# Two local inequalities

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**Abstract.** We prove two new local inequalities for divisors on smooth surfaces and consider several applications of these inequalities.

**Keywords:** Tian's $\alpha$-invariant, del Pezzo surface, Cremona group.

*To the memory of Vasilii Alekseevich Iskovskikh* (1939–2009)

## § 1. Introduction

All varieties are assumed to be projective, normal and defined over $\mathbb{C}$.

In many algebro-geometric problems it is required to prove that a log pair consisting of a variety (possibly singular) and an effective divisor (called the *boundary*) is log canonical provided that the divisor satisfies certain numerical conditions. In contrast to the intersection multiplicities of effective divisors, the property of being log canonical is not easily derived from global numerical conditions. Therefore one should use local inequalities that relate the intersection multiplicities of the boundary components and log canonicity of the pair. We illustrate this with an example.

Let $S$ be a surface, $O$ a smooth point of $S$, and $\Delta_1$ and $\Delta_2$ curves on $S$ such that $O \in \Delta_1 \cap \Delta_2$, both $\Delta_1$ and $\Delta_2$ are irreducible, reduced and smooth at $O$, and $\Delta_1$ intersects $\Delta_2$ transversally at $O$. Let $a_1$ and $a_2$ be rational numbers.

**Theorem 1.1** ([1], Corollary 5.57). *Let $D$ be an effective $\mathbb{Q}$-divisor on $S$ such that the log pair $(S, D + \Delta_1)$ is not log canonical at $O$. Suppose that $\Delta_1 \not\subseteq \operatorname{Supp}(D)$ and $a_1 \leqslant 1$. Then $\operatorname{mult}_O(D \cdot \Delta_1) > 1$.*

**Corollary 1.2.** *Let $D$ be an effective $\mathbb{Q}$-divisor on $S$ such that the log pair $(S, D + a_1\Delta_1 + a_2\Delta_2)$ is not log canonical at $O$. Suppose that $\Delta_1 \not\subseteq \operatorname{Supp}(D) \not\supseteq \Delta_2$, $a_1 \geqslant 0$, $a_2 \geqslant 0$. Then $\operatorname{mult}_O(D \cdot \Delta_1) > 1 - a_2$ or $\operatorname{mult}_O(D \cdot \Delta_2) > 1 - a_1$.*

The following analogue of Corollary 1.2 is obtained implicitly in [2].

**Theorem 1.3.** *Let $D$ be an effective $\mathbb{Q}$-divisor on $S$ such that the log pair $(S, D + a_1\Delta_1 + a_2\Delta_2)$ is not log canonical at $O$. Suppose that $\Delta_1 \not\subseteq \operatorname{Supp}(D) \not\supseteq \Delta_2$, $a_1 \geqslant 0$, $a_2 \geqslant 0$. Then $\operatorname{mult}_O(D \cdot \Delta_1) > 2a_1 - a_2$ or $\operatorname{mult}_O(D \cdot \Delta_2) > \frac{3}{2}a_2 - a_1$ provided that $a_1 + \frac{a_2}{2} \leqslant 1$.*

The area of application of Theorem 1.3 is rather limited. Our first goal is to prove the following generalization of Theorem 1.3 and give some applications.

**Theorem 1.4.** *Let* $D$ *be an effective* $\mathbb{Q}$-*divisor on* $S$ *such that the log pair* $(S, D + a_1\Delta_1 + a_2\Delta_2)$ *is not log canonical at* $O$. *Suppose that* $\Delta_1 \not\subseteq \operatorname{Supp}(D) \not\supseteq \Delta_2$, $a_1 \geqslant 0$, $a_2 \geqslant 0$. *Then* $\operatorname{mult}_O(D \cdot \Delta_1) > M + Aa_1 - a_2$ *or* $\operatorname{mult}_O(D \cdot \Delta_2) > N + Ba_2 - a_1$ *provided that* $\alpha a_1 + \beta a_2 \leqslant 1$, *where* $A$, $B$, $M$, $N$, $\alpha$, $\beta$ *are non-negative rational numbers with* $A(B-1) \geqslant 1 \geqslant \max(M, N)$, $\alpha(A+M-1) \geqslant A^2(B+N-1)\beta$, $\alpha(1 - M) + A\beta \geqslant A$, *and either* $2M + AN \leqslant 2$ *or* $\alpha(B + 1 - MB - N) + \beta(A + 1 - AN - M) \geqslant AB - 1$ (*or both*).

In birational geometry one often encounters log pairs whose boundary components are mobile linear systems. Despite their somewhat abstract appearance, such log pairs can always be treated as ordinary ones. Moreover, one can bound the intersection multiplicity of general elements in the mobile boundary components in terms of the log canonicity of the log pair.

**Theorem 1.5** ([3], Theorem 3.1, [4], Lemma 3.3). *Let* $\mathcal{M}$ *be a linear system without fixed components on* $S$ *and let* $\varepsilon$ *be a positive rational number. Suppose that the log pair* $(S, \varepsilon\mathcal{M} + a_1\Delta_1 + a_2\Delta_2)$ *is not Kawamata log terminal at* $O$. *Let* $M_1$ *and* $M_2$ *be sufficiently general curves in* $\mathcal{M}$. *Then we have*

$$\operatorname{mult}_O(M_1 \cdot M_2) \geqslant \begin{cases} \dfrac{4(1 - a_1)(1 - a_2)}{\varepsilon^2} & \text{if } a_1 \geqslant 0 \ \text{ or } \ a_2 \geqslant 0, \\ \dfrac{4(1 - a_1 - a_2)}{\varepsilon^2} & \text{if } a_1 \leqslant 0 \ \text{ and } \ a_2 \leqslant 0. \end{cases} \tag{1.1}$$

*Moreover, if* (1.1) *is an equality, then* $\operatorname{mult}_O(\mathcal{M}) = \frac{2}{\varepsilon}(a_1 - 1)$, *the log pair* $(S, \varepsilon\mathcal{M} + a_1\Delta_1 + a_2\Delta_2)$ *is log canonical and* $a_1 = a_2 \geqslant 0$.

Our second goal is to prove the following analogue of Theorem 1.5 and give an application.

**Theorem 1.6.** *Let* $\mathcal{M}$ *be a linear system without fixed components on* $S$, *and let* $\varepsilon$ *be a positive rational number. Suppose that the log pair* $(S, \varepsilon\mathcal{M} + a_1\Delta_1)$ *is not terminal at* $O$ *and* $a_1 \leqslant 0$. *Let* $M_1$ *and* $M_2$ *be sufficiently general curves in* $\mathcal{M}$. *Then we have*

$$\operatorname{mult}_O(M_1 \cdot M_2) \geqslant \begin{cases} \dfrac{1 - 2a_1}{\varepsilon^2} & \text{if } a_1 \geqslant -\dfrac{1}{2}, \\ \dfrac{-4a_1}{\varepsilon^2} & \text{if } a_1 \leqslant -\dfrac{1}{2}. \end{cases} \tag{1.2}$$

*Moreover, if* (1.2) *is an equality, then* $(S, \varepsilon\mathcal{M} + a_1\Delta_1)$ *is canonical and either* $-a_1 \in \mathbb{N}$ *and* $\operatorname{mult}_O(\mathcal{M}) = \frac{2}{\varepsilon}$, *or* $a_1 = 0$ *and* $\operatorname{mult}_O(\mathcal{M}) = \frac{1}{\varepsilon}$.

## § 2. Preliminaries

Let $\varphi$, $\psi$ and $v_1, \ldots, v_\gamma$ be non-zero polynomials in $\mathbb{C}[z_1, \ldots, z_n]$ such that the locus in $\mathbb{C}^n$ given by the equations $v_1 = \cdots = v_\gamma = 0$ has dimension at most $n - 2$. We denote the origin in $\mathbb{C}^n$ by $O$. Let $a$, $b$ and $c$ be non-negative rational numbers.

We define a function

$$\Omega = \frac{|\psi|^b}{|\varphi|^a(|v_1| + \cdots + |v_\gamma|)^c}.$$

**Question 2.1.** When is $\Omega^2$ locally integrable in a neighbourhood of $O$?

The answer to Question 2.1 is given in Example 2.8. By putting

$$c_O(\Omega) = \sup\{\varepsilon \in \mathbb{Q} \mid \Omega^{2\varepsilon} \text{ is locally integrable near } O \in \mathbb{C}^n\},$$

we see that the function $\Omega^2$ is locally integrable in a neighbourhood of $O$ if and only if $c_0(\Omega) > 1$. Note that $c_0(\Omega) \in \mathbb{Q}_{\geqslant 0} \cup \{+\infty\}$.

**Example 2.2.** Let $m_1, \ldots, m_n$ be positive integers. Then

$$\min\left(1, \sum_{i=1}^n \frac{1}{m_i}\right) = c_0\left(\frac{1}{|\sum_{i=1}^n z_i^{m_i}|}\right) \geqslant c_0\left(\frac{1}{|\prod_{i=1}^n z_i^{m_i}|}\right)$$

$$= \min\left(\frac{1}{m_1}, \frac{1}{m_2}, \ldots, \frac{1}{m_n}\right).$$

Let $X$ be a variety with at most rational singularities. We consider a formal linear combination $B_X = \sum_{i=1}^r a_i B_i + \sum_{i=1}^l c_i \mathcal{M}_i$, where $a_i$ and $c_i$ are rational numbers (not necessary positive), $B_i$ is a prime Weil divisor on $X$, and $\mathcal{M}_i$ is a linear system without fixed components on $X$. We assume that $B_i \neq B_j$ and $\mathcal{M}_i \neq \mathcal{M}_j$ for $i \neq j$.

*Remark* 2.3. Let $k$ be any sufficiently large positive integer. For every $i \in \{1, \ldots, l\}$ and every $j \in \{1, \ldots, k\}$ let $M_i^j$ be a general element in $\mathcal{M}_i$. Then, replacing every $\mathcal{M}_i$ by the divisor

$$\frac{M_i^1 + M_i^2 + \cdots + M_i^k}{k},$$

we can always regard $B_X$ as a $\mathbb{Q}$-divisor on $X$.

Suppose that $K_X + B_X$ is a $\mathbb{Q}$-Cartier divisor.

**Definition 2.4.** We say that $B_X$ is the *boundary* of the log pair $(X, B_X)$. The boundary $B_X$ is said to be *mobile* (resp. *effective*) if $a_i = 0$ for every $i \in \{1, \ldots, r\}$ (resp. $a_i \geqslant 0$ for every $i \in \{1, \ldots, r\}$ and $c_j \geqslant 0$ for every $j \in \{1, \ldots, l\}$).

Let $\pi \colon \bar{X} \to X$ be a birational morphism such that $\bar{X}$ is smooth. We put

$$B_{\bar{X}} = \sum_{i=1}^r a_i \bar{B}_i + \sum_{i=1}^l c_i \overline{\mathcal{M}}_i,$$

where $\bar{B}_i$ is the proper transform of a divisor $B_i$ on $\bar{X}$, and $\overline{\mathcal{M}}_i$ is the proper transform of a linear system $\mathcal{M}_i$ on $\bar{X}$. Then

$$K_{\bar{X}} + B_{\bar{X}} \equiv \pi^*(K_X + B_X) + \sum_{i=1}^m d_i E_i,$$

where $d_i \in \mathbb{Q}$, and $E_i$ is an exceptional divisor of $\pi$. We additionally assume that $\sum_{i=1}^{r} \overline{B}_i + \sum_{i=1}^{m} E_i$ is a divisor with simple normal crossings and $\overline{\mathcal{M}}_i$ is base-point-free for every $i \in \{1, \ldots, l\}$. Put $B^{\overline{X}} = B_{\overline{X}} - \sum_{i=1}^{m} d_i E_i$ and take a rational number $\varepsilon$ such that $1 \geqslant \varepsilon \geqslant 0$. Then $(\overline{X}, B^{\overline{X}})$ is called the *log pullback* of $(X, B_X)$.

**Definition 2.5.** The log pair $(X, B_X)$ is said to be *$\varepsilon$-log canonical* (resp. *$\varepsilon$-log terminal*) if $a_i \leqslant 1 - \varepsilon$ (resp. $a_i < 1 - \varepsilon$) for all $i \in \{1, \ldots, r\}$ and $d_j \geqslant -1 + \varepsilon$ (resp. $d_j > -1 + \varepsilon$) for all $j \in \{1, \ldots, m\}$.

We say that $(X, B_X)$ *has log canonical singularities* (resp. *log terminal singularities*) if the log pair $(X, B_X)$ is 0-log canonical (resp. 0-log terminal).

*Remark* 2.6. Let $P$ be a point of $X$ and $\Delta$ an effective divisor on $X$ such that $\Delta = \sum_{i=1}^{r} \varepsilon_i B_i \equiv B_X$, where the $\varepsilon_i$ are non-negative rational numbers. Suppose that the boundary $B_X$ is effective, the divisor $\Delta$ is a $\mathbb{Q}$-Cartier divisor, the log pair $(X, \Delta)$ is log canonical at $P \in X$, and the log pair $(X, B_X)$ is not log canonical at $P \in X$. We put

$$\alpha = \min\left\{ \frac{a_i}{\varepsilon_i} \,\middle|\, \varepsilon_i \neq 0 \right\}$$

and note that $\alpha$ is well defined because $\varepsilon_i \neq 0$ for some $i$. Then $\alpha < 1$ and the log pair

$$\left( X, \sum_{i=1}^{r} \frac{a_i - \alpha \varepsilon_i}{1 - \alpha} B_i + \sum_{i=1}^{l} c_i \mathcal{M}_i \right)$$

is not log canonical at $P \in X$. We also have

$$\sum_{i=1}^{r} \frac{a_i - \alpha \varepsilon_i}{1 - \alpha} B_i + \sum_{i=1}^{l} c_i \mathcal{M}_i \equiv B_X \equiv \Delta$$

and at least one irreducible component of $\mathrm{Supp}(\Delta)$ is not contained in

$$\mathrm{Supp}\left( \sum_{i=1}^{r} \frac{a_i - \alpha \varepsilon_i}{1 - \alpha} B_i \right).$$

Let $D_X$ be any boundary on $X$ such that $K_X + B_X + D_X$ is a $\mathbb{Q}$-Cartier divisor. Let $Z$ be a closed subvariety of $X$.

**Definition 2.7.** The *$\varepsilon$-log canonical threshold of $D_X$ along $Z$* is

$$\mathrm{c}_Z^\varepsilon(X, B_X, D_X) = \sup\left\{ \lambda \in \mathbb{Q} \,\middle|\, \begin{array}{l} \text{the pair } (X, B_X + \lambda D_X) \text{ is } \varepsilon\text{-log canonical} \\ \text{at each point of } Z \end{array} \right\}$$

$$\in \mathbb{Q} \cup \{\pm\infty\}.$$

The number $\mathrm{c}_Z^\varepsilon(X, B_X, D_X)$ with $\varepsilon = 0$ plays a very important role in geometry. Therefore we put $\mathrm{c}_Z(X, B_X, D_X) = \mathrm{c}_Z^0(X, B_X, D_X)$ for simplicity of notation. We similarly put $\mathrm{c}^\varepsilon(X, B_X, D_X) = \mathrm{c}_X^\varepsilon(X, B_X, D_X)$ and $\mathrm{c}(X, B_X, D_X) = \mathrm{c}_X^0(X, B_X, D_X)$. Note that $\mathrm{c}_Z^\varepsilon(X, B_X, D_X) = -\infty$ if, for example, the log pair

$(X, B_X)$ is not $\varepsilon$-log canonical along $Z$ and $D_X = 0$. If $B_X = 0$, then we put $c_Z^\varepsilon(X, D_X) = c_Z^\varepsilon(X, B_X, D_X)$ for simplicity of notation. For the same reason, we put $c^\varepsilon(X, D_X) = c_X^\varepsilon(X, D_X)$, $c_Z(X, D_X) = c_Z(X, B_X, D_X)$ and $c(X, D_X) = c_X(X, D_X)$.

**Example 2.8.** It follows from [5] that $c_O(\Omega) = c_O(\mathbb{C}^n, a(\psi = 0) - b(\varphi = 0) + c\mathcal{B})$, and the following conditions are equivalent.

1) The function $\Omega^2$ is locally integrable in a neighbourhood of $O \in \mathbb{C}^n$.

2) The singularities of the log pair $\left(\mathbb{C}^n, a(\varphi=0) - b(\varphi=0) + \frac{c}{k}\sum_{i=1}^k (\sum_{j=1}^\gamma \lambda_j^i v_j = 0)\right)$ are log terminal in a neighbourhood of $O \in \mathbb{C}^n$, where $[\lambda_1^i : \ldots : \lambda_\gamma^i]$ is a general point of $\mathbb{P}^{\gamma-1}$ and $k$ is any sufficiently large positive integer.

3) The singularities of the log pair $(\mathbb{C}^n, a(\varphi = 0) - b(\varphi = 0) + c\mathcal{B})$ are at most log terminal in a neighbourhood of $O \in \mathbb{C}^n$, where $\mathcal{B}$ is the linear system generated by the divisors $v_1 = 0, \ldots, v_r = 0$.

We say that the log pair $(X, B_X)$ has *canonical singularities* (resp. *terminal singularities*) if the log pair $(X, B_X)$ is 1-log canonical (resp. 1-log terminal).

*Remark* 2.9. Suppose that $B_X$ is effective and $(X, B_X)$ is canonical. Then $a_1 = \cdots = a_r = 0$.

One can show that Definition 2.5 is independent of the choice of $\pi$. We put

$$\mathrm{LCS}_\varepsilon(X, B_X) = \left(\bigcup_{a_i \geqslant 1-\varepsilon} B_i\right) \cup \left(\bigcup_{d_i \leqslant -1+\varepsilon} \pi(E_i)\right) \subsetneq X,$$

$\mathrm{LCS}(X, B_X) = \mathrm{LCS}_0(X, B_X)$ and $\mathrm{CS}(X, B_X) = \mathrm{LCS}_1(X, B_X)$. The subsets $\mathrm{LCS}_\varepsilon(X, B_X)$, $\mathrm{LCS}(X, B_X)$, $\mathrm{CS}(X, B_X)$ are called the *loci* of $\varepsilon$-log canonical, log canonical, and canonical singularities of $(X, B_X)$ respectively.

**Definition 2.10.** A proper irreducible subvariety $Y \subsetneq X$ is called a *centre* of $\varepsilon$-log canonical singularities of the log pair $(X, B_X)$ if one of the following conditions holds for some choice of the birational morphism $\pi \colon \overline{X} \to X$.

1) The inequality $a_i \geqslant 1 - \varepsilon$ holds and $Y = B_i$ for some $i \in \{1, \ldots, r\}$.
2) The inequality $d_i \leqslant -1 + \varepsilon$ holds and $Y = \pi(E_i)$ for some $i \in \{1, \ldots, m\}$.

Let $\mathbb{LCS}_\varepsilon(X, B_X)$ be the set of all centres of $\varepsilon$-log canonical singularities of $(X, B_X)$. Then $Y \in \mathbb{LCS}_\varepsilon(X, B_X) \Longrightarrow Y \subseteq \mathrm{LCS}_\varepsilon(X, B_X)$ and $\mathbb{LCS}_\varepsilon(X, B_X) = \varnothing \Longleftrightarrow \mathrm{LCS}_\varepsilon(X, B_X) = \varnothing \Longleftrightarrow$ the log pair $(X, B_X)$ is $\varepsilon$-log terminal.

*Remark* 2.11. Let $\mathcal{H}$ be a base-point-free linear system on $X$, $H$ a sufficiently general divisor in $\mathcal{H}$, and $Y \subsetneq X$ an irreducible subvariety. We write $Y \cap H = \sum_{i=1}^k Z_i$, where the $Z_i$ are irreducible subvarieties of $H$. It follows from Definition 2.10 (see Theorem 2.17) that $Y \in \mathbb{LCS}_\varepsilon(X, B_X)$ if and only if $\{Z_1, \ldots, Z_k\} \subseteq \mathbb{LCS}_\varepsilon(H, B_X|_H)$.

We put $\mathbb{LCS}(X, B_X) = \mathbb{LCS}_0(X, B_X)$ and $\mathbb{CS}(X, B_X) = \mathbb{LCS}_1(X, B_X)$. The elements of $\mathbb{LCS}(X, B_X)$ are called centres of log canonical singularities of the log pair $(X, B_X)$. The elements of $\mathbb{CS}(X, B_X)$ are called centres of canonical singularities of $(X, B_X)$.

**Example 2.12.** Let $\chi\colon \widehat{X} \to X$ be the blow-up of a smooth point $P \in X$. We put $B_{\widehat{X}} = \sum_{i=1}^{r} a_i \widehat{B}_i + \sum_{i=1}^{l} c_i \widehat{\mathcal{M}}_i$, where $\widehat{B}_i$ is the proper transform of a divisor $B_i$ on $\widehat{X}$ and $\widehat{\mathcal{M}}_i$ is the proper transform of a linear system $\mathcal{M}_i$ on $\widehat{X}$. Then

$$K_{\widehat{X}} + B_{\widehat{X}} \equiv \pi^*(K_X + B_X) + (\dim(X) - 1 - \mathrm{mult}_P(B_X))E,$$

where $E$ is the exceptional divisor of $\chi$ and $\mathrm{mult}_P(B_X) \in \mathbb{Q}$. Hence the log pair $(X, B_X)$ is $\varepsilon$-log canonical in a neighbourhood of the point $P \in X$ if and only if the log pair

$$\left(\widehat{X}, B_{\widehat{X}} + (\mathrm{mult}_P(B_X) - \dim(X) + 1)E\right)$$

is $\varepsilon$-log canonical in a neighbourhood of $E$. In particular, if $\mathrm{mult}_P(B_X) \geqslant \dim(X) - \varepsilon$, then $P \in \mathbb{LCS}_\varepsilon(X, B_X)$. If the boundary $B_X$ is effective and $\mathrm{mult}_P(B_X) < 1 - \varepsilon$, then $P \notin \mathrm{LCS}_\varepsilon(X, B_X)$. If the boundary $B_X$ is effective and mobile and $\mathrm{mult}_P(B_X) < 1$, then $P \notin \mathrm{CS}(X, B_X)$. In the case when $\dim(X) = 2$ and $B_X$ is effective, we have $P \in \mathbb{CS}(X, B_X)$ if and only if $\mathrm{mult}_P(B_X) \geqslant 1$.

If the boundary $B_X$ is effective, then the locus $\mathrm{LCS}(X, B_X) \subset X$ can be equipped with the structure of a subscheme (see [6]) in a natural way. Indeed, if $B_X$ is effective, we can put

$$\mathcal{I}(X, B_X) = \pi_* \left( \sum_{i=1}^{m} \lceil d_i \rceil E_i - \sum_{i=1}^{r} \lfloor a_i \rfloor \overline{B}_i \right).$$

We write $\mathcal{L}(X, B_X)$ for the subscheme that corresponds to the ideal sheaf $\mathcal{I}(X, B_X)$.

**Definition 2.13.** If the boundary $B_X$ is effective, then we say that $\mathcal{L}(X, B_X)$ is the *subscheme of log canonical singularities* of the log pair $(X, B_X)$, and $\mathcal{I}(X, B_X)$ is the *multiplier ideal sheaf* of the log pair $(X, B_X)$.

If the boundary $B_X$ is effective, then it follows from the construction of $\mathcal{L}(X, B_X)$ that $\mathrm{Supp}(\mathcal{L}(X, B_X)) = \mathrm{LCS}(X, B_X) \subset X$.

**Theorem 2.14** ([7], Theorem 9.4.8). *Let $H$ be a nef and big $\mathbb{Q}$-divisor on $X$ such that $K_X + B_X + H \equiv D$ for some Cartier divisor $D$ on $X$. Suppose that $B_X$ is effective. Then $H^i(\mathcal{I}(X, B_X) \otimes D) = 0$ for every $i \geqslant 1$.*

**Corollary 2.15.** *Suppose that $B_X$ is effective and $-(K_X + B_X)$ is nef and big. Then the locus $\mathrm{LCS}(X, B_X)$ is connected.*

Corollary 2.15 is a special case of the following result.

**Theorem 2.16** ([6], Lemma 5.7). *Let $\psi\colon X \to Z$ be a surjective morphism with connected fibres. Suppose that the boundary $B_X$ is effective and the divisor $-(K_X + B_X)$ is $\psi$-nef and $\psi$-big. Then the locus $\mathrm{LCS}(\overline{X}, B^{\overline{X}})$ is connected in a neighbourhood of every fibre of the morphism $\psi \circ \pi\colon \overline{X} \to Z$.*

Applying Theorem 2.16, one can easily prove the following result.

**Theorem 2.17** ([5], Theorem 7.5). *Suppose that the divisor $K_X$ is $\mathbb{Q}$-Cartier, the boundary $B_X$ is effective, $a_1 = 1$, and $B_1$ is a Cartier divisor with at most log terminal singularities. Then the log pair $(X, B_X)$ is canonical in a neighbourhood of $B_1$ if and only if the singularities of the log pair $(B_1, \sum_{i=2}^{r} a_i B_i|_{B_1})$ are log canonical.*

Note that Theorem 1.1 is a simple corollary of Theorem 2.17.

**Definition 2.18.** We say that the log pair $(X, B_X)$ has *purely log terminal singularities* if $a_i \leqslant 1$ for every $i \in \{1, \ldots, r\}$ and $d_j > -1$ for every $j \in \{1, \ldots, m\}$.

**Theorem 2.19** ([5], Theorem 7.5). *Suppose that $B_X$ is effective, $a_1 = 1$, and $B_1$ is a Cartier divisor. Then $(X, B_X)$ has purely log terminal singularities in a neighbourhood of $B_1$ if and only if $B_1$ has rational singularities and the log pair $(B_1, \sum_{i=2}^{r} a_i B_i|_{B_1})$ is log terminal.*

Suppose in addition that the boundary $B_X$ is effective and movable. Thus we have $B_X = \sum_{i=1}^{l} c_i \mathcal{M}_i$, where the $c_i$ are non-negative rational numbers and the $\mathcal{M}_i$ are linear systems without fixed components on $X$.

**Definition 2.20.** We say that a log pair $(Y, B_Y)$ is *birationally equivalent* to $(X, B_X)$ if the boundary $B_Y$ is effective and mobile and there is a birational map $\xi \colon X \dashrightarrow Y$ such that $B_Y = \sum_{i=1}^{l} c_i \xi(\mathcal{M}_i)$, where $\xi(\mathcal{M}_i)$ is the proper transform of the linear system $\mathcal{M}_i$ on $Y$.

Thus the log pairs $(\overline{X}, B_{\overline{X}})$ and $(X, B_X)$ are birationally equivalent.

**Definition 2.21.** Let $D$ be a $\mathbb{Q}$-Weil divisor on $X$. We say that $D$ is $\mathbb{Q}$-*effective* if there is a positive integer $n$ such that the linear system $|nmD|$ is non-empty, where $m$ is a positive integer such that $mD$ is an integral Weil divisor.

**Definition 2.22.** The *Kodaira dimension* of the log pair $(X, B_X)$ is the number

$$
\kappa(X, B_X) = \begin{cases} \sup_{n \in \mathbb{N}} (\dim(\varphi_{|nm(K_{\overline{X}} + B_{\overline{X}})|}(X))) & \text{if} \quad K_{\overline{X}} + B_{\overline{X}} \text{ is } \mathbb{Q}\text{-effective,} \\ -\infty & \text{if} \quad K_{\overline{X}} + B_{\overline{X}} \text{ is not } \mathbb{Q}\text{-effective,} \end{cases}
$$

where $m \in \mathbb{N}$ is such that $m(K_{\overline{X}} + B_{\overline{X}})$ is a Cartier divisor.

One can show that the Kodaira dimension $\kappa(X, B_X)$ is independent of the choice of $\pi$ (see [8], Lemma 1.3.6).

**Lemma 2.23** ([8], Lemma 1.3.6). *Let $(Y, B_Y)$ be a log pair that is birationally equivalent to $(X, B_X)$. Then $\kappa(X, B_X) = \kappa(Y, B_Y)$.*

If the singularities of the log pair $(X, B_X)$ are canonical, then it follows from [8], Lemma 1.3.6, that

$$
\kappa(X, B_X) = \begin{cases} \sup_{n \in \mathbb{N}} (\dim(\varphi_{|nm(K_X + B_X)|}(X))) & \text{if} \quad K_X + B_X \text{ is } \mathbb{Q}\text{-effective,} \\ -\infty & \text{if} \quad K_X + B_X \text{ is not } \mathbb{Q}\text{-effective,} \end{cases}
$$

where $m \in \mathbb{N}$ is such that $m(K_X + B_X)$ is a Cartier divisor.

**Corollary 2.24.** *If $(X, B_X)$ has at most canonical singularities and $K_X + B_X$ is $\mathbb{Q}$-effective, then $\kappa(X, B_X) \geqslant 0$.*

It follows from Definition 2.22 that

$$\kappa\left(X, \sum_{i=1}^{l} c_i' \mathcal{M}_i\right) \geqslant \kappa(X, B_X) = \kappa\left(X, \sum_{i=1}^{l} c_i \mathcal{M}_i\right)$$

in the case when $c_i' \geqslant c_i$ for every $i \in \{1, \ldots, l\}$.

**Definition 2.25.** The log pair $(X, B_X)$ is called a *canonical model* if the divisor $K_X + B_X$ is ample and the log pair $(X, B_X)$ has canonical singularities.

It follows from Definition 2.22 that $\kappa(X, B_X) = \dim(X)$ if $(X, B_X)$ is a canonical model.

**Definition 2.26.** A log pair $(Y, B_Y)$ is a *canonical model of the log pair* $(X, B_X)$ if the log pair $(Y, B_Y)$ is a canonical model and the log pairs $(Y, B_Y)$ and $(X, B_X)$ are birationally equivalent.

Note that the log pair $(X, B_X)$ has no canonical model if $\kappa(X, B_X) < \dim(X)$.

**Theorem 2.27** ([8], Theorem 1.3.20). *A canonical model is unique whenever it exists.*

It follows from [9] that $(X, B_X)$ has a canonical model if and only if $\kappa(X, B_X) = \dim(X)$.

## § 3. The first inequality

The purpose of this section is to prove Theorem 1.4.

Let $X$ be a surface, $O$ a smooth point of $X$, and $\Delta_1$ and $\Delta_2$ curves on $X$ such that $O \in \Delta_1 \cap \Delta_2$, both $\Delta_1$ and $\Delta_2$ are irreducible and reduced, both $\Delta_1$ and $\Delta_2$ are smooth at $O$, and $\Delta_1$ intersects $\Delta_2$ transversally at $O$. Let $D$ be an effective $\mathbb{Q}$-divisor on $X$ whose support contains neither $\Delta_1$ nor $\Delta_2$, and let $a_1$, $a_2$ be non-negative rational numbers. Suppose that the log pair $(X, D + a_1\Delta_1 + a_2\Delta_2)$ is not log canonical at $O$. Let $A$, $B$, $M$, $N$, $\alpha$, $\beta$ be non-negative rational numbers with the following properties.

(i) $\alpha a_1 + \beta a_2 \leqslant 1$.
(ii) $A(B-1) \geqslant 1 \geqslant \max(M, N)$.
(iii) $\alpha(A+M-1) \geqslant A^2(B+D-1)\beta$ and $\alpha(1-M) + A\beta \geqslant A$.
(iv) Either $2M + AN \leqslant 2$, or

$$\alpha(B+1-MB-N) + \beta(A+1-AN-M) \geqslant AB - 1.$$

**Lemma 3.1.** *We have* $A + M \geqslant 1$, $B > 1$, *and*

$$\alpha(B+1-MB-N) + \beta(A+1-AN-M) \geqslant AB - 1,$$
$$\beta(1-N) + B\alpha \geqslant B, \quad \frac{\alpha(2-M)}{A+1} + \frac{\beta(2-N)}{B+1} \geqslant 1,$$
$$\alpha(2-M)B + \beta(1-N)(A+1) \geqslant B(A+1).$$

*Proof.* The inequality $B > 1$ follows from the inequality $A(B-1) \geqslant 1$. Then

$$\frac{\alpha}{A+1} + \frac{\beta}{B+1} \geqslant \frac{\alpha}{A+1} + \frac{\beta}{2B} \geqslant \frac{1}{2}$$

because $2B \geqslant B + 1$. We similarly see that $A + M \geqslant 1$ because

$$\frac{\alpha(A+M-1)}{A^2(B+D-1)} \geqslant \beta \geqslant 0$$

and $B + D - 1 \geqslant 0$. The inequality $\beta(1-N) + B\alpha \geqslant B$ follows from the inequalities

$$\alpha + \frac{\beta(1-N)}{B} \geqslant \frac{2-M}{A+1}\alpha + \frac{\beta(1-N)}{B} \geqslant 1$$

because $A + 1 \geqslant 2 - M$.

We now show that $\alpha(2-M)B + \beta(1-N)(A+1) \geqslant B(A+1)$ using the inequality $A(B-1) \geqslant 1$. Let $L_1$ be the line in $\mathbb{R}^2$ given by the equation

$$x(2-M)B + y(1-N)(A+1) - B(A+1) = 0,$$

and let $L_2$ be the line given by the equation $x(1-M) + Ay - A = 0$, where $(x, y)$ are coordinates in $\mathbb{R}^2$. Then $L_1$ intersects the line $y = 0$ at the point $(\frac{A+1}{2-M}, 0)$, and $L_2$ intersects the line $y = 0$ at the point $(\frac{A}{1-M}, 0)$. On the other hand, we have

$$\frac{A+1}{2-M} < \frac{A}{1-M},$$

whence we see that $\alpha(2-M)B + \beta(1-N)(A+1) \geqslant B(A+1)$ if

$$A^2\beta_0(B+N-1) \geqslant \alpha_0(A+M-1),$$

where $(\alpha_0, \beta_0)$ is the point of intersection of $L_1$ and $L_2$. Moreover,

$$(\alpha_0, \beta_0) = \left( \frac{A(A+1)(B+N-1)}{\Delta}, \frac{B(A-1+M)}{\Delta} \right),$$

where $\Delta = 2AB - ABM - A + AM - 1 + M + NA - NAM + N - NM$. We have

$$A^2(B(A-1+M))(B+N-1) \geqslant (A(A+1)(B+N-1))(A+M-1)$$

because $A(B-1) \geqslant 1$. It follows that $A^2\beta_0(B+N-1) \geqslant \alpha_0(A+M-1)$.

To complete the proof of the lemma, we must show that $\alpha(B+1-MB-N) + \beta(A+1-AN-M) \geqslant AB - 1$.

Let $L_1'$ be the line in $\mathbb{R}^2$ given by the equation

$$x(B+1-MB-N) + y(A+1-AN-M) - AB + 1 = 0,$$

and let $L_2$ be the line given by the equation $x(1-M) + Ay - A = 0$, where $(x, y)$ are the coordinates in $\mathbb{R}^2$. Then $L_1'$ intersects the line $y = 0$ at the point

$(\frac{AB-1}{B+1-MB-N},0)$, and $L_2$ intersects the line $y=0$ at the point $(\frac{A}{1-M},0)$. On the other hand, we have

$$\frac{AB-1}{B+1-MB-N} < \frac{A}{1-M},$$

whence $\alpha(B+1-MB-N)+\beta(A+1-AN-M) \geqslant AB-1$ provided that $A^2\beta_1(B+N-1) \geqslant \alpha_1(A+M-1)$, where $(\alpha_1,\beta_1)$ is the point of intersection of $L_1'$ and $L_2$. Note that

$$(\alpha_1,\beta_1) = \left(\frac{A(AB-A-2+NA+M)}{\Delta'}, \frac{A+1-NA-M}{\Delta'}\right),$$

where $\Delta' = AB-1-ABM+AM+2M-NAM-M^2$. Thus, to complete the proof, we must show that

$$A^2(A+1-NA-M)(B+N-1) \geqslant \big(A(AB-A-2+NA+M)\big)(A+M-1).$$

This inequality is equivalent to the inequality

$$(2-M)(A+M-1) \geqslant A(AN+2M-2)(B+N-1),$$

which indeed holds because $M \leqslant 1$ and $AN+2M-2 \leqslant 0$. $\square$

Suppose that $\mathrm{mult}_O(D \cdot \Delta_1) \leqslant M+Aa_1-a_2$ and $\mathrm{mult}_O(D \cdot \Delta_2) \leqslant N+Ba_2-a_1$. We shall see that this assumption leads to a contradiction. This will prove Theorem 1.4.

**Lemma 3.2.** *We have* $a_1 > \frac{1-M}{A}$ *and* $a_2 > \frac{1-N}{B}$.

*Proof.* If $a_1 > 1$, then $a_1 > \frac{1-M}{A}$ by Lemma 3.1. Suppose that $a_1 \leqslant 1$. Then the log pair $(X, D+\Delta_1+a_2\Delta_2)$ is not log canonical at $O$. Thus it follows from Theorem 1.1 that $M+Aa_1-a_2 \geqslant \mathrm{mult}_O(D \cdot \Delta_1) > 1-a_2$, whence $a_1 > \frac{1-M}{A}$. We similarly have $a_2 > \frac{1-N}{B}$. $\square$

**Lemma 3.3.** *We have* $a_1 < 1$ *and* $a_2 < 1$.

*Proof.* By Lemma 3.2, $a_1 > \frac{1-M}{A}$ and $a_2 > \frac{1-N}{B}$. On the other hand, $\alpha a_1+\beta a_2 \leqslant 1$, whence $a_1\alpha < 1-\beta\frac{1-N}{B}$ and $a_2\beta < 1-\alpha\frac{1-M}{A}$. Therefore $a_1 < 1$ and $a_2 < 1$ because $\beta(1-N)+B\alpha \geqslant B$ by Lemma 3.1 and $\alpha(1-M)+A\beta \geqslant A$ by assumption. $\square$

Put $m_0 = \mathrm{mult}_O(D)$. Then $m_0$ is a positive rational number.

**Lemma 3.4.** *We have* $m_0 \leqslant M+Aa_1-a_2$ *and* $m_0 \leqslant N+Ba_2-a_1$.

*Proof.* This follows from the inequalities $m_0 \leqslant \mathrm{mult}_O(D \cdot \Delta_1)$ and $m_0 \leqslant \mathrm{mult}_O(D \cdot \Delta_2)$. $\square$

**Lemma 3.5.** *We have* $m_0+a_1+a_2 \leqslant 2$.

*Proof.* It is known that $m_0+a_1+a_2 \leqslant M+(A+1)a_1$ and $m_0+a_1+a_2 \leqslant N+(B+1)a_2$. Then

$$(m_0+a_1+a_2)\left(\frac{\alpha}{A+1}+\frac{\beta}{B+1}\right) \leqslant \alpha a_1+\beta a_2+\frac{\alpha M}{A+1}+\frac{\beta N}{B+1} \leqslant 1+\frac{\alpha M}{A+1}+\frac{\beta N}{B+1}.$$

It follows that $m_0+a_1+a_2 \leqslant 2$ because $\frac{\alpha(2-M)}{A+1}+\frac{\beta(2-N)}{B+1} \geqslant 1$ by Lemma 3.1. $\square$

Let $\pi_1\colon X_1 \to X$ be the blow-up of $O$, and let $F_1$ be the exceptional curve for $\pi_1$. Then

$$K_{X_1} + D^1 + a_1\Delta_1^1 + a_2\Delta_2^1 + (m_0 + a_1 + a_2 - 1)F_1 \equiv \pi_1^*(K_X + D + a_1\Delta_1 + a_2\Delta_2),$$

where $D^1$, $\Delta_1^1$, $\Delta_2^1$ are the proper transforms on $X_1$ of $D$, $\Delta_1$, $\Delta_2$ respectively. In this case the log pair $(X_1, D^1 + a_1\Delta_1^1 + a_2\Delta_2^1 + (m_0 + a_1 + a_2 - 1)F_1)$ is not log canonical at some point $O_1 \in F_1$. Note that $m_0 + a_1 + a_2 - 1 \geqslant 0$.

**Lemma 3.6.** *Either* $O_1 = F_1 \cap \Delta_1^1$ *or* $O_1 = F_1 \cap \Delta_2^1$.

*Proof.* Suppose that $O_1 \notin \Delta_1^1 \cup \Delta_2^1$. Then the log pair $(X_1, D^1 + (m_0 + a_1 + a_2 - 1)F_1)$ is not log canonical at $O_1$. On the other hand, $m_0 = D^1 \cdot F_1 > 1$ by Theorem 1.1 because $m_0 + a_1 + a_2 \leqslant 2$ by Lemma 3.5. Then

$$m_0\left(\frac{\beta + B\alpha}{AB - 1} + \frac{\alpha + A\beta}{AB - 1}\right) \leqslant (M + Aa_1 - a_2)\frac{\beta + B\alpha}{AB - 1} + (N + Ba_2 - a_1)\frac{\alpha + A\beta}{AB - 1}$$

because $m_0 \leqslant M + Aa_1 - a_2$ and $m_0 \leqslant N + Ba_2 - a_1$. Moreover, we have

$$(M + Aa_1 - a_2)\frac{\beta + B\alpha}{AB - 1} + (N + Ba_2 - a_1)\frac{\alpha + A\beta}{AB - 1} \leqslant 1 + \frac{M\beta + MB\alpha + N\alpha + AN\beta}{AB - 1}$$

because $\alpha a_1 + \beta a_2 \leqslant 1$ and $AB - 1 > 0$. But we have already proved that $m_0 > 1$. Thus we see that $\beta + B\alpha + \alpha + A\beta \leqslant AB - 1 + M\beta + MB\alpha + N\alpha + AN\beta$. But this contradicts Lemma 3.1. $\square$

**Lemma 3.7.** *We have* $O_1 \neq F_1 \cap \Delta_1^1$.

*Proof.* Suppose that $O_1 = F_1 \cap \Delta_1^1$. Then the log pair $(X_1, D^1 + a_1\Delta_1^1 + (m_0 + a_1 + a_2 - 1)F_1)$ is not log canonical at $O_1$. Therefore,

$$M + Aa_1 - a_2 - m_0 = D^1 \cdot \Delta_1^1 > 1 - (m_0 + a_1 + a_2 - 1)$$

by Theorem 1.1 because $a_1 < 1$ by Lemma 3.3. We have $a_1 > \frac{2-M}{A+1}$. Then

$$\frac{2 - M\alpha}{A + 1} + \frac{\beta(1 - N)}{B} < \alpha a_1 + \beta a_2 \leqslant 1$$

because $a_2 > \frac{1-N}{B}$ by Lemma 3.3. Hence we see that $\frac{2-M\alpha}{A+1} + \frac{\beta(1-N)}{B} < 1$. But this contradicts Lemma 3.1. $\square$

Thus we see that $O_1 = F_1 \cap \Delta_2^1$. Then $(X_1, D^1 + a_1\Delta_1^1 + a_2\Delta_2^1 + (m_0 + a_1 + a_2 - 1)F_1)$ is not log canonical at $O_1$. It is also known that $1 \geqslant m_0 + a_1 + a_2 - 1 \geqslant 0$.

We already have the blow-up $\pi_1\colon X_1 \to X$. For every positive integer $n$ we consider a sequence of blow-ups

$$X_n \xrightarrow{\pi_n} X_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_3} X_2 \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X$$

such that $\pi_{i+1}$ is the blow-up of the point $F_i \cap \Delta_2^i$ for every $i \in \{1, \ldots, n-1\}$, where $F_i$ is the exceptional curve for $\pi_i$, and $\Delta_2^i$ is the proper transform of $\Delta_2$ on $X_i$.

For every $k \in \{1, \ldots, n\}$ and every $i \in \{1, \ldots, k\}$ we put $O_k = F_k \cap \Delta_2^k$ and denote the proper transforms of $D$, $\Delta_1$ and $F_i$ on $X_k$ by $D^k$, $\Delta_1^k$ and $F_i^k$ respectively. Then

$$K_{X_n} + D^n + a_1 \Delta_1^n + a_2 \Delta_2^n + \sum_{i=1}^{n} \left( a_1 + i a_2 - i + \sum_{j=0}^{i-1} m_j \right) F_i^n$$

$$\equiv \pi^*(K_X + D + a_1 \Delta_1 + a_2 \Delta_2),$$

where $\pi = \pi_n \circ \cdots \circ \pi_2 \circ \pi_1$ and $m_i = \mathrm{mult}_{O_i}(D^i)$ for all $i \in \{1, \ldots, n\}$. Hence the log pair

$$\left( X_n, D^n + a_1 \Delta_1^n + a_2 \Delta_2^n + \sum_{i=1}^{n} \left( a_1 + i a_2 - i + \sum_{j=0}^{i-1} m_j \right) F_i^n \right) \qquad (3.1)$$

is not log canonical at some point of the set $F_1^n \cup F_2^n \cup \cdots \cup F_n^n$.

**Lemma 3.8.** *For every $i \in \{1, \ldots, n\}$ we have $1 \geqslant a_1 + i a_2 - i + \sum_{j=0}^{i-1} m_j \geqslant 0$ and the log pair* (3.1) *is log canonical at every point of the set* $(F_1^n \cup F_2^n \cup \cdots \cup F_n^n) \setminus O_n$.

We now see that Lemma 3.8 contradicts itself. Indeed, it follows from Lemma 3.8 that there is a positive integer $n$ such that $a_1 + n a_2 - n + \sum_{j=0}^{n-1} m_j > 1$. But this contradicts Lemma 3.8. Therefore, to complete the proof of Theorem 1.4, it suffices to prove Lemma 3.8. This will be done by induction on $n \in \mathbb{N}$. Note that the case $n = 1$ is already proved.

By induction, we may assume that $n \geqslant 2$ and, for every $k \in \{1, \ldots, n-1\}$, we have $1 \geqslant a_1 + k a_2 - k + \sum_{j=0}^{k-1} m_j \geqslant 0$ and the log pair $(X_k, D^k + a_1 \Delta_1^k + a_2 \Delta_2^k + \sum_{i=1}^{k} (a_1 + i a_2 - i + \sum_{j=0}^{i-1} m_j) F_i^k)$ is log canonical at every point of the set $(F_1^k \cup F_2^k \cup \cdots \cup F_k^k) \setminus O_k$. Hence the singularities of this log pair are not log canonical at $O_k$.

**Lemma 3.9.** *We have $a_2 > \frac{n-N}{B+n-1}$.*

*Proof.* The singularities of the log pair $(X_{n-1}, D^{n-1} + a_2 \Delta_2^k + (a_1 + (n-1)a_2 - (n-1) + \sum_{j=0}^{n-2} m_j) F_{n-1}^n)$ are not log canonical at $O_{n-1}$. Then

$$N - B a_2 - a_1 - \sum_{j=0}^{n-2} m_j = D^{n-1} \cdot \Delta_2^{n-1} > 1 - \left( a_1 + (n-1)a_2 - (n-1) + \sum_{j=0}^{n-2} m_j \right)$$

by Theorem 1.1 because $a_2 < 1$ by Lemma 3.3. We have $a_2 > \frac{n-N}{B+n-1}$, as required. $\square$

**Lemma 3.10.** *We have $1 \geqslant a_1 + n a_2 - n + \sum_{j=0}^{n-1} m_j \geqslant 0$.*

*Proof.* The inequality $a_1 + n a_2 - n + \sum_{j=0}^{n-1} m_j \geqslant 0$ follows from the fact that the log pair $(X_{n-1}, D^{n-1} + a_2 \Delta_2^k + (a_1 + (n-1)a_2 - (n-1) + \sum_{j=0}^{n-2} m_j) F_{n-1}^n)$ is not log canonical at $O_{n-1}$. Suppose that $a_1 + n a_2 - n + \sum_{j=0}^{n-1} m_j > 1$. Then $m_0 + a_2 \leqslant M + A a_1$ by Lemma 3.4. Thus we see that

$$a_1 + nM + nAa_1 - n \geqslant a_1 + n a_2 - n + n m_0 \geqslant a_1 + n a_2 - n + \sum_{j=0}^{n-1} m_j > 1,$$

whence we immediately get $a_1 > \frac{n+1-Mn}{nA+1}$. On the other hand, we know that $a_2 > \frac{n-N}{B+n-1}$ by Lemma 3.9. Therefore,

$$\left(\frac{\alpha - M}{A} + \beta\right) + \alpha\frac{A - 1 + M}{A(An+1)} + \beta\frac{1 - B - N}{B + n - 1}$$

$$= \alpha\frac{n+1-Mn}{nA+1} + \beta\frac{n-N}{B+n-1} < \alpha a_1 + \beta a_2 \leqslant 1,$$

where $\frac{\alpha(1-M)}{A} + \beta \geqslant 1$. Hence we get $\alpha\frac{A+M-1}{A(An+1)} < \beta\frac{B+N-1}{B+n-1}$, where $n \geqslant 2$. On the other hand, it follows from Lemmas 3.2, 3.3 that $A + M > 1$ and $B > 1$. Thus we see that $\frac{A(An+1)}{\alpha(A+M-1)} > \frac{B+n-1}{\beta(B+N-1)}$, but $A^2(B + N - 1)\beta \leqslant \alpha(A + M - 1)$ by assumption. Then

$$\frac{A}{\alpha(A + M - 1)} - \frac{B - 1}{\beta(B + N - 1)}$$

$$\geqslant \left(\frac{A^2}{\alpha(A + M - 1)} - \frac{1}{\beta(B + M - 1)}\right)n + \frac{A}{\alpha(A + M - 1)} - \frac{B - 1}{\beta(B + N - 1)} > 0,$$

whence $\beta A(B + N - 1) > \alpha(B - 1)(A + M - 1)$. We have

$$\frac{\alpha(A + M - 1)}{A} \geqslant \beta A(B + N - 1) > \alpha(B - 1)(A + M - 1)$$

because $A^2(B + N - 1)\beta \leqslant \alpha(A + M - 1)$ by assumption. Then $\alpha \neq 0$ and $A(B - 1) < 1$, which is a contradiction since $A(B - 1) \geqslant 1$ by assumption. $\square$

**Lemma 3.11.** *The singularities of the log pair* (3.1) *are log canonical at every point of the set* $F_n \setminus ((F_n \cap F_{n-1}^n) \cup (F_n \cap \Delta_2^n))$.

*Proof.* Suppose that there is a point $Q \in F_n$ such that $F_n \cap F_{n-1}^n \neq Q \neq F_n \cap \Delta_2^n$ and the pair (3.1) is not log canonical at $Q$. Then the log pair $\left(X_n, D^n + \left(a_1 + na_2 - n + \sum_{j=0}^{n-1} m_j\right)F_n\right)$ is not canonical at $Q$. We have $m_0 \geqslant m_{n-1} = D^n \cdot F_n > 1$ by Theorem 1.1 because $a_1 + na_2 - n + \sum_{j=0}^{n-1} m_j \leqslant 1$ by Lemma 3.10. Then

$$m_0\left(\frac{\beta + B\alpha}{AB - 1} + \frac{\alpha + A\beta}{AB - 1}\right) \leqslant (M + Aa_1 - a_2)\frac{\beta + B\alpha}{AB - 1} + (N + Ba_2 - a_1)\frac{\alpha + A\beta}{AB - 1}$$

because $m_0 \leqslant M + Aa_1 - a_2$ and $m_0 \leqslant N + Ba_2 - a_1$ by Lemma 3.4. We have

$$(M + Aa_1 - a_2)\frac{\beta + B\alpha}{AB - 1} + (N + Ba_2 - a_1)\frac{\alpha + A\beta}{AB - 1} \leqslant 1 + \frac{M\beta + MB\alpha + N\alpha + AN\beta}{AB - 1}$$

because $\alpha a_1 + \beta a_2 \leqslant 1$ and $AB - 1 > 0$. On the other hand, $m_0 > 1$. Thus we see that $\beta + B\alpha + \alpha + A\beta \leqslant AB - 1 + M\beta + MB\alpha + N\alpha + AN\beta$. This contradicts our initial assumption. $\square$

**Lemma 3.12.** *The log pair* (3.1) *is log canonical at the point* $F_n \cap F_{n-1}^n$.

*Proof.* Suppose that (3.1) is not log canonical at $F_n \cap F_{n-1}^n$. Then the log pair

$$\left( X_n, D^n + \left( a_1 + (n-1)a_2 - (n-1) + \sum_{j=0}^{n-2} m_j \right) F_{n-1}^n + \left( a_1 + na_2 - n + \sum_{j=0}^{n-1} m_j \right) F_n \right)$$

is not log canonical at the point $F_n \cap F_{n-1}^n$. Therefore,

$$m_{n-2} - m_{n-1} = D^n \cdot F_{n-2} > 1 - \left( a_1 + na_2 - n + \sum_{j=0}^{n-1} m_j \right)$$

by Theorem 1.1 because $a_1 + (n-1)a_2 - (n-1) + \sum_{j=0}^{n-2} m_j$. Note that

$$M + Aa_1 - a_2 - m_0 \geqslant \mathrm{mult}_O(D \cdot \Delta_1) - m_0 \geqslant D \cdot \Delta_1 - m_0 = D^1 \cdot \Delta_1^1 \geqslant 0,$$

whence $m_0 + a_2 \leqslant Aa_1 + M$. Then

$$nM + nAa_1 - na_2 \geqslant nm_0 \geqslant (n+1)m_0 - m_{n-1}$$

$$\geqslant m_{n-2} - m_{n-1} + \sum_{j=0}^{n-1} m_j > n + 1 - a_1 - na_2.$$

It follows that $a_1 > \frac{n+1-nM}{An+1}$. Arguing as in the proof of Lemma 3.10, we arrive at a contradiction. $\square$

This proves Lemma 3.8 and thus completes the proof of Theorem 1.4.

## §4. The second inequality

The goal of this section is to prove Theorem 1.6.

Let $X$ be a surface, $O$ a smooth point of $X$, $\mathcal{M}$ a linear system without fixed components on $X$, and $\Delta_1$ an irreducible reduced curve on $X$ such that $O \in \Delta_1 \setminus \mathrm{Sign}(\Delta_1)$. Let $\varepsilon$ and $a_1$ be rational numbers such that $\varepsilon > 0$ and $a_1 \leqslant 0$. Suppose that the log pair $(X, \varepsilon\mathcal{M} + a_1\Delta_1)$ is not terminal at $O$. Then there is a birational morphism $\pi \colon \bar{X} \to X$ (a composite of blow-ups of smooth points) that contracts $m$ irreducible curves $E_1, E_2, \ldots, E_m$ to the point $O$ and induces an isomorphism $\bar{X} \setminus \bigcup_{i=1}^m E_i \cong X \setminus O$. For some rational numbers $d_1, d_2, \ldots, d_m$ we have

$$K_{\bar{X}} + \varepsilon\overline{\mathcal{M}} + a_1\bar{\Delta}_1 \equiv \pi^*(K_X + \varepsilon\mathcal{M} + a_1\Delta_1) + \sum_{i=1}^m d_i E_i,$$

where $\overline{\mathcal{M}}$ and $\bar{\Delta}_1$ are the proper transforms on $\bar{X}$ of $\mathcal{M}$ and $\Delta_1$ respectively, $d_m \leqslant 0$, and either $m = 1$, or we have $m \geqslant 2$ and $d_i > 0$ for every $i \in \{1, \ldots, m-1\}$.

**Lemma 4.1.** *Suppose that* $a_1 \geqslant 0$. *Then* $m = 1$.

*Proof.* This is well known and easy to prove. $\square$

Let $\chi\colon \widehat{X} \to X$ be the blow-up of $O$ and $E$ the exceptional curve of $\chi$. Then

$$K_{\widehat{X}} + \varepsilon\widehat{\mathcal{M}} + a_1\widehat{\Delta}_1 + (\varepsilon\operatorname{mult}_O(\mathcal{M}) + a_1 - 1)E \equiv \pi^*(K_X + \varepsilon\mathcal{M} + a_1\Delta_1),$$

where $\widehat{\mathcal{M}}$ and $\widehat{\Delta}_1$ are the proper transforms on $\widehat{X}$ of $\mathcal{M}$ and $\Delta_1$ respectively.

**Lemma 4.2.** *Suppose that $m = 1$. Then the inequality* (1.2) *holds. Moreover, if* (1.2) *is an equality, then either* $-a_1 \in \mathbb{N}$ *and* $\operatorname{mult}_O(\mathcal{M}) = \frac{2}{\varepsilon}$, *or* $a_1 = 0$ *and* $\operatorname{mult}_O(\mathcal{M}) = \frac{1}{\varepsilon}$.

*Proof.* Note that $\pi = \chi$ and $d_1 = 1 - \varepsilon\operatorname{mult}_O(\mathcal{M}) - a_1 \leqslant 0$. Hence we have

$$\operatorname{mult}_O(M_1 \cdot M_1) \geqslant \operatorname{mult}_P^2(\mathcal{M}) \geqslant \frac{(1 - a_1)^2}{\varepsilon^2} \geqslant \max\left(\frac{1 - 2a_1}{\varepsilon^2}, \frac{-4a_1}{\varepsilon^2}\right),$$

which yields (1.2). Moreover, if (1.2) is an equality, then either $a_1 = -1$ and $\operatorname{mult}_O(\mathcal{M}) = \frac{2}{\varepsilon}$, or $a_1 = 0$ and $\operatorname{mult}_O(\mathcal{M}) = \frac{1}{\varepsilon}$. $\square$

*Proof of Theorem* 1.6. The proof is by induction on $m$. We may assume that $m \geqslant 2$. Then $a_1 < 0$ and the log pair $(\widehat{X}, a_1\widehat{\Delta}_1 + \varepsilon\widehat{\mathcal{M}} + (\varepsilon\operatorname{mult}_O(\mathcal{M}) + a_1 - 1)E)$ is not terminal at some point $Q \in E$. Note that $d_1 = 1 - \varepsilon\operatorname{mult}_O(\mathcal{M}) - a_1$.

Let $\widehat{M}_1$ and $\widehat{M}_2$ be the proper transforms on $\widehat{X}$ of the curves $M_1$ and $M_2$ respectively. Then

$$\operatorname{mult}_O(M_1 \cdot M_2) \geqslant \operatorname{mult}_P^2(\mathcal{M}) + \operatorname{mult}_Q(\widehat{M}_1 \cdot \widehat{M}_2),$$

where $\operatorname{mult}_O(\mathcal{M}) \geqslant \frac{1}{\varepsilon}$. On the other hand, we have $d_1 = 1 - \varepsilon\operatorname{mult}_O(\mathcal{M}) - a_1 \leqslant 0$ and the log pair $(\widehat{X}, \varepsilon\widehat{\mathcal{M}} + (\varepsilon\operatorname{mult}_O(\mathcal{M}) + a_1 - 1)E)$ is not terminal at $Q$. By induction we have

$$\operatorname{mult}_Q(\widehat{M}_1 \cdot \widehat{M}_2) \geqslant \begin{cases} \dfrac{3 - 2\varepsilon\operatorname{mult}_O(\mathcal{M}) - 2a_1}{\varepsilon^2} & \text{if } \varepsilon\operatorname{mult}_O(\mathcal{M}) + a_1 \geqslant \dfrac{1}{2}, \\[3mm] \dfrac{4 - 4\varepsilon\operatorname{mult}_O(\mathcal{M}) - 4a_1}{\varepsilon^2} & \text{if } \varepsilon\operatorname{mult}_O(\mathcal{M}) + a_1 \leqslant \dfrac{1}{2}. \end{cases}$$

$$(4.1)$$

If (4.1) is an equality, then either $-a_1 - \varepsilon\operatorname{mult}_O(\mathcal{M}) + 1 \in \mathbb{N}$ and $\operatorname{mult}_Q(\widehat{M}_1) = \operatorname{mult}_Q(\widehat{M}_2) = \frac{2}{\varepsilon}$, or $\varepsilon\operatorname{mult}_O(\mathcal{M}) + a_1 = 1$ and $\operatorname{mult}_Q(\widehat{M}_1) = \operatorname{mult}_Q(\widehat{M}_2) = \frac{1}{\varepsilon}$.

**Lemma 4.3.** *Suppose that* $\varepsilon\operatorname{mult}_O(\mathcal{M}) + a_1 \leqslant \frac{1}{2}$. *Then* (1.2) *holds. Moreover, if* (1.2) *is an equality, then either* $-a_1 \in \mathbb{N}$ *and* $\operatorname{mult}_O(\mathcal{M}) = \frac{2}{\varepsilon}$, *or* $a_1 = 0$ *and* $\operatorname{mult}_O(\mathcal{M}) = \frac{1}{\varepsilon}$.

*Proof.* Since $a_1 \leqslant \frac{1}{2} - \varepsilon\operatorname{mult}_O(\mathcal{M}) \leqslant -\frac{1}{2}$ and

$$\operatorname{mult}_O(M_1 \cdot M_2) \geqslant \operatorname{mult}_P^2(\mathcal{M}) + \frac{4 - 4\varepsilon\operatorname{mult}_O(\mathcal{M}) - 4a_1}{\varepsilon^2}$$

$$= -\frac{4a_1}{\varepsilon^2} + \left(\operatorname{mult}_O(\mathcal{M}) - \frac{2}{\varepsilon}\right)^2 \geqslant -\frac{4a_1}{\varepsilon^2},$$

we get (1.2). Moreover, if (1.2) is an equality, then $\operatorname{mult}_O(\mathcal{M}) = \frac{2}{\varepsilon}$, whence (4.1) is also an equality. By induction we see that $-a_1 - 1 = -a_1 - \varepsilon\operatorname{mult}_O(\mathcal{M}) + 1 \in \mathbb{N}$, whence $-a_1 \in \mathbb{N}$. $\square$

Thus, to complete the proof of the theorem, we may assume that $\varepsilon \operatorname{mult}_O(\mathcal{M}) + a_1 \geqslant \frac{1}{2}$.

**Lemma 4.4.** *Suppose that $a_1 \geqslant -\frac{1}{2}$. Then the inequality* (1.2) *holds and is not an equality.*

*Proof.* It follows from the inequalities $\varepsilon \operatorname{mult}_O(\mathcal{M}) + a_1 \geqslant \frac{1}{2}$ and $a_1 \geqslant -\frac{1}{2}$ that

$$\operatorname{mult}_O(M_1 \cdot M_1) \geqslant \operatorname{mult}_P^2(\mathcal{M}) + \frac{3 - 2\varepsilon \operatorname{mult}_O(\mathcal{M}) - 2a_1}{\varepsilon^2} \geqslant \frac{2 - 2a_1}{\varepsilon^2}.$$

$\square$

Thus, to complete the proof of the theorem, we may assume that $a_1 \leqslant -\frac{1}{2}$. Then it follows from the inequality $\operatorname{mult}_O(\mathcal{M}) \geqslant \frac{1/2 - a_1}{\varepsilon}$ that

$$\operatorname{mult}_O(M_1 \cdot M_1) \geqslant \operatorname{mult}_P^2(\mathcal{M}) + \frac{3 - 2\varepsilon \operatorname{mult}_O(\mathcal{M}) - 2a_1}{\varepsilon^2} \geqslant \frac{9/4 - a_1 + a_1^2}{\varepsilon^2} \geqslant \frac{-4a_1}{\varepsilon^2}.$$

This immediately yields (1.2).

**Lemma 4.5.** *The inequality* (1.2) *is not an equality.*

*Proof.* Assume that (1.2) is an equality. Then $a_1 = -\frac{3}{2}$ and $\operatorname{mult}_O(\mathcal{M}) = \frac{1/2 - a_1}{\varepsilon}$, whence $\operatorname{mult}_O(\mathcal{M}) = \frac{2}{\varepsilon}$. Thus (4.1) must also be an equality. In this case we obtain by induction that $\frac{1}{2} = \varepsilon \operatorname{mult}_O(\mathcal{M}) + a_1 = 1$. The resulting contradiction completes the proof of the lemma. $\square$

Thus Theorem 1.6 is proved. $\square$

## § 5. Del Pezzo orbifolds

The purpose of this section is to show how to apply Theorem 1.4. All results proved in this section are well known to experts. Nevertheless, Theorem 1.4 enables one to shorten their proofs considerably.

Let $X$ be a Fano variety with at most quotient singularities. The real number

$$\operatorname{lct}(X) = \sup\left\{ \lambda \in \mathbb{Q} \ \middle| \ \begin{array}{l} \text{the log pair } (X, \lambda D) \text{ is log canonical} \\ \text{for every effective } \mathbb{Q}\text{-divisor } D \equiv -K_X \text{ on } X \end{array} \right\}$$

is called the *global log canonical threshold* of the Fano variety $X$ (see Definition A.22 in Appendix A).

**Theorem 5.1** ([10]–[12], Appendix A). *The variety $X$ possesses an orbifold Kähler–Einstein metric if $\operatorname{lct}(X) > \frac{\dim(X)}{\dim(X)+1}$.*

We now show how to use Theorem 1.4 along with Theorem 5.1 to prove the existence of orbifold Kähler–Einstein metrics on some Fano orbifolds.

**Theorem 5.2** [2]. *Let $X$ be a hypersurface of degree 6 in $\mathbb{P}(1, 1, 2, 3)$. If the set $\operatorname{Sign}(X)$ consists of Du Val singular points of type $\mathbb{A}_3$, then $\operatorname{lct}(X) \geqslant \frac{5}{6}$.*

*Proof.* Suppose that the set $\mathrm{Sign}(X)$ consists of Du Val singular points of type $\mathbb{A}_3$. Put $\omega = \frac{5}{6}$. Suppose that $\mathrm{lct}(X) < \omega$. Then there is an effective $\mathbb{Q}$-divisor $D$ on $X$ such that $D \equiv -K_X$, and the log pair $(X, \omega D)$ is not log canonical at some point $P \in X$.

Let $C$ be a curve in $|-K_X|$ such that $P \in C$. Then $C$ is irreducible and the log pair $(X, \omega C)$ is log canonical. By Remark 2.6 we may assume that $C \not\subset \mathrm{Supp}(D)$.

Note that $X$ must be singular at $P$ since otherwise we get a contradiction:

$$1 = K_X^2 = C \cdot D \geqslant \mathrm{mult}_P(D) > \frac{1}{\omega} = \frac{6}{5} > 1.$$

Let $\pi \colon \widetilde{X} \to X$ be a birational morphism that contracts three irreducible curves $E_1$, $E_2$, $E_3$ to the point $P \in X$ and induces an isomorphism $\bar{X} \setminus (E_1 \cup E_2 \cup E_3) \cong X \setminus P$. The surface $\bar{X}$ is smooth along the curves $E_1$, $E_2$, $E_3$.

Note that $E_1^2 = E_2^2 = E_3^2 = -2$. We may assume without loss of generality that $E_1 \cdot E_3 = 0$ and $E_1 \cdot E_2 = E_2 \cdot E_3 = 1$.

Let $\bar{C}$ be the proper transform of $C$ on the surface $\bar{X}$. Then $\bar{C} \equiv \pi^*(C) - E_1 - E_2 - E_3$. Let $\bar{D}$ be the proper transform of $D$ on $\bar{X}$. Then $\bar{D} \equiv \pi^*(D) - a_1 E_1 - a_2 E_2 - a_3 E_3$, where $a_1$, $a_2$, $a_3$ are positive rational numbers. We have

$$1 - a_1 - a_3 = \bar{D} \cdot \bar{C} \geqslant 0, \qquad 2a_1 - a_2 = \bar{D} \cdot E_1 \geqslant 0,$$
$$2a_2 - a_1 - a_3 = \bar{D} \cdot E_2 \geqslant 0, \qquad 2a_3 - a_2 = \bar{D} \cdot E_3 \geqslant 0,$$

whence $1 \geqslant a_1 + a_3$, $2a_1 \geqslant a_2$, $3a_2 \geqslant 2a_3$, $2a_3 \geqslant a_2$, $3a_2 \geqslant 2a_1$, $a_1 \leqslant \frac{3}{4}$, $a_2 \leqslant 1$, $a_3 \leqslant \frac{3}{4}$. On the other hand, since

$$K_{\bar{X}} + \bar{D} + a_1 E_1 + a_2 E_2 + a_3 E_3 \equiv \pi^*(K_X + D),$$

it follows that there is a point $Q \in E_1 \cup E_2 \cup E_3$ such that the log pair $(\bar{X}, \bar{D} + a_1 E_1 + a_2 E_2 + a_3 E_3)$ is not log canonical at $Q$ because $(X, D)$ is not log canonical at $P \in X$.

Suppose that $Q \in E_1$ and $Q \notin E_2$. Then $(\bar{X}, \bar{D} + E_1)$ is not log canonical at $Q$. Hence,

$$2a_1 - a_2 = \bar{D} \cdot E_1 \geqslant \mathrm{mult}_Q(\bar{D} \cdot E_1) > 1$$

by Theorem 1.1. Therefore we see that $1 \geqslant \frac{4}{3} a_1 \geqslant 2a_1 - \frac{2}{3} a_1 \geqslant 2a_1 - a_2 > 1$, a contradiction.

Suppose that $Q \in E_2$ and $Q \notin E_1 \cup E_3$. Then $(\bar{X}, \bar{D} + E_2)$ is not log canonical at $Q$. Hence,

$$2a_2 - a_1 - a_3 = \bar{D} \cdot E_2 \geqslant \mathrm{mult}_Q(\bar{D} \cdot E_2) > 1$$

by Theorem 1.1. Therefore we see that $1 \geqslant a_2 = 2a_2 - \frac{a_2}{2} - \frac{a_2}{2} \geqslant 2a_2 - a_1 - a_3 > 1$, a contradiction.

There is no loss of generality in assuming that $Q = E_1 \cap E_2$. Then $(\bar{X}, \bar{D} + a_1 E_1 + a_2 E_2)$ is not log canonical at $Q$. On the other hand, we have $a_1 + \frac{a_2}{2} \leqslant a_1 + a_3 \leqslant 1$ because $2a_3 \geqslant a_2$. Hence we can apply Theorem 1.4 to the log pair $(\bar{X}, \bar{D} + a_1 E_1 + a_2 E_2)$ with $M = N = 0$, $A = 2$, $B = \frac{3}{2}$, $\alpha = 1$ and $\beta = \frac{1}{2}$. Thus we have

$$2a_2 - a_1 - a_3 = \mathrm{mult}_O(\bar{D} \cdot E_2) > \frac{3}{2} a_2 - a_1$$

because $\bar{D} \cdot E_1 = 2a_1 - a_2$. Then $a_2 > 2a_3$ contrary to the inequality $2a_3 \geqslant a_2$. $\square$

**Theorem 5.3** [2]. *Let $X$ be a singular hypersurface of degree 6 in $\mathbb{P}(1, 1, 2, 3)$ such that the set $\mathrm{Sign}(X)$ consists of Du Val singular points of type $\mathbb{A}_4$. Then $\mathrm{lct}(X) = \frac{4}{5}$.*

*Proof.* Let $P$ be a singular point of $X$ and let $\pi\colon \widetilde{X} \to X$ be a birational morphism that contracts four irreducible curves $E_1$, $E_2$, $E_3$, $E_4$ to the point $P \in X$ and induces an isomorphism $\bar{X} \setminus (E_1 \cup E_2 \cup E_3 \cup E_4) \cong X \setminus P$. The surface $\bar{X}$ is smooth along the curves $E_1$, $E_2$, $E_3$, $E_4$. We have $E_1^2 = E_2^2 = E_3^2 = E_4^2 = -2$. We may assume that $E_1 \cdot E_3 = E_1 \cdot E_4 = E_2 \cdot E_4 = 0$ and $E_1 \cdot E_2 = E_2 \cdot E_3 = E_3 \cdot E_4 = 1$.

There is a unique smooth irreducible curve $\bar{Z} \subset \bar{X}$ such that $\pi(\bar{Z}) \sim -2K_X$ and $E_2 \cap E_3 \in Z$. Put $\omega = \frac{4}{5}$ and $Z = \pi(\bar{Z})$. Then $Z \sim \pi^*(-2K_X) - E_1 - 2E_2 - 2E_3 - E_4$, which implies that $(X, \omega Z)$ is log canonical and not log terminal. In particular, we have $\mathrm{lct}(X) \leqslant \omega$.

Suppose that $\mathrm{lct}(X) < \omega$. Then there is an effective $\mathbb{Q}$-divisor $D$ on $X$ such that $D \equiv -K_X$ and the log pair $(X, \omega D)$ is not log canonical at some point $O \in X$. Remark 2.6 enables us to assume that $Z \not\subset \mathrm{Supp}(D)$ because $Z$ is irreducible. Arguing as in the proof of Theorem 5.2, we can also assume that $O = P$.

Let $\bar{D}$ be the proper transform of $D$ on $\bar{X}$. Then $\bar{D} \equiv \pi^*(D) - a_1 E_1 - a_2 E_2 - a_3 E_3 - a_4 E_4$, where $a_1$, $a_2$, $a_3$ and $a_4$ are positive rational numbers. Then the equivalence

$$K_{\bar{X}} + \bar{D} + \sum_{i=1}^{4} a_i E_i \equiv \pi^*(K_X + D)$$

yields the existence of a point $Q \in E_1 \cup E_2 \cup E_3 \cup E_4$ such that the log pair $(\bar{X}, \bar{D} + a_1 E_1 + a_2 E_2 + a_3 E_3 + a_4 E_4)$ is not log canonical at $Q$ because $(X, D)$ is not log canonical at $P \in X$.

Let $C$ be the curve through $P$ in the linear system $|-K_X|$. Then $C$ is irreducible and the log pair $(X, \omega C)$ is log canonical. Remark 2.6 enables us to assume that $C \not\subset \mathrm{Supp}(D)$. Let $\bar{C}$ be the proper transform of $C$ on $\bar{X}$. Then $\bar{C} \equiv \pi^*(C) - E_1 - E_2 - E_3 - E_4$ and $\bar{D} \cdot \bar{C} \geqslant 0$. Thus we have $1 - a_1 - a_4 = \bar{D} \cdot \bar{C} \geqslant 0$, whence $a_1 + a_4 \leqslant 1$. We similarly have

$$2a_1 - a_2 = \bar{D} \cdot E_1 \geqslant 0, \qquad 2a_2 - a_1 - a_3 = \bar{D} \cdot E_2 \geqslant 0,$$
$$2a_3 - a_2 - a_4 = \bar{D} \cdot E_3 \geqslant 0, \qquad 2a_4 - a_3 = \bar{D} \cdot E_4 \geqslant 0,$$

whence $a_1 \leqslant \frac{4}{5}$, $a_2 \leqslant \frac{6}{5}$, $a_3 \leqslant \frac{6}{5}$ and $a_4 \leqslant \frac{4}{5}$. It follows from Theorem 2.16 that $\mathbb{LCS}(\bar{X}, \bar{D} + a_1 E_1 + a_2 E_2 + a_3 E_3 + a_4 E_4) = \{Q\}$ because $\omega < \frac{5}{6}$. We similarly see that $\frac{4}{5}a_1 + \frac{2}{5}a_2 = \omega a_1 + \frac{\omega a_2}{2} \leqslant 1$.

Suppose that $Q \in E_1$ and $Q \notin E_2$. Then $(\bar{X}, \bar{D} + E_1)$ is not log canonical at $Q$. Hence $2a_1 - a_2 = \bar{D} \cdot E_1 \geqslant \mathrm{mult}_Q(\bar{D} \cdot E_1) > 1$ by Theorem 1.1. It follows that $1 \geqslant \frac{5}{4}a_1 \geqslant 2a_1 - \frac{3}{4}a_1 \geqslant 2a_1 - a_2 > 1$, a contradiction.

Suppose that $Q \in E_2$ and $Q \notin E_1 \cup E_3$. Then $2a_2 - a_1 - a_3 = \bar{D} \cdot E_2 \geqslant \mathrm{mult}_Q(\bar{D} \cdot E_2) > 1$ by Theorem 1.1. Hence we have $1 \geqslant \frac{5}{6}a_2 \geqslant 2a_2 - \frac{a_2}{2} - \frac{2}{3}a_2 \geqslant 2a_2 - a_1 - a_3 > 1$, a contradiction.

Suppose that $Q = E_1 \cap E_2$. Then we can apply Theorem 1.4 to the log pair $(\bar{X}, \omega \bar{D} + \omega a_1 E_1 + \omega a_2 E_2)$ with $M = N = 0$, $A = 2$, $B = \frac{3}{2}$, $\alpha = 1$ and $\beta = \frac{1}{2}$.

Thus we have $2a_2 - a_1 - a_3 = \text{mult}_O(\overline{D} \cdot E_2) > \frac{3}{2}a_2 - a_1$. Since $\overline{D} \cdot E_1 = 2a_1 - a_2$, we get $a_2 > 2a_3$. This easily leads to a contradiction.

To complete the proof, we may assume that $Q = E_2 \cap E_3$. Then $(\overline{X}, \omega\overline{D} + \omega a_2 E_2 + \omega a_3 E_3)$ is not log canonical at $Q$. Hence,

$$2a_2 - \frac{1}{2}a_2 - a_3 \geqslant 2a_2 - a_1 - a_3 = \overline{D} \cdot E_2 \geqslant \text{mult}_Q(\overline{D} \cdot E_2) > \frac{5}{4} - a_3$$

by Theorem 1.1. We similarly see that

$$2a_3 - a_2 - a_4 = \overline{D} \cdot E_3 \geqslant \text{mult}_Q(\overline{D} \cdot E_3) > \frac{5}{4} - a_2,$$

whence $a_2 > \frac{5}{6}$ and $a_3 > \frac{5}{6}$.

Let $\xi \colon \widetilde{X} \to \overline{X}$ be the blow-up of $Q$, $E$ the exceptional curve of $\xi$, and $\widetilde{D}$ the proper transform of $\overline{D}$ on $\widetilde{X}$. We put $m = \text{mult}_Q(\overline{D})$. Then $\widetilde{D} \equiv \xi^*(\overline{D}) - mE$. Let $\widetilde{E}_1$, $\widetilde{E}_2$, $\widetilde{E}_3$, $\widetilde{E}_4$ be the proper transforms on $\widetilde{X}$ of the curves $E_1$, $E_2$, $E_3$, $E_4$ respectively. Then

$$K_{\widetilde{X}} + \omega\widetilde{D} + \omega a_2 \widetilde{E}_2 + \omega a_3 \widetilde{E}_3 + (\omega a_2 + \omega a_3 + \omega m - 1)E$$
$$\equiv \xi^*(K_{\overline{X}} + \omega\overline{D} + \omega a_2 E_2 + \omega a_3 E_3).$$

Hence there is a point $R \in E$ such that the log pair $(\widetilde{X}, \omega\widetilde{D} + \omega a_2 \widetilde{E}_2 + \omega a_3 \widetilde{E}_3 + (\omega a_2 + \omega a_3 + \omega \text{mult}_Q(\overline{D}) - 1)E)$ is not log canonical at $R$.

Let $\widetilde{Z}$ be the proper transform of $\overline{Z}$ on $\widetilde{X}$. Then

$$0 \leqslant \widetilde{Z} \cdot \widetilde{D} = 2 - a_2 - a_3 - \text{mult}_Q(\overline{D}) = 2 - a_2 - a_3 - m,$$

whence $m + a_2 + a_3 \leqslant 2$. In particular, we see that $\omega a_2 + \omega a_3 + \omega m - 1 \leqslant 2\omega - 1 \leqslant \frac{3}{5}$. It follows that $\mathbb{LCS}(\widetilde{X}, \omega\widetilde{D} + \omega a_2 \widetilde{E}_2 + \omega a_3 \widetilde{E}_3 + (\omega a_2 + \omega a_3 + \omega \text{mult}_Q(\overline{D}) - 1)E) = \{R\}$ by Theorem 2.16. We similarly see that

$$2a_3 - a_2 - a_4 - m = \widetilde{E}_3 \cdot \widetilde{D} \geqslant 0, \qquad 2a_2 - a_1 - a_3 - m = \widetilde{E}_2 \cdot \widetilde{D} \geqslant 0,$$

whence $E \cdot \widetilde{D} = m \leqslant \frac{1}{2}$.

Suppose that $Q \notin E_2 \cup E_3$. Then the log pair

$$(\widetilde{X}, \omega\widetilde{D} + (\omega a_2 + \omega a_3 + \omega \text{mult}_Q(\overline{D}) - 1)E)$$

is not log canonical at $R$. Therefore,

$$\frac{5}{4} > \frac{1}{2} \geqslant m = \widetilde{D} \cdot E \geqslant \text{mult}_R(\widetilde{D} \cdot E) > \frac{5}{4}$$

by Theorem 1.1. The resulting contradiction shows that either $R = \widetilde{E}_2 \cap E$, or $R = \widetilde{E}_3 \cap E$.

We may assume without loss of generality that $R = \widetilde{E}_2 \cap E$. Then the singularities of the log pair $(\widetilde{X}, \omega\widetilde{D} + \omega a_2 \widetilde{E}_2 + (\omega a_2 + \omega a_3 + \omega \text{mult}_Q(\overline{D}) - 1)E)$ are not log canonical at the point $R$. Hence it follows from Theorem 1.1 that

$$\frac{5}{4} - a_2 > \frac{6}{5} - a_2 = 2 - \frac{4}{5} - a_2 \geqslant 2 - a_2 - a_3 \geqslant m = \widetilde{D} \cdot E \geqslant \text{mult}_R(\widetilde{D} \cdot E) > \frac{5}{4} - a_2$$

because $m + a_2 + a_3 \leqslant 2$. The resulting contradiction completes the proof. $\square$

**Theorem 5.4** [13]. *Let $X$ be a quasi-smooth hypersurface of degree 95 in the weighted projective space $\mathbb{P}(11, 21, 29, 37)$. Then $\operatorname{lct}(X) = \frac{11}{4}$.*

*Proof.* We may assume that $X$ is defined by a quasi-homogeneous equation,

$$t^2 y + tz^2 + xy^4 + x^6 z = 0 \subset \operatorname{Proj}(\mathbb{C}[x, y, z, t]),$$

where $\operatorname{wt}(x) = 11$, $\operatorname{wt}(y) = 21$, $\operatorname{wt}(z) = 29$, $\operatorname{wt}(t) = 37$. Let $O_x$ be the point of $X$ given by the equations $y = z = t = 0$. We similarly define the singular points $O_y$, $O_z$ and $O_t$ of $X$. Then $O_x$, $O_y$, $O_z$, $O_t$ are singular points of $X$ of types $\frac{1}{11}(5, 2)$, $\frac{1}{21}(1, 2)$, $\frac{1}{29}(11, 21)$, $\frac{1}{37}(11, 29)$ respectively.

Let $C_x$ be the curve cut out on $X$ by the equation $x = 0$. Then $C_x = L_{xt} + R_x$, where $L_{xt}$ and $R_x$ are irreducible reduced curves on $X$ such that $L_{xt}$ is given by the equations $x = t = 0$ and $R_x$ by the equations $x = yt + z^2 = 0$. We have $\operatorname{lct}(X, \frac{3}{11}C_x) = \frac{11}{4}$, whence $\operatorname{lct}(X) \leqslant \frac{11}{4}$. Let $C_y$ be the curve cut out on $X$ by the equation $y = 0$. Then $C_y = L_{yz} + R_y$, where $L_{yz}$ and $R_y$ are irreducible reduced curves on $X$ such that $L_{yz}$ is given by the equations $y = z = 0$ and $R_y$ by the equations $y = zt + x^6 = 0$. Let $C_z$ be the curve cut out on $X$ by the equation $z = 0$. Then $C_z = L_{yz} + R_z$, where $R_z$ is the irreducible reduced curve given by the equations $z = xy^3 + t^2 = 0$. Let $C_t$ be the curve cut out on $X$ by the equation $t = 0$. Then $C_t = L_{xt} + R_t$, where $R_t$ is the irreducible reduced curve given by the equations $t = y^4 + x^5 z = 0$. It is easy to compute the intersection numbers of the divisors $D$, $L_{xt}$, $L_{yz}$, $R_x$, $R_y$, $R_z$, $R_t$ on $X$:

$$D \cdot L_{xt} = \frac{1}{7 \cdot 29}, \quad D \cdot R_x = \frac{2}{7 \cdot 37}, \quad D \cdot R_y = \frac{18}{29 \cdot 37},$$

$$D \cdot L_{yz} = \frac{3}{11 \cdot 37}, \quad D \cdot R_z = \frac{2}{7 \cdot 11}, \quad D \cdot R_t = \frac{12}{11 \cdot 29},$$

$$L_{xt} \cdot R_x = \frac{2}{21}, \quad L_{yz} \cdot R_y = \frac{6}{37}, \quad L_{yz} \cdot R_z = \frac{2}{11}, \quad L_{xt} \cdot R_t = \frac{4}{29},$$

$$R_x^2 = -\frac{52}{21 \cdot 37}, \quad L_{xt}^2 = -\frac{47}{21 \cdot 29}, \quad R_y^2 = -\frac{48}{29 \cdot 37},$$

$$L_{yz}^2 = -\frac{45}{11 \cdot 37}, \quad R_z^2 = \frac{16}{11 \cdot 21}, \quad R_t^2 = \frac{104}{11 \cdot 29}.$$

We have $L_{xt} \cap R_x = \{O_y\}$, $L_{yz} \cap R_y = \{O_t\}$, $L_{yz} \cap R_z = \{O_x\}$, $L_{xt} \cap R_t = \{O_z\}$, and

$$\min\left(\operatorname{lct}\left(X, \frac{3}{21}C_y\right), \operatorname{lct}\left(X, \frac{3}{29}C_z\right), \operatorname{lct}\left(X, \frac{3}{37}C_t\right)\right) \geqslant \frac{11}{4}.$$

Put $\omega = \frac{11}{4}$. Suppose that $\operatorname{lct}(X) < \omega$. Then there is an effective $\mathbb{Q}$-divisor $D \equiv -K_X$ on $X$ such that the log pair $(X, \omega D)$ is not log canonical at some point $P \in X$.

By Remark 2.6 we may assume that the support of $D$ does not contain at least one irreducible component of each of the divisors $C_x$, $C_y$, $C_z$, $C_t$. Since $D \cdot L_{xt} = \frac{3}{29 \cdot 21}$ and $D \cdot R_x = \frac{6}{37 \cdot 21}$, we see that $P \neq O_y$. We similarly have $P \neq O_x$. Since $D \cdot L_{xt} = \frac{1}{7 \cdot 29}$ and $D \cdot R_t = \frac{12}{11 \cdot 29}$, we see that $P \neq O_z$ because the curve $R_t$ is singular at $O_z$ and its orbifold multiplicity at $O_z$ is equal to 4.

We put $D = mL_{xt} + \Omega$, where $m$ is a non-negative rational number and $\Omega$ is an effective $\mathbb{Q}$-divisor whose support does not contain $L_{xt}$. Then $m \leqslant \frac{4}{11}$ because the log pair $(X, \omega D)$ is log canonical at $O_y$. Therefore we have

$$(D - mL_{xt}) \cdot L_{xt} = \frac{3 + 47m}{21 \cdot 29} \leqslant \frac{4}{11},$$

whence $P \in L_{xt}$ by Theorem 1.1. We similarly see that either $P = O_t$ or $P \notin C_x \cup C_y \cup C_z \cup C_t$.

Suppose that $P \neq O_t$. Let $\mathcal{L}$ be the pencil cut out on $X$ by the equation $\lambda yt + \mu z^2 = 0$, where $[\lambda : \mu] \in \mathbb{P}^1$. Then the base locus of $\mathcal{L}$ consists of the curve $L_{yz}$ and the point $O_y$. Let $E$ be the unique curve through $P$ in $\mathcal{L}$. Then $E$ is cut out on $X$ by the equation $z^2 = \alpha yt$ for some non-zero $\alpha \in \mathbb{C}$ because $P \notin C_x \cup C_y \cup C_z \cup C_t$.

Suppose that $\alpha \neq -1$. Then $E$ is isomorphic to the curve given by the equations $yt - z^2 = t^2 y + xy^4 + x^6 z = 0$. Therefore $E$ is smooth at $P$. Note that $E = L_{yz} + C$, where $C$ is an irreducible reduced curve and it is known that $P \in C$. We have $D \cdot C = D \cdot E - D \cdot L_{yz} = \frac{169}{7 \cdot 11 \cdot 37}$ and $C \cdot C > 0$. This easily leads to conclusions that contradict Theorem 1.1.

Suppose that $\alpha = -1$. Then $E = L_{yz} + R_x + M$, where $M$ is an irreducible reduced curve containing $P$. It is easy to see that $M$ is non-singular at $P$. We have $D \cdot M = D \cdot E - D \cdot L_{yz} - D \cdot R_x = \frac{147}{7 \cdot 11 \cdot 37}$ and $M \cdot M > 0$. Arguing as in the previous case, we again easily arrive at a contradiction to Theorem 1.1.

Thus we see that $P = O_t$. Put $D = \Delta + aL_{yz} + bR_x$, where $a$ and $b$ are non-negative rational numbers and $\Delta$ is an effective $\mathbb{Q}$-divisor whose support contains neither $L_{yz}$ nor $R_x$. Then $a > 0$. Indeed, otherwise we obtain a contradiction: $\frac{3}{11} = 37D \cdot L_{yz} \geqslant \mathrm{mult}_{O_t}(D) > \frac{4}{11}$.

Note that $R_y \not\subset \mathrm{Supp}(\Delta)$ because $a > 0$. If $b > 0$, then $L_{xt}$ is not contained in the support of $D$. It follows that $\frac{3}{21 \cdot 29} = D \cdot L_{xt} \geqslant b(R_x \cdot L_{xt}) = \frac{2b}{21}$, whence $b \leqslant \frac{3}{58}$. We similarly see that

$$\frac{18}{29 \cdot 37} = D \cdot R_y \geqslant \frac{6a}{37} + \frac{b}{37} + \frac{\mathrm{mult}_{O_t}(D) - a - b}{37} > \frac{5a}{37} + \frac{4}{11 \cdot 37},$$

whence $a < \frac{82}{1595}$. One can easily check that $\Delta \cdot L_{yz} = \frac{3}{11 \cdot 37} + a\frac{45}{11 \cdot 37} - \frac{b}{37}$ and $\Delta \cdot R_x = \frac{2}{7 \cdot 37} + b\frac{52}{21 \cdot 37} - \frac{a}{37}$. Using Theorem 1.4 for the log pair $(X, \omega\Delta + \omega aL_{yz} + \omega bR_x)$ with $M = \frac{3}{11}$, $N = \frac{2}{7}$, $A = \frac{45}{11}$, $B = \frac{52}{21}$, $\alpha = \frac{675}{197}$ and $\beta = \frac{77}{197}$, we see that $\frac{24681}{45704} \geqslant \alpha\omega a + \beta\omega b > 1$, a contradiction. $\square$

**Theorem 5.5** [13]. *Let $X$ be a quasi-smooth hypersurface of degree 79 in the weighted projective space $\mathbb{P}(13, 14, 23, 33)$. Then $\mathrm{lct}(X) = \frac{65}{32}$.*

*Proof.* The surface $X$ can be defined by a quasi-homogeneous equation

$$z^2 t + y^4 z + xt^2 + x^5 y = 0 \subset \mathrm{Proj}(\mathbb{C}[x, y, z, t]),$$

where $\mathrm{wt}(x) = 13$, $\mathrm{wt}(y) = 14$, $\mathrm{wt}(z) = 23$, $\mathrm{wt}(t) = 33$. Let $O_x$ be the point on $X$ given by the equations $y = z = t = 0$. We similarly define the singular points $O_y$, $O_z$ and $O_t$ of $X$. Then $O_x$, $O_y$, $O_z$, $O_t$ are singular points of $X$ of types $\frac{1}{13}(1, 2)$, $\frac{1}{14}(13, 5)$, $\frac{1}{23}(13, 14)$, $\frac{1}{33}(14, 23)$ respectively.

Let $C_x$ be the curve cut out on $X$ by the equation $x = 0$. Then $C_x = L_{xz} + R_x$, where $L_{xz}$ and $R_x$ are irreducible reduced curves on $X$ such that $L_{xt}$ is given by the equations $x = z = 0$ and $R_x$ by the equations $x = y^4 + zt = 0$. Let $C_y$ be the curve cut out on $X$ by the equation $y = 0$. Then $C_y = L_{yt} + R_y$, where $L_{yt}$ and $R_y$ are irreducible reduced curves on $X$ such that $L_{yt}$ is given by $y = t = 0$ and $R_y$ by $y = z^2 + xt = 0$. Let $C_z$ be the curve cut out on $X$ by the equation $z = 0$. Then $C_z = L_{xz} + R_z$, where $R_z$ is an irreducible reduced curve given by the equations $z = x^4 y + t^2 = 0$. Let $C_t$ be the curve cut out on $X$ by the equation $t = 0$. Then $C_t = L_{yt} + R_t$, where $R_t$ is the irreducible reduced curve given by the equations $t = x^5 + y^3 z = 0$. It is easy to compute the intersection numbers of the divisors $D$, $L_{xz}$, $L_{yt}$, $R_x$, $R_y$, $R_z$, $R_t$ on $X$:

$$L_{xz}^2 = -\frac{43}{14 \cdot 33}, \quad R_x^2 = -\frac{40}{23 \cdot 33}, \quad L_{xz} \cdot R_x = \frac{4}{33},$$

$$D \cdot L_{xz} = \frac{4}{14 \cdot 33}, \quad D \cdot R_x = \frac{16}{23 \cdot 33},$$

$$L_{yt}^2 = -\frac{32}{13 \cdot 23}, \quad R_y^2 = -\frac{38}{13 \cdot 33}, \quad L_{yt} \cdot R_y = \frac{2}{13},$$

$$D \cdot L_{yt} = \frac{4}{13 \cdot 23}, \quad D \cdot R_y = \frac{8}{13 \cdot 33},$$

$$R_z^2 = \frac{20}{13 \cdot 14}, \quad L_{xz} \cdot R_z = \frac{2}{14}, \quad D \cdot R_z = \frac{8}{13 \cdot 14}, \quad R_t^2 = \frac{95}{14 \cdot 13},$$

$$L_{yt} \cdot R_t = \frac{5}{23}, \quad D \cdot R_t = \frac{20}{14 \cdot 23}, \quad R_y \cdot R_x = L_{xz} \cdot R_y = \frac{1}{33}, \quad R_x \cdot L_{yt} = \frac{1}{23}.$$

We have $L_{xz} \cap R_x = \{O_t\}$, $L_{xz} \cap R_z = \{O_y\}$, $L_{yt} \cap R_y = \{O_x\}$, $L_{yt} \cap R_t = \{O_z\}$. Then

$$\mathrm{lct}\left(X, \frac{4}{13}C_x\right) = \frac{65}{32} < \mathrm{lct}\left(X, \frac{4}{14}C_y\right)$$

$$= \frac{21}{8} < \mathrm{lct}\left(X, \frac{4}{25}C_t\right) = \frac{33}{10} < \mathrm{lct}\left(X, \frac{4}{23}C_z\right) = \frac{69}{20}.$$

In particular, it follows that $\mathrm{lct}(X) \leqslant \frac{65}{32}$.

Put $\omega = \frac{65}{32}$. Suppose that $\mathrm{lct}(X) < \omega$. Then there is an effective $\mathbb{Q}$-divisor $D \equiv -K_X$ on $X$ such that the log pair $(X, \omega D)$ is not log canonical at some point $P \in X$.

By Remark 2.6 we may assume that $\mathrm{Supp}(D)$ does not contain at least one irreducible component of each of the curves $C_x$, $C_y$, $C_z$, $C_t$. Arguing as in the proof of Theorem 5.4, we obtain that $P = O_t$ (see [13]). The curve $L_{xz}$ must be contained in $\mathrm{Supp}(D)$ since otherwise we have a contradiction: $\mathrm{mult}_{O_t}(D) \leqslant 33(D \cdot L_{xz}) = \frac{2}{7} < \frac{32}{65}$. Thus we see that $R_x \not\subset \mathrm{Supp}(D)$. Put $D = \Delta + a L_{xz} + b R_y$, where $a$ and $b$ are non-negative rational numbers and $\Delta$ is an effective $\mathbb{Q}$-divisor whose support contains neither $L_{xz}$ nor $R_y$. Then we have

$$\frac{16}{23 \cdot 33} = D \cdot R_x \geqslant a(L_{xz} \cdot R_x) + \frac{\mathrm{mult}_{O_t}(D) - a}{33} > \frac{3a}{33} + \frac{32}{33 \cdot 65},$$

whence $a < \frac{304}{4485}$. If $b \neq 0$, then $L_{yt} \not\subset \mathrm{Supp}(D)$. It follows that $\frac{4}{13 \cdot 23} = D \cdot L_{yt} \geqslant b(R_y \cdot L_{yt}) = \frac{2b}{13}$, whence $b \leqslant \frac{2}{23}$. Note that $33(\Delta \cdot L_{xz}) = \frac{4}{14} + a\frac{43}{14} - b$ and $33(\Delta \cdot R_y) = \frac{8}{13} + b\frac{38}{23} - a$. Put $M = \frac{4}{14}$, $N = \frac{8}{13}$, $A = \frac{43}{14}$, $B = \frac{38}{23}$, $\alpha = \frac{700771}{301108}$ and $\beta = \frac{69069}{150554}$ and apply Theorem 1.4 to the log pair $(X, \omega\Delta + \omega a L_{xz} + \omega b R_y)$. Then $\frac{66727051}{166211616} \geqslant \alpha\omega a + \beta\omega b > 1$, a contradiction. $\square$

**Theorem 5.6.** *Let $X$ be a quasi-smooth hypersurface of degree 79 in $\mathbb{P}(11, 17, 24, 31)$. Then $\mathrm{lct}(X) = \frac{33}{16}$.*

*Proof.* The surface $X$ can be defined by a quasi-homogeneous equation,

$$t^2 y + tz^2 + xy^4 + x^5 z = 0 \subset \mathrm{Proj}(\mathbb{C}[x, y, z, t]),$$

where $\mathrm{wt}(x) = 11$, $\mathrm{wt}(y) = 17$, $\mathrm{wt}(z) = 24$, $\mathrm{wt}(t) = 31$. We define points $O_x$, $O_y$, $O_z$, $O_t$ as in the proof of Theorem 5.4. Then $O_x$, $O_y$, $O_z$, $O_t$ are singular points of $X$ of types $\frac{1}{11}(2, 3)$, $\frac{1}{17}(1, 2)$, $\frac{1}{24}(11, 17)$, $\frac{1}{31}(11, 24)$ respectively.

Let $C_x$ be the curve cut out on $X$ by the equation $x = 0$. Then $C_x = L_{xt} + R_x$, where $L_{xt}$ and $R_x$ are irreducible reduced curves on $X$ such that $L_{xt}$ is given by the equations $x = t = 0$ and $R_x$ by the equations $x = yt + z^2 = 0$. Let $C_y$ be the curve cut out on $X$ by the equation $y = 0$. Then $C_y = L_{yz} + R_y$, where $L_{yz}$ and $R_y$ are irreducible reduced curves on $X$ such that $L_{yz}$ is given by the equations $y = z = 0$ and $R_y$ by the equations $y = zt + x^5 = 0$. Let $C_z$ be the curve cut out on $X$ by the equation $z = 0$. Then $C_z = L_{yz} + R_z$, where $R_z$ is an irreducible reduced curve given by the equations $z = xy^3 + t^2 = 0$. Let $C_t$ be the curve cut out on $X$ by the equation $t = 0$. Then $C_t = L_{xt} + R_t$, where $R_t$ is an irreducible reduced curve given by the equations $t = y^4 + x^4 z = 0$. It is easy to compute the intersection numbers of the divisors $D$, $L_{xt}$, $L_{yz}$, $R_x$, $R_y$, $R_z$, $R_t$ on $X$:

$$D \cdot L_{xt} = \frac{1}{6 \cdot 17}, \quad D \cdot R_x = \frac{8}{17 \cdot 31}, \quad D \cdot R_y = \frac{5}{6 \cdot 31},$$

$$D \cdot L_{yz} = \frac{4}{11 \cdot 31}, \quad D \cdot R_z = \frac{8}{11 \cdot 17},$$

$$D \cdot R_t = \frac{2}{3 \cdot 11}, \quad L_{xt} \cdot R_x = \frac{2}{17}, \quad L_{yz} \cdot R_y = \frac{5}{31},$$

$$L_{yz} \cdot R_z = \frac{2}{11}, \quad L_{xt} \cdot R_t = \frac{1}{6}, \quad L_{xt}^2 = -\frac{37}{17 \cdot 24},$$

$$R_x^2 = -\frac{40}{17 \cdot 31}, \quad R_y^2 = -\frac{35}{24 \cdot 31}, \quad L_{yz}^2 = -\frac{38}{11 \cdot 31},$$

$$R_z^2 = \frac{14}{11 \cdot 17}, \quad R_t^2 = \frac{10}{3 \cdot 11}.$$

We have $L_{xt} \cap R_x = \{O_y\}$, $L_{yz} \cap R_y = \{O_t\}$, $L_{yz} \cap R_z = \{O_x\}$, $L_{xt} \cap R_t = \{O_z\}$. Then

$$\mathrm{lct}\left(X, \frac{4}{11}C_x\right) = \frac{33}{16} < \min\left(\mathrm{lct}\left(X, \frac{4}{17}C_y\right), \mathrm{lct}\left(X, \frac{4}{24}C_z\right), \mathrm{lct}\left(X, \frac{4}{31}C_t\right)\right).$$

In particular, it follows that $\mathrm{lct}(X) \leqslant \frac{33}{16}$.

Put $\omega = \frac{33}{16}$. Suppose that $\mathrm{lct}(X) < \omega$. Then there is an effective $\mathbb{Q}$-divisor $D \equiv -K_X$ on $X$ such that the log pair $(X, \omega D)$ is not log canonical at some point $P \in X$.

By Remark 2.6, we may assume that $\mathrm{Supp}(D)$ does not contain at least one irreducible component of each of the curves $C_x$, $C_y$, $C_z$, $C_t$. Arguing as in the proof of Theorem 5.4, we obtain that $P = O_t$ (see [13]).

Note that the curve $L_{yz}$ must be contained in $\mathrm{Supp}(D)$ since otherwise we have $31(D \cdot L_{yz}) = \frac{4}{11} < \frac{16}{33}$: a contradiction. Thus we see that $R_y \not\subset \mathrm{Supp}(D)$. Put $D = \Delta + aL_{yz} + bR_x$, where $a$ and $b$ are non-negative rational numbers and $\Delta$ is an effective divisor whose support does not contain the curves $L_{yz}$ and $R_x$. Then $\frac{1}{6 \cdot 17} = D \cdot L_{xt} \geqslant b(R_x \cdot L_{xt}) = \frac{2b}{17}$, when $b \leqslant \frac{1}{12}$. On the other hand, we have

$$\frac{5}{6 \cdot 31} = D \cdot R_y \geqslant \frac{5a}{31} + \frac{b}{31} + \frac{\mathrm{mult}_{O_t}(D) - a - b}{31} > \frac{4a}{31} + \frac{16}{31 \cdot 33},$$

whence $a < \frac{23}{264}$. It is known that $31(\Delta \cdot L_{yz}) = \frac{4}{11} + a\frac{38}{11} - b$ and $31(\Delta \cdot R_x) = \frac{8}{17} + b\frac{40}{17} - a$. Thus we can apply Theorem 1.4 to the log pair $(X, \omega\Delta + \omega a L_{yz} + \omega b R_x)$ with $M = \frac{4}{11}$, $N = \frac{8}{17}$, $A = \frac{38}{11}$, $B = \frac{40}{17}$, $\alpha = \frac{1444}{453}$ and $\beta = \frac{187}{453}$. We get $\frac{6221}{9664} \geqslant \alpha\omega a + \beta\omega b > 1$, a contradiction. $\square$

**Theorem 5.7** [13]. *Let $X$ be a quasi-smooth surface of degree* 95 *in the weighted projective space* $\mathbb{P}(13, 17, 27, 41)$. *Then* $\mathrm{lct}(X) = \frac{65}{24}$.

*Proof.* The surface $X$ can be defined by a quasi-homogeneous equation

$$z^2 t + y^4 z + x t^2 + x^6 y = 0 \subset \mathrm{Proj}(\mathbb{C}[x, y, z, t]),$$

where $\mathrm{wt}(x) = 13$, $\mathrm{wt}(y) = 17$, $\mathrm{wt}(z) = 27$, $\mathrm{wt}(t) = 41$. We define points $O_x$, $O_y$, $O_z$, $O_t$ as in the proof of Theorem 5.4. They are singular points of $X$ of types $\frac{1}{13}(1, 2)$, $\frac{1}{17}(13, 7)$, $\frac{1}{27}(13, 17)$, $\frac{1}{41}(17, 27)$ respectively.

Let $C_x$ be the curve cut out on $X$ by the equation $x = 0$. Then $C_x = L_{xz} + R_x$, where $L_{xz}$ and $R_x$ are irreducible reduced curves on $X$ such that $L_{xt}$ is given by the equations $x = z = 0$ and $R_x$ by the equations $x = y^4 + zt = 0$. Let $C_y$ be the curve cut out on $X$ by the equation $y = 0$. Then $C_y = L_{yt} + R_y$, where $L_{yt}$ and $R_y$ are irreducible reduced curves on $X$ such that $L_{yt}$ is given by the equations $y = t = 0$ and $R_y$ by the equations $y = z^2 + xt = 0$. Let $C_z$ be the curve cut out on $X$ by the equation $z = 0$. Then $C_z = L_{xz} + R_z$, where $R_z$ is an irreducible reduced curve given by $z = t^2 + x^5 y = 0$. Let $C_t$ be the curve cut out on $X$ by the equation $t = 0$. Then $C_t = L_{yt} + R_t$, where $R_t$ is an irreducible reduced curve given by $t = x^6 + y^3 z = 0$. It is easy to compute the intersection numbers of the divisors $D$, $L_{xz}$, $L_{yt}$, $R_x$, $R_y$, $R_z$, $R_t$ on $X$:

$$D \cdot L_{xz} = \frac{3}{17 \cdot 41}, \quad D \cdot L_{yt} = \frac{1}{9 \cdot 13}, \quad D \cdot R_x = \frac{4}{9 \cdot 41},$$

$$D \cdot R_y = \frac{6}{13 \cdot 41}, \quad D \cdot R_z = \frac{6}{13 \cdot 17},$$

$$D \cdot R_t = \frac{2}{3 \cdot 17}, \quad L_{xz}^2 = -\frac{55}{17 \cdot 41}, \quad L_{yt}^2 = -\frac{37}{13 \cdot 27},$$

$$R_x^2 = -\frac{56}{27 \cdot 41}, \quad R_y^2 = -\frac{48}{13 \cdot 41},$$

$$R_z^2 = \frac{28}{13 \cdot 17}, \quad R_t^2 = \frac{16}{3 \cdot 17}, \quad L_{xz} \cdot R_x = \frac{4}{41},$$

$$L_{yt} \cdot R_y = \frac{2}{13}, \quad L_{xz} \cdot R_z = \frac{2}{17}, \quad L_{yt} \cdot R_t = \frac{2}{9}.$$

We have $L_{xz} \cap R_x = \{O_t\}$, $L_{xz} \cap R_z = \{O_y\}$, $L_{yt} \cap R_y = \{O_x\}$, $L_{yt} \cap R_t = \{O_z\}$. Then

$$\frac{65}{24} = \mathrm{lct}\left(X, \frac{3}{13}C_x\right) < \frac{51}{12}$$

$$= \mathrm{lct}\left(X, \frac{3}{17}C_y\right) < \frac{41}{8} = \mathrm{lct}\left(X, \frac{3}{41}C_t\right) < \frac{21}{4} = \mathrm{lct}\left(X, \frac{3}{27}C_z\right).$$

It follows that $\mathrm{lct}(X) \leqslant \frac{65}{24}$.

Put $\omega = \frac{65}{24}$. Suppose that $\mathrm{lct}(X) < \omega$. Then there is an effective $\mathbb{Q}$-divisor $D \equiv -K_X$ on $X$ such that the log pair $(X, \omega D)$ is not log canonical at some point $P \in X$.

By Remark 2.6, we may assume that $\mathrm{Supp}(D)$ does not contain at least one component of each of the curves $C_x$, $C_y$, $C_z$, $C_t$. Arguing as in the proof of Theorem 5.4, we have $P = O_t$ (see [13]). If $L_{xz} \not\subseteq \mathrm{Supp}(D)$, then $\frac{24}{65} < \mathrm{mult}_{O_t}(D) \leqslant 41(D \cdot L_{xz}) = \frac{3}{17} < \frac{24}{65}$, a contradiction. Therefore $L_{xz} \subseteq \mathrm{Supp}(D)$, whence $R_x \not\subseteq \mathrm{Supp}(D)$. We put $D = aL_{xz} + bR_y + \Delta$, where $a$ and $b$ are non-negative rational numbers and $\Delta$ is an effective $\mathbb{Q}$-divisor whose support contains neither $L_{xz}$ nor $R_y$. Then we have

$$\frac{4}{9 \cdot 41} = D \cdot R_x \geqslant a(L_{xz} \cdot R_x) + \frac{\mathrm{mult}_{O_t}(D) - a}{41} > \frac{3a}{41} + \frac{24}{41 \cdot 65},$$

whence $a \leqslant \frac{44}{585}$. If $b \neq 0$, then $\frac{1}{9 \cdot 13} = D \cdot L_{yt} \geqslant b(R_y \cdot L_{yt}) = \frac{2b}{13}$. It follows that $b \leqslant \frac{1}{18}$. On the other hand, $41(\Delta \cdot L_{xz}) = \frac{3}{17} + a\frac{55}{17} - b$ and $41(\Delta \cdot R_y) = \frac{6}{13} + b\frac{48}{41} - a$. Therefore one can apply Theorem 1.4 to the log pair $(X, \omega\Delta + \omega a L_{xz} + \omega b R_y)$ with $M = \frac{6}{13}$, $N = \frac{3}{17}$, $A = \frac{48}{41}$, $B = \frac{55}{17}$, $\alpha = \frac{29952}{19505}$, $\beta = \frac{5729}{19505}$. We get

$$\frac{306379}{1053270} \geqslant \alpha\omega b + \beta\omega a > 1,$$

a contradiction. $\square$

**Theorem 5.8** [13]. *Let $X$ be a quasi-smooth hypersurface of degree 99 in the weighted projective space $\mathbb{P}(14, 17, 29, 41)$. Then $\mathrm{lct}(X) = \frac{51}{10}$.*

*Proof.* The surface $X$ can be defined by a quasi-homogeneous equation,

$$t^2 y + tz^2 + xy^5 + x^5 z = 0 \subset \mathrm{Proj}(\mathbb{C}[x, y, z, t]),$$

where $\mathrm{wt}(x) = 14$, $\mathrm{wt}(y) = 17$, $\mathrm{wt}(z) = 29$, $\mathrm{wt}(t) = 41$. We define points $O_x$, $O_y$, $O_z$, $O_t$ as in the proof of Theorem 5.4. Then $O_x$, $O_y$, $O_z$, $O_t$ are singular points of $X$ of types $\frac{1}{14}(3, 13)$, $\frac{1}{17}(12, 7)$, $\frac{1}{29}(11, 17)$, $\frac{1}{41}(14, 29)$ respectively.

Let $C_x$ be the curve cut out on $X$ by the equation $x = 0$. Then $C_x = L_{xt} + R_x$, where $L_{xt}$ and $R_x$ are irreducible reduced curves on $X$ such that $L_{xt}$ is given by the equations $x = t = 0$ and $R_x$ by the equations $x = yt + z^2 = 0$. Let $C_y$ be the

curve cut out on $X$ by the equation $y = 0$. Then $C_y = L_{yz} + R_y$, where $L_{yz}$ and $R_y$ are irreducible reduced curves on $X$ such that $L_{yz}$ is given by the equations $y = z = 0$ and $R_y$ by the equations $y = zt + x^5 = 0$. Let $C_z$ be the curve cut out on $X$ by the equation $z = 0$. Then $C_z = L_{yz} + R_z$, where $R_z$ is an irreducible reduced curve given by the equations $z = xy^4 + t^2 = 0$. Let $C_t$ be the curve cut out on $X$ by the equation $t = 0$. Then $C_t = L_{xt} + R_t$, where $R_t$ is an irreducible reduced curve given by the equations $t = y^5 + x^4 z = 0$. It is easy to compute the intersection numbers of the divisors $D$, $L_{xt}$, $L_{yz}$, $R_x$, $R_y$, $R_z$, $R_t$ on $X$:

$$D \cdot L_{xt} = \frac{2}{17 \cdot 29}, \quad D \cdot R_x = \frac{4}{17 \cdot 41}, \quad D \cdot R_y = \frac{10}{29 \cdot 41},$$

$$D \cdot L_{yz} = \frac{1}{7 \cdot 41}, \quad D \cdot R_z = \frac{2}{7 \cdot 17},$$

$$D \cdot R_t = \frac{5}{7 \cdot 29}, \quad L_{xt} \cdot R_x = \frac{2}{17}, \quad L_{yz} \cdot R_y = \frac{5}{41},$$

$$L_{yz} \cdot R_z = \frac{1}{7}, \quad L_{xt} \cdot R_t = \frac{5}{29}, \quad L_{xt}^2 = -\frac{44}{17 \cdot 29},$$

$$R_x^2 = -\frac{54}{17 \cdot 41}, \quad R_y^2 = -\frac{60}{29 \cdot 41}, \quad L_{yz}^2 = -\frac{53}{14 \cdot 41},$$

$$R_z^2 = \frac{12}{7 \cdot 17}, \quad R_t^2 = \frac{135}{14 \cdot 29}.$$

We have $L_{xt} \cap R_x = \{O_y\}$, $L_{yz} \cap R_y = \{O_t\}$, $L_{yz} \cap R_z = \{O_x\}$, $L_{xt} \cap R_t = \{O_z\}$. Then

$$\mathrm{lct}\left(X, \frac{2}{17} C_y\right) = \frac{51}{10} < \min\left(\mathrm{lct}\left(X, \frac{1}{7} C_x\right), \mathrm{lct}\left(X, \frac{2}{29} C_z\right), \mathrm{lct}\left(X, \frac{2}{41} C_t\right)\right).$$

It follows that $\mathrm{lct}(X) \leqslant \frac{51}{10}$.

Put $\omega = \frac{51}{10}$. Suppose that $\mathrm{lct}(X) < \omega$. Then there is an effective $\mathbb{Q}$-divisor $D \equiv -K_X$ on $X$ such that the log pair $(X, \omega D)$ is not log canonical at some point $P \in X$.

By Remark 2.6 we may assume that the support of $D$ does not contain at least one component of each of the curves $C_x$, $C_y$, $C_z$, $C_t$. Arguing as in the proof of Theorem 5.4, we see that $P = O_t$. Put $D = aL_{yz} + bR_x + \Delta$, where $a$ and $b$ are non-negative rational numbers and $\Delta$ is an effective $\mathbb{Q}$-divisor whose support contains neither $L_{yz}$ nor $R_x$. If $a = 0$, then $\frac{1}{7} = 41(D \cdot L_{yz}) \geqslant \mathrm{mult}_{O_t}(D) > \frac{10}{51}$, a contradiction. Therefore $a > 0$. Hence we have $R_y \not\subset \mathrm{Supp}(\Delta)$. If $b > 0$, then $\frac{2}{17 \cdot 29} = D \cdot L_{xt} \geqslant b(R_x \cdot L_{xt}) = \frac{2b}{17}$, whence $b \leqslant \frac{1}{19}$. We similarly see that

$$\frac{10}{29 \cdot 41} = D \cdot R_y \geqslant \frac{5a}{41} + \frac{b}{41} + \frac{\mathrm{mult}_{O_t}(D) - a - b}{41} > \frac{4a}{41} + \frac{4}{21 \cdot 41},$$

whence $a < \frac{47}{1218}$. One has $41(\Delta \cdot L_{yz}) = \frac{2}{14} + a\frac{53}{14} - b$ and $41(\Delta \cdot R_x) = \frac{4}{17} + b\frac{54}{17} - a$. Putting $M = \frac{1}{7}$, $N = \frac{4}{17}$, $A = \frac{53}{14}$, $B = \frac{54}{17}$, $\alpha = \frac{2809}{874}$, $\beta = \frac{119}{437}$ and applying Theorem 1.4 to the log pair $(X, \omega \Delta + \omega a L_{yz} + \omega b R_x)$, we get

$$\frac{2414323}{3548440} \geqslant \alpha \omega a + \beta \omega b > 1,$$

a contradiction. $\square$

Note that every del Pezzo surface satisfying the hypotheses of at least one of Theorems 5.2–5.8 admits a Kähler–Einstein metric by Theorem 5.1.

## §6. The icosahedral group

The purpose of this section is to give an application of Theorem 1.6.

We fix embeddings of groups $\mathbb{A}_5 \cong G_1 \subset \mathrm{Aut}(\mathbb{P}^1)$, $\mathbb{A}_5 \cong G_2 \subset \mathrm{Aut}(\mathbb{P}^2)$ and the induced embedding

$$\mathbb{A}_5 \times \mathbb{A}_5 \cong G_1 \times G_2 \subset \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^2) \cong \mathbb{PSL}(2,\mathbb{C}) \times \mathbb{PSL}(3,\mathbb{C}).$$

Put $G = G_1 \times G_2$ and $X = \mathbb{P}^1 \times \mathbb{P}^2$. Let $\pi_1 \colon X \to \mathbb{P}^1$ and $\pi_2 \colon X \to \mathbb{P}^2$ be the natural projections. Then $\mathcal{P}(X,G) \supseteq \{\pi_1, \pi_2\}$, where $\mathcal{P}(X,G)$ is the $G$-pliability of $X$ (see Definition A.9). Here is the main result.

**Theorem 6.1.** *We have* $\mathcal{P}(X,G) = \{\pi_1, \pi_2\}$.

**Corollary 6.2.** *Let* $\gamma \colon X \dashrightarrow \mathbb{P}^3$ *be an arbitrary birational map. Then the subgroup* $\gamma \circ G \circ \gamma^{-1} \subset \mathrm{Bir}(\mathbb{P}^3)$ *is not conjugate to any subgroup of* $\mathbb{PSL}(3,\mathbb{C}) \cong \mathrm{Aut}(\mathbb{P}^3) \subset \mathrm{Bir}(\mathbb{P}^3)$.

**Theorem 6.3.** *There is a group isomorphism* $\mathrm{Bir}^G(X) \cong \mathbb{A}_5 \times \mathbb{S}_5$.

*Proof.* Let $\xi$ be an arbitrary $G$-equivariant birational automorphism of $X$. Using the notation in the proof of Theorem 6.1, we put $\overline{X} = X$, $\overline{\pi} = \pi_1$ and $\nu = \xi$. Arguing as in the proof of Theorem 6.1 (see below), we obtain a commutative diagram

$$\begin{array}{ccc} X & \dashrightarrow^{\ \xi\ } & X \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ \mathbb{P}^1 & \xrightarrow{\ \sigma\ } & \mathbb{P}^1 \end{array}$$

where $\sigma \in \mathrm{Aut}^{G_1}(\mathbb{P}^1)$. Then it follows from Remark A.26 and Theorem B.10 that either $\xi$ induces an isomorphism of the generic fibre of $\pi_1$, or $\xi \circ \tau$ induces an isomorphism of the generic fibre of $\pi_1$ for some birational automorphism $\tau \in \mathrm{Bir}^{G_2}(\mathbb{P}^2)$. It follows that either $\xi$ is biregular or $\xi \circ \tau$ is biregular (see [14], Theorem 1.5, and [15], Example 5.4). Then

$$\mathrm{Bir}^G(X) \cong \mathrm{Aut}^{G_1}(\mathbb{P}^1) \times \mathrm{Bir}^{G_2}(\mathbb{P}^2).$$

Moreover, $\mathrm{Aut}^{G_1}(\mathbb{P}^1) \cong \mathbb{A}_5$ and it follows from Theorem B.10 that $\mathrm{Bir}^{G_2}(\mathbb{P}^2) \cong \mathbb{S}_5$. $\square$

*Proof of Theorem 6.1.* We suppose that $\mathcal{P}(X,G) \neq \{\pi_1, \pi_2\}$ and derive a contradiction. One can find a $G$-Mori fibration $\overline{\pi} \colon \overline{X} \to \overline{S}$ which is not square birationally equivalent to $\pi_1$ or $\pi_2$ but admits a $G$-equivariant birational map $\nu \colon \overline{X} \dashrightarrow X$. In the case when $\dim(\overline{S}) \neq 0$, we put $\mathcal{M}_{\overline{X}} = |\overline{\pi}^*(D)|$ for some very ample divisor $D$ on $\overline{S}$. In the case when $\dim(\overline{S}) = 0$, we put $\mathcal{M}_{\overline{X}} = |-mK_{\overline{X}}|$ for a sufficiently large and divisible positive integer $m$. Put $\mathcal{M}_X = \nu(\mathcal{M}_{\overline{X}})$.

**Lemma 6.4.** *For each of* $i \in \{1,2\}$ *there is a* $\lambda_i \in \mathbb{Q}$ *such that* $\lambda_i > 0$ *and* $K_X + \lambda_i \mathcal{M}_X \equiv \pi_i^*(H_i)$, *where* $H_i$ *is a* $\mathbb{Q}$-*divisor on* $\mathbb{P}^i$.

*Proof.* The existence of such positive rational numbers $\lambda_1$ and $\lambda_2$ is obvious in the case when $\dim(\overline{S}) = 0$. Hence we may assume that either $\overline{S} \cong \mathbb{P}^1$ or $\dim(\overline{S}) = 2$.

The fibration $\overline{\pi}$ is not square birational to $\pi_1$. Thus the existence of $\lambda_1$ is obvious. The fibration $\overline{\pi}$ is not birationally equivalent to $\pi_2$, whence $\lambda_2$ exists in the case when $\dim(\overline{S}) = 2$. Therefore we may assume that $\dim(\overline{S}) = 1$. Then $\overline{S} \cong \mathbb{P}^1$.

Suppose that $\lambda_2$ does not exist. Then there is a commutative diagram

$$
\begin{array}{ccc}
\overline{X} & \dashrightarrow^{\nu} & X \\
{\scriptstyle\overline{\pi}}\downarrow & & \downarrow{\scriptstyle\pi_2} \\
\mathbb{P}^1 & \dashleftarrow_{\zeta} & \mathbb{P}^2
\end{array}
$$

where $\zeta$ is a rational dominant map. The normalization of a general fibre of $\zeta$ is a rational curve since the general fibre of $\overline{\pi}$ is a rational surface. On the other hand, the map $\zeta$ is $G_2$-equivariant. This contradicts Theorem B.10, which says that $\mathbb{P}^2$ is $G_2$-birationally rigid. $\square$

Let $D$ be a general fibre of $\pi_1$. Then it follows from Theorems A.15, B.10 that either the log pair $(D, \lambda_1 \mathcal{M}_X|_D)$ has canonical singularities, or there is a birational involution $\tau \in \mathrm{Bir}^G(X)$ such that the log pair $(D, \lambda_1' \tau(\mathcal{M}_X)|_D)$ has canonical singularities, where $\lambda_1' \in \mathbb{Q}$ is such that $K_X + \lambda_1' \tau(\mathcal{M}_X) \equiv \pi_1^*(H_1')$ for some $\mathbb{Q}$-divisor $H_1'$ on $\mathbb{P}^1$.

**Corollary 6.5.** *We may assume that $(D, \lambda_1 \mathcal{M}_X|_D)$ has canonical singularities.*

There is a commutative diagram

$$
\begin{array}{ccc}
 & W & \\
{\scriptstyle\alpha}\swarrow & & \searrow{\scriptstyle\beta} \\
X & \dashleftarrow_{\nu} & \overline{X}
\end{array}
$$

where $\alpha$ and $\beta$ are $G$-equivariant birational morphisms and $W$ is a smooth variety. Let $\mathcal{M}_W$ be the proper transform of $\mathcal{M}_{\overline{X}}$ on $W$. Take a rational number $\varepsilon$. Then

$$
\alpha^*(K_X + \varepsilon\mathcal{M}_X) + \sum_{i=1}^{k} a_i^\varepsilon F_i \equiv K_W + \varepsilon\mathcal{M}_W \equiv \beta^*(K_{\overline{X}} + \varepsilon\mathcal{M}_{\overline{X}}) + \sum_{i=1}^{r} b_i^\varepsilon E_i,
$$

where $a_i^\varepsilon$ is a rational number, $b_i^\varepsilon$ is a positive rational number, $F_i$ is an exceptional prime divisor for the morphism $\alpha$, and $E_i$ is an exceptional prime divisor of the morphism $\beta$.

**Lemma 6.6.** *We have $\lambda_1 > \lambda_2$.*

*Proof.* Suppose that $\lambda_2 \geqslant \lambda_1$. Then the divisor $H_2$ is $\mathbb{Q}$-effective and numerically effective. By Lemmas A.13, A.14 we may assume that there is an $l \in \{1, \ldots, k\}$ such that $F_1, \ldots, F_l$ are all $\alpha$-exceptional divisors which are not exceptional divisors for $\beta$, and all the rational numbers $a_1^{\lambda_2}, \ldots, a_l^{\lambda_2}$ are negative. Put $Z_i = \alpha(F_i)$ for every $i \in \{1, \ldots, r\}$. Then $\pi_2(Z_i)$ is either an irreducible curve or a closed point.

Suppose that $\dim(\pi_2(Z_i)) = 1$. Let $\Gamma_i$ be the fibre of $\pi_2$ over a generic point of $\pi_2(Z_i)$. Then

$$2 = \lambda_2 \mathcal{M}_X \cdot \Gamma_i \geqslant \lambda_2 \operatorname{mult}_{Z_i}(\mathcal{M}_X)|Z_i \cap \Gamma_i| \geqslant 12\lambda_2 \operatorname{mult}_{Z_i}(\mathcal{M}_X)$$

because each $G_1$-orbit in $\mathbb{P}^1$ consists of at least 12 points (see [16]). It follows that $i > l$.

Thus we have shown that $\pi_2(Z_i)$ is a closed point for every $i \in \{1, \ldots, l\}$. It follows from Proposition 1 in [17] that there is a commutative diagram
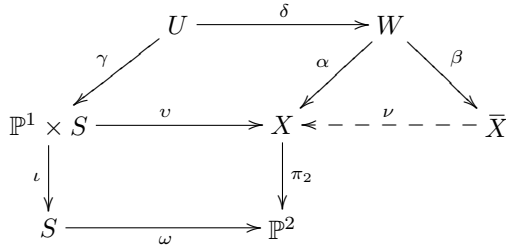


where $S$ is a smooth surface, $\omega$ is a $G_2$-invariant birational morphism, $\upsilon$ is the induced birational morphism, $U$ is a smooth threefold, $\gamma$ and $\delta$ are $G$-invariant birational morphisms, and $\dim(\iota \circ \gamma(\overline{F}_i)) = 1$ for every $i \in \{1, \ldots, k\}$, where $\overline{F}_i$ is the proper transform of $F_i$ on $U$. We put $V = \mathbb{P}^1 \times S$.

Let $\mathcal{M}_U$ and $\mathcal{M}_V$ be the proper transforms of $\mathcal{M}_X$ on the varieties $U$ and $V$ respectively. Then

$$K_U + \lambda_2 \mathcal{M}_U \equiv \gamma^*(K_V + \lambda_2 \mathcal{M}_V) + \sum_{i=1}^{k} c_i^{\lambda_2} \overline{F}_i + \sum_{i=1}^{s} d_i^{\lambda_2} B_i,$$

where $c_i^{\lambda_2}$ and $d_i^{\lambda_2}$ are rational numbers and $B_i$ is a $\delta$-exceptional divisor. Note that $\overline{F}_i$ need not be exceptional for $\gamma$. Namely, if $\overline{F}_i$ is not $\gamma$-exceptional, then $c_i^{\lambda_2} = 0$.

Let $R$ be the $G_2$-invariant divisor on $S$ such that $K_V + \lambda_2 \mathcal{M}_V \equiv \iota^*(R)$. We have $D \cong \mathbb{P}^2$. Identifying $S$ with the proper transform of $D$ on $V$, we have $R \equiv K_S + \lambda_2 \mathcal{M}_V|_S$, and the linear system $\mathcal{M}_V|_S$ is the proper transform of $\mathcal{M}_X|_D$. On the other hand, $\kappa(D, \lambda_2 \mathcal{M}_X|_D) \geqslant 0$ because the singularities of the log pair $(D, \lambda_1 \mathcal{M}_X|_D)$ are canonical and $\lambda_2 \geqslant \lambda_1$. Then the divisor $R$ is $\mathbb{Q}$-effective.

By Lemmas A.13, A.14 we have $c_t^{\lambda_2} < 0$ for some $t \in \{1, \ldots, l\}$. On the other hand, we have $\dim(\iota \circ \gamma(\overline{F}_t)) = 1$. Arguing as in the case when $\dim(\pi_2(Z_i)) = 1$, we arrive at a contradiction. $\square$

We thus see that $H_1 \equiv rF$ for some positive rational number $r$ because $\lambda_2 < \lambda_1$. Then

$$K_W + \lambda_1 \mathcal{M}_W \equiv \alpha^*(rD) + \sum_{i=1}^{k} a_i^{\lambda_1} F_i.$$

We put $a_i = a_i^{\lambda_1}$ for every $i \in \{1, \ldots, k\}$ and define

$$\mathcal{J} = \{P \in \mathbb{P}^1 \mid \exists\, i \in \{1, \ldots, k\} \colon P \in \pi_1 \circ \alpha(F_i),\ \pi_1 \circ \alpha(F_i) \neq \mathbb{P}^1,\ a_i < 0\}.$$

Let $D_\lambda$ be the fibre of $\pi_1$ over the point $\lambda \in \mathbb{P}^1$. Then $D_\lambda \cong \mathbb{P}^2$ and we have $\alpha^*(D_\lambda) \equiv \overline{D}_\lambda + \sum_{i=1}^{k} b_i^\lambda F_i$, where $\overline{D}_\lambda$ is the proper transform of $D_\lambda$ on $W$ and $b_i^\lambda$ is a non-negative integer. Note that $b_i^\lambda \neq 0$ if and only if there is a $\lambda \in \mathcal{J}$ with $\alpha(F_i) \subset D_\lambda$. Note also that $\mathcal{J} \neq \varnothing$ by Corollary 6.5 and Lemmas A.13, A.14. For every $\lambda \in \mathcal{J}$ we put

$$\delta_\lambda = \max\left\{-\frac{a_i}{b_i^\lambda} \,\middle|\, \alpha(F_i) \subset D_\lambda,\, a_i < 0\right\} > 0.$$

We also put $\delta_\lambda = 0$ for all $\lambda \in \mathbb{P}^1 \setminus \mathcal{J}$. Then it follows from Definition 2.7 and Corollary 6.5 that $\delta_\lambda = -\mathrm{c}_{D_\lambda}^1(X, \lambda_1 \mathcal{M}_X, D_\lambda)$.

**Lemma 6.7.** *We have* $\sum_{\lambda \in \mathcal{J}} \delta_\lambda \geqslant r$.

*Proof.* Suppose that $\sum_{\lambda \in \mathcal{J}} \delta_\lambda < r$. Then

$$K_W + \lambda_1 \mathcal{M}_W \equiv \alpha^*\left(\left(r - \sum_{\lambda \in \mathcal{J}} \delta_\lambda\right)D\right) + \sum_{\lambda \in \mathcal{J}} \delta_\lambda \overline{D}_\lambda + \sum_{i=1}^{k}(a_i + \delta_\lambda b_i^\lambda)F_i,$$

where $a_i + \delta_\lambda b_i^\lambda \geqslant 0$ for every $i \in \{1, \dots, k\}$. It follows that $\kappa(\overline{X}, \lambda_1 M_{\overline{X}}) = \kappa(X, \lambda_1 M_X) = \kappa(W, \lambda_1 M_W) = 1$. This is a contradiction because we have either $\kappa(\overline{X}, \lambda_1 M_{\overline{X}}) \leqslant 0$ or $\kappa(\overline{X}, \lambda_1 M_{\overline{X}}) = 3$. $\square$

**Corollary 6.8.** *For every set* $\{c_\lambda\}_{\lambda \in \mathcal{J}}$ *of rational numbers with* $\sum_{\lambda \in \mathcal{J}} c_\lambda \leqslant r$ *we have* $\mathbb{CS}\big(X, \lambda_1 \mathcal{M}_X - \sum_{\lambda \in \mathcal{J}} c_\lambda D_\lambda\big) \neq \varnothing$.

Let $Z \cong \mathbb{P}^1$ be a fibre of $\pi_2$, and let $C$ be a line in $D \cong \mathbb{P}^2$. Then $\mathbb{NE}(X) = \mathbb{R}_{\geqslant 0}Z \oplus \mathbb{R}_{\geqslant 0}C$ and $K_X^2 \equiv 9Z + 12C$. Let $M_1$ and $M_2$ be general divisors in $\mathcal{M}_X$. We put $T_0 = \lambda_1^2 M_1 \cdot M_2$. Then

$$T_0 = Z_X + \sum_{\lambda \in \mathbb{P}^1} C_\lambda \equiv 9Z + (12 + 6r)C,$$

where $C_\lambda$ is an effective cycle whose components lie in the fibre of $\pi_1$ over a point $\lambda \in \mathbb{P}^1$, and $Z_X$ is an effective cycle none of whose components lie in a fibre of $\pi_1$. We have $12 + 6r \leqslant 12 + 6\sum_{\lambda \in \mathcal{J}} \delta_\lambda$ by Lemma 6.7. Take $\beta_\lambda \in \mathbb{Q}_{\geqslant 0}$ such that $C_\lambda \equiv \beta_\lambda C$. Then

$$\sum_{\lambda \in \mathcal{J}} \beta_\lambda \leqslant \sum_{\lambda \in \mathbb{P}^1} \beta_\lambda \leqslant 12 + 6r \leqslant 12 + 6\sum_{\lambda \in \mathcal{J}} \delta_\lambda$$

because $Z$ and $C$ generate the cone of effective cycles $\mathbb{NE}(X)$. We denote the $G_1$-orbit of the point $\lambda \in \mathbb{P}^1$ by $O_\lambda^1$. Then

$$\sum_{O_\lambda^1,\, \lambda \in \mathcal{J}} \beta_\lambda |O_\lambda^1| \leqslant 12 + 6r \leqslant 12 + 6\sum_{O_\lambda^1,\, \lambda \in \mathcal{J}} \delta_\lambda |O_\lambda^1|,$$

whence there is a point $\omega \in \mathbb{P}^1$ such that $\beta_\omega |O_\omega^1| \leqslant 12 + 6\delta_\omega |O_\omega^1|$, where $|O_\omega^1| \geqslant 12$ (see [16]).

**Corollary 6.9.** *There is a $t \in \{1, \ldots, k\}$ such that $\alpha(F_t) \subset D_\omega$ and $\beta_\omega \leqslant 1 - 6\frac{a_t}{b_t^\omega}$, where*

$$\frac{a_t}{b_t^\omega} = -\delta_\omega = \min\left\{ \frac{a_i}{b_i^\omega} \;\middle|\; \alpha(F_i) \subset D_\omega, \, a_i < 0 \right\} = c_{D_\omega}^1(X, \lambda_1 \mathcal{M}_X, D_\omega),$$

*the log pair $(X, \lambda_1 \mathcal{M}_X - \delta_\omega D_\omega)$ is canonical along $D_\omega$, but $\alpha(F_t) \in \mathbb{CS}(X, \lambda_1 \mathcal{M}_X + \delta_\omega D_\omega)$.*

Arguing as in the proof of Lemma B.13, we see that there are no $G_2$-invariant curves on $D_\omega \cong \mathbb{P}^2$ of degrees 1, 3 or 5. Moreover, there is a unique $G_2$-invariant conic $\Gamma_2 \subset D_\omega \cong \mathbb{P}^2$. $\Gamma_2$ is non-singular and irreducible. The action of the group $G_2$ on the curve $\Gamma_2 \cong \mathbb{P}^1$ induces an embedding $G_2 \subset \mathrm{Aut}(\Gamma_2) \cong \mathbb{PGL}(2, \mathbb{C})$. Every $G_2$-orbit in $D_\omega \cong \mathbb{P}^2$ consists of at least 6 points. Every $G_2$-orbit in $\Gamma_2$ consists of at least 12 points (see [16]).

**Lemma 6.10.** *We have $\dim(\alpha(F_t)) = 0$.*

*Proof.* Suppose that $\alpha(F_t)$ is an irreducible curve. Put $\Lambda = \alpha(F_t)$. Then $\mathrm{mult}_\Lambda(\mathcal{M}_X) > \frac{1}{\lambda_1}$. Replacing, if necessary, the divisor $F_t$ by its $G$-orbit, we may assume that the subvariety $\Lambda$ is a $G_2$-invariant curve. Let $L$ be a general line in $D_\omega \cong \mathbb{P}^2$. Then we have

$$\frac{3}{\lambda_1} = L \cdot \mathcal{M}_X > \frac{L \cdot \Lambda}{\lambda_1},$$

whence $\Lambda = \Gamma_2$. We put $C_\omega = m\Lambda + \Delta$, where $\Delta$ is an effective cycle such that $\Lambda \nsubseteq \mathrm{Supp}(\Delta)$. Therefore

$$m \geqslant \begin{cases} 1 + 2\delta_\omega & \text{if} \quad \delta_\omega \leqslant \frac{1}{2}, \\ 4\delta_\omega & \text{if} \quad \delta_\omega \geqslant \frac{1}{2} \end{cases}$$

by Theorem 1.6. On the other hand, it was shown above that $\beta_\omega \geqslant 2m$ and $\beta_\omega \leqslant 1 + 6\delta_\omega$ (see Corollary 6.9). We have

$$\frac{1}{2} + 3\delta_\omega \geqslant \begin{cases} 1 + 2\delta_\omega & \text{if} \quad \delta_\omega \leqslant \frac{1}{2}, \\ 4\delta_\omega & \text{if} \quad \delta_\omega \geqslant \frac{1}{2}, \end{cases}$$

whence $\delta_\omega = \frac{1}{2}$ and $m = 2$. This contradicts Theorem 1.6. $\square$

We put $P = \alpha(F_t)$ and denote the $G_2$-orbit of the point $P \in D_\omega$ by $O_P^2$.

**Lemma 6.11.** *We have $\mathrm{mult}_P(Z_X) \leqslant \frac{3}{2}$ and $\mathrm{mult}_P(C_\omega) \leqslant \frac{\beta_\omega}{2}$.*

*Proof.* Put $r = |O_P^2|$. Then $r \geqslant 6$. We have

$$9 = \left( Z_X + \sum_{\lambda \in \mathbb{P}^1} C_\lambda \right) \cdot D_\omega = Z_X \cdot D_\omega \geqslant \sum_{Q \in O_P^2} \mathrm{mult}_Q(Z_X)$$

$$= r \, \mathrm{mult}_P(Z_X) \geqslant 6 \, \mathrm{mult}_P(Z_X),$$

whence $\mathrm{mult}_P(Z_X) \leqslant \frac{3}{2}$. Let us show that $\mathrm{mult}_P(C_\omega) \leqslant \frac{\beta_\omega}{2}$.

The curve $C_\omega$ may be regarded as an effective $G_2$-invariant $\mathbb{Q}$-divisor of degree $\beta_\omega$ on $\mathbb{P}^2 \cong D_\omega$. Then $C_\omega = m\Gamma_2 + \Delta$, where $m \in \mathbb{Q}$ is such that $\frac{\beta_\omega}{2} \geqslant m \geqslant 0$, and $\Delta$ is an effective $\mathbb{Q}$-divisor whose support does not contain $\Gamma_2$.

Suppose that $P \in \Gamma_2$. Then

$$2(\beta_\omega - 2m) = \Gamma_2 \cdot \Delta \geqslant \sum_{Q \in O_P} \mathrm{mult}_Q(\Delta) = r(\mathrm{mult}_P(C_\omega) - m) \geqslant r(\mathrm{mult}_P(C_\omega) - m)$$

and $r \geqslant 12$ (see [16]). Therefore we have

$$\mathrm{mult}_P(C_\omega) \leqslant \frac{2\beta_\omega + (r-4)m}{r} \leqslant \frac{2\beta_\omega + (r-4)\beta_\omega/2}{r} = \frac{\beta_\omega}{2}$$

because $r \geqslant 4$. Thus, to complete the proof, we may assume that $P \notin \Gamma_2$.

Suppose that $\mathrm{mult}_P(C_\omega) > \frac{\beta_\omega}{2}$. Then $\mathrm{mult}_P(\Delta) > \frac{\beta_\omega}{2}$ and there is a rational number $\mu$ such that $\mathrm{mult}_P(\mu\Delta) \geqslant 2$ and $\mu < \frac{4}{\beta_\omega}$. In particular, we see that $O_P^2 \subseteq \mathrm{LCS}(\mathbb{P}^2, \mu\Delta)$.

Suppose that there is a $G_2$-invariant reduced curve $\Omega \subset \mathbb{P}^2$ such that $\mu\Delta = \upsilon\Omega + \Upsilon$, where $\upsilon \geqslant 1$ and $\Upsilon$ is an effective $\mathbb{Q}$-divisor with $\Omega \nsubseteq \mathrm{Supp}(\Upsilon)$. Then

$$4 > \mu\beta_\omega - 2m\mu = \mu\Delta \cdot H = (\upsilon\Omega + \Upsilon) \cdot H \geqslant \upsilon\Omega \cdot H \geqslant \Omega \cdot H,$$

where $H$ is a general line on $\mathbb{P}^2$. Hence $\Omega = \Gamma_2$, a contradiction.

Thus we see that the scheme $\mathcal{L}(\mathbb{P}^2, \mu\Delta)$ is zero-dimensional and its support contains $O_P^2$. By Theorem 2.14 there is an exact sequence of groups

$$\mathbb{C}^3 \cong H^0(\mathcal{O}_{\mathbb{P}^2}(1)) \longrightarrow H^0(\mathcal{O}_{\mathcal{L}(\mathbb{P}^2, \mu\Delta)}) \longrightarrow 0,$$

whence $r \leqslant 3$. On the other hand, we have shown above that $r \geqslant 6$. The resulting contradiction proves the lemma. $\square$

**Lemma 6.12.** *We have* $\mathrm{mult}_P(\mathcal{M}_X) < \frac{2+\delta_\omega}{\lambda_1}$.

*Proof.* Suppose that $\mathrm{mult}_P(\mathcal{M}_X) \geqslant \frac{2+\delta_\omega}{\lambda_1}$. Then

$$\frac{\beta_\omega}{2} \geqslant \mathrm{mult}_P(C_\omega) \geqslant \lambda_1^2 \mathrm{mult}_P(M_1 \cdot M_2) - \mathrm{mult}_P(Z_X) \geqslant \lambda_1^2 \mathrm{mult}_P^2(\mathcal{M}_X) - \frac{3}{2}$$

$$\geqslant \delta_\omega^2 + 2\delta_\omega + \frac{5}{2}$$

by Lemma 6.11. But $\beta_\omega \leqslant 1 + 6\delta_\omega$ by Corollary 6.9, a contradiction. $\square$

Put $X_0 = X$ and $\Theta_0 = P$. Then one can find positive integers $N$ and $K$, $N \geqslant K \geqslant 2$, such that there is a sequence of blow-ups

$$X_N \xrightarrow{\psi_{N,N-1}} X_{N-1} \xrightarrow{\psi_{N-1,N-2}} \cdots \xrightarrow{\psi_{3,2}} X_2 \xrightarrow{\psi_{2,1}} X_1 \xrightarrow{\psi_{1,0}} X_0$$

with the following properties. The morphism $\psi_{1,0}$ blows up the point $\Theta_0$ and, for every $i \in \{2, \ldots, K\}$, $\psi_{i,i-1}$ blows up a point $\Theta_{i-1} \in G_{i-1} \cong \mathbb{P}^2$, where $G_{i-1}$ is the

exceptional divisor for $\psi_{i-1,i-2}$. The morphism $\psi_{K+1,K}$ blows up an irreducible curve $\Theta_K \subset G_K \cong \mathbb{P}^2$ and, for every $i \in \{K+2, \ldots, N\}$, $\psi_{i,i-1}$ blows up an irreducible curve $\Theta_{i-1} \subset G_{i-1}$ with $\psi_{i-1,i-2}(\Theta_{i-1}) = \Theta_{i-1}$, where $G_{i-1}$ is the exceptional divisor of $\psi_{i-1,i-2}$. The divisors $G_N$ and $F_t$ induce the same discrete valuation of the field of rational functions on $X$. For $N \geqslant j \geqslant i \geqslant 0$ we denote the proper transforms of $G_i$, $\mathcal{M}_X$ and $D_\omega$ on the variety $X_j$ by $G_i^j$, $\mathcal{M}_{X_j}$ and $D_\omega^j$ respectively and put $\psi_{j,i} = \psi_{i+1,i} \circ \cdots \circ \psi_{j,j-1}$, where $\psi_{j,j} = \mathrm{id}_{X_j}$. Then

$$K_{X_N} + \lambda_1 \mathcal{M}_{X_N} \equiv \psi_{N,0}^*(rD_\omega) + \sum_{i=1}^{N} c_i G_i^N,$$

where $c_i \in \mathbb{Q}$ and $c_N = a_t$. We similarly have $\psi_{N,0}^*(D_\omega) \equiv D_\omega^N + \sum_{i=1}^{N} d_i G_i^N$, where $d_i \in \mathbb{N}$ and $d_N = b_t^\omega$. Note that $c_N < 0$ and $\delta_\omega = -\frac{c_N}{d_N}$. We may assume that $-\delta_\omega = \frac{a_t}{b_t^\omega} = \frac{c_N}{d_N} < \frac{c_i}{d_i}$ for every $i < N$. The curves $\Theta_i$, $i \geqslant K$, might *a priori* be singular. But Lemma 6.12 yields that $\Theta_K$ is a line in $G_K \cong \mathbb{P}^2$ and, for every $i > K$, $\Theta_i$ is a section of the induced morphism $\psi_{i-1,i-2}|_{G_{i-1}} \colon G_{i-1} \to \Theta_{i-2} \cong \mathbb{P}^1$, whence, in particular, $\Theta_i \cong \mathbb{P}^1$ for $i \geqslant K$.

Let $\Gamma$ be a directed graph whose set of vertices consists of the exceptional divisors $G_1, \ldots, G_N$ and the set of edges is defined by the formulae

$$(G_j, G_i) \in \Gamma \quad \Longleftrightarrow \quad j > i, \; \Theta_{j-1} \subset G_i^{j-1} \subset X_{j-1},$$

where $(G_j, G_i)$ is an edge from the vertex $G_j$ to the vertex $G_i$. Let $P_i$ be the number of directed paths from $G_N$ to $G_i$ in $\Gamma$. Then

$$c_N = \sum_{i=1}^{K} P_i(2 - \nu_i) + \sum_{i=K+1}^{N} P_i(1 - \nu_i),$$

where $\nu_i = \lambda_1 \mathrm{mult}_{P_{i-1}}(\mathcal{M}_{X_{i-1}})$. Note that $(G_i, G_{i-1}) \in \Gamma$ for every $i \in \{1, \ldots, N\}$. Put $\Sigma_0 = \sum_{i=1}^{K} P_i$ and $\Sigma_1 = \sum_{i=K+1}^{N} P_i$. Let $M$ be the largest positive integer such that $M \leqslant K$ and $P_{M-1} \in F_\omega^{M-1}$. We put $\Sigma_0' = \sum_{i=1}^{M} P_i$. Then

$$\mathrm{mult}_P(Z_X)\Sigma_0 + \mathrm{mult}_P(C_\omega)\Sigma_0' \geqslant \frac{(2\Sigma_0 + \Sigma_1 - c_N)^2}{\Sigma_0 + \Sigma_1}$$

by [18]. Note that $d_N \geqslant \Sigma_0' \leqslant \Sigma_0$. On the other hand,

$$\mathrm{mult}_P(C_\omega) \leqslant \frac{\beta_\omega}{2} \leqslant \frac{1}{2} + 3\delta_\omega = \frac{1}{2} - 3\frac{c_N}{d_N}$$

by Lemma 6.11 and Corollary 6.9. Therefore we see that

$$\left(\mathrm{mult}_P(Z_X) + \frac{1}{2}\right)\Sigma_0 - 3c_N \geqslant \frac{(2\Sigma_0 + \Sigma_1 - c_N)^2}{\Sigma_0 + \Sigma_1},$$

where $c_N < 0$ and $\mathrm{mult}_P(Z_X) \leqslant \frac{3}{2}$. Thus $2\Sigma_0 - 3c_N \geqslant \frac{(2\Sigma_0 + \Sigma_1 - c_N)^2}{\Sigma_0 + \Sigma_1}$.

**Lemma 6.13.** *We have* $\mathrm{mult}_P(Z_X) \leqslant \frac{3}{4}$.

*Proof.* Since the log pair $(X, \lambda_1 \mathcal{M}_X)$ is not log canonical at the point $P \in D_\omega$, the log pair $(X, \lambda_1 \mathcal{M}_X + D_\omega)$ is not log canonical at $P$. Hence it follows from Theorem 2.17 that $(D_\omega, \lambda_1 \mathcal{M}_X|_{D_\omega})$ is not log canonical at $P$. There is a rational number $\mu$ such that $0 < \mu < \lambda_1$ and the log pair $(D_\omega, \mu \mathcal{M}_X|_{D_\omega})$ is not log canonical at $P$. Then $\mathrm{LCS}(D_\omega, \mu \mathcal{M}_X|_{D_\omega})$ is connected by Theorem 2.16. Arguing as in the proof of Lemma 6.11, we see that either the subscheme $\mathcal{L}(D_\omega, \lambda_1 \mathcal{M}_X|_{D_\omega})$ is zero-dimensional or $P \in \Gamma_2$. If $\mathcal{L}(D_\omega, \lambda_1 \mathcal{M}_X|_{D_\omega})$ is zero-dimensional, then $|O_P^2| = 1$ by connectedness. This contradicts the inequality $|O_P^2| \geqslant 6$. Hence we see that $P \in \Gamma_2$. Then $|O_P^2| \geqslant 12$ (see [16]). We have

$$9 = \left( Z_X + \sum_{\lambda \in \mathbb{P}^1} C_\lambda \right) \cdot D_\omega$$

$$= Z_X \cdot D_\omega \geqslant \sum_{Q \in O_P^2} \mathrm{mult}_Q(Z_X) = |O_P^2| \, \mathrm{mult}_P(Z_X) \geqslant 12 \, \mathrm{mult}_P(Z_X),$$

whence $\mathrm{mult}_P(Z_X) \leqslant \frac{3}{4}$. $\square$

Thus we see that $c_N < 0$, $\Sigma_0 \geqslant 1$, $\Sigma_1 \geqslant 1$ and

$$\frac{5}{4} \Sigma_0 - 3 c_N \geqslant \frac{(2\Sigma_0 + \Sigma_1 - c_N)^2}{\Sigma_0 + \Sigma_1}.$$

This is easily seen to be a contradiction. Theorem 6.1 is proved.

## Appendix A. Non-rationality

Let $X$ be a variety, $\pi \colon X \to S$ a morphism, and $G$ a finite subgroup of $\mathrm{Aut}(X)$.

**Definition A.1.** The morphism $\pi$ is called a *G-Mori fibration* if the following conditions hold. The morphism $\pi$ is $G$-equivariant, the variety $X$ has terminal singularities, every $G$-invariant Weil divisor on $X$ is a $\mathbb{Q}$-Cartier divisor, $\pi$ is surjective and $\pi_*(\mathcal{O}_X) = \mathcal{O}_S$, the divisor $-K_X$ is $\pi$-ample, $\dim(S) < \dim(X)$ and $\dim_{\mathbb{Q}}(\mathrm{Pic}^G(X/S) \otimes \mathbb{Q}) = 1$.

Suppose that $\pi$ is a $G$-Mori fibration. Then $X$ is rationally connected if and only if $S$ is rationally connected (see [19], [20]).

*Remark A.2.* $G$ acts naturally on $S$, but this action need not be faithful. One can show that every $G$-invariant Weil divisor on $S$ is a $\mathbb{Q}$-Cartier divisor (see [21]).

Suppose additionally that $X$ is rationally connected. Then

$$\dim_{\mathbb{Q}}(\mathrm{Pic}^G(X) \otimes \mathbb{Q}) = \dim_{\mathbb{Q}}(\mathrm{Pic}^G(S) \otimes \mathbb{Q}) + 1.$$

**Definition A.3.** The fibration $\pi$ is *G-birationally rigid* if, given any $G$-equivariant birational map $\xi \colon X \dashrightarrow X'$ to a $G$-Mori fibration $\pi' \colon X' \to S'$, there is a commutative diagram

$$
\begin{array}{ccccc}
X & \dashrightarrow^{\rho} & X & \dashrightarrow^{\xi} & X' \\
\downarrow{\scriptstyle \pi} & & & & \downarrow{\scriptstyle \pi'} \\
S & \dashrightarrow & & {\scriptstyle \sigma} & S'
\end{array}
$$

where $\sigma$ is a birational map and $\rho \in \mathrm{Bir}^G(X)$ is such that the rational map $\xi \circ \rho$ induces an isomorphism of generic fibres of the $G$-Mori fibrations $\pi$ and $\pi'$.

**Definition A.4.** The fibration $\pi$ is $G$-*birationally superrigid* if, given any $G$-equivariant birational map $\xi\colon X \dashrightarrow X'$ to a $G$-Mori fibration $\pi'\colon X' \to S'$, there is a commutative diagram

$$
\begin{array}{ccc}
X & \overset{\xi}{\dashrightarrow} & X' \\
{\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi'} \\
S & \underset{\sigma}{\dashrightarrow} & S'
\end{array}
$$

where $\sigma$ is a birational map and $\xi$ induces an isomorphism of generic fibres of the $G$-Mori fibrations $\pi$ and $\pi'$.

For simplicity we say that $X$ is $G$-*birationally rigid* (resp. $G$-*birationally superrigid*) if $\dim(S) = 0$ and $\pi\colon X \to S$ is $G$-birationally rigid (resp. $G$-birationally superrigid).

*Remark A*.5. Suppose that $\dim(S) = 0$. Then $X$ is $G$-birationally superrigid if and only if $X$ is $G$-birationally rigid and $\mathrm{Bir}^G(X) = \mathrm{Aut}^G(X)$.

We also say that a fibration $\pi\colon X \to S$ is *birationally rigid* (resp. *birationally superrigid*) if it is $G$-birationally rigid (resp. $G$-birationally superrigid) when $G$ is the trivial group.

**Example A.6.** It was shown in [18] that $\pi\colon X \to S$ is birationally rigid if the following conditions hold. The variety $X$ is smooth, $\dim(X) = 3$, $\dim(S) = 1$, $K_X^2$ is not an interior point of the closed cone of effective cycles on $X$ and $K_X^2 \cdot F \leqslant 2$, where $F$ is a general fibre of $\pi$.

We say that $X$ is *birationally rigid* (resp. *birationally superrigid*) if $\dim(S) = 0$ and the fibration $\pi\colon X \to S$ is birationally rigid (resp. birationally superrigid).

**Example A.7.** It was shown in [22] that a general hypersurface of degree $n \geqslant 4$ in $\mathbb{P}^n$ is birationally superrigid.

It follows from Definition A.3 that if the $G$-Mori fibration $\pi\colon X \to S$ is $G$-birationally superrigid and $X \not\cong \mathbb{P}^n$, then there is no $G$-equivariant birational map $X \dashrightarrow \mathbb{P}^n$, where $n = \dim(X)$.

**Definition A.8.** The fibration $\pi$ is *square birationally equivalent* to a $G$-Mori fibration $\pi'\colon X' \to S'$ if there is a commutative diagram

$$
\begin{array}{ccc}
X & \overset{\xi}{\dashrightarrow} & X' \\
{\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi'} \\
S & \underset{\sigma}{\dashrightarrow} & S'
\end{array}
$$

where $\sigma$ is a birational map and $\xi$ is a $G$-equivariant birational map that induces an isomorphism of generic fibres of the $G$-Mori fibrations $\pi$ and $\pi'$.

The following definition was introduced in [23].

**Definition A.9.** Let $V$ be a variety and let $\Gamma$ be a finite subgroup of $\mathrm{Aut}(V)$. Then the set

$$\mathcal{P}(V, G) = \left\{ \tau \colon Y \to T \;\middle|\; \begin{matrix} \text{there is a } G\text{-equivariant} \\ \text{birational map } Y \dashrightarrow V \end{matrix} \right\} / \bigstar$$

where $\tau$ is a $G$-Mori fibration, is called the $G$-*pliability* of $V$. Here the equivalence relation $\bigstar$ is determined by the square birational equivalence of $G$-Mori fibrations.

We put $\mathcal{P}(V) = \mathcal{P}(V, G)$ if $G$ is trivial. The following conditions are equivalent: the fibration $\pi \colon X \to S$ is $G$-birationally rigid, the set $\mathcal{P}(X, G)$ consists of the fibration $\pi \colon X \to S$, we have $|\mathcal{P}(X, G)| = 1$.

*Remark A*.10. In the notation and under hypotheses of Definition A.9, it follows from [9] that the following conditions are equivalent: $\mathcal{P}(V, \Gamma) \neq \varnothing$, the divisor $-K_V$ is not pseudo-effective, the variety $V$ is uniruled.

**Example A.11.** Let $X$ be a quartic in $\mathbb{P}^4$ given by the equation

$$w^2 x^2 + wyz + xg_3(y, z, t, w) + g_4(y, z, t, w) = 0 \subset \mathbb{P}^4 \cong \mathrm{Proj}(\mathbb{C}[x, y, z, t, w]),$$

where $g_i$ is a sufficiently general homogeneous polynomial of degree $i$. It was shown in [23] that $|\mathcal{P}(X)| = 2$.

Let $\overline{\pi} \colon \overline{X} \to \overline{S}$ be a $G$-Mori fibration such that there is a $G$-equivariant birational map $\nu \colon \overline{X} \dashrightarrow X$. If $\dim(\overline{S}) \neq 0$, then we put $\mathcal{M}_{\overline{X}} = |\overline{\pi}^*(D)|$ for an arbitrary very ample divisor $D$ (whose class in $\mathrm{Pic}(\overline{S})$ is $G$-invariant) on $\overline{S}$. If $\dim(\overline{S}) = 0$, then we put $\mathcal{M}_{\overline{X}} = |-mK_{\overline{X}}|$ for a sufficiently large and sufficiently divisible positive integer $m$. Put $\mathcal{M}_X = \nu(\mathcal{M}_{\overline{X}})$.

**Lemma A.12.** *Suppose that* $\dim(\overline{S}) \neq 0$. *Then either there is a commutative diagram*

$$\begin{array}{ccc} \overline{X} & \dashrightarrow{\;\nu\;} & X \\ {\scriptstyle\overline{\pi}}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ \overline{S} & \dashleftarrow[\zeta] & S \end{array} \tag{A.1}$$

*where $\zeta$' is a rational dominant map, or there is a $\lambda \in \mathbb{Q}$ such that $K_X + \lambda \mathcal{M}_X \equiv \pi^*(H)$, where $H$ is a $G$-invariant $\mathbb{Q}$-divisor on $S$ such that either the log pair $(X, \lambda \mathcal{M}_X)$ is not canonical or the divisor $H$ is not $\mathbb{Q}$-effective.*

*Proof.* Note that the commutative diagram (A.1) exists if and only if the linear system $\mathcal{M}_X$ lies in the fibres of $\pi$. Hence we may assume that $\mathcal{M}_X$ does not lie in the fibres of $\pi$. Then there is a $\lambda \in \mathbb{Q}$ such that $K_X + \lambda \mathcal{M}_X \equiv \pi^*(H)$, where $H$ is a $G$-invariant $\mathbb{Q}$-divisor on $S$. By Lemma 2.23, $\kappa(K_X + \lambda \mathcal{M}_X) = -\infty$. But we have $\kappa(K_X + \lambda \mathcal{M}_X) = 0$ in the case when $(X, \lambda \mathcal{M}_X)$ is canonical and $H$ is $\mathbb{Q}$-effective (see Corollary 2.24). $\square$

Let $\varepsilon$ be a positive rational number. There is a commutative diagram

$$
\begin{array}{ccc}
 & W & \\
{\scriptstyle\alpha}\swarrow & & \searrow{\scriptstyle\beta} \\
X & \xleftarrow{\ \ \nu\ \ } & \overline{X}
\end{array}
$$

where $\alpha$ and $\beta$ are $G$-equivariant birational morphisms and $W$ is a smooth variety. Let $\mathcal{M}_W$ be the proper transform of $\mathcal{M}_{\overline{X}}$ on $W$. Then

$$
\alpha^*(K_X + \varepsilon\mathcal{M}_X) + \sum_{i=1}^{k} a_i^\varepsilon F_i \equiv K_W + \varepsilon\mathcal{M}_W \equiv \beta^*(K_{\overline{X}} + \varepsilon\mathcal{M}_{\overline{X}}) + \sum_{i=1}^{r} b_i^\varepsilon E_i,
$$

where $a_i^\varepsilon$ is a rational number, $b_i^\varepsilon$ is a positive rational number, and $F_i$ and $E_i$ stand for the $G$-orbits of exceptional prime divisors for the morphisms $\alpha$ and $\beta$ respectively.

**Lemma A.13.** *Suppose that* $\dim(\overline{S}) \neq 0$. *Then either there is a commutative diagram*

$$
\begin{array}{ccc}
\overline{X} & \dashrightarrow^{\ \nu\ } & X \\
{\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\
\overline{S} & \xleftarrow{\ \ \zeta\ \ } & S
\end{array}
$$

*where $\zeta$ is a rational dominant map, or there is a $\lambda \in \mathbb{Q}$ such that $K_X + \lambda\mathcal{M}_X \equiv \pi^*(H)$, where $H$ is a $G$-invariant $\mathbb{Q}$-divisor on $S$ such that either there is an $i \in \{1, \ldots, k\}$ for which $a_i^\lambda < 0$ and $F_i$ is not $\beta$-exceptional, or the divisor $H$ is not $\mathbb{Q}$-effective.*

*Proof.* Arguing as in the proof of Lemma A.12, we may assume that there is a $\lambda \in \mathbb{Q}$ such that $K_X + \lambda\mathcal{M}_X \equiv \pi^*(H)$, where $H$ is a $G$-invariant $\mathbb{Q}$-divisor on $S$. Suppose that $H$ is not $\mathbb{Q}$-effective and we have $a_i^\lambda \geqslant 0$ for all $i \in \{1, \ldots, k\}$ such that $F_i$ is not $\beta$-exceptional. Then

$$
\beta\left((\pi \circ \alpha)^*(H) + \sum_{i=1}^{k} a_i^\lambda F_i\right) \equiv \beta\left((\pi \circ \alpha)^*(H) + \sum_{a_i^\lambda \geqslant 0} a_i^\lambda F_i\right) \equiv K_{\overline{X}} + \lambda\mathcal{M}_{\overline{X}}, \ \ (\text{A.2})
$$

whence $K_{\overline{X}} + \lambda\mathcal{M}_{\overline{X}}$ is $\mathbb{Q}$-effective, a contradiction. $\square$

We note that Lemma A.13 is an analogue of Proposition 2 in [17].

**Lemma A.14.** *Suppose that* $\dim(\overline{S}) = 0$. *Then there is a $\lambda \in \mathbb{Q}$ such that $K_X + \lambda\mathcal{M}_X \equiv \pi^*(H)$, where $H$ is a $G$-invariant $\mathbb{Q}$-divisor (whose class in $\mathrm{Pic}(S)$ is $G$-invariant) on $S$. Moreover, either $\dim(S) = 0$ and $\nu$ is an isomorphism. or there is an $i \in \{1, \ldots, k\}$ such that $a_i^\lambda < 0$ and $F_i$ is not $\beta$-exceptional, or else the divisor $H$ is not $\mathbb{Q}$-effective.*

*Proof.* Recall that $\mathcal{M}_{\overline{X}} = |-mK_{\overline{X}}|$, where $m$ is a sufficiently large and sufficiently divisible positive integer. Clearly, $\mathcal{M}_X$ does not lie in the fibres of $\pi$. In particular, there is a $\lambda \in \mathbb{Q}$ such that $K_X + \lambda \mathcal{M}_X \equiv \pi^*(H)$, where $H$ is a $G$-invariant $\mathbb{Q}$-divisor on $S$. Then

$$\dim(S) \geqslant \kappa(X, \lambda \mathcal{M}_X) = \kappa(\overline{X}, \lambda \mathcal{M}_{\overline{X}}) = \begin{cases} \dim(\overline{X}) & \text{if } \lambda > \frac{1}{m}, \\ 0 & \text{if } \lambda = \frac{1}{m}, \\ -\infty & \text{if } \mu < \frac{1}{m}, \end{cases}$$

by Lemma 2.23. Thus we see that $\lambda \leqslant \frac{1}{m}$.

Suppose that $H$ is $\mathbb{Q}$-effective and $a_i^\lambda \geqslant 0$ for every $i \in \{1, \ldots, k\}$ such that $F_i$ is not $\beta$-exceptional. Then the numerical equivalence (A.2) holds and, therefore, $K_{\overline{X}} + \lambda \mathcal{M}_{\overline{X}}$ is $\mathbb{Q}$-effective. Thus we see that $\lambda = \frac{1}{m}$. It follows that

$$(\pi \circ \alpha)^*(H) + \sum_{i=1}^{k} a_i^\lambda F_i \equiv \sum_{i=1}^{r} b_i^\lambda E_i.$$

Suppose that $\dim(S) = 0$. It follows from Lemma 2.19 in [24] that $\sum_{i=1}^{k} a_i^\lambda F_i = \sum_{i=1}^{r} b_i^\lambda E_i$, whence the log pair $(X, \lambda \mathcal{M}_X)$ is terminal. Therefore $\nu$ is an isomorphism by Theorem 2.27.

To complete the proof, we may assume that $\dim(S) \neq 0$. Then

$$\dim_{\mathbb{Q}}(\text{Pic}^G(\overline{S}) \otimes \mathbb{Q}) + 1 + k = \dim_{\mathbb{Q}}(\text{Pic}^G(X) \otimes \mathbb{Q}) + k = \dim_{\mathbb{Q}}(\text{Pic}^G(W) \otimes \mathbb{Q}) = 1 + r,$$

whence $k \leqslant r - 1$. For every $i \in \{1, \ldots, r\}$, we see that either $E_i$ is $\alpha$-exceptional or $\alpha(E_i)$ is a divisor that lies in the fibres of $\pi$.

Suppose that $F_1$ is not $\beta$-exceptional. Then $\langle F_1, E_1, \ldots, E_r \rangle = \text{Pic}^G(W) \otimes \mathbb{Q}$, where $\langle F_1, E_1, \ldots, E_r \rangle$ is the linear span of $F_1, E_1, \ldots, E_r$. We have

$$\dim_{\mathbb{Q}}(\langle E_1, \ldots, E_r \rangle \cap \langle F_1, \ldots, F_k \rangle) = k - 1$$

since the divisors $F_1, \ldots, F_k$ are linearly independent in $\text{Pic}^G(\overline{W}) \otimes \mathbb{Q}$. Therefore,

$$\dim_{\mathbb{Q}}(\langle \alpha(E_1), \ldots, \alpha(E_r) \rangle) = r - k + 1 = \dim_{\mathbb{Q}}(\text{Pic}^G(X) \otimes \mathbb{Q}),$$

where we assume that $\alpha(E_i) = 0$ in the case when $E_i$ is $\alpha$-exceptional. On the other hand, $\langle \alpha(E_1), \ldots, \alpha(E_r) \rangle \neq \text{Pic}^G(X) \otimes \mathbb{Q}$ because $\alpha(E_i)$ lies in the fibres of $\pi$ if $E_i$ is not $\alpha$-exceptional.

Thus we see that $F_1$ is $\beta$-exceptional. We similarly see that all the divisors $F_1, \ldots, F_k$ are $\beta$-exceptional. There is no loss of generality in assuming that $F_i = E_i$ for every $i \in \{1, \ldots, k\}$. Then $\dim_{\mathbb{Q}}(\langle \alpha(E_{k+1}), \ldots, \alpha(E_r) \rangle) = r - k = \dim_{\mathbb{Q}}(\text{Pic}^G(S) \otimes \mathbb{Q})$ and $\alpha(E_{k+1}), \ldots, \alpha(E_r)$ lie in the fibres of $\pi$.

Let $M$ be a general very ample divisor on $S$, and let $N$ be the proper transform of $M$ on $W$. Then $\pi^*(M) \in \langle \alpha(E_{k+1}), \ldots, \alpha(E_r) \rangle$ and $N \sim (\pi \circ \alpha)^*(M)$. Note that the divisor $N$ is not $\beta$-exceptional. On the other hand, $(\pi \circ \alpha)^*(M) \in \langle E_1, \ldots, E_r \rangle$, a contradiction. $\square$

Suppose additionally that $\dim(S) = 0$.

**Theorem A.15** ([12], Theorem 1.26; see Lemmas A.13, A.14). *The following conditions are equivalent. The Fano variety $X$ is $G$-birationally rigid. For every $G$-invariant linear system $\mathcal{M}$ without fixed components on $X$, there is a $G$-equivariant birational automorphism $\xi \in \mathrm{Bir}^G(X)$ such that the log pair $(X, \lambda\xi(\mathcal{M}))$ has canonical singularities, where $\lambda \in \mathbb{Q}$ is such that $K_X + \lambda\xi(\mathcal{M}) \equiv 0$.*

**Corollary A.16.** *The following conditions are equivalent. The Fano variety $X$ is $G$-birationally rigid. For every $G$-invariant linear system $\mathcal{M}$ without fixed components on $X$, there is a $G$-equivariant birational automorphism $\xi \in \mathrm{Bir}^G(X)$ such that $\kappa(X, \lambda\xi(\mathcal{M})) \geqslant 0$, where $\lambda \in \mathbb{Q}$ is such that $K_X + \lambda\xi(\mathcal{M}) \equiv 0$.*

**Corollary A.17.** *The following conditions are equivalent. The Fano variety $X$ is $G$-birationally superrigid. For every $G$-invariant linear system $\mathcal{M}$ without fixed components on $X$, the log pair $(X, \lambda\mathcal{M})$ has canonical singularities, where $\lambda \in \mathbb{Q}$ is such that $K_X + \lambda\mathcal{M} \equiv 0$.*

We now give an elementary application of Corollary A.17.

**Lemma A.18.** *Suppose that $X$ is a smooth del Pezzo surface and $|\Sigma| \geqslant K_X^2$ for every $G$-orbit $\Sigma \subset X$. Then $X$ is $G$-birationally superrigid.*

*Proof.* Suppose that $X$ is not $G$-birationally superrigid. Then, by Corollary A.17, there is a $G$-invariant linear system $\mathcal{M}$ without fixed components on $X$ such that the log pair $(X, \lambda\mathcal{M})$ is not canonical at some point $O \in X$, where $\lambda \in \mathbb{Q}$ is such that $K_X + \lambda\mathcal{M} \equiv 0$.

We denote the $G$-orbit of $O$ by $\Sigma$. Then $\mathrm{mult}_P(\mathcal{M}) > \frac{1}{\lambda}$ for every point $P \in \Sigma$. We have

$$\frac{K_X^2}{\lambda^2} = M_1 \cdot M_2 \geqslant \sum_{P \in \Sigma} \mathrm{mult}_P(M_1 \cdot M_2) \geqslant \sum_{P \in \Sigma} \mathrm{mult}_P^2(\mathcal{M}) > \frac{|\Sigma|}{\lambda^2} \geqslant \frac{K_X^2}{\lambda^2},$$

where $M_1$ and $M_2$ are sufficiently general curves in $\mathcal{M}$, a contradiction. $\square$

Let us show how to apply Lemma A.18.

**Theorem A.19.** *Let $G$ be a finite subgroup of $\mathrm{Aut}(\mathbb{P}^2) \cong \mathbb{PGL}(3, \mathbb{C})$ such that $G \cong \mathbb{A}_6$. Then $\mathbb{P}^2$ is $G$-birationally superrigid.*

*Proof.* Let $\Sigma$ be an arbitrary $G$-orbit in $\mathbb{P}^2$. Then it follows from [25] that $|\Sigma| \geqslant 12$. Hence $\mathbb{P}^2$ is $G$-birationally superrigid by Lemma A.18. $\square$

Let $\Gamma$ be a subset of $\mathrm{Bir}^G(X)$.

**Definition A.20.** The subset $\Gamma$ *untwists all $G$-maximal singularities* if, for every $G$-invariant linear system $\mathcal{M}$ without fixed components on $X$, there is a $\xi \in \Gamma$ such that the log pair $(X, \lambda\xi(\mathcal{M}))$ has canonical singularities, where $\lambda$ is a rational number such that $K_X + \lambda\xi(\mathcal{M}) \equiv 0$.

Lemmas A.13, A.14 yield the following corollary.

**Corollary A.21.** *Suppose that $\Gamma$ untwists all $G$-maximal singularities. Then $X$ is $G$-birationally rigid, and the group $\mathrm{Bir}^G(X)$ is generated by $\Gamma$ and $\mathrm{Aut}^G(X)$.*

It follows from Theorem A.15 that $X$ is $G$-birationally rigid if and only if the group $\mathrm{Bir}^G(X)$ untwists all $G$-maximal singularities.

**Definition A.22.** The *global $G$-invariant log canonical threshold* of a variety $X$ is the number

$$\mathrm{lct}(X, G) = \sup \left\{ \lambda \in \mathbb{Q} \;\middle|\; \begin{array}{l} \text{the log pair } (X, \lambda D) \text{ has log canonical singularities} \\ \text{for every } G\text{-invariant effective } \mathbb{Q}\text{-divisor } D \equiv -K_X \end{array} \right\}.$$

We give an example of the calculation of the number $\mathrm{lct}(X, G)$ (see [15], Lemma 5.7).

**Theorem A.23.** *Let $X$ be a smooth del Pezzo surface with $K_X^2 = 5$. Then $\mathrm{Aut}(X) \cong \mathbb{S}_5$, we have $\mathrm{lct}(X, \mathbb{A}_5) = 2$, and $X$ is $\mathbb{A}_5$-birationally superrigid, where $\mathrm{Pic}^{\mathbb{A}_5}(X) \cong \mathbb{Z}$.*

*Proof.* The isomorphisms $\mathrm{Aut}(X) \cong \mathbb{S}_5$ and $\mathrm{Pic}^{\mathbb{A}_5}(X) \cong \mathbb{Z}$ are well known (see [26], [27]). Let $P$ be a point of $X$ and let $H \subset \mathbb{A}_5$ be the stabilizer of $P$. We denote the $\mathbb{A}_5$-orbit of $P$ by $\Sigma$. Then $|H| = \frac{|\mathbb{A}_5|}{|\Sigma|} = \frac{60}{|\Sigma|}$ and $H$ acts faithfully on the tangent space of $X$ at $P$. It follows that $|\Sigma| \neq 5$ because $\mathbb{A}_4$ has no faithful two-dimensional representations. Since $\mathbb{A}_5$ is simple, we have $|\Sigma| \geqslant 6$. In particular, $X$ is $G$-birationally superrigid by Lemma A.18. This also follows from [28] or [29].

The surface $X$ contains 10 smooth rational curves $L_1, L_2, \ldots, L_{10}$ such that $L_1 \cdot L_1 = L_2 \cdot L_2 = \cdots = L_{10} \cdot L_{10} = -1$ and $\sum_{i=1}^{10} L_i \sim -2K_X$. In particular, we have $\mathrm{lct}(X, \mathbb{A}_5) \leqslant 2$ because the divisor $\sum_{i=1}^{10} L_i$ is $\mathbb{A}_5$-invariant.

It follows from [30] that there is a birational morphism $\chi \colon X \to \mathbb{P}^2$ which blows up the four singular points of the curve $W \subset \mathbb{P}^2$ given by the equation

$$x^6 + y^6 + z^6 + (x^2 + y^2 + z^2)(x^4 + y^4 + z^4) = 12x^2 y^2 z^2 \subset \mathbb{P}^2 \cong \mathrm{Proj}(\mathbb{C}[x, y, z]).$$

Let $Z$ be the proper transform of $W$ on $X$. Then $Z \sim 2K_X$, and the only $\mathbb{S}_5$-invariant curves in the linear system $|-2K_X|$ are $Z$ and $\sum_{i=1}^{10} L_i$. Let $\mathcal{P}$ be the pencil on $X$ generated by the curves $Z$ and $\sum_{i=1}^{10} L_i$. Then every curve in $\mathcal{P}$ is $\mathbb{A}_5$-invariant (see [30]).

Suppose additionally that $|\Sigma| = 6$. Let $T$ be a curve in $\mathcal{P}$ such that $\Sigma \cap T \neq \varnothing$. Then $\Sigma \subset \mathrm{Sign}(T)$ because $H$ is non-Abelian and, therefore, has no faithful one-dimensional representations. It follows from [30] that $\mathcal{P}$ contains five singular curves: the curve $\sum_{i=1}^{10} L_i$, two irreducible rational curves $R_1$ and $R_2$ having 6 nodes each, and two reduced curves $F_1$ and $F_2$ consisting of 5 smooth rational curves each. Thus we see that either $\Sigma = \mathrm{Sign}(R_1)$, or $\Sigma = \mathrm{Sign}(R_2)$.

The six-dimensional representation of $\mathbb{A}_5$ induced by the action of $\mathbb{A}_5$ on $|-K_X|$ is the sum of two inequivalent irreducible three-dimensional representations (see [26]). This yields $\mathbb{A}_5$-equivariant projections $\varphi \colon X \dashrightarrow \mathbb{P}^2$ and $\psi \colon X \dashrightarrow \mathbb{P}^2$ respectively. It follows from [26] that $\varphi$ and $\psi$ are morphisms of degree 5, and the actions of $\mathbb{A}_5$ on $\mathbb{P}^2$ induced by $\varphi$ and $\psi$ coincide with the actions of $\mathbb{A}_5$ induced by the two non-isomorphic irreducible three-dimensional representations (see the proof of Lemma B.13) respectively.

Suppose that $\mathrm{lct}(X, \mathbb{A}_5) \neq 2$. Then there is an effective $\mathbb{A}_5$-invariant $\mathbb{Q}$-divisor $D$ on $X$ such that $\mathbb{LCS}(X, \lambda D) \neq \varnothing$ and $D \equiv -K_X$, where $\lambda$ is a positive rational

number and $\lambda < 2$. Hence $h^1(\mathcal{O}_X(-K_X) \otimes \mathcal{I}(X, \lambda D)) = 0$ by Theorem 2.14, where $\mathcal{I}(X, \lambda D)$ is the multiplier ideal sheaf of the log pair $(X, \lambda D)$ (see Definition 2.13).

Suppose that $X$ contains an $\mathbb{A}_5$-invariant reduced curve $C$ with $\lambda D = \mu C + \Omega$, where $\mu$ is a positive rational number, $\mu \geqslant 1$, and $\Omega$ is an effective $\mathbb{Q}$-divisor on $X$ whose support contains no components of $C$. Then $C \sim -mK_X$ for some positive integer $m$. Therefore we have $5 = -K_X \cdot D \geqslant \frac{5\mu}{\lambda} > \frac{5m}{2}$, whence $m = 1$. On the other hand, the equality $m = 1$ is impossible.

Thus we obtain that the set $\mathbb{LCS}(X, \lambda D)$ contains no curves (see Definition 2.10). Since $h^1(\mathcal{O}_X(-K_X) \otimes \mathcal{I}(X, \lambda D)) = 0$, it follows that $|\operatorname{LCS}(X, \lambda D)| \leqslant 6$, but the set $\operatorname{LCS}(X, \lambda D)$ is $\mathbb{A}_5$-invariant. In particular, we have either $\operatorname{LCS}(X, \lambda D) = \operatorname{Sign}(R_1)$, or $\operatorname{LCS}(X, \lambda D) = \operatorname{Sign}(R_2)$.

To complete the proof, we may assume that $\operatorname{Supp}(D)$ contains neither $R_1$ nor $R_2$ (see Remark 2.6). We may also assume that $\operatorname{LCS}(X, \lambda D) = \operatorname{Sign}(R_1) = \Sigma$. Write $\operatorname{Sign}(R_1) = \{O_1, O_2, O_3, O_4, O_5, O_6\}$, where the $O_i$ are the singular points of $R_1$. Suppose that $P = O_1$. Then we have

$$10 = R_1 \cdot D \geqslant \sum_{i=1}^{6} \operatorname{mult}_{O_i}(D) \operatorname{mult}_{O_i}(R_1) \geqslant 12 \operatorname{mult}_P(D),$$

whence $\operatorname{mult}_P(D) \leqslant \frac{5}{6}$.

Let $\pi \colon U \to X$ be the blow-up of the points $O_1, O_2, O_3, O_4, O_5, O_6$. Then

$$K_U + \lambda \overline{D} + \sum_{i=1}^{6} (\lambda \operatorname{mult}_{O_i}(D) - 1) E_i \equiv \pi^*(K_X + \lambda D),$$

where $E_i$ is an exceptional curve of $\pi$ with $\pi(E_i) = O_i$ and $\overline{D}$ is the proper transform of $D$ on $U$. Moreover, the log pair $\left(U, \lambda \overline{D} + \sum_{i=1}^{6} (\lambda \operatorname{mult}_{O_i}(D) - 1) E_i\right)$ is not log canonical at some point $Q \in E_1$. It follows from Theorem 2.16 that

$$\operatorname{LCS}\left(U, \lambda \overline{D} + \sum_{i=1}^{6} (\lambda \operatorname{mult}_{O_i}(D) - 1) E_i\right) \cap E_1 = Q.$$

On the other hand, the $\mathbb{A}_5$-orbit of $Q$ contains at least two points of the exceptional curve $E_1$ since the stabilizer $H$ acts faithfully on the tangent space of $X$ at $P$, a contradiction. $\square$

The number $\operatorname{lct}(X, G)$ plays an important role in geometry (see Theorem 5.1). For simplicity we put $\operatorname{lct}(X) = \operatorname{lct}(X, G)$ in the case when $G$ is trivial. Then $\operatorname{lct}(\mathbb{P}^n) = \frac{1}{n+1}$ (see [12]).

**Example A.24.** Let $X$ be a general hypersurface of degree $n \geqslant 6$ in $\mathbb{P}^n$. It was shown in [17] that $\operatorname{lct}(X) = 1$.

Note that Definition A.3 makes sense in the case when $X$ is defined over an arbitrary perfect field, not necessarily algebraically closed.

**Definition A.25.** The variety $X$ is said to be *universally $G$-birationally rigid* if $X \otimes \operatorname{Spec}(\mathbb{C}(U))$ is $G$-birationally rigid for every variety $U$. Here $\mathbb{C}(U)$ is the field of rational functions on $U$, and we regard $G$ as a subgroup of $\operatorname{Aut}(X \otimes \operatorname{Spec}(\mathbb{C}(U)))$.

It follows from Definition A.4 that all $G$-birationally superrigid varieties $X$ are universally $G$-birationally rigid because the variety $X \otimes \mathrm{Spec}(\mathbb{C}(U))$ is $G$-birationally superrigid for every variety $U$.

*Remark A.26.* Suppose that $\dim(X) \neq 1$ and $X$ is $G$-birationally rigid. Then the group $\mathrm{Aut}^G(X)$ is finite (see [31]). Moreover, $X$ is universally $G$-birationally rigid if, for example, the group $\mathrm{Bir}^G(X)$ is countable (see [32]).

Take a positive integer $r \geq 2$. For every $i \in \{1, 2, \ldots, r\}$ let $X_i$ be a Fano variety and $G_i$ a finite subgroup of $\mathrm{Aut}(X_i)$ such that $\dim_{\mathbb{Q}}(\mathrm{Pic}^{G_i}(X_i) \otimes \mathbb{Q}) = 1$ and every $G_i$-invariant Weil divisor on $X_i$ is a $\mathbb{Q}$-Cartier divisor. Suppose that each $X_i$ is $G_i$-birationally rigid and has at most terminal singularities. Put $\mathcal{X} = X_1 \times \cdots \times X_r$, $\mathcal{G} = G_1 \times \cdots \times G_r$, $\mathcal{S}_1 = X_2 \times \cdots \times X_r$, $\mathcal{S}_r = X_1 \times \cdots \times X_r$ and $\mathcal{S}_i = X_1 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_r$ for every $i \in \{2, \ldots, r-1\}$.

The natural projection $\pi_i \colon \mathcal{X} \to \mathcal{S}_i$ is a $\mathcal{G}$-Mori fibration and we have $\{\pi_1, \ldots, \pi_r\} \subseteq \mathcal{P}(\mathcal{X}, \mathcal{G})$, where $\mathcal{P}(\mathcal{X}, \mathcal{G})$ is the $\mathcal{G}$-pliability of $\mathcal{X}$ (see Definition A.9).

**Theorem A.27.** *For every $i \in \{1, 2, \ldots, r\}$ suppose that $X_i$ is universally $G_i$-birationally rigid and $\mathrm{lct}(X_i, G_i) \geq 1$. Then $\mathcal{P}(\mathcal{X}, \mathcal{G}) = \{\pi_1, \ldots, \pi_r\}$.*

*Proof.* Suppose that there is a $\mathcal{G}$-equivariant birational map $\nu \colon \overline{X} \dashrightarrow \mathcal{X}$ for some $\mathcal{G}$-Mori fibration $\overline{\pi} \colon \overline{X} \to \overline{S}$. We must prove the existence of an $i \in \{1, 2, \ldots, r\}$ such that $\dim(\overline{S}) = \dim(\mathcal{S}_i)$ and there is a commutative diagram

$$
\begin{array}{ccccc}
\overline{X} & \dashrightarrow^{\nu} & \mathcal{X} & \dashrightarrow^{\rho} & \mathcal{X} \\
\downarrow{\overline{\pi}} & & & & \downarrow{\pi_i} \\
\overline{S} & \dashrightarrow & & {}^{\sigma} & \mathcal{S}_i
\end{array}
\tag{A.3}
$$

where $\rho$ is a $\mathcal{G}$-equivariant birational automorphism and $\sigma$ is a birational map. Arguing as in the proofs of Theorem 1 in [17] and Theorem 6.5 in [33] and using Lemma A.14 instead of Proposition 2 in [17], we see that $\dim(\overline{S}) \neq 0$ (compare with the proof of Theorem 6.1). Then, arguing as in the proofs of Theorem 1 in [17] and Theorem 6.5 in [33] and using Lemma A.13 instead of Proposition 2 in [17], we obtain the existence of the commutative diagram (A.3). $\square$

We now give some applications of Theorem A.27 (see Examples A.7, A.24).

**Example A.28.** It is known that the simple group $\mathbb{A}_6$ is a group of automorphisms of the curve

$$10x^3y^3 + 9zx^5 + 9zy^5 + 27z^6 = 45x^2y^2z^2 + 135xyz^4 \subset \mathbb{P}^2 \cong \mathrm{Proj}(\mathbb{C}[x, y, z]).$$

Hence there is a monomorphism $\mathbb{A}_6 \times \mathbb{A}_6 \to \mathrm{Aut}(\mathbb{P}^2 \times \mathbb{P}^2)$, which in turn induces a monomorphism $\varphi \colon \mathbb{A}_6 \times \mathbb{A}_6 \to \mathrm{Bir}(\mathbb{P}^4)$ since the variety $\mathbb{P}^2 \times \mathbb{P}^2$ is rational. Then it follows from Theorem A.27 that the subgroup $\varphi(\mathbb{A}_6 \times \mathbb{A}_6)$ is not conjugate to a subgroup of $\mathrm{Aut}(\mathbb{P}^4)$ because $\mathbb{P}^2$ is $\mathbb{A}_6$-birationally superrigid by Theorem A.19 and we have $\mathrm{lct}(\mathbb{P}^2, \mathbb{A}_6) = 2$ (see [15]).

For further applications of Theorem A.27 see [17] and [33].

## Appendix B. The Cremona group

Let $G$ be a finite group. We put $\mathrm{Cr}_2(\mathbb{C}) = \mathrm{Bir}(\mathbb{P}^2)$ and assume that there is a monomorphism $\varphi\colon G \to \mathrm{Cr}_2(\mathbb{C})$.

**Problem B.1.** Find all subgroups of $\mathrm{Cr}_2(\mathbb{C})$ (up to conjugacy) that are isomorphic to $\varphi(G)$.

The purpose of this appendix is to solve Problem B.1 in the case when $G$ is a simple non-Abelian subgroup of $\mathrm{Cr}_2(\mathbb{C})$ (see Theorem B.7).

**Theorem B.2.** *Up to conjugacy,* $\mathrm{Cr}_2(\mathbb{C})$ *contains precisely* 3, 1, 2 *subgroups isomorphic to* $\mathbb{A}_5$, $\mathbb{A}_6$, $\mathbb{PSL}(2,\mathbb{F}_7)$ *respectively. An explicit description of these subgroups follows from Theorems B.8–B.10.*

Let $\varphi'\colon G \to \mathrm{Cr}_2(\mathbb{C})$ be a monomorphism. We note that if $\varphi(G) = \varphi'(G)$, then there is an automorphism $\chi \in \mathrm{Aut}(G)$ such that $\varphi' = \varphi \circ \chi$.

**Definition B.3.** The pairs $(G,\varphi)$ and $(G,\varphi')$ are said to be *conjugate* if there is a birational automorphism $\varepsilon \in \mathrm{Cr}_2(\mathbb{C})$ and a commutative diagram

$$
\begin{array}{ccc}
& G & \\
{\scriptstyle\varphi}\swarrow & & \searrow{\scriptstyle\varphi'} \\
\mathrm{Cr}_2(\mathbb{C}) & \xrightarrow{\ \omega_\varepsilon\ } & \mathrm{Cr}_2(\mathbb{C})
\end{array}
$$

where $\omega_\varepsilon$ is the inner automorphism $\omega_\varepsilon(g) = \varepsilon \circ g \circ \varepsilon^{-1}$ for all $g \in \mathrm{Cr}_2(\mathbb{C})$.

If $\varphi' = \varphi \circ \chi$ for some inner automorphism $\chi \in \mathrm{Aut}(G)$, then the pairs $(G,\varphi)$ and $(G,\varphi')$ are conjugate.

**Problem B.4.** For every $\sigma \in \mathrm{Aut}(G)$ decide whether $(G,\varphi)$ and $(G,\varphi \circ \sigma)$ are conjugate.

It follows from [27], Lemma 3.5, that there is a smooth rational surface $X$, a monomorphism $\upsilon\colon G \to \mathrm{Aut}(X)$ and a birational map $\xi\colon X \dashrightarrow \mathbb{P}^2$ such that $\varphi(g) = \xi \circ \upsilon(g) \circ \xi^{-1} \in \mathrm{Bir}(\mathbb{P}^2)$ for all $g \in G$. Note that the triple $(X,\xi,\upsilon)$ is not uniquely determined by the pair $(G,\varphi)$.

**Definition B.5.** The triple $(X,\xi,\upsilon)$ is called a *regularization* of the pair $(G,\varphi)$.

Let $(X',\xi',\upsilon')$ be a regularization of $(G,\varphi')$.

**Theorem B.6** ([27], Lemma 3.4). *The pairs* $(G,\varphi)$ *and* $(G,\varphi')$ *are conjugate if and only if there is a birational map* $\rho\colon X \dashrightarrow X'$ *such that* $\upsilon'(g) = \rho \circ \upsilon(g) \circ \rho^{-1} \in \mathrm{Aut}(X')$ *for all* $g \in G$.

In particular, Theorem B.6 implies that to solve Problems B.1, B.4, we may assume the existence of a morphism $\pi\colon X \to S$, which is a $\upsilon(G)$-Mori fibration. Note that either $S \cong \mathbb{P}^1$, or $S$ is a point.

**Theorem B.7** ([27]). *Let* $G$ *be a non-Abelian simple group. Then* $G$ *is isomorphic to one of the following groups:* $\mathbb{A}_5$, $\mathbb{A}_6$, $\mathbb{PSL}(2,\mathbb{F}_7)$.

We identify the subgroup $\upsilon(G)$ of $\mathrm{Aut}(X)$ with $G$.

**Theorem B.8.** *Suppose that $G \cong \mathbb{PSL}(2, \mathbb{F}_7)$. Then $S$ is a point and the surface $X$ is $G$-birationally superrigid. Moreover, either $X \cong \mathbb{P}^2$ and $G$ is conjugate to a subgroup of $\mathrm{Aut}(X)$ that leaves the smooth curve*

$$x^3 y + y^3 z + z^3 x = 0 \subset \mathrm{Proj}(\mathbb{C}[x, y, z]) \cong \mathbb{P}^2 \qquad (\mathrm{B.4})$$

*invariant, or $X$ is a double covering of $\mathbb{P}^2$ branched along the curve* (B.4). *For every $\sigma \in \mathrm{Aut}(G)$, the pairs $(G, \varphi)$ and $(G, \varphi \circ \sigma)$ are conjugate if and only if the automorphism $\sigma$ is inner.*

*Proof.* It is easy to see that $S$ is a point and either $X \cong \mathbb{P}^2$ and $G$ is conjugate to a subgroup of $\mathrm{Aut}(X)$ that leaves the curve (B.4) invariant, or $X$ is a double covering of $\mathbb{P}^2$ branched along the curve (B.4).

Note that any $G$-orbit in $X$ consists of at least two points. Indeed, since $G$ is simple, it has no faithful two-dimensional representations and no subgroups of index two. If $X \cong \mathbb{P}^2$, then any $G$-orbit in $X$ contains at least 12 points (see [16], [25]). In particular, we see that $X$ is $G$-birationally superrigid by Lemma A.18.

Consider an arbitrary automorphism $\sigma \in \mathrm{Aut}(G)$ such that the pairs $(G, \varphi)$ and $(G, \varphi \circ \sigma)$ are conjugate. By Theorem B.6, there is a birational map $\rho \colon X \dashrightarrow X$ such that $\upsilon \circ \sigma(g) = \rho \circ \upsilon(g) \circ \rho^{-1}$ for all $g \in G$. We have $\rho \in \mathrm{Aut}(X)$ because $X$ is $G$-birationally superrigid. Put $\widehat{G} = \langle G, \rho \rangle$ (where, as above, we identify $G$ with the subgroup $\upsilon(G)$ of $\mathrm{Aut}(X)$). Then $\widehat{G}$ is a finite subgroup of $\mathrm{Aut}^G(X)$. It follows that $\widehat{G} = G$ because $\mathrm{Aut}^G(X) = G$ if $X \cong \mathbb{P}^2$, and $\mathrm{Aut}(X) = \mathrm{Aut}^G(X) \cong \mathbb{PSL}(2, \mathbb{F}_7) \times \mathbb{Z}_2$ if $X \not\cong \mathbb{P}^2$. In particular, we see that $\rho \in G$ and, therefore, $\sigma$ is an inner automorphism of $G$. $\square$

**Theorem B.9.** *Suppose that $G \cong \mathbb{A}_6$. Then $X \cong \mathbb{P}^2$ and $X$ is $G$-birationally superrigid. For every $\sigma \in \mathrm{Aut}(G)$, the pairs $(G, \varphi)$ and $(G, \varphi \circ \sigma)$ are conjugate if and only if $\sigma$ is an inner automorphism. Moreover, the subgroup $G$ is conjugate to a subgroup of $\mathrm{Aut}(X)$ that leaves invariant the curve*

$$10x^3 y^3 + 9zx^5 + 9zy^5 + 27z^6 = 45x^2 y^2 z^2 + 135xyz^4 \subset \mathbb{P}^2 \cong \mathrm{Proj}(\mathbb{C}[x, y, z]). \ (\mathrm{B.5})$$

*Proof.* Denote the curve (B.5) by $C$. Arguing as in the proof of Theorem B.8, we see that $X \cong \mathbb{P}^2$ and $G$ is conjugate to a subgroup that leaves $C$ invariant. Hence we may assume that $C$ is $G$-invariant. By Theorem A.19, $X$ is $G$-birationally superrigid.

Let $\rho$ be any element of $\mathrm{Aut}^G(X)$ and $g$ any element of $G$. Then $g(\rho(C)) = \rho(g'(C)) = \rho(C)$ for some $g' \in G$. On the other hand, $C$ is the only $G$-invariant sextic in $\mathbb{P}^2$. Thus we see that $\rho(C) = C$. It follows that $\rho \in G$ because $G \cong \mathrm{Aut}(C)$. We complete the proof by arguing as in the proof of Theorem B.8. $\square$

**Theorem B.10.** *Suppose that $G \cong \mathbb{A}_5$. Then the pairs $(G, \varphi)$ and $(G, \varphi \circ \sigma)$ are conjugate for every $\sigma \in \mathrm{Aut}(G)$. Moreover, one of the following possibilities holds.*

1) *$X$ is a blow-up of $\mathbb{P}^2$ at any four points in general position, $S$ is a point, $X$ is $G$-birationally superrigid, and $\mathrm{Aut}(X) \cong \mathbb{S}_5$.*

2) $X \cong \mathbb{P}^2$, $X$ is $G$-birationally rigid, and

$$\mathbb{A}_5 \cong \mathrm{Aut}^{\mathbb{A}_5}(\mathbb{P}^2) \subsetneq \mathrm{Bir}^{\mathbb{A}_5}(\mathbb{P}^2) \cong \mathbb{S}_5.$$

3) $X \cong \mathbb{F}_n$, where $n \in \mathbb{N} \cup \{0\}$ is even, there is a birational map $\rho \colon X \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ inducing a monomorphism $\overline{\upsilon} \colon G \to \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ such that $\overline{\upsilon}(g) = \rho \circ \upsilon(g) \circ \rho^{-1}$ for all $g \in G$, and $\overline{\upsilon}$ is induced by the natural action of $\mathbb{A}_5 \times \mathrm{id}_{\mathbb{P}^1}$ on $\mathbb{P}^1 \times \mathbb{P}^1$.

Note that Theorems B.8–B.10 completely solve Problems B.1 and B.4 in the case when $G \in \{\mathbb{A}_5, \mathbb{A}_6, \mathbb{PSL}(2, \mathbb{F}_7)\}$.

*Proof of Theorem B.10.* Suppose that $G \cong \mathbb{A}_5$.

**Lemma B.11.** *If $S$ is a point, then either $K_X^2 = 5$ or $X \cong \mathbb{P}^2$. If $S$ is not a point, then $S \cong \mathbb{P}^1$ and $X \cong \mathbb{F}_n$, where $n \in \mathbb{N} \cup \{0\}$.*

*Proof.* If $S$ is a point, then either $K_X^2 = 5$ or $X \cong \mathbb{P}^2$ (see [27]). If $S \cong \mathbb{P}^1$, then $X \cong \mathbb{F}_n$ by Lemma 5.6 in [27], where $n \in \mathbb{N} \cup \{0\}$. □

Note that if $K_X^2 = 5$ and $S$ is a point, then $\mathrm{Aut}(X) \cong \mathbb{S}_5$ and $X$ is a blow-up of $\mathbb{P}^2$ at four arbitrary points, no three of which lie on a line (see [26], [27]).

**Lemma B.12.** *Suppose that $K_X^2 = 5$ and $S$ is a point. Then $X$ is $G$-birationally superrigid. Moreover, the pairs $(G, \varphi)$ and $(G, \varphi \circ \sigma)$ are conjugate for every $\sigma \in \mathrm{Aut}(G)$.*

*Proof.* The surface $X$ is $G$-birationally superrigid by Theorem A.23. It is well known that $\mathrm{Aut}(G) \cong \mathbb{S}_5$ and every element of $\mathrm{Aut}(G)$ is induced by an inner automorphism of $\mathbb{S}_5$. Therefore the pairs $(G, \varphi)$ and $(G, \varphi \circ \sigma)$ are conjugate for every $\sigma \in \mathrm{Aut}(G)$. □

Note that if $X \cong \mathbb{P}^2$, then the embedding $\mathbb{A}_5 \cong G \subset \mathrm{Aut}(X) \cong \mathbb{PSL}(3, \mathbb{C})$ is induced by a non-trivial three-dimensional representation of $\mathbb{A}_5$.

**Lemma B.13.** *Suppose that $X \cong \mathbb{P}^2$. Then $\mathbb{A}_5 \cong \mathrm{Aut}^G(X) \subsetneq \mathrm{Bir}^G(X) \cong \mathbb{S}_5$ and $X$ is $G$-birationally rigid. Moreover, the pairs $(G, \varphi)$ and $(G, \varphi \circ \sigma)$ are conjugate for every $\sigma \in \mathrm{Aut}(G)$.*

*Proof.* By [25] there are no $G$-invariant curves of degrees 1, 3 or 5 in $\mathbb{P}^2$. On the other hand, there is a unique $G$-invariant conic in $\mathbb{P}^2$. We denote this conic by $C$. Note that $C$ is irreducible. The action of $G \cong \mathbb{A}_5$ on the curve $C \cong \mathbb{P}^1$ induces an embedding $C \subset \mathrm{Aut}(C)$.

It is well known that there is a unique $G$-invariant curve in $\mathbb{P}^2$ which is the union of 6 distinct lines. We denote these lines by $L_1, \ldots, L_6$. Then $C \cap (L_1 \cup \cdots \cup L_6)$ is the unique $G$-orbit in $\mathbb{P}^2$ that lies in $C$ and consists of 12 points (see [16]). It is also well known that every $G$-orbit in $\mathbb{P}^2$ consists of at least 6 points, and there is a unique $G$-orbit of 6 points.

Take any element $\sigma$ of $\mathrm{Aut}(G)$. Then $\sigma$ is induced by an inner automorphism of $\mathbb{S}_5$. Thus, if $\mathrm{Bir}^G(X) \cong \mathbb{S}_5$, then the pairs $(G, \varphi)$ and $(G, \varphi \circ \sigma)$ are conjugate.

Let $\rho$ be an arbitrary element of $\mathrm{Aut}^G(X)$, and $g$ an arbitrary element of $G$. Then $g \circ \rho = \rho \circ g'$ for some $g' \in G$. Hence we have $g(\rho(C)) = \rho(g'(C)) = \rho(C)$ and, therefore, $\rho(C) = C$. Let $\Lambda$ be the unique $G$-orbit in $\mathbb{P}^2$ that lies in $C$ and consists of 12 points. Then $g(\rho(\Lambda)) = \rho(g'(\Lambda)) = \rho(\Lambda)$ and, therefore, $\rho(\Lambda) = \Lambda$. Note that the subgroup of those elements of $\mathrm{Aut}(C)$ that leave $\Lambda$ invariant is finite. Hence $\rho \in G$. Thus we see that $\mathrm{Aut}^G(X) = G$.

To complete the proof, we must show that $X$ is $G$-birationally rigid and $\mathrm{Bir}^G(X) \cong \mathbb{S}_5$.

Let $\Sigma$ be the $G$-orbit of some point in $\mathbb{P}^2$. Suppose that $|\Sigma| \leqslant 9$. Then, by [16] or [25], we have $\Sigma \cap C = \varnothing$ and $|\Sigma| = 6$ (see § 6). Note that $\Sigma$ is uniquely determined by the equality $|\Sigma| = 6$.

Let $\gamma \colon W \to X$ be the blow-up of all points of $\Sigma$. Then it follows from Proposition 1 in [34] that $\mathrm{Aut}(W) \cong \mathbb{S}_5$ and $W$ is isomorphic to the Clebsch cubic surface (see [27]). We put $\tau = \gamma \circ \theta \circ \gamma^{-1}$, where $\theta$ is an odd involution in $\mathrm{Aut}(W) \cong \mathbb{S}_5$. Then $\tau \notin \mathrm{Aut}(\mathbb{P}^2)$. Note that the involution $\tau$ induces a monomorphism $\upsilon' \colon G \to \mathrm{Aut}(\mathbb{P}^2)$ such that $\upsilon'(g) = \tau \circ \upsilon(g) \circ \tau^{-1}$ for every $g \in G$. Then $\upsilon'$ is induced by a three-dimensional irreducible representation of $\mathbb{A}_5$ and this representation is not isomorphic to the representation that induces $\upsilon$ (see [27], § 9).

Let $E$ be a reduced $\gamma$-exceptional divisor with $\gamma(E) = \Sigma$. Then there is a commutative diagram

$$
\begin{array}{ccc}
 & W & \\
{}^{\gamma}\swarrow & & \searrow^{\psi} \\
\mathbb{P}^2 \dashrightarrow_{\tau} & & \mathbb{P}^2
\end{array}
$$
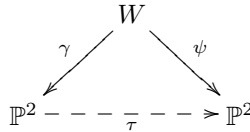
where $\psi$ blows down the curve $\theta(E)$ to the set $\Sigma$ (see the proof of Theorem A.23). If the group generated by $\tau$ untwists all $G$-maximal singularities (see Definition A.20), then $\mathrm{Bir}^G(X) = \langle \mathrm{Aut}^G(X), \tau \rangle \cong \mathbb{S}_5$ by Corollary A.21 and $X$ is $G$-birationally rigid by Corollary A.16. Hence, to complete the proof, we must show that the group generated by $\tau$ untwists all $G$-maximal singularities.

Let $\mathcal{M}$ be a $G$-invariant linear system without fixed curves on $X$ such that the log pair $(X, \mu\mathcal{M})$ is not canonical at some point $O \in X$, where $\mu \in \mathbb{Q}$ and $K_X + \mu\mathcal{M} \equiv 0$. We denote the $G$-orbit of $O$ by $\Delta$. Then $\mathrm{mult}_P(\mathcal{M}) > \frac{1}{\mu}$ for every point $P \in \Delta$. Let $M_1$ and $M_2$ be general curves in $\mathcal{M}$. Then we have

$$
\frac{9}{\mu^2} = \frac{K_X^2}{\mu^2} = M_1 \cdot M_2 \geqslant \sum_{P \in \Delta} \mathrm{mult}_P(M_1 \cdot M_2)
$$

$$
\geqslant \sum_{P \in \Delta} \mathrm{mult}_P(M_1)\,\mathrm{mult}_P(M_2) = \sum_{P \in \Delta} \mathrm{mult}_P^2(\mathcal{M}) > \frac{|\Delta|}{\mu^2},
$$

whence $|\Delta| < 9$. Therefore $\Delta = \Sigma$. Putting $H = \gamma^*(\mathcal{O}_{\mathbb{P}^2}(1))$, we have

$$
\theta^*(H) \sim 5H - 2E, \qquad \theta^*(E) \sim 12H - 5E \tag{B.6}
$$

since the involution $\theta$ acts non-trivially on $\mathrm{Pic}(W)$. Put $\mathcal{M}' = \tau(\mathcal{M})$. Then $K_X + \mu'\mathcal{M}' \equiv 0$ for some positive rational number $\mu'$. Therefore it follows from (B.6) that

$$\mu' = \frac{3}{15/\mu - 12\,\mathrm{mult}_O(\mathcal{M})},$$

whence $\mu' > \mu$. We similarly deduce from (B.6) that $\mathrm{mult}_P(\mathcal{M}') = \frac{6}{\mu} - 5\,\mathrm{mult}_O(\mathcal{M})$ for every point $P \in \Sigma$. This proves that the log pair $(X, \mu'\mathcal{M}')$ has canonical singularities at every point of $\Sigma$. Arguing as at the beginning of the proof of the lemma, we see that the singularities of the log pair $(X, \mu'\mathcal{M}')$ are canonical everywhere. It follows that the group generated by $\tau$ untwists all $G$-maximal singularities. $\square$

Hence, to complete the proof of Theorem B.10, we may assume that $X \cong \mathbb{F}_n$, where $n \in \mathbb{N} \cup \{0\}$.

**Lemma B.14.** *Suppose that there are a monomorphism $\iota\colon G \to \mathrm{Aut}(\mathbb{P}^1)$, an outer isomorphism $\nu \in \mathrm{Aut}(G)$ and a birational map $\chi\colon X \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ such that*

$$\chi \circ \upsilon(g) \circ \chi^{-1}(a, b) = (\iota(g)(a), \iota \circ \nu(g)(b))$$

*for all $g \in G$ and $(a, b) \in \mathbb{P}^1 \times \mathbb{P}^1$. Then the pairs $(G, \varphi)$ and $(G, \varphi \circ \sigma)$ are conjugate for every $\sigma \in \mathrm{Aut}(G)$.*

*Proof.* The birational map $\chi$ induces a monomorphism $\grave{\upsilon}\colon G \to \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ such that $\grave{\upsilon}(g) = \chi \circ \upsilon(g) \circ \chi^{-1}$ for all $g \in G$. In this case $\grave{\upsilon}$ is induced by the twisted diagonal action of $\mathbb{A}_5$ on $\mathbb{P}^1 \times \mathbb{P}^1$. We identify the subgroup $\grave{\upsilon}(G)$ with $G$.

Take an automorphism $\tau$ of $\mathbb{P}^1 \times \mathbb{P}^1$ such that $\tau(a, b) = (b, a)$ for all $(a, b) \in \mathbb{P}^1 \times \mathbb{P}^1$. Then $\langle G, \tau \rangle \cong \mathbb{S}_5$. It follows that the pairs $(G, \varphi)$ and $(G, \varphi \circ \sigma)$ are conjugate for every $\sigma \in \mathrm{Aut}(G)$ because every automorphism of $\mathbb{A}_5$ is induced by an inner automorphism of $\mathbb{S}_5$. $\square$

Thus, to complete the proof of Theorem B.10, we may ignore any difference between the monomorphisms $\upsilon$ and $\upsilon \circ \sigma$, where $\sigma \in \mathrm{Aut}(G)$.

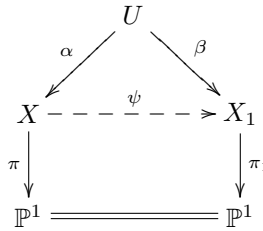**Lemma B.15.** *Suppose that $n \neq 0$. Then $n$ is even and there is a commutative diagram*

$$
\begin{array}{ccc}
X & \dashrightarrow{\;\;\rho\;\;} & \mathbb{P}^1 \times \mathbb{P}^1 \\
{\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\
\mathbb{P}^1 & =\!=\!=\!= & \mathbb{P}^1
\end{array}
$$

*where $\rho$ is a birational map that induces a monomorphism $\overline{\upsilon}\colon G \to \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ satisfying $\overline{\upsilon}(g) = \rho \circ \upsilon(g) \circ \rho^{-1}$ for every $g \in G$, and $\overline{\upsilon}$ is induced by the natural action of the group $\mathbb{A}_5 \times \mathrm{id}_{\mathbb{P}^1}$ on $\mathbb{P}^1 \times \mathbb{P}^1$.*

*Proof.* Let $Z$ be the unique section of $\pi$ such that $Z^2 = -n$. Then $Z$ is a $G$-invariant curve. It is well known that $Z$ contains $G$-orbits consisting of 12, 20 and 30 points (see [16]).

Let $\Sigma$ be a $G$-orbit such that $\Sigma \subset Z$ and $|\Sigma| = 30$. Then there is a commutative diagram

$$
\begin{array}{ccc}
 & U & \\
{\scriptstyle\alpha}\swarrow & & \searrow{\scriptstyle\beta} \\
X \dashrightarrow{\psi} & & X_1 \\
{\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi_1} \\
\mathbb{P}^1 & = & \mathbb{P}^1
\end{array}
$$

where $\psi$ is a birational map, $\pi_1$ is a $\mathbb{P}^1$-bundle, $\alpha$ is the blow-up of $\Sigma$, and $\beta$ is the blow-down of the proper transforms of all fibres of $\pi$ that pass through points of $\Sigma$.
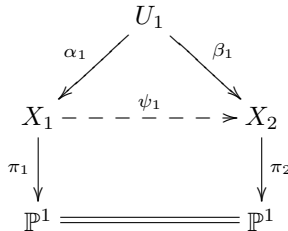
The birational map $\psi$ induces a monomorphism $\upsilon_1 \colon G \to \mathrm{Aut}(X_1)$ such that $\upsilon_1(g) = \psi \circ \upsilon(g) \circ \psi^{-1}$ for all $g \in G$. We identify the subgroup $\upsilon_1(G)$ with $G$. Put $Z_1 = \psi(Z)$. Then we have

$$
Z_1 \cdot Z_1 = Z \cdot Z - 30 = -n - 30 < 0,
$$

whence $X_1 \cong \mathbb{F}_{n+30}$ and $Z_1$ is $G$-invariant. The curve $Z_1$ contains $G$-orbits consisting of 12, 20 and 30 points (see [16]).

Let $\Sigma_1$ be a $G$-orbit with $\Sigma_1 \subset Z_1$ and $|\Sigma_1| = 20$, $P_1$ a point of $\Sigma_1$, and $H_1$ the stabilizer of $P_1$ in $G$. Then $H_1 \cong \mathbb{Z}_3$. Let $L_1$ be the fibre of $\pi_1$ such that $P_1 \in L_1$. Then $h_1(L_1) = L_1$ for every $h_1 \in H_1$. Thus there is a point $Q_1 \in L_1 \setminus P_1$ such that $h_1(Q_1) = Q_1$ for every $h_1 \in H_1$.

We denote the $G$-orbit of $Q_1$ by $\Lambda_1$. Then $|\Lambda_1| = 20$ and $\Lambda_1 \cap Z_1 = \varnothing$ because $Z_1$ is $G$-invariant and $Q_1 \notin Z_1$. Thus there is a commutative diagram

$$
\begin{array}{ccc}
 & U_1 & \\
{\scriptstyle\alpha_1}\swarrow & & \searrow{\scriptstyle\beta_1} \\
X_1 \dashrightarrow{\psi_1} & & X_2 \\
{\scriptstyle\pi_1}\downarrow & & \downarrow{\scriptstyle\pi_2} \\
\mathbb{P}^1 & = & \mathbb{P}^1
\end{array}
$$

where $\psi_1$ is a birational map, $\pi_2$ is a $\mathbb{P}^1$-bundle, $\alpha_1$ is the blow-up of $\Lambda_1$, and $\beta_1$ is the blow-down of the proper transform of the fibres of $\pi_1$ passing through the points of $\Lambda_1$.

The birational map $\psi_1$ induces a monomorphism $\upsilon_2 \colon G \to \mathrm{Aut}(X_2)$ such that $\upsilon_2(g) = \psi_1 \circ \upsilon_1(g) \circ \psi_1^{-1}$ for every $g \in G$. We identify the subgroup $\upsilon_2(G)$ with $G$. Put $Z_2 = \psi_1(Z_1)$. Then we have

$$
Z_2 \cdot Z_2 = -n - 10 < 0,
$$

whence $X_2 \cong \mathbb{F}_{n+10}$ and $Z_2$ is $G$-invariant. The curve $Z_2$ contains $G$-orbits consisting of 12, 20 and 30 points (see [16]).

Let $\Sigma_2$ be the unique $G$-orbit with $\Sigma_2 \subset Z_2$ and $|\Sigma_2| = 12$, $P_2$ a point of $\Sigma_2$, and $H_2$ the stabilizer of $P_2$ in $G$. Then $H_2 \cong \mathbb{Z}_5$. Let $L_2$ be the fibre of $\pi_2$ such

that $P_2 \in L_2$. Then $h_2(L_2) = L_2$ for every $h_2 \in H_2$. Hence there is a point $Q_2 \in L_2 \setminus P_2$ such that $h_2(Q_2) = Q_2$ for every $h_2 \in H_2$.

We denote the $G$-orbit of $Q_2$ by $\Lambda_2$. Then $|\Lambda_2| = 12$ and $\Lambda_2 \cap Z_2 = \varnothing$ because $Z_2$ is $G$-invariant and $Q_2 \notin Z_2$. Thus there is a commutative diagram

$$
\begin{array}{ccc}
& U_2 & \\
\alpha_2 \swarrow & & \searrow \beta_2 \\
X_2 \dashrightarrow\!\!\!\!\!\!\!\!\overset{\psi_2}{\phantom{xx}}\!\!\!\!\!\!\!\!\dashrightarrow & & X' \\
\pi_2 \downarrow & & \downarrow \pi' \\
\mathbb{P}^1 & =\!=\!=\!=\!= & \mathbb{P}^1
\end{array}
$$

where $\psi_2$ is a birational map, $\pi'$ is a $\mathbb{P}^1$-bundle, $\alpha_2$ is the blow-up of $\Lambda_2$, and $\beta_2$ is the blow-down of the proper transforms of the fibres of $\pi_2$ that pass through the points of $\Lambda_2$.

The birational map $\psi_2$ induces a monomorphism $\upsilon' \colon G \to \mathrm{Aut}(X')$ such that $\upsilon'(g) = \psi_2 \circ \upsilon_3(g) \circ \psi_2^{-1}$ for every $g \in G$. We identify the subgroup $\upsilon'(G)$ with $G$. Put $Z' = \psi_2(Z_2)$. Then $Z' \cdot Z' = -n + 2 \leqslant 0$, and the curve $Z'$ is a $G$-invariant section of $\pi'$. Note that $X' \cong \mathbb{F}_{n-2}$.

We put $\nu = \psi_2 \circ \psi_1 \circ \psi$. Then the triple $(X', \xi \circ \nu, \upsilon')$ is a regularization of the pair $(G, \varphi)$. Thus we have constructed a commutative diagram

$$
\begin{array}{ccc}
X \dashrightarrow\!\!\!\!\!\!\!\!\overset{\nu}{\phantom{xx}}\!\!\!\!\!\!\!\!\dashrightarrow & & X' \\
\pi \downarrow & & \downarrow \pi' \\
\mathbb{P}^1 & =\!=\!=\!=\!= & \mathbb{P}^1
\end{array}
$$

where $\nu$ is a birational map, $\pi'$ is a $\mathbb{P}^1$-bundle, $X' \cong \mathbb{F}_{n-2}$, and there is a section $Z'$ of $\pi'$ such that $Z' \cdot Z' = -n + 2$ and the curve $Z'$ is $G$-invariant. If $n = 2$, then we are done. We similarly see that $n \neq 3$. Hence $n \geqslant 4$. Repeating the construction above $\lfloor n/2 \rfloor$ times, we complete the proof. $\square$

To complete the proof of Theorem B.10, we may put $X \cong \mathbb{P}^1 \times \mathbb{P}^1$ and assume that $\pi \colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ is the projection onto the first factor.

**Lemma B.16.** *Suppose that there is a $G$-invariant section $Z$ of the fibration $\pi$. Then there is a commutative diagram*

$$
\begin{array}{ccc}
X \dashrightarrow\!\!\!\!\!\!\!\!\overset{\rho}{\phantom{xx}}\!\!\!\!\!\!\!\!\dashrightarrow & & \mathbb{P}^1 \times \mathbb{P}^1 \\
\pi \downarrow & & \downarrow \overline{\pi} \\
\mathbb{P}^1 & =\!=\!=\!=\!= & \mathbb{P}^1
\end{array}
$$

*where $\rho$ is a birational map inducing a monomorphism $\overline{\upsilon} \colon G \to \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ such that $\overline{\upsilon}(g) = \rho \circ \upsilon(g) \circ \rho^{-1}$ for every $g \in G$, and $\overline{\upsilon}$ is induced by the natural action of the group $\mathbb{A}_5 \times \mathrm{id}_{\mathbb{P}^1}$ on $\mathbb{P}^1 \times \mathbb{P}^1$.*

*Proof.* Let $\Sigma$ be a $G$-orbit such that $\Sigma \subset Z$ and $|\Sigma| = 60$. Then there is a commutative diagram
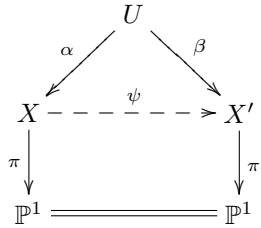


where $\psi$ is a birational map, $\pi'$ is a $\mathbb{P}^1$-bundle, $\alpha$ is the blow-up of $\Sigma$, and $\beta$ is the blow-down of the proper transforms of the fibres of $\pi$ that pass through the points of $\Sigma$. The birational map $\psi$ induces a monomorphism $\upsilon' \colon G \to \mathrm{Aut}(X')$ such that $\upsilon'(g) = \psi \circ \upsilon(g) \circ \psi^{-1}$ for every $g \in G$. We identify the subgroup $\upsilon'(G)$ with $G$. Put $Z' = \psi(Z)$. Then $Z' \cdot Z' = Z \cdot Z - 60$ and the curve $Z'$ is a $G$-invariant section of $\pi'$. We put $m = -Z' \cdot Z'$. If $m \geqslant 0$, then $X' \cong \mathbb{F}_m$ and we are done by Lemma B.15. If $m < 0$, then we can repeat the construction above $\lceil m/60 \rceil$ times to complete the proof. $\square$

We define a biregular involution $\tau$ of the surface $\mathbb{P}^1 \times \mathbb{P}^1$ by putting $\tau(a, b) = (b, a)$ for all $(a, b) \in \mathbb{P}^1 \times \mathbb{P}^1$. Then $\tau$ induces a monomorphism $\upsilon' \colon G \to \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ such that $\upsilon'(g) = \tau \circ \upsilon(g) \circ \tau$ for every $g \in G$. Hence the triple $(\mathbb{P}^1 \times \mathbb{P}^1, \xi \circ \tau, \upsilon')$ is a regularization of the pair $(G, \varphi)$.

**Lemma B.17.** *The monomorphism $\upsilon$ is induced by the natural action of the group $\mathrm{id}_{\mathbb{P}^1} \times \mathbb{A}_5$ if and only if $\upsilon'$ is induced by the natural action of $\mathbb{A}_5 \times \mathrm{id}_{\mathbb{P}^1}$.*

*Proof.* This is obvious. $\square$

We fix a monomorphism $\iota \colon G \to \mathrm{Aut}(\mathbb{P}^1)$ and an outer automorphism $\nu \in \mathrm{Aut}(G)$. It follows from Lemmas B.14, B.16, B.17 that we can assume that $\upsilon(g)(a, b) = (\iota(g)(a), \iota \circ \nu(g)(b))$ for every $g \in G$ and every $(a, b) \in \mathbb{P}^1 \times \mathbb{P}^1$.

**Lemma B.18.** *There is a $G$-invariant section of $\pi$.*

*Proof.* An explicit calculation shows that there is a $G$-invariant curve $Z \subset \mathbb{P}^1 \times \mathbb{P}^1$ such that

$$Z \sim \pi^*(\mathcal{O}_{\mathbb{P}^1}(7)) \otimes (\pi \circ \tau)^*(\mathcal{O}_{\mathbb{P}^1}(1)).$$

In particular, it follows that $Z$ is a $G$-invariant section of $\pi$. $\square$

Theorem B.10 is proved.

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