

K-STABILITY OF FANO 3-FOLDS OF PICARD RANK 3 AND DEGREE 22

IVAN CHELTSOV

To the memory of Sasha Ananin

ABSTRACT. We prove K-stability of smooth Fano 3-folds of Picard rank 3 and degree 22 that satisfy very explicit generality condition.

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1. INTRODUCTION

Let L is a line in \mathbb{P}^3 , let C_4 be a smooth quartic elliptic curve in \mathbb{P}^3 such that $L \cap C_4 = \emptyset$, and let $\pi: X \rightarrow \mathbb{P}^3$ be the blow up of these two curves. Then X is a smooth Fano 3-fold of degree 22. Moreover, all smooth Fano 3-folds of Picard rank 3 and degree 22 can be obtained in this way.

Choosing appropriate coordinates x_0, x_1, x_2, x_3 on \mathbb{P}^3 , we may assume that

$$C_4 = \{x_0^2 + x_1^2 + \lambda(x_2^2 + x_3^2) = 0, \lambda(x_0^2 - x_1^2) + x_2^2 - x_3^2 = 0\} \subset \mathbb{P}^3$$

for a complex number λ such that $\lambda \notin \{0, \pm 1, \pm i\}$. Further, we may assume that

$$L = \{a_0x_0 + a_1x_1 + a_2x_2 = 0, b_1x_1 + b_2x_2 + b_3x_3 = 0\} \subset \mathbb{P}^3$$

for some $[a_0 : a_1 : a_2]$ and $[b_1 : b_2 : b_3]$ in \mathbb{P}^2 . The following result is proved in [3].

Lemma 1 ([3, Lemma 5.73]). *If $L = \{x_0 - x_2 = 0, x_1 - x_3 = 0\}$, then X is K-stable.*

This implies that the Fano 3-fold X is K-stable if L and C_4 are chosen to be sufficiently general, because K-stability is an open property [4, Theorem 4.5] (see [11] for basics facts about K-stability). Actually, we expect that X is always K-stable. However, we are unable to prove this at the moment. In this paper, we prove that X is K-stable if it satisfies the following condition:

for every plane $\Pi \subset \mathbb{P}^3$ passing through L ,
 the intersection $\Pi \cap C_4$ contains at most one multiple point,
 and the multiplicity of this point is at most three.

Throughout this paper, all varieties are assumed to be projective and defined over \mathbb{C} .

One can check that this generality condition holds in case when L is the line $\{x_0 - x_2 = 0, x_1 - x_3 = 0\}$. Hence, our main theorem is a generalization of Lemma 1, but their proofs are very different.

To state our main theorem in a more natural way, note that we have commutative diagram

$$\begin{array}{ccccc}
 & & \mathbb{P}^1 \times \mathbb{P}^1 & & \\
 & \swarrow \text{pr}_1 & \downarrow \eta & \searrow \text{pr}_2 & \\
 \mathbb{P}^1 & \xleftarrow{\sigma} & X & \xrightarrow{\phi} & \mathbb{P}^1 \\
 & \nwarrow \varsigma & \downarrow \pi & \nearrow \varphi & \\
 & & \mathbb{P}^3 & &
 \end{array}$$

where ς is given by $[x_0 : x_1 : x_2 : x_3] \mapsto [a_0 x_0 + a_1 x_1 + a_2 x_2 : b_1 x_1 + b_2 x_2 + b_3 x_3]$, the map φ is given by

$$[x_0 : x_1 : x_2 : x_3] \mapsto [x_0^2 + x_1^2 + \lambda(x_2^2 + x_3^2) : \lambda(x_0^2 - x_1^2) + x_2^2 - x_3^2],$$

the map σ is a fibration into quintic del Pezzo surfaces, ϕ is a fibration into sextic del Pezzo surfaces, the map η is a conic bundle, pr_1 and pr_2 are projections to the first and the second factors, respectively. Now, we can state our main theorem as follows:

Theorem 1. *Suppose that every singular fiber of the fibration $\sigma: X \rightarrow \mathbb{P}^1$ has one singular point, and this point is either a singular point of type \mathbb{A}_1 or a singular point of type \mathbb{A}_2 . Then X is K-stable.*

Using basic geometric facts about quintic del Pezzo surfaces with at most Du Val singularities [6], one can show that the generality condition in Theorem 1 is equivalent to the following condition:

every fiber of the conic bundle $\eta: X \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ is reduced.

But this condition is equivalent to the smoothness of the discriminant curve of the conic bundle η .

Corollary 1. *If the discriminant curve of the conic bundle η is smooth, then X is K-stable.*

One can show that the discriminant curve of the conic bundle η is given in $\mathbb{P}^1 \times \mathbb{P}^1$ by

$$\begin{aligned}
 & \lambda(b_1^2 - \lambda b_2^2 + \lambda b_3^2)y_0^2 z_0^3 - (\lambda^3 b_2^2 + \lambda^3 b_3^2 + b_1^2)y_0^2 z_0^2 z_1 - (\lambda^3 b_1^2 - b_2^2 + b_3^2)y_0^2 z_0 z_1^2 - \\
 & - (\lambda^3 b_1^2 - b_2^2 + b_3^2)y_0^2 z_1^3 - 2\lambda(a_1 b_1 - \lambda a_2 b_2)y_0 y_1 z_0^3 + 2(\lambda^3 a_2 b_2 + a_1 b_1)y_0 y_1 z_0^2 z_1 + \\
 & + 2(\lambda^3 a_1 b_1 - a_2 b_2)y_0 y_1 z_0 z_1^2 - 2\lambda(\lambda a_1 b_1 + a_2 b_2)y_0 y_1 z_1^3 - \lambda(\lambda a_2^2 + a_0^2 - a_1^2)y_1^2 z_0^3 - \\
 & - (\lambda^3 a_2^2 + a_0^2 + a_1^2)y_1^2 z_0^2 z_1 + (\lambda^3 a_0^2 - \lambda^3 a_1^2 + a_2^2)y_1^2 z_0 z_1^2 + \lambda(\lambda a_0^2 + \lambda a_1^2 + a_2^2)y_1^2 z_1^3 = 0,
 \end{aligned}$$

where $([y_0 : y_1], [z_0 : z_1])$ are coordinates on $\mathbb{P}^1 \times \mathbb{P}^1$.

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2. THE PROOF

Let us use all assumptions and notations introduced in Section 1. To prove Theorem 1, we suppose that each singular fiber of the fibration σ has one singular point, and this point is either a singular point of type \mathbb{A}_1 or a singular point of type \mathbb{A}_2 . Set $H = \pi^*(\mathcal{O}_{\mathbb{P}^3}(1))$. Let E and R be the exceptional surfaces of the blow up π such that $\pi(E) = C_4$ and $\pi(R) = L$. Then

- the quintic del Pezzo fibration σ is given by the pencil $|H - R|$,
- the sextic del Pezzo fibration φ is given by the pencil $|2H - E|$,
- the conic bundle η is given by $|3H - E - R|$.

Note that $\text{Eff}(X) = \langle E, R, H - R, 2H - E \rangle$, and the cone $\overline{\text{NE}(X)}$ is generated by the classes of curves contracted by the blow up $\pi: X \rightarrow \mathbb{P}^3$ and the conic bundle $\eta: X \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$.

By the Fujita–Li valuation criterion [8, 10], the Fano 3-fold X is K-stable if and only if

$$\beta(\mathbf{F}) = A_X(\mathbf{F}) - S_X(\mathbf{F}) > 0$$

for every prime divisor \mathbf{F} over X , where $A_X(\mathbf{F})$ is the log discrepancy of the divisor \mathbf{F} , and

$$S_X(\mathbf{F}) = \frac{1}{(-K_X)^3} \int_0^\infty \text{vol}(-K_X - u\mathbf{F}) du.$$

To show this, we fix a prime divisor \mathbf{F} over X . Then we set $Z = C_X(\mathbf{F})$. If Z is an irreducible surface, then it follows from [9] that $\beta(\mathbf{F}) > 0$, see also [3, Theorem 3.17]. Therefore, we may assume that

- either Z is an irreducible curve in X ,
- or Z is a point in X .

In both cases, we fix a point $P \in Z$. Let S be one of the following two surfaces:

- (1) the surface in the pencil $|H - R|$ that contains P ,
- (2) the surface in the pencil $|2H - E|$ that contains P .

Then S is a del Pezzo surface with at most Du Val singularities. Set

$$\tau = \sup \left\{ u \in \mathbb{R}_{\geq 0} \mid \text{the divisor } -K_X - uS \text{ is pseudo-effective} \right\}.$$

For $u \in [0, \tau]$, let $P(u)$ be the positive part of the Zariski decomposition of the divisor $-K_X - uS$, and let $N(u)$ be its negative part. If $S \in |H - R|$, then $\tau = 2$,

$$P(u) \sim_{\mathbb{R}} \begin{cases} (4-u)H - E + (u-1)R & \text{if } 0 \leq u \leq 1, \\ (4-u)H - E & \text{if } 1 \leq u \leq 2, \end{cases}$$

and

$$N(u) = \begin{cases} 0 & \text{if } 0 \leq u \leq 1, \\ (u-1)R & \text{if } 1 \leq u \leq 2, \end{cases}$$

which gives $S_X(S) = \frac{1}{22} \int_0^2 P(u)^3 du = \frac{67}{88}$. Similarly, if $S \in |2H - E|$, then $\tau = \frac{3}{2}$,

$$P(u) \sim_{\mathbb{R}} \begin{cases} (4-2u)H + (u-1)E - R & \text{if } 0 \leq u \leq 1, \\ (4-2u)H - R & \text{if } 1 \leq u \leq \frac{3}{2}, \end{cases}$$

and

$$N(u) = \begin{cases} 0 & \text{if } 0 \leq u \leq 1, \\ (u-1)E & \text{if } 1 \leq u \leq \frac{3}{2}, \end{cases}$$

which gives $S_X(S) = \frac{109}{176}$. Now, for every prime divisor F over the surface S , we set

$$S(W_{\bullet,\bullet}^S; F) = \frac{3}{(-K_X)^3} \int_0^\tau \text{ord}_F(N(u)|_S) (P(u)|_S)^2 du + \frac{3}{(-K_X)^3} \int_0^\tau \int_0^\infty \text{vol}(P(u)|_S - vF) dv du.$$

Then, following [1, 3], we let

$$\delta_P(S, W_{\bullet,\bullet}^S) = \inf_{\substack{F/S \\ P \in C_S(F)}} \frac{A_S(F)}{S(W_{\bullet,\bullet}^S; F)},$$

where the infimum is taken by all prime divisors over the surface S whose center on S contains P . Then it follows from [1, 3] that

$$(\star) \quad \frac{A_X(\mathbf{F})}{S_X(\mathbf{F})} \geq \min \left\{ \frac{1}{S_X(S)}, \delta_P(S, W_{\bullet,\bullet}^S) \right\}.$$

Therefore, if $\beta(\mathbf{F}) \leq 0$, then $\delta_P(S, W_{\bullet,\bullet}^S) \leq 1$.

Remark 1. If Z is a point and $\beta(\mathbf{F}) \leq 0$, then it follows from [1, 3] that $\delta_P(S, W_{\bullet,\bullet}^S) < 1$.

To estimate $\delta_P(S, W_{\bullet,\bullet}^S)$, we set $D = P(u)|_S$. Then D is ample for $u \in [0, \tau]$, and

$$D^2 = \begin{cases} (2-u)(6-u) & \text{if } S \in |H - R|, \\ 2(3-2u)(5-2u) & \text{if } S \in |2H - E|. \end{cases}$$

For $u \in [0, \tau]$, set

$$\delta_P(S, D) = \inf_{\substack{F/S \\ P \in C_S(F)}} \frac{A_S(F)}{S_D(F)},$$

where

$$S_D(F) = \frac{1}{D^2} \int_0^\infty \text{vol}(D - vF) dv,$$

and the infimum is taken by all prime divisors over S whose center on S contains P .

Lemma 2 (cf. [5, Lemma 27]). *Let $f: [1, \tau] \rightarrow \mathbb{R}_{>0}$ be a continuous positive function such that*

$$\delta_P(S, D) \geq f(u)$$

for every $u \in [1, \tau]$. If $S \in |H - R|$ and $S \notin R$, then

$$\delta_P(S, W_{\bullet,\bullet}^S) \geq \frac{1}{\frac{15}{22f(1)} + \frac{3}{22} \int_1^2 \frac{(2-u)(6-u)}{f(u)} du}.$$

If $S \in |H - R|$ and $S \in R$, then

$$\delta_P(S, W_{\bullet,\bullet}^S) \geq \frac{1}{\frac{9}{88} + \frac{15}{22f(1)} + \frac{3}{22} \int_1^2 \frac{(2-u)(6-u)}{f(u)} du}.$$

If $S \in |2H - E|$ and $S \notin E$, then

$$\delta_P(S, W_{\bullet,\bullet}^S) \geq \frac{1}{\frac{9}{11f(1)} + \frac{3}{22} \int_1^{\frac{3}{2}} \frac{2(3-2u)(5-2u)}{f(u)} du}.$$

If $S \in |H - R|$ and $S \in E$, then

$$\delta_P(S, W_{\bullet,\bullet}^S) \geq \frac{1}{\frac{5}{176} + \frac{9}{11f(1)} + \frac{3}{22} \int_1^{\frac{3}{2}} \frac{2(3-2u)(5-2u)}{f(u)} du}.$$

Proof. The proof is the same as the proof of [5, Lemma 27]. Namely, fix a prime divisor F over S . Write $S(W_{\bullet,\bullet}^S; F) = R_1 + R_2 + R_3$ for

$$\begin{aligned} R_1 &= \frac{3}{22} \int_0^1 \int_0^\infty \text{vol}(P(u)|_S - vF) dv du, \\ R_2 &= \frac{3}{22} \int_1^\tau \int_0^\infty \text{vol}(P(u)|_S - vF) dv du, \\ R_3 &= \frac{3}{22} \int_1^\tau \text{ord}_F(N(u)|_S)(P(u)|_S)^2 du. \end{aligned}$$

Then

$$R_1 \leq \frac{3(-K_S)^2}{22\delta_P(S, -K_S)} A_S(F) \leq \frac{3(-K_S)^2}{22f(1)} A_S(F).$$

Similarly, we see that

$$R_2 \leq A_S(F) \frac{3}{22} \int_1^\tau \frac{D^2}{f(u)} du.$$

Finally, observe that $\text{ord}_F(N(u)|_S) \leq (u-1)A_S(F)$, since $\text{Supp}(N(u)|_S)$ is a smooth curve contained in the smooth locus of the surface S for $u \in (1, \tau]$. Hence, we have

$$R_3 \leq A_S(F) \frac{3}{22} \int_1^\tau (u-1)(P(u)|_S)^2 du = \begin{cases} \frac{9}{88} A_S(F) & \text{if } S \in |H-R|, \\ \frac{5}{176} A_S(F) & \text{if } S \in |2H-E|. \end{cases}$$

If $P \notin \text{Supp}(N(u))$ for $u \in (1, \tau]$, then $R_3 = 0$. Combining our estimates, we complete the proof. \square

To use Lemma 2, we must bound $\delta_P(S, D)$ for $u \in [1, \tau]$. This is done in the next four propositions.

Proposition 1. *Suppose that $S \in |H-R|$, and the surface S is smooth. Then*

$$\delta_P(S, D) \geq \frac{3(6-u)}{u^2 - 10u + 22}$$

for every $u \in [1, 2]$.

Proof. Use Corollary A.1 with $a = 4-u$. If $u = 1$, this follows from [2, Proposition 2.7] or [7, § 5.1]. \square

Proposition 2. *Let S be the surface in $|H-R|$ containing P . Suppose that S has one singular point, which is a singular point of type \mathbb{A}_1 . Let C be the fiber of the conic bundle $\phi|_S: S \rightarrow \mathbb{P}^1$ that contains the singular point of the surface S . If $P \notin C$ and $P \notin E$, then*

$$\delta_P(S, D) \geq \frac{3(6-u)}{u^2 - 10u + 22}$$

for every $u \in [1, 2]$. Similarly, if $P \notin C$ and $P \in E$, then

$$\delta_P(S, D) \geq \begin{cases} \frac{1}{2-u} & \text{if } 1 \leq u \leq \frac{7-\sqrt{21}}{2}, \\ \frac{3(6-u)}{u^2 - 10u + 22} & \text{if } \frac{7-\sqrt{21}}{2} \leq u < 2. \end{cases}$$

Proof. Use Corollary A.2 with $a = 4-u$. If $u = 1$, this follows from [2, Proposition 2.1] or [7, § 5.2]. \square

Proposition 3. Let S be the surface in $|H - R|$ containing P . Suppose that S has one singular point, which is a singular point of type \mathbb{A}_2 . Let C be the fiber of the conic bundle $\phi|_S: S \rightarrow \mathbb{P}^1$ that contains the singular point of the surface S . If $P \notin C$ and $P \notin E$, then

$$\delta_P(S, D) \geq \begin{cases} \frac{6(6-u)}{(2-u)(22+u)} & \text{if } 1 \leq u \leq \frac{1+\sqrt{21}}{5}, \\ \frac{4(6-u)}{u^2-14u+28} & \text{if } \frac{1+\sqrt{21}}{5} \leq u \leq \sqrt{5}-1, \\ \frac{2(6-u)}{u^2-6u+12} & \text{if } \sqrt{5}-1 \leq u < 2. \end{cases}$$

Similarly, if $P \notin C$ and $P \in E$, then

$$\delta_P(S, D) \geq \begin{cases} \frac{6(6-u)}{(2-u)(38-7u)} & \text{if } 1 \leq u \leq \frac{7-\sqrt{17}}{2}, \\ \frac{6(6-u)}{u^2-10u+28} & \text{if } \frac{7-\sqrt{17}}{2} \leq u < 2. \end{cases}$$

Proof. Use Corollary A.3 with $a = 4 - u$. If $u = 1$, this follows from [2, Proposition 2.4] or [7, § 5.6]. \square

Proposition 4. Let S be the surface in $|2H - E|$ containing P . Suppose that S is smooth. Then

$$\delta_P(S, D) \geq \begin{cases} \frac{1}{2-u} & \text{if } P \text{ is contained in a } (-1)\text{-curve and } 1 \leq u < \frac{3}{2}, \\ \frac{2(5-2u)}{4u^2-18u+19} & \text{if } P \text{ is not contained in a } (-1)\text{-curve and } 1 \leq u \leq \frac{9-\sqrt{21}}{4}, \\ \frac{3(5-2u)}{4u^2-18u+21} & \text{if } P \text{ is not contained in a } (-1)\text{-curve and } \frac{9-\sqrt{21}}{4} \leq u < \frac{3}{2}. \end{cases}$$

Proof. Apply Lemmas A.12, A.13 and A.14 with $a = 4 - 2u$. If $u = 1$, the required inequality follows from [2, Proposition 3.6] or [7, § 4.1]. \square

Now, applying Lemma 2 together with Proposition 1, we get

Corollary 2. Let S be the surface in $|H - R|$ such that $P \in S$. If S is smooth, then $\delta_P(S, W_{\bullet,\bullet}^S) > 1$.

Similarly, applying Lemma 2 together with Propositions 2 and 3, we get

Corollary 3. Let S be the surface in $|H - R|$ that contains P . Suppose that S has one singular point. Let C be the fiber of the conic bundle $\phi|_S: S \rightarrow \mathbb{P}^1$ that contains the singular point of the surface S . Suppose that $P \notin C$. Then $\delta_P(S, W_{\bullet,\bullet}^S) > 1$.

Finally, applying Lemma 2 together with Proposition 4, we get

Corollary 4. Let S be the surface in $|2H - E|$ that contains P . Suppose that S is smooth. Then

$$\delta_P(S, W_{\bullet,\bullet}^S) \geq 1.$$

Moreover, if $P \notin E$ or P is not contained in a (-1) -curve in S , then $\delta_P(S, W_{\bullet,\bullet}^S) > 1$.

Now, we are ready to show that $\beta(\mathbf{F}) > 0$. Suppose that $\beta(\mathbf{F}) \leq 0$. Let us seek for a contradiction. We may assume that P is a general point in Z . Let S be the surface in $|H - R|$ that contains P . If S is smooth, $\delta_P(S, W_{\bullet,\bullet}^S) > 1$ by Corollary 2, which contradicts (\star) . Therefore, the surface S is singular. This implies that $Z \subset S$.

Recall that S has one singular point, and this singular point is either a singular point of type \mathbb{A}_1 or a singular point of type \mathbb{A}_2 . Set $O = \text{Sing}(S)$, let S' be the surface in $|2H - E|$ that contains O , and set $C = S \cap S'$. Then C is a fiber of the conic bundle $\phi|_S: S \rightarrow \mathbb{P}^1$. Moreover, we have

$$Z \subset C$$

since otherwise $\delta_P(S, W_{\bullet,\bullet}^S) > 1$ by Corollary 3, which is impossible by (\star) .

We claim that S' is smooth. Indeed, it follows from [6] that $C = \ell_1 + \ell_2$, where ℓ_1 and ℓ_2 are smooth rational curves that intersect transversally at O . Note that

- $\pi(S)$ is a plane such that $L \subset \pi(S)$,
- $\pi(S')$ is a quadric surface such that $C_4 \subset \pi(S')$.
- $\pi(\ell_1)$ and $\pi(\ell_2)$ are lines such that $\pi(O) = \pi(\ell_1) \cap \pi(\ell_2)$.

Since the surface S is singular at O , the plane $\pi(S)$ is tangent to C_4 at the point $\pi(O)$. In particular, we see that $\pi(O) \in C_4$, which implies that S' is smooth at $\pi(O)$, because C_4 is smooth. Since

$$\pi(S) \cap \pi(S') = \pi(\ell_1) \cup \pi(\ell_2),$$

we conclude that the quadric surface $\pi(S')$ is smooth. Moreover, since

$$L \cap \pi(S') = (L \cap \pi(\ell_1)) \cup (L \cap \pi(\ell_2)),$$

we see that the line L intersects the quadric $\pi(S')$ transversally by the points $L \cap \pi(\ell_1)$ and $L \cap \pi(\ell_2)$, which implies that the surface S' is smooth.

We see that S' is a smooth sextic del Pezzo surface. Observe that $E \cap S'$ is a smooth elliptic curve. Moreover, applying Corollary 4 and (\star) to the surface S' , we see that P is one of finitely many intersection points of this elliptic curve with six (-1) -curves in S' . This shows that Z is a point, so that $Z = P$. Since $\delta_P(S', W_{\bullet,\bullet}^{S'}) \geq 1$ by Corollary 4, it follows from (\star) that

$$1 \geq \frac{A_X(\mathbf{F})}{S_X(\mathbf{F})} \geq \min \left\{ \frac{1}{S_X(S')}, \delta_P(S, W_{\bullet,\bullet}^{S'}) \right\} = \min \left\{ \frac{176}{109}, \delta_P(S', W_{\bullet,\bullet}^{S'}) \right\} = \min \left\{ \frac{176}{109}, 1 \right\} = 1.$$

Now, using Remark 1, we obtain a contradiction, because Z is a point in S' . Theorem 1 is proved.

APPENDIX A. δ -INVARIANTS OF SOME POLARIZED DEL PEZZO SURFACES

Let S be a del Pezzo surface with at most Du Val singularities, let D be an ample divisor on S . For every prime divisor F over S , set

$$S_D(F) = \frac{1}{D^2} \int_0^\infty \text{vol}(D - vF) dv.$$

Let P be a smooth point in S , and let

$$\delta_P(S, D) = \inf_{\substack{F/S \\ P \in C_S(F)}} \frac{A_S(F)}{S_D(F)},$$

where the infimum is taken by all prime divisors over S whose center on S contains P . Set

$$\delta(S, D) = \inf_{P \in S} \delta_P(S, D).$$

In this appendix, we estimate $\delta_P(S, D)$ in some cases similar to what is done in [2, 7] for $D = -K_S$.

To explain how to estimate $\delta_P(S, D)$, fix a smooth curve $C \subset S$ that passes through P . Set

$$\tau = \sup \left\{ u \in \mathbb{R}_{\geq 0} \mid \text{the divisor } D - vC \text{ is pseudo-effective} \right\}.$$

For $v \in [0, \tau]$, let $P(v)$ be the positive part of the Zariski decomposition of the divisor $D - vC$, and let $N(v)$ be its negative part. Then

$$S_D(C) = \frac{1}{D^2} \int_0^\infty \text{vol}(D - vC) dv = \frac{1}{D^2} \int_0^\tau P(v)^2 dv.$$

Note that $\delta_P(S, D) \leq \frac{1}{S_D(C)}$, since $A_S(C) = 1$. To estimate $\delta_P(S, D)$ from below, set

$$S(W_{\bullet,\bullet}^C; P) = \frac{2}{D^2} \int_0^\tau \text{ord}_P(N(v)|_C)(P(v) \cdot C) dv + \frac{1}{D^2} \int_0^\tau (P(v) \cdot C)^2 dv.$$

Then it follows from [1, 3] that

$$(\heartsuit) \quad \delta_P(S, D) \geq \min \left\{ \frac{1}{S_D(C)}, \frac{1}{S(W_{\bullet,\bullet}^C; P)} \right\}.$$

Usually, (\heartsuit) gives a very good estimate for $\delta_P(S, D)$ when $C^2 < 0$. If P is not contained in any curve with negative self-intersection, we have to blow up the surface S at the point P , and apply similar arguments to the exceptional curve of this blow up.

Namely, let $f: \tilde{S} \rightarrow S$ be the blow up of S at the point P , and let E be the f -exceptional curve. In all applications, the surface \tilde{S} will be a del Pezzo surface with at most Du Val singularities. Set

$$\tilde{\tau} = \sup \left\{ u \in \mathbb{R}_{\geq 0} \mid \text{the divisor } f^*(D) - vE \text{ is pseudo-effective} \right\}.$$

For $v \in [0, \tilde{\tau}]$, let $\tilde{P}(v)$ be the positive part of the Zariski decomposition of the divisor $f^*(D) - vE$, and let $\tilde{N}(v)$ be its negative part. Then

$$S_D(E) = \frac{1}{D^2} \int_0^\infty \text{vol}(f^*(D) - vC) dv = \frac{1}{D^2} \int_0^{\tilde{\tau}} \tilde{P}(v)^2 dv.$$

Note that $\delta_P(S, D) \leq \frac{2}{S_D(E)}$, since $A_S(E) = 2$. Now, for every point $O \in E$, we set

$$S(W_{\bullet,\bullet}^E; O) = \frac{2}{D^2} \int_0^{\tilde{\tau}} \text{ord}_O(\tilde{N}(v)|_E)(\tilde{P}(v)|_E) dv + \frac{1}{D^2} \int_0^{\tilde{\tau}} (\tilde{P}(v) \cdot E)^2 dv.$$

Then it follows from [1, 3] that

$$(\diamondsuit) \quad \delta_P(S, D) \geq \min \left\{ \frac{2}{S_D(E)}, \inf_{O \in E} \frac{1}{S(W_{\bullet,\bullet}^E; O)} \right\}.$$

In the next four subsections, we will apply (\heartsuit) and (\diamondsuit) using notations introduced here.

A.1. Smooth quintic del Pezzo surface. Let S be a smooth del Pezzo surface such that $K_S^2 = 5$. There is a birational morphism $\pi: S \rightarrow \mathbb{P}^2$ that blows up 4 points. Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ be the exceptional curves of the morphism π , and let $\mathbf{h} = \pi^*(\mathcal{O}_{\mathbb{P}^2}(1))$. Set

$$D = a\mathbf{h} - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3 - \mathbf{e}_4$$

for $a \in (2, 3]$. Then D is ample and $D^2 = a^2 - 4$.

Lemma A.1. *Let P be a point in $\mathbf{e}_1 \cup \mathbf{e}_2 \cup \mathbf{e}_3 \cup \mathbf{e}_4$. Then*

$$\delta_P(S, D) \geq \frac{3(a+2)}{a^2 + 2a - 2}.$$

Proof. We may assume that $P \in \mathbf{e}_1$. Set $C = \mathbf{e}_1$. Then $\tau = 2a - 4$. Moreover, we have

$$P(v) \sim_{\mathbb{R}} \begin{cases} a\mathbf{h} - (1+v)\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3 - \mathbf{e}_4 & \text{if } 0 \leq v \leq a-2, \\ (4a-3v-6)\mathbf{h} + (2v+5-3a)\mathbf{e}_1 + (1-a+v)(\mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4) & \text{if } a-2 \leq v \leq 2a-4, \end{cases}$$

and

$$N(v) = \begin{cases} 0 & \text{if } 0 \leq v \leq a-2, \\ (v+2-a)(\mathbf{l}_{12} + \mathbf{l}_{13} + \mathbf{l}_{14}) & \text{if } a-2 \leq v \leq 2a-4, \end{cases}$$

where $\mathbf{l}_{12}, \mathbf{l}_{13}, \mathbf{l}_{14}$ are (-1) -curves in $|\mathbf{h} - \mathbf{e}_1 - \mathbf{e}_2|, |\mathbf{h} - \mathbf{e}_1 - \mathbf{e}_3|, |\mathbf{h} - \mathbf{e}_1 - \mathbf{e}_4|$, respectively. Then

$$P(v)^2 = \begin{cases} a^2 - v^2 - 2v - 4 & \text{if } 0 \leq v \leq a-2, \\ 2(2a-v-4)(a-v-1) & \text{if } a-2 \leq v \leq 2a-4, \end{cases}$$

and

$$P(v) \cdot C = \begin{cases} 1+v & \text{if } 0 \leq v \leq a-2, \\ 3a-2v-5 & \text{if } a-2 \leq v \leq 2a-4. \end{cases}$$

Integrating, we get $S_D(C) = \frac{(a-2)(a+10)}{3(a+2)}$ and

$$S(W_{\bullet,\bullet}^C; P) = \frac{2}{a^2-4} \int_{a-2}^{2a-4} \text{ord}_P(N(v)|_C)(P(v) \cdot C) dv + \frac{2a^2-5a+8}{3(a+2)}.$$

Thus, if $P \notin \mathbf{l}_{12} \cup \mathbf{l}_{13} \cup \mathbf{l}_{14}$, then $S(W_{\bullet,\bullet}^C; P) = \frac{2a^2-5a+8}{3(a+2)}$. Similarly, if $P \in \mathbf{l}_{12} \cup \mathbf{l}_{13} \cup \mathbf{l}_{14}$, then

$$S(W_{\bullet,\bullet}^C; P) = \frac{2}{a^2-4} \int_{a-2}^{2a-4} (v+2-a)(P(v) \cdot C) dv + \frac{2a^2-5a+8}{3a+6} = \frac{a^2+2a-2}{3(a+2)}.$$

Now, using (\heartsuit) , we obtain the required assertion. \square

As in the proof of Lemma A.1, let $\mathbf{l}_{12}, \mathbf{l}_{13}, \mathbf{l}_{14}, \mathbf{l}_{23}, \mathbf{l}_{24}, \mathbf{l}_{34}$ be (-1) -curves in $|\mathbf{h} - \mathbf{e}_1 - \mathbf{e}_2|, |\mathbf{h} - \mathbf{e}_1 - \mathbf{e}_3|, |\mathbf{h} - \mathbf{e}_1 - \mathbf{e}_4|, |\mathbf{h} - \mathbf{e}_2 - \mathbf{e}_3|, |\mathbf{h} - \mathbf{e}_2 - \mathbf{e}_4|, |\mathbf{h} - \mathbf{e}_3 - \mathbf{e}_4|$, respectively.

Lemma A.2. *Let P be a point in $\mathbf{l}_{12} \cup \mathbf{l}_{13} \cup \mathbf{l}_{14} \cup \mathbf{l}_{23} \cup \mathbf{l}_{24} \cup \mathbf{l}_{34}$. Then*

$$\delta_P(S, D) \geq \frac{3(a+2)}{a^2+2a-2}.$$

Proof. We may assume that $P \in \mathbf{l}_{12}$. By Lemma A.1, we may also assume that $P \notin \mathbf{e}_1 \cup \mathbf{e}_2 \cup \mathbf{e}_3 \cup \mathbf{e}_4$. Set $C = \mathbf{l}_{12}$. Then $\tau = a-1$. Moreover, we have

$$P(v) \sim_{\mathbb{R}} \begin{cases} (a-v)\mathbf{h} - (1-v)(\mathbf{e}_1 + \mathbf{e}_2) - \mathbf{e}_3 - \mathbf{e}_4 & \text{if } 0 \leq v \leq a-2, \\ (2a-2v-2)\mathbf{h} + (v-1)(\mathbf{e}_1 + \mathbf{e}_2) + (1-a+v)(\mathbf{e}_3 + \mathbf{e}_4) & \text{if } a-2 \leq v \leq 1, \\ (2a-2v-2)\mathbf{h} + (1-a+v)(\mathbf{e}_3 + \mathbf{e}_4) & \text{if } 1 \leq v \leq a-1, \end{cases}$$

and

$$N(v) = \begin{cases} 0 & \text{if } 0 \leq v \leq a-2, \\ (v+2-a)\mathbf{l}_{34} & \text{if } a-2 \leq v \leq 1, \\ (v+2-a)\mathbf{l}_{34} + (v-1)(\mathbf{e}_1 + \mathbf{e}_2) & \text{if } 1 \leq v \leq a-1. \end{cases}$$

This gives

$$P(v)^2 = \begin{cases} a^2 - 2av - v^2 + 4v - 4 & \text{if } 0 \leq v \leq a-2, \\ 2(a-2)(a-2v) & \text{if } a-2 \leq v \leq 1, \\ 2(a-v-1)^2 & \text{if } 1 \leq v \leq a-1, \end{cases},$$

and

$$P(v) \cdot C = \begin{cases} a+v-2 & \text{if } 0 \leq v \leq a-2, \\ 2a-4 & \text{if } a-2 \leq v \leq 1, \\ 2a-2v-2 & \text{if } 1 \leq v \leq a-1. \end{cases},$$

Now, integrating, we get $S_D(C) = \frac{a^2+2a-2}{3(a+2)}$, which gives $\delta_P(S, D) \leq \frac{3(a+2)}{a^2+2a-2}$. Similarly, we get

$$S(W_{\bullet,\bullet}^C; P) = \frac{2}{a^2-4} \int_{a-2}^{a-1} \text{ord}_P(N(v)|_C)(P(v) \cdot C) dv + \frac{(a-2)(14-a)}{3(a+2)}.$$

Thus, if $P \notin \mathbf{l}_{34}$, then $S(W_{\bullet,\bullet}^C; P) = \frac{(a-2)(14-a)}{3(a+2)}$. Similarly, if $P \in \mathbf{l}_{34}$, then

$$S(W_{\bullet,\bullet}^C; P) = \frac{2}{a^2-4} \int_{a-2}^{a-1} (v+2-a)(P(v) \cdot C) dv + \frac{(a-2)(14-a)}{3(a+2)} = \frac{a^2+2a-2}{3(a+2)} = S_D(C),$$

so that $\delta_P(S, D) \geq \frac{3(a+2)}{a^2+2a-2}$ by (\heartsuit) . \square

Finally, we prove

Lemma A.3. *Let P be a point in $S \setminus (\mathbf{e}_1 \cup \mathbf{e}_2 \cup \mathbf{e}_3 \cup \mathbf{e}_4 \cup \mathbf{l}_{12} \cup \mathbf{l}_{13} \cup \mathbf{l}_{14} \cup \mathbf{l}_{23} \cup \mathbf{l}_{24} \cup \mathbf{l}_{34})$. Then*

$$\delta_P(S, D) \geq \begin{cases} \frac{2(a+2)}{a^2-2a+4} & \text{if } 2 < a \leq 5 - \sqrt{5}, \\ \frac{2(2a+4)}{a^2+6a-12} & \text{if } 5 - \sqrt{5} \leq a \leq 3. \end{cases}$$

Proof. Recall that $f: \tilde{S} \rightarrow S$ is a blow up of S at the point P , and E is the f -exceptional curve. Note that \tilde{S} is a del Pezzo surface of degree 4. Let $\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3, \tilde{\mathbf{e}}_4, \tilde{\mathbf{l}}_{12}, \tilde{\mathbf{l}}_{13}, \tilde{\mathbf{l}}_{14}, \tilde{\mathbf{l}}_{23}, \tilde{\mathbf{l}}_{24}, \tilde{\mathbf{l}}_{34}$ be the strict transforms on \tilde{S} of the (-1) -curves $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{l}_{12}, \mathbf{l}_{13}, \mathbf{l}_{14}, \mathbf{l}_{23}, \mathbf{l}_{24}, \mathbf{l}_{34}$, respectively. Set $\tilde{\mathbf{h}} = f^*(\mathbf{h})$. Let $\mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4$ be the (-1) -curves in $|2\tilde{\mathbf{h}} - E - \tilde{\mathbf{e}}_1 - \tilde{\mathbf{e}}_2 - \tilde{\mathbf{e}}_3 - \tilde{\mathbf{e}}_4|, |\tilde{\mathbf{h}} - E - \tilde{\mathbf{e}}_1|, |\tilde{\mathbf{h}} - E - \tilde{\mathbf{e}}_2|, |\tilde{\mathbf{h}} - E - \tilde{\mathbf{e}}_3|, |\tilde{\mathbf{h}} - E - \tilde{\mathbf{e}}_4|$, respectively. Then

$$\mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4, \tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3, \tilde{\mathbf{e}}_4, \tilde{\mathbf{l}}_{12}, \tilde{\mathbf{l}}_{13}, \tilde{\mathbf{l}}_{14}, \tilde{\mathbf{l}}_{23}, \tilde{\mathbf{l}}_{24}, \tilde{\mathbf{l}}_{34}, E$$

are all (-1) -curves in \tilde{S} . These curves generates the Mori cone of the surface \tilde{S} .

We compute $\tilde{\tau} = \frac{3a-4}{2}$. Similarly, we see that

$$\tilde{P}(v) \sim_{\mathbb{R}} \begin{cases} a\tilde{\mathbf{h}} - \tilde{\mathbf{e}}_1 - \tilde{\mathbf{e}}_2 - \tilde{\mathbf{e}}_3 - \tilde{\mathbf{e}}_4 - vE & \text{if } 0 \leq v \leq 2a-4, \\ (5a-2v-8)\tilde{\mathbf{h}} + (3-2a+v)(\tilde{\mathbf{e}}_1 + \tilde{\mathbf{e}}_2 + \tilde{\mathbf{e}}_3 + \tilde{\mathbf{e}}_4) - (2a-4)E & \text{if } 2a-4 \leq v \leq a-1, \\ (3a-2v-4)(3\tilde{\mathbf{h}} - \tilde{\mathbf{e}}_1 - \tilde{\mathbf{e}}_2 - \tilde{\mathbf{e}}_3 - \tilde{\mathbf{e}}_4 - 2E) & \text{if } a-1 \leq v \leq \frac{3a-4}{2}, \end{cases}$$

and

$$\tilde{N}(v) = \begin{cases} 0 & \text{if } 0 \leq v \leq 2a-4, \\ (v+4-2a)\mathbf{c}_0 & \text{if } 2a-4 \leq v \leq a-1, \\ (v+4-2a)\mathbf{c}_0 + (v+1-a)(\mathbf{c}_1 + \mathbf{c}_2 + \mathbf{c}_3 + \mathbf{c}_4) & \text{if } a-1 \leq v \leq \frac{3a-4}{2}. \end{cases}$$

This gives

$$\tilde{P}(v)^2 = \begin{cases} a^2 - v^2 - 4 & \text{if } 0 \leq v \leq 2a-4, \\ (a-2)(5a-4v-6) & \text{if } 2a-4 \leq v \leq a-1, \\ (3a-2v-4)^2 & \text{if } a-1 \leq v \leq \frac{3a-4}{2}, \end{cases}$$

and

$$\tilde{P}(v) \cdot E = \begin{cases} v & \text{if } 0 \leq v \leq 2a - 4, \\ 2a - 4 & \text{if } 2a - 4 \leq v \leq a - 1, \\ 6a - 4v - 8 & \text{if } a - 1 \leq v \leq \frac{3a - 4}{2}. \end{cases}$$

Now, integrating, we get $S_D(E) = \frac{a^2 + 6a - 12}{2(a+2)}$.

Let O be a point in E . Then

$$S(W_{\bullet,\bullet}^E; O) = \frac{2}{a^2 - 4} \int_{2a-4}^{\frac{3a-4}{2}} \text{ord}_O(\tilde{N}(v)|_E)(\tilde{P}(v) \cdot E) dv + \frac{2(8-a)(a-2)}{3(a+2)}.$$

Thus, if $O \notin \mathbf{c}_0 \cup \mathbf{c}_1 \cup \mathbf{c}_2 \cup \mathbf{c}_3 \cup \mathbf{c}_4$, then $S(W_{\bullet,\bullet}^E; O) = \frac{2(8-a)(a-2)}{3(a+2)}$. Similarly, if $O \in \mathbf{c}_0$, then

$$S(W_{\bullet,\bullet}^E; O) = \frac{2}{a^2 - 4} \int_{2a-4}^{\frac{3a-4}{2}} (v + 4 - 2a)(\tilde{P}(v) \cdot E) dv + \frac{2(8-a)(a-2)}{3(a+2)} = \frac{a^2 - 2a + 4}{2(a+2)}.$$

Likewise, if $O \in \mathbf{c}_1 \cup \mathbf{c}_2 \cup \mathbf{c}_3 \cup \mathbf{c}_4$, then

$$S(W_{\bullet,\bullet}^E; O) = \frac{2}{a^2 - 4} \int_{a-1}^{\frac{3a-4}{2}} (v + 1 - a)(\tilde{P}(v) \cdot E) dv + \frac{2(8-a)(a-2)}{3(a+2)} = \frac{(a-2)(10-a)}{2(a+2)}.$$

Since $\frac{a^2 - 2a + 4}{2(a+2)} \geq \frac{(a-2)(10-a)}{2(a+2)}$ for $a \in (2, 3]$, we have

$$\inf_{O \in E} \frac{1}{S(W_{\bullet,\bullet}^E; O)} = \frac{2(a+2)}{a^2 - 2a + 4}.$$

Therefore, it follows from (\diamond) that

$$\delta_P(S, D) \geq \min \left\{ \frac{2(2a+4)}{a^2 + 6a - 12}, \frac{2(a+2)}{a^2 - 2a + 4} \right\},$$

which gives the required assertion. \square

Combining Lemmas A.1, A.2, A.3, we obtain

Corollary A.1. *One has $\delta(S, D) \geq \frac{3(a+2)}{a^2 + 2a - 2}$ for every $a \in (2, 3]$.*

In fact, the proofs of Lemmas A.1, A.2, A.3 give $\delta(S, D) = \frac{3(a+2)}{a^2 + 2a - 2}$ for every $a \in (2, 3]$.

A.2. Quintic del Pezzo surface with singular point of type \mathbb{A}_1 . Let S be a del Pezzo surface such that $K_S^2 = 5$, and S has one singular point, which is a singular point of type \mathbb{A}_1 . Set $O = \text{Sing}(S)$. It follows from [6] that there exists a birational morphism $\pi: S \rightarrow \mathbb{P}^2$ that contracts three smooth irreducible rational curves $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ such that \mathbf{e}_1 and \mathbf{e}_2 are (-1) -curves contained in the smooth locus of the del Pezzo surface S , the curve \mathbf{e}_3 contains the point O , and $\mathbf{e}_3^2 = -\frac{1}{2}$. Set $\mathbf{h} = \pi^*(\mathcal{O}_{\mathbb{P}^2}(1))$. Let $\mathbf{l}_1, \mathbf{l}_2, \mathbf{l}_3, \mathbf{l}_4$ be irreducible curves in $|\mathbf{h} - \mathbf{e}_1 - \mathbf{e}_3|, |\mathbf{h} - \mathbf{e}_2 - \mathbf{e}_3|, |\mathbf{h} - 2\mathbf{e}_3|, |\mathbf{h} - \mathbf{e}_1 - \mathbf{e}_2|$, respectively. Observe that $O = \mathbf{l}_1 \cap \mathbf{l}_2 \cap \mathbf{e}_3$, and $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{l}_1, \mathbf{l}_2, \mathbf{l}_3, \mathbf{l}_4$ are all curves in S that have negative self-intersections. These curves are smooth, and their intersections are given in the following table:

•	\mathbf{e}_1	\mathbf{e}_2	\mathbf{e}_3	\mathbf{l}_1	\mathbf{l}_2	\mathbf{l}_3	\mathbf{l}_4
\mathbf{e}_1	-1	0	0	1	0	0	1
\mathbf{e}_2	0	-1	0	0	1	0	1
\mathbf{e}_3	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	0
\mathbf{l}_1	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	0	0
\mathbf{l}_2	0	1	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	0
\mathbf{l}_3	0	0	1	0	0	-1	1
\mathbf{l}_4	1	1	0	0	0	1	-1

Set $D = a\mathbf{h} - \mathbf{e}_1 - \mathbf{e}_2 - 2\mathbf{e}_3$ for $a \in (2, 3]$. Then D is ample and $D^2 = a^2 - 4$.

Lemma A.4. *Let P be a point in \mathbf{e}_3 . Then*

$$\delta_P(S, D) \geq \begin{cases} \frac{3(a+2)}{a^2+2a-2} & \text{if } 2 < a \leq \frac{1+\sqrt{21}}{2}, \\ \frac{1}{a-2} & \text{if } \frac{1+\sqrt{21}}{2} \leq a \leq 3. \end{cases}$$

Proof. Set $C = \mathbf{e}_3$. Then $\tau = 2a - 4$. Moreover, we have

$$P(v) \sim_{\mathbb{R}} \begin{cases} a\mathbf{h} - \mathbf{e}_1 - \mathbf{e}_2 - (2+v)\mathbf{e}_3 & \text{if } 0 \leq v \leq a-2, \\ (2a-v-2)\mathbf{h} - \mathbf{e}_1 - \mathbf{e}_2 + (v+2-2a)\mathbf{e}_3 & \text{if } a-2 \leq v \leq 2a-4, \end{cases}$$

and

$$N(v) = \begin{cases} 0 & \text{if } 0 \leq v \leq a-2, \\ (v-a+2)\mathbf{l}_3 & \text{if } a-2 \leq v \leq 2a-4, \end{cases}$$

which gives

$$P(v)^2 = \begin{cases} \frac{2a^2 - v^2 - 4v - 8}{2} & \text{if } 0 \leq v \leq a-2, \\ \frac{(2a-v)(2a-v-4)}{2} & \text{if } a-2 \leq v \leq 2a-4, \end{cases}$$

and

$$P(v) \cdot C = \begin{cases} \frac{2+v}{2} & \text{if } 0 \leq v \leq a-2, \\ \frac{2a-v-2}{2} & \text{if } a-2 \leq v \leq 2a-4. \end{cases}$$

Integrating, we get $S_D(C) = a-2$ and

$$S(W_{\bullet,\bullet}^C; P) = \frac{2}{a^2-4} \int_{a-2}^{2a-4} \text{ord}_P(N(v)|_C)(P(v) \cdot C) dv + \frac{a^2+2a+4}{6(a+2)}.$$

Thus, if $P \notin \mathbf{l}_3$, then $S(W_{\bullet,\bullet}^C; P) = \frac{a^2+2a+4}{6(a+2)}$. Similarly, if $P \in \mathbf{l}_3$, then

$$S(W_{\bullet,\bullet}^C; P) = \frac{2}{a^2-4} \int_{a-2}^{2a-4} (v+2-a)(P(v) \cdot C) dv + \frac{a^2+2a+4}{6(a+2)} = \frac{a^2+2a-2}{3(a+2)}.$$

Now, using (\heartsuit) , we obtain the required assertion. \square

Lemma A.5. Let P be a point in $\mathbf{e}_1 \cup \mathbf{e}_2$ such that $P \notin \mathbf{l}_1 \cup \mathbf{l}_2$. Then

$$\delta_P(S, D) \geq \frac{3(a+2)}{a^2 + 2a - 2}.$$

Proof. We may assume that $P \in \mathbf{e}_1$. Set $C = \mathbf{e}_1$. Then $\tau = 2a - 4$. Moreover, we have

$$P(v) \sim_{\mathbb{R}} \begin{cases} a\mathbf{h} - (1+v)\mathbf{e}_1 - \mathbf{e}_2 - 2\mathbf{e}_3 & \text{if } 0 \leq v \leq a-2, \\ (4a-3v-6)\mathbf{h} + (2v+5-3a)\mathbf{e}_1 + (1-a+v)(\mathbf{e}_2 + 2\mathbf{e}_3) & \text{if } a-2 \leq v \leq 2a-4, \end{cases}$$

and

$$N(v) = \begin{cases} 0 & \text{if } 0 \leq v \leq a-2, \\ (v-a+2)(2\mathbf{l}_1 + \mathbf{l}_4) & \text{if } a-2 \leq v \leq 2a-4, \end{cases}$$

which gives

$$P(v)^2 = \begin{cases} a^2 - v^2 - 2v - 4 & \text{if } 0 \leq v \leq a-2, \\ 2(2a-v-4)(a-v-1) & \text{if } a-2 \leq v \leq 2a-4, \end{cases}$$

and

$$P(v) \cdot C = \begin{cases} 1+v & \text{if } 0 \leq v \leq a-2, \\ 3a-5-2v & \text{if } a-2 \leq v \leq 2a-4. \end{cases}$$

Integrating, we get $S_D(C) = \frac{(a+10)(a-2)}{3(a+2)}$ and

$$S(W_{\bullet,\bullet}^C; P) = \frac{2}{a^2 - 4} \int_{a-2}^{2a-4} \text{ord}_P(N(v)|_C)(P(v) \cdot C) dv + \frac{2a^2 - 5a + 8}{3(a+2)}.$$

Thus, if $P \notin \mathbf{l}_4$, then $S(W_{\bullet,\bullet}^C; P) = \frac{2a^2 - 5a + 8}{3(a+2)}$. Similarly, if $P \in \mathbf{l}_4$, then

$$S(W_{\bullet,\bullet}^C; P) = \frac{2}{a^2 - 4} \int_{a-2}^{2a-4} (v+2-a)(P(v) \cdot C) dv + \frac{2a^2 - 5a + 8}{3(a+2)} = \frac{a^2 + 2a - 2}{3(a+2)}.$$

Since $\frac{a^2 + 2a - 2}{3(a+2)} \geq \frac{(a+10)(a-2)}{3(a+2)}$ for $a \in (2, 3]$, the required assertion follows from (\heartsuit) . \square

Lemma A.6. Let P be a point in \mathbf{l}_3 such that $P \notin \mathbf{e}_3$. Then

$$\delta_P(S, D) \geq \frac{3(a+2)}{a^2 + 2a - 2}.$$

Proof. Set $C = \mathbf{l}_3$. Then $\tau = a - 1$. Moreover, we have

$$P(v) \sim_{\mathbb{R}} \begin{cases} (a-v)\mathbf{h} - \mathbf{e}_1 - \mathbf{e}_2 - (2-2v)\mathbf{e}_3 & \text{if } 0 \leq v \leq a-2, \\ (2a-2v-2)\mathbf{h} + (1-a+v)(\mathbf{e}_1 + \mathbf{e}_2) + (2v-2)\mathbf{e}_3 & \text{if } a-2 \leq v \leq 1, \\ (2a-2v-2)\mathbf{h} + (1-a+v)(\mathbf{e}_1 + \mathbf{e}_2) & \text{if } 1 \leq v \leq a-1, \end{cases}$$

and

$$N(v) = \begin{cases} 0 & \text{if } 0 \leq v \leq a-2, \\ (v+2-a)\mathbf{l}_4 & \text{if } a-2 \leq v \leq 1, \\ (v+2-a)\mathbf{l}_4 + 2(v-1)\mathbf{e}_3 & \text{if } 1 \leq v \leq a-1. \end{cases}$$

This gives

$$P(v)^2 = \begin{cases} a^2 - 2av - v^2 + 4v - 4 & \text{if } 0 \leq v \leq a-2, \\ 2(a-2)(a-2v) & \text{if } a-2 \leq v \leq 1, \\ 2(a-v-1)^2 & \text{if } 1 \leq v \leq a-1, \end{cases},$$

and

$$P(v) \cdot C = \begin{cases} a+v-2 & \text{if } 0 \leq v \leq a-2, \\ 2a-4 & \text{if } a-2 \leq v \leq 1, \\ 2a-2v-2 & \text{if } 1 \leq v \leq a-1. \end{cases},$$

Now, integrating, we get $S_D(C) = \frac{a^2+2a-2}{3(a+2)}$, which gives $\delta_P(S, D) \leq \frac{3(a+2)}{a^2+2a-2}$. Similarly, we get

$$S(W_{\bullet,\bullet}^C; P) = \frac{2}{a^2-4} \int_{a-2}^{a-1} \text{ord}_P(N(v)|_C)(P(v) \cdot C) dv + \frac{(a-2)(14-a)}{3(a+2)}.$$

Thus, if $P \notin \mathbf{l}_4$, then $S(W_{\bullet,\bullet}^C; P) = \frac{(a-2)(14-a)}{3(a+2)}$. Similarly, if $P \in \mathbf{l}_4$, then

$$S(W_{\bullet,\bullet}^C; P) = \frac{2}{a^2-4} \int_{a-2}^{a-1} (v+2-a)(P(v) \cdot C) dv + \frac{(a-2)(14-a)}{3(a+2)} = \frac{a^2+2a-2}{3(a+2)} = S_D(C),$$

so the required assertion follows from (\heartsuit) . \square

Lemma A.7. *Let P be a point in \mathbf{l}_4 such that $P \notin \mathbf{e}_1 \cup \mathbf{e}_2 \cup \mathbf{l}_3$. Then*

$$\delta_P(S, D) \geq \frac{3(a+2)}{a^2+2a-2}.$$

Proof. Set $C = \mathbf{l}_4$. Then $\tau = a - 1$. Moreover, we have

$$P(v) \sim_{\mathbb{R}} \begin{cases} (a-v)\mathbf{h} - (1-v)(\mathbf{e}_1 + \mathbf{e}_2) - 2\mathbf{e}_3 & \text{if } 0 \leq v \leq a-2, \\ (2a-2v-2)\mathbf{h} + (v-1)(\mathbf{e}_1 + \mathbf{e}_2) + (2+2v-2a)\mathbf{e}_3 & \text{if } a-2 \leq v \leq 1, \\ (2a-2v-2)\mathbf{h} + (2+2v-2a)\mathbf{e}_3 & \text{if } 1 \leq v \leq a-1, \end{cases}$$

and

$$N(v) = \begin{cases} 0 & \text{if } 0 \leq v \leq a-2, \\ (v+2-a)\mathbf{l}_3 & \text{if } a-2 \leq v \leq 1, \\ (v+2-a)\mathbf{l}_3 + (v-1)(\mathbf{e}_1 + \mathbf{e}_2) & \text{if } 1 \leq v \leq a-1. \end{cases}$$

This gives

$$P(v)^2 = \begin{cases} a^2 - 2av - v^2 + 4v - 4 & \text{if } 0 \leq v \leq a-2, \\ 2(a-2)(a-2v) & \text{if } a-2 \leq v \leq 1, \\ 2(a-v-1)^2 & \text{if } 1 \leq v \leq a-1, \end{cases},$$

and

$$P(v) \cdot C = \begin{cases} a+v-2 & \text{if } 0 \leq v \leq a-2, \\ 2a-4 & \text{if } a-2 \leq v \leq 1, \\ 2a-2v-2 & \text{if } 1 \leq v \leq a-1. \end{cases},$$

Now, integrating, we get $S_D(C) = \frac{a^2+2a-2}{3(a+2)}$, which gives $\delta_P(S, D) \leq \frac{3(a+2)}{a^2+2a-2}$. Similarly, we get

$$S(W_{\bullet,\bullet}^C; P) = \frac{2}{a^2-4} \int_{a-2}^{a-1} \text{ord}_P(N(v)|_C)(P(v) \cdot C) dv + \frac{(a-2)(14-a)}{3(a+2)} = \frac{(a-2)(14-a)}{3(a+2)},$$

because $P \notin \mathbf{e}_1 \cup \mathbf{e}_2 \cup \mathbf{l}_3$. Since $\frac{(a-2)(14-a)}{3(a+2)} \leq S_D(C)$, we have $\delta_P(S, D) = \frac{3(a+2)}{a^2+2a-2}$ by (\heartsuit) . \square

Lemma A.8. Suppose that $P \notin \mathbf{e}_1 \cup \mathbf{e}_2 \cup \mathbf{e}_3 \cup \mathbf{l}_1 \cup \mathbf{l}_2 \cup \mathbf{l}_3 \cup \mathbf{l}_4$. Then

$$\delta_P(S, D) \geq \begin{cases} \frac{2(a+2)}{a^2 - 2a + 4} & \text{if } 2 < a \leq 5 - \sqrt{5}, \\ \frac{2(2a+4)}{a^2 + 6a - 12} & \text{if } 5 - \sqrt{5} \leq u \leq 3. \end{cases}$$

Proof. Let $\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3, \tilde{\mathbf{l}}_1, \tilde{\mathbf{l}}_2, \tilde{\mathbf{l}}_3, \tilde{\mathbf{l}}_4$ be the strict transforms on \tilde{S} of the curves $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{l}_1, \mathbf{l}_2, \mathbf{l}_3, \mathbf{l}_4$, respectively. Set $\mathbf{h} = f^*(\mathbf{h})$. Let $\mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$ be the curves in $|2\tilde{\mathbf{h}} - E - \tilde{\mathbf{e}}_1 - \tilde{\mathbf{e}}_2 - 2\tilde{\mathbf{e}}_3|, |\mathbf{h} - E - \tilde{\mathbf{e}}_1|, |\tilde{\mathbf{h}} - E - \tilde{\mathbf{e}}_2|, |\tilde{\mathbf{h}} - E - \tilde{\mathbf{e}}_3|$, respectively. Then

$$\mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3, \tilde{\mathbf{l}}_1, \tilde{\mathbf{l}}_2, \tilde{\mathbf{l}}_3, \tilde{\mathbf{l}}_4, E$$

are all curves in \tilde{S} that have negative self-intersections [6].

We compute $\tilde{\tau} = \frac{3a-4}{2}$. Similarly, we see that

$$\tilde{P}(v) \sim_{\mathbb{R}} \begin{cases} a\tilde{\mathbf{h}} - \tilde{\mathbf{e}}_1 - \tilde{\mathbf{e}}_2 - 2\tilde{\mathbf{e}}_3 - vE & \text{if } 0 \leq v \leq 2a-4, \\ (5a-2v-8)\tilde{\mathbf{h}} + (3-2a+v)(\tilde{\mathbf{e}}_1 + \tilde{\mathbf{e}}_2 + 2\tilde{\mathbf{e}}_3) - (2a-4)E & \text{if } 2a-4 \leq v \leq a-1, \\ (3a-2v-4)(3\tilde{\mathbf{h}} - \tilde{\mathbf{e}}_1 - \tilde{\mathbf{e}}_2 - 2\tilde{\mathbf{e}}_3 - 2E) & \text{if } a-1 \leq v \leq \frac{3a-4}{2}, \end{cases}$$

and

$$\tilde{N}(v) = \begin{cases} 0 & \text{if } 0 \leq v \leq 2a-4, \\ (v+4-2a)\mathbf{c}_0 & \text{if } 2a-4 \leq v \leq a-1, \\ (v+4-2a)\mathbf{c}_0 + (v+1-a)(\mathbf{c}_1 + \mathbf{c}_2 + 2\mathbf{c}_3) & \text{if } a-1 \leq v \leq \frac{3a-4}{2}. \end{cases}$$

This gives

$$\tilde{P}(v)^2 = \begin{cases} a^2 - v^2 - 4 & \text{if } 0 \leq v \leq 2a-4, \\ (a-2)(5a-4v-6) & \text{if } 2a-4 \leq v \leq a-1, \\ (3a-2v-4)^2 & \text{if } a-1 \leq v \leq \frac{3a-4}{2}, \end{cases}$$

and

$$\tilde{P}(v) \cdot E = \begin{cases} v & \text{if } 0 \leq v \leq 2a-4, \\ 2a-4 & \text{if } 2a-4 \leq v \leq a-1, \\ 6a-4v-8 & \text{if } a-1 \leq v \leq \frac{3a-4}{2}. \end{cases}$$

Now, integrating, we get $S_D(E) = \frac{a^2+6a-12}{2(a+2)}$.

Let O be a point in E . Then

$$S(W_{\bullet,\bullet}^E; O) = \frac{2}{a^2 - 4} \int_{2a-4}^{\frac{3a-4}{2}} \text{ord}_O(\tilde{N}(v)|_E)(\tilde{P}(v) \cdot E) dv + \frac{2(8-a)(a-2)}{3(a+2)}.$$

Thus, if $O \notin \mathbf{c}_0 \cup \mathbf{c}_1 \cup \mathbf{c}_2 \cup \mathbf{c}_3$, then $S(W_{\bullet,\bullet}^E; O) = \frac{2(8-a)(a-2)}{3(a+2)}$. Similarly, if $O \in \mathbf{c}_0$, then

$$S(W_{\bullet,\bullet}^E; O) = \frac{2}{a^2 - 4} \int_{2a-4}^{\frac{3a-4}{2}} (v+4-2a)(\tilde{P}(v) \cdot E) dv + \frac{2(8-a)(a-2)}{3(a+2)} = \frac{a^2 - 2a + 4}{2(a+2)}.$$

Likewise, if $O \in \mathbf{c}_1 \cup \mathbf{c}_2$, then

$$S(W_{\bullet,\bullet}^E; O) = \frac{2}{a^2 - 4} \int_{a-1}^{\frac{3a-4}{2}} (v+1-a)(\tilde{P}(v) \cdot E) dv + \frac{2(8-a)(a-2)}{3(a+2)} = \frac{(a-2)(10-a)}{2(a+2)}.$$

Finally, if $O \in \mathbf{c}_3$, then

$$S(W_{\bullet,\bullet}^E; O) = \frac{2}{a^2 - 4} \int_{a-1}^{\frac{3a-4}{2}} 2(v+1-a)(\tilde{P}(v) \cdot E) dv + \frac{2(8-a)(a-2)}{3(a+2)} = \frac{(a-2)(14-a)}{3(a+2)}.$$

Therefore, using (\diamond) , we get

$$\delta_P(S, D) \geq \min \left\{ \frac{4(a+2)}{a^2 + 6a - 12}, \frac{2(a+2)}{a^2 - 2a + 4}, \frac{2(a+2)}{(a-2)(10-a)}, \frac{3(a+2)}{(a-2)(14-a)} \right\},$$

which implies the required assertion. \square

Combining Lemmas A.4, A.5, A.6, A.7, A.8, we obtain

Corollary A.2. *Let P be a point in S such that $P \notin \mathbf{l}_1 \cup \mathbf{l}_2$. If $P \in \mathbf{e}_1 \cup \mathbf{e}_2 \cup \mathbf{e}_3$, then*

$$\delta_P(S, D) \geq \begin{cases} \frac{3(a+2)}{a^2 + 2a - 2} & \text{if } 2 < a \leq \frac{1+\sqrt{21}}{2}, \\ \frac{1}{a-2} & \text{if } \frac{1+\sqrt{21}}{2} \leq u \leq 3. \end{cases}$$

If $P \notin \mathbf{e}_1 \cup \mathbf{e}_2 \cup \mathbf{e}_3$, then $\delta_P(S, D) \geq \frac{3(a+2)}{a^2 + 2a - 2}$ for every $a \in (2, 3]$.

A.3. Quintic del Pezzo surface with singular point of type \mathbb{A}_2 . Let S be a del Pezzo surface such that $K_S^2 = 5$, and S has one singular point, which is a singular point of type \mathbb{A}_1 . Set $O = \text{Sing}(S)$. Then it follows from [6] that there exists a birational morphism $\pi: S \rightarrow \mathbb{P}^2$ that contracts smooth irreducible rational curves \mathbf{e}_1 and \mathbf{e}_2 such that $\mathbf{e}_1 \cap \mathbf{e}_2 = \emptyset$, $O \notin \mathbf{e}_1$, $O \in \mathbf{e}_2$, $\mathbf{e}_1^2 = -1$, and $\mathbf{e}_3^2 = -\frac{1}{3}$. Set $\mathbf{h} = \pi^*(\mathcal{O}_{\mathbb{P}^2}(1))$. Let \mathbf{l}_1 and \mathbf{l}_2 be irreducible curves in $|\mathbf{h} - 2\mathbf{e}_2|$ and $|\mathbf{h} - \mathbf{e}_1 - \mathbf{e}_2|$, respectively. Then \mathbf{l}_1 and \mathbf{l}_2 are smooth and rational, $O = \mathbf{l}_1 \cap \mathbf{l}_2 \cap \mathbf{e}_2$, and $\mathbf{e}_1, \mathbf{e}_2, \mathbf{l}_1, \mathbf{l}_2$ are all curves in S that have negative self-intersections. They intersections are given in the following table:

•	\mathbf{e}_1	\mathbf{e}_2	\mathbf{l}_1	\mathbf{l}_2
\mathbf{e}_1	-1	0	0	1
\mathbf{e}_2	0	$-\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$
\mathbf{l}_1	0	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$
\mathbf{l}_2	1	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$

Set $D = a\mathbf{h} - \mathbf{e}_1 - 3\mathbf{e}_2$ for $a \in (2, 3]$. Then D is ample and $D^2 = a^2 - 4$.

Lemma A.9. *Let P be a point in \mathbf{e}_2 such that $P \neq O$. Then*

$$\delta_P(S, D) \geq \begin{cases} \frac{6(a+2)}{a^2 + 2a + 4} & \text{if } 2 < a \leq \frac{1+\sqrt{17}}{2}, \\ \frac{6(a+2)}{(7a+10)(a-2)} & \text{if } \frac{1+\sqrt{17}}{2} \leq a \leq 3. \end{cases}$$

Proof. Set $C = \mathbf{e}_2$. Then $\tau = 2a - 4$. Moreover, we have

$$P(v) \sim_{\mathbb{R}} \begin{cases} a\mathbf{h} - \mathbf{e}_1 - (3+v)\mathbf{e}_2 & \text{if } 0 \leq v \leq \frac{3a-6}{2}, \\ (4a-6-2v)\mathbf{h} - \mathbf{e}_1 + (3v+9-6a)\mathbf{e}_2 & \text{if } \frac{3a-6}{2} \leq v \leq 2a-4, \end{cases}$$

and

$$N(v) = \begin{cases} 0 & \text{if } 0 \leq v \leq \frac{3a-6}{2}, \\ (6-3a+2v)\mathbf{l}_1 & \text{if } \frac{3a-6}{2} \leq v \leq 2a-4, \end{cases}$$

which gives

$$P(v)^2 = \begin{cases} \frac{3a^2 - v^2 - 6v - 12}{3} & \text{if } 0 \leq v \leq \frac{3a-6}{2}, \\ (2a-v-2)(2a-v-4) & \text{if } \frac{3a-6}{2} \leq v \leq 2a-4, \end{cases}$$

and

$$P(v) \cdot C = \begin{cases} \frac{3+v}{3} & \text{if } 0 \leq v \leq \frac{3a-6}{2}, \\ 2a-3-v & \text{if } \frac{3a-6}{2} \leq v \leq 2a-4. \end{cases}$$

Integrating, we get $S_D(C) = \frac{(7a+10)(a-2)}{6(a+2)}$. Similarly, we compute $S(W_{\bullet,\bullet}^C; P) = \frac{a^2+2a+4}{6(a+2)}$, since $P \notin \mathbf{l}_1$. Now, using (\heartsuit) , we obtain the required inequality. \square

Lemma A.10. *Let P be a point in \mathbf{e}_1 such that $P \notin \mathbf{l}_2$. Then*

$$\delta_P(S, D) \geq \begin{cases} \frac{3(a+2)}{2a^2 - 5a + 8} & \text{if } 2 < a \leq \frac{13 + \sqrt{57}}{2}, \\ \frac{3(a+2)}{(a+10)(a-2)} & \text{if } \frac{13 + \sqrt{57}}{2} \leq a \leq 3. \end{cases}$$

Proof. Set $C = \mathbf{e}_1$. Then $\tau = 2a - 4$. Moreover, we have

$$P(v) \sim_{\mathbb{R}} \begin{cases} a\mathbf{h} - (1+v)\mathbf{e}_1 - 3\mathbf{e}_2 & \text{if } 0 \leq v \leq a-2, \\ (4a-3v-6)\mathbf{h} + (2v+5-3a)\mathbf{e}_1 + 3(1-a+v)\mathbf{e}_2 & \text{if } a-2 \leq v \leq 2a-4, \end{cases}$$

and

$$N(v) = \begin{cases} 0 & \text{if } 0 \leq v \leq a-2, \\ 3(v-a+2)\mathbf{l}_2 & \text{if } a-2 \leq v \leq 2a-4, \end{cases}$$

which gives

$$P(v)^2 = \begin{cases} a^2 - v^2 - 2v - 4 & \text{if } 0 \leq v \leq a-2, \\ 2(2a-v-4)(a-v-1) & \text{if } a-2 \leq v \leq 2a-4, \end{cases}$$

and

$$P(v) \cdot C = \begin{cases} 1+v & \text{if } 0 \leq v \leq a-2, \\ 3a-5-2v & \text{if } a-2 \leq v \leq 2a-4. \end{cases}$$

This gives $S_D(C) = \frac{(a+10)(a-2)}{3(a+2)}$ and $S(W_{\bullet,\bullet}^C; P) = \frac{2a^2-5a+8}{3(a+2)}$, so (\heartsuit) implies the required assertion. \square

Lemma A.11. Suppose that $P \notin \mathbf{e}_1 \cup \mathbf{e}_2 \cup \mathbf{l}_1 \cup \mathbf{l}_2$. Then

$$\delta_P(S, D) \geq \begin{cases} \frac{2(a+2)}{a^2 - 2a + 4} & \text{if } 2 < a \leq 5 - \sqrt{5}, \\ \frac{2(2a+4)}{a^2 + 6a - 12} & \text{if } 5 - \sqrt{5} \leq a \leq \frac{19 - \sqrt{21}}{5}, \\ \frac{6(a+2)}{(a-2)(26-a)} & \text{if } \frac{19 - \sqrt{21}}{5} \leq a \leq 3. \end{cases}$$

Proof. Let $\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{l}}_1$ and $\tilde{\mathbf{l}}_2$, be the strict transforms on \tilde{S} of the curves $\mathbf{e}_1, \mathbf{e}_2, \mathbf{l}_1$ and \mathbf{l}_2 , respectively. Set $\tilde{\mathbf{h}} = f^*(\mathbf{h})$. Let $\mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2$ be the curves in $|2\tilde{\mathbf{h}} - E - \tilde{\mathbf{e}}_1 - 3\tilde{\mathbf{e}}_2|, |\tilde{\mathbf{h}} - E - \tilde{\mathbf{e}}_1|, |\tilde{\mathbf{h}} - E - \tilde{\mathbf{e}}_2|$, respectively. Then $\mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2, \tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{l}}_1, \tilde{\mathbf{l}}_2, E$ are all curves in \tilde{S} that have negative self-intersections [6].

We compute $\tilde{\tau} = \frac{3a-4}{2}$. Similarly, we see that

$$\tilde{P}(v) \sim_{\mathbb{R}} \begin{cases} a\tilde{\mathbf{h}} - \tilde{\mathbf{e}}_1 - 3\tilde{\mathbf{e}}_2 - vE & \text{if } 0 \leq v \leq 2a-4, \\ (5a-2v-8)\tilde{\mathbf{h}} + (3-2a+v)(\tilde{\mathbf{e}}_1 + 3\tilde{\mathbf{e}}_2) + (4-2a)E & \text{if } 2a-4 \leq v \leq a-1, \\ (3a-2v-4)(3\tilde{\mathbf{h}} - \tilde{\mathbf{e}}_1 - 3\tilde{\mathbf{e}}_2 - 2E) & \text{if } a-1 \leq v \leq \frac{3a-4}{2}, \end{cases}$$

and

$$\tilde{N}(v) = \begin{cases} 0 & \text{if } 0 \leq v \leq 2a-4, \\ (v+4-2a)\mathbf{c}_0 & \text{if } 2a-4 \leq v \leq a-1, \\ (v+4-2a)\mathbf{c}_0 + (v+1-a)(\mathbf{c}_1 + 3\mathbf{c}_2) & \text{if } a-1 \leq v \leq \frac{3a-4}{2}. \end{cases}$$

This gives

$$\tilde{P}(v)^2 = \begin{cases} a^2 - v^2 - 4 & \text{if } 0 \leq v \leq 2a-4, \\ (a-2)(5a-4v-6) & \text{if } 2a-4 \leq v \leq a-1, \\ (3a-2v-4)^2 & \text{if } a-1 \leq v \leq \frac{3a-4}{2}, \end{cases}$$

and

$$\tilde{P}(v) \cdot E = \begin{cases} v & \text{if } 0 \leq v \leq 2a-4, \\ 2a-4 & \text{if } 2a-4 \leq v \leq a-1, \\ 6a-4v-8 & \text{if } a-1 \leq v \leq \frac{3a-4}{2}. \end{cases}$$

Now, integrating, we get $S_D(E) = \frac{a^2+6a-12}{2(a+2)}$.

Let O be a point in E . Then

$$S(W_{\bullet,\bullet}^E; O) = \frac{2}{a^2 - 4} \int_{2a-4}^{\frac{3a-4}{2}} \text{ord}_O(\tilde{N}(v)|_E)(\tilde{P}(v) \cdot E) dv + \frac{2(8-a)(a-2)}{3(a+2)}.$$

Thus, if $O \notin \mathbf{c}_0 \cup \mathbf{c}_1 \cup \mathbf{c}_2$, then $S(W_{\bullet,\bullet}^E; O) = \frac{2(8-a)(a-2)}{3(a+2)}$. Similarly, if $O \in \mathbf{c}_0$, then

$$S(W_{\bullet,\bullet}^E; O) = \frac{2}{a^2 - 4} \int_{2a-4}^{\frac{3a-4}{2}} (v+4-2a)(\tilde{P}(v) \cdot E) dv + \frac{2(8-a)(a-2)}{3(a+2)} = \frac{a^2 - 2a + 4}{2(a+2)}.$$

Likewise, if $O \in \mathbf{c}_1$, then

$$S(W_{\bullet,\bullet}^E; O) = \frac{2}{a^2 - 4} \int_{a-1}^{\frac{3a-4}{2}} (v+1-a)(\tilde{P}(v) \cdot E) dv + \frac{2(8-a)(a-2)}{3(a+2)} = \frac{(a-2)(10-a)}{2(a+2)}.$$

Finally, if $O \in \mathbf{c}_2$, then

$$S(W_{\bullet,\bullet}^E; O) = \frac{2}{a^2 - 4} \int_{a-1}^{\frac{3a-4}{2}} 3(v+1-a)(\tilde{P}(v) \cdot E) dv + \frac{2(8-a)(a-2)}{3(a+2)} = \frac{(a-2)(26-a)}{6(a+2)}.$$

Therefore, using (\diamond) , we get

$$\delta_P(S, D) \geq \min \left\{ \frac{4(a+2)}{a^2 + 6a - 12}, \frac{2(a+2)}{a^2 - 2a + 4}, \frac{2(a+2)}{(a-2)(10-a)}, \frac{6(a+2)}{(a-2)(26-a)} \right\},$$

which implies the required assertion. \square

Combining Lemmas A.9, A.10, A.11, we obtain

Corollary A.3. *Let P be a point in S such that $P \notin \mathbf{l}_1 \cup \mathbf{l}_2$. If $P \in \mathbf{e}_1 \cup \mathbf{e}_2$, then*

$$\delta_P(S, D) \geq \begin{cases} \frac{6(a+2)}{a^2 + 2a + 4} & \text{if } 2 < a \leq \frac{1 + \sqrt{17}}{2}, \\ \frac{6(a+2)}{(7a+10)(a-2)} & \text{if } \frac{1 + \sqrt{17}}{2} \leq a \leq 3. \end{cases}$$

If $P \notin \mathbf{e}_1 \cup \mathbf{e}_2$, then

$$\delta_P(S, D) \geq \begin{cases} \frac{2(a+2)}{a^2 - 2a + 4} & \text{if } 2 < a \leq 5 - \sqrt{5}, \\ \frac{2(2a+4)}{a^2 + 6a - 12} & \text{if } 5 - \sqrt{5} \leq a \leq \frac{19 - \sqrt{21}}{5}, \\ \frac{6(a+2)}{(a-2)(26-a)} & \text{if } \frac{19 - \sqrt{21}}{5} \leq a \leq 3. \end{cases}$$

A.4. Smooth sextic del Pezzo surface. Let $\ell_1, \ell_2, \ell_3, \ell_4$ be four distinct rulings of $\mathbb{P}^1 \times \mathbb{P}^1$ such that $\ell_1 \cap \ell_3 = \emptyset$, $\ell_2 \cap \ell_4 = \emptyset$, and each intersection $\ell_1 \cap \ell_2, \ell_2 \cap \ell_3, \ell_3 \cap \ell_4, \ell_4 \cap \ell_1$ consists of one point, let $\pi: S \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ be the blow up of the points $\ell_1 \cap \ell_2$ and $\ell_3 \cap \ell_4$, let \mathbf{e}_1 and \mathbf{e}_2 be the π -exceptional curves such that $\pi(\mathbf{e}_1) = \ell_1 \cap \ell_2$ and $\pi(\mathbf{e}_2) = \ell_3 \cap \ell_4$, let $\mathbf{l}_1, \mathbf{l}_2, \mathbf{l}_3, \mathbf{l}_4$ be the strict transforms on S of the curves $\ell_1, \ell_2, \ell_3, \ell_4$, respectively. Then S is a del Pezzo surface of degree 6, and $\mathbf{e}_1, \mathbf{e}_2, \mathbf{l}_1, \mathbf{l}_2, \mathbf{l}_3, \mathbf{l}_4$ are all (-1) -curves in S . Set $\mathbf{h}_1 = \pi^*(\ell_1)$ and $\mathbf{h}_2 = \pi^*(\ell_2)$. Set

$$D = a(\mathbf{h}_1 + \mathbf{h}_2) - \mathbf{e}_1 - \mathbf{e}_2$$

for $a \in (1, 2]$. Then D is ample and $D^2 = 2a^2 - 2$.

Lemma A.12. *Let P be a point in $\mathbf{e}_1 \cup \mathbf{e}_2$. If $P \in \mathbf{l}_1 \cup \mathbf{l}_2 \cup \mathbf{l}_3 \cup \mathbf{l}_4$, then $\delta_P(S, D) \geq \frac{2}{a}$ for $a \in (1, 2]$. Similarly, if $P \notin \mathbf{l}_1 \cup \mathbf{l}_2 \cup \mathbf{l}_3 \cup \mathbf{l}_4$, then*

$$\delta_P(S, D) \geq \begin{cases} \frac{3(a+1)}{a^2 + a + 1} & \text{if } 1 < a \leq \frac{1 + \sqrt{33}}{4}, \\ \frac{1}{a-1} & \text{if } \frac{1 + \sqrt{33}}{4} \leq a \leq 2. \end{cases}$$

Proof. We may assume that $P \in \mathbf{e}_1$. Set $C = \mathbf{e}_1$. Then $\tau = 2a - 2$. Moreover, we have

$$P(v) \sim_{\mathbb{R}} \begin{cases} a(\mathbf{h}_1 + \mathbf{h}_2) - (1+v)\mathbf{e}_1 - \mathbf{e}_2 & \text{if } 0 \leq v \leq a-1, \\ (2a-v-1)(\mathbf{h}_1 + \mathbf{h}_2) + (v+1-2a)\mathbf{e}_1 - \mathbf{e}_2 & \text{if } a-1 \leq v \leq 2a-2, \end{cases}$$

and

$$N(v) = \begin{cases} 0 & \text{if } 0 \leq v \leq a-1, \\ (v+1-a)(\mathbf{l}_1 + \mathbf{l}_2) & \text{if } a-1 \leq v \leq 2a-2, \end{cases}$$

which gives

$$P(v)^2 = \begin{cases} 2a^2 - v^2 - 2v - 2 & \text{if } 0 \leq v \leq a-1, \\ (2a-v)(2a-v-2) & \text{if } a-1 \leq v \leq 2a-2, \end{cases}$$

and

$$P(v) \cdot C = \begin{cases} 1+v & \text{if } 0 \leq v \leq a-1, \\ 2a-v-1 & \text{if } a-1 \leq v \leq 2a-2. \end{cases}$$

Integrating, we get $S_D(C) = a-1$. Similarly, we get

$$S(W_{\bullet,\bullet}^C; P) = \frac{1}{a^2-1} \int_{a-1}^{2a-2} \text{ord}_P(N(v)|_C)(P(v) \cdot C) dv + \frac{a^2+a+1}{3(a+1)}.$$

Thus, if $P \notin \mathbf{l}_1 \cup \mathbf{l}_2$, then $S(W_{\bullet,\bullet}^C; P) = \frac{a^2+a+1}{3(a+1)}$, so that (\heartsuit) gives

$$\delta_P(S, D) \geq \min \left\{ \frac{1}{a-1}, \frac{3(a+1)}{a^2+a+1} \right\} = \begin{cases} \frac{3(a+1)}{a^2+a+1} & \text{if } 1 < a \leq \frac{1+\sqrt{33}}{4}, \\ \frac{1}{a-1} & \text{if } \frac{1+\sqrt{33}}{4} \leq a \leq 2. \end{cases}$$

Similarly, if $P \in \mathbf{l}_1 \cup \mathbf{l}_2$, then

$$S(W_{\bullet,\bullet}^C; P) = \frac{1}{a^2-1} \int_{a-1}^{2a-2} (v+1-a)(P(v) \cdot C) dv + \frac{a^2+a+1}{3(a+1)} = \frac{a}{2},$$

so that $S(W_{\bullet,\bullet}^C; P) \leq \frac{a}{2}$ and $S_D(C) \leq \frac{a}{2}$, which gives $\delta_P(S, D) \geq \frac{2}{a}$ by (\heartsuit) . \square

Lemma A.13. *Let P be a point in $\mathbf{l}_1 \cup \mathbf{l}_2 \cup \mathbf{l}_3 \cup \mathbf{l}_4$. Then $\delta_P(S, D) \geq \frac{2}{a}$ for every $a \in (1, 2]$.*

Proof. We may assume that $P \in \mathbf{l}_1$. Moreover, by Lemma A.12, we may assume that $P \notin \mathbf{e}_1 \cup \mathbf{e}_2$. Set $C = \mathbf{l}_1$. Then $\tau = a$. Moreover, we have

$$P(v) \sim_{\mathbb{R}} \begin{cases} (a-v)\mathbf{h}_1 + a\mathbf{h}_2 + (v-1)\mathbf{e}_1 - \mathbf{e}_2 & \text{if } 0 \leq v \leq a-1, \\ (a-v)\mathbf{h}_1 + (2a-v-1)\mathbf{h}_2 + (v-1)\mathbf{e}_1 + (v-a)\mathbf{e}_2 & \text{if } a-1 \leq v \leq 1, \\ (a-v)\mathbf{h}_1 + (2a-v-1)\mathbf{h}_2 + (v-a)\mathbf{e}_2 & \text{if } 1 \leq v \leq a, \end{cases}$$

and

$$N(v) = \begin{cases} & \text{if } 0 \leq v \leq a-1, \\ (v+1-a)\mathbf{l}_4 & \text{if } a-1 \leq v \leq 1, \\ (v+1-a)\mathbf{l}_4 + (v-1)\mathbf{e}_1 & \text{if } 1 \leq v \leq a, \end{cases}$$

which gives

$$P(v)^2 = \begin{cases} 2a^2 - 2av - v^2 + 2v - 2 & \text{if } 0 \leq v \leq a-1, \\ (a-1)(3a-4v+1) & \text{if } a-1 \leq v \leq 1, \\ (3a-v-2)(a-v) & \text{if } 1 \leq v \leq a, \end{cases}$$

and

$$P(v) \cdot C = \begin{cases} a + v - 1 & \text{if } 0 \leq v \leq a - 1, \\ 2a - 2 & \text{if } a - 1 \leq v \leq 1, \\ 2a - v - 1 & \text{if } 1 \leq v \leq a. \end{cases}$$

Integrating, we get $S_D(C) = \frac{a}{2}$. Similarly, we get

$$S(W_{\bullet,\bullet}^C; P) = \frac{1}{a^2 - 1} \int_{a-1}^a \text{ord}_P(N(v)|_C)(P(v) \cdot C) dv + \frac{(a+5)(a-1)}{3(a+1)}.$$

Thus, if $P \notin \mathbf{l}_4 \cup \mathbf{e}_1$, then $S(W_{\bullet,\bullet}^C; P) = \frac{(a+5)(a-1)}{3(a+1)}$. Similarly, if $P \in \mathbf{l}_4$, then

$$S(W_{\bullet,\bullet}^C; P) = \frac{1}{a^2 - 1} \int_{a-1}^a (v + 1 - a)(P(v) \cdot C) dv + \frac{(a+5)(a-1)}{3(a+1)} = \frac{a}{2}.$$

Hence, we see that $S(W_{\bullet,\bullet}^C; P) \leq S_D(C) \geq \frac{a}{2}$, so that $\delta_P(S, D) = \frac{2}{a}$ by (\heartsuit) . \square

Lemma A.14. Suppose that $P \notin \mathbf{e}_1 \cup \mathbf{e}_2 \cup \mathbf{l}_1 \cup \mathbf{l}_2 \cup \mathbf{l}_3 \cup \mathbf{l}_4$. Then

$$\delta_P(S, D) \geq \begin{cases} \frac{3(a+1)}{a^2 + a + 1} & \text{if } 1 < a \leq \frac{\sqrt{21}-1}{2}, \\ \frac{2(a+1)}{a^2 + a - 1} & \text{if } \frac{\sqrt{21}-1}{2} \leq a \leq 2. \end{cases}$$

Proof. Recall that \tilde{S} is a smooth del Pezzo surface of degree 5. Let $\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{l}}_1, \tilde{\mathbf{l}}_2, \tilde{\mathbf{l}}_3, \tilde{\mathbf{l}}_4$ be the strict transforms on \tilde{S} of the (-1) -curves $\mathbf{e}_1, \mathbf{e}_2, \mathbf{l}_1, \mathbf{l}_2, \mathbf{l}_3, \mathbf{l}_4$, respectively. Set $\tilde{\mathbf{h}}_1 = f^*(\mathbf{h}_1)$ and $\tilde{\mathbf{h}}_2 = f^*(\mathbf{h}_2)$. Let $\mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2$ be the curves in $|\tilde{\mathbf{h}}_1 + \tilde{\mathbf{h}}_2 - \tilde{\mathbf{e}}_1 - \tilde{\mathbf{e}}_2 - E|, |\tilde{\mathbf{h}}_1 - E|, |\tilde{\mathbf{h}}_2 - E|$, respectively. Then

$$\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{l}}_1, \tilde{\mathbf{l}}_2, \tilde{\mathbf{l}}_3, \tilde{\mathbf{l}}_4, \mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2, E$$

are all (-1) -curves in \tilde{S} . We compute $\tilde{\tau} = 2a - 1$. Similarly, we see that

$$\tilde{P}(v) \sim_{\mathbb{R}} \begin{cases} a(\tilde{\mathbf{h}}_1 + \tilde{\mathbf{h}}_2) - \tilde{\mathbf{e}}_1 - \tilde{\mathbf{e}}_2 - vE & \text{if } 0 \leq v \leq 2a - 2, \\ (3a - v - 2)(\tilde{\mathbf{h}}_1 + \tilde{\mathbf{h}}_2) + (1 - 2a + v)(\tilde{\mathbf{e}}_1 + \tilde{\mathbf{e}}_2) + (2 - 2a)E & \text{if } 2a - 2 \leq v \leq a, \\ (2a - 1 - v)(2\tilde{\mathbf{h}}_1 + 2\tilde{\mathbf{h}}_2 - \tilde{\mathbf{e}}_1 - \tilde{\mathbf{e}}_2 - 2E) & \text{if } a \leq v \leq 2a - 1, \end{cases}$$

and

$$\tilde{N}(v) = \begin{cases} 0 & \text{if } 0 \leq v \leq 2a - 2, \\ (v + 2 - 2a)\mathbf{c}_0 & \text{if } 2a - 2 \leq v \leq a, \\ (v + 2 - 2a)\mathbf{c}_0 + (v - a)(\mathbf{c}_1 + \mathbf{c}_2) & \text{if } a \leq v \leq 2a - 1. \end{cases}$$

This gives

$$\tilde{P}(v)^2 = \begin{cases} 2a^2 - v^2 - 2 & \text{if } 0 \leq v \leq 2a - 2, \\ 2(a-1)(3a - 2v - 1) & \text{if } 2a - 2 \leq v \leq a, \\ 2(2a - 1 - v)^2 & \text{if } a \leq v \leq 2a - 1, \end{cases}$$

and

$$\tilde{P}(v) \cdot E = \begin{cases} v & \text{if } 0 \leq v \leq 2a - 2, \\ 2a - 2 & \text{if } 2a - 2 \leq v \leq a, \\ 4a - 2v - 2 & \text{if } a \leq v \leq 2a - 1. \end{cases}$$

Now, integrating, we get $S_D(E) = \frac{a^2+a-1}{a+1}$.

Let O be a point in E . Then

$$S(W_{\bullet,\bullet}^E; O) = \frac{1}{a^2 - 1} \int_{2a-2}^{2a-1} \text{ord}_O(\tilde{N}(v)|_E)(\tilde{P}(v) \cdot E) dv + \frac{2(a-1)}{a+1}.$$

Thus, if $O \notin \mathbf{c}_0 \cup \mathbf{c}_1 \cup \mathbf{c}_2$, then $S(W_{\bullet,\bullet}^E; O) = \frac{2(a-1)}{a+1}$. Similarly, if $O \in \mathbf{c}_0$, then

$$S(W_{\bullet,\bullet}^E; O) = \frac{1}{a^2 - 1} \int_{2a-2}^{2a-1} (v + 4 - 2a)(\tilde{P}(v) \cdot E) dv + \frac{2(a-1)}{a+1} = \frac{a^2 + a + 1}{3(a+1)}.$$

Likewise, if $O \in \mathbf{c}_1 \cup \mathbf{c}_2$, then

$$S(W_{\bullet,\bullet}^E; O) = \frac{1}{a^2 - 1} \int_a^{2a-1} (v + 1 - a)(\tilde{P}(v) \cdot E) dv + \frac{2(a-1)}{a+1} = \frac{(a+5)(a-1)}{3(a+1)}.$$

Therefore, using (\diamond) , we get

$$\delta_P(S, D) \geq \min \left\{ \frac{2(a+1)}{a^2 + a - 1}, \frac{3(a+1)}{a^2 + a + 1}, \frac{3(a+1)}{(a+5)(a-1)} \right\},$$

which implies the required assertion. \square

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Ivan Cheltsov

University of Edinburgh, Edinburgh, Scotland

I.Cheltsov@ed.ac.uk