{e}_1+$$
$$+\frac{\left(6+14n+7n^2\right)\left(59+112n+56n^2-u\left(57+112n+56n^2\right)-7v(1+n)(3+2n)\right)}{7(1+n)(3+2n)}B_{n,3}+$$
$$+\frac{(1+n)\left(11+7n\right)\left(59+112n+56n^2-u\left(57+112n+56n^2\right)-7v(1+n)(3+2n)\right)}{7(1+n)(3+2n)}B_{n,4}$$

and

$$N(u,v) = \left(u\left(14+42n+28n^2\right)+v\left(6+14n+7n^2\right)-15-42n-28n^2\right)B_{n,3}+$$
$$+\left(u\left(30+58n+28n^2\right)+v\left(11+18n+7n^2\right)-31-58n-28n^2\right)B_{n,4}.$$

The same holds if $u \in I''_{u,3}$ and $v \in \left[\frac{15+42n+28n^2-u\left(14+42n+28n^2\right)}{6+14n+7n^2}, \frac{59+112n+56n^2-u\left(57+112n+56n^2\right)}{7(1+n)(3+2n)}\right]$. In both cases, we have

$$P(u,v)\cdot\mathbf{e}_1 = 14(1+n)(3+2n)\left(59+112n+56n^2-u\left(57+112n+56n^2\right)-7v(1+n)(3+2n)\right).$$

and

$$\big(P(u,v)\big)^2 = 2\left(59+112n+56n^2-u\left(57+112n+56n^2\right)-7v(1+n)(3+2n)\right)^2.$$

Therefore, if $u \in I_{n,3}$, then

$$t(u) = \frac{59+112n+56n^2-u\left(57+112n+56n^2\right)}{7(1+n)(3+2n)}.$$

Finally, we deal with $u \in I_{n,4}$. If $u \in I'_{n,4}$ and $v \in \left[0, \frac{31+58n+28n^2-u\left(30+58n+28n^2\right)}{11+18n+7n^2}\right]$, then

$$P(u,v) = \frac{103+182n+84n^2-u\left(100+182n+84n^2\right)-v\left(36+56n+21n^2\right)}{36+56n+21n^2}\mathbf{e}_1+$$
$$+\frac{17+32n+14n^2-u\left(20+34n+14n^2\right)}{36+56n+21n^2}B_{n,4}+\frac{u\left(48+98n+49n^2\right)-39-91n-49n^2}{36+56n+21n^2}B_{n+1,1}$$

and $N(u,v)=0$. The same holds when $u \in I''_{n,4}$ and $v \in \left[0, \frac{44+70n+28n^2-u\left(43+70n+28n^2\right)}{15+21n+7n^2}\right]$. In both cases, we compute

$$\begin{aligned}\big(P(u,v)\big)^2 &= 10-12u+2u^2-2v-v^2,\\ P(u,v)\cdot\mathbf{e}_1 &= 1+v.\end{aligned}$$

If $u \in I'_{n,4}$ and $v \in \left[\frac{31+58n+28n^2-u\left(30+58n+28n^2\right)}{11+18n+7n^2}, \frac{44+70n+28n^2-u\left(43+70n+28n^2\right)}{15+21n+7n^2}\right]$, then

$$P(u,v) = \frac{103+182n+84n^2-u(100+182n+84n^2)-v(36+56n+21n^2)}{36+56n+21n^2}\mathbf{e}_1+$$
$$+\frac{(1+n)(11+7n)\big(103+182n+84n^2-u(100+182n+84n^2)-v(36+56n+21n^2)\big)}{36+56n+21n^2}B_{n,4}+$$
$$+\frac{u(48+98n+49n^2)-39-91n-49n^2}{36+56n+21n^2}B_{n+1,1},$$
$$N(u,v) = \left(u\left(30+58n+28n^2\right)+v\left(11+18n+7n^2\right)-31-58n-28n^2\right)B_{n,4},$$

$$\big(P(u,v)\big)^2 = 10-12u+2u^2-2v-v^2+$$
$$+7\left(u\left(30+58n+28n^2\right)+v\left(11+18n+7n^2\right)-31-58n-28n^2\right)^2,$$

$$P(u,v)\cdot\mathbf{e}_1 = 2388+8372n+10983n^2+6370n^2+1372n^4-$$
$$-14u(1+n)^2(11+7n)(15+14n)-$$
$$-v\left(846+2772n+3346n^2+1764n^3+343n^4\right).$$

If $u \in I''_{n,4}$ and $v \in \left[\frac{44+70n+28n^2-u\left(43+70n+28n^2\right)}{15+21n+7n^2}, \frac{31+58n+28n^2-u\left(30+58n+28n^2\right)}{11+18n+7n^2}\right]$, then

$$P(u,v) = \frac{103+182n+84n^2-u(100+182n+84n^2)-v(36+56n+21n^2)}{36+56n+21n^2}\mathbf{e}_1+$$
$$+\frac{(15+21n+7n^2)\big(103+182n+84n^2-u(100+182n+84n^2)-v(36+56n+21n^2)\big)}{36+56n+21n^2}B_{n+1,1}+$$
$$+\frac{17+32n+14n^2-u\left(20+34n+14n^2\right)}{36+56n+21n^2}B_{n,4}$$

and

$$N(u,v) = \left(u\left(43+70n+28n^2\right)+v\left(15+21n+7n^2\right)-44-70n-28n^2\right)B_{n+1,1}.$$

Moreover, in this case, we have

$$\big(P(u,v)\big)^2 = 10-12u+2u^2-2v-v^2+ \\ +2\left(u\left(43+70n+28n^2\right)+v\left(15+21n+7n^2\right)-44-70n-28n^2\right)^2$$

and

$$P(u,v)\cdot\mathbf{e}_1 = 1321+3948n+4396n^2+2156n^3+392n^4- \\ -2u\left(15+21n+7n^2\right)\left(43+70n+28n^2\right)- \\ -v\left(449+1260n+1302n^2+588n^3+98n^4\right).$$

If $u\in I'_{n,4}$ and $v\in\left[\frac{44+70n+28n^2-u\left(43+70n+28n^2\right)}{15+21n+7n^2},\frac{103+182n+84n^2-u\left(100+182n+84n^2\right)}{36+56n+21n^2}\right]$, then

$$P(u,v)=\frac{103+182n+84n^2-u(100+182n+84n^2)-v(36+56n+21n^2)}{36+56n+21n^2}\mathbf{e}_1+ \\ +\frac{(1+n)(11+7n)\big(103+182n+84n^2-u(100+182n+84n^2)-v(36+56n+21n^2)\big)}{36+56n+21n^2}B_{n,4}+ \\ +\frac{(15+21n+7n^2)\big(103+182n+84n^2-u(100+182n+84n^2)-v(36+56n+21n^2)\big)}{36+56n+21n^2}B_{n+1,1}$$

and

$$N(u,v)=\big(u(30+58n+28n^2)+v(11+18n+7n^2)-31-58n-28n^2\big)B_{n,4}+ \\ +\big(u(43+70n+28n^2)+v(15+21n+7n^2)-44-70n-28n^2\big)B_{n+1,1}.$$

The same holds when $u\in I''_{n4}$ and $v\in\left[\frac{31+58n+28n^2-u\left(30+58n+28n^2\right)}{11+18n+7n^2},\frac{103+182n+84n^2-u\left(100+182n+84n^2\right)}{36+56n+21n^2}\right]$. Moreover, in both cases, we have

$$P(u,v)\cdot\mathbf{e}_1=(36+56n+21n^2)\big(103+182n+84n^2-u(100+182n+84n^2)-v(36+56n+21n^2)\big)$$

and

$$\big(P(u,v)\big)^2=\big(103+182n+84n^2-u(100+182n+84n^2)-v(36+56n+21n^2)\big)^2$$

Thus, if $u\in I_{n,4}$, then

$$t(u)=\frac{103+182n+84n^2-u\left(100+182n+84n^2\right)}{36+56n+21n^2}.$$

Now, we are ready to compute $S(W^T_{\bullet,\bullet};\mathbf{e}_1)$. Namely, for every $i\in\{1,2,3,4\}$, we set

$$S_{n,i}=\frac{3}{14}\int_{I_{n,i}}\int_0^{t(u)}\big(P(u,v)\big)^2dvdu.$$

Then

$$S\big(W^T_{\bullet,\bullet};\mathbf{e}_1\big)=\sum_{n=0}^{\infty}\Big(S_{n,1}+S_{n,2}+S_{n,3}+S_{n,4}\Big).$$

On the other hand, integrating, we get

$$S_{n,1}=\begin{cases}\dfrac{84365}{114688} \text{ if } n=0,\\ \dfrac{(8+28n+21n^2)A_{n,1}}{448n^4(1+n)(2+7n)^4(3+7n)^4(4+7n)^4(1+7n+7n^2)} \text{ if } n\geqslant 1,\end{cases}$$

where

$$\begin{aligned}A_{n,1}=&\,1536+109312n+2935552n^2+42681728n^3+386407488n^4+2335296292n^5+\\&+9789648099n^6+29038364761n^7+61312905318n^8+91454579804n^9+\\&+94035837280n^{10}+63317750608n^{11}+25088413952n^{12}+4427367168n^{13}.\end{aligned}$$

Similarly, we get

$$S_{n,2}=\frac{(1+2n)A_{n,2}}{4(1+n)(2+7n)^4(3+7n)^4(4+7n)^4(6+14n+7n^2)},$$

where

$$\begin{aligned}A_{n,2}=&\,1618654+31459234n+271069253n^2+1362423916n^3+4419070194n^4+9654348284n^5+\\&+14368501182n^6+14362052096n^7+9209328422n^8+3412762192n^9+553420896n^{10}\end{aligned}$$

Likewise, we get

$$S_{n,3}=\frac{(3+2n)A_{n,3}}{4(1+n)(3+7n)^4(6+7n)^4(8+7n)^4(11+7n)(6+14n+7n^2)},$$

where

$$\begin{aligned}A_{n,3}=&\,1167997914+15454923336n+91492878645n^2+319934133575n^3+\\&+734395997090n^4+1162203105378n^5+1294197714054n^6+1014406754242n^7+\\&+548632346402n^8+195059453722n^9+41045383120n^{10}+3873946272n^{11}.\end{aligned}$$

Finally, we get

$$S_{n,4}=\frac{(36+56n+21n^2)A_{n,4}}{448(1+n)^4(6+7n)^4(8+7n)^4(10+7n)^4(11+7n)(15+21n+7n^2)},$$

where

$$\begin{aligned}A_{n,4}=&\,365613573312+4021500121920n+20341847967024n^2+\\&+62650071283024n^3+131072047236004n^4+196698030664492n^5+217823761840153n^6+\\&+180219167765455n^7+111395400841326n^8+50802960251820n^9+16615457209344n^{10}+\\&+3690223711216n^{11}+498816700928n^{12}+30991570176n^{13}.\end{aligned}$$

Then, adding, we get

$$S\big(W^T_{\bullet,\bullet};\mathbf{e}_1\big)=\sum_{n=0}^{\infty}\Big(S_{n,1}+S_{n,2}+S_{n,3}+S_{n,4}\Big)\approx 0.976712233\ldots<1.$$

Finally, let us compute $S(W^{T,\mathbf{e}_1}_{\bullet,\bullet,\bullet};P)$. For every $i\in\{1,2,3,4\}$, we set

$$M'_{n,i}=\frac{3}{14}\int_{I'_{n,i}}\int_0^{t(u)}\Big(\big(P(u,v)\cdot\mathbf{e}_1\big)\Big)^2dvdu,$$

$$M''_{n,i}=\frac{3}{14}\int_{I''_{n,i}}\int_0^{t(u)}\Big(\big(P(u,v)\cdot\mathbf{e}_1\big)\Big)^2dvdu.$$

Then

$$S\big(W^{T,\mathbf{e}_1}_{\bullet,\bullet,\bullet};P\big)=\sum_{n=0}^{\infty}\sum_{i=1}^{4}\Big(M'_{n,i}+M''_{n,i}\Big)+\frac{3}{7}\int_0^1\int_0^{t(u)}\big(P(u,v)\cdot\mathbf{e}_1\big)\mathrm{ord}_P\Big(N(u,v)\big|_{\mathbf{e}_1}\Big)dvdu.$$

On the other hand, integrating, we get

$$M'_{n,1}=\begin{cases}\dfrac{1403}{22268} \text{ if } n=0,\\ \dfrac{(1+n)A'_{n,1}}{448n^4(1+3n)^4(3+7n)^4(1+7n+7n^2)} \text{ if } n\geqslant 1,\end{cases}$$

where

$$A'_{n,1}=1+81n+2535n^2+37209n^3+301046n^4+1459736n^5+ \\ +4420190n^6+8425410n^7+9821448n^8+6392736n^9+1778112n^{10}.$$

Similarly, we get

$$M''_{n,1}=\frac{(1+7n+7n^2)A''_{n,1}}{28(1+n)(1+3n)^4(2+7n)^4(3+7n)^4(4+7n)^4},$$

where

$$A''_{n,1}=480574+12906866n+157271760n^2+1149521334n^3+5612285145n^4+ \\ +19278934535n^5+47770884833n^6+86016481159n^7+111679016743n^8+ \\ +101939513907n^9+62077730148n^{10}+22635902898n^{11}+3735591048n^{12}.$$

Likewise, we have

$$M'_{n,2}=\frac{(6+14n+7n^2)A'_{n,2}}{224(1+n)(1+2n)^3(2+7n)^4(3+7n)^4(4+7n)^4},$$

and

$$M''_{n,2}=\frac{11780+111142n+430951n^2+875637n^3+978656n^4+566832n^5+131712n^6}{224(1+2n)^3(3+7n)^4(6+14n+7n^2)}.$$

where

$$A'_{n,2}=1561176+35176776n+356105548n^2+2137950448n^3+ \\ +8458603286n^4+23158717414n^5+44778314889n^6+61151030584n^7+ \\ +57807289939n^8+36026947376n^9+13321631568n^{10}+2213683584n^{11}.$$

Similarly, we have

$$M'_{n,3}=\frac{(11+7n)A'_{n,3}}{224(1+n)^3(3+7n)^4(13+14n)^4(6+14n+7n^2)},$$

where

$$A'_{n,3} = 13726028 + 164541190n + 859036123n^2 + 2564002455n^3 + 4823323519n^4+ \\ + 5933644367n^5 + 4776917782n^6 + 2428774768n^7 + 708314208n^8 + 90354432n^9.$$

Likewise, we have

$$M''_{n,3} = \frac{(6 + 14n + 7n^2)A''_{n,3}}{224(1 + n)^3(6 + 7n)^4(8 + 7n)^4(11 + 7n)(13 + 14n)^4},$$

where

$$A''_{n,3} = 67760261208 + 703706084640n + 3313300067388n^2 + 9335574166156n^3+ \\ + 17489294547578n^4 + 22873117200584n^5 + 21308562209725n^6 + 14139587568253n^7+ \\ + 6548997703738n^8 + 2016283621072n^9 + 371345421216n^{10} + 30991570176n^{11}.$$

Similarly, we see that

$$M'_{n,4} = \frac{(15 + 21n + 7n^2)A'_{n,4}}{28(1 + n)^4(6 + 7n)^4(8 + 7n)^4(11 + 7n)(23 + 21n)^4},$$

where

$$A'_{n,4} = 88135013250 + 967134809574n + 4853884596732n^2+ \\ + 14732868828434n^3 + 30120687035243n^4 + 43697011451345n^5 + 46124583653603n^6+ \\ + 35692827118809n^7 + 20096052100397n^8 + 8028312817917n^9+ \\ + 2160120347280n^{10} + 351456857766n^{11} + 26149137336n^{12}.$$

Finally, we have

$$M''_{n,4} = \frac{(11 + 7n)A''_{n,4}}{448(1 + n)^4(10 + 7n)^4(23 + 21n)^4(15 + 21n + 7n^2)},$$

where

$$A''_{n,4} = 7582266167 + 59702225967n + 210973884925n^2 + 440580768679n^3+ \\ + 602090743422n^4 + 562572998512n^5 + 363945674554n^6 + 160955181870n^7+ \\ + 46566357768n^8 + 7957643904n^9 + 609892416n^{10}.$$

Now, adding terms together, we see that

$$S\big(W^{T,\mathbf{e}_1}_{\bullet,\bullet,\bullet}; P\big) \leqslant 0.974 + \frac{3}{7}\int_0^1 \int_0^{t(u)} \big(P(u,v)\cdot \mathbf{e}_1\big)\mathrm{ord}_P\Big(N(u,v)\big|_{\mathbf{e}_1}\Big)dvdu. \tag{6.8}$$

Now, for every $i \in \{1,2,3,4\}$ and any irreducible component ℓ of the curve $B_{n,i}$, we let

$$
\begin{aligned}
F_{n,i} = \frac{3}{7}\int_0^1\int_0^{t(u)} \big(P(u,v)\cdot\mathbf{e}_1\big)\mathrm{ord}_\ell\big(N(u,v)\big)\big(\ell\cdot\mathbf{e}_1\big)dvdu = \\
= \frac{3}{7}\int_{I_{n,i-1}}\int_0^{t(u)} \big(P(u,v)\cdot\mathbf{e}_1\big)\mathrm{ord}_\ell\big(N(u,v)\big)\big(\ell\cdot\mathbf{e}_1\big)dvdu + \\
+ \frac{3}{7}\int_{I_{n,i}}\int_0^{t(u)} \big(P(u,v)\cdot\mathbf{e}_1\big)\mathrm{ord}_\ell\big(N(u,v)\big)\big(\ell\cdot\mathbf{e}_1\big)dvdu + \\
+ \frac{3}{7}\int_{I_{n,i+1}}\int_0^{t(u)} \big(P(u,v)\cdot\mathbf{e}_1\big)\mathrm{ord}_\ell\big(N(u,v)\big)\big(\ell\cdot\mathbf{e}_1\big)dvdu,
\end{aligned}
$$

where $I_{0,0} = \varnothing$ and $I_{n,5} = I_{n+1,1}$. Since irreducible components of $B_{n,i}$ are disjoint, we get

$$
\frac{3}{7}\int_0^1\int_0^{t(u)} \big(P(u,v)\cdot\mathbf{e}_1\big)\mathrm{ord}_P\Big(N(u,v)\big|_{\mathbf{e}_1}\Big)dvdu \leqslant \sum_{n=0}^{\infty}\sum_{i=1}^{4} F_{n,i}.
$$

On the other hand, each $F_{n,i}$ is not difficult to compute. For instance, we have

$$
\begin{aligned}
F_{0,1} = \frac{3}{7}\int_0^{\frac{1}{3}}\int_{2-u}^{\frac{7-6u}{3}} (v+u-2)(5-2u-v)dvdu + \\
+ \frac{3}{7}\int_0^{\frac{1}{3}}\int_{\frac{7-6u}{3}}^{\frac{19-16u}{8}} 8(u+v-2)(19-16u-8v)dvdu + \\
+ \frac{3}{7}\int_{\frac{1}{3}}^{\frac{3}{8}}\int_{2-u}^{\frac{19-16u}{8}} 8(u+v-2)(19-16u-8v)dvdu = \frac{281}{32256}.
\end{aligned}
$$

Similarly, we see that

$$
F_{n,1} = \frac{3(1+7n+7n^2)^2}{2n^2(1+3n)(-1+7n)(1+7n)(2+7n)(3+7n)^2(4+7n)(2+21n)}
$$

for $n \geqslant 1$. Likewise, we get

$$
F_{n,2} = \begin{cases} \dfrac{5}{3584} \text{ if } n=0, \\ \dfrac{1+n}{112n(1+2n)(1+3n)(2+7n)(3+7n)^2(4+7n)} \text{ if } n \geqslant 1. \end{cases}
$$

Likewise, for every $n \geqslant 0$, we have

$$
F_{n,3} = \frac{15(6+14n+7n^2)^2}{4(1+n)^2(1+2n)(2+7n)(3+7n)^2(4+7n)(6+7n)(8+7n)(13+14n)}
$$

and

$$
F_{n,4} = \frac{(11+7n)^2}{112(1+n)^2(3+7n)(6+7n)(8+7n)(10+7n)(13+14n)(23+21n)}.
$$

Now, one can easily check that the total sum of all $F_{n,1}$, $F_{n,2}$, $F_{n,3}$, $F_{n,4}$ is at most 0.014. This and (6.8) give $S(W^{T,\mathbf{e}_1}_{\bullet,\bullet,\bullet};P) \leqslant 0.974 + 0.014 = 0.988$. Using (6.7), we get $\delta_P(X) > 1$.

Corollary 6.11. *All smooth Fano threefolds in the family №2.7 are K-stable.*

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