ON MAXIMALLY NON-FACTORIAL NODAL FANO THREEFOLDS

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Abstract. We classify non-factorial nodal Fano threefolds with 1 node and class group of rank 2.

Let X be a Fano threefold that has at worst isolated ordinary double points (nodes). Then both the Picard group $\mathrm{Pic}(X)$ and the class group $\mathrm{Cl}(X)$ are torsion-free of finite rank, and the number

$$\mathrm{rk}\,\mathrm{Cl}(X) - \mathrm{rk}\,\mathrm{Pic}(X)$$

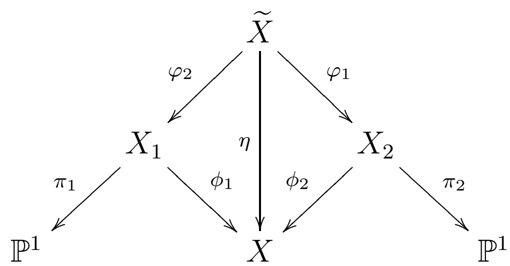
is known as the *defect* of X [14, 19, 20, 32]. If the defect is zero, we say that X is *factorial* [7, 8]. Factoriality imposes significant constraints on the geometry of the Fano threefold [9, 11, 37, 46].

The defect of the Fano threefold X does not exceed the number of its singular points [40]. If

$$\mathrm{rk}\,\mathrm{Cl}(X) - \mathrm{rk}\,\mathrm{Pic}(X) = |\mathrm{Sing}(X)|,$$

then X is said to be $\mathbb{Q}$-*maximally non-factorial* [36, Definition 6.10]. If X has a single node, then the threefold X is $\mathbb{Q}$-maximally non-factorial if and only if it is non-factorial.

Example. Let X be the quadric cone in $\mathbb{P}^4$ with one node. Then X is $\mathbb{Q}$-maximally non-factorial nodal Fano threefold. Let $\eta\colon \widetilde{X} \to X$ be the blow up of the singular point of the threefold X, and let E be the η-exceptional surface. Then $E \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $E|_E \cong \mathcal{O}_E(-1,-1)$, which implies that there exists the following commutative diagram:



where φ_1 and φ_2 are contractions of the surface E to curves such that $\varphi_1 \circ \varphi_2^{-1}$ is an Atiyah flop, both ϕ_1 and ϕ_2 are small projective resolutions, and both π_1 and π_2 are $\mathbb{P}^2$-bundles.

$\mathbb{Q}$-maximally non-factorial nodal Fano threefolds are very special from the perspective of derived categories of coherent sheaves, in particular on the derived categories level they behave almost as if they were smooth [31, 36, 40]. On the other hand, $\mathbb{Q}$-maximally non-factorial Fano threefolds are rather rare among all nodal Fano threefolds. It seems natural to pose the following problem.

Problem. *Classify all $\mathbb{Q}$-maximally non-factorial nodal Fano threefolds.*

The goal of this paper is to partially solve this problem. Namely, we aim to classify $\mathbb{Q}$-maximally non-factorial nodal Fano threefolds of Picard rank one that have exactly one singular point (node). Before we present our classification, let us remind the following construction of Yuri Prokhorov.

Construction ([44, § 3.4 Case 4^o]). Let $\overline{E} = \{z_1 = z_2 = 0\} \subset \mathbb{P}^2_{x_1,y_1,z_1} \times \mathbb{P}^2_{x_2,y_2,z_2}$, and let

$$\overline{X} = \big\{z_1 f(x_1, y_1, z_1; x_2, y_2, z_2) = z_2 g(x_1, y_1, z_1; x_2, y_2, z_2)\big\},$$

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Throughout this paper, all varieties are assumed to be projective and defined over $\mathbb{C}$.

where f and g are some sufficiently general polynomials of bi-degrees $(1,2)$ and $(2,1)$, respectively. Then $\overline{X}$ is a singular Verra threefold (a hypersurface of bidegree $(2,2)$ in $\mathbb{P}^2\times\mathbb{P}^2$) with 5 nodes. Note that $\overline{E}\cong\mathbb{P}^1\times\mathbb{P}^1$, $\overline{E}\subset\overline{X}$ and

$$\mathrm{Sing}(\overline{X})=\big\{z_1=z_2=f=g=0\big\}\subset\overline{E}.$$

Let $\rho\colon\mathbb{P}^2_{x_1,y_1,z_1}\times\mathbb{P}^2_{x_2,y_2,z_2}\dashrightarrow\mathbb{P}^4_{x,y,z,t,w}$ be the rational map given by

$$\big([x_1:y_1:z_1],[x_2:y_2:z_2]\big)\mapsto\big[x_1z_2:y_1z_2:x_2z_1:y_2z_1:z_1z_2\big].$$

Then ρ is birational, and the inverse map ρ^{-1} is given by $[x:y:z:t:w]\mapsto([x:y:w],[z:t:w])$. Let $\xi\colon W\to\mathbb{P}^2_{x_1,y_1,z_1}\times\mathbb{P}^2_{x_2,y_2,z_2}$ be the blow up of the surface $\overline{E}$, let $\mathscr{E}$ be its exceptional divisor, let $\overline{G}_1=\{z_1=0\}$ and $\overline{G}_2=\{z_2=0\}$, let G_1 and G_2 be proper transforms on W of $\overline{G}_1$ and $\overline{G}_2$. Then we have the following commutative diagram:

$$\begin{array}{ccc} & W & \\ {\scriptstyle\xi}\swarrow & & \searrow{\scriptstyle\theta} \\ \mathbb{P}^2_{x_1,y_1,z_1}\times\mathbb{P}^2_{x_2,y_2,z_2} & \overset{\rho}{\dashrightarrow} & \mathbb{P}^4_{x,y,z,t,w}\end{array}$$

where θ blows down G_1 and G_2 to the lines $\ell_1=\{z=t=w=0\}$ and $\ell_2=\{x=y=w=0\}$. Note that $\theta(\mathscr{E})$ is the hyperplane $\{w=0\}$ — the unique hyperplane containing the lines ℓ_1 and ℓ_2. Set $V=\rho(\overline{X})$. Then V is a smooth cubic threefold in $\mathbb{P}^4_{x,y,z,t,w}$. Moreover, we have

$$V=\big\{f(x,y,w;z,t,w)=g(x,y,w;z,t,w)\big\}\subset\mathbb{P}^4_{x,y,z,t,w}.$$

Now, let $\widehat{X}$ be the strict transform of the threefold $\overline{X}$ on W, let $\varsigma\colon\widehat{X}\to\overline{X}$ be the morphism induced by ξ, and let $\nu\colon\widehat{X}\to V$ be the morphism induced by θ. Then $\widehat{X}$ is smooth, ς is a small projective resolution, and we have the following commutative diagram:

$$\begin{array}{ccc} & \widehat{X} & \\ {\scriptstyle\varsigma}\swarrow & & \searrow{\scriptstyle\nu} \\ \overline{X} & \overset{\rho|_{\overline{X}}}{\dashrightarrow} & V\end{array}\qquad.$$

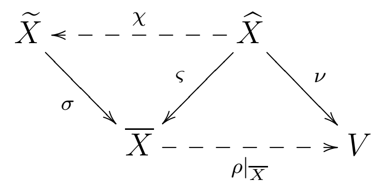
Note that ν is a blow up of the cubic threefold V along the lines ℓ_1 and ℓ_2. Let $\widehat{E}=\mathscr{E}|_{\widehat{X}}$. Then

- the induced map $\varsigma|_{\widehat{E}}\colon\widehat{E}\to\overline{E}$ is a blow up of the points $\mathrm{Sing}(\overline{X})$,
- $\widehat{E}$ is isomorphic to a smooth cubic surface,
- $\nu(\widehat{E})$ is the hyperplane section $\{w=0\}\cap V$.

Now, we complement the last commutative diagram by the following commutative diagram:

$$\begin{array}{ccccc} & & V & & \\ & {\scriptstyle\psi_1}\nearrow & \uparrow{\scriptstyle\nu} & \nwarrow{\scriptstyle\psi_2} & \\ V_1 & \overset{\nu_2}{\longleftarrow} & \widehat{X} & \overset{\nu_1}{\longrightarrow} & V_2 \\ \downarrow{\scriptstyle\upsilon_1} & & \downarrow{\scriptstyle\varsigma} & & \downarrow{\scriptstyle\upsilon_2} \\ \mathbb{P}^2_{x_1,y_1,z_1} & \overset{\mathrm{pr}_1}{\longleftarrow} & \overline{X} & \overset{\mathrm{pr}_2}{\longrightarrow} & \mathbb{P}^2_{x_2,y_2,z_2}\end{array}$$

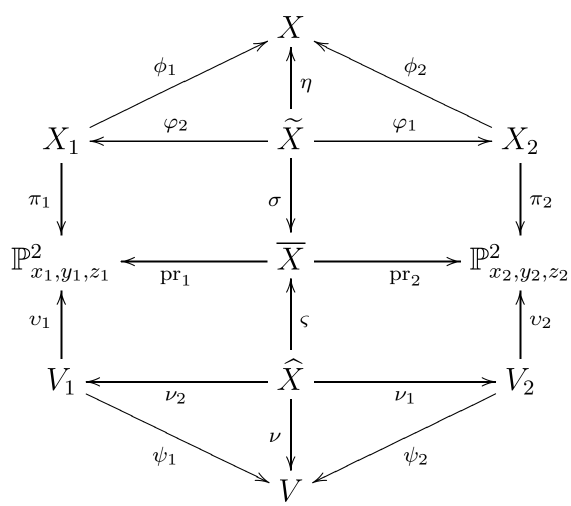
where ψ_1 and ψ_2 are blow ups of the lines ℓ_1 and ℓ_2, respectively, ν_1 and ν_2 are blow ups of the strict transforms of the lines ℓ_1 and ℓ_2, respectively, both υ_1 and υ_2 are standard conic bundles [42], and pr_1 and pr_2 are natural projections. Let Δ_1 and Δ_2 be the discriminant curves of the conic bundles υ_1 and υ_2, respectively. Then Δ_1 and Δ_2 are quintic curves with at most nodal singularities. Since ς is a flopping contraction, there exists a composition of flops $\chi\colon \widehat{X} \dashrightarrow \widetilde{X}$ of all curves contracted by ς. Then $\widetilde{X}$ is smooth and projective, and we have another commutative diagram:



where σ is a small resolution. Let $E = \chi(\widehat{E})$. Then χ induces a morphism $\widehat{E} \to E$ that blows down all five curves contracted by ς, which implies that σ induces an isomorphism $E \cong \overline{E} \cong \mathbb{P}^1 \times \mathbb{P}^1$. Note that $E|_E \sim \mathcal{O}_E(-1,-1)$, and there exists a birational morphism $\eta\colon \widetilde{X} \to X$ that blows down the surface E to an ordinary double point of the threefold X. We have $-K_X^3 = -K_{\widetilde{X}}^2 - 2 = 14$ and

$$1 = \operatorname{rk}\operatorname{Pic}(X) < \operatorname{rk}\operatorname{Cl}(X) = 1 + \big|\operatorname{Sing}(X)\big| = 2.$$

Therefore, the threefold X is a $\mathbb{Q}$-maximally non-factorial nodal Fano threefold that has one node. Summarizing, we have the following commutative diagram:



where ϕ_1 and ϕ_2 are two small resolutions such that the composition $\phi_1^{-1} \circ \phi_2$ is an Atiyah flop, both φ_1 and φ_2 are contractions of the surface E to curves, π_1 and π_2 are standard conic bundles whose discriminant curves are Δ_1 and Δ_2, respectively. Note that X is irrational as it is birational to a smooth cubic threefold [15], and

$$h^{1,2}(X_1) = h^{1,2}(X_2) = h^{1,2}(\widetilde{X}) = h^{1,2}(\widehat{X}) = h^{1,2}(V) = 5.$$

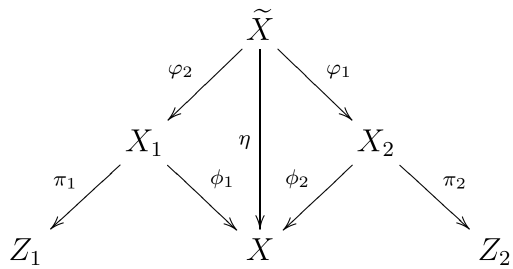
Instead of using the Verra threefold $\overline{X}$ containing $\overline{E}$, we can construct the nodal threefold X using the birational map ρ^{-1}, and the smooth cubic threefold V containing the lines ℓ_1 and ℓ_2.

Now, we are ready to present the main result of this paper. To do this, we suppose that

- the nodal Fano threefold X has one node,
- the rank of the Picard group $\operatorname{Pic}(X)$ is one,
- the rank of the class group $\operatorname{Cl}(X)$ is two.

Let $\eta\colon \widetilde{X} \to X$ be the blow up of the node of the threefold X, let E be the η-exceptional surface. Then $\widetilde{X}$ is smooth, $E \cong \mathbb{P}^1 \times \mathbb{P}^1$, $E|_E \simeq \mathcal{O}_E(-1,-1)$, and it follows from [16] that X uniquely determines the following Sarkisov link:

(★)



where φ_1 and φ_2 are contractions of the surface E to curves such that $\varphi_1 \circ \varphi_2^{-1}$ is an Atiyah flop, both ϕ_1 and ϕ_2 are small projective resolutions, and both π_1 and π_2 are extremal contractions [38]. Note that $-K_{X_1} \sim \phi_1^*(-K_X)$ and $-K_{X_2} \sim \phi_2^*(-K_X)$, so that

$$-K_{X_1}^3 = -K_{X_2}^3 = -K_X^3.$$

It follows from [39, 30] that X admits a smoothing $X \rightsquigarrow X_s$, where X_s is a smooth Fano threefold, $-K_X^3 = -K_{X_s}^3$, and the rank of the Picard group $\mathrm{Pic}(X_s)$ is 1. We also know from [14] that

(✠) $$h^{1,2}(\widetilde{X}) = h^{1,2}(X_1) = h^{1,2}(X_2) = h^{1,2}(X_s),$$

which imposes a significant constraint on the link (★). We set

$$d = -K_X^3,$$
$$h^{1,2} = h^{1,2}(X_s),$$

and

$$I = \max\big\{n \in \mathbb{Z}_{>0} \text{ such that } -K_{X_s} \sim nH \text{ for } H \in \mathrm{Pic}(X_s)\big\}.$$

Then I is the *index* of the Fano threefold X_s, which is also the index of the Fano threefold X [30].

In the remaining part of this paper, we prove the following theorem.

Theorem. *All possibilities for (★), up to swapping the left and right sides of the diagram, are described in the table at the end of the paper.*

Each Sarkisov link in the table exists and can be described explicitly (we provide the relevant references in the table). For the case of $-K_X^3 = 22$, our theorem follows from [41, Theorem 1.2]. Upon circulating a draft of our paper, we were informed that A. Kuznetsov and Yu. Prokhorov had independently obtained the same classification but their results are not publicly available yet.

Remark. It should be pointed out that it follows from our classification that one-nodal $\mathbb{Q}$-maximally non-factorial degenerations of smooth Fano threefolds of Picard rank one have the same rationality as their smoothing (in the cases **2** and **7** we need to assume that the Fano threefolds are general). Indeed, this can be verified case by case, using the rationality results from [3, 10, 15, 23, 42, 49].

Observation. If X is a del Pezzo threefold ($I = 2$) of Picard rank one such that $-K_X^3 \leqslant 32$, then the nodal Fano threefold X is never $\mathbb{Q}$-maximally non-factorial. This follows from [19, 20, 31, 40]. Therefore, the only options for X when $I > 1$ are these two Fano threefolds:

- the nodal quadric threefold in $\mathbb{P}^3$ ($I = 3$, $-K_X^3 = 54$, the Sarkisov link **17**);
- a quintic del Pezzo threefold ($I = 2$, $-K_X^3 = 40$, the Sarkisov link **16**).

We prove the theorem by analyzing the possible links (★) in the following order:

(1) π_1 is a del Pezzo fibration, and π_2 is arbitrary;
(2) both π_1 and π_2 are birational;
(3) π_1 is a conic bundle and π_2 is arbitrary.

This covers all possible Mori fiber spaces arising in (★), up to swapping π_1 and π_2.

Note that all possibilities for the smooth Fano variety X_s are known and can be found in [25]. Using this classification, we list the possible values of $h^{1,2}$ as follows.

(d, I)	$(2,1)$	$(4,1)$	$(6,1)$	$(8,1)$	$(10,1)$	$(12,1)$	$(14,1)$,	$(16,1)$	$(18,1)$	$(22,1)$
$h^{1,2}$	52	30	20	14	10	7	5	3	2	0

(d, I)	$(8,2)$	$(16,2)$	$(24,2)$	$(32,2)$	$(40,2)$	$(54,3)$	$(64,4)$
$h^{1,2}$	21	10	5	2	0	0	0

Possibilities for (★) are studied in [1, 4, 17, 18, 21, 22, 26, 27, 28, 29, 30, 33, 41, 43, 44, 47, 48, 50]. Using some of these results, we immediately obtain the following corollary.

Corollary. *Suppose that π_1 is a fibration into del Pezzo surfaces. Then (★) is one of the links*

1, **2**, **3**, **4**, **5**, **6**, **8**, **9**, **10**, **12**, **15**, **16**, **17**

in the table at the end of the paper.

Proof. If π_1 is a fibration into del Pezzo surfaces of degree 6, the assertion follows from [21, 22], in which case we get the link **15**. In the remaining cases, the required assertion follows from [48]. □

Therefore, we may assume that neither π_1 nor π_2 is a fibration into del Pezzo surfaces.

Proposition. *Suppose that π_1 and π_2 are birational. Then (★) is the link* **13** *in the table.*

Proof. Both Z_1 and Z_2 are (possibly singular) Fano threefolds, and $\operatorname{rk}\operatorname{Pic}(Z_1) = \operatorname{rk}\operatorname{Pic}(Z_2) = 1$.

Suppose that Z_1 is smooth (i.e. π_1 is a contraction of type E_1 or E_2 in [34, Theorem 1.32] and π_2 is a contraction of type $E_1 - E_5$). Then all possibilities for $h^{1,2}(Z_1)$ are listed in the two tables presented above. Using [18], we obtain all possible values of $h^{1,2}(X_1)$. Now, using (✠), in combination with the list of Sarkisov links in [18, Tables 1–7] we see, carrying out a case-by-case analysis, that

$$Z_1 \cong Z_2 \cong \mathbb{P}^3,$$

and both π_1 and π_2 are blow ups of smooth rational curves of degree 5. Alternatively, one can run a short computer programme exhausting all the possibilities for Z_1 and Z_2 and reach the same conclusion: both morphisms π_1 and π_2 are blow ups of $\mathbb{P}^3$ along smooth rational curves of degree 5. Observe also that none of these curves are contained in a quadric surface, because the birational morphisms ϕ_1 and ϕ_2 are small by construction. Therefore, the Sarkisov link (★) is the link **13** in the table presented at the end of the paper.

We may assume that Z_1 and Z_2 are singular. Now, using [18, Tables 8–9], we get $-K_X^3 \in \{2, 4\}$. Hence, if $|-K_X|$ does not have base points, then X is one of the following threefolds:

(1) sextic hypersurface in $\mathbb{P}(1,1,1,1,3)$,
(2) quartic hypersurface in $\mathbb{P}^4$,
(3) complete intersection of a quadric cone and a quartic hypersurface in $\mathbb{P}(1,1,1,1,1,2)$.

Indeed, if $|-K_X|$ does not have base points, then $|-K_X|$ gives a morphism $\phi\colon X \to \mathbb{P}^N$ such that the induced map $\varphi\colon X \to \phi(X)$ is finite, and

$$\deg\big(\phi(X)\big) \cdot \deg\big(\varphi\big) = -K_X^3.$$

If $-K_X^3 = 3$, then $N = 3$, $\phi(X) = \mathbb{P}^3$, and the morphism φ is a double cover ramified at a sextic hypersurface (by Hurwitz's formula), thus giving the first case. Similarly, if $-K_X^3 = 4$, then we obtain one of the last two cases. Now, studying the defect in each of these three cases, we see that the Fano threefold X must be factorial [6, 7, 8, 9, 46], which contradicts our initial assumption. This shows that the linear system $|-K_X|$ has base points.

Now, using [29, Theorem 1.1], we see that $-K_X^3 = 2$, and

$$X = \{x_0x_1 - x_2x_3 = 0, f_6(x_0, x_1, x_2, x_3, x_4) - x_5^2 = 0\} \subset \mathbb{P}(1, 1, 1, 1, 2, 3),$$

where f_6 is a quasi-homogeneous polynomial of degree 6, x_0, x_1, x_2, x_3 are coordinates of weight 1, and x_4 and x_5 are coordinates of weights 2 and 3, respectively. After a small resolution, the map

$$[x_0 : x_1 : x_2 : x_3 : x_4 : x_5] \mapsto [x_0 : x_2]$$

gives a fibration into del Pezzo surfaces of degree 1. Similarly, the map

$$[x_0 : x_1 : x_2 : x_3 : x_4 : x_5] \mapsto [x_0 : x_3]$$

gives another fibration into del Pezzo surfaces of degree 1. This implies that (★) is the link **1** in the table, so that π_2 is not birational, which contradicts our assumption. □

Thus, we may assume that π_1 is a conic bundle, and either π_2 is birational, or π_2 is a conic bundle. Then the surface Z_1 is smooth [38, (3.5.1)], which implies that $Z_1 = \mathbb{P}^2$, since X_1 has Picard rank 2. Let d_1 be the degree of the discriminant curve of the conic bundle π_1. Then [45, 1.6 Main Theorem] implies $d_1 \leqslant 11$, where $d_1 = 0$ if and only if π_1 is a $\mathbb{P}^1$-bundle. By [3, 51], we get

(♠) $$h^{1,2}(X_1) = \frac{d_1(d_1 - 3)}{2},$$

so $d_1 \notin \{1, 2\}$. Using (✠) and the list of possible values of $h^{1,2}$ presented in tables above, we get

$$d_1 \in \{0, 3, 4, 5, 7, 8\}.$$

Using the Observation above, *for the remaining part of the proof* we will always assume that $I = 1$. Therefore we have

(♢) $$(d, h^{1,2}, d_1) \in \{(6, 20, 8), (8, 14, 7), (14, 5, 5), (18, 1, 4), (22, 0, 0), (22, 0, 3)\}.$$

Let D_2 be a Cartier divisor on X_2, let D_1 be its strict transform on X_1, and let H_1 be a sufficiently general surface in $|\pi_1^*(\mathcal{O}_{\mathbb{P}^2}(1))|$. Then

$$D_1 \sim_{\mathbb{Q}} a(-K_{X_1}) - bH_1$$

for some rational numbers a and b. Moreover, if $d_1 \neq 0$, then both numbers a and b are integers. Similarly, if $d_1 = 0$, then $2a$ and $2b$ are integers. But we have (e.g. see [12, Lemma A.3])

$$-K_{X_1} \cdot D_1^2 = -K_{X_2} \cdot D_2^2,$$
$$\big(-K_{X_1}\big)^2 \cdot D_1 = \big(-K_{X_2}\big)^2 \cdot D_2.$$

Moreover, we have [5, Proposition 6]

$$-K_{X_1}^3 = d,\ (-K_{X_1})^2 \cdot H_1 = 12 - d_1,\ -K_{X_1} \cdot H_1^2 = 2,\ H_1^3 = 0.$$

This gives

(♡) $$\begin{cases} da^2 - 2(12 - d_1)ab + 2b^2 = -K_{X_2} \cdot D_2^2, \\ da - (12 - d_1)b = \big(-K_{X_2}\big)^2 \cdot D_2. \end{cases}$$

Lemma. *Suppose that π_2 is birational. Then (★) is either the link* **11** *or the link* **14** *in the table.*

Proof. To prove the lemma, we let D_2 be the π_2-exceptional surface. Then $a = D_1 \cdot H_1^2 \geqslant 0$.

If $\pi_2(D_2)$ is a point, it follows from [38, Theorem (3.3)] that one of the following cases holds:

(A) $D_2 = \mathbb{P}^2$ and $D_2|_{E_2}$ is a line bundle of degree -1,
(B) $D_2 = \mathbb{P}^2$ and $D_2|_{E_2}$ is a line bundle of degree -2,
(C) D_2 is an irreducible quadric surface in $\mathbb{P}^3$,

which implies that

$$-K_{X_2}\cdot D_2^2=\begin{cases}-2 \text{ in the case (A)},\\ -4 \text{ in the case (B)},\\ -2 \text{ in the case (C)},\end{cases}$$

and

$$\big(-K_{X_2}\big)^2\cdot D_2=\begin{cases}4 \text{ in the case (A)},\\ 1 \text{ in the case (B)},\\ 2 \text{ in the case (C)}.\end{cases}$$

Now, solving (♡) for each triple $(d,h^{1,2},d_1)$ listed in (♢), we see that $2a$ is never a non-negative integer. This shows that $\pi_2(D_2)$ is not a point.

We see that Z_2 is a smooth Fano threefold of Picard rank 1, and $\pi_2(D_2)$ is a smooth curve in Z_2. Then it follows from [28, Theorem 7.14] and (✠) that (★) is one of the Sarkisov links **11** and **14**, which would complete the proof of the lemma.

Note, however, that [28] has gaps [13, Remark 1.18]. For instance, the link in the construction contradicts [28, Theorem 7.4], and few examples constructed in [50] contradict [28, Proposition 7.2]. Keeping this in mind, let us complete the proof of the lemma without using [28, Theorem 7.14].

Set $C_2=\pi_2(D_2)$. Let $d_2=-K_{Z_2}\cdot C_2$, and let g_2 be the genus of the curve C_2. Then

$$h^{1,2}(Z_2)+g_2=h^{1,2}\in\big\{0,2,5,14,20\big\},$$

where the latter follows from (♠) and the further limitation imposed by $I=1$.

As a result, using the classification of smooth Fano threefolds, we get

$$h^{1,2}(Z_2)\in\big\{0,2,3,5,7,10,14,20\big\}.$$

In fact, we can say a bit more. Let $e=-K_{Z_2}^3$, let i be the index of the Fano threefold Z_2. Then

- $(e,i)=(64,4)\iff Z_2=\mathbb{P}^3$,
- $(e,i)=(54,3)\iff Z_2$ is a smooth quadric threefold in $\mathbb{P}^4$.

Moreover, the possible values of $h^{1,2}(Z_2)\leqslant 20$ can be listed as follows.

(e,i)	$(6,1)$	$(8,1)$	$(10,1)$	$(12,1)$	$(14,1)$,	$(16,1)$	$(18,1)$	$(22,1)$
$h^{1,2}(Z_2)$	20	14	10	7	5	3	2	0

(e,i)	$(16,2)$	$(24,2)$	$(32,2)$	$(40,2)$	$(54,3)$	$(64,4)$
$h^{1,2}(Z_2)$	10	5	2	0	0	0

This leaves not so many possibilities for the genus $g_2=h^{1,2}-h^{1,2}(Z_2)$.

One the other hand, it follows from [25, Lemma 4.1.2] that

$$\begin{aligned}-K_{X_2}\cdot D_2^2&=2g_2-2,\\ -(-K_{X_2})^2\cdot D_2&=d_2+2-2g_2,\\ -K_{X_2}^3&=e-2+2g_2-2d_2,\end{aligned}$$

so that (♡) gives

$$\begin{cases}da^2-2(12-d_1)ab+2b^2=2g_2-2,\\ da-(12-d_1)b=d_2+2-2g_2,\\ d=e-2+2g_2-2d_2.\end{cases}$$

Now, solving this system of equations for each triple $(d, I, h^{1,2}, d_1)$ listed in ($\diamondsuit$), and each possible triple $(e, i, g_2) = (e, i, h^{1,2} - h^{1,2}(Z_2))$, we obtain the following three cases:

(I) $d = 18$, $I = 1$, $h^{1,2} = 2$, $d_1 = 4$, $Z_2 = \mathbb{P}^3$, $d_2 = 24$, $g_2 = 2$, $a = 3$, $b = 4$;

(II) $d = 22$, $I = 1$, $h^{1,2} = 0$, $d_1 = 3$, Z_2 is a smooth quadric in $\mathbb{P}^4$, $d_2 = 15$, $g_2 = 0$, $a = 3$, $b = 4$;

In the case (I), ($\bigstar$) is the link **11** in the table. In the case (II), ($\bigstar$) is the link **14** in the table. $\square$

Therefore, we may assume that π_2 is also a conic bundle. Then $Z_2 = \mathbb{P}^2$, and the discriminant curve of the conic bundle π_2 must also have degree d_1, since

$$\frac{d_2(d_2-3)}{2} = h^{1,2}(X_2) = h^{1,2}(X_1) = \frac{d_1(d_1-3)}{2}.$$

Now, we let D_2 be a general surface in $|\pi_2^*(\mathcal{O}_{\mathbb{P}^2}(1))|$. Then ($\heartsuit$) simplifies as

$$\begin{cases} da^2 - 2(12-d_1)ab + 2b^2 = 2, \\ da - (12-d_1)b = 12 - d_1. \end{cases}$$

Solving these equations for each quadruple $(d, h^{1,2}, d_1)$ listed in ($\diamondsuit$), we get the following cases:

(1) $a = 0$, $b = -1$;

(2) $d = 14$, $I = 1$, $h^{1,2} = 5$, $d_1 = 5$, $a = 1$, $b = 1$.

In the case (1), the composition $\varphi_1 \circ \varphi_2^{-1}$ is biregular. But this contradicts our initial assumption. So, the case (2) holds. Then ($\bigstar$) is the link **7** in the table, which proves the theorem.

Let us conclude this paper by showing that the Sarkisov link **7** in the table is always obtained using Prokhorov's construction [44, § 3.4 Case 4^o] revisited above. Let C_1 and C_2 be the curves contracted by ϕ_1 and ϕ_2, respectively. Then it follows from [12, Lemma A.3] that

$$-1 = \big(-K_X - H_1\big)^3 = \big(a(-K_X) - bH_1\big)^3 = D_1^3 = D_2^3 - \big(D_2 \cdot C_2\big)^3 = -\big(D_2 \cdot C_2\big)^3,$$

so that $D_2 \cdot C_2 = 1$. Similarly, get $H_1 \cdot C_1 = 1$. Using this and $D_2 \sim -K_X - H_1$, we see that

$$-K_{\widetilde{X}} \sim \varphi_1^*(H_1) + \varphi_2^*(D_2).$$

Note that

$$-K_{\widetilde{X}}^3 = 12,\ h^{1,2}(\widetilde{X}) = 5,\ \operatorname{rk}\operatorname{Pic}(\widetilde{X}) = 3,$$

which implies that the divisor $-K_{\widetilde{X}}$ is not ample, because smooth Fano threefolds with these discrete invariants do not exists [25].

Combining $\pi_1 \circ \varphi_1$ and $\pi_2 \circ \varphi_2$, we obtain a morphism $\widetilde{X} \to \mathbb{P}^2 \times \mathbb{P}^2$. Let $\overline{X}$ be its image, and let $\sigma\colon \widetilde{X} \to \overline{X}$ be the induced morphism. Then one of the following two cases holds:

- either $\overline{X}$ is a divisor of degree $(2,2)$, and σ is birational,
- or $\overline{X}$ is a divisor of degree $(1,1)$, and σ is generically two-to-one.

In the former case, it follows from the subadjunction formula that the threefold $\overline{X}$ is normal, because hypersurface singularities are normal if and only if they are smooth in codimension two. In the latter case, the threefold $\overline{X}$ is also normal — it is either smooth or has one node.

Set $\overline{E} = \sigma(E)$. Let $\mathrm{pr}_1\colon \overline{X} \to \mathbb{P}^2$ and $\mathrm{pr}_2\colon \overline{X} \to \mathbb{P}^2$ be the projections to the first and the second factors of the fourfold $\mathbb{P}^2 \times \mathbb{P}^2$, respectively. Then $\mathrm{pr}_1(\overline{E})$ and $\mathrm{pr}_2(\overline{E})$ are lines, so we can choose coordinates $([x_1 : y_1 : z_1], [x_2 : y_2 : z_2])$ on $\mathbb{P}^2 \times \mathbb{P}^2$ such that

$$\overline{E} = \big\{z_1 = z_2 = 0\big\}.$$

Since $\overline{E} \subset \overline{X}$, we see that $\overline{X}$ is singular. Note also that σ induces an isomorphism $E \cong \overline{E}$.

Claim. *The threefold $\overline{X}$ is a divisor of degree $(2,2)$, and σ is a small birational morphism.*

Proof. If σ contracts a divisor F, then

$$F \sim a_1\varphi_1^*(H_1) + a_2\varphi_2^*(D_2) + a_3E$$

for some integers a_1, a_2, a_3, because $\varphi_1^*(H_1)$, $\varphi_2^*(D_2)$ and E freely generate the group $\mathrm{Pic}(\widetilde{X})$. Thus, in this case, we have

$$\begin{aligned} 2a_2 &= F \cdot \varphi_1^*(H_1) \cdot \varphi_1^*(H_1) = 0, \\ 2a_1 &= F \cdot \varphi_1^*(D_2) \cdot \varphi_1^*(D_2) = 0, \\ 2a_1 + 2a_2 + a_3 &= F \cdot \varphi_1^*(H_1) \cdot \varphi_1^*(D_2) = 0, \end{aligned}$$

which gives $a_1 = 0$, $a_2 = 0$, $a_3 = 0$. This shows that σ does not contract any divisors.

The Stein factorization of σ is the following commutative diagram:

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\alpha} & \widehat{X} \\ & {\scriptstyle\sigma}\searrow \quad \swarrow{\scriptstyle\beta} & \\ & \overline{X} & \end{array}$$

where α is a birational morphism, and β is either an isomorphism or a (ramified) double cover. Since σ does not contract divisors and $-K_{\widetilde{X}}$ is not ample, we see that α is a flopping contraction, and $\widehat{X}$ has terminal Gorenstein singularities. We must show that β is an isomorphism.

Suppose β is a double cover. Its Galois involution induces a birational involution $\tau \in \mathrm{Bir}(\widetilde{X})$. Then τ acts naturally on $\mathrm{Pic}(\widetilde{X})$ such that

$$\begin{aligned} \tau_*(\varphi_1^*(H_1)) &\sim \varphi_1^*(H_1), \\ \tau_*(\varphi_1^*(D_2)) &\sim \varphi_1^*(D_2), \\ \tau_*(E) &\sim b_1\varphi_1^*(H_1) + b_2\varphi_2^*(D_2) + b_3E \end{aligned}$$

for some integers b_1, b_2, b_3. Then

$$\begin{aligned} 2b_2 &= \tau_*(E) \cdot \varphi_1^*(H_1) \cdot \varphi_1^*(H_1) = E \cdot \varphi_1^*(H_1) \cdot \varphi_1^*(H_1) = 0, \\ 2b_1 &= \tau_*(E) \cdot \varphi_1^*(D_2) \cdot \varphi_1^*(D_2) = E \cdot \varphi_1^*(D_2) \cdot \varphi_1^*(D_2) = 0, \\ b_2b_1 + 2b_2 + b_3 &= \tau(E)_* \cdot \varphi_1^*(H_1) \cdot \varphi_1^*(D_2) = E \cdot \varphi_1^*(H_1) \cdot \varphi_1^*(D_2) = 1, \end{aligned}$$

which gives $b_1 = 0$, $b_2 = 0$, $b_3 = 1$, so $\tau_*(E) \sim E$, which gives $\tau(E) = E$, since E is η-exceptional.

Since $\tau(E) = E$ and σ induces an isomorphism $E \cong \overline{E}$, we see that the surface $\overline{E}$ is contained in the ramification divisor of the double cover β. This implies that $\widehat{X}$ has non-isolated singularities, which is impossible, since $\widehat{X}$ has terminal singularities. Thus, we see that β is an isomorphism. $\square$

We see that $\overline{X}$ is a divisor in $\mathbb{P}^2 \times \mathbb{P}^2$ of degree $(2, 2)$, and σ is a flopping contraction. Then

$$\overline{X} = \big\{z_1 f(x_1, y_1, z_1; x_2, y_2, z_2) = z_2 g(x_1, y_1, z_1; x_2, y_2, z_2)\big\}$$

for some polynomials f and g of bi-degree $(1, 2)$ and $(2, 1)$, respectively, and X can be obtained using Prokhrov's construction presented earlier in the paper.

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Table describing all possibilities for the Sarkisov link (★).

№	d	I	$h^{1,2}$	$\pi_1 \colon X_1 \to Z_1$	$\pi_2 \colon X_2 \to Z_2$	References
1	2	1	52	$Z_1 = \mathbb{P}^1$, π_1 is a fibration into del Pezzo surfaces of degree 1	$Z_2 = \mathbb{P}^1$, π_2 is is a fibration into del Pezzo surfaces of degree 1	[23, 24, 29], [48, (2.5.2)]
2	6	1	20	$Z_1 = \mathbb{P}^1$, π_1 is a fibration into del Pezzo surfaces of degree 2	Z_2 is a del Pezzo threefold of degree 1 that has one singular double point, π_2 is a blow up of the singular point	[10, Proposition 5.6], [23, 24], [43, Example 4.3], [48, (2.7.3)]
3	8	1	14	$Z_1 = \mathbb{P}^1$, π_1 is a fibration into cubic surfaces	$Z_2 \cong \mathbb{P}^2$, π_2 is a conic bundle with septic discriminant curve	[10, Proposition 5.9], [43, Example 4.6], [48, (2.9.4)]
4	10	1	10	$Z_1 = \mathbb{P}^1$, π_1 is a fibration into cubic surfaces	Z_2 is a smooth del Pezzo threefold of degree 2, π_2 is blow up of a smooth rational curve that has anticanonical degree 2	[10, Example 1.11], [44, § 3.12 Case 11^o], [48, (2.9.3)]
5	12	1	7	$Z_1 = \mathbb{P}^1$, π_1 is a fibration into quartic del Pezzo surfaces	$Z_2 \cong \mathbb{P}^3$, π_2 is a blow up of a smooth curve of degree 8 and genus 7	[28, Proposition 3.16], [48, (2.11.5)]
6	14	1	5	$Z_1 = \mathbb{P}^1$, π_1 is a fibration into quartic del Pezzo surfaces	Z_2 is a smooth cubic threefold, π_2 is a blow up of a smooth conic	[28, Proposition 3.16], [44, § 3.13 Case 12^o], [48, (2.11.4)]
7	14	1	5	$Z_1 = \mathbb{P}^2$, π_1 is a conic bundle with quntic discriminant curve	$Z_2 = \mathbb{P}^2$, π_1 is a conic bundle with quntic discriminant curve	[44, § 3.4 Case 4^o], Construction, Claim
8	16	1	3	$Z_1 = \mathbb{P}^1$, π_1 is a fibration into quintic del Pezzo surfaces	Z_2 is a smooth quadric in $\mathbb{P}^4$, π_2 is a blow up of a smooth curve of degree 7 and genus 3	[28, Proposition 3.16], [48, (2.13.4)]

9	16	1	3	$Z_1 = \mathbb{P}^1$, π_1 is a quadric bundle	$Z_2 = \mathbb{P}^1$, π_2 is a fibration into quartic del Pezzo surfaces	[2, Example 4.9], [48, (2.3.8)], [48, (2.11.2)]
10	18	1	2	$Z_1 = \mathbb{P}^1$, π_1 is a fibration into quintic del Pezzo surfaces	Z_2 is a smooth complete intersection of two quadrics in $\mathbb{P}^5$, π_2 is a blow up of a twisted cubic	[28, Proposition 3.16], [48, (2.13.3)]
11	18	1	2	$Z_1 \cong \mathbb{P}^2$, π_1 is a conic bundle with quartic discriminant curve	$Z_2 = \mathbb{P}^3$, π_2 is a blow up of a smooth curve of degree 6 and genus 2	[28, Proposition 4.14], [4, Example 4.8], [28, Theorem 7.14]
12	22	1	0	$Z_1 = \mathbb{P}^1$, π_1 is a fibration into quintic del Pezzo surfaces	$Z_2 \cong \mathbb{P}^2$, π_2 is a $\mathbb{P}^1$-bundle	[41, (IV)], [48, (2.13.1)]
13	22	1	0	$Z_1 = \mathbb{P}^3$, π_1 is a blow up of a smooth rational curve of degree 5 that is not contained in a quadric	$Z_2 = \mathbb{P}^3$, π_1 is a blow up of a smooth rational curve of degree 5 that is not contained in a quadric	[18, Proposition 2.11], [41, (I)]
14	22	1	0	$Z_1 \cong \mathbb{P}^2$, π_1 is a conic bundle with cubic discriminant curve	Z_2 is a smooth quadric threefold, π_2 is a blow up of a smooth rational quintic curve	[28, Proposition 4.14], [41, (II)]
15	22	1	0	$Z_1 \cong \mathbb{P}^1$, π_1 is a fibration into sextic del Pezzo surfaces	$Z_2 \cong V_5$, π_2 is a blow up of a rational quartic curve	[28, Theorem 7.14], [41, (III)]
16	40	2	0	$Z_1 = \mathbb{P}^1$, π_1 is a quadric bundle	$Z_2 = \mathbb{P}^2$, π_2 is a $\mathbb{P}^1$-bundle	[26, Theorem 3.5], [48, (2.3.2)]
17	54	3	0	$Z_1 = \mathbb{P}^1$, π_1 is a $\mathbb{P}^2$-bundle	$Z_2 = \mathbb{P}^1$, π_2 is a $\mathbb{P}^2$-bundle	Example

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