K-STABLE FANO 3-FOLDS IN THE FAMILIES №2.18 AND №3.4

IVAN CHELTSOV, KENTO FUJITA, TAKASHI KISHIMOTO, JIHUN PARK

Abstract. We prove that smooth Fano 3-folds in the families №2.18 and №3.4 are K-stable.

Contents

Throughout this paper, all varieties are assumed to be projective and defined over $\mathbb{C}$.

1. Introduction

Smooth Fano threefolds are classified into 105 families labeled as №1.1, №1.2, №1.3, ..., №10.1. For the description of these families, see [18]. It has been proved in [3, 14, 16] that the families

№2.23, №2.26, №2.28, №2.30, №2.31, №2.33, №2.35, №2.36, №3.14,
№3.16, №3.18, №3.21, №3.22, №3.23, №3.24, №3.26, №3.28, №3.29,
№3.30, №3.31, №4.5, №4.8, №4.9, №4.10, №4.11, №4.12, №5.2

do not have smooth K-polystable members, and general members of other families are K-polystable. For 56 families, K-polystable smooth Fano 3-folds are described in [2, 3, 4, 6, 8, 11, 17, 20, 22, 24, 7]. The remaining 21 deformation families are:

№1.9, №1.10, №2.5, №2.9, №2.10, №2.11, №2.12, №2.13, №2.14, №2.16,
№2.17, №2.18, №2.19, №2.20, №3.2, №3.4, №3.5, №3.6, №3.7, №3.8, №3.11.

The families №1.10, №2.20, №3.5, №3.8 contain both K-polystable and non-K-polystable members, and all smooth Fano threefolds in the families

№1.9, №2.5, №2.9, №2.10, №2.11, №2.12, №2.13, №2.14,
№2.16, №2.17, №2.18, №2.19, №3.2, №3.4, №3.6, №3.7, №3.11

are conjectured to be K-stable [3]. In this paper, we verify this conjecture for two families:

Main Theorem. *All smooth Fano 3-folds in the families №2.18 and №3.4 are K-stable.*

Hence, to find all smooth K-polystable Fano 3-folds, one have to deal with 19 families

№1.9, №1.10, №2.5, №2.9, №2.10, №2.11, №2.12, №2.13, №2.14,
№2.16, №2.17, №2.19, №2.20, №3.2, №3.5, №3.6, №3.7, №3.8, №3.11.

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To describe smooth Fano 3-folds in the families №2.18 and №3.4, let $V \to \mathbb{P}^1 \times \mathbb{P}^2$ be a double cover branched along a smooth surface of degree $(2, 2)$, let $V \to \mathbb{P}^2$ be the composition of this double cover and the projection $\mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^2$, and let $X \to V$ be the blow up of a smooth fiber of this composition morphism. Then we have the following commutative diagram:

$$
\begin{array}{ccc}
X & \longrightarrow & V \\
\downarrow & & \downarrow \\
\mathbb{P}^1 \times \mathbb{F}_1 & \longrightarrow & \mathbb{P}^1 \times \mathbb{P}^2
\end{array}
$$

where $\mathbb{F}_1$ is the first Hirzebruch surface, the morphism $X \to \mathbb{P}^1 \times \mathbb{F}_1$ is a double cover ramified over the proper transform on $\mathbb{P}^1 \times \mathbb{F}_1$ of the ramification surface of the double cover $V \to \mathbb{P}^1 \times \mathbb{P}^2$, and $\mathbb{P}^1 \times \mathbb{F}_1 \to \mathbb{P}^1 \times \mathbb{P}^2$ is a birational morphism induced by the blow up $\mathbb{F}_1 \to \mathbb{P}^2$. Then

- V is a smooth Fano 3-fold in the deformation family №2.18,
- X is a smooth Fano 3-fold in the deformation family №3.4.

Furthermore, all smooth Fano 3-folds in these deformation families can be obtained in this way.

Let us say few words about the proof of Main Theorem. To prove that V is K-stable, we recall from [12, 15, 21, 25] that

$$\text{the Fano 3-fold } V \text{ is K-stable} \iff \text{the log Fano pair } \Big(\mathbb{P}^1 \times \mathbb{P}^2, \tfrac{1}{2}R\Big) \text{ is K-stable,}$$

where R is the ramification surface of the double cover $V \to \mathbb{P}^1 \times \mathbb{P}^2$. In Section 2, we prove that the log Fano pair $(\mathbb{P}^1 \times \mathbb{P}^2, cR)$ is K-stable for every $c \in (0, 1) \cap \mathbb{Q}$ using Abban–Zhuang theory and the technique developed in [3, 16]. We refer the reader to [3, § 1.7] and [16, § 4] for details. Similarly, to prove that X is K-stable, we prove that the log Fano pair $(\mathbb{P}^1 \times \mathbb{F}_1, \frac{1}{2}R)$ is K-stable, where now R is the ramification surface of the double cover $X \to \mathbb{P}^1 \times \mathbb{F}_1$. The proof is much more involved in this case, because we have to resolve two deadlocks arising when R is quite special. To overcome these difficulties, we apply Abban–Zhuang theory to exceptional surfaces of toric weighted blow ups of the 3-fold $\mathbb{P}^1 \times \mathbb{F}_1$, and use toric geometry to compute Zariski decompositions. This is a new approach, which can resolve deadlocks in similar problems.

The structure of this paper is simple: we prove Main Theorem for the family №2.18 in Section 2, and we prove Main Theorem for the family №3.4 in Section 3. In Appendix A, we put all the tables necessary for the Zariski decompositions discussed in Section 3.

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2. Smooth Fano 3-folds in the family №2.18

Let $Y = \mathbb{P}^1 \times \mathbb{P}^2$, let R be a smooth surface in Y of degree $(2, 2)$, and let $V \to Y$ be the double cover branched over R. Then $\mathrm{Aut}(V)$ is finite [9], so V is K-stable if and only if V is K-polystable. On the other hand, it follows from [12, 15, 21, 25] that

$$V \text{ is K-polystable} \iff (Y, \tfrac{1}{2}R) \text{ is K-polystable.}$$

Let $\Delta_Y = cR$ for $c \in [0, 1) \cap \mathbb{Q}$. Then (Y, Δ_Y) is a log Fano pair for every $c \in [0, 1) \cap \mathbb{Q}$.

Theorem 2.1. *The log Fano pair (Y, Δ_Y) is K-stable for every $c \in (0, 1) \cap \mathbb{Q}$.*

Let us prove Theorem 2.1. Set $L=-K_Y-\Delta_Y$. Then L is a divisor of degree $(2-2c,3-2c)$, so

$$L^3=6(1-c)(3-2c)^2.$$

Fix $c\in(0,1)\in\mathbb{Q}$. Let P be a point in Y. Recall that

$$\delta_P(Y,\Delta_Y)=\inf_{\substack{\mathbf{E}/Y\\ P\in C_Y(\mathbf{E})}}\frac{A_{Y,\Delta_Y}(\mathbf{E})}{S_L(\mathbf{E})},$$

where the infimum is taken over all prime divisors $\mathbf{E}$ over Y whose centers on Y contain P, and

$$S_L(\mathbf{E})=\frac{1}{L^3}\int\limits_0^\infty \mathrm{vol}\big(L-u\mathbf{E}\big)du.$$

By [19, 14], to prove that (Y,Δ_Y) is K-stable, it is enough to show that $\delta_P(Y,\Delta_Y)>1$.

Lemma 2.2. *Suppose that $P\not\in R$. Then $\delta_P(Y,\Delta_Y)>1$.*

Proof. Let S be the surface in Y of degree $(1,0)$ that contains P, let $R_S=R|_S$, and let $\Delta_S=cR_S$. Take $u\in\mathbb{R}_{\geqslant 0}$. Then $L-uS$ is pseudoeffective $\iff$ $L-uS$ is nef $\iff$ $u\in[0,2-2c]$. This gives

$$S_L(S)=\frac{1}{L^3}\int\limits_0^{2-2c}\big(L-uS\big)^3du=\frac{1}{L^3}\int\limits_0^{2-2c}3(3-2c)^2(2-2c-u)du=1-c<1.$$

Note that $S\cong\mathbb{P}^2$. Let ℓ be a general line in S that passes through P, and let v be a non-negative real number. Then $(L-uS)|_S-v\ell$ is a divisor of degree $3-2c-v$. Thus, we have

$$(L-uS)|_S-v\ell \text{ is pseudoeffective} \iff (L-uS)|_S-v\ell \text{ is nef} \iff v\in[0,3-2c].$$

Now, following [1, 3, 16], we set

$$S_L\big(W^S_{\bullet,\bullet};\ell\big)=\frac{3}{L^3}\int\limits_0^{2-2c}\int\limits_0^{3-2c}\big((L-uS)\big|_S-v\ell\big)^2dvdu$$

and

$$S_L\big(W^{S,\ell}_{\bullet,\bullet,\bullet};P\big)=\frac{3}{L^3}\int\limits_0^{2-2c}\int\limits_0^{3-2c}\Big(\big((L-uS)\big|_S-v\ell\big)\cdot\ell\Big)^2dvdu.$$

Integrating, we get $S_L(W^S_{\bullet,\bullet};\ell)=S_L(W^{S,\ell}_{\bullet,\bullet,\bullet};P)=\frac{3-2c}{3}$. Thus, it follows from [1, 3, 16] that

$$\delta_P(Y,\Delta_Y)\geqslant\min\left\{\frac{1-\mathrm{ord}_P\big(\Delta_S\big|_\ell\big)}{S_L(W^{S,\ell}_{\bullet,\bullet,\bullet};P)},\frac{1}{S_L(W^S_{\bullet,\bullet};\ell)},\frac{1}{S_L(S)}\right\}=\frac{3}{3-2c}>1,$$

since $\mathrm{ord}_P(\Delta_S|_\ell)=0$, because $P\not\in R$ by assumption. □

Thus, to prove Theorem 2.1, we may assume that $P\in R$.

Lemma 2.3. *Let $\mathbf{f}$ be the fiber of the projection $Y\to\mathbb{P}^2$ such that $P\in\mathbf{f}$. Suppose that $\mathbf{f}\not\subset R$. Then $\delta_P(Y,\Delta_Y)>1$.*

Proof. Let S be a general surface in Y of degree $(0,1)$ that contains $\mathbf{f}$, let $R_S=R|_S$, let $\Delta_S=cR_S$. Then $S\cong\mathbb{P}^1\times\mathbb{P}^1$, and R_S is a smooth curve such that $R_S\sim 2\mathbf{s}+2\mathbf{f}$, where $\mathbf{s}$ is the smooth curve in the surface S such that $\mathbf{s}^2=0$, $\mathbf{s}\cdot\mathbf{f}=1$ and $P\in\mathbf{s}$. Note that $L|_S\sim_{\mathbb{R}}(2-2c)\mathbf{s}+(3-2c)\mathbf{f}$.

Take $u\in\mathbb{R}_{\geqslant 0}$. Then $L-uS$ is pseudoeffective $\iff$ $L-uS$ is nef $\iff$ $u\in[0,3-2c]$, so

$$S_L(S)=\frac{1}{L^3}\int\limits_0^{3-2c}\big(L-uS\big)^3du=\frac{1}{L^3}\int\limits_0^{3-2c}6(1-c)(3-2c-u)^2du=\frac{3-2c}{3}<1.$$

Note that $(L-uS)|_S\sim_{\mathbb{R}}(2-2c)\mathbf{s}+(3-2c-u)\mathbf{f}$.

Now, let $\alpha\colon\widetilde{S}\to S$ be the blow up of the point P, let $\mathbf{e}$ be the exceptional curve of the blow up α, let $\widetilde{\mathbf{s}}$, $\widetilde{\mathbf{f}}$ and $R_{\widetilde{S}}$ be the proper transforms on $\widetilde{S}$ of the curves $\mathbf{s}$, $\mathbf{f}$ and R_S, respectively. Set $\Delta_{\widetilde{S}}=cR_{\widetilde{S}}$. Then $\widetilde{S}$ is the smooth del Pezzo surface of degree 7, $\widetilde{\mathbf{s}}\cap\widetilde{\mathbf{f}}=\varnothing$, and $\widetilde{\mathbf{s}}$, $\widetilde{\mathbf{f}}$, $\mathbf{e}$ are (-1)-curves in $\widetilde{S}$. Let v be a non-negative real number. Then

$$\alpha^*\big((L-uS)\big|_S\big)-v\mathbf{e}\sim_{\mathbb{R}}(2-2c)\widetilde{\mathbf{s}}+(3-2c-u)\widetilde{\mathbf{f}}+(5-4c-u-v)\mathbf{e},$$

and it is pseudoeffective $\iff$ $v\leqslant 5-4c-u$. For $v\in[0,5-4c-u]$, we let $\widetilde{P}(u,v)$ be the positive part of the Zariski decomposition of $\alpha^*((L-uS)\big|_S)-v\mathbf{e}$, and we let $\widetilde{N}(u,v)$ be its negative part. As in the proof of Lemma 2.2, we set

$$S_L\big(W^S_{\bullet,\bullet};\mathbf{e}\big)=\frac{3}{L^3}\int\limits_0^{3-2c}\int\limits_0^{5-4c-u}\big(\widetilde{P}(u,v)\big)^2dvdu.$$

Similarly, for every point $O\in\mathbf{e}$, we set

$$S\big(W^{\widetilde{S},\mathbf{e}}_{\bullet,\bullet,\bullet};O\big)=\frac{3}{L^3}\int\limits_0^{3-2c}\int\limits_0^{5-4c-u}\big(\widetilde{P}(u,v)\cdot\mathbf{e}\big)^2dvdu+F_O\big(W^{\widetilde{S},\mathbf{e}}_{\bullet,\bullet,\bullet}\big),$$

where

$$F_O\big(W^{\widetilde{S},\mathbf{e}}_{\bullet,\bullet,\bullet}\big)=\frac{6}{L^3}\int\limits_0^{3-2c}\int\limits_0^{5-4c-u}\big(\widetilde{P}(u,v)\cdot\mathbf{e}\big)\cdot\mathrm{ord}_O\big(\widetilde{N}(u,v)\big|_{\mathbf{e}}\big)dvdu.$$

Then it follows from [1, 3, 16] that

$$\delta_P(Y,\Delta_Y)\geqslant\min\left\{\min_{O\in\mathbf{e}}\frac{1-\mathrm{ord}_O\big(\Delta_{\widetilde{S}}\big|_{\mathbf{e}}\big)}{S_L\big(W^{\widetilde{S},\mathbf{e}}_{\bullet,\bullet,\bullet};O\big)},\frac{A_{S,\Delta_S}(\mathbf{e})}{S_L\big(W^S_{\bullet,\bullet};\mathbf{e}\big)},\frac{1}{S_L(S)}\right\},\tag{2.1}$$

where $A_{S,\Delta_S}(\mathbf{e})=2-c$. On the other hand, if $0\leqslant u\leqslant 1$, then

$$\widetilde{P}(u,v)\sim_{\mathbb{R}}\begin{cases}(2-2c)\widetilde{\mathbf{s}}+(3-2c-u)\widetilde{\mathbf{f}}+(5-4c-u-v)\mathbf{e} & \text{if } 0\leqslant v\leqslant 2-2c,\\ (2-2c)\widetilde{\mathbf{s}}+(5-4c-u-v)\big(\widetilde{\mathbf{f}}+\mathbf{e}\big) & \text{if } 2-2c\leqslant v\leqslant 3-2c-u,\\ (5-4c-u-v)\big(\widetilde{\mathbf{s}}+\widetilde{\mathbf{f}}+\mathbf{e}\big) & \text{if } 3-2c-u\leqslant v\leqslant 5-4c-u,\end{cases}$$

and

$$\widetilde{N}(u,v)=\begin{cases}0 & \text{if } 0\leqslant v\leqslant 2-2c,\\ (v+2c-2)\widetilde{\mathbf{f}} & \text{if } 2-2c\leqslant v\leqslant 3-2c-u,\\ (v+2c-2)\widetilde{\mathbf{f}}+(v+u-3+2c)\widetilde{\mathbf{s}} & \text{if } 3-2c-u\leqslant v\leqslant 5-4c-u,\end{cases}$$

which gives

$$\big(\widetilde{P}(u,v)\big)^2=\begin{cases}8c^2+4cu-v^2-20c-4u+12 & \text{if } 0\leqslant v\leqslant 2-2c,\\ 4(1-c)(4-3c-u-v) & \text{if } 2-2c\leqslant v\leqslant 3-2c-u,\\ (5-4c-u-v)^2 & \text{if } 3-2c-u\leqslant v\leqslant 5-4c-u,\end{cases}$$

and

$$\widetilde{P}(u,v)\cdot\mathbf{e}=\begin{cases}v \ \text{ if } 0\leqslant v\leqslant 2-2c,\\ 2-2c \ \text{ if } 2-2c\leqslant v\leqslant 3-2c-u,\\ 5-4c-u-v \ \text{ if } 3-2c-u\leqslant v\leqslant 5-4c-u.\end{cases}$$

Similarly, if $1\leqslant u\leqslant 3-2c$, then

$$\widetilde{P}(u,v)\sim_{\mathbb{R}}\begin{cases}(2-2c)\widetilde{\mathbf{s}}+(3-2c-u)\widetilde{\mathbf{f}}+(5-4c-u-v)\mathbf{e} \ \text{ if } 0\leqslant v\leqslant 3-2c-u,\\ (5-4c-u-v)\big(\widetilde{\mathbf{s}}+\mathbf{e}\big)+(3-2c-u)\widetilde{\mathbf{f}} \ \text{ if } 3-2c-u\leqslant v\leqslant 2-2c,\\ (5-4c-u-v)\big(\widetilde{\mathbf{s}}+\widetilde{\mathbf{f}}+\mathbf{e}\big) \ \text{ if } 2-2c\leqslant v\leqslant 5-4c-u,\end{cases}$$

and

$$\widetilde{N}(u,v)=\begin{cases}0 \ \text{ if } 0\leqslant v\leqslant 3-2c-u,\\ (v+u-3+2c)\widetilde{\mathbf{s}} \ \text{ if } 3-2c-u\leqslant v\leqslant 2-2c,\\ (v+2c-2)\widetilde{\mathbf{f}}+(v+u-3+2c)\widetilde{\mathbf{s}} \ \text{ if } 2-2c\leqslant v\leqslant 5-4c-u,\end{cases}$$

which gives

$$\big(\widetilde{P}(u,v)\big)^2=\begin{cases}8c^2+4cu-v^2-20c-4u+12 \ \text{ if } 0\leqslant v\leqslant 3-2c-u,\\ (3-2c-u)(7-6c-u-2v) \ \text{ if } 3-2c-u\leqslant v\leqslant 2-2c,\\ (5-4c-u-v)^2 \ \text{ if } 2-2c\leqslant v\leqslant 5-4c-u,\end{cases}$$

and

$$\widetilde{P}(u,v)\cdot\mathbf{e}=\begin{cases}v \ \text{ if } 0\leqslant v\leqslant 3-2c-u,\\ 3-2c-u \ \text{ if } 3-2c-u\leqslant v\leqslant 2-2c,\\ 5-4c-u-v \ \text{ if } 2-2c\leqslant v\leqslant 5-4c-u.\end{cases}$$

Thus, integrating, we get $S_L(W^S_{\bullet,\bullet};\mathbf{e})=\frac{6-5c}{3}$ and

$$S_L\big(W^{\widetilde{S},\mathbf{e}}_{\bullet,\bullet,\bullet};O\big)=\begin{cases}1-c-\dfrac{2(5-3c)(1-c)^2}{3(3-2c)^2} \ \text{ if } O\not\in\widetilde{\mathbf{f}}\cup\widetilde{\mathbf{s}},\\ 1-c \ \text{ if } O\in\widetilde{\mathbf{s}},\\ \dfrac{3-2c}{3} \ \text{ if } O\in\widetilde{\mathbf{f}}.\end{cases}$$

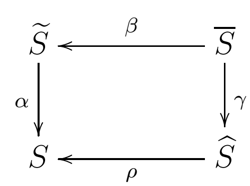
Therefore, if $\widetilde{\mathbf{s}}\cap R_{\widetilde{S}}\cap\mathbf{e}=\varnothing$ and $\widetilde{\mathbf{f}}\cap R_{\widetilde{S}}\cap\mathbf{e}=\varnothing$, then (2.1) gives $\delta_P(Y,\Delta_Y)>1$.

Thus, to complete the proof, we may assume that either $\widetilde{\mathbf{s}}\cap R_{\widetilde{S}}\cap\mathbf{e}\neq\varnothing$ or $\widetilde{\mathbf{f}}\cap R_{\widetilde{S}}\cap\mathbf{e}\neq\varnothing$. Then exactly one of the following two (mutually excluding) cases holds:

($\heartsuit$) the curve $\widetilde{\mathbf{s}}$ contains the point $R_{\widetilde{S}}\cap\mathbf{e}$, i.e. the curves $\mathbf{s}$ and R_S are tangent at P,

($\diamondsuit$) the curve $\widetilde{\mathbf{f}}$ contains the point $R_{\widetilde{S}}\cap\mathbf{e}$, i.e. the curves $\mathbf{f}$ and R_S are tangent at P.

In both cases, we consider the following commutative diagram:



where β is the blow up of the intersection point $R_{\widetilde{S}}\cap\mathbf{e}$, the map γ is the contraction of the proper transform of the curve $\mathbf{e}$ to an ordinary double point of the surface $\widehat{S}$, and ρ is the contraction of the proper transform of the β-exceptional curve. Then $\widehat{S}$ is a singular del Pezzo surface of degree 6, and ρ is a weighted blow up of the point P with weights $(1,2)$.

Let $\widehat{\mathbf{f}}$, $\widehat{\mathbf{s}}$ and $R_{\widehat{S}}$ be the proper transforms on the surface $\widehat{S}$ of the curves $\mathbf{f}$, $\mathbf{s}$ and R_S, respectively, and let $\mathbf{z}$ be the ρ-exceptional curve. In the case ($\heartsuit$), we have

$$\rho^*\big((L-uS)|_S\big) - v\mathbf{z} \sim_{\mathbb{R}} (2-2c)\widehat{\mathbf{s}} + (3-2c-u)\widehat{\mathbf{f}} + (7-6c-u-v)\mathbf{z},$$

and the intersections of the curves $\mathbf{z}$, $\widehat{\mathbf{f}}$ and $\widehat{\mathbf{s}}$ are given in the following table:

	$\mathbf{z}$	$\widehat{\mathbf{f}}$	$\widehat{\mathbf{s}}$
$\mathbf{z}$	$-\frac{1}{2}$	$\frac{1}{2}$	1
$\widehat{\mathbf{f}}$	$\frac{1}{2}$	$-\frac{1}{2}$	0
$\widehat{\mathbf{s}}$	1	0	-2

Similarly, in the case ($\diamondsuit$), we have

$$\rho^*\big((L-uS)|_S\big) - v\mathbf{z} \sim_{\mathbb{R}} (2-2c)\widehat{\mathbf{s}} + (3-2c-u)\widehat{\mathbf{f}} + (8-6c-2u-v)\mathbf{z},$$

and the intersections of the curves $\mathbf{z}$, $\widehat{\mathbf{f}}$ and $\widehat{\mathbf{s}}$ are given in the following table:

	$\mathbf{z}$	$\widehat{\mathbf{f}}$	$\widehat{\mathbf{s}}$
$\mathbf{z}$	$-\frac{1}{2}$	1	$\frac{1}{2}$
$\widehat{\mathbf{f}}$	1	-2	0
$\widehat{\mathbf{s}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$

In both cases, let $\widehat{t}(u)$ be the largest $v \in \mathbb{R}_{\geqslant 0}$ such that $\rho^*\big(P(u)\big|_S\big) - v\mathbf{z}$ is pseudoeffective. Then

$$\widehat{t}(u) = \begin{cases} 7-6c-u & \text{in the case } (\heartsuit), \\ 8-6c-2u & \text{in the case } (\diamondsuit). \end{cases}$$

For each $v \in [0, \widehat{t}(u)]$, let $\widehat{P}(u,v)$ be the positive part of the Zariski decomposition of this divisor, and let $\widehat{N}(u,v)$ be its negative part. Set

$$S_L\big(W^S_{\bullet,\bullet};\mathbf{z}\big) = \frac{3}{L^3}\int\limits_0^{3-2c}\int\limits_0^{\widehat{t}(u)} \big(\widehat{P}(u,v)\big)^2 dvdu.$$

Similarly, for every point $O \in \mathbf{z}$, we set

$$S\big(W^{\widehat{S},\mathbf{z}}_{\bullet,\bullet,\bullet};O\big) = \frac{3}{L^3}\int\limits_0^{3-2c}\int\limits_0^{\widehat{t}(u)} \big(\widehat{P}(u,v)\cdot\mathbf{z}\big)^2 dvdu + F_O\big(W^{\widehat{S},\mathbf{z}}_{\bullet,\bullet,\bullet}\big),$$

where

$$F_O\big(W^{\widehat{S},\mathbf{z}}_{\bullet,\bullet,\bullet}\big) = \frac{6}{L^3}\int\limits_0^{3-2c}\int\limits_0^{\widehat{t}(u)} \big(\widehat{P}(u,v)\cdot\mathbf{z}\big)\cdot \mathrm{ord}_O\big(\widehat{N}(u,v)\big|_{\mathbf{z}}\big) dvdu.$$

Let Q be the singular point of the surface $\widehat{S}$. Then $Q \notin R_{\widehat{S}}$, since

$$Q = \begin{cases} \widehat{\mathbf{f}}\cap\mathbf{z} & \text{in the case } (\heartsuit), \\ \widehat{\mathbf{s}}\cap\mathbf{z} & \text{in the case } (\diamondsuit). \end{cases}$$

Let $\Delta_{\widehat{S}}=cR_{\widehat{S}}$ and $\Delta_{\mathbf{z}}=\frac{1}{2}Q+\Delta_{\widehat{S}}|_{\mathbf{z}}$. Then it follows from [1, 3, 16] that

$$\delta_P(Y,\Delta_Y)\geqslant\min\left\{\min_{O\in\mathbf{z}}\frac{A_{\mathbf{z},\Delta_{\mathbf{z}}}(O)}{S_L(W^{\widehat{S},\mathbf{z}}_{\bullet,\bullet,\bullet};O)},\frac{A_{S,\Delta_S}(\mathbf{z})}{S_L(W^S_{\bullet,\bullet};\mathbf{z})},\frac{A_{Y,\Delta_Y}(S)}{S_L(S)}\right\},\tag{2.2}$$

where $A_{Y,\Delta_Y}(S)=1$, $A_{S,\Delta_S}(\mathbf{z})=3-2c$ and $A_{\mathbf{z},\Delta_{\mathbf{z}}}(O)=1-\mathrm{ord}_O(\Delta_{\mathbf{z}})$ for every point $O\in\mathbf{z}$.

Let us compute $S_L(W^S_{\bullet,\bullet};\mathbf{z})$ and $S_L(W^{\widehat{S},\mathbf{z}}_{\bullet,\bullet,\bullet};O)$ for every point $O\in\mathbf{z}$.

First, we deal with the case ($\heartsuit$). In this case, if $c\leqslant\frac{1}{2}$ or if $c>\frac{1}{2}$ and $2c-1\leqslant u\leqslant 3-2c$, then

$$\widehat{P}(u,v)\sim_{\mathbb{R}}\begin{cases}(2-2c)\widehat{\mathbf{s}}+(3-2c-u)\widehat{\mathbf{f}}+(7-6c-u-v)\mathbf{z} & \text{if } 0\leqslant v\leqslant 3-2c-u,\\ \dfrac{7-6c-u-v}{2}\big(\widehat{\mathbf{s}}+2\mathbf{z}\big)+(3-2c-u)\widehat{\mathbf{f}} & \text{if } 3-2c-u\leqslant v\leqslant 4-4c,\\ \dfrac{7-6c-u-v}{2}\big(\widehat{\mathbf{s}}+2\mathbf{z}+2\widehat{\mathbf{f}}\big) & \text{if } 4-4c\leqslant v\leqslant 7-6c-u,\end{cases}$$

and

$$\widehat{N}(u,v)=\begin{cases}0 & \text{if } 0\leqslant v\leqslant 3-2c-u,\\ \dfrac{v+u+2c-3}{2}\widehat{\mathbf{s}} & \text{if } 3-2c-u\leqslant v\leqslant 4-4c,\\ \dfrac{v+u+2c-3}{2}\widehat{\mathbf{s}}+(v-4+4c)\widehat{\mathbf{f}} & \text{if } 4-4c\leqslant v\leqslant 7-6c-u,\end{cases}$$

which gives

$$\big(\widehat{P}(u,v)\big)^2=\begin{cases}8c^2+4cu+12-20c-4u-\dfrac{v^2}{2} & \text{if } 0\leqslant v\leqslant 3-2c-u,\\ \dfrac{(3-2c-u)(11-10c-u-2v)}{2} & \text{if } 3-2c-u\leqslant v\leqslant 4-4c,\\ \dfrac{(7-6c-u-v)^2}{2} & \text{if } 4-4c\leqslant v\leqslant 7-6c-u,\end{cases}$$

and

$$\widehat{P}(u,v)\cdot\mathbf{z}=\begin{cases}\dfrac{v}{2} & \text{if } 0\leqslant v\leqslant 3-2c-u,\\ \dfrac{3-u-2c}{2} & \text{if } 3-2c-u\leqslant v\leqslant 4-4c,\\ \dfrac{7-6c-u-v}{2} & \text{if } 4-4c\leqslant v\leqslant 7-6c-u.\end{cases}$$

Similarly, in the case ($\heartsuit$), if $c>\frac{1}{2}$ and $0\leqslant u\leqslant 2c-1$, then

$$\widehat{P}(u,v)\sim_{\mathbb{R}}\begin{cases}(2-2c)\widehat{\mathbf{s}}+(3-2c-u)\widehat{\mathbf{f}}+(7-6c-u-v)\mathbf{z} & \text{if } 0\leqslant v\leqslant 4-4c,\\ (2-2c)\widehat{\mathbf{s}}+(7-6c-u-v)\big(\widehat{\mathbf{f}}+\widehat{\mathbf{z}}\big) & \text{if } 4-4c\leqslant v\leqslant 3-2c-u,\\ \dfrac{7-6c-u-v}{2}\big(\widehat{\mathbf{s}}+2\mathbf{z}+2\widehat{\mathbf{f}}\big) & \text{if } 3-2c-u\leqslant v\leqslant 7-6c-u,\end{cases}$$

and

$$\widehat{N}(u,v)=\begin{cases}0 & \text{if } 0\leqslant v\leqslant 4-4c,\\ (v-4+4c)\widehat{\mathbf{f}} & \text{if } 4-4c\leqslant v\leqslant 3-2c-u,\\ \dfrac{v+u+2c-3}{2}\widehat{\mathbf{s}}+(v-4+4c)\widehat{\mathbf{f}} & \text{if } 3-2c-u\leqslant v\leqslant 7-6c-u,\end{cases}$$

which gives

$$\big(\widehat{P}(u,v)\big)^2=\begin{cases}8c^2+4cu+12-20c-4u-\dfrac{v^2}{2} & \text{if } 0\leqslant v\leqslant 4-4c,\\ 4(1-c)(5-4c-u-v) & \text{if } 4-4c\leqslant v\leqslant 3-2c-u,\\ \dfrac{(7-6c-u-v)^2}{2} & \text{if } 3-2c-u\leqslant v\leqslant 7-6c-u,\end{cases}$$

and

$$\widehat{P}(u,v)\cdot\mathbf{z}=\begin{cases}\dfrac{v}{2} & \text{if } 0\leqslant v\leqslant 4-4c,\\ 2-2c & \text{if } 4-4c\leqslant v\leqslant 3-2c-u,\\ \dfrac{7-6c-u-v}{2} & \text{if } 3-2c-u\leqslant v\leqslant 7-6c-u.\end{cases}$$

Now, integrating, we get $S_L(W^S_{\bullet,\bullet};\mathbf{z})=3-\frac{8}{3}c<3-2c=A_{S,\Delta_S}(\mathbf{z})$ and

$$S_L\big(W^{\widehat{S},\mathbf{z}}_{\bullet,\bullet,\bullet};O\big)=\begin{cases}1-c-\dfrac{68c^2-124c+57}{96(1-c)} & \text{if } O\notin\widehat{\mathbf{f}}\cup\widehat{\mathbf{s}} \text{ and } c\leqslant\dfrac{1}{2},\\ 1-c-\dfrac{8(2-c)(1-c)^2}{3(3-2c)^2} & \text{if } O\notin\widehat{\mathbf{f}}\cup\widehat{\mathbf{s}} \text{ and } c>\dfrac{1}{2},\\ 1-c & \text{if } O\in\widehat{\mathbf{s}},\\ \dfrac{1}{2}-\dfrac{c}{3} & \text{if } O\in\widehat{\mathbf{f}}.\end{cases}$$

Hence, using (2.2), we obtain $\delta_P(Y,\Delta_Y)>1$.

Now, we deal with the case ($\diamondsuit$). If $0\leqslant u\leqslant 2-c$, then

$$\widehat{P}(u,v)\sim_{\mathbb{R}}\begin{cases}(2-2c)\widehat{\mathbf{s}}+(3-2c-u)\widehat{\mathbf{f}}+(8-6c-2u-v)\mathbf{z} & \text{if } 0\leqslant v\leqslant 2-2c,\\ (2-2c)\widehat{\mathbf{s}}+\dfrac{8-6c-2u-v}{2}\big(\widehat{\mathbf{f}}+2\mathbf{z}\big) & \text{if } 2-2c\leqslant v\leqslant 6-4c-2u,\\ \dfrac{8-6c-2u-v}{2}\big(2\widehat{\mathbf{s}}+\widehat{\mathbf{f}}+2\mathbf{z}\big) & \text{if } 6-4c-2u\leqslant v\leqslant 8-6c-2u,\end{cases}$$

and

$$\widehat{N}(u,v)=\begin{cases}0 & \text{if } 0\leqslant v\leqslant 2-2c,\\ \dfrac{v-2+2c}{2}\widehat{\mathbf{f}} & \text{if } 2-2c\leqslant v\leqslant 6-4c-2u,\\ \dfrac{v-2+2c}{2}\widehat{\mathbf{f}}+(v+2u-6+4c)\widehat{\mathbf{s}} & \text{if } 6-4c-2u\leqslant v\leqslant 8-6c-2u,\end{cases}$$

which gives

$$\big(\widehat{P}(u,v)\big)^2=\begin{cases}8c^2+4cu+12-20c-4u-\dfrac{v^2}{2} & \text{if } 0\leqslant v\leqslant 2-2c,\\ 2(1-c)(7-5c-2u-v) & \text{if } 2-2c\leqslant v\leqslant 6-4c-2u,\\ \dfrac{(8-6c-2u-v)^2}{2} & \text{if } 6-4c-2u\leqslant v\leqslant 8-6c-2u,\end{cases}$$

and

$$\widehat{P}(u,v)\cdot\mathbf{z}=\begin{cases}\dfrac{v}{2} & \text{if } 0\leqslant v\leqslant 2-2c,\\ 1-c & \text{if } 2-2c\leqslant v\leqslant 6-4c-2u,\\ \dfrac{8-6c-2u-v}{2} & \text{if } 6-4c-2u\leqslant v\leqslant 8-6c-2u.\end{cases}$$

Similarly, if $2-c\leqslant u\leqslant 3-2c$, then

$$\widehat{P}(u,v)\sim_{\mathbb{R}}\begin{cases}(2-2c)\widehat{\mathbf{s}}+(3-2c-u)\widehat{\mathbf{f}}+(8-6c-2u-v)\mathbf{z} & \text{if } 0\leqslant v\leqslant 6-4c-2u,\\ (8-6c-2u-v)\big(\widehat{\mathbf{s}}+\mathbf{z}\big)+(3-2c-u)\widehat{\mathbf{f}} & \text{if } 6-4c-2u\leqslant v\leqslant 2-2c,\\ \dfrac{8-6c-2u-v}{2}\big(2\widehat{\mathbf{s}}+\widehat{\mathbf{f}}+2\mathbf{z}\big) & \text{if } 2-2c\leqslant v\leqslant 8-6c-2u,\end{cases}$$

and

$$\widehat{N}(u,v)=\begin{cases}0 & \text{if } 0\leqslant v\leqslant 6-4c-2u,\\ (v+2u-6+4c)\widehat{\mathbf{s}} & \text{if } 6-4c-2u\leqslant v\leqslant 2-2c,\\ \dfrac{v-2+2c}{2}\widehat{\mathbf{f}}+(v+2u-6+4c)\widehat{\mathbf{s}} & \text{if } 2-2c\leqslant v\leqslant 8-6c-2u,\end{cases}$$

which gives

$$\big(\widehat{P}(u,v)\big)^2=\begin{cases}8c^2+4cu+12-20c-4u-\dfrac{v^2}{2} & \text{if } 0\leqslant v\leqslant 6-4c-2u,\\ 2(3-2c-u)(5-4c-u-v) & \text{if } 6-4c-2u\leqslant v\leqslant 2-2c,\\ \dfrac{(8-6c-2u-v)^2}{2} & \text{if } 2-2c\leqslant v\leqslant 8-6c-2u,\end{cases}$$

and

$$\widehat{P}(u,v)\cdot\mathbf{z}=\begin{cases}\dfrac{v}{2} & \text{if } 0\leqslant v\leqslant 6-4c-2u,\\ 3-2c-u & \text{if } 6-4c-2u\leqslant v\leqslant 2-2c,\\ \dfrac{8-6c-2u-v}{2} & \text{if } 2-2c\leqslant v\leqslant 8-6c-2u.\end{cases}$$

Now, integrating, we get $S_L(W^S_{\bullet,\bullet};\mathbf{z})=3-\frac{7}{3}c<3-2c=A_{S,\Delta_S}(\mathbf{z})$ and

$$S_L\big(W^{\widehat{S},\mathbf{z}}_{\bullet,\bullet,\bullet};O\big)=\begin{cases}1-c-\dfrac{(1-c)(31c^2-90c+65)}{12(3-2c)^2} & \text{if } O\notin\widehat{\mathbf{f}}\cup\widehat{\mathbf{s}},\\ \dfrac{1}{2}-\dfrac{c}{2} & \text{if } O\in\widehat{\mathbf{s}},\\ 1-\dfrac{2c}{3} & \text{if } O\in\widehat{\mathbf{f}}.\end{cases}$$

Hence, using (2.2), we get $\delta_P(Y,\Delta_Y)>1$. This completes the proof of the lemma. □

Finally, we prove

Lemma 2.4. *Let* $\mathbf{f}$ *be the fiber of the projection* $Y\to\mathbb{P}^2$ *such that* $P\in\mathbf{f}$. *Suppose that* $\mathbf{f}\subset R$. *Then* $\delta_P(Y,\Delta_Y)>1$.

Proof. Let $\nu\colon\mathscr{Y}\to Y$ be the blow up of the smooth curve $\mathbf{f}$, let E be the ν-exceptional surface. Take $u\in\mathbb{R}_{\geqslant 0}$. Then $\nu^*(L)-uE$ is pseudoeffective $\iff$ $\nu^*(L)-uE$ is nef $\iff$ $u\leqslant 3-2c$, so

$$S_L(E)=\frac{1}{L^3}\int\limits_0^{3-2c}\big(\nu^*(L)-uE\big)^3du=\frac{1}{6(1-c)(3-2c)^2}\int\limits_0^{3-2c}6(1-c)(3-2c-u)(3-2c+u)du=2-\frac{4}{3}c,$$

which gives

$$\delta_P(Y,\Delta_Y)\leqslant\frac{A_{Y,\Delta_Y}(E)}{S_L(E)}=1+\frac{c}{2(3-2c)}.$$

Now, let $R_{\mathscr{Y}}$ be the proper transform on $\mathscr{Y}$ of the surface R, let $R_E = R_{\mathscr{Y}}|_E$, let $\Delta_E = cR_E$, and let $\mathbf{l}$ be the fiber of the projection $E \to \mathbf{f}$ such that $\nu(\mathbf{l}) = P$. Then R_E is a smooth curve, which implies that (E, Δ_E) has Kawamata log terminal singularities. For every point $\mathscr{P} \in \mathbf{l}$, set

$$\delta_{\mathscr{P}}\big(E,\Delta_E;W^E_{\bullet,\bullet}\big) = \inf_{\substack{F/E,\\ \mathscr{P}\in C_E(F)}} \frac{A_{E,\Delta_E}(F)}{S(W^E_{\bullet,\bullet};F)},$$

where the infimum is taken over all prime divisors F over E whose centers on E contain $\mathscr{P}$, and

$$S\big(W^E_{\bullet,\bullet};F\big) = \frac{3}{L^3}\int_0^{3-2c}\int_0^{\infty}\mathrm{vol}\Big(\big(\nu^*(L)-uE\big)\big|_E - vF\Big)dvdu.$$

Then it follows from [1, 3, 16] that

$$\delta_P(Y,\Delta_Y) \geqslant \min\left\{\inf_{\mathscr{P}\in\mathbf{l}}\delta_{\mathscr{P}}\big(E,\Delta_E;W^E_{\bullet,\bullet}\big), \frac{A_{Y,\Delta_Y}(E)}{S_L(E)}\right\} = \min\left\{\inf_{\mathscr{P}\in\mathbf{l}}\delta_{\mathscr{P}}\big(E,\Delta_E;W^E_{\bullet,\bullet}\big), 1+\frac{c}{2(3-2c)}\right\}.$$

Thus, to complete the proof, it is enough to show that $\delta_{\mathscr{P}}(E,\Delta_E;W^E_{\bullet,\bullet}) > 1$ for every point $\mathscr{P} \in \mathbf{l}$.

Fix a point $\mathscr{P} \in \mathbf{l}$. Let $\mathbf{s}$ be the smooth curve in $E \cong \mathbb{P}^1 \times \mathbb{P}^1$ such that $\mathbf{s}^2 = 0$, $\mathbf{s}\cdot\mathbf{l} = 1$, $\mathscr{P} \in \mathbf{s}$. Then $E|_E \sim -\mathbf{s}$, $R_E \sim 2\mathbf{l} + \mathbf{s}$, and $(\nu^*(L) - uE)\big|_E \sim_{\mathbb{R}} (2-2c)\mathbf{l} + u\mathbf{s}$.

Let $\alpha\colon \widetilde{E} \to E$ be the blow up of the point $\mathscr{P}$, let $\mathbf{e}$ be the exceptional curve of the blow up α, and let $\widetilde{\mathbf{s}}$, $\widetilde{\mathbf{l}}$, $R_{\widetilde{E}}$ be the proper transforms on $\widetilde{E}$ of the curves $\mathbf{s}$, $\mathbf{l}$, R_E, respectively. Set $\Delta_{\widetilde{E}} = cR_{\widetilde{E}}$. Then $\widetilde{E}$ is a smooth del Pezzo surface of degree 7, $\widetilde{\mathbf{s}} \cap \widetilde{\mathbf{l}} = \varnothing$, and $\widetilde{\mathbf{s}}$, $\widetilde{\mathbf{l}}$, $\mathbf{e}$ are all (-1)-curves in $\widetilde{E}$. Let v be a non-negative real number. Then

$$\alpha^*\Big(\big(\nu^*(L)-uE\big)\big|_E\Big) - v\mathbf{e} \sim_{\mathbb{R}} (2-2c)\widetilde{\mathbf{l}} + u\widetilde{\mathbf{s}} + (2-2c+u-v)\mathbf{e},$$

and it is pseudoeffective $\iff$ $v \leqslant 2-2c+u$. For $v \in [0, 2-2c+u]$, let $\widetilde{P}(u,v)$ be the positive part of the Zariski decomposition of this divisor, and let $\widetilde{N}(u,v)$ be its negative part. Set

$$S_L\big(W^E_{\bullet,\bullet};\mathbf{e}\big) = \frac{3}{L^3}\int_0^{3-2c}\int_0^{2-2c+u}\big(\widetilde{P}(u,v)\big)^2dvdu.$$

Likewise, for every point $O \in \mathbf{e}$, we set

$$S\big(W^{\widetilde{E},\mathbf{e}}_{\bullet,\bullet,\bullet};O\big) = \frac{3}{L^3}\int_0^{3-2c}\int_0^{2-2c+u}\big(\widetilde{P}(u,v)\cdot\mathbf{e}\big)^2dvdu + F_O\big(W^{\widetilde{E},\mathbf{e}}_{\bullet,\bullet,\bullet}\big),$$

where

$$F_O\big(W^{\widetilde{E},\mathbf{e}}_{\bullet,\bullet,\bullet}\big) = \frac{6}{L^3}\int_0^{3-2c}\int_0^{2-2c+u}\big(\widetilde{P}(u,v)\cdot\mathbf{e}\big)\cdot\mathrm{ord}_O\big(\widetilde{N}(u,v)\big|_{\mathbf{e}}\big)dvdu.$$

Then it follows from [1, 3, 16] that

$$\delta_{\mathscr{P}}\big(E,\Delta_E;W^E_{\bullet,\bullet}\big) \geqslant \min\left\{\min_{O\in\mathbf{e}}\frac{1-\mathrm{ord}_O\big(\Delta_{\widetilde{E}}\big|_{\mathbf{e}}\big)}{S_L(W^{\widetilde{E},\mathbf{e}}_{\bullet,\bullet,\bullet};O)}, \frac{A_{E,\Delta_E}(\mathbf{e})}{S_L(W^E_{\bullet,\bullet};\mathbf{e})}\right\}, \tag{2.3}$$

where

$$A_{E,\Delta_E}(\mathbf{e}) = \begin{cases} 2-c & \text{if } \mathscr{P} \in R_E, \\ 2 & \text{if } \mathscr{P} \notin R_E. \end{cases}$$

On the other hand, if $0 \leqslant u \leqslant 2-2c$, then

$$\widetilde{P}(u,v) \sim_{\mathbb{R}} \begin{cases} (2-2c)\widetilde{\mathbf{l}} + u\widetilde{\mathbf{s}} + (2-2c+u-v)\mathbf{e} & \text{if } 0 \leqslant v \leqslant u, \\ (2-2c+u-v)\big(\mathbf{e}+\widetilde{\mathbf{l}}\big) + u\widetilde{\mathbf{s}} & \text{if } u \leqslant v \leqslant 2-2c, \\ (2-2c+u-v)\big(\mathbf{e}+\widetilde{\mathbf{l}}+\widetilde{\mathbf{s}}\big) & \text{if } 2-2c \leqslant v \leqslant 2-2c+u, \end{cases}$$

and

$$\widetilde{N}(u,v) = \begin{cases} 0 & \text{if } 0 \leqslant v \leqslant u, \\ (v-u)\widetilde{\mathbf{l}} & \text{if } u \leqslant v \leqslant 2-2c, \\ (v-u)\widetilde{\mathbf{l}} + (v-2+2c)\widetilde{\mathbf{s}} & \text{if } 2-2c \leqslant v \leqslant 2-2c+u, \end{cases}$$

which gives

$$\big(\widetilde{P}(u,v)\big)^2 = \begin{cases} (4-4c)u - v^2 & \text{if } 0 \leqslant v \leqslant u, \\ u(4-4c+u-2v) & \text{if } u \leqslant v \leqslant 2-2c, \\ (2-2c+u-v)^2 & \text{if } 2-2c \leqslant v \leqslant 2-2c+u, \end{cases}$$

and

$$\widetilde{P}(u,v)\cdot\mathbf{e} = \begin{cases} v & \text{if } 0 \leqslant v \leqslant u, \\ u & \text{if } u \leqslant v \leqslant 2-2c, \\ 2-2c+u-v & \text{if } 2-2c \leqslant v \leqslant 2-2c+u. \end{cases}$$

Similarly, if $2-2c \leqslant u \leqslant 3-2c$, then

$$\widetilde{P}(u,v) \sim_{\mathbb{R}} \begin{cases} (2-2c)\widetilde{\mathbf{l}} + u\widetilde{\mathbf{s}} + (2-2c+u-v)\mathbf{e} & \text{if } 0 \leqslant v \leqslant 2-2c, \\ (2-2c)\widetilde{\mathbf{l}} + (2-2c+u-v)\big(\mathbf{e}+\widetilde{\mathbf{s}}\big) & \text{if } 2-2c \leqslant v \leqslant u, \\ (2-2c+u-v)\big(\mathbf{e}+\widetilde{\mathbf{l}}+\widetilde{\mathbf{s}}\big) & \text{if } u \leqslant v \leqslant 2-2c+u, \end{cases}$$

and

$$\widetilde{N}(u,v) = \begin{cases} 0 & \text{if } 0 \leqslant v \leqslant 2-2c, \\ (v-2+2c)\widetilde{\mathbf{s}} & \text{if } 2-2c \leqslant v \leqslant u, \\ (v-u)\widetilde{\mathbf{l}} + (v-2+2c)\widetilde{\mathbf{s}} & \text{if } u \leqslant v \leqslant 2-2c+u, \end{cases}$$

which gives

$$\big(\widetilde{P}(u,v)\big)^2 = \begin{cases} (4-4c)u - v^2 & \text{if } 0 \leqslant v \leqslant 2-2c, \\ 4(1-c)(1-c+u-v) & \text{if } 2-2c \leqslant v \leqslant u, \\ (2-2c+u-v)^2 & \text{if } u \leqslant v \leqslant 2-2c+u, \end{cases}$$

and

$$\widetilde{P}(u,v)\cdot\mathbf{e} = \begin{cases} v & \text{if } 0 \leqslant v \leqslant 2-2c, \\ 2-2c & \text{if } 2-2c \leqslant v \leqslant u, \\ 2-2c+u-v & \text{if } u \leqslant v \leqslant 2-2c+u. \end{cases}$$

Thus, integrating, we get $S_L(W^{E}_{\bullet,\bullet};\mathbf{e}) = 2-\frac{5}{3}c < 2-c$ and

$$S_L\big(W^{\widetilde{E},\mathbf{e}}_{\bullet,\bullet,\bullet};O\big) = \begin{cases} 1-c-\dfrac{2(5-3c)(1-c)^2}{3(3-2c)^2} & \text{if } O \notin \widetilde{\mathbf{l}}\cup\widetilde{\mathbf{s}}, \\ 1-\dfrac{2}{3}c & \text{if } O \in \widetilde{\mathbf{s}}, \\ 1-c & \text{if } O \in \widetilde{\mathbf{l}}. \end{cases}$$

Therefore, if $\mathscr{P}\not\in R_E$, then (2.3) gives $\delta_{\mathscr{P}}(E,\Delta_E;W^E_{\bullet,\bullet})>1$. Similarly, if $\mathscr{P}\in R_E$, then $\widetilde{\mathbf{l}}\cap R_{\widetilde{E}}=\varnothing$, the set $\widetilde{\mathbf{s}}\cap R_{\widetilde{E}}\cap\mathbf{e}$ consists of at most 1 point, and (2.3) gives $\delta_{\mathscr{P}}(E,\Delta_E;W^E_{\bullet,\bullet})>1$ if $\widetilde{\mathbf{s}}\cap R_{\widetilde{E}}\cap\mathbf{e}=\varnothing$.

To complete the proof, we may assume that the intersection $\widetilde{\mathbf{s}}\cap R_{\widetilde{E}}\cap\mathbf{e}$ consists of one point, which means that the curves $\mathbf{s}$ and R_E are tangent at the point P. As in the proof of Lemma 2.3, let us consider the following commutative diagram:

$$\begin{array}{ccc} \widetilde{E} & \xleftarrow{\ \beta\ } & \overline{E} \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\gamma} \\ E & \xleftarrow[\ \rho\ ]{} & \widehat{E} \end{array}$$

where β is the blow up of the point $\widetilde{\mathbf{s}}\cap R_{\widetilde{E}}\cap\mathbf{e}$, the morphism γ is the contraction of the proper transform of the curve $\mathbf{e}$ to an ordinary double point of the surface $\widehat{E}$, and ρ is the contraction of the proper transform of the β-exceptional curve.

Let $\widehat{\mathbf{s}}$, $\widehat{\mathbf{l}}$, $R_{\widehat{E}}$ be the proper transforms on $\widehat{E}$ of the curves $\mathbf{s}$, $\mathbf{l}$, R_E, respectively. Then $R_{\widehat{E}}\cap\widehat{\mathbf{s}}=\varnothing$, and the curves $\widehat{\mathbf{s}}$, $\widehat{\mathbf{l}}$, $R_{\widehat{E}}$ are smooth. Let $\mathbf{z}$ be the ρ-exceptional curve. Then $R_{\widehat{E}}\cap\widehat{\mathbf{l}}\cap\mathbf{z}=\varnothing$, $\mathbf{z}\cong\mathbb{P}^1$, and the intersections of the curves $\mathbf{z}$, $\widehat{\mathbf{s}}$ and $\widehat{\mathbf{l}}$ are given in the following table:

	$\mathbf{z}$	$\widehat{\mathbf{s}}$	$\widehat{\mathbf{l}}$
$\mathbf{z}$	$-\frac{1}{2}$	1	$\frac{1}{2}$
$\widehat{\mathbf{s}}$	1	-2	0
$\widehat{\mathbf{l}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$

Furthermore, we have

$$\rho^*\Big(\big(\nu^*(L)-uE\big)\big|_E\Big)-v\mathbf{z}\sim_{\mathbb{R}}(2-2c)\widehat{\mathbf{l}}+u\widehat{\mathbf{s}}+(2-2c+2u-v)\mathbf{z},$$

and it is pseudoeffective $\iff$ $v\leqslant 2-2c+2u$. For $v\in[0,2-2c+2u]$, let $\widehat{P}(u,v)$ be the positive part of the Zariski decomposition of this divisor, and let $\widehat{N}(u,v)$ be its negative part. Set

$$S_L\big(W^E_{\bullet,\bullet};\mathbf{z}\big)=\frac{3}{L^3}\int\limits_0^{3-2c}\int\limits_0^{2-2c+2u}\big(\widehat{P}(u,v)\big)^2dvdu.$$

Similarly, for every point $O\in\mathbf{z}$, we set

$$S\big(W^{\widehat{E},\mathbf{z}}_{\bullet,\bullet,\bullet};O\big)=\frac{3}{L^3}\int\limits_0^{3-2c}\int\limits_0^{2-2c+2u}\big(\widehat{P}(u,v)\cdot\mathbf{z}\big)^2dvdu+F_O\big(W^{\widehat{E},\mathbf{z}}_{\bullet,\bullet,\bullet}\big),$$

where

$$F_O\big(W^{\widehat{E},\mathbf{z}}_{\bullet,\bullet,\bullet}\big)=\frac{6}{L^3}\int\limits_0^{3-2c}\int\limits_0^{2-2c+2u}\big(\widehat{P}(u,v)\cdot\mathbf{z}\big)\cdot\mathrm{ord}_O\big(\widehat{N}(u,v)\big|_{\mathbf{z}}\big)dvdu.$$

Let Q be the singular point of the surface $\widehat{E}$. Then $Q=\widehat{\mathbf{l}}\cap\mathbf{z}$. Let $\Delta_{\widehat{E}}=cR_{\widehat{E}}$ and $\Delta_{\mathbf{z}}=\frac{1}{2}Q+\Delta_{\widehat{E}}|_{\mathbf{z}}$. Then it follows from [1, 3, 16] that

$$\delta_{\mathscr{P}}\big(E,\Delta_E;W^E_{\bullet,\bullet}\big)\geqslant\min\left\{\min_{O\in\mathbf{z}}\frac{A_{\mathbf{z},\Delta_{\mathbf{z}}}(O)}{S_L(W^{\widehat{E},\mathbf{z}}_{\bullet,\bullet,\bullet};O)},\frac{A_{E,\Delta_E}(\mathbf{z})}{S_L(W^E_{\bullet,\bullet};\mathbf{z})}\right\},\tag{2.4}$$

where $A_{E,\Delta_E}(\mathbf{z}) = 3-2c$ and $A_{\mathbf{z},\Delta_{\mathbf{z}}}(O) = 1-\mathrm{ord}_O(\Delta_{\mathbf{z}})$. On the other hand, if $0 \leqslant u \leqslant 1-c$, then

$$\widehat{P}(u,v) \sim_{\mathbb{R}} \begin{cases} (2-2c)\widehat{\mathbf{l}} + u\widehat{\mathbf{s}} + (2-2c+2u-v)\mathbf{z} & \text{if } 0 \leqslant v \leqslant 2u, \\ (2-2c+2u-v)\big(\mathbf{z}+\widehat{\mathbf{l}}\big) + u\widehat{\mathbf{s}} & \text{if } 2u \leqslant v \leqslant 2-2c, \\ (2-2c+2u-v)\big(\mathbf{z}+\widehat{\mathbf{l}}+\widehat{\mathbf{s}}\big) & \text{if } 2-2c \leqslant v \leqslant 2-2c+2u, \end{cases}$$

and

$$\widehat{N}(u,v) = \begin{cases} 0 & \text{if } 0 \leqslant v \leqslant 2u, \\ (v-2u)\widehat{\mathbf{l}} & \text{if } 2u \leqslant v \leqslant 2-2c, \\ (v-2u)\widehat{\mathbf{l}} + \dfrac{v-2+2c}{2}\widehat{\mathbf{s}} & \text{if } 2-2c \leqslant v \leqslant 2-2c+2u, \end{cases}$$

which gives

$$\big(\widehat{P}(u,v)\big)^2 = \begin{cases} (4-4c)u - \dfrac{v^2}{2} & \text{if } 0 \leqslant v \leqslant 2u, \\ 2u(2-2c+u-v) & \text{if } 2u \leqslant v \leqslant 2-2c, \\ \dfrac{(2-2c+2u-v)^2}{2} & \text{if } 2-2c \leqslant v \leqslant 2-2c+2u, \end{cases}$$

and

$$\widehat{P}(u,v)\cdot\mathbf{z} = \begin{cases} \dfrac{v}{2} & \text{if } 0 \leqslant v \leqslant 2u, \\ u & \text{if } 2u \leqslant v \leqslant 2-2c, \\ \dfrac{2-2c+2u-v}{2} & \text{if } 2-2c \leqslant v \leqslant 1+u. \end{cases}$$

Similarly, if $1-c \leqslant u \leqslant 3-2c$, then

$$\widehat{P}(u,v) \sim_{\mathbb{R}} \begin{cases} (2-2c)\widehat{\mathbf{l}} + u\widehat{\mathbf{s}} + (2-2c+2u-v)\mathbf{z} & \text{if } 0 \leqslant v \leqslant 2-2c, \\ \dfrac{2-2c+2u-v}{2}\big(2\mathbf{z}+\widehat{\mathbf{s}}\big) + (2-2c)\widehat{\mathbf{l}} & \text{if } 2-2c \leqslant v \leqslant 2u, \\ \dfrac{2-2c+2u-v}{2}\big(2\mathbf{z}+2\widehat{\mathbf{l}}+\widehat{\mathbf{s}}\big) & \text{if } 2u \leqslant v \leqslant 2-2c+2u, \end{cases}$$

and

$$\widehat{N}(u,v) = \begin{cases} 0 & \text{if } 0 \leqslant v \leqslant 2-2c, \\ \dfrac{v-2+2c}{2}\widehat{\mathbf{s}} & \text{if } 2-2c \leqslant v \leqslant 2u, \\ (v-2u)\widehat{\mathbf{l}} + \dfrac{v-2+2c}{2}\widehat{\mathbf{s}} & \text{if } 2u \leqslant v \leqslant 2-2c+2u, \end{cases}$$

which gives

$$\big(\widehat{P}(u,v)\big)^2 = \begin{cases} (4-4c)u - \dfrac{v^2}{2} & \text{if } 0 \leqslant v \leqslant 2-2c, \\ 2(1-c)(1-c+2u-v) & \text{if } 2-2c \leqslant v \leqslant 2u, \\ \dfrac{(2-2c+2u-v)^2}{2} & \text{if } 2u \leqslant v \leqslant 1+2u, \end{cases}$$

and

$$\widehat{P}(u,v)\cdot\mathbf{z} = \begin{cases} \dfrac{v}{2} & \text{if } 0 \leqslant v \leqslant 2-2c, \\ 1-c & \text{if } 2-2c \leqslant v \leqslant 2u, \\ \dfrac{2-2c+2u-v}{2} & \text{if } 2u \leqslant v \leqslant 1+u. \end{cases}$$

Now, integrating, we get $S_L(W^{E}_{\bullet,\bullet};\mathbf{z}) = 3 - \frac{7}{3}c < 3 - 2c = A_{E,\Delta_E}(\mathbf{z})$ and

$$S_L\big(W^{\widehat{E},\mathbf{z}}_{\bullet,\bullet,\bullet};O\big) = \begin{cases} 1 - c - \dfrac{(1-c)(31c^2-90c+65)}{12(3-2c)^2} & \text{if } O \notin \widehat{\mathbf{l}} \cup \widehat{\mathbf{s}}, \\ \dfrac{1}{2} - \dfrac{c}{2} & \text{if } O \in \widehat{\mathbf{l}}, \\ 1 - \dfrac{2c}{3} & \text{if } O \in \widehat{\mathbf{s}}. \end{cases}$$

Hence, using (2.4), we get $\delta_{\mathscr{P}}(E,\Delta_E;W^{E}_{\bullet,\bullet}) > 1$, which gives $\delta_P(Y,\Delta_Y) > 1$. □

Now, combining Lemmas 2.2, 2.3 and 2.4, we obtain Theorem 2.1.

3. Smooth Fano 3-folds in the family №3.4

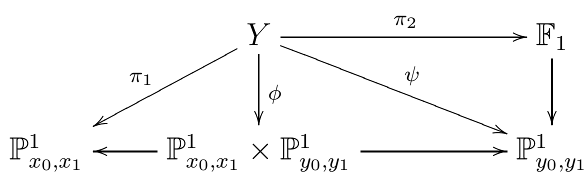
Let $Y = \mathbb{P}^1 \times \mathbb{F}_1$. Identify $Y = (\mathbb{A}^2 \setminus 0)^3/\mathbb{G}_m^3$ for the $\mathbb{G}_m^3$-action

$$\big((x_0,x_1),(y_0,y_1),(z_0,z_1)\big) \mapsto \left(\big(\lambda x_0,\lambda x_1\big),\Big(\big(\mu y_0,\mu y_1\big),\frac{\nu z_0}{\mu},\nu z_1\Big),\right)$$

where $(\lambda,\mu,\nu) \in \mathbb{G}_m^3$, and $((x_0,x_1),(y_0,y_1),(z_0,z_1))$ are coordinates on $(\mathbb{A}^2)^3$. We will use

- $([x_0:x_1],[y_0:y_1;z_0:z_1])$ as coordinates on $\mathbb{P}^1 \times \mathbb{F}_1$,
- $[x_0:x_1]$ as coordinates on the first factor of $Y = \mathbb{P}^1 \times \mathbb{F}_1$,
- $[y_0:y_1;z_0:z_1]$ as coordinates on the second factor of $Y = \mathbb{P}^1 \times \mathbb{F}_1$,
- $[y_0:y_1]$ as coordinates on the base of the natural projection $\mathbb{F}_1 \to \mathbb{P}^1$.

To distinguish the first factor of $Y = \mathbb{P}^1 \times \mathbb{F}_1$ and the base of the natural projection $\mathbb{F}_1 \to \mathbb{P}^1$, we will use notations $\mathbb{P}^1_{x_0,x_1}$ and $\mathbb{P}^1_{y_0,y_1}$ for them, respectively. Then $Y = \mathbb{P}^1_{x_0,x_1} \times \mathbb{F}_1$, and we have the following commutative diagram:



where π_1 and π_2 are projections to the first and the second factors, respectively, ϕ is the $\mathbb{P}^1$-bundle

$$\big([x_0:x_1],[y_0:y_1;z_0:z_1]\big) \mapsto ([x_0:x_1],[y_0:y_1]),$$

the morphism ψ is the $\mathbb{P}^1\times\mathbb{P}^1$-bundle $([x_0:x_1],[y_0:y_1;z_0:z_1]) \mapsto [y_0:y_1]$, and all other morphisms are natural projections. Let F be a fiber of the morphism π_1, let S be a fiber of the morphism ψ, let E be the exceptional surface of the birational contraction $Y \to \mathbb{P}^1_{x_0,x_1} \times \mathbb{P}^2$ given by

$$\big([x_0:x_1],[y_0:y_1;z_0:z_1]\big) \mapsto \big([x_0:x_1];[y_0z_0:y_1z_0:z_1]\big),$$

let R be a smooth surface in $|2F+2E+2S|$, and let $\eta\colon X \to \mathbb{P}^1_{x_0,x_1}\times\mathbb{F}_1$ be a double cover ramified in the surface R. Then X is a smooth Fano threefold in the family №3.4.

Recall that X is K-stable $\iff$ X is K-polystable, because $\mathrm{Aut}(X)$ is finite [9]. Let $\Delta_Y = \frac{1}{2}R$. Then it follows from [12, 15, 21, 25] that

$$X \text{ is K-polystable} \iff (Y,\Delta_Y) \text{ is K-polystable.}$$

The goal of this section is to prove the following result.

Theorem 3.1. *The log Fano pair (Y,Δ_Y) is K-stable.*

Before proving Theorem 3.1, observe that $E=\{z_0=0\}\subset Y$, and R is given in Y by

$$
\begin{aligned}
(3.1)\quad & x_0^2\Big(\big(a_0y_0^2+b_0y_0y_1+c_0y_1^2\big)z_0^2+\big(d_0y_0+e_0y_1\big)z_0z_1+f_0z_1^2\Big)+\\
&+x_0x_1\Big(\big(a_1y_0^2+b_1y_0y_1+c_1y_1^2\big)z_0^2+\big(d_1y_0+e_1y_1\big)z_0z_1+f_1z_1^2\Big)+\\
&+x_1^2\Big(\big(a_2y_0^2+b_2y_0y_1+c_2y_1^2\big)z_0^2+\big(d_2y_0+e_2y_1\big)z_0z_1+f_2z_1^2\Big)=0,
\end{aligned}
$$

where a_0, b_0, c_0, d_0, e_0, f_0, a_1, b_1, c_1, d_1, e_1, f_1, a_2, b_2, c_2, d_2, e_2, f_2 are some numbers.

Lemma 3.2. *Set $R_E=R|_E$, $R_S=R|_S$, $R_F=R|_F$. Then*

(i) *R_E is a disjoint union of two fibers of the projection $\pi_1|_E\colon E\to\mathbb{P}^1_{x_0,x_1}$,*
(ii) *the curve R_S is reduced,*
(iii) *if R_F is reduced, then it has one or two ordinary double points,*
(iv) *if R_F is not reduced, then $\mathrm{Sing}(R_F)=F\cap E$.*

Let P be a point in $F\cap S$ such that $P\notin E$ and $P\in R$, let Z be the fiber of ϕ that contains P, and let C be the fiber of π_2 that contains P. Then

(v) *if $Z\subset R$, then R_F and R_S are singular at some points in Z,*
(vi) *if $C\subset R$, then R_S is singular at some point in C.*
(vii) *at least one of the surfaces R_F and R_S is smooth at P,*
(viii) *if R_S is singular at P, and $Z\not\subset R$, then R_F is smooth.*

Proof. First, let us choose appropriate coordinates on Y such that $F=\{x_1=0\}$ and $S=\{y_1=0\}$. To prove (i), observe that

$$R_E=\big\{z_0=0, f_0x_0^2+f_1x_0x_1+f_2x_1^2=0\big\}\subset Y.$$

Moreover, if $f_0x_0^2+f_1x_0x_1+f_2x_1^2$ is a square, then R is singular. This proves (i).

Let us prove (ii). Using (3.1), we see that $R_S=\{f=0\}\subset S$ for

$$f=x_0^2(a_0z_0^2+d_0z_0z_1+f_0z_1^2)+x_0x_1(a_1z_0^2+d_1z_0z_1+f_1z_1^2)+x_1^2(a_2z_0^2+d_2z_0z_1+f_2z_1^2),$$

where we consider $([x_0:x_1],[z_0:z_1])$ as coordinates on $S\cong\mathbb{P}^1\times\mathbb{P}^1$. Hence, if R_S is not reduced, then $f=gh^2$ for a non-constant polynomial h and a polynomial g. Then we can rewrite (3.1) as

$$y_1\Big(x_0^2\big((b_0y_0+c_0y_1)z_0^2+e_0z_0z_1\big)+x_0x_1\big((b_1y_0+c_1y_1)z_0^2+e_1z_0z_1\big)+x_1^2\big((b_2y_0+c_2y_1)z_0^2+e_2z_0z_1\big)\Big)+gh^2=0,$$

which implies that the surface R is singular at every point of the non-empty subset

$$\big\{y_1=0, x_0^2\big(b_0y_0z_0^2+e_0z_0z_1\big)+x_0x_1\big(b_1y_0z_0^2+e_1z_0z_1\big)+x_1^2\big(b_2y_0z_0^2+e_2z_0z_1\big), h=0\big\}\subset Y,$$

which is impossible by assumption. Hence, we see that R_S is reduced. This proves (ii).

Let us prove (iii) and (iv). Identify $F=\mathbb{F}_1$ with coordinates $[y_0:y_1;z_0:z_1]$. Then

$$R_F=\big\{(a_0y_0^2+b_0y_0y_1+c_0y_1^2)z_0^2+(d_0y_0+e_0y_1)z_0z_1+f_0z_1^2=0\big\}\subset F.$$

Let $\upsilon\colon\mathbb{F}_1\to\mathbb{P}^2$ be the blow up $[y_0:y_1;z_0:z_1]\mapsto[y_0z_0:y_1z_0:z_1]$, and let $\mathbf{e}$ be its exceptional curve. Then $\upsilon(R_F)$ is a reduced conic. Furthermore, if $f_0\neq 0$, then $R_F\cap\mathbf{e}=\varnothing$, and either R_F is smooth, or the curve R_F is a union of two smooth irreducible curves intersecting transversally at one point. Thus, we may assume that $f_0=0$. Then $\upsilon(R_F)$ contains $\upsilon(\mathbf{e})$, and $R_F=\mathbf{e}+R_F'$, where

$$R_F'=\big\{(a_0y_0^2+b_0y_0y_1+c_0y_1^2)z_0+(d_0y_0+e_0y_1)z_1=0\big\}\subset F.$$

If $d_0\neq 0$ or $e_0\neq 0$, then R_F' is the proper transform of the conic $\upsilon(R_F)$, which is smooth at $\upsilon(\mathbf{e})$. In this case, if $\upsilon(R_F)$ is irreducible, then the curve R_F' is smooth, and R_F has one ordinary double point — the intersection point $\mathbf{e}\cap R_F'$. Similarly, if $\upsilon(R_F)$ is reducible, then R_F has two ordinary double points — the intersection point $\mathbf{e}\cap R_F'$, and the unique singular point of the curve R_F'.

Finally, if $d_0 = 0$ and $e_0 = 0$, then $R_F = 2\mathbf{e} + \mathbf{l} + \mathbf{l}'$, where $\mathbf{l} + \mathbf{l}' = \{a_0 y_0^2 + b_0 y_0 y_1 + c_0 y_1^2 = 0\} \subset F$, so that $\mathbf{l}$ and $\mathbf{l}'$ are distinct fibers of the projection $\mathbb{F}_1 \to \mathbb{P}^1_{y_0,y_1}$. This proves (iii) and (iv).

Now, choosing appropriate coordinates on Y, we may assume that $P = ([1:0],[1:0;1:0])$. Then $a_0 = 0$, since $P \in R$. Note also that $Z = \{x_1 = 0, y_1 = 0\}$ and $C = \{y_1 = 0, z_1 = 0\}$.

Both assertions (v) and (vi) are obvious. Now, let us prove (vii). In the affine chart $x_0 y_0 z_0 \neq 0$, the surface R is given by

$$a_1 x + b_0 y + d_0 z + \text{higher order terms} = 0,$$

where $x = \frac{x_1}{x_0}$, $y = \frac{y_1}{y_0}$, $z = \frac{z_1}{z_0}$. which implies that $(a_1, b_0, d_0) \neq (0,0,0)$, because R is smooth at P. If R_F is singular at P, then $b_0 = 0$ and $d_0 = 0$. If R_S is singular at P, then $a_1 = 0$ and $d_0 = 0$. Hence, if both R_F and R_S are singular at P, then $(a_1, b_0, d_0) = (0,0,0)$. This proves (vii).

Let's prove (viii). Suppose that R_S is singular at P, and $Z \not\subset R$. Then $a_1 = d_0 = 0$ and $b_0 f_0 \neq 0$. Observe that $R_F \cap \mathbf{e} = \varnothing$, since $f_0 \neq 0$. Now, computing the defining equation of the conic $\upsilon(R_F)$, we see that this conic is smooth, because $b_0 f_0 \neq 0$. Then R_F is also smooth. This proves (viii). $\square$

3.1. **The proof.** Set $L = -(K_Y + \Delta_Y)$. Then $L \sim_{\mathbb{Q}} F + E + 2S$ and $L^3 = 9$. To prove Theorem 3.1, we must show that $\beta_{Y,\Delta_Y}(\mathbf{E}) = A_{Y,\Delta_Y}(\mathbf{E}) - S_L(\mathbf{E}) > 0$ for every prime divisor $\mathbf{E}$ over Y, where

$$S_L(\mathbf{E}) = \frac{1}{L^3} \int\limits_0^\infty \mathrm{vol}\big(L - u\mathbf{E}\big) du.$$

Fix a prime divisor $\mathbf{F}$ over Y. Let us show that $\beta_{Y,\Delta_Y}(\mathbf{F}) > 0$. Set $\mathfrak{C} = C_Y(\mathbf{F})$. Then

(1) either $\mathfrak{C}$ is a point,
(2) or $\mathfrak{C}$ is an irreducible curve,
(3) or $\mathfrak{C}$ is an irreducible surface.

In each case, let P be some point in $\mathfrak{C}$. If $\beta_{Y,\Delta_Y}(\mathbf{F}) \leqslant 0$, then $\delta_P(Y, \Delta_Y) \leqslant 1$, where

$$\delta_P(Y, \Delta_Y) = \inf_{\substack{\mathbf{E}/Y \\ P \in C_Y(\mathbf{E})}} \frac{A_{Y,\Delta_Y}(\mathbf{E})}{S_L(\mathbf{E})},$$

where the infimum is taken over all prime divisors $\mathbf{E}$ over Y whose centers on Y contain P.

Changing coordinates on Y, we may assume that $P = ([1:0],[1:0;a:b])$ for some $[a:b] \in \mathbb{P}^1$ such that $ab = 0$. Thus, we have the following two possibilities:

(♣) $P = ([1:0],[1:0;0:1]) \in E$,
(♠) $P = ([1:0],[1:0;1:0]) \notin E$.

Moreover, we can choose S to be the fiber of the morphism $\psi\colon Y \to \mathbb{P}^1_{y_0,y_1}$ that contains the point P, and we can choose F to be the fiber of the morphism $\pi_1\colon Y \to \mathbb{P}^1_{x_0,x_1}$ that contains P. Then

$$E = \{z_0 = 0\} \cong \mathbb{P}^1 \times \mathbb{P}^1,$$
$$S = \{y_1 = 0\} \cong \mathbb{P}^1 \times \mathbb{P}^1,$$
$$F = \{x_1 = 0\} \cong \mathbb{F}_1.$$

Lemma 3.3. *Suppose that $\mathfrak{C}$ is a surface. Then $\beta_{Y,\Delta_Y}(\mathbf{F}) > 0$.*

Proof. Since $\mathfrak{C} \sim n_F F + n_E E + n_S S$ for some non-negative integers n_F, n_E, n_S that are not all zero, we have

$$\beta_{Y,\Delta_Y}(\mathbf{F}) = \beta_{Y,\Delta_Y}(\mathfrak{C}) \geqslant \min\big\{\beta_{Y,\Delta_Y}(F), \beta_{Y,\Delta_Y}(E), \beta_{Y,\Delta_Y}(S)\big\},$$

but $\beta_{Y,\Delta_Y}(F)=\frac{1}{2}$, $\beta_{Y,\Delta_Y}(E)=\frac{4}{9}$, $\beta_{Y,\Delta_Y}(S)=\frac{2}{9}$. Indeed, let us compute $\beta_{Y,\Delta_Y}(E)$. Take $u\in\mathbb{R}_{\geqslant 0}$. Then $L-uE$ is pseudoeffective $\iff$ $L-uE$ is nef $\iff$ $u\in[0,1]$. Using this, we compute

$$\beta_{Y,\Delta_Y}(E)=1-S_L(E)=1-\frac{1}{L^3}\int\limits_0^1(L-uE)^3du=1-\frac{1}{9}\int\limits_0^1 6u(1+u)du=\frac{4}{9}.$$

Similarly, we compute $\beta_{Y,\Delta_Y}(F)=\frac{1}{2}$ and $\beta_{Y,\Delta_Y}(S)=\frac{2}{9}$. □

Let $R_E=R|_E$ and $\Delta_E=\frac{1}{2}R_E$. Then, by Lemma 3.2, the curve R_E is a union of two distinct fibers of the morphisms $\pi_1|_E\colon E\to\mathbb{P}^1_{x_0,x_1}$.

Lemma 3.4. *Suppose that $P\in E$. Then $\delta_P(Y,\Delta_Y)\geqslant 1$. Moreover, if $\mathfrak{C}\subset E$, then $\beta_{Y,\Delta_Y}(\mathbf{F})>0$.*

Proof. Take $u\in\mathbb{R}_{\geqslant 0}$. From the proof of Lemma 3.3, we know that

$$L-uE \text{ is pseudoeffective} \iff L-uE \text{ is nef} \iff u\in[0,1].$$

Let $\mathbf{l}$ and $\mathbf{s}$ be some fibers of the morphisms $\pi_1|_E\colon E\to\mathbb{P}^1_{x_0,x_1}$ and $\psi|_E\colon E\to\mathbb{P}^1_{y_0,y_1}$, respectively. Choose $\mathbf{l}$ and $\mathbf{s}$ such that $P\in\mathbf{l}\cap\mathbf{s}$. Take $v\in\mathbb{R}_{\geqslant 0}$. Then $(L-uE)|_E-v\mathbf{l}\sim_{\mathbb{R}}(1-v)\mathbf{l}+(1+u)\mathbf{s}$, and this divisor is pseudoeffective $\iff$ it is nef $\iff$ $v\in[0,1]$. Now, following [1, 3, 16], we set

$$S_L\big(W^E_{\bullet,\bullet};\mathbf{l}\big)=\frac{3}{L^3}\int\limits_0^1\int\limits_0^1\big((L-uE)\big|_E-v\mathbf{l}\big)^2dvdu$$

and

$$S_L\big(W^{E,\mathbf{l}}_{\bullet,\bullet,\bullet};P\big)=\frac{3}{L^3}\int\limits_0^1\int\limits_0^1\Big(\big((L-uE)\big|_E-v\mathbf{l}\big)\cdot\mathbf{l}\Big)^2dvdu.$$

Integrating, we get $S_L(W^E_{\bullet,\bullet};\mathbf{l})=\frac{1}{2}$ and $S_L(W^{E,\mathbf{l}}_{\bullet,\bullet,\bullet};P)=\frac{7}{9}$.

If $\mathbf{l}$ is not an irreducible component of the curve R_E, then it follows from [1, 3, 16] that

$$\frac{A_{Y,\Delta_Y}(\mathbf{F})}{S_L(\mathbf{F})}\geqslant\delta_P(Y,\Delta_Y)\geqslant\min\left\{\frac{1}{S_L(W^{E,\mathbf{l}}_{\bullet,\bullet,\bullet};P)},\frac{1}{S_L(W^E_{\bullet,\bullet};\mathbf{l})},\frac{1}{S_L(E)}\right\}=\frac{9}{7},$$

because we computed $S_L(E)=\frac{5}{9}$ in the proof of Lemma 3.3. Similarly, if $\mathbf{l}\subset\mathrm{Supp}(R_E)$, then

$$\frac{A_{Y,\Delta_Y}(\mathbf{F})}{S_L(\mathbf{F})}\geqslant\delta_P(Y,\Delta_Y)\geqslant\min\left\{\frac{1}{S_L(W^{E,\mathbf{l}}_{\bullet,\bullet,\bullet};P)},\frac{1-\mathrm{ord}_{\mathbf{l}}(\Delta_E)}{S_L(W^E_{\bullet,\bullet};\mathbf{f})},\frac{1}{S_L(E)}\right\}=1.$$

Moreover, if $\mathfrak{C}=P$, then it follows from [1, 3, 16] that $\beta_{Y,\Delta_Y}(\mathbf{F})>0$.

Thus, we see that $\delta_P(Y,\Delta_Y)\geqslant 1$. In particular, we have $\beta_{Y,\Delta_Y}(\mathbf{F})\geqslant 0$.

To complete the proof, we may assume that $\mathfrak{C}$ is a curve in E. Let us show that $\beta_{Y,\Delta_Y}(\mathbf{F})>0$. Suppose that $\beta_{Y,\Delta_Y}(\mathbf{F})=0$. Let us seek for a contradiction. As above, we let

$$S_L\big(W^E_{\bullet,\bullet};\mathfrak{C}\big)=\frac{3}{L^3}\int\limits_0^1\int\limits_0^\infty\mathrm{vol}\big(L|_E-v\mathfrak{C}\big)dvdu.$$

Then it follows from [1, 3, 16] that

$$1=\frac{A_{Y,\Delta_Y}(\mathbf{F})}{S_L(\mathbf{F})}>\frac{1-\mathrm{ord}_{\mathfrak{C}}(\Delta_E)}{S_L(W^E_{\bullet,\bullet};\mathfrak{C})}.$$

If $\mathfrak{C}$ is an irreducible component of the curve R_E, then $\mathfrak{C}=\mathbf{l}$, so $S_L(W^E_{\bullet,\bullet};\mathbf{l})=\frac{1}{2}$ and $\mathrm{ord}_{\mathbf{l}}(\Delta_E)=\frac{1}{2}$, which gives us a contradiction. Thus, we have $\mathrm{ord}_{\mathfrak{C}}(\Delta_E)=0$, which gives $S_L(W^E_{\bullet,\bullet};\mathfrak{C})>1$. But

$$S_L\big(W^E_{\bullet,\bullet};\mathfrak{C}\big)\leqslant\min\Big\{S_L\big(W^E_{\bullet,\bullet};\mathbf{l}\big),S_L\big(W^E_{\bullet,\bullet};\mathbf{s}\big)\Big\},$$

because $|\mathfrak{C}-\mathbf{l}|\neq\varnothing$ or $|\mathfrak{C}-\mathbf{s}|\neq\varnothing$. Hence, we conclude that $S_L(W^E_{\bullet,\bullet};\mathbf{s})>1$.

Let us compute $S_L(W^E_{\bullet,\bullet};\mathbf{s})$. For $v\in\mathbb{R}_{\geqslant 0}$, we have $(L-uE)|_E-v\mathbf{s}\sim_{\mathbb{R}}\mathbf{l}+(1+u-v)\mathbf{s}$, and this divisor is pseudoeffective $\iff$ it is nef $\iff$ $v\in[0,1+u]$. Hence, we have

$$1<S_L\big(W^E_{\bullet,\bullet};\mathbf{s}\big)=\frac{3}{L^3}\int\limits_0^1\int\limits_0^{1-u}\big(\mathbf{l}+(1+u-v)\mathbf{s}\big)^2dvdu=\frac{3}{L^3}\int\limits_0^1\int\limits_0^{1+u}2(1+u-v)dvdu=\frac{7}{9},$$

which is a contradiction. ∎

Let $R_F=R|_F$ and $\Delta_F=\frac{1}{2}R_F$. Set $Z=S\cdot F$. Then $Z=\{x_1=0,y_1=0\}\subset Y$.

Lemma 3.5. *Suppose that R_F is smooth. Then $\delta_P(Y,\Delta_Y)\geqslant 1$. If $\mathfrak{C}=P$, then $\beta_{Y,\Delta_Y}(\mathbf{F})>0$.*

Proof. We recall that $F=\{x_1=0\}\subset Y$. Let us identify $F=\mathbb{F}_1$ with coordinates $[y_0:y_1;z_0:z_1]$. Let $\upsilon\colon F\to\mathbb{P}^2$ be the blow up $[y_0:y_1;z_0:z_1]\mapsto[y_0z_0:y_1z_0:z_1]$, and let $\mathbf{e}$ be its exceptional curve. Then $R_F\cap\mathbf{e}=\varnothing$, and $\upsilon(R_F)$ is a smooth conic in $\mathbb{P}^2$. Moreover, we have

$$R_F\sim 2\big(Z+\mathbf{e}\big),$$

and Z is the fiber of the natural projection $F\to\mathbb{P}^1_{y_0,y_1}$ over the point $[0:1]$.

Take $u\in\mathbb{R}_{\geqslant 0}$. Then $L-uF$ is pseudoeffective $\iff$ $L-uF$ is nef $\iff$ $u\leqslant 1$. Set

$$\delta_P\big(F,\Delta_F;W^F_{\bullet,\bullet}\big)=\inf_{\substack{\mathbf{f}/F,\\ P\in C_F(\mathbf{f})}}\frac{A_{F,\Delta_F}(\mathbf{f})}{S_L(W^F_{\bullet,\bullet};\mathbf{f})},$$

where

$$S_L\big(W^F_{\bullet,\bullet};\mathbf{f}\big)=\frac{3}{L^3}\int\limits_0^1\int\limits_0^{\infty}\mathrm{vol}\big((L-uF)\big|_F-v\mathbf{f}\big)dvdu,$$

and the infimum is taken over all prime divisors $\mathbf{f}$ over the surface F whose centers on F contain P. Then it follows from [1, 3, 16] that

$$\frac{A_{Y,\Delta_Y}(\mathbf{F})}{S_L(\mathbf{F})}\geqslant\delta_P(Y,\Delta_Y)\geqslant\min\left\{\delta_P\big(F,\Delta_F;W^F_{\bullet,\bullet}\big),\frac{1}{S_L(F)}\right\}.$$

Further, if both these inequalities are equalities and $\mathfrak{C}=P$, then [1, 3, 16] gives $\delta_P(Y,\Delta_Y)=\frac{1}{S_L(F)}$. Moreover, we know from the proof of Lemma 3.3 that $S_L(F)=\frac{1}{2}$. Hence, to complete the proof, it is enough to show that $\delta_P(F,\Delta_F;W^F_{\bullet,\bullet})\geqslant 1$. Let us do this.

Note that (F,Δ_F) is a log Fano pair. Recall from [3] that its δ-invariant is the number

$$\delta(F,\Delta_F)=\inf_{\mathbf{f}/F}\frac{A_{F,\Delta_F}(\mathbf{f})}{S_{F,\Delta_F}(\mathbf{f})},$$

where

$$S_{F,\Delta_F}\big(\mathbf{f}\big)=\frac{1}{(K_F+\Delta_F)^2}\int\limits_0^{\infty}\mathrm{vol}\big(-(K_F+\Delta_F)-v\mathbf{f}\big)dv,$$

and the infimum is taken over all prime divisors $\mathbf{f}$ over the surface F. We claim that $\delta(F,\Delta_F)\geqslant 1$. Indeed, either one can check this explicitly similar to what is done in [3, § 2], or one can use the fact that the double cover of the surface F branched over the curve R_F is a smooth del Pezzo of degree 6, which is known to be K-polystable, so (F,Δ_F) is also K-polystable [15], which gives $\delta(F,\Delta_F)\geqslant 1$. Then, using the idea of the proof of [7, Nemuro Lemma], we get

$$S_L\big(W^F_{\bullet,\bullet};\mathbf{f}\big)=\frac{3}{L^3}\int\limits_0^1\int\limits_0^\infty \mathrm{vol}\big((L-uF)\big|_F-v\mathbf{f}\big)dvdu=\frac{3}{L^3}\int\limits_0^1\int\limits_0^\infty \mathrm{vol}\big(L\big|_F-v\mathbf{f}\big)dvdu=$$
$$=\frac{3}{L^3}\int\limits_0^\infty \mathrm{vol}\big(L\big|_F-v\mathbf{f}\big)dv=\frac{1}{(K_F+\Delta_F)^2}\int\limits_0^\infty \mathrm{vol}\big(-(K_F+\Delta_F)-v\mathbf{f}\big)dvdu\leqslant A_{F,\Delta_F}(\mathbf{f})$$

for every divisor $\mathbf{f}$ over the surface F. This exactly means that $\delta_P(F,\Delta_F;W^F_{\bullet,\bullet})\geqslant 1$. ☐

Let $R_S=R|_S$ and $\Delta_S=\frac{1}{2}R_S$. Recall that $S=\{y_1=0\}$ and $Z=\{x_1=0,y_1=0\}\subset S$. Set
$$C=\{y_1=0,az_1=bz_0\}\subset Y.$$
Then Z and C are rulings of the surface $S\cong\mathbb{P}^1\times\mathbb{P}^1$ such that $P=Z\cap C$.

Lemma 3.6. *Suppose that $P\notin E$. Then*
1. *if $\mathfrak{C}\subset S$ and $\mathfrak{C}$ is a curve, then $\beta_{Y,\Delta_Y}(\mathbf{F})>0$,*
2. *if $P\notin R$, then $\delta_P(Y,\Delta_Y)>1$,*
3. *if $P\in R$ and R_S is smooth at P, then $\delta_P(Y,\Delta_Y)\geqslant 1$,*
4. *if $P\in R$, R_S is smooth at P, and $\mathfrak{C}=P$, then $\beta_{Y,\Delta_Y}(\mathbf{F})>0$,*
5. *if $P\in R$, R_S is smooth at P, and $Z\not\subset\mathrm{Supp}(R_S)$, then $\delta_P(Y,\Delta_Y)>1$.*

Proof. Let u be a non-negative real number. Then $L-uS$ is pseudoeffective if and only if $u\leqslant 2$. For $u\in[0,2]$, let $P(u)$ be the positive part of the Zariski decomposition of the divisor $L-uS$, and let $N(u)$ be the negative part of the Zariski decomposition of the divisor $L-uS$. Then
$$P(u)\sim_{\mathbb{R}}\begin{cases}F+E+(2-u)S & \text{for } 0\leqslant u\leqslant 1,\\ F+(2-u)(E+S) & \text{for } 1\leqslant u\leqslant 2,\end{cases}$$
and
$$N(u)=\begin{cases}0 & \text{for } 0\leqslant u\leqslant 1,\\ (u-1)E & \text{for } 1\leqslant u\leqslant 2.\end{cases}$$
Observe that $R_S\sim 2(Z+C)$ and
$$P(u)\big|_S\sim_{\mathbb{R}}\begin{cases}Z+C & \text{for } 0\leqslant u\leqslant 1,\\ Z+(2-u)C & \text{for } 1\leqslant u\leqslant 2.\end{cases}$$

Let G be an irreducible curve in S that passes though P. Take $v\in\mathbb{R}_{\geqslant 0}$. Set
$$t(u)=\inf\Big\{v\in\mathbb{R}_{\geqslant 0}\ \Big|\ \text{the divisor } P(u)\big|_S-vG \text{ is pseudoeffective}\Big\}.$$
Since $S\cong\mathbb{P}^1\times\mathbb{P}^1$, the divisor $P(u)|_S-vG$ is nef $\iff$ $v\leqslant t(u)$. Set
$$S_L\big(W^S_{\bullet,\bullet};G\big)=\frac{3}{L^3}\int\limits_0^2\int\limits_0^{t(u)}\big(P(u)\big|_S-vG\big)^2dvdu$$
and
$$S_L\big(W^{S,G}_{\bullet,\bullet,\bullet};P\big)=\frac{3}{L^3}\int\limits_0^2\int\limits_0^{t(u)}\Big(\big(P(u)|_S-vG\big)\cdot G\Big)^2dvdu.$$

If $G=\mathfrak{C}$ is a curve in S, it follows from [1, 3, 16] that

$$\frac{A_{Y,\Delta_Y}(\mathbf{F})}{S_L(\mathbf{F})}\geqslant \min\left\{\frac{1-\mathrm{ord}_{\mathfrak{C}}(\Delta_S)}{S_L(W^S_{\bullet,\bullet};G)},\frac{1}{S_L(S)}\right\}.$$

Moreover, if this inequality is an equality, it further follows from [1, 3, 16] that

$$\frac{A_{Y,\Delta_Y}(\mathbf{F})}{S_L(\mathbf{F})}=\frac{1}{S_L(S)}.$$

On the other hand, we know from the proof of Lemma 3.3 that $S_L(S)=\frac{7}{9}$. Moreover, we have

$$S_L\big(W^S_{\bullet,\bullet};G\big)\leqslant \min\big\{S_L\big(W^S_{\bullet,\bullet};Z\big),S_L\big(W^S_{\bullet,\bullet};C\big)\big\}.$$

Therefore, to prove assertion (1), it is enough to check that $S_L(W^S_{\bullet,\bullet};Z)\leqslant\frac{1}{2}$ and $S_L(W^S_{\bullet,\bullet};C)\leqslant\frac{1}{2}$. This is not difficult. Indeed, if $G=Z$, then $t(u)=1$ for every $u\in[0,2]$, and

$$S_L(W^S_{\bullet,\bullet};Z)=\frac{1}{3}\int\limits_0^1\int\limits_0^1 2(1-v)dudu+\frac{1}{3}\int\limits_1^2\int\limits_0^1 2(1-v)(2-u)dudu=\frac{1}{2}.$$

Similarly, if $G=C$, then

$$t(u)=\begin{cases}1 & \text{for } 0\leqslant u\leqslant 1,\\ 2-u & \text{for } 1\leqslant u\leqslant 2,\end{cases}$$

and

$$S_L(W^S_{\bullet,\bullet};C)=\frac{1}{3}\int\limits_0^1\int\limits_0^1 2(1-v)dudu+\frac{1}{3}\int\limits_1^2\int\limits_0^{2-u} 2(2-u-v)dudu=\frac{4}{9}.$$

This proves (1).

Let G be one of the curves Z or C. If $G\not\subset\mathrm{Supp}(R_S)$, then it follows from [1, 3, 16] that

$$\frac{A_{Y,\Delta_Y}(\mathbf{F})}{S_L(\mathbf{F})}\geqslant\delta_P(Y,\Delta_Y)\geqslant\min\left\{\frac{1-\mathrm{ord}_P(\Delta_S|_G)}{S_L(W^{S,G}_{\bullet,\bullet,\bullet};P)},\frac{1}{S_L(W^S_{\bullet,\bullet};G)},\frac{1}{S_L(S)}\right\}. \tag{3.2}$$

On the other hand, we compute

$$S_L(W^{S,G}_{\bullet,\bullet,\bullet};P)=\begin{cases}\frac{4}{9} & \text{if } G=Z,\\ \frac{1}{2} & \text{if } G=C.\end{cases}$$

If $P\not\in R$, then $Z\not\subset\mathrm{Supp}(R_S)$ and $C\not\subset\mathrm{Supp}(R_S)$, so (3.2) gives $\delta_P(Y,\Delta_Y)\geqslant\frac{9}{7}$. This proves (2).

Now, we suppose that $P\in R$ and R_S is smooth at P. Then $Z\not\subset\mathrm{Supp}(R_S)$ or $C\not\subset\mathrm{Supp}(R_S)$. Moreover, if $Z\not\subset\mathrm{Supp}(R_S)$ and R_S intersects Z transversally at P, then (3.2) gives $\delta_P(Y,\Delta_Y)\geqslant\frac{9}{8}$. Therefore, to prove (3), (4) and (5) we may assume that

- either Z is an irreducible component of the curve R_S,
- or the curve R_S is tangent to Z at the point P.

Then C is not an irreducible component of the curve R_S, and R_S intersects C transversally at P. Hence, using (3.2), we obtain $\delta_P(Y,\Delta_Y)\geqslant 1$. This proves (3).

We have $\beta_{Y,\Delta_Y}(\mathbf{F})\geqslant 0$. If $\mathfrak{C}=P$ and $\beta_{Y,\Delta_Y}(\mathbf{F})=0$, then both inequalities in (3.2) are equalities. In this case, it follows from [1, 3, 16] that $\delta_P(Y,\Delta_Y)=\frac{1}{S_L(S)}=\frac{9}{7}$, which contradicts $\beta_{Y,\Delta_Y}(\mathbf{F})\leqslant 0$. Therefore, if $\mathfrak{C}=P$, then $\beta_{Y,\Delta_Y}(\mathbf{F})>0$. This proves (4).

Finally, let us prove (5). We suppose that Z is not an irreducible component of the curve R_S. Then R_S is tangent to the curve Z at the point P. Let $\alpha\colon\widetilde{S}\to S$ be the blow up of the point P, and let $\beta\colon \overline{S}\to\widetilde{S}$ be the blow up of the intersection point of the α-exceptional curve and the proper transform of the curve Z. Then there exists the following commutative diagram:

$$\begin{array}{ccc} \widetilde{S} & \xleftarrow{\ \beta\ } & \overline{S} \\ {\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\gamma} \\ S & \xleftarrow[\ \rho\ ]{} & \widehat{S} \end{array}$$

where γ is the contraction of the proper transform of the α-exceptional curve to an ordinary double point of the surface $\widehat{S}$, and ρ is the contraction of the proper transform of the β-exceptional curve. Then $\widehat{S}$ is a singular del Pezzo surface of degree 6, and ρ is a weighted blow up with weights $(1,2)$.

Denote by $\widehat{Z}$, $\widehat{C}$, $R_{\widehat{S}}$ the proper transforms on $\widehat{S}$ via ρ of the curves Z, C, R_S, respectively. Let $\mathbf{e}$ be the ρ-exceptional curve, and let

$$\widehat{t}(u)=\inf\Big\{v\in\mathbb{R}_{\geqslant 0}\ \big|\ \text{the divisor } \rho^*\big(P(u)\big|_S\big)-v\mathbf{e}\text{ is pseudoeffective}\Big\}.$$

Observe that

$$\rho^*\big(P(u)\big|_S\big)-v\mathbf{e}\sim_{\mathbb{R}}\begin{cases}\widehat{Z}+\widehat{C}+(3-v)\mathbf{e}\text{ for } u\in[0,1],\\ \widehat{Z}+(2-u)\widehat{C}+(4-u-v)\mathbf{e}\text{ for } u\in[1,2].\end{cases}$$

Thus, we conclude that

$$\widehat{t}(u)=\begin{cases}3\text{ for } u\in[0,1],\\ 4-u\text{ for } u\in[1,2].\end{cases}$$

Now, for every $u\in[0,2]$ and every $v\in[0,\widehat{t}(u)]$, we let $\widehat{P}(u,v)$ be the positive part of the Zariski decomposition of the divisor $\rho^*(P(u)|_S)-v\mathbf{e}$, and let $\widehat{N}(u,v)$ be its negative part. Let

$$S_L\big(W^S_{\bullet,\bullet};\mathbf{e}\big)=\frac{3}{L^3}\int\limits_0^2\int\limits_0^{\widehat{t}(u)}\big(\widehat{P}(u,v)\big)^2dvdu.$$

For every point $O\in\mathbf{e}$, let

$$S\big(W^{\widehat{S},\mathbf{e}}_{\bullet,\bullet,\bullet};O\big)=\frac{3}{L^3}\int\limits_0^2\int\limits_0^{\widehat{t}(u)}\big(\widehat{P}(u,v)\cdot\mathbf{e}\big)^2dvdu+F_O\big(W^{\widehat{S},\mathbf{e}}_{\bullet,\bullet,\bullet}\big),$$

where

$$F_O\big(W^{\widehat{S},\mathbf{e}}_{\bullet,\bullet,\bullet}\big)=\frac{6}{L^3}\int\limits_0^2\int\limits_0^{\widehat{t}(u)}\big(\widehat{P}(u,v)\cdot\mathbf{e}\big)\cdot\mathrm{ord}_O\big(\widehat{N}(u,v)\big|_{\mathbf{e}}\big)dvdu.$$

Let Q be the singular point of the surface $\widehat{S}$. Then $Q=\widehat{C}\cap\mathbf{e}$. Let $\Delta_{\widehat{S}}=\frac{1}{2}R_{\widehat{S}}$ and $\Delta_{\mathbf{e}}=\frac{1}{2}Q+\Delta_{\widehat{S}}|_{\mathbf{e}}$. Then it follows from [1, 3, 16] that

$$\delta_P(Y,\Delta_Y)\geqslant\min\left\{\min_{O\in\mathbf{e}}\frac{A_{\mathbf{e},\Delta_{\mathbf{e}}}(O)}{S_L(W^{\widehat{S},\mathbf{e}}_{\bullet,\bullet,\bullet};O)},\frac{A_{S,\Delta_S}(\mathbf{e})}{S_L(W^S_{\bullet,\bullet};\mathbf{e})},\frac{A_{Y,\Delta_Y}(S)}{S_L(S)}\right\},\tag{3.3}$$

where $A_{Y,\Delta_Y}(S)=1$, $A_{S,\Delta_S}(\mathbf{e})=2$ $A_{\mathbf{e},\Delta_{\mathbf{e}}}(O)=1-\mathrm{ord}_O(\Delta_{\mathbf{e}})$. Moreover, if $0\leqslant u\leqslant 1$, then

$$\widehat{P}(u,v)\sim_{\mathbb{R}}\begin{cases}\widehat{Z}+\widehat{C}+(3-v)\mathbf{e} & \text{if } 0\leqslant v\leqslant 1,\\ \widehat{C}+\dfrac{3-v}{2}\big(\widehat{Z}+2\mathbf{e}\big) & \text{if } 1\leqslant v\leqslant 2,\\ \dfrac{3-v}{2}\big(2\widehat{C}+\widehat{Z}+2\mathbf{e}\big) & \text{if } 2\leqslant v\leqslant 3,\end{cases}$$

and

$$\widehat{N}(u,v)=\begin{cases}0 & \text{if } 0\leqslant v\leqslant 1,\\ \dfrac{v-1}{2}\widehat{Z} & \text{if } 1\leqslant v\leqslant 2,\\ \dfrac{v-1}{2}\widehat{Z}+(v-2)\widehat{C} & \text{if } 2\leqslant v\leqslant 3,\end{cases}$$

which gives

$$\big(\widehat{P}(u,v)\big)^2=\begin{cases}2-\dfrac{v^2}{2} & \text{if } 0\leqslant v\leqslant 1,\\ \dfrac{5}{2}-v & \text{if } 1\leqslant v\leqslant 2,\\ \dfrac{(3-v)^2}{2} & \text{if } 2\leqslant v\leqslant 3,\end{cases}$$

and

$$\widehat{P}(u,v)\cdot\mathbf{e}=\begin{cases}\dfrac{v}{2} & \text{if } 0\leqslant v\leqslant 1,\\ \dfrac{1}{2} & \text{if } 1\leqslant v\leqslant 2,\\ \dfrac{3-v}{2} & \text{if } 2\leqslant v\leqslant 3.\end{cases}$$

Similarly, if $1\leqslant u\leqslant 2$, then

$$\widehat{P}(u,v)\sim_{\mathbb{R}}\begin{cases}\widehat{Z}+(2-u)\widehat{C}+(4-u-v)\mathbf{e} & \text{if } 0\leqslant v\leqslant 2-u,\\ \dfrac{4-u-v}{2}\big(\widehat{Z}+2\mathbf{e}\big)+(2-u)\widehat{C} & \text{if } 2-u\leqslant v\leqslant 2,\\ \dfrac{4-u-v}{2}\big(2\widehat{C}+\widehat{Z}+2\mathbf{e}\big) & \text{if } 2\leqslant v\leqslant 4-u,\end{cases}$$

and

$$\widehat{N}(u,v)=\begin{cases}0 & \text{if } 0\leqslant v\leqslant 2-u,\\ \dfrac{v+u-2}{2}\widehat{Z} & \text{if } 2-u\leqslant v\leqslant 2,\\ \dfrac{v+u-2}{2}\widehat{Z}+(v-2)\widehat{C} & \text{if } 2\leqslant v\leqslant 4-u,\end{cases}$$

which gives

$$\big(\widehat{P}(u,v)\big)^2=\begin{cases}4-2u-\dfrac{v^2}{2} & \text{if } 0\leqslant v\leqslant 2-u,\\ \dfrac{(u-2)(u+2v-6)}{2} & \text{if } 2-u\leqslant v\leqslant 2,\\ \dfrac{(4-u-v)^2}{2} & \text{if } 2\leqslant v\leqslant 4-u,\end{cases}$$

and

$$\widehat{P}(u,v)\cdot\mathbf{e}=\begin{cases}\dfrac{v}{2} & \text{if } 0\leqslant v\leqslant 2-u,\\ 1-\dfrac{u}{2} & \text{if } 2-u\leqslant v\leqslant 2,\\ \dfrac{4-u-v}{2} & \text{if } 2\leqslant v\leqslant 4-u.\end{cases}$$

Now, integrating, we get $S_L(W^S_{\bullet,\bullet};\mathbf{e})=\frac{13}{9}<2=A_{S,\Delta_S}(\mathbf{e})$ and

$$S_L\big(W^{\widehat{S},\mathbf{e}}_{\bullet,\bullet,\bullet};O\big)=\begin{cases}\dfrac{3}{16} & \text{if } O\notin\widehat{Z}\cup\widehat{C},\\ \dfrac{2}{9} & \text{if } O\in\widehat{C},\\ \dfrac{1}{2} & \text{if } O\in\widehat{Z}.\end{cases}$$

Hence, using (3.3), we obtain $\delta_P(Y,\Delta_Y)>1$ as required. □

Now, we are ready to prove

Lemma 3.7. *Suppose that $\beta_{Y,\Delta_Y}(\mathbf{F})\leqslant 0$. Then $\mathfrak{C}$ is a point.*

Proof. Suppose $\mathfrak{C}$ is not a point. By Lemma 3.3, the center $\mathfrak{C}$ is not a surface. Then $\mathfrak{C}$ is a curve. We may assume that P is a general point in $\mathfrak{C}$. By Lemma 3.4, we have $\mathfrak{C}\not\subset E$, so $P\notin E$ either.

If $\psi(\mathfrak{C})=\mathbb{P}^1_{y_0,y_1}$, then S is a general fiber of the morphism ψ, which implies that R_S is smooth, so that $Z\not\subset R$ by Lemma 3.2. Then $\delta_P(Y,\Delta_Y)>1$ by Lemma 3.6, which contradicts $\beta_{Y,\Delta_Y}(\mathbf{F})\leqslant 0$. Thus, we see that $\psi(\mathfrak{C})$ is point in $\mathbb{P}^1_{y_0,y_1}$. This means that $\mathfrak{C}\subset S$.

Now, applying Lemma 3.6, we get $\beta_{Y,\Delta_Y}(\mathbf{F})>0$, which is a contradiction. □

Now, we suppose that $\beta_{Y,\Delta_Y}(\mathbf{F})\leqslant 0$. Let us seek for a contradiction. First, applying Lemma 3.7, we see that the center $\mathfrak{C}=C_Y(\mathbf{F})$ is a point. Using notations we introduced earlier, we have $P=\mathfrak{C}$. Moreover, applying Lemmas 3.2, 3.4, 3.5, 3.6, we obtain the following assertions:

- $P\notin E$ by Lemma 3.4,
- R_F is singular by Lemma 3.5,
- $P\in R$ by Lemma 3.6,
- R_S is singular at P by Lemma 3.6,
- R_F is smooth at P by Lemma 3.2,
- $Z\subset R$ by Lemma 3.6.

In particular, the curve R_S is reducible. Namely, we have $R_S=Z+T$, where T is a possibly reducible reduced curve in $|Z+2C|$ such that $P\in T$.

Lemma 3.8. *The curve R_S does not have an ordinary double singularity at P.*

Proof. Suppose that R_S has an ordinary double singularity at P. Let us seek for a contradiction. Let us use notations introduced in the proof of Lemma 3.6. Then we have

$$P(u)\big|_S\sim_{\mathbb{R}}\begin{cases}C+Z & \text{for } 0\leqslant u\leqslant 1,\\ (2-u)C+Z & \text{for } 1\leqslant u\leqslant 2.\end{cases}$$

Let $\alpha\colon\widetilde{S}\to S$ be the blow up of the point P, let $\mathbf{e}$ be the α-exceptional curve. For $u\in[0,2]$, let

$$\widetilde{t}(u)=\max\Big\{v\in\mathbb{R}_{\geqslant 0}\ \big|\ \alpha^*\big(P(u)|_S\big)-v\mathbf{e}\text{ is pseudoeffective}\Big\}.$$

For $v\in[0,\widetilde{t}(u)]$, let $\widetilde{P}(u,v)$ be the positive part of the Zariski decomposition of $\alpha^*(P(u)|_S)-v\mathbf{e}$, and let $\widetilde{N}(u,v)$ be the negative part of the Zariski decomposition of this divisor. Set

$$S\big(W^S_{\bullet,\bullet};\mathbf{e}\big)=\frac{3}{L^3}\int\limits_0^2\int\limits_0^{\widetilde{t}(u)}\widetilde{P}(u,v)^2dvdu.$$

Then, for every point $O\in\mathbf{e}$, we set

$$S\big(W^{\widetilde{S},\mathbf{e}}_{\bullet,\bullet,\bullet};O\big)=\frac{3}{L^3}\int\limits_0^2\int\limits_0^{\widetilde{t}(u)}\Big(\widetilde{P}(u,v)\cdot\mathbf{e}\Big)^2dvdu+F_O\big(W^{\widetilde{S},\mathbf{e}}_{\bullet,\bullet,\bullet}\big),$$

where

$$F_O\big(W^{\widetilde{S},\mathbf{e}}_{\bullet,\bullet,\bullet}\big)=\frac{6}{L^3}\int\limits_0^2\int\limits_0^{\widetilde{t}(u)}\big(\widetilde{P}(u,v)\cdot\mathbf{e}\big)\cdot\mathrm{ord}_O\Big(\widetilde{N}(u,v)\big|_{\mathbf{e}}\Big)dvdu.$$

Let $\widetilde{C}$, $\widetilde{Z}$, $\widetilde{T}$ be the proper transforms on $\widetilde{S}$ of the curves C, Z, T, respectively. Set $\Delta_{\widetilde{S}}=\frac{1}{2}\widetilde{Z}+\frac{1}{2}\widetilde{T}$. Then $\widetilde{Z}$ and $\widetilde{T}$ intersect $\mathbf{e}$ transversally at two distinct points, since T and Z do not tangent at P. Set $\Delta_{\mathbf{e}}=\Delta_{\widetilde{S}}|_{\mathbf{e}}$. Then it follows from [1, 3, 16] that

$$1\geqslant\frac{A_{Y,\Delta_Y}(\mathbf{F})}{S_L(\mathbf{F})}\geqslant\delta_P(Y,\Delta_Y)\geqslant\min\left\{\min_{O\in\mathbf{e}}\frac{A_{\mathbf{e},\Delta_{\mathbf{e}}}(O)}{S(W^{\widetilde{S},\mathbf{e}}_{\bullet,\bullet,\bullet};O)},\frac{A_{S,\Delta_S}(\mathbf{e})}{S(W^S_{\bullet,\bullet};\mathbf{e})},\frac{A_{Y,\Delta_Y}(S)}{S_L(S)}\right\},$$

and not all inequalities here are equalities. Note that $A_{Y,\Delta_Y}(S)=1$, $A_{S,\Delta_S}(\mathbf{e})=1$, and

$$A_{\mathbf{e},\Delta_{\mathbf{e}}}(O)=1-\mathrm{ord}_O(\Delta_{\mathbf{e}})=\begin{cases}\dfrac{1}{2}\ \text{ if } O=\widetilde{Z}\cap\mathbf{e},\\ \dfrac{1}{2}\ \text{ if } O=\widetilde{T}\cap\mathbf{e},\\ 1\ \text{ if } O\not\in\widetilde{Z}\cup\widetilde{T}.\end{cases}$$

Since $S_L(S)=\frac{7}{9}$, we conclude that $S(W^S_{\bullet,\bullet};\mathbf{e})>1$ or there exists a point $O\in\mathbf{e}$ such that

$$S\big(W^{\widetilde{S},\mathbf{e}}_{\bullet,\bullet,\bullet};O\big)>1-\mathrm{ord}_O\big(\Delta_{\mathbf{e}}\big).$$

Let us compute $S(W^S_{\bullet,\bullet};\mathbf{e})$, and let us compute $S(W^{\widetilde{S},\mathbf{e}}_{\bullet,\bullet,\bullet};O)$ for every point $O\in\mathbf{e}$.

Let v be a non-negative real number. Then

$$\alpha^*\big(P(u)|_S\big)-v\mathbf{e}\sim_{\mathbb{R}}\begin{cases}\widetilde{C}+\widetilde{Z}+(2-v)\mathbf{e}\ \text{ for } 0\leqslant u\leqslant 1,\\ (2-u)\widetilde{C}+\widetilde{Z}+(3-u-v)\mathbf{e}\ \text{ for } 1\leqslant u\leqslant 2.\end{cases}$$

Since $\widetilde{Z}$ and $\widetilde{C}$ are disjoint (-1)-curves in $\widetilde{S}$, we have

$$\widetilde{t}(u)=\begin{cases}2\ \text{ for } 0\leqslant u\leqslant 1,\\ 3-u\ \text{ for } 1\leqslant u\leqslant 2.\end{cases}$$

Furthermore, if $0\leqslant u\leqslant 1$, then

$$\widetilde{P}(u,v)\sim_{\mathbb{R}}\begin{cases}\widetilde{C}+\widetilde{Z}+(2-v)\mathbf{e}\ \text{ for } 0\leqslant v\leqslant 1,\\ (2-v)\big(\widetilde{C}+\widetilde{Z}+\mathbf{e}\big)\ \text{ for } 1\leqslant v\leqslant 2,\end{cases}$$

and

$$\widetilde{N}(u,v)=\begin{cases}0 \text{ for } 0\leqslant v\leqslant 1,\\ (v-1)\big(\widetilde{C}+\widetilde{Z}\big) \text{ for } 1\leqslant v\leqslant 2,\end{cases}$$

which gives

$$\big(\widetilde{P}(u,v)\big)^2=\begin{cases}2-v^2 \text{ for } 0\leqslant v\leqslant 1,\\ (2-v)^2 \text{ for } 1\leqslant v\leqslant 2,\end{cases}$$

and

$$\widetilde{P}(u,v)\cdot\mathbf{e}=\begin{cases}v \text{ for } 0\leqslant v\leqslant 1,\\ 2-v \text{ for } 1\leqslant v\leqslant 2.\end{cases}$$

Similarly, if $1\leqslant u\leqslant 2$, then

$$\widetilde{P}(u,v)\sim_{\mathbb{R}}\begin{cases}(2-u)\widetilde{C}+\widetilde{Z}+(3-u-v)\mathbf{e} \text{ for } 0\leqslant v\leqslant 2-u,\\ (2-u)\widetilde{C}+(3-u-v)\big(\widetilde{Z}+\mathbf{e}\big) \text{ for } 2-u\leqslant v\leqslant 1,\\ (3-u-v)\big(\widetilde{C}+\widetilde{Z}+\mathbf{e}\big) \text{ for } 1\leqslant v\leqslant 3-u,\end{cases}$$

and

$$\widetilde{N}(u,v)=\begin{cases}0 \text{ for } 0\leqslant v\leqslant 2-u,\\ (v+u-2)\widetilde{Z} \text{ for } 2-u\leqslant v\leqslant 1,\\ (v+u-2)\widetilde{Z}+(v-1)\widetilde{C} \text{ for } 1\leqslant v\leqslant 3-u,\end{cases}$$

which gives

$$\big(\widetilde{P}(u,v)\big)^2=\begin{cases}4-2u-v^2 \text{ for } 0\leqslant v\leqslant 2-u,\\ (2-u)(4-u-2v) \text{ for } 2-u\leqslant v\leqslant 1,\\ (3-u-v)^2 \text{ for } 1\leqslant v\leqslant 3-u,\end{cases}$$

and

$$\widetilde{P}(u,v)\cdot\mathbf{e}=\begin{cases}v \text{ for } 0\leqslant v\leqslant 2-u,\\ 2-u \text{ for } 2-u\leqslant v\leqslant 1,\\ 3-u-v \text{ for } 1\leqslant v\leqslant 3-u.\end{cases}$$

Therefore, integrating, we get $S(W^{\widetilde{S}}_{\bullet,\bullet};\mathbf{e})=\frac{17}{18}<1$ and

$$S\big(W^{\widetilde{S},\mathbf{e}}_{\bullet,\bullet,\bullet};O\big)=\frac{11}{36}+F_O\big(W^{\widetilde{S},\mathbf{e}}_{\bullet,\bullet,\bullet}\big)=\begin{cases}\dfrac{11}{36} \text{ if } O\not\in\widetilde{C}\cup\widetilde{Z}\\ \dfrac{4}{9} \text{ if } O\in\widetilde{C},\\ \dfrac{1}{2} \text{ if } O\in\widetilde{Z},\end{cases}$$

which gives $S(W^{\widetilde{S},\mathbf{e}}_{\bullet,\bullet,\bullet};O)\leqslant 1-\mathrm{ord}_O(\Delta_{\mathbf{e}})$ for every point $O\in\mathbf{e}$. This is a contradiction. □

Now, using Lemma 3.8, we see that one of the following two remaining cases occurs:

($\mathbb{A}_3$) $R_S=Z+T$, where T is a smooth curve in $|Z+2C|$ that is tangent to Z at the point P,
($\mathbb{D}_4$) $R_S=Z+T=Z+C+T'$, where T' is a smooth curve in $|Z+C|$ such that $P\in T'$.

This imposes certain constraints on the equation (3.1), which can be listed as follows:

- $a_0=0$, since $P=([1:0],[1:0;1:0])\in R$,
- $a_1=0$ and $d_0=0$, since R_S is singular at P,
- $f_0=0$ and $d_0=0$, since $Z\subset R$,
- $d_1=0$, since R_S does not have ordinary double point at P.

Changing coordinates on Y, we can simplified (3.1) a bit more. First, we may assume that $b_0 = 1$, since R is smooth at P. Second, we have $R \cap E = \{z_0 = 0, x_1(f_2x_0 + f_1x_1) = 0\}$, but $R \cap E$ is smooth. Hence, we can change the coordinate x_0 such that $f_2 = 0$ and $f_1 = 1$. This simplifies (3.1) as

$$
\begin{aligned}
(3.4) \qquad & x_0^2\big((c_0y_1^2 + y_0y_1)z_0^2 + e_0y_1z_0z_1\big) + \\
& + x_0x_1\big((b_1y_0y_1 + c_1y_1^2)z_0^2 + e_1y_1z_0z_1 + z_1^2\big) + \\
& + x_1^2\big((a_2y_0^2 + b_2y_0y_1 + c_2y_1^2)z_0^2 + (d_2y_0 + e_2y_1)z_0z_1\big) = 0.
\end{aligned}
$$

Recall that $S = \{y_1 = 0\} \subset Y$, so we can identify $S = \mathbb{P}^1 \times \mathbb{P}^1$ with coordinates $([x_0 : x_1], [z_0 : z_1])$. Using this identification, we see that $Z = \{x_1 = 0\} \subset S$, $C = \{z_1 = 0\} \subset S$, and

$$
T = \big\{a_2x_1z_0^2 + d_2x_1z_0z_1 + x_0z_1^2 = 0\big\} \subset S.
$$

that T is irreducible $\iff$ $a_2 \neq 0$. Further, if $a_2 = 0$, then $T = C + T'$ for $T' = \{d_2sx + ty = 0\}$, where $d_2 \neq 0$, since R_S is reduced. Thus, the cases $(\mathbb{A}_3)$ and $(\mathbb{D}_4)$ can be described as follows:

- $(\mathbb{A}_3)$ $a_2 \neq 0$,
- $(\mathbb{D}_4)$ $a_2 = 0$ and $d_2 \neq 0$.

We will exclude the remaining cases $(\mathbb{D}_4)$ and $(\mathbb{A}_3)$ in Sections 3.2 and 3.3, respectively.

3.2. **Exclusion of the case $(\mathbb{D}_4)$.** Let us continue the proof of Theorem 3.1 started in Section 3.1. Now, we assume that the surface R is given by (3.4) and we have $a_2 = 0$, i.e. we are in the case $(\mathbb{D}_4)$. In the chart $\mathbb{A}^3_{x,y,z} = \{x_0y_0z_0 \neq 0\}$ with coordinates $x = \frac{x_1}{x_0}$, $y = \frac{y_1}{y_0}$, $z = \frac{z_1}{z_0}$, we have $P = (0, 0, 0)$, and the surface R is given by the following equation:

$$
y + xz^2 + d_2x^2z + \big(b_1xy + e_0yz + b_2x^2y + e_1xyz + e_2x^2yz + c_0y^2 + c_1xy^2 + c_2x^2y^2\big) = 0,
$$

where $y + xz^2 + d_2x^2z$ is the smallest degree term for the weights $\mathrm{wt}(x) = 1$, $\mathrm{wt}(y) = 3$, $\mathrm{wt}(z) = 1$. Let $\lambda\colon W_0 \to Y$ be the corresponding weighted blow up of the point P with weights $(1, 3, 1)$, and let G be the λ-exceptional surface. Then $G \cong \mathbb{P}(1, 3, 1)$.

Let R_{W_0}, F_{W_0} and S_{W_0} be the proper transforms on Y of the surfaces R, S and F, respectively. Set $R_G = R_{W_0}|_G$, $\Delta_G = \frac{1}{2}R_G$ and $\Delta_{W_0} = \frac{1}{2}R_{W_0}$. Note that

$$
\big(K_{W_0} + \Delta_{W_0} + G\big)\big|_G \sim_{\mathbb{Q}} K_G + \Delta_G.
$$

Let us also consider (x, y, z) as coordinates on $G \cong \mathbb{P}(1, 3, 1)$ with $\mathrm{wt}(x) = 1$, $\mathrm{wt}(y) = 3$, $\mathrm{wt}(z) = 1$. Then $F_{W_0}|_G = \{x = 0\}$, $S_{W_0}|_G = \{y = 0\}$, and

$$
R_G = \big\{y + xz^2 + d_2x^2z = 0\big\} \subset R.
$$

Recall from the end of Section 3.1 that $d_2 \neq 0$. Since $\mathrm{ord}_G(R) = 3$, we have $A_{Y,\Delta_Y}(G) = \frac{7}{2}$. Then

$$
\delta_P(Y, \Delta_Y) \leqslant \frac{A_{Y,\Delta_Y}(G)}{S_L(G)} = \frac{7}{2S_L(G)},
$$

where

$$
S_L(G) = \frac{1}{L^3}\int\limits_0^\infty \mathrm{vol}\big(\lambda^*(L) - uG\big)du.
$$

Let us compute $S_L(G)$. To do this, note that Y is toric, and the blow up $\lambda\colon W_0 \to Y$ is also toric for the torus action on Y with an open orbit $\{x_0y_0z_0x_1y_1z_1 \neq 0\} \subset Y$, so the threefold W_0 is toric, and G is a torus invariant divisor. Let us present toric data for the threefolds Y and W_0.

Let Σ_Y be the simplicial fan in $\mathbb{R}^3$ defined by the following data:

- the list of primitive generators of rays of Σ_Y is

$$
v_1 = (1, 0, 0), v_2 = (0, 0, 1), v_3 = (0, 1, 0), v_4 = (0, 0, -1), v_5 = (0, -1, 1), v_6 = (-1, 0, 0);
$$

- the list of maximal cones of Σ_Y is

$$[1,2,3], [1,3,4], [1,4,5], [1,2,5], [2,3,6], [3,4,6], [4,5,6], [2,5,6],$$

where $[i,j,k]$ is the cone generated by the rays v_i, v_j, and v_k.

Then Y is defined by Σ_Y. Let Σ_{W_0} be the simplicial fan in $\mathbb{R}^3$ defined by the following data:

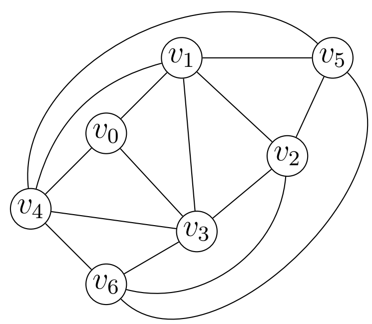
- the list of primitive generators of rays in Σ_{W_0} is

$$\begin{array}{llll} v_0 = (1,3,-1), & v_1 = (1,0,0), & v_2 = (0,0,1), & v_3 = (0,1,0), \\ v_4 = (0,0,-1), & v_5 = (0,-1,1), & v_6 = (-1,0,0); & \end{array}$$

- the list of maximal cones in Σ_{W_0} is

$$[0,1,3], [0,1,4], [0,3,4], [1,2,3], [1,2,5], [1,4,5], [2,3,6], [2,5,6], [3,4,6], [4,5,6].$$

Then the toric threefold W_0 is given by the fan Σ_{W_0}, which can be diagramed as follows:



Let us compute $S_L(G)$. Let P_L be the convex polytope in the dual space of $\mathbb{R}^3$ associated to L. Then, since L corresponds to the lattice point $(1,2,1)$, we have

$$P_L = \big\{x_1 \geqslant -1, x_3 \geqslant -1, x_2 \geqslant -2, -x_3 \geqslant 0, -x_2 + x_3 \geqslant 0, -x_1 \geqslant 0\big\}.$$

Thus, since G corresponds to $v_0 = (1,3,-1)$, it follows from [5, Corollary 7.7] that

$$S_L(G) = -\min_{v \in P_L \cap \mathbb{Z}^3} v \cdot (1,3,-1) + \frac{3!}{L^3} \iiint_{P_L} (x_1,x_2,x_3) \cdot (1,3,-1) dx_1 dx_2 dx_3 = \frac{59}{18}.$$

where $\cdot$ is the standard inner product in $\mathbb{R}^3$. Consequently, we obtain $\frac{A_{Y,\Delta_Y}(G)}{S_L(G)} = \frac{63}{58}$

Now, let us exclude the case $(\mathbb{D}_4)$ using the results obtained in [1, 3, 16]. To do this, we must find the Zariski decomposition of the divisor $\lambda^*(L) - uG$ for every $u \in \mathbb{R}_{\geqslant 0}$. First, let us compute intersections of torus invariant divisors in W_0. Let T_i be the torus invariant divisor corresponding to the ray v_i. Then $T_0 = G$, and it follows from [10, §6.4] that

$$T_iT_jT_k = \begin{cases} \dfrac{1}{|[i,j,k]|} & \text{if } [i,j,k] \text{ belongs to the list of maximal cones in } \Sigma_{W_0} \\ 0 \quad \text{otherwise,} & \end{cases} \tag{3.5}$$

where $|[i,j,k]|$ stands for the absolute value of the determinant of the 3×3 matrix given by v_i, v_j, v_k. This gives $T_0T_1T_4 = \frac{1}{3}$ and

$$T_0T_1T_3 = T_0T_3T_4 = T_1T_2T_3 = T_1T_4T_5 = T_1T_2T_5 = T_2T_3T_6 = T_3T_4T_6 = T_4T_5T_6 = T_2T_5T_6 = 1,$$

while all other $T_iT_jT_k = 0$ with distinct indices i, j, k. The characters χ_1, χ_2, χ_3 corresponding to the lattice points $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ in the dual lattice generate the following relations among the torus invariant divisors:

$$
\begin{aligned}
0 &\sim \mathrm{div}(\chi_1) = T_0 + T_1 - T_6,\\
0 &\sim \mathrm{div}(\chi_2) = 3T_0 + T_3 - T_5,\\
0 &\sim \mathrm{div}(\chi_3) = -T_0 + T_2 - T_4 + T_5.
\end{aligned} \tag{3.6}
$$

Now, using these relations, we can determine the intersection numbers $T_i^2T_j$ for $i \neq j$. For instance, we have $T_3^2T_6 = (T_5 - 3T_0)T_3T_6 = 0$ and $T_2^2T_6 = (T_0 + T_4 - T_5)T_2T_6 = -1$.

For all possible indices $i \neq j$, let us denote by T_iT_j the torus invariant curve that is given by the intersection of the divisors T_i and T_j provided that $T_i \cap T_j \neq \varnothing$. Note that

$$T_i \cap T_j \neq \varnothing \iff \text{the 2-dimensional cone generated by } v_i \text{ and } v_j \text{ belongs to the fan } \Sigma_{W_0}.$$

If $T_i \cap T_j \neq \varnothing$, then T_iT_j is not necessarily reduced, but its support coincides with the torus invariant curve that corresponds to the 2-dimensional cone generated by the rays v_i and v_j, which we will denote by $\lfloor T_iT_j \rfloor$.

Let u be a non-negative real number. For simplicity, set $L_u = \lambda^*(L) - uT_0$. Then

$$L_u = (7-u)T_0 + T_1 + T_2 + 2T_3.$$

Now, we can compute the intersection of the $\mathbb{R}$-divisor L_u with each torus invariant curve in W_0. For instance we have

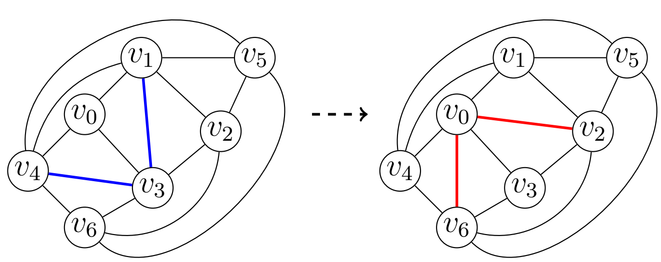
$$
\begin{aligned}
L_uT_0T_1 &= \left((7-u)T_0 + T_1 + T_2 + 2T_3\right)T_0T_1 = (7-u)T_0^2T_1 + T_0T_1^2 + T_0T_1T_2 + 2T_0T_1T_3 = \frac{u}{3},\\
L_uT_0T_3 &= \left((7-u)T_0 + T_1 + T_2 + 2T_3\right)T_0T_3 = (7-u)T_0^2T_3 + T_0T_1T_3 + T_0T_2T_3 + 2T_0T_3^2 = u,\\
L_uT_0T_4 &= \left((7-u)T_0 + T_1 + T_2 + 2T_3\right)T_0T_4 = (7-u)T_0^2T_4 + T_0T_1T_4 + T_0T_2T_4 + 2T_0T_3T_4 = \frac{u}{3}.
\end{aligned}
$$

Similarly, we compute

$$
\begin{aligned}
&L_uT_1T_2 = 1, \quad L_uT_1T_3 = 1-u, \quad L_uT_1T_4 = \frac{6-u}{3}, \quad L_uT_1T_5 = 1, \quad L_uT_2T_3 = 1, \quad L_uT_2T_5 = 1,\\
&L_uT_2T_6 = 1, \quad L_uT_3T_4 = 1-u, \quad L_uT_3T_6 = 1, \quad L_uT_4T_5 = 1, \quad L_uT_4T_6 = 2, \quad L_uT_5T_6 = 1.
\end{aligned}
$$

Therefore, we see that L_u is nef for $0 \leqslant u \leqslant 1$.

To find Zariski decomposition of the divisor L_u for small $u > 1$, we must perform a small birational map $W_0 \dashrightarrow W_1$ along the two torus invariant curves $\lfloor T_1T_3 \rfloor$ and $\lfloor T_3T_4 \rfloor$, because these are the only curves that intersect L_u negatively for small $u > 1$. The corresponding change of fans can be diagramed as follows:



The toric 3-fold W_1 is defined by the simplicial fan Σ_{W_1} in $\mathbb{R}^3$ determined by the following data:

- the list of primitive generators of rays of Σ_{W_1} is

$$v_0 = (1,3,-1), \quad v_1 = (1,0,0), \quad v_2 = (0,0,1), \quad v_3 = (0,1,0),$$
$$v_4 = (0,0,-1), \quad v_5 = (0,-1,1), \quad v_6 = (-1,0,0);$$

- the list of maximal cones of Σ_{W_1} is

$$[0,1,2],[0,2,3],[0,3,6],[0,4,6],[0,1,4],[1,4,5],[1,2,5],[2,3,6],[4,5,6],[2,5,6].$$

On the 3-fold W_1, we use the same notations for the transformed L_u, the torus invariant divisors and curves as on W_0. Since the formula (3.5) is valid on W_1, we get

$$T_0T_1T_2 = T_0T_1T_4 = T_0T_4T_6 = \frac{1}{3},$$
$$T_0T_2T_3 = T_0T_3T_6 = T_1T_4T_5 = T_1T_2T_5 = T_2T_3T_6 = T_4T_5T_6 = T_2T_5T_6 = 1,$$

and all other $T_iT_jT_k = 0$ with distinct indices i,j,k. Since we have the same list of primitive generators of rays as on Σ_{W_0}, the relations (3.6) are valid on W_1. This gives

$$L_uT_0T_1 = \frac{1}{3}, \quad L_uT_0T_2 = \frac{u-1}{3}, \quad L_uT_0T_3 = 2-u, \quad L_uT_0T_4 = \frac{1}{3}, \quad L_uT_0T_6 = \frac{u-1}{3},$$
$$L_uT_1T_2 = \frac{4-u}{3}, \quad L_uT_1T_4 = \frac{6-u}{3}, \quad L_uT_1T_5 = 1, \quad L_uT_2T_3 = 2-u, \quad L_uT_2T_5 = 1,$$
$$L_uT_2T_6 = 1, \quad L_uT_3T_6 = 2-u, \quad L_uT_4T_5 = 1, \quad L_uT_4T_6 = \frac{7-u}{3}, \quad L_uT_5T_6 = 1.$$

Therefore, the divisor L_u is nef for $1 \leqslant u \leqslant 2$.

The unique torus invariant surface T_3 that contains $\lfloor T_0T_3 \rfloor$, $\lfloor T_2T_3 \rfloor$, $\lfloor T_3T_6 \rfloor$ is of Picard rank 1. Since $(L_u - aT_3)T_0T_3 = 2-u+3a$ for any non-negative real number a, the Nakayama–Zariski decomposition $L_u = P(u) + N(u)$ for $u > 2$ on the 3-fold W_1 must satisfy

$$N(u) \geqslant \frac{u-2}{3}T_3,$$

where $P(u)$ is the positive part of the decomposition, and $N(u)$ is the negative part. Set

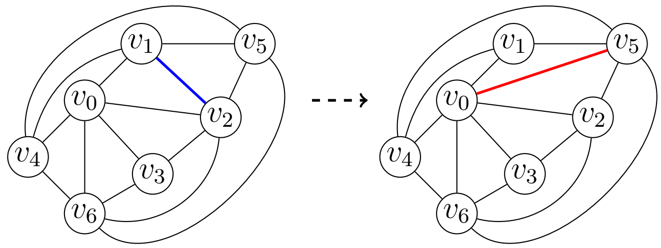
$$P_u^1 = L_u - \frac{u-2}{3}T_3 = (7-u)T_0 + T_1 + T_2 + \frac{8-u}{3}T_3.$$

Then

$$P_u^1T_0T_1 = \frac{1}{3}, \quad P_u^1T_0T_2 = \frac{1}{3}, \quad P_u^1T_0T_3 = 0, \quad P_u^1T_0T_4 = \frac{1}{3}, \quad P_u^1T_0T_6 = \frac{1}{3},$$
$$P_u^1T_1T_2 = \frac{4-u}{3}, \quad P_u^1T_1T_4 = \frac{6-u}{3}, \quad P_u^1T_1T_5 = 1, \quad P_u^1T_2T_3 = 0, \quad P_u^1T_2T_5 = 1,$$
$$P_u^1T_2T_6 = \frac{5-u}{3}, \quad P_u^1T_3T_6 = 0, \quad P_u^1T_4T_5 = 1, \quad P_u^1T_4T_6 = \frac{7-u}{3}, \quad P_u^1T_5T_6 = 1.$$

Therefore, if $2 \leqslant u \leqslant 4$, then P_u^1 is nef, and hence $L_u = P_u^1 + \frac{u-2}{3}T_3$ is the Zariski decomposition, i.e. P_u^1 is the positive part, and $\frac{u-2}{3}T_3$ is the negative part.

For small enough $u > 4$, the curve $\lfloor T_1T_2 \rfloor$ is the only curve in W_1 that intersects P_u^1 negatively. Let $W_1 \dashrightarrow W_2$ be the small birational map of this curve. Then the change of fans can be diagramed as follows:



The toric 3-fold W_2 is defined by the simplicial fan Σ_{W_2} in $\mathbb{R}^3$ determined by the following data:

- the list of primitive generators of rays of Σ_{W_2} is

$$v_0 = (1,3,-1), \quad v_1 = (1,0,0), \quad v_2 = (0,0,1), \quad v_3 = (0,1,0),$$
$$v_4 = (0,0,-1), \quad v_5 = (0,-1,1), \quad v_6 = (-1,0,0);$$

- the list of maximal cones of Σ_{W_2} is

$$[0,1,5], [0,2,5], [0,2,3], [0,3,6], [0,4,6], [0,1,4], [1,4,5], [2,3,6], [4,5,6], [2,5,6].$$

As before, we keep the same notations for the transformed L_u and P^1_u, the torus invariant divisors and curves on W_2. It follows from (3.5) that

$$T_0T_4T_6 = T_0T_1T_4 = \frac{1}{3},$$
$$T_0T_1T_5 = \frac{1}{2},$$
$$T_0T_2T_5 = T_0T_2T_3 = T_0T_3T_6 = T_1T_4T_5 = T_2T_3T_6 = T_4T_5T_6 = T_2T_5T_6 = 1,$$

and all other $T_iT_jT_k = 0$ with distinct i, j, k. We have

$$P^1_u = (7-u)T_0 + T_1 + T_2 + \frac{8-u}{3}T_3,$$

and we compute

$$P^1_uT_0T_1 = \frac{6-u}{6}, \quad P^1_uT_0T_2 = \frac{5-u}{3}, \quad P^1_uT_0T_3 = 0, \quad P^1_uT_0T_4 = \frac{1}{3}, \quad P^1_uT_0T_5 = \frac{u-4}{2},$$
$$P^1_uT_0T_6 = \frac{1}{3}, \quad P^1_uT_1T_4 = \frac{6-u}{3}, \quad P^1_uT_1T_5 = \frac{6-u}{2}, \quad P^1_uT_2T_3 = 0, \quad P^1_uT_2T_5 = 5-u,$$
$$P^1_uT_2T_6 = \frac{5-u}{3}, \quad P^1_uT_3T_6 = 0, \quad P^1_uT_4T_5 = 1, \quad P^1_uT_4T_6 = \frac{7-u}{3}, \quad P^1_uT_5T_6 = 1.$$

Hence, if $u \in [4,5]$, then P^1_u is nef on W_2, so $L_u = P^1_u + \frac{u-2}{3}T_3$ is the required Zariski decomposition.

Observe that T_2 is the unique torus invariant surface that contains the curves T_0T_2, T_2T_5, T_2T_6, and T_0T_2 is nef on T_2, since $(T_0|_{T_2})^2 = T_0^2T_2 = 0$. For non-negative real numbers a and b, we have

$$(P^1_u - aT_2 - bT_3)T_0T_2 = \frac{5-u}{3} + a - b,$$
$$(P^1_u - aT_2 - bT_3)T_0T_3 = -a + 3b.$$

These intersections are non-negative for $a \geqslant \frac{u-5}{2}$ and $b \geqslant \frac{u-5}{6}$. Therefore, the Nakayama-Zariski decomposition $L_u = P(u) + N(u)$ on W_2 satisfies

$$N(u) \geqslant \frac{u-2}{3}T_3 + \left(\frac{u-5}{2}T_2 + \frac{u-5}{6}T_3\right) = \frac{u-5}{2}T_2 + \frac{u-3}{2}T_3,$$

where $P(u)$ stands for the positive part, and $N(u)$ stands for the negative part. Put

$$P^2_u = P^1_u - \left(\frac{u-5}{2}T_2 + \frac{u-5}{6}T_3\right).$$

Then

$$\begin{array}{lllll}
P^2_u T_0T_1 = \frac{6-u}{6}, & P^2_u T_0T_2 = 0, & P^2_u T_0T_3 = 0, & P^2_u T_0T_4 = \frac{1}{3}, & P^2_u T_0T_5 = \frac{1}{2}, \\
P^2_u T_0T_6 = \frac{7-u}{6}, & P^2_u T_1T_4 = \frac{6-u}{3}, & P^2_u T_1T_5 = \frac{6-u}{2}, & P^2_u T_2T_3 = 0, & P^2_u T_2T_5 = 0, \\
P^2_u T_2T_6 = 0, & P^2_u T_3T_6 = 0, & P^2_u T_4T_5 = 1, & P^2_u T_4T_6 = \frac{7-u}{3}, & P^2_u T_5T_6 = \frac{7-u}{2}.
\end{array}$$

Hence, the divisor P^2_u is nef for $u \in [5,6]$, which implies that $P(u) = P^2_u$ and

$$N(u) = \frac{u-5}{2}T_2 + \frac{u-3}{2}T_3.$$

This gives the Zariski decomposition of the divisor L_u on the 3-fold W_2 for $u \in [5,6]$.

The surface T_1 is the unique torus invariant surface that contains the curves T_0T_1, T_1T_4, T_1T_5, it has Picard rank 1, and it is disjoint from T_2 and T_3. But

$$(P^2_u - aT_1)T_0T_1 = \frac{6-u}{6} + \frac{a}{6}.$$

Therefore, the Nakayama-Zariski decomposition $L_u = P(u) + N(u)$ on W_2 for $u > 6$ satisfies

$$N(u) \geqslant (u-6)T_1 + \frac{u-5}{2}T_2 + \frac{u-3}{2}T_3,$$

where $P(u)$ is the positive part, and $N(u)$ is the negative part. Set $P^3_u = P^2_u - (u-6)T_1$. Then

$$\begin{array}{lllll}
P^3_u T_0T_1 = 0, & P^3_u T_0T_2 = 0, & P^3_u T_0T_3 = 0, & P^3_u T_0T_4 = \frac{7-u}{3}, & P^3_u T_0T_5 = \frac{7-u}{2}, \\
P^3_u T_0T_6 = \frac{7-u}{6}, & P^3_u T_1T_4 = 0, & P^3_u T_1T_5 = 0, & P^3_u T_2T_3 = 0, & P^3_u T_2T_5 = 0, \\
P^3_u T_2T_6 = 0, & P^3_u T_3T_6 = 0, & P^3_u T_4T_5 = 7-u, & P^3_u T_4T_6 = \frac{7-u}{3}, & P^3_u T_5T_6 = \frac{7-u}{2}.
\end{array}$$

Then $P(u) = P^3_u$ is the positive part of the Zariski decomposition of L_u on W_2 for $u \in [6,7]$, and the negative part is

$$N(u) = (u-6)T_1 + \frac{u-5}{2}T_2 + \frac{u-3}{2}T_3.$$

If $u > 7$, then L_u is not pseudoeffective.

Remark 3.9. The toric varieties W_0, W_1, W_2 are projective. Indeed, the variety W_0 is obtained by taking a weighted blowup of a projective variety. On W_1, the transformed $L_{\frac{3}{2}}$ is an ample divisor. On W_2, we can obtain an ample divisor from $P^1_{\frac{9}{2}} + \frac{1}{m}T_2$ by taking sufficiently large integer m.

To apply [1, 3, 16], we must consider a common partial resolution of the 3-folds W_0, W_1, W_2. Namely, let $\widetilde{W}$ be the toric 3-fold defined by the simplicial fan $\Sigma_{\widetilde{W}}$ in $\mathbb{R}^3$ given by

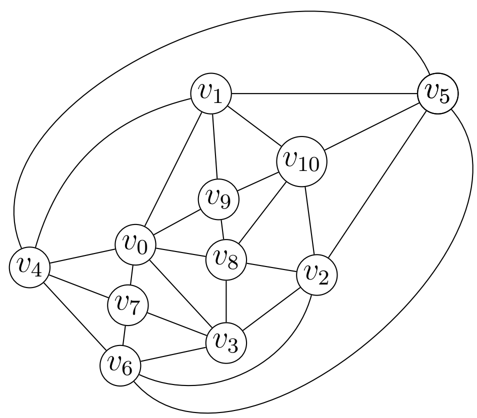
- the list of primitive generators of rays of $\Sigma_{\widetilde{W}}$ is

$v_0 = (1,3,-1)$, $v_1 = (1,0,0)$, $v_2 = (0,0,1)$, $v_3 = (0,1,0)$, $v_4 = (0,0,-1)$, $v_5 = (0,-1,1)$, $v_6 = (-1,0,0)$, $v_7 = (0,3,-1)$, $v_8 = (1,3,0)$, $v_9 = (1,2,0)$, $v_{10} = (1,0,2)$;

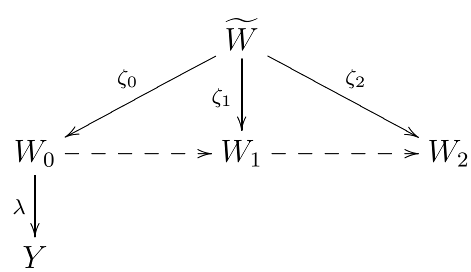
- the list of maximal cones of $\Sigma_{\widetilde{W}}$ is

$$
\begin{array}{lllllllll}
[0,1,4], & [0,1,9], & [0,3,7], & [0,3,8], & [0,4,7], & [0,8,9], & [1,4,5], & [1,5,10], & [1,9,10], \\
[2,3,6], & [2,3,8], & [2,5,6], & [2,5,10], & [2,8,10], & [3,6,7], & [4,5,6], & [4,6,7], & [8,9,10].
\end{array}
$$

The fan $\Sigma_{\widetilde{W}}$ can be diagramed as follows:



Then there exists the following commutative diagram:



where ζ_0, ζ_1 and ζ_2 are toric birational morphisms.

Let us denote by $\widetilde{T}_i$ the torus invariant divisor on $\widetilde{W}$ corresponding to the ray v_i in the fan $\Sigma_{\widetilde{W}}$. Then the formula (3.5) implies that

$$
\begin{aligned}
&\widetilde{T}_1\widetilde{T}_9\widetilde{T}_{10} = \frac{1}{4},\\
&\widetilde{T}_0\widetilde{T}_1\widetilde{T}_4 = \widetilde{T}_0\widetilde{T}_4\widetilde{T}_7 = \widetilde{T}_2\widetilde{T}_8\widetilde{T}_{10} = \widetilde{T}_4\widetilde{T}_6\widetilde{T}_7 = \frac{1}{3},\\
&\widetilde{T}_0\widetilde{T}_1\widetilde{T}_9 = \widetilde{T}_1\widetilde{T}_5\widetilde{T}_{10} = \widetilde{T}_8\widetilde{T}_9\widetilde{T}_{10} = \frac{1}{2},\\
&\widetilde{T}_0\widetilde{T}_3\widetilde{T}_7 = \widetilde{T}_0\widetilde{T}_3\widetilde{T}_8 = \widetilde{T}_0\widetilde{T}_8\widetilde{T}_9 = \widetilde{T}_1\widetilde{T}_4\widetilde{T}_5 = \widetilde{T}_2\widetilde{T}_3\widetilde{T}_6 = 1,\\
&\widetilde{T}_2\widetilde{T}_3\widetilde{T}_8 = \widetilde{T}_2\widetilde{T}_5\widetilde{T}_6 = \widetilde{T}_2\widetilde{T}_5\widetilde{T}_{10} = \widetilde{T}_3\widetilde{T}_6\widetilde{T}_7 = \widetilde{T}_4\widetilde{T}_5\widetilde{T}_6 = 1,
\end{aligned}
\tag{3.7}
$$

and other $\widetilde{T}_i\widetilde{T}_j\widetilde{T}_k$ with distinct indices i,j,k are 0. Further, the characters χ_1, χ_2, χ_3 corresponding to the lattice points $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ in the dual lattice yield the following relations:

$$
\begin{aligned}
0 &\sim \mathrm{div}(\chi_1) = \widetilde{T}_0 + \widetilde{T}_1 - \widetilde{T}_6 + \widetilde{T}_8 + \widetilde{T}_9 + \widetilde{T}_{10},\\
0 &\sim \mathrm{div}(\chi_2) = 3\widetilde{T}_0 + \widetilde{T}_3 - \widetilde{T}_5 + 3\widetilde{T}_7 + 3\widetilde{T}_8 + 2\widetilde{T}_9,\\
0 &\sim \mathrm{div}(\chi_3) = -\widetilde{T}_0 + \widetilde{T}_2 - \widetilde{T}_4 + \widetilde{T}_5 - \widetilde{T}_7 + 2\widetilde{T}_{10}.
\end{aligned}
\tag{3.8}
$$

Moreover, we have

$$
\begin{aligned}
&\zeta_0^*(T_0) = \widetilde{T}_0, && \zeta_0^*(T_1) = \widetilde{T}_1 + \widetilde{T}_8 + \widetilde{T}_9 + \widetilde{T}_{10},\\
&\zeta_0^*(T_2) = \widetilde{T}_2 + 2\widetilde{T}_{10}, && \zeta_0^*(T_3) = \widetilde{T}_3 + 3\widetilde{T}_7 + 3\widetilde{T}_8 + 2\widetilde{T}_9,\\
&\zeta_1^*(T_0) = \widetilde{T}_0 + \widetilde{T}_7 + \widetilde{T}_8 + \tfrac{2}{3}\widetilde{T}_9, && \zeta_1^*(T_1) = \widetilde{T}_1 + \tfrac{1}{3}\widetilde{T}_9 + \widetilde{T}_{10},\\
&\zeta_1^*(T_2) = \widetilde{T}_2 + \widetilde{T}_8 + \tfrac{2}{3}\widetilde{T}_9 + 2\widetilde{T}_{10}, && \zeta_1^*(T_3) = \widetilde{T}_3,\\
&\zeta_2^*(T_0) = \widetilde{T}_0 + \widetilde{T}_7 + \widetilde{T}_8 + \widetilde{T}_9 + \widetilde{T}_{10}, && \zeta_2^*(T_1) = \widetilde{T}_1,\\
&\zeta_2^*(T_2) = \widetilde{T}_2 + \widetilde{T}_8, && \zeta_2^*(T_3) = \widetilde{T}_3.
\end{aligned}
$$

Let us briefly explain how we get these expressions. For instance, the divisor T_0 on W_0 does not contain centers of ζ_0-exceptional surfaces, so $\zeta_0^*(T_0) = \widetilde{T}_0$. Similarly, the divisor T_0 on W_2 contains centers of the following ζ_2-exceptional divisors: $\widetilde{T}_7$, $\widetilde{T}_8$, $\widetilde{T}_9$, $\widetilde{T}_{10}$, which implies that

$$\zeta_2^*(T_0) = \widetilde{T}_0 + a_7\widetilde{T}_7 + a_8\widetilde{T}_8 + a_9\widetilde{T}_9 + a_{10}\widetilde{T}_{10}$$

for some positive rational numbers a_7, a_8, a_9, a_{10}. Then we obtain

$$
\begin{aligned}
0 &= \Big(\widetilde{T}_0 + a_7\widetilde{T}_7 + a_8\widetilde{T}_8 + a_9\widetilde{T}_9 + a_{10}\widetilde{T}_{10}\Big)\widetilde{T}_3\widetilde{T}_7 = 1 - a_7,\\
0 &= \Big(\widetilde{T}_0 + a_7\widetilde{T}_7 + a_8\widetilde{T}_8 + a_9\widetilde{T}_9 + a_{10}\widetilde{T}_{10}\Big)\widetilde{T}_3\widetilde{T}_8 = 1 - a_8,\\
0 &= \Big(\widetilde{T}_0 + a_7\widetilde{T}_7 + a_8\widetilde{T}_8 + a_9\widetilde{T}_9 + a_{10}\widetilde{T}_{10}\Big)\widetilde{T}_8\widetilde{T}_{10} = -\frac{1}{3}a_8 + \frac{1}{2}a_9 - \frac{1}{6}a_{10},\\
0 &= \Big(\widetilde{T}_0 + a_7\widetilde{T}_7 + a_8\widetilde{T}_8 + a_9\widetilde{T}_9 + a_{10}\widetilde{T}_{10}\Big)\widetilde{T}_1\widetilde{T}_{10} = \frac{1}{4}a_8 - \frac{1}{4}a_{10},
\end{aligned}
$$

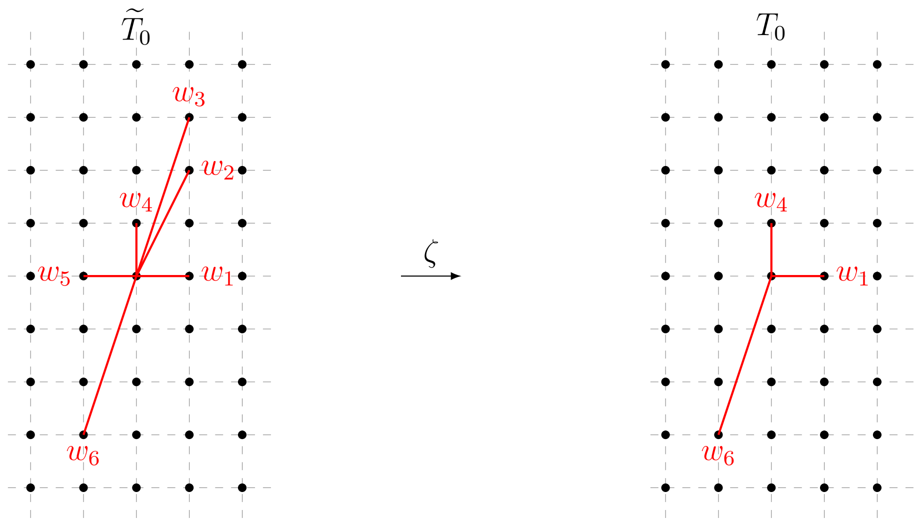
which gives $a_7 = a_8 = a_9 = a_{10} = 1$. Here, all intersections are derived from (3.7) and (3.8).

For every $u \in [0,7]$, the Zariski decomposition of the divisor $\zeta_0^*(L_u)$ exists on the 3-fold $\widetilde{W}$. Let $P_{\widetilde{W}}(u)$ and $N_{\widetilde{W}}(u)$ be its positive and negative parts, respectively. Then their expressions as linear combinations of the torus invariant divisors on $\widetilde{W}$ are given in Table 1.

We now consider the toric surface $\widetilde{T}_0$. Its fan is the image of the fan $\Sigma_{\widetilde{W}}$ under the quotient lattice homomorphism $\mathbb{Z}^3 \to \mathbb{Z}^3/\mathbb{Z}v_0 \cong \mathbb{Z}^2$. We may assume that $v_1 \mapsto w_1 = (1,0)$ and $v_3 \mapsto w_4 = (0,1)$, which determines the quotient homomorphism. Then the list of primitive generators of the rays in the fan consists of

$$w_1 = (1,0), w_2 = (1,2), w_3 = (1,3), w_4 = (0,1), w_5 = (-1,0), w_6 = (-1,-3).$$

Let ζ be the restriction morphism $\zeta_0|_{\widetilde{T}_0}\colon \widetilde{T}_0 \to T_0$. Then ζ contracts the torus invariant curves defined by w_5, w_3, w_2, since ζ_0 contracts $\lfloor \widetilde{T}_0\widetilde{T}_7 \rfloor$, $\lfloor \widetilde{T}_0\widetilde{T}_8 \rfloor$, $\lfloor \widetilde{T}_0\widetilde{T}_9 \rfloor$. This can be illustrated as follows.



Let $\alpha_1, \dots, \alpha_6$ be the torus invariant curves in $\widetilde{T}_0$ defined by the rays $w_1, \dots, w_6$, respectively. Set $\overline{\alpha}_1 = \zeta(\alpha_1)$, $\overline{\alpha}_4 = \zeta(\alpha_4)$, $\overline{\alpha}_6 = \zeta(\alpha_6)$. Note that $\alpha_1+\alpha_2+\alpha_3 = \alpha_5+\alpha_6$ and $2\alpha_2+3\alpha_3+\alpha_4 = 3\alpha_6$. With these relations, [10, §6.4] yields the following intersection matrix:

$$A := (\alpha_i\alpha_j) = \begin{pmatrix} -\frac{1}{6} & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{3} \\ \frac{1}{2} & -\frac{3}{2} & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 \end{pmatrix}.$$

It follows from [10, Lemma 12.5.2] that

$$\widetilde{T}_1\big|_{\widetilde{T}_0} = \alpha_1, \widetilde{T}_3\big|_{\widetilde{T}_0} = \alpha_4, \widetilde{T}_4\big|_{\widetilde{T}_0} = \alpha_6, \widetilde{T}_7\big|_{\widetilde{T}_0} = \alpha_5, \widetilde{T}_8\big|_{\widetilde{T}_0} = \alpha_3, \widetilde{T}_9\big|_{\widetilde{T}_0} = \alpha_2.$$

Moreover, (3.8) implies

$$\widetilde{T}_0\big|_{\widetilde{T}_0} = -\big(\widetilde{T}_1 - \widetilde{T}_6 + \widetilde{T}_8 + \widetilde{T}_9 + \widetilde{T}_{10}\big)\big|_{\widetilde{T}_0} = -\big(\alpha_1 + \alpha_2 + \alpha_3\big).$$

Set $\widetilde{P}(u) = P_{\widetilde{W}}(u)|_{\widetilde{T}_0}$ and $\widetilde{N}(u) = N_{\widetilde{W}}(u)|_{\widetilde{T}_0}$. Then we can express $\widetilde{P}(u)$ and $\widetilde{N}(u)$ as linear combinations of the curves α_1, α_2, α_3, α_4, α_5, α_6. These expressions are presented in Table 2.

We are ready to apply [1, 3, 16] to estimate $\delta_P(Y, \Delta_Y)$ from below. Let Q be a point in $G = T_0$, let C be a smooth curve in G such that $Q \in C \not\subset \Delta_G$, and let $\widetilde{C}$ be its proper transform on $\widetilde{T}_0$. Then ζ induces an isomorphism $\widetilde{C} \cong C$. For every $u \in [0, 7]$, let

$$t(u) = \inf\Big\{v \in \mathbb{R}_{\geqslant 0} \;\big|\; \widetilde{P}(u) - vC \text{ is pseudoeffective}\Big\}.$$

For every $v \in [0, t(u)]$, let $P(u, v)$ be the positive part of the Zariski decomposition of $\widetilde{P}(u) - vC$, and let $N(u, v)$ be its negative part. Set

$$S_L\big(W^G_{\bullet,\bullet}; C\big) = \frac{3}{L^3}\int\limits_0^7 \big(\widetilde{P}(u)\big)^2 \mathrm{ord}_C\big(\widetilde{N}(u)\big)du + \frac{3}{L^3}\int\limits_0^7\int\limits_0^{t(u)} \big(P(u,v)\big)^2 dv du.$$

Now, we write $\zeta^*(C) = \widetilde{C} + \Sigma$ for an effective $\mathbb{R}$-divisor Σ on the surface $\widetilde{T}_0$. For every $u \in [0, 7]$, write $\widetilde{N}(u) = d(u)C + N'(u)$, where $d(u) = \mathrm{ord}_C(\widetilde{N}(u))$, and $N'(u)$ is an effective $\mathbb{R}$-divisor on $\widetilde{T}_0$.

Now, as in [15, Definition 4.16], we set

$$F_Q\big(W^{G,C}_{\bullet,\bullet,\bullet}\big)=\frac{6}{L^3}\int\limits_0^7\int\limits_0^{t(u)}\big(P(u,v)\cdot\widetilde{C}\big)\cdot\mathrm{ord}_Q\Big(\big(N'(u)+N(u,v)-(v+d(u))\Sigma\big)\big|_{\widetilde{C}}\Big)dvdu,$$

where we consider Q as a point in $\widetilde{C}$ using the isomorphism $\widetilde{C}\cong C$ induced by ζ. Finally, we set

$$S\big(W^{G,C}_{\bullet,\bullet,\bullet};Q\big)=\frac{3}{L^3}\int\limits_0^7\int\limits_0^{t(u)}\big(P(u,v)\cdot\widetilde{C}\big)^2dvdu+F_Q\big(W^{G,C}_{\bullet,\bullet,\bullet}\big).$$

We have $(K_G+C+\Delta_G)|_C\sim_{\mathbb{R}}K_C+\Delta_C$ for an effective divisor Δ_C known as the different [23], which can be computed locally near any point in C. Using [16, Corollary 4.18], we obtain

$$\delta_P(Y,\Delta_Y)\geqslant\min\left\{\frac{A_{Y,\Delta_Y}(G)}{S_L(G)},\inf_{Q\in G}\min\left\{\frac{A_{G,\Delta_G}(C)}{S_L(W^G_{\bullet,\bullet};C)},\frac{A_{C,\Delta_C}(Q)}{S(W^{G,C}_{\bullet,\bullet,\bullet};Q)}\right\}\right\},$$

where $A_{G,\Delta_G}(C)=1$, because $C\not\subset\Delta_G$ by assumption. On the other hand, we assumed that there exists a prime divisor $\mathbf{F}$ over Y such that $\beta_{Y,\Delta_Y}(\mathbf{F})\leqslant 0$. Moreover, we proved that $C_Y(\mathbf{F})=P$, so

$$1\geqslant\frac{A_{Y,\Delta_Y}(\mathbf{F})}{S_L(\mathbf{F})}\geqslant\delta_P(Y,\Delta_Y)\geqslant\min\left\{\frac{A_{Y,\Delta_Y}(G)}{S_L(G)},\inf_{Q\in G}\min\left\{\frac{A_{G,\Delta_G}(C)}{S_L(W^G_{\bullet,\bullet};C)},\frac{A_{C,\Delta_C}(Q)}{S(W^{G,C}_{\bullet,\bullet,\bullet};Q)}\right\}\right\}.$$

Therefore, since $\frac{A_{Y,\Delta_Y}(G)}{S_L(G)}=\frac{63}{58}$, it follows from [16, Corollary 4.18] and [1, Theorem 3.3] that

$$\inf_{Q\in G}\min\left\{\frac{A_{G,\Delta_G}(C)}{S_L(W^G_{\bullet,\bullet};C)},\frac{A_{C,\Delta_C}(Q)}{S(W^{G,C}_{\bullet,\bullet,\bullet};Q)}\right\}<1.$$

Therefore, to exclude the case $(\mathbb{D}_4)$, it is enough to show that for every point $Q\in G$, there exists a smooth irreducible curve $C\subset G$ such that $Q\in C\not\subset\Delta_G$ and

$$S_L\big(W^G_{\bullet,\bullet};C\big)\leqslant 1\leqslant\frac{A_{C,\Delta_C}(Q)}{S(W^{G,C}_{\bullet,\bullet,\bullet};Q)}\tag{3.9}$$

This is what we will do in the rest of this section.

Let Q be a point in $G=T_0\cong\mathbb{P}(1,3,1)$. Recall that $\overline{\alpha}_1$, $\overline{\alpha}_4$, $\overline{\alpha}_6$ are all torus invariant curves in G. Let $Q_{14}=\overline{\alpha}_1\cap\overline{\alpha}_4$, $Q_{16}=\overline{\alpha}_1\cap\overline{\alpha}_6$, $Q_{46}=\overline{\alpha}_4\cap\overline{\alpha}_6$, where Q_{16} is the singular point of the surface G. Recall that R_G meets the curve $\overline{\alpha}_4$ transversally at three distinct points including Q_{14} and Q_{46}. Let us denote by Q_4 the point in $R_G\cap\overline{\alpha}_4$ that is different from Q_{14} and Q_{46}.

Now, let us choose the curve C. If $Q\in\overline{\alpha}_1\cup\overline{\alpha}_4\cup\overline{\alpha}_6$, we choose C as follows:

- if $Q\in\overline{\alpha}_1$, $Q\neq Q_{14}$, $Q\neq Q_{16}$, we let $C=\overline{\alpha}_1$,
- if $Q\in\overline{\alpha}_4$, $Q\neq Q_{14}$, $Q\neq Q_{46}$, we let $C=\overline{\alpha}_4$,
- if $Q\in\overline{\alpha}_6$, $Q\neq Q_{16}$, $Q\neq Q_{46}$, we let $C=\overline{\alpha}_6$,
- if $Q=Q_{14}$, we let $C=\overline{\alpha}_1$ or $C=\overline{\alpha}_4$,
- if $Q=Q_{16}$, we let $C=\overline{\alpha}_1$ or $C=\overline{\alpha}_6$,
- if $Q=Q_{46}$, we let $C=\overline{\alpha}_4$ or $C=\overline{\alpha}_6$.

Similarly, if $Q\notin\overline{\alpha}_1\cup\overline{\alpha}_4\cup\overline{\alpha}_6$, there exists a unique curve $\overline{\alpha}_0\in|\mathcal{O}_G(1)|$ such that $\overline{\alpha}_0$ contains Q. In this case, we let $C=\overline{\alpha}_0$, and we let α_0 be the proper transform of the curve $\overline{\alpha}_0$ on the surface $\widetilde{T}_0$. Then the divisor Σ and the different Δ_C can be described as follows:

($\overline{\alpha}_1$) if $C=\overline{\alpha}_1$, then $\Sigma=\alpha_2+\alpha_3$ and $\Delta_C=\frac{2}{3}Q_{16}+\frac{1}{2}Q_{14}$,
($\overline{\alpha}_4$) if $C=\overline{\alpha}_4$, then $\Sigma=2\alpha_2+3\alpha_3+3\alpha_5$ and $\Delta_C=\frac{1}{2}Q_{14}+\frac{1}{2}Q_{46}+\frac{1}{2}Q_4$,
($\overline{\alpha}_6$) if $C=\overline{\alpha}_6$, then $\Sigma=\alpha_5$ and $\Delta_C=\frac{1}{2}Q_{46}+\frac{2}{3}Q_{16}$,
($\overline{\alpha}_0$) if $C=\overline{\alpha}_0$, then $\Sigma=0$ and $\Delta_C=\Delta_G|_C+\frac{2}{3}Q_{16}$.

In the last case, we have $\mathrm{ord}_Q(\Delta_C) \leqslant \frac{1}{2}$, because the curves $\overline{\alpha}_0$ and R_G meet transversally.

In each possible case, we compute $t(u)$ as follows in Table 3.

For each $u \in [0,7]$ and $v \in [0,t(u)]$, we can express the divisors $P(u,v)$ and $N(u,v)$ as linear combinations of the curves α_1, α_2, α_3, α_4, α_5, α_6. These expressions are listed in Tables 4, 5, 6, 7.

We now regard the divisor $P(u,v)$ as a row vector $\mathbf{p}(u,v) \in \mathbb{R}^6$ defined as

$$\mathbf{p}(u,v) = \big(c_1(u,v), c_2(u,v), c_3(u,v), c_4(u,v), c_5(u,v), c_6(u,v)\big),$$

where $P(u,v) = c_1(u,v)\alpha_1 + c_2(u,v)\alpha_2 + c_3(u,v)\alpha_3 + c_4(u,v)\alpha_4 + c_5(u,v)\alpha_5 + c_6(u,v)\alpha_6$. Then

$$\big(P(u,v)\big)^2 = \mathbf{p}(u,v)A\mathbf{p}(u,v)^T.$$

Thus, we have

$$S_L\big(W^G_{\bullet,\bullet};C\big) = \frac{3}{9}\int_0^7 \mathbf{p}(u,0)A\mathbf{p}(u,0)^T \cdot d(u)du + \frac{3}{9}\int_0^7\int_0^{t(u)} \mathbf{p}(u,v)A\mathbf{p}(u,v)^T dvdu.$$

Now, integrating we get

$$S_L\big(W^G_{\bullet,\bullet};C\big) = \begin{cases} \frac{1}{2} & \text{if } C = \overline{\alpha}_1, \\ \frac{7}{9} & \text{if } C = \overline{\alpha}_4, \\ \frac{4}{9} & \text{if } C = \overline{\alpha}_6, \\ \frac{11}{36} & \text{if } C = \overline{\alpha}_0. \end{cases}$$

In each case, we have $S_L(W^G_{\bullet,\bullet};C) < 1$ as required for (3.9).

To present a formula for $S_L(W^{G,C}_{\bullet,\bullet,\bullet};Q)$, let $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$, $\boldsymbol{e}_4$, $\boldsymbol{e}_5$, $\boldsymbol{e}_6$ be the standard basis for $\mathbb{R}^6$, and let $\boldsymbol{e}_0 = \boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3$. If $C = \overline{\alpha}_i$ for $i \in \{1,4,6,0\}$, then

$$S_L\big(W^{G,C}_{\bullet,\bullet,\bullet};Q\big) = \frac{3}{9}\int_0^7\int_0^{t(u)} \Big(\mathbf{p}(u,v)A\boldsymbol{e}_i^T\Big)^2 dvdu + F_Q\big(W^{G,C}_{\bullet,\bullet,\bullet}\big),$$

where

$$F_Q\big(W^{G,C}_{\bullet,\bullet,\bullet}\big) = \frac{6}{9}\int_0^7\int_0^{t(u)} \Big(\mathbf{p}(u,v)A\boldsymbol{e}_i^T\Big) \cdot \mathrm{ord}_Q\Big(\big(N'(u) + N(u,v) - (v+d(u))\Sigma\big)\big|_{\widetilde{C}} dvdu.$$

In particular, if $Q \notin \overline{\alpha}_1 \cup \overline{\alpha}_4 \cup \overline{\alpha}_6$, then $C = \overline{\alpha}_0$, so that

$$S_L\big(W^{G,C}_{\bullet,\bullet,\bullet};Q\big) = \frac{3}{9}\int_0^7\int_0^{t(u)} \Big(\mathbf{p}(u,v)A\boldsymbol{e}_0^T\Big)^2 dvdu = \frac{5}{24} < \frac{1}{2} \leqslant 1 - \mathrm{ord}_Q(\Delta_C) = A_{C,\Delta_C}(Q),$$

which gives (3.9). Similarly, if $Q \in \overline{\alpha}_1$ and $C = \overline{\alpha}_1$, then

$$S_L\big(W^{G,C}_{\bullet,\bullet,\bullet};Q\big) = \frac{3}{9}\int_0^7\int_0^{t(u)} \Big(\mathbf{p}(u,v)A\boldsymbol{e}_1^T\Big)^2 dvdu + F_Q\big(W^{G,C}_{\bullet,\bullet,\bullet}\big) = \frac{4}{27} + F_Q\big(W^{G,C}_{\bullet,\bullet,\bullet}\big) = \begin{cases} \frac{83}{108} & \text{if } Q = Q_{14}, \\ \frac{4}{27} & \text{if } Q \neq Q_{14}, \end{cases}$$

while $\Delta_C = \frac{2}{3}Q_{16} + \frac{1}{2}Q_{14}$. This gives (3.9) for $Q \in \overline{\alpha}_1 \setminus \{Q_{14}\}$. If $Q \in \overline{\alpha}_6 \setminus \{Q_{16}\}$ and $C = \overline{\alpha}_6$, then

$$S_L\big(W^{G,C}_{\bullet,\bullet,\bullet};Q\big) = \frac{3}{9}\int_0^7\int_0^{t(u)}\Big(\mathbf{p}(u,v)A\boldsymbol{e}_6^T\Big)^2 dvdu + F_Q\big(W^{G,C}_{\bullet,\bullet,\bullet}\big) == \begin{cases} \dfrac{126}{162} & \text{if } Q = Q_{46}, \\ \dfrac{25}{162} & \text{if } Q \neq Q_{46}, \end{cases}$$

while $\Delta_C = \frac{1}{2}Q_{46} + \frac{2}{3}Q_{16}$, which gives (3.9) for $Q \in \overline{\alpha}_6 \setminus \{Q_{46}, Q_{16}\}$. If $Q \in \overline{\alpha}_4$ and $C = \overline{\alpha}_4$, then

$$S_L\big(W^{G,C}_{\bullet,\bullet,\bullet};Q\big) = \frac{3}{9}\int_0^7\int_0^{t(u)}\Big(\mathbf{p}(u,v)A\boldsymbol{e}_4^T\Big)^2 dvdu + F_Q\big(W^{G,C}_{\bullet,\bullet,\bullet}\big) = \begin{cases} \dfrac{1}{2} & \text{if } Q = Q_{46}, \\ \dfrac{8}{18} & \text{if } Q = Q_{14}, \\ \dfrac{11}{36} & \text{if } Q \neq Q_{46} \text{ and } Q \neq Q_{14}, \end{cases}$$

while $\Delta_C = \frac{1}{2}Q_{14} + \frac{1}{2}Q_{46} + \frac{1}{2}Q_4$. This gives (3.9) for $Q \in \overline{\alpha}_4$.

Therefore, we see that (3.9) holds for every $Q \in G$ for an appropriate choice of the curve C, which excludes the case $(\mathbb{D}_4)$ as we explained earlier.

3.3. **Exclusion of the case $(\mathbb{A}_3)$.** Let us finish the proof of Theorem 3.1. Now, we assume that the surface R is given by the equation (3.4) with $a_2 \neq 0$. In the chart $\mathbb{A}^3_{x,y,z} = \{x_0y_0z_0 \neq 0\}$ with coordinates $x = \frac{x_1}{x_0}$, $y = \frac{y_1}{y_0}$, $z = \frac{z_1}{z_0}$, we have $P = (0,0,0)$, and the surface R is given by

$$y + xz^2 + a_2x^2 + \big(e_0yz + d_2x^2z + b_1xy + e_1xyz + c_0y^2 + b_2x^2y + e_2x^2yz + c_1xy^2 + c_2x^2y^2\big) = 0,$$

where $y + xz^2 + a_2x^2$ is the smallest degree term for the weights $\mathrm{wt}(x) = 2$, $\mathrm{wt}(y) = 4$, $\mathrm{wt}(z) = 1$. Let $\lambda\colon W_0 \to Y$ be the corresponding weighted blow up of the point P with weights $(2,4,1)$, and let G be the λ-exceptional surface. Then $G \cong \mathbb{P}(1,2,1)$, and we can also consider (x,y,z) as global coordinates on G with $\mathrm{wt}(x) = 1$, $\mathrm{wt}(y) = 2$, $\mathrm{wt}(z) = 1$.

Let R_{W_0}, F_{W_0} and S_{W_0} be the proper transforms on W_0 of the surfaces R, S and F, respectively. Set $R_G = R_{W_0}|_G$, let n_G be the curve $\{z = 0\} \subset G$, set $\Delta_G = \frac{1}{2}R_G + \frac{1}{2}n_G$ and $\Delta_{W_0} = \frac{1}{2}R_{W_0}$. Then

$$\big(K_{W_0} + \Delta_{W_0} + G\big)\big|_G \sim_{\mathbb{Q}} K_G + \Delta_G.$$

Note that $F_{W_0}|_G = \{x = 0\}$, $S_{W_0}|_G = \{y = 0\}$ and $R_G = \{y + xz + a_2x^2 = 0\}$.

The remaining part of this subsection is very similar to what has been done in Section 3.2, so we will omit some details here. We have $A_{Y,\Delta_Y}(G) = 5$. Using [5, Corollary 7.7], we get $S_{Y,\Delta_Y}(G) = \frac{41}{9}$.

Both 3-folds Y and W_0 are toric, and the weighted blow up λ is also toric. Let Σ_Y and Σ_{W_0} be the fans of the 3-folds Y and W_0, respectively. Then the fan Σ_Y is presented in Section 3.2, and the fan Σ_{W_0} is the simplicial fan in $\mathbb{R}^3$ defined by the following data:

- the list of primitive generators of rays in Σ_{W_0} is

  $v_0 = (2,4,-1)$, $v_1 = (1,0,0)$, $v_2 = (0,0,1)$, $v_3 = (0,1,0)$,
  $v_4 = (0,0,-1)$, $v_5 = (0,-1,1)$, $v_6 = (-1,0,0)$;

- the list of maximal cones in Σ_{W_0} is

  $[0,1,3], [0,1,4], [0,3,4], [1,2,3], [1,2,5], [1,4,5], [2,3,6], [2,5,6], [3,4,6], [4,5,6],$

  where $[i,j,k]$ is the cone generated by the rays v_i, v_j, and v_k.

As in Section 3.2, let us denote by T_i the torus invariant divisor that corresponds to the ray v_i. Note that T_0 is the exceptional divisor G.

Take $u \in \mathbb{R}_{\geqslant 0}$. As in Section 3.2, we let $L_u = \lambda^*(L) - uT_0$. Then

$$L_u \sim_{\mathbb{R}} (10-u)T_0 + T_1 + T_2 + 2T_3,$$

which implies that L_u is pseudoeffective if and only if $u \in [0, 10]$.

Let W_1, W_2, W_3 be the toric 3-folds defined by the simplicial fans Σ_{W_1}, Σ_{W_2}, Σ_{W_3} in $\mathbb{R}^3$, respectively, which are determined by the following data:

- the list of primitive generators of rays of the fans Σ_{W_1}, Σ_{W_2}, Σ_{W_3} is

$$v_0 = (2, 4, -1), \quad v_1 = (1, 0, 0), \quad v_2 = (0, 0, 1), \quad v_3 = (0, 1, 0),$$
$$v_4 = (0, 0, -1), \quad v_5 = (0, -1, 1), \quad v_6 = (-1, 0, 0);$$

- the list of maximal cones of Σ_{W_1} is

$$[0, 1, 2], [0, 2, 3], [0, 1, 4], [0, 3, 4], [1, 4, 5], [1, 2, 5], [2, 3, 6], [3, 4, 6], [4, 5, 6], [2, 5, 6];$$

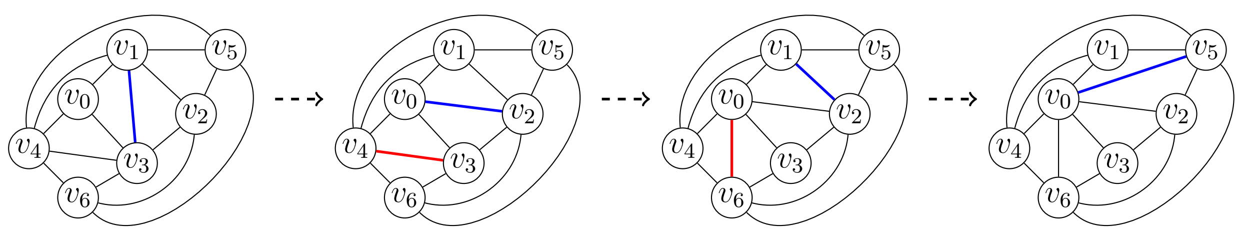
- the list of maximal cones of Σ_{W_2} is

$$[0, 3, 4], [0, 4, 6], [0, 1, 2], [0, 2, 3], [0, 1, 4], [1, 4, 5], [1, 2, 5], [2, 3, 6], [4, 5, 6], [2, 5, 6];$$

- the list of maximal cones of Σ_{W_3} is

$$[0, 1, 5], [0, 2, 5], [0, 3, 6], [0, 4, 6], [0, 2, 3], [0, 1, 4], [1, 4, 5], [2, 3, 6], [4, 5, 6], [2, 5, 6].$$

Then W_1, W_2, W_3 are projective, and there are small birational maps $W_0 \dashrightarrow W_1 \dashrightarrow W_2 \dashrightarrow W_3$, which can be illustrated by the following self-explanatory toric diagrams:



As in Section 3.2, let us use the same notations for the corresponding torus invariant divisors and torus invariant curves on each 3-fold W_i. Similarly, we will use the same notation for the strict transforms of the divisor L_u on each 3-fold W_i. As in Section 3.2, we see that

- L_u is nef on W_0 for $u \in [0, 1]$;
- L_u is nef on W_1 for $u \in [1, 2]$;
- L_u is nef on W_2 for $u \in [2, 3]$.

Moreover, the Zariski decomposition of the divisor L_u exists on the 3-fold W_2 for each $u \in [3, 5]$, and the Zariski decomposition exists on W_3 for $u \in [5, 10]$. Let us denote by $P(u)$ its positive part, and let us denote by $N(u)$ its negative part. Then $P(u) = L_u - N(u)$, where

$$N(u) = \begin{cases} \dfrac{u-3}{4} T_3 \ \text{ for } \ u \in [3, 5], \\ \dfrac{u-3}{4} T_3 \ \text{ for } \ u \in [5, 7], \\ \dfrac{u-7}{3} T_2 + \dfrac{u-4}{3} T_3 \ \text{ for } \ u \in [7, 8], \\ \dfrac{u-8}{2} T_1 + \dfrac{u-7}{3} T_2 + \dfrac{u-4}{3} T_3 \ \text{ for } \ u \in [8, 10]. \end{cases}$$

Here, the divisor $L_u - \frac{u-3}{4} T_3$ is nef on W_2 for $u \in [3, 5]$, and it is nef on W_3 for $[5, 7]$.

Now, let us consider a common partial toric resolution $\widetilde{W}$ of the toric 3-folds W_0, W_1, W_2 and W_3. Namely, let $\widetilde{W}$ be the toric 3-fold defined by the simplicial fan $\Sigma_{\widetilde{W}}$ in $\mathbb{R}^3$ given by the following data:

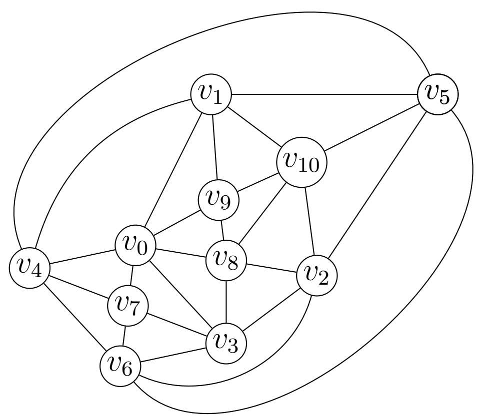
- the list of primitive generators of rays of $\Sigma_{\widetilde{W}}$ is

$$v_0 = (2,4,-1),\quad v_1 = (1,0,0),\quad v_2 = (0,0,1),\quad v_3 = (0,1,0),\quad v_4 = (0,0,-1),\quad v_5 = (0,-1,1),$$
$$v_6 = (-1,0,0),\quad v_7 = (0,4,-1),\quad v_8 = (1,2,0),\quad v_9 = (2,3,0),\quad v_{10} = (2,0,3);$$

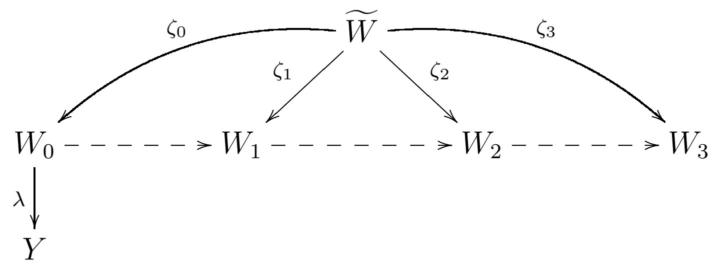
- the list of maximal cones of $\Sigma_{\widetilde{W}}$ is

$$[0,1,4],\quad [0,1,9],\quad [0,3,7],\quad [0,3,8],\quad [0,4,7],\quad [0,8,9],\quad [1,4,5],\quad [1,5,10],\quad [1,9,10],$$
$$[2,3,6],\quad [2,3,8],\quad [2,5,6],\quad [2,5,10],\quad [2,8,10],\quad [3,6,7],\quad [4,5,6],\quad [4,6,7],\quad [8,9,10].$$

The fan $\Sigma_{\widetilde{W}}$ can be diagramed as follows:



Then there exists the following toric commutative diagram



where ζ_0, ζ_1, ζ_2, ζ_3 are toric birational morphisms.

Let $\widetilde{T}_i$ be the torus invariant divisor on $\widetilde{W}$ corresponding to the ray v_i in the fan $\Sigma_{\widetilde{W}}$. Then

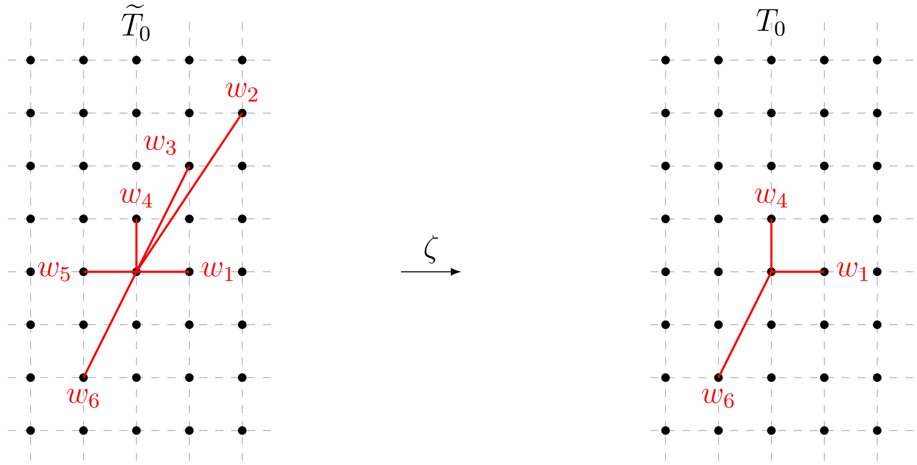
$$\begin{aligned}
\zeta_0^*(T_0) &= \widetilde{T}_0, & \zeta_0^*(T_1) &= \widetilde{T}_1 + \widetilde{T}_8 + 2\widetilde{T}_9 + 2\widetilde{T}_{10},\\
\zeta_0^*(T_2) &= \widetilde{T}_2 + 3\widetilde{T}_{10}, & \zeta_0^*(T_3) &= \widetilde{T}_3 + 4\widetilde{T}_7 + 2\widetilde{T}_8 + 3\widetilde{T}_9,\\
\zeta_1^*(T_0) &= \widetilde{T}_0 + \tfrac{1}{2}\widetilde{T}_8 + \tfrac{3}{4}\widetilde{T}_9, & \zeta_1^*(T_1) &= \widetilde{T}_1 + \tfrac{1}{2}\widetilde{T}_9 + 2\widetilde{T}_{10},\\
\zeta_1^*(T_2) &= \widetilde{T}_2 + \tfrac{1}{2}\widetilde{T}_8 + \tfrac{3}{4}\widetilde{T}_9 + 3\widetilde{T}_{10}, & \zeta_1^*(T_3) &= \widetilde{T}_3 + 4\widetilde{T}_7,\\
\zeta_2^*(T_0) &= \widetilde{T}_0 + \widetilde{T}_7 + \tfrac{1}{2}\widetilde{T}_8 + \tfrac{3}{4}\widetilde{T}_9, & \zeta_2^*(T_1) &= \widetilde{T}_1 + \tfrac{1}{2}\widetilde{T}_9 + 2\widetilde{T}_{10},\\
\zeta_2^*(T_2) &= \widetilde{T}_2 + \tfrac{1}{2}\widetilde{T}_8 + \tfrac{3}{4}\widetilde{T}_9 + 3\widetilde{T}_{10}, & \zeta_2^*(T_3) &= \widetilde{T}_3,\\
\zeta_3^*(T_0) &= \widetilde{T}_0 + \widetilde{T}_7 + \tfrac{1}{2}\widetilde{T}_8 + \widetilde{T}_9 + \widetilde{T}_{10}, & \zeta_3^*(T_1) &= \widetilde{T}_1,\\
\zeta_3^*(T_2) &= \widetilde{T}_2 + \tfrac{1}{2}\widetilde{T}_8, & \zeta_3^*(T_3) &= \widetilde{T}_3.
\end{aligned}$$

On the 3-fold $\widetilde{W}$, the Zariski decomposition of the divisor $\zeta_0^*(L_u)$ does exist for every $u \in [0, 10]$. Let $P_{\widetilde{W}}(u)$ be its positive part, and let $N_{\widetilde{W}}(u)$ be its negative part. We can express them as linear combinations of the torus invariant divisors. These expressions are presented in Table 8.

Fix the quotient homomorphism $\mathbb{Z}^3 \to \mathbb{Z}^3/\mathbb{Z}v_0 \cong \mathbb{Z}^2$ such that $v_1 \mapsto (1, 0)$ and $v_3 \mapsto (0, 1)$. Then $\Sigma_{\widetilde{W}}$ is mapped to the fan in $\mathbb{R}^2$ whose rays are generated by the following vectors:

$$w_1 = (1, 0), w_2 = (2, 3), w_3 = (1, 2), w_4 = (0, 1), w_5 = (-1, 0), w_6 = (-1, -2).$$

This two-dimensional fan defines the surface $\widetilde{T}_0$. Let $\zeta = \zeta_0|_{\widetilde{T}_0} \colon \widetilde{T}_0 \to T_0$ be the restriction map. Then ζ is described by a map from the fan of the toric surface $\widetilde{T}_0$ to the fan of the surface T_0, which can be illustrated by the following toric picture:



It contracts the curves of the rays w_5, w_3, w_2 to points on the surface T_0.

Let $\alpha_1, \ldots, \alpha_6$ be the torus invariant curves in $\widetilde{T}_0$ defined by $w_1, \ldots, w_6$, respectively. Then

$$\widetilde{T}_1\big|_{\widetilde{T}_0} = \alpha_1, \widetilde{T}_3\big|_{\widetilde{T}_0} = \alpha_4, \widetilde{T}_4\big|_{\widetilde{T}_0} = \frac{1}{2}\alpha_6, \widetilde{T}_7\big|_{\widetilde{T}_0} = \frac{1}{2}\alpha_5, \widetilde{T}_8\big|_{\widetilde{T}_0} = \alpha_3, \widetilde{T}_9\big|_{\widetilde{T}_0} = \alpha_2, \widetilde{T}_0\big|_{\widetilde{T}_0} = -\frac{1}{2}(\alpha_1+2\alpha_2+\alpha_3).$$

Set $\overline{\alpha}_1 = \zeta(\alpha_1)$, $\overline{\alpha}_4 = \zeta(\alpha_4)$, $\overline{\alpha}_6 = \zeta(\alpha_6)$. Then $\overline{\alpha}_1 = \{x = 0\}$, $\overline{\alpha}_4 = \{y = 0\}$, $\overline{\alpha}_6 = n_G = \{z = 0\}$. Set $Q_{14} = \overline{\alpha}_1 \cap \overline{\alpha}_4$, $Q_{16} = \overline{\alpha}_1 \cap \overline{\alpha}_6$, $Q_{46} = \overline{\alpha}_4 \cap \overline{\alpha}_6$. Then Q_{16} is the singular point of the surface G. Note that the curve R_G meets $\overline{\alpha}_1$ transversally at Q_{14}, it meets the curve $\overline{\alpha}_4$ transversally at two distinct points (one of them is Q_{14}), and R_G meets the curve $\overline{\alpha}_6$ transversally at a single point, which is different from Q_{16} and Q_{46}. Let Q_4 be the point in $R_G \cap \overline{\alpha}_4$ that is different from Q_{14}, and let Q_6 be the intersection point $R_G \cap \overline{\alpha}_6$.

Arguing as in Section 3.2, we obtain the following intersection matrix:

$$A := (\alpha_i\alpha_j) = \begin{pmatrix} -\frac{1}{6} & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & -\frac{2}{3} & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}$$

Now, set $\widetilde{P}(u) = P_{\widetilde{W}}(u)|_{\widetilde{T}_0}$ and $\widetilde{N}(u) = N_{\widetilde{W}}(u)|_{\widetilde{T}_0}$. We can express $\widetilde{P}(u)$ and $\widetilde{N}(u)$ as linear combinations of the curves α_1, α_2, α_3, α_4, α_5, α_6. These expressions are presented in Table 9.

Let Q be a point in the surface $G=T_0$, let C be a smooth curve in G that passes through P, and let $\widetilde{C}$ be its proper transform on $\widetilde{T}_0$. For every $u\in[0,10]$, let

$$t(u)=\inf\Big\{v\in\mathbb{R}_{\geqslant 0}\ \Big|\ \widetilde{P}(u)-vC\text{ is pseudoeffective}\Big\}.$$

For every $v\in[0,t(u)]$, let $P(u,v)$ be the positive part of the Zariski decomposition of $\widetilde{P}(u)-vC$, and let $N(u,v)$ be its negative part. Set

$$S_L\big(W^G_{\bullet,\bullet};C\big)=\frac{3}{L^3}\int\limits_0^{10}\big(\widetilde{P}(u)\big)^2\mathrm{ord}_C\big(\widetilde{N}(u)\big)du+\frac{3}{L^3}\int\limits_0^{10}\int\limits_0^{t(u)}\big(P(u,v)\big)^2dvdu.$$

Now, we write $\zeta^*(C)=\widetilde{C}+\Sigma$ for an effective $\mathbb{R}$-divisor Σ on the surface $\widetilde{T}_0$. For every $u\in[0,10]$, write $\widetilde{N}(u)=d(u)C+N'(u)$, where $d(u)=\mathrm{ord}_C(\widetilde{N}(u))$, and $N'(u)$ is an effective divisor on $\widetilde{T}_0$. Set

$$S\big(W^{G,C}_{\bullet,\bullet,\bullet};Q\big)=\frac{3}{L^3}\int\limits_0^{10}\int\limits_0^{t(u)}\big(P(u,v)\cdot\widetilde{C}\big)^2dvdu+F_Q\big(W^{G,C}_{\bullet,\bullet,\bullet}\big)$$

for

$$F_Q\big(W^{G,C}_{\bullet,\bullet,\bullet}\big)=\frac{6}{L^3}\int\limits_0^{10}\int\limits_0^{t(u)}\big(P(u,v)\cdot\widetilde{C}\big)\cdot\mathrm{ord}_Q\Big(\big(N'(u)+N(u,v)-(v+d(u))\Sigma\big)\big|_{\widetilde{C}}\Big)dvdu,$$

where we consider Q as a point in $\widetilde{C}$ using the isomorphism $\widetilde{C}\cong C$ induced by ζ.

If $C\not\subset\mathrm{Supp}(\Delta_G)$, we have $(K_G+C+\Delta_G)|_C\sim_{\mathbb{R}}K_C+\Delta_C$, where Δ_C is an effective divisor known as the different. If $C\not\subset\mathrm{Supp}(\Delta_G)$, we still can define the different Δ_C using

$$\big(K_G+C+\Delta_G-\mathrm{ord}_C(\Delta_G)\big)\big|_C\sim_{\mathbb{R}}K_C+\Delta_C.$$

The different Δ_C can be computed locally near any point in C. Now, arguing as in Section 3.2, we see that to exclude the case ($\mathbb{A}_3$), it is enough to show that for every point $Q\in G$, there exists a smooth irreducible curve $C\subset G$ passing through Q such that

$$S_L\big(W^G_{\bullet,\bullet};C\big)\leqslant A_{G,\Delta_G}(C)\tag{3.10}$$

and

$$S(W^{G,C}_{\bullet,\bullet,\bullet};Q)\leqslant A_{C,\Delta_C}(Q).\tag{3.11}$$

Let us do this in the rest of this section, which would complete the proof of Theorem 3.1.

Let Q be a point in $G=T_0\cong\mathbb{P}(1,2,1)$. Let us choose the curve C as follows. If $Q\in\overline{\alpha}_1\cup\overline{\alpha}_4\cup\overline{\alpha}_6$, we let C be a curve among $\overline{\alpha}_1$, $\overline{\alpha}_4$, $\overline{\alpha}_6$ that contains Q. If $Q\not\in\overline{\alpha}_1\cup\overline{\alpha}_4\cup\overline{\alpha}_6$, then there is a unique curve $\overline{\alpha}_0\in|\mathcal{O}_G(1)|$ that contains Q. In this case, we let $C=\overline{\alpha}_0$, and we denote by α_0 the proper transform of the curve $\overline{\alpha}_0$ on the surface $\widetilde{T}_0$. Then Σ and Δ_C can be described as follows:

- ($\overline{\alpha}_1$) if $C=\overline{\alpha}_1$, then $\Sigma=2\alpha_2+\alpha_3$ and $\Delta_C=\frac{1}{2}Q_{16}+\frac{1}{2}Q_{14}$,
- ($\overline{\alpha}_4$) if $C=\overline{\alpha}_4$, then $\Sigma=3\alpha_2+2\alpha_3+2\alpha_5$ and $\Delta_C=\frac{1}{2}Q_{14}+\frac{1}{2}Q_4$,
- ($\overline{\alpha}_6$) if $C=\overline{\alpha}_6$, then $\Sigma=\alpha_5$ and $\Delta_C=\frac{3}{4}Q_{16}+\frac{1}{2}Q_6$,
- ($\overline{\alpha}_0$) if $C=\overline{\alpha}_0$, then $\Sigma=0$ and $\Delta_C=\Delta_G|_C+\frac{3}{4}Q_{16}$.

In the last case, we have $\mathrm{ord}_Q(\Delta_C)\leqslant\frac{1}{2}$, because $\overline{\alpha}_0$ and R_G meet transversally.

In each possible case, we compute $t(u)$ in Table 10.

For each $u\in[0,10]$ and $v\in[0,t(u)]$, we can express both divisors $P(u,v)$ and $N(u,v)$ as linear combinations of the curves α_1, α_2, α_3, α_4, α_5, α_6. They are listed in Tables 11, 12, 13, 14.

Now, arguing as in Section 3.2, we compute

$$S_L\big(W^G_{\bullet,\bullet};C\big)=\begin{cases}\dfrac{1}{2} & \text{if } C=\overline{\alpha}_1,\\ \dfrac{7}{9} & \text{if } C=\overline{\alpha}_4,\\ \dfrac{2}{9} & \text{if } C=\overline{\alpha}_6,\\ \dfrac{3}{16} & \text{if } C=\overline{\alpha}_0.\end{cases}$$

This gives (3.10). Note that $A_{G,\Delta_G}(\overline{\alpha}_6)=\frac{1}{2}$.

If $Q\in\alpha_1\setminus\{Q_{14}\}$, let $C=\overline{\alpha}_1$, then $S_L(W^{G,\overline{\alpha}_1}_{\bullet,\bullet,\bullet};Q)=\frac{1}{9}$. If $Q\in\overline{\alpha}_4\setminus\{Q_{46}\}$, let $C=\overline{\alpha}_4$, then

$$S_L\big(W^{G,\overline{\alpha}_4}_{\bullet,\bullet,\bullet};Q\big)=\begin{cases}\dfrac{1}{2} & \text{if } Q=Q_{14},\\ \dfrac{3}{16} & \text{if } Q\neq Q_{14}.\end{cases}$$

If $Q\in\overline{\alpha}_6\setminus\{Q_{16}\}$, we let $C=\overline{\alpha}_1$, which gives

$$S_L\big(W^{G,\overline{\alpha}_6}_{\bullet,\bullet,\bullet};Q\big)=\begin{cases}\dfrac{7}{9} & \text{if } Q=Q_{46},\\ \dfrac{2}{9} & \text{if } Q\neq Q_{46}.\end{cases}$$

If $Q\notin\overline{\alpha}_1\cup\overline{\alpha}_4\cup\overline{\alpha}_6)$, we let $C=\overline{\alpha}_0$, which gives $S_L(W^{G,\overline{\alpha}_0}_{\bullet,\bullet,\bullet};Q)=\frac{1729}{6912}$. In each case we get (3.11). This excludes the case ($\mathbb{A}_3$), and completes the proof of Theorem 3.1.

Appendix A. Tables

Table 1: Zariski decomposition of the divisor $\zeta_0^*(L_u)$

u	$P_{\widetilde{W}}(u)$ & $N_{\widetilde{W}}(u)$	$\widetilde{T}_0$	$\widetilde{T}_1$	$\widetilde{T}_2$	$\widetilde{T}_3$	$\widetilde{T}_7$	$\widetilde{T}_8$	$\widetilde{T}_9$	$\widetilde{T}_{10}$
$[0,1]$	$P_{\widetilde{W}}(u)$	$7-u$	1	1	2	6	7	5	3
	$N_{\widetilde{W}}(u)$	0	0	0	0	0	0	0	0
$[1,2]$	$P_{\widetilde{W}}(u)$	$7-u$	1	1	2	$7-u$	$8-u$	$\frac{17-2u}{3}$	3
	$N_{\widetilde{W}}(u)$	0	0	0	0	$u-1$	$u-1$	$\frac{2}{3}(u-1)$	0
$[2,4]$	$P_{\widetilde{W}}(u)$	$7-u$	1	1	$\frac{8-u}{3}$	$7-u$	$8-u$	$\frac{17-2u}{3}$	3
	$N_{\widetilde{W}}(u)$	0	0	0	$\frac{u-2}{3}$	$u-1$	$u-1$	$\frac{2}{3}(u-1)$	0
$[4,5]$	$P_{\widetilde{W}}(u)$	$7-u$	1	1	$\frac{8-u}{3}$	$7-u$	$8-u$	$7-u$	$7-u$
	$N_{\widetilde{W}}(u)$	0	0	0	$\frac{u-2}{3}$	$u-1$	$u-1$	$u-2$	$u-4$
$[5,6]$	$P_{\widetilde{W}}(u)$	$7-u$	1	$\frac{7-u}{2}$	$\frac{7-u}{2}$	$7-u$	$\frac{3}{2}(7-u)$	$7-u$	$7-u$
	$N_{\widetilde{W}}(u)$	0	0	$\frac{u-5}{2}$	$\frac{u-3}{2}$	$u-1$	$\frac{3u-7}{2}$	$u-2$	$u-4$
$[6,7]$	$P_{\widetilde{W}}(u)$	$7-u$	$7-u$	$\frac{7-u}{2}$	$\frac{7-u}{2}$	$7-u$	$\frac{3}{2}(7-u)$	$7-u$	$7-u$
	$N_{\widetilde{W}}(u)$	0	$u-6$	$\frac{u-5}{2}$	$\frac{u-3}{2}$	$u-1$	$\frac{3u-7}{2}$	$u-2$	$u-4$

Table 2: Expressions for $\widetilde{P}(u)$ and $\widetilde{N}(u)$

u		α_1	α_2	α_3	α_4	α_5	α_6
$[0,1]$	$\widetilde{P}(u)$	$u-6$	$u-2$	u	2	6	0
	$\widetilde{N}(u)$	0	0	0	0	0	0
$[1,2]$	$\widetilde{P}(u)$	$u-6$	$\frac{u-4}{3}$	1	2	$7-u$	0
	$\widetilde{N}(u)$	0	$\frac{2}{3}(u-1)$	$u-1$	0	$u-1$	0
$[2,4]$	$\widetilde{P}(u)$	$u-6$	$\frac{u-4}{3}$	1	$\frac{8-u}{3}$	$7-u$	0
	$\widetilde{N}(u)$	0	$\frac{2}{3}(u-1)$	$u-1$	$\frac{u-2}{3}$	$u-1$	0
$[4,5]$	$\widetilde{P}(u)$	$u-6$	0	1	$\frac{8-u}{3}$	$7-u$	0
	$\widetilde{N}(u)$	0	$u-2$	$u-1$	$\frac{u-2}{3}$	$u-1$	0
$[5,6]$	$\widetilde{P}(u)$	$6-u$	0	$\frac{7-u}{2}$	$\frac{7-u}{2}$	$7-u$	0
	$\widetilde{N}(u)$	0	$u-2$	$\frac{3u-7}{2}$	$\frac{u-3}{2}$	$u-1$	0
$[6,7]$	$\widetilde{P}(u)$	0	0	$\frac{7-u}{2}$	$\frac{7-u}{2}$	$7-u$	0
	$\widetilde{N}(u)$	$u-6$	$u-2$	$\frac{3u-7}{2}$	$\frac{u-3}{2}$	$u-1$	0

Table 3: Values of $t(u)$

C \ u	$[0,1]$	$[1,2]$	$[2,4]$	$[4,5]$	$[5,6]$	$[6,7]$
$\overline{\alpha}_1$	u	1	1	1	1	$7-u$
$\overline{\alpha}_4$	$\frac{u}{3}$	$\frac{u}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{9-u}{6}$	$\frac{7-u}{2}$
$\overline{\alpha}_6$	u	1	1	1	$\frac{7-u}{2}$	$\frac{7-u}{2}$
$\overline{\alpha}_0$	u	1	1	$\frac{7-u}{3}$	$\frac{7-u}{3}$	$\frac{7-u}{3}$

Table 4: Expressions for $P(u,v)$ and $N(u,v)$ in the case $C=\overline{\alpha}_1$

u	v	$P(u,v)$ & $N(u,v)$	α_1	α_2	α_3	α_4	α_5	α_6
$[0,1]$	$[0,u]$	$P(u,v)$	$u-6-v$	$u-2-v$	$u-v$	2	6	0
		$N(u,v)$	0	v	v	0	0	0
$[1,2]$	$[0,u-1]$	$P(u,v)$	$u-6-v$	$\frac{u-4-v}{3}$	1	2	$7-u$	0
		$N(u,v)$	0	$\frac{v}{3}$	0	0	0	0
$[1,2]$	$[u-1,1]$	$P(u,v)$	$u-6-v$	$u-2-v$	$u-v$	2	$7-u$	0
		$N(u,v)$	0	$\frac{3v-2u+2}{3}$	$v-u+1$	0	0	0

$[2,4]$	$[0,1]$	$P(u,v)$	$u-6-v$	$\frac{u-4-v}{3}$	1	$\frac{8-u}{3}$	$7-u$	0
		$N(u,v)$	0	$\frac{v}{3}$	0	0	0	0
$[4,5]$	$[0,u-4]$	$P(u,v)$	$u-6-v$	0	1	$\frac{8-u}{3}$	$7-u$	0
		$N(u,v)$	0	0	0	0	0	0
$[4,5]$	$[u-4,1]$	$P(u,v)$	$u-6-v$	$\frac{u-4-v}{3}$	1	$\frac{8-u}{3}$	$7-u$	0
		$N(u,v)$	0	$\frac{u-4-v}{6}$	0	0	0	0
$[5,6]$	$[0,1]$	$P(u,v)$	$6-u-v$	0	$\frac{7-u}{2}$	$\frac{7-u}{2}$	$7-u$	0
		$N(u,v)$	0	0	0	0	0	0
$[6,7]$	$[0,7-u]$	$P(u,v)$	$-v$	0	$\frac{7-u}{2}$	$\frac{7-u}{2}$	$7-u$	0
		$N(u,v)$	0	0	0	0	0	0

Table 5: Expressions for $P(u,v)$ and $N(u,v)$ in the case $C=\overline{\alpha}_4$

u	v	$P(u,v)$ & $N(u,v)$	α_1	α_2	α_3	α_4	α_5	α_6
$[0,1]$	$[0,\frac{u}{3}]$	$P(u,v)$	$u-6$	$u-2-2v$	$u-3v$	$2-v$	$6-3v$	0
		$N(u,v)$	0	$2v$	$3v$	0	$3v$	0
$[1,2]$	$[0,\frac{u-1}{3}]$	$P(u,v)$	$u-6$	$\frac{u-4}{3}$	1	$2-v$	$7-u$	0
		$N(u,v)$	0	0	0	0	0	0
$[1,2]$	$[\frac{u-1}{3},\frac{u}{3}]$	$P(u,v)$	$u-6$	$u-2-2v$	$u-3v$	$2-v$	$6-3v$	0
		$N(u,v)$	0	$\frac{6v-2u+2}{3}$	$3v-u+1$	0	$3v-u+1$	0
$[2,4]$	$[0,\frac{1}{3}]$	$P(u,v)$	$u-6$	$\frac{u-4}{3}$	1	$\frac{8-u-3v}{3}$	$7-u$	0
		$N(u,v)$	0	0	0	0	0	0
$[2,4]$	$[\frac{1}{3},\frac{2}{3}]$	$P(u,v)$	$u-6$	$\frac{u-2-6v}{3}$	$2-3v$	$\frac{8-u-3v}{3}$	$8-u-3v$	0
		$N(u,v)$	0	$\frac{6v-2}{3}$	$3v-1$	0	$3v-1$	0
$[4,5]$	$[0,\frac{5-u}{3}]$	$P(u,v)$	$u-6$	0	1	$\frac{8-u-3v}{3}$	$7-u$	0
		$N(u,v)$	0	0	0	0	0	0
$[4,5]$	$[\frac{5-u}{3},\frac{1}{3}]$	$P(u,v)$	$u-6$	0	$\frac{8-u-3v}{3}$	$\frac{8-u-3v}{3}$	$7-u$	0
		$N(u,v)$	0	0	$\frac{3v+u-5}{3}$	0	0	0
$[4,5]$	$[\frac{1}{3},\frac{u-2}{6}]$	$P(u,v)$	$u-6$	0	$\frac{8-u-3v}{3}$	$\frac{8-u-3v}{3}$	$8-u-3v$	0
		$N(u,v)$	0	0	$\frac{3v+u-5}{3}$	0	$3v-1$	0
$[4,5]$	$[\frac{u-2}{6},\frac{2}{3}]$	$P(u,v)$	$u-6$	$\frac{u-2-6v}{3}$	$2-3v$	$\frac{8-u-3v}{3}$	$8-u-3v$	0
		$N(u,v)$	0	$\frac{6v+2-u}{3}$	$3v-1$	0	$3v-1$	0

$[5,6]$	$[0, \frac{7-u}{6}]$	$P(u,v)$	$u-6$	0	$\frac{7-u-2v}{2}$	$\frac{7-u-2v}{2}$	$7-u$	0
		$N(u,v)$	0	0	v	0	0	0
$[5,6]$	$[\frac{7-u}{6}, \frac{1}{2}]$	$P(u,v)$	$u-6$	0	$\frac{7-u-2v}{2}$	$\frac{7-u-2v}{2}$	$\frac{21-3u-6v}{2}$	0
		$N(u,v)$	0	0	v	0	$\frac{6v+u-7}{2}$	0
$[5,6]$	$[\frac{1}{2}, \frac{9-u}{6}]$	$P(u,v)$	$u-6$	$1-2v$	$\frac{9-u-6v}{2}$	$\frac{7-u-2v}{2}$	$\frac{21-3u-6v}{2}$	0
		$N(u,v)$	0	$2v-1$	$3v-1$	0	$\frac{6v+u-7}{2}$	0
$[6,7]$	$[0, \frac{7-u}{6}]$	$P(u,v)$	0	0	$\frac{7-u-2v}{2}$	$\frac{7-u-2v}{2}$	$7-u$	0
		$N(u,v)$	0	0	v	0	0	0
$[6,7]$	$[\frac{7-u}{6}, \frac{7-u}{2}]$	$P(u,v)$	0	0	$\frac{7-u-2v}{2}$	$\frac{7-u-2v}{2}$	$\frac{21-3u-6v}{2}$	0
		$N(u,v)$	0	0	0	0	$\frac{6v+u-7}{2}$	0

Table 6: Expressions for $P(u,v)$ and $N(u,v)$ in the case $C = \overline{\alpha}_6$

u	v	$P(u,v)$ & $N(u,v)$	α_1	α_2	α_3	α_4	α_5	α_6
$[0,1]$	$[0,u]$	$P(u,v)$	$u-6$	$u-2$	u	2	$6-v$	$-v$
		$N(u,v)$	0	0	0	0	v	0
$[1,2]$	$[0,u-1]$	$P(u,v)$	$u-6$	$\frac{u-4}{3}$	1	2	$7-u$	$-v$
		$N(u,v)$	0	0	0	0	0	0
$[1,2]$	$[u-1,1]$	$P(u,v)$	$u-6$	$\frac{u-4}{3}$	1	2	$6-v$	$-v$
		$N(u,v)$	0	0	0	0	$v-u+1$	0
$[2,4]$	$[0,1]$	$P(u,v)$	$u-6$	$\frac{u-4}{3}$	1	$\frac{8-u}{3}$	$7-u$	$-v$
		$N(u,v)$	0	0	0	0	0	0
$[4,5]$	$[0, \frac{6-u}{2}]$	$P(u,v)$	$u-6$	0	1	$\frac{8-u}{3}$	$7-u$	$-v$
		$N(u,v)$	0	0	0	0	0	0
$[4,5]$	$[\frac{6-u}{2}, 1]$	$P(u,v)$	$-2v$	0	1	$\frac{8-u}{3}$	$7-u$	$-v$
		$N(u,v)$	$2v-6+u$	0	0	0	0	0
$[5,6]$	$[0, \frac{6-u}{2}]$	$P(u,v)$	$6-u$	0	$\frac{7-u}{2}$	$\frac{7-u}{2}$	$7-u$	$-v$
		$N(u,v)$	0	0	0	0	0	0
$[5,6]$	$[\frac{6-u}{2}, \frac{7-u}{2}]$	$P(u,v)$	$-2v$	0	$\frac{7-u}{2}$	$\frac{7-u}{2}$	$7-u$	$-v$
		$N(u,v)$	$2v+u-6$	0	0	0	0	0
$[6,7]$	$[0, \frac{7-u}{2}]$	$P(u,v)$	$-2v$	0	$\frac{7-u}{2}$	$\frac{7-u}{2}$	$7-u$	$-v$
		$N(u,v)$	$2v$	0	0	0	0	0

Table 7: Expressions for $P(u,v)$ and $N(u,v)$ in the case $C = \overline{\alpha}_0$

u	v	$P(u,v)$ & $N(u,v)$	α_1	α_2	α_3	α_4	α_5	α_6
$[0,1]$	$[0,u]$	$P(u,v)$	$u-6-v$	$u-2-v$	$u-v$	2	6	0
		$N(u,v)$	0	0	0	0	0	0
$[1,2]$	$[0,2-u]$	$P(u,v)$	$u-6-v$	$\frac{u-4-3v}{3}$	$1-v$	2	$7-u$	0
		$N(u,v)$	0	0	0	0	0	0
$[1,2]$	$[2-u,1]$	$P(u,v)$	$u-6-v$	$\frac{u-4-3v}{3}$	$1-v$	$\frac{8-u-v}{3}$	$7-u$	0
		$N(u,v)$	0	0	0	$\frac{v+u-2}{3}$	0	0
$[2,4]$	$[0,1]$	$P(u,v)$	$u-6-v$	$\frac{u-4-3v}{3}$	$1-v$	$\frac{8-u-v}{3}$	$7-u$	0
		$N(u,v)$	0	0	0	$\frac{v}{3}$	0	0
$[4,5]$	$[0,5-u]$	$P(u,v)$	$u-6-v$	$-v$	$1-v$	$\frac{8-u-v}{3}$	$7-u$	0
		$N(u,v)$	0	0	0	$\frac{v}{3}$	0	0
$[4,5]$	$[5-u,\frac{6-u}{2}]$	$P(u,v)$	$u-6-v$	$-v$	$\frac{7-u-3v}{2}$	$\frac{7-u-v}{2}$	$7-u$	0
		$N(u,v)$	0	0	$\frac{u+v-5}{2}$	$\frac{u+3v-5}{6}$	0	0
$[4,5]$	$[\frac{6-u}{2},\frac{7-u}{3}]$	$P(u,v)$	$-3v$	$-v$	$\frac{7-u-3v}{2}$	$\frac{7-u-v}{2}$	$7-u$	0
		$N(u,v)$	$u-6+2v$	0	$\frac{u+v-5}{2}$	$\frac{u+3v-5}{6}$	0	0
$[5,6]$	$[0,\frac{6-u}{2}]$	$P(u,v)$	$u-6-v$	$-v$	$\frac{7-u-3v}{2}$	$\frac{7-u-v}{2}$	$7-u$	0
		$N(u,v)$	0	0	$\frac{v}{2}$	$\frac{v}{2}$	0	0
$[5,6]$	$[\frac{6-u}{2},\frac{7-u}{3}]$	$P(u,v)$	$-3v$	$-v$	$\frac{7-u-3v}{2}$	$\frac{7-u-v}{2}$	$7-u$	0
		$N(u,v)$	$u-6+2v$	0	$\frac{v}{2}$	$\frac{v}{2}$	0	0
$[6,7]$	$[0,\frac{7-u}{3}]$	$P(u,v)$	$-2v$	$-v$	$\frac{7-u-3v}{2}$	$\frac{7-u-v}{2}$	$7-u$	0
		$N(u,v)$	$2v$	0	$\frac{v}{2}$	$\frac{v}{2}$	0	0

Table 8: Zariski decomposition of the divisor $\zeta_0^*(L_u)$

u	$P_{\widetilde{W}}(u)$ & $N_{\widetilde{W}}(u)$	$\widetilde{T}_0$	$\widetilde{T}_1$	$\widetilde{T}_2$	$\widetilde{T}_3$	$\widetilde{T}_7$	$\widetilde{T}_8$	$\widetilde{T}_9$	$\widetilde{T}_{10}$
$[0,1]$	$P_{\widetilde{W}}(u)$	$10-u$	1	1	2	8	5	8	5
	$N_{\widetilde{W}}(u)$	0	0	0	0	0	0	0	0
$[1,2]$	$P_{\widetilde{W}}(u)$	$10-u$	1	1	2	8	$\frac{11-u}{2}$	$\frac{35-3u}{4}$	5
	$N_{\widetilde{W}}(u)$	0	0	0	0	0	$\frac{u-1}{2}$	$\frac{3(u-1)}{4}$	0
$[2,3]$	$P_{\widetilde{W}}(u)$	$10-u$	1	1	2	$10-u$	$\frac{11-u}{2}$	$\frac{35-3u}{4}$	5
	$N_{\widetilde{W}}(u)$	0	0	0	0	$u-2$	$\frac{u-1}{2}$	$\frac{3(u-1)}{4}$	0

$[3,5]$	$P_{\widetilde{W}}(u)$	$10-u$	1	1	$\frac{11-u}{4}$	$10-u$	$\frac{11-u}{2}$	$\frac{35-3u}{4}$	5
	$N_{\widetilde{W}}(u)$	0	0	0	$\frac{u-3}{4}$	$u-2$	$\frac{u-1}{2}$	$\frac{3(u-1)}{4}$	0
$[5,7]$	$P_{\widetilde{W}}(u)$	$10-u$	1	1	$\frac{11-u}{4}$	$10-u$	$\frac{11-u}{2}$	$10-u$	$10-u$
	$N_{\widetilde{W}}(u)$	0	0	0	$\frac{u-3}{4}$	$u-2$	$\frac{u-1}{2}$	$u-2$	$u-5$
$[7,8]$	$P_{\widetilde{W}}(u)$	$10-u$	1	$\frac{10-u}{3}$	$\frac{10-u}{3}$	$10-u$	$\frac{2(10-u)}{3}$	$10-u$	$10-u$
	$N_{\widetilde{W}}(u)$	0	0	$\frac{u-7}{3}$	$\frac{u-4}{3}$	$u-2$	$\frac{2u-5}{3}$	$u-2$	$u-5$
$[8,10]$	$P_{\widetilde{W}}(u)$	$10-u$	$\frac{10-u}{2}$	$\frac{10-u}{3}$	$\frac{10-u}{3}$	$10-u$	$\frac{2(10-u)}{3}$	$10-u$	$10-u$
	$N_{\widetilde{W}}(u)$	0	$\frac{u-8}{2}$	$\frac{u-7}{3}$	$\frac{u-4}{3}$	$u-2$	$\frac{2u-5}{3}$	$u-2$	$u-5$

Table 9: Expressions for $\widetilde{P}(u)$ and $\widetilde{N}(u)$

u	$\widetilde{P}(u)$ & $\widetilde{N}(u)$	α_1	α_2	α_3	α_4	α_5	α_6
$[0,1]$	$\widetilde{P}(u)$	$\frac{u-8}{2}$	$u-2$	$\frac{u}{2}$	2	4	0
	$\widetilde{N}(u)$	0	0	0	0	0	0
$[1,2]$	$\widetilde{P}(u)$	$\frac{u-8}{2}$	$\frac{u-5}{4}$	$\frac{1}{2}$	2	4	0
	$\widetilde{N}(u)$	0	$\frac{3(u-1)}{4}$	$\frac{u-1}{2}$	0	0	0
$[2,3]$	$\widetilde{P}(u)$	$\frac{u-8}{2}$	$\frac{u-5}{4}$	$\frac{1}{2}$	2	$\frac{10-u}{2}$	0
	$\widetilde{N}(u)$	0	$\frac{3(u-1)}{4}$	$\frac{u-1}{2}$	0	$\frac{u-2}{2}$	0
$[3,5]$	$\widetilde{P}(u)$	$\frac{u-8}{2}$	$\frac{u-5}{4}$	$\frac{1}{2}$	$\frac{11-u}{4}$	$\frac{10-u}{2}$	0
	$\widetilde{N}(u)$	0	$\frac{3(u-1)}{4}$	$\frac{u-1}{2}$	$\frac{u-3}{4}$	$\frac{u-2}{2}$	0
$[5,7]$	$\widetilde{P}(u)$	$\frac{u-8}{2}$	0	$\frac{1}{2}$	$\frac{11-u}{4}$	$\frac{10-u}{2}$	0
	$\widetilde{N}(u)$	0	$u-2$	$\frac{u-1}{2}$	$\frac{u-3}{4}$	$\frac{u-2}{2}$	0
$[7,8]$	$\widetilde{P}(u)$	$\frac{u-8}{2}$	0	$\frac{10-u}{6}$	$\frac{10-u}{3}$	$\frac{10-u}{2}$	0
	$\widetilde{N}(u)$	0	$u-2$	$\frac{2u-5}{3}$	$\frac{u-4}{3}$	$\frac{u-2}{2}$	0
$[8,10]$	$\widetilde{P}(u)$	0	0	$\frac{10-u}{6}$	$\frac{10-u}{3}$	$\frac{10-u}{2}$	0
	$\widetilde{N}(u)$	$\frac{u-8}{2}$	$u-2$	$\frac{2u-5}{3}$	$\frac{u-4}{3}$	$\frac{u-2}{2}$	0

Table 10: Values of $t(u)$

C \ u	$[0,1]$	$[1,2]$	$[2,3]$	$[3,5]$	[5,6]	$[6,7]$	$[7,8]$	$[8,10]$
$\overline{\alpha}_1$	$\frac{u}{2}$	$\frac{u}{2}$	1	1	1	1	1	$\frac{10-u}{2}$
$\overline{\alpha}_4$	$\frac{u}{4}$	$\frac{u}{4}$	$\frac{u}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{16-u}{12}$	$\frac{10-u}{3}$

$\overline{\alpha}_6$	$\frac{u}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{10-u}{6}$	$\frac{10-u}{6}$
$\overline{\alpha}_0$	$\frac{u}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{10-u}{8}$	$\frac{10-u}{8}$	$\frac{10-u}{8}$

Table 11: Expressions for $P(u,v)$ and $N(u,v)$ in the case $C=\overline{\alpha}_1$

u	v	$P(u,v)$ & $N(u,v)$	α_1	α_2	α_3	α_4	α_5	α_6
$[0,1]$	$[0,\frac{u}{2}]$	$P(u,v)$	$\frac{u-8}{2}-v$	$u-2-2v$	$\frac{u}{2}-v$	2	4	0
		$N(u,v)$	0	$2v$	v	0	0	0
$[1,2]$	$[0,\frac{u-1}{2}]$	$P(u,v)$	$\frac{u-8}{2}-v$	$\frac{u-5-2v}{4}$	$\frac{1}{2}$	2	4	0
		$N(u,v)$	0	$\frac{v}{2}$	0	0	0	0
$[1,2]$	$[\frac{u-1}{2},\frac{u}{2}]$	$P(u,v)$	$\frac{u-8}{2}-v$	$u-2-2v$	$\frac{u-2v}{2}$	2	4	0
		$N(u,v)$	0	$\frac{3-3u+8v}{4}$	$\frac{2v-u+1}{2}$	0	0	0
$[2,3]$	$[0,\frac{u-1}{2}]$	$P(u,v)$	$\frac{u-8}{2}-v$	$\frac{u-5-2v}{4}$	$\frac{1}{2}$	2	$\frac{10-u}{2}$	0
		$N(u,v)$	0	$\frac{v}{2}$	0	0	0	0
$[2,3]$	$[\frac{u-1}{2},1]$	$P(u,v)$	$\frac{u-8}{2}-v$	$u-2-2v$	$\frac{u-2v}{2}$	2	$\frac{10-u}{2}$	0
		$N(u,v)$	0	$\frac{3-3u+8v}{4}$	$\frac{2v-u+1}{2}$	0	0	0
$[3,5]$	$[0,1]$	$P(u,v)$	$\frac{u-8}{2}-v$	$\frac{u-5-2v}{4}$	$\frac{1}{2}$	$\frac{11-u}{4}$	$\frac{10-u}{2}$	0
		$N(u,v)$	0	$\frac{v}{2}$	0	0	0	0
$[5,7]$	$[0,\frac{u-5}{2}]$	$P(u,v)$	$\frac{u-8}{2}-v$	0	$\frac{1}{2}$	$\frac{11-u}{4}$	$\frac{10-u}{2}$	0
		$N(u,v)$	0	0	0	0	0	0
$[5,7]$	$[\frac{u-5}{2},1]$	$P(u,v)$	$\frac{u-8}{2}-v$	$\frac{u-5-2v}{4}$	$\frac{1}{2}$	$\frac{11-u}{4}$	$\frac{10-u}{2}$	0
		$N(u,v)$	0	$\frac{2v-u+5}{4}$	0	0	0	0
$[7,8]$	$[0,1]$	$P(u,v)$	$\frac{u-8}{2}-v$	0	$\frac{10-u}{6}$	$\frac{10-u}{3}$	$\frac{10-u}{2}$	0
		$N(u,v)$	0	0	0	0	0	0
$[8,10]$	$[0,\frac{10-u}{2}]$	$P(u,v)$	$-v$	0	$\frac{10-u}{6}$	$\frac{10-u}{3}$	$\frac{10-u}{2}$	0
		$N(u,v)$	0	0	0	0	0	0

Table 12: Expressions for $P(u,v)$ and $N(u,v)$ in the case $C=\overline{\alpha}_4$

u	v	$P(u,v)$ & $N(u,v)$	α_1	α_2	α_3	α_4	α_5	α_6
$[0,1]$	$[0,\frac{u}{4}]$	$P(u,v)$	$\frac{u-8}{2}$	$u-2-3v$	$\frac{u}{2}-2v$	$2-v$	$4-2v$	0
		$N(u,v)$	0	$3v$	$2v$	0	$2v$	0

$[1,2]$	$[0, \frac{u-1}{4}]$	$P(u,v)$	$\frac{u-8}{2}$	$\frac{u-5}{4}$	$\frac{1}{2}$	$2-v$	$4-2v$	0
		$N(u,v)$	0	0	0	0	$2v$	0
$[1,2]$	$[\frac{u-1}{4}, \frac{u}{4}]$	$P(u,v)$	$\frac{u-8}{2}$	$u-2-3v$	$\frac{u}{2}-2v$	$2-v$	$4-2v$	0
		$N(u,v)$	0	$\frac{3(4v-u+1)}{4}$	$\frac{4v-u+1}{2}$	0	$2v$	0
$[2,3]$	$[0, \frac{u-2}{4}]$	$P(u,v)$	$\frac{u-8}{2}$	$\frac{u-5}{4}$	$\frac{1}{2}$	$2-v$	$\frac{10-u}{2}$	0
		$N(u,v)$	0	0	0	0	0	0
$[2,3]$	$[\frac{u-2}{4}, \frac{u-1}{4}]$	$P(u,v)$	$\frac{u-8}{2}$	$\frac{u-5}{4}$	$\frac{1}{2}$	$2-v$	$4-2v$	0
		$N(u,v)$	0	0	0	0	$\frac{4v-u+2}{2}$	0
$[2,3]$	$[\frac{u-1}{4}, \frac{u}{4}]$	$P(u,v)$	$\frac{u-8}{2}$	$u-2-3v$	$\frac{u}{2}-2v$	$2-v$	$4-2v$	0
		$N(u,v)$	0	$\frac{3(4v-u+1)}{4}$	$\frac{4v-u+1}{2}$	0	$\frac{4v-u+2}{2}$	0
$[3,5]$	$[0, \frac{1}{4}]$	$P(u,v)$	$\frac{u-8}{2}$	$\frac{u-5}{4}$	$\frac{1}{2}$	$\frac{11-u-4v}{4}$	$\frac{10-u}{2}$	0
		$N(u,v)$	0	0	0	0	0	0
$[3,5]$	$[\frac{1}{4}, \frac{1}{2}]$	$P(u,v)$	$\frac{u-8}{2}$	$\frac{u-5}{4}$	$\frac{1}{2}$	$\frac{11-u-4v}{4}$	$\frac{11-u-4v}{2}$	0
		$N(u,v)$	0	0	0	0	$\frac{4v-1}{2}$	0
$[3,5]$	$[\frac{1}{2}, \frac{3}{4}]$	$P(u,v)$	$\frac{u-8}{2}$	$\frac{u+1-12v}{4}$	$\frac{3-2v}{2}$	$\frac{11-u-4v}{4}$	$\frac{11-u-4v}{2}$	0
		$N(u,v)$	0	$\frac{3(2v-1)}{2}$	$2v-1$	0	$\frac{4v-1}{2}$	0
$[5,6]$	$[0, \frac{1}{4}]$	$P(u,v)$	$\frac{u-8}{2}$	0	$\frac{1}{2}$	$\frac{11-u-4v}{4}$	$\frac{10-u}{2}$	0
		$N(u,v)$	0	0	0	0	0	0
$[5,6]$	$[\frac{1}{4}, \frac{7-u}{4}]$	$P(u,v)$	$\frac{u-8}{2}$	0	$\frac{1}{2}$	$\frac{11-u-4v}{4}$	$\frac{11-u-4v}{2}$	0
		$N(u,v)$	0	0	0	0	$\frac{4v-1}{2}$	0
$[5,6]$	$[\frac{7-u}{4}, \frac{1+u}{12}]$	$P(u,v)$	$\frac{u-8}{2}$	0	$\frac{11-u-4v}{8}$	$\frac{11-u-4v}{4}$	$\frac{11-u-4v}{2}$	0
		$N(u,v)$	0	0	$\frac{4v+u-7}{8}$	0	$\frac{4v-1}{2}$	0
$[5,6]$	$[\frac{1+u}{12}, \frac{3}{4}]$	$P(u,v)$	$\frac{u-8}{2}$	$\frac{1+u-6v}{4}$	$\frac{3-4v}{2}$	$\frac{11-u-4v}{4}$	$\frac{11-u-4v}{2}$	0
		$N(u,v)$	0	$\frac{12v+u-1}{4}$	$2v-1$	0	$\frac{4v-1}{2}$	0
$[6,7]$	$[0, \frac{7-u}{4}]$	$P(u,v)$	$\frac{u-8}{2}$	0	$\frac{1}{2}$	$\frac{11-u-4v}{4}$	$\frac{10-u}{2}$	0
		$N(u,v)$	0	0	0	0	0	0
$[6,7]$	$[\frac{7-u}{4}, \frac{1}{4}]$	$P(u,v)$	$\frac{u-8}{2}$	0	$\frac{11-u-4v}{8}$	$\frac{11-u-4v}{4}$	$\frac{10-u}{2}$	0
		$N(u,v)$	0	0	$\frac{4v+u-7}{8}$	0	0	0
$[6,7]$	$[\frac{1}{4}, \frac{1+u}{12}]$	$P(u,v)$	$\frac{u-8}{2}$	0	$\frac{11-u-4v}{8}$	$\frac{11-u-4v}{4}$	$\frac{11-u-4v}{2}$	0
		$N(u,v)$	0	0	$\frac{4v+u-7}{8}$	0	$\frac{4v-1}{2}$	0
$[6,7]$	$[\frac{1+u}{12}, \frac{3}{4}]$	$P(u,v)$	$\frac{u-8}{2}$	$\frac{1+u-6v}{4}$	$\frac{3-4v}{2}$	$\frac{11-u-4v}{4}$	$\frac{11-u-4v}{2}$	0
		$N(u,v)$	0	$\frac{12v+u-1}{4}$	$2v-1$	0	$\frac{4v-1}{2}$	0

$[7,8]$	$[0, \frac{10-u}{12}]$	$P(u,v)$	$\frac{u-8}{2}$	0	$\frac{10-u-3v}{6}$	$\frac{10-u-3v}{3}$	$\frac{10-u}{2}$	0
		$N(u,v)$	0	0	$\frac{v}{2}$	0	0	0
$[7,8]$	$[\frac{10-u}{12}, \frac{2}{3}]$	$P(u,v)$	$\frac{u-8}{2}$	0	$\frac{10-u-3v}{6}$	$\frac{10-u-3v}{3}$	$\frac{2(10-u-3v)}{3}$	0
		$N(u,v)$	0	0	$\frac{v}{2}$	0	$\frac{12v+u-10}{6}$	0
$[7,8]$	$[\frac{2}{3}, \frac{16-u}{12}]$	$P(u,v)$	$\frac{u-8}{2}$	$2-3v$	$\frac{16-u-12v}{6}$	$\frac{10-u-3v}{3}$	$\frac{2(10-u-3v)}{3}$	0
		$N(u,v)$	0	$3v-2$	$2v-1$	0	$\frac{12v+u-10}{6}$	0
$[8,10]$	$[0, \frac{10-u}{12}]$	$P(u,v)$	0	0	$\frac{10-u-3v}{6}$	$\frac{10-u-3v}{3}$	$\frac{10-u}{2}$	0
		$N(u,v)$	0	0	$\frac{v}{2}$	0	0	0
$[8,10]$	$[\frac{10-u}{12}, \frac{10-u}{3}]$	$P(u,v)$	0	0	$\frac{10-u-3v}{6}$	$\frac{10-u-3v}{3}$	$\frac{2(10-u-3v)}{3}$	0
		$N(u,v)$	0	0	$\frac{v}{2}$	0	$\frac{12v+u-10}{6}$	0

Table 13: Expressions for $P(u,v)$ and $N(u,v)$ in the case $C = \overline{\alpha}_6$

u	v	$P(u,v)$ & $N(u,v)$	α_1	α_2	α_3	α_4	α_5	α_6
$[0,1]$	$[0, \frac{u}{2}]$	$P(u,v)$	$\frac{u-8}{2}$	$u-2$	$\frac{u}{2}$	2	$4-v$	$-v$
		$N(u,v)$	0	0	0	0	v	0
$[1,2]$	$[0, \frac{1}{2}]$	$P(u,v)$	$\frac{u-8}{2}$	$\frac{u-5}{4}$	$\frac{u}{2}$	2	$4-v$	$-v$
		$N(u,v)$	0	0	0	0	v	0
$[2,3]$	$[0, \frac{u-2}{2}]$	$P(u,v)$	$\frac{u-8}{2}$	$\frac{u-5}{4}$	$\frac{1}{2}$	2	$\frac{10-u}{2}$	$-v$
		$N(u,v)$	0	0	0	0	0	0
$[2,3]$	$[\frac{u-2}{2}, \frac{1}{2}]$	$P(u,v)$	$\frac{u-8}{2}$	$\frac{u-5}{4}$	$\frac{1}{2}$	$\frac{10-u}{2}$	$4-v$	$-v$
		$N(u,v)$	0	0	0	0	$\frac{2v+2-u}{2}$	0
$[3,5]$	$[0, \frac{1}{2}]$	$P(u,v)$	$\frac{u-8}{2}$	$\frac{u-5}{4}$	$\frac{1}{2}$	$\frac{11-u}{4}$	$\frac{10-u}{2}$	$-v$
		$N(u,v)$	0	0	0	0	0	0
$[5,7]$	$[0, \frac{8-u}{6}]$	$P(u,v)$	$\frac{u-8}{2}$	0	$\frac{1}{2}$	$\frac{11-u}{4}$	$\frac{10-u}{2}$	$-v$
		$N(u,v)$	0	0	0	0	0	0
$[5,7]$	$[\frac{8-u}{6}, \frac{1}{2}]$	$P(u,v)$	$-3v$	0	$\frac{1}{2}$	$\frac{11-u}{4}$	$\frac{10-u}{2}$	$-v$
		$N(u,v)$	$\frac{6v+u-8}{2}$	0	0	0	0	0
$[7,8]$	$[0, \frac{8-u}{6}]$	$P(u,v)$	$\frac{u-8}{2}$	0	$\frac{10-u}{6}$	$\frac{10-u}{3}$	$\frac{10-u}{2}$	$-v$
		$N(u,v)$	0	0	0	0	0	0
$[7,8]$	$[\frac{8-u}{6}, \frac{10-u}{6}]$	$P(u,v)$	$-3v$	0	$\frac{10-u}{6}$	$\frac{10-u}{3}$	$\frac{10-u}{2}$	$-v$
		$N(u,v)$	$\frac{6v+u-8}{2}$	0	0	0	0	0

$[8,10]$	$[0,\frac{10-u}{6}]$	$P(u,v)$	$-3v$	0	$\frac{10-u}{6}$	$\frac{10-u}{3}$	$\frac{10-u}{2}$	$-v$
		$N(u,v)$	$3v$	0	0	0	0	0

Table 14: Expressions for $P(u,v)$ and $N(u,v)$ in the case $C=\overline{\alpha}_0$

u	v	$P(u,v)$ & $N(u,v)$	α_1	α_2	α_3	α_4	α_5	α_6
$[0,1]$	$[0,\frac{u}{2}]$	$P(u,v)$	$\frac{u-8}{2}-v$	$u-2-2v$	$\frac{u}{2}-v$	2	4	0
		$N(u,v)$	0	0	0	0	0	0
$[1,2]$	$[0,\frac{1}{2}]$	$P(u,v)$	$\frac{u-8}{2}-v$	$\frac{u-5-8v}{4}$	$\frac{1}{2}-v$	2	4	0
		$N(u,v)$	0	0	0	0	0	0
$[2,3]$	$[0,\frac{3-u}{2}]$	$P(u,v)$	$\frac{u-8}{2}-v$	$\frac{u-5-8v}{4}$	$\frac{1}{2}-v$	2	$\frac{10-u}{2}$	0
		$N(u,v)$	0	0	0	0	0	0
$[2,3]$	$[\frac{3-u}{2},\frac{1}{2}]$	$P(u,v)$	$\frac{u-8}{2}-v$	$\frac{u-5-8v}{4}$	$\frac{1}{2}-v$	$\frac{11-u-2v}{4}$	$\frac{10-u}{2}$	0
		$N(u,v)$	0	0	0	$\frac{2v+u-3}{4}$	0	0
$[3,5]$	$[0,\frac{1}{2}]$	$P(u,v)$	$\frac{u-8}{2}-v$	$\frac{u-5-8v}{4}$	$\frac{1}{2}-v$	$\frac{11-u-2v}{4}$	$\frac{10-u}{2}$	0
		$N(u,v)$	0	0	0	$\frac{v}{2}$	0	0
$[5,6]$	$[0,\frac{8-u}{6}]$	$P(u,v)$	$\frac{u-8}{2}-v$	$-2v$	$\frac{1}{2}-v$	$\frac{11-u-2v}{4}$	$\frac{10-u}{2}$	0
		$N(u,v)$	0	0	0	$\frac{v}{2}$	0	0
$[5,6]$	$[\frac{8-u}{6},\frac{1}{2}]$	$P(u,v)$	$-4v$	$-2v$	$\frac{1}{2}-v$	$\frac{11-u-2v}{4}$	$\frac{10-u}{2}$	0
		$N(u,v)$	$\frac{6v+u-8}{2}$	0	0	$\frac{v}{2}$	0	0
$[6,\frac{13}{2}]$	$[0,\frac{8-u}{6}]$	$P(u,v)$	$\frac{u-8}{2}-v$	$-2v$	$\frac{1}{2}-v$	$\frac{11-u-2v}{4}$	$\frac{10-u}{2}$	0
		$N(u,v)$	0	0	0	$\frac{v}{2}$	0	0
$[6,\frac{13}{2}]$	$[\frac{8-u}{6},\frac{7-u}{2}]$	$P(u,v)$	$-4v$	$-2v$	$\frac{1}{2}-v$	$\frac{11-u-2v}{4}$	$\frac{10-u}{2}$	0
		$N(u,v)$	$\frac{6v+u-8}{2}$	0	0	$\frac{v}{2}$	0	0
$[6,\frac{13}{2}]$	$[\frac{7-u}{2},\frac{10-u}{8}]$	$P(u,v)$	$-4v$	$-2v$	$\frac{10-u-8v}{6}$	$\frac{10-u-2v}{3}$	$\frac{10-u}{2}$	0
		$N(u,v)$	$\frac{6v+u-8}{2}$	0	$\frac{2v+u-7}{6}$	$\frac{8v+u-7}{12}$	0	0
$[\frac{13}{2},7]$	$[0,\frac{7-u}{2}]$	$P(u,v)$	$\frac{u-8}{2}-v$	$-2v$	$\frac{1}{2}-v$	$\frac{11-u-2v}{4}$	$\frac{10-u}{2}$	0
		$N(u,v)$	0	0	0	$\frac{v}{2}$	0	0
$[\frac{13}{2},7]$	$[\frac{7-u}{2},\frac{8-u}{6}]$	$P(u,v)$	$\frac{u-8}{2}-v$	$-2v$	$\frac{10-u-8v}{6}$	$\frac{10-u-2v}{3}$	$\frac{10-u}{2}$	0
		$N(u,v)$	0	0	$\frac{2v+u-7}{6}$	$\frac{8v+u-7}{12}$	0	0
$[\frac{13}{2},7]$	$[\frac{8-u}{6},\frac{10-u}{8}]$	$P(u,v)$	$-4v$	$-2v$	$\frac{10-u-8v}{6}$	$\frac{10-u-2v}{3}$	$\frac{10-u}{2}$	0
		$N(u,v)$	$\frac{6v+u-8}{2}$	0	$\frac{2v+u-7}{6}$	$\frac{8v+u-7}{12}$	0	0

$[7,8]$	$[0,\frac{8-u}{6}]$	$P(u,v)$	$\frac{u-8}{2}-v$	$-2v$	$\frac{10-u-8v}{6}$	$\frac{10-u-2v}{3}$	$\frac{10-u}{2}$	0
		$N(u,v)$	0	0	$\frac{v}{3}$	$\frac{2v}{3}$	0	0
$[7,8]$	$[\frac{8-u}{6},\frac{10-u}{8}]$	$P(u,v)$	$-4v$	$-2v$	$\frac{10-u-8v}{6}$	$\frac{10-u-2v}{3}$	$\frac{10-u}{2}$	0
		$N(u,v)$	$\frac{6v+u-8}{2}$	0	$\frac{v}{3}$	$\frac{2v}{3}$	0	0
$[8,10]$	$[0,\frac{10-u}{8}]$	$P(u,v)$	$-4v$	$-2v$	$\frac{10-u-8v}{6}$	$\frac{10-u-2v}{3}$	$\frac{10-u}{2}$	0
		$N(u,v)$	$3v$	0	$\frac{v}{3}$	$\frac{2v}{3}$	0	0

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Ivan Cheltsov
University of Edinburgh, Edinburgh, Scotland
i.cheltsov@ed.ac.uk

Kento Fujita
Osaka University, Osaka, Japan
fujita@math.sci.osaka-u.ac.jp

Takashi Kishimoto
Saitama University, Saitama, Japan
kisimoto.takasi@gmail.com

Jihun Park
Institute for Basic Science, Pohang, Korea
POSTECH, Pohang, Korea
wlog@postech.ac.kr