Available online at www.sciencedirect.com
ScienceDirect
Advances in Mathematics 217 (2008) 97–124
ADVANCES IN Mathematics
www.elsevier.com/locate/aim

Fano varieties with many selfmaps

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Received 19 January 2007; accepted 18 June 2007

Available online 29 June 2007

Communicated by Michael J. Hopkins

Abstract

We study global log canonical thresholds of anticanonically embedded quasismooth weighted Fano threefold hypersurfaces having terminal quotient singularities to prove the existence of a Kähler–Einstein metric on most of them, and to produce examples of Fano varieties with infinite discrete groups of birational automorphisms.

Keywords: Fano threefold; Log canonical threshold; Weighted hypersurface; Kähler–Einstein metric; Birational automorphism; Non-rationality

1. Introduction

Let X be a Fano variety[1] of dimension n that has at most log terminal singularities.

Definition 1.1. The global log canonical threshold of the variety X is the number

$$\mathrm{lct}(X)=\sup\left\{\lambda\in\mathbb{Q}\;\middle|\;\begin{array}{l}\text{the log pair } (X,\lambda H) \text{ has log canonical singularities}\\ \text{for every effective } \mathbb{Q}\text{-divisor } H \text{ such that } H\equiv -K_X\end{array}\right\}\geqslant 0.$$

It follows from [6,11,13] that the Fano variety X has an orbifold Kähler–Einstein metric in the case when X has quotient singularities and the inequality $\mathrm{lct}(X)>n/(n+1)$ holds.[2]

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[1] We assume that all varieties are projective, normal, and defined over $\mathbb{C}$.

[2] The number $\mathrm{lct}(X)$ is an algebraic counterpart of the α-invariant introduced in [13].

0001-8708/$ – see front matter
doi:10.1016/j.aim.2007.06.010

Example 1.2. Let X be a general hypersurface in $\mathbb{P}(1^4, 3)$ of degree 6. Then $\mathrm{lct}(X) = 1$ by [12].

Quasismooth anticanonically embedded weighted Fano threefold hypersurfaces with terminal singularities are studied extensively in [2–5]. In this paper we prove the following result.

Theorem 1.3. *Let X be a general quasismooth hypersurface in $\mathbb{P}(1, a_1, \ldots, a_4)$ of degree $\sum_{i=1}^{4} a_i$ having at most terminal singularities such that $-K_X^3 \leqslant 1$. Then* $\mathrm{lct}(X) = 1$.

The proof of Theorem 1.3 is algebro-geometric, but Theorem 1.3 implies the following result.

Corollary 1.4. *With the assumptions of Theorem* 1.3*, the variety X has a Kähler–Einstein metric.*

It follows from [5,12] that Theorem 1.3 also implies the following result (see Theorem 6.5).

Corollary 1.5. *Let $X_1, \ldots, X_r$ be varieties that satisfy all hypotheses of Theorem* 1.3. *Then*

$$\mathrm{Bir}(X_1 \times \cdots \times X_r) = \left\langle \prod_{i=1}^{r} \mathrm{Bir}(X_i),\ \mathrm{Aut}(X_1 \times \cdots \times X_r) \right\rangle,$$

the variety $X_1 \times \cdots \times X_r$ is non-rational, and for any dominant map $\rho : X_1 \times \cdots \times X_r \dashrightarrow Y$ whose general fiber is rationally connected, there is a commutative diagram

$$\begin{tikzcd} X_1 \times \cdots \times X_r \arrow[r, dashed, "\sigma"] \arrow[d, "\pi"'] & X_1 \times \cdots \times X_r \arrow[d, dashed, "\rho"] \\ X_{i_1} \times \cdots \times X_{i_k} \arrow[r, dashed, "\xi"'] & Y, \end{tikzcd}$$

where ξ and σ are birational maps, and π is a projection for some $\{i_1, \ldots, i_k\} \subsetneq \{1, \ldots, r\}$.

Unlike those of dimension three, no Fano varieties of dimension four or higher having infinite groups of birational automorphisms whose birational automorphisms are well understood have been known so far. However, we can now easily obtain the following example.

Example 1.6. Let X be a general hypersurface in $\mathbb{P}(1, 1, 4, 5, 10)$ of degree 20. Then it immediately follows from [3,5] and Corollary 1.5 that there is an exact sequence of groups

$$1 \longrightarrow \prod_{i=1}^{m} (\mathbb{Z}_2 * \mathbb{Z}_2) \longrightarrow \mathrm{Bir}(\underbrace{X \times \cdots \times X}_{m \text{ times}}) \longrightarrow \mathrm{S}_m \longrightarrow 1,$$

where $\mathbb{Z}_2 * \mathbb{Z}_2$ is the infinite dihedral group.

The assertion of Theorem 1.3 may fail without the generality assumption.

Example 1.7. Let X be a hypersurface in $\mathbb{P}(1,1,2,6,9)$ of degree 18 given by the equation

$$w^2 = t^3 + z^9 + y^{18} + x^{18} \subset \mathbb{P}(1,1,2,6,9) \cong \operatorname{Proj}\big(\mathbb{C}[x,y,z,t,w]\big),$$

where $\mathrm{wt}(x) = \mathrm{wt}(y) = 1$, $\mathrm{wt}(z) = 2$, $\mathrm{wt}(t) = 6$, $\mathrm{wt}(w) = 9$. The hypersurface X has terminal quotient singularities, and $-K_X^3 = 1/6$. Arguing as in the proof of Theorem 1.3, we see that

$$\mathrm{lct}(X) = \sup\big\{\lambda \in \mathbb{Q} \,\big|\, \text{the log pair } (X, \lambda D) \text{ is log canonical for every Weil divisor } D \in |-K_X|\big\},$$

which easily implies that $\mathrm{lct}(X) = 17/18$ by Lemma 8.12 and Proposition 8.14 in [9].

Nevertheless, the proof of Theorem 1.3 in [5] and the proof of Theorem 1.3 can also be used to construct explicit examples of Fano threefolds to which Corollaries 1.4 and 1.5 can be applied.

Example 1.8. Let X be a hypersurface in $\mathbb{P}(1,2,2,3,7)$ of degree 14 given by the equation

$$w^2 = t^4 z + y^7 - z^7 + x^{14} \subset \mathbb{P}(1,2,2,3,7) \cong \operatorname{Proj}\big(\mathbb{C}[x,y,z,t,w]\big),$$

where $\mathrm{wt}(x) = 1$, $\mathrm{wt}(y) = \mathrm{wt}(z) = 2$, $\mathrm{wt}(t) = 3$, $\mathrm{wt}(w) = 7$. The hypersurface X has terminal quotient singularities, and $-K_X^3 = 1/6$. Arguing as in the proof of Theorem 1.3, we see that

$$\mathrm{lct}(X) = \sup\big\{\lambda \in \mathbb{Q} \,\big|\, \text{the log pair } (X, \lambda D) \text{ is log canonical}\big\},$$

where D is the unique Weil divisor in $|-K_X|$. Then $\mathrm{lct}(X) = 1$ by Lemma 8.12 and Proposition 8.14 in [9]. The threefold X has a Kähler–Einstein metric, and the group $\mathrm{Bir}(X \times X)$ is finite.

The proof of Theorem 1.3 is based on the results obtained in [2–5], but it is lengthy, because the hypotheses of Theorem 1.3 are satisfied for general members of 90 out of 95 families of quasismooth terminal anticanonically embedded weighted Fano threefold hypersurfaces (see [7]).

For the convenience of the reader, we organize this paper in the following way:

- we prove Theorem 1.3 in Section 2 omitting the proofs of Lemmas 2.3, 2.10, 2.11;
- we prove auxiliary technical Lemmas 2.3, 2.10, 2.11 in Sections 3, 4, 5, respectively;
- we consider one important generalization of Corollary 1.5 in Section 6.

2. The proof of main result

Let X be a general quasismooth hypersurface in $\mathbb{P}(1, a_1, a_2, a_3, a_4)$ of degree $d = \sum_{i=1}^4 a_i$ with terminal singularities, and let $\beth \in \{1, \ldots, 95\}$ be the ordinal number of (a_1, a_2, a_3, a_4) in the notation of Table 5 in [7], where $a_1 \leqslant a_2 \leqslant a_3 \leqslant a_4$. Then $-K_X^3 \leqslant 1 \Leftrightarrow \beth \geqslant 6$.

We suppose that $\beth \geqslant 6$, but there is $D \in |-nK_X|$ such that $(X, \frac{1}{n}D)$ is not log canonical, where n is a natural number. Then to prove Theorem 1.3 it is enough to derive a contradiction, because the class group of the hypersurface X is generated by the divisor $-K_X$.

Remark 2.1. Let V be a variety, let B and B' be effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors on V such that the singularities of the log pairs (V, B) and (V, B') are log canonical, and let α be a rational number such that $0 \leqslant \alpha \leqslant 1$. Then the log pair $(V, \alpha B + (1-\alpha)B')$ is log canonical.

Thus, we may assume that D is an irreducible surface due to Remark 2.1.

Lemma 2.2. *The inequality $n \neq 1$ holds.*

Proof. Suppose that $n = 1$. Then the log pair (X, D) is log canonical at every singular point of the threefold X by Lemma 8.12 and Proposition 8.14 in [9]. Thus, the equality $a_1 = 1$ holds, because the linear system $|-K_X|$ consists of a single surface in the case when $a_1 \neq 1$.

The equality $a_1 = 1$ holds for 36 values of $\daleth \in \{6, 7, \dots, 95\}$, but all possible cases are very similar. So for the sake of simplicity, we assume that $\daleth = 14$. Then there is a natural double cover $\pi : X \to \mathbb{P}(1, 1, 1, 4)$ branched over a general hypersurface $F \subset \mathbb{P}(1, 1, 1, 4)$ of degree 12.

Suppose that the singularities of the log pair (X, D) are not log canonical at some smooth point P of the threefold X. Let us show that this assumption leads to a contradiction.

Put $\bar{D} = \pi(D)$ and $\bar{P} = \pi(P)$. Counting parameters, we see that $\mathrm{mult}_{\bar{P}}(F|_{\bar{D}}) \leqslant 2$, which is a contradiction, because $(\bar{D}, \frac{1}{2}F|_{\bar{D}})$ is not log canonical at $\bar{P}$ by Lemma 8.12 in [9]. □

Lemma 2.3. *The log pair $(X, \frac{1}{n}D)$ is log canonical at smooth points of the threefold X.*

Proof. See Section 3. □

Therefore, there is a singular point O of the threefold X such that $(X, \frac{1}{n}D)$ is not canonical at the point O. It follows from [7] that O is a singular point of type $\frac{1}{r}(1, a, r - a)$, where a and r are coprime natural numbers such that $r > 2a - 1$ (see Table 5 in [7] for the values of a and r).

Let $\alpha : U \to X$ be a blow up of O with weights $(1, a, r - a)$. Then

$$-K_U^3 = -K_X^3 - \frac{1}{r^3}E^3 = -K_X^3 - \frac{1}{ra(r-a)} = \frac{\sum_{i=1}^4 a_i}{a_1a_2a_3a_4} - \frac{1}{ra(r-a)}, \tag{2.4}$$

where E is the exceptional divisor of α. There is a rational number μ such that

$$\bar{D} \equiv \alpha^*(D) - \mu E \equiv -nK_U + (n/r - \mu)E$$

where $\bar{D}$ is the proper transform of D on U. Then it follows from [8] that $\mu > n/r$.

Lemma 2.5. *The inequality $-K_U^3 \geqslant 0$ holds.*

Proof. Suppose that $-K_U^3 < 0$. Let C be a curve in E. Then the curve C generates an extremal ray of the cone $\mathbb{NE}(U)$. Moreover, it follows from Corollary 5.4.6 in [5] that there is an irreducible curve $\Gamma \subset U$ such that Γ generates the extremal ray of $\mathbb{NE}(U)$ that is different from $\mathbb{R}_{\geqslant 0}C$, and

$$\Gamma \equiv -K_U \cdot (-bK_U + cE),$$

where $b > 0$ and $c \geqslant 0$ are integers (see Remark 5.4.7 in [5]).

Let T be a divisor in $|-K_U|$. Then $\bar{D} \cdot T$ is effective, because $\bar{D} \neq T$. However we have

$$\bar{D} \cdot T \equiv -K_U \cdot \big(-nK_U + (n/r - \mu)E\big) \notin \mathbb{NE}(U),$$

because $\mu > n/r$, $b > 0$, and $c \geqslant 0$. So we have a contradiction. □

Taking into account the possible values of (a_1, a_2, a_3, a_4), we see that $\gimel \notin \{75, 84, 87, 93\}$.

Lemma 2.6. *The inequality $-K_U^3 \neq 0$ holds.*

Proof. Firstly, suppose that $-K_U^3 = 0$ and $\gimel \neq 82$. Then the linear system $|-rK_U|$ does not have base points for $r \gg 0$ and induces a morphism $\eta : U \to \mathbb{P}(1, a_1, a_2)$ such that the diagram

$$\begin{tikzcd} & U \arrow[dl, "\alpha"'] \arrow[dr, "\eta"] & \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1, a_1, a_2) \end{tikzcd}$$

is commutative, where ψ is a natural projection. The morphism η is an elliptic fibration. Thus

$$\bar{D} \cdot C = -nK_U \cdot C + (n/r - \mu)E \cdot C = (n/r - \mu)E \cdot C < 0,$$

where C is a general fiber of η, which is a contradiction.

Suppose that $-K_U^3 = 0$ and $\gimel = 82$. Then X is a hypersurface in $\mathbb{P}(1, 1, 5, 12, 18)$ of degree 36, whose singularities consist of two points P and Q of types $\frac{1}{5}(1, 2, 3)$ and $\frac{1}{6}(1, 1, 5)$, respectively.

We see that either $P = O$, or $Q = O$. The hypersurface X can be given by the equation

$$z^7 y + \sum_{i=0}^{6} z^i f_{36-5i}(x, y, z, t) = 0 \subset \mathbb{P}(1, 1, 5, 12, 18) \cong \mathrm{Proj}\big(\mathbb{C}[x, y, z, t, w]\big),$$

where $\mathrm{wt}(x) = \mathrm{wt}(y) = 1$, $\mathrm{wt}(z) = 5$, $\mathrm{wt}(t) = 12$, $\mathrm{wt}(w) = 18$, and f_i is a quasihomogeneous polynomial of degree i. Then P is given by the equations $x = z = t = w = 0$.

Suppose that $Q = O$. Then the linear system $|-rK_U|$ has no base points for $r \gg 0$, which leads to a contradiction as in the case when $\gimel \neq 82$. So we see that $P = O$.

Let $\bar{S}$ be the proper transform on U of the surface that is cut out on X by $y = 0$. Then

$$\bar{S} \equiv \alpha^*(-K_X) - \frac{6}{5}E,$$

and the base locus of the pencil $|-K_U|$ consists of two irreducible curves L and C such that the curve L is contained in the α-exceptional surface E, and the curve $\pi(C)$ is the unique base curve of the pencil $|-K_X|$. Then $-K_U \cdot C = -1/6$ and $-K_U \cdot L > 0$. We have $\mu \leqslant n/5$ due to

$$n/5 - \mu = \big(-K_U + \alpha^*(-5K_X)\big) \cdot \bar{S} \cdot \bar{D} \geqslant 0,$$

because it follows from Lemma 8.12 and Proposition 8.14 in [9] that $\bar{D} \neq \bar{S}$. However we know that the inequality $\mu > n/5$ holds by [8]. So again we have a contradiction. □

Thus, taking into account the equality (2.4) and possible values of (a_1, a_2, a_3, a_4), we see that

$\daleth \notin \{11, 14, 19, 22, 28, 34, 37, 39, 49, 52, 53, 57, 59, 64, 66, 70, 72, 73, 78, 80, 81, 86, 88, 89, 90, 92, 94, 95\}$

by Lemma 2.6. So the assertion of Theorem 1.3 is proved for 32 values of (a_1, a_2, a_3, a_4).

Lemma 2.7. *The groups* $\mathrm{Bir}(X)$ *and* $\mathrm{Aut}(X)$ *do not coincide.*

Proof. Suppose that $\mathrm{Bir}(X) = \mathrm{Aut}(X)$. Let $\bar{S}$ be a general surface in $|-K_U|$. Then it follows from Lemma 5.4.5 in [5] that there is an irreducible surface $\bar{T} \subset U$ such that

- the equivalence $\bar{T} \sim c\bar{S} - bE$ holds, where $c \geqslant 1$ and $b \geqslant 1$ are natural numbers,
- the scheme-theoretic intersection $\bar{T} \cdot \bar{S}$ is an irreducible and reduced curve Γ,
- the curve Γ generates an extremal ray of the cone $\mathbb{NE}(U)$.

The surface $\bar{T}$ is easy to construct explicitly (see [5]), and the possible values for the natural numbers c and b can be found in [5]. The surface $\bar{T}$ is determined uniquely by the point O.

Put $T = \alpha(\bar{T})$. Then it follows from Lemma 8.12 and Proposition 8.14 in [9] that the singularities of the log pair $(X, \frac{1}{c}T)$ are log canonical. Therefore, we have $D \neq T$.

Let $\mathcal{P}$ be the pencil generated by the effective divisors nT and cD. Then the singularities of the log pair $(X, \frac{1}{cn}\mathcal{P})$ are not canonical, which is impossible due to [5]. $\square$

It follows from [5] and Lemma 2.7 that $\daleth \notin \{21, 29, 35, 50, 51, 55, 62, 63, 67, 71, 77, 82, 83, 85, 91\}$.

Lemma 2.8. *The divisor* $-K_U$ *is nef.*

Proof. Suppose that $-K_U$ is not nef. Then it follows from [5] that $\daleth = 47$ and O is a singular point of type $\frac{1}{5}(1, 2, 3)$. The hypersurface X can be given by the equation

$$z^4y + \sum_{i=0}^{3} z^i f_{21-5i}(x, y, w, t) = 0 \subset \mathbb{P}(1, 1, 5, 7, 8) \cong \mathrm{Proj}\big(\mathbb{C}[x, y, z, t, w]\big),$$

where $\mathrm{wt}(x) = \mathrm{wt}(y) = 1$, $\mathrm{wt}(z) = 5$, $\mathrm{wt}(t) = 7$, $\mathrm{wt}(w) = 8$, and f_i is a general quasihomogeneous polynomial of degree i. Let S be the surface on X that is cut out by the equation $y = 0$, and $\bar{S}$ be the proper transform of the surface S on the threefold U. Then

$$\bar{S} \equiv \alpha^*(-K_X) - \frac{6}{5}E,$$

but the divisor $-3K_U + \alpha^*(-5K_X)$ is nef (see [2]). Thus, the inequality $\mu \leqslant n/5$ holds due to

$$n/5 - \mu = \frac{1}{3}\big(-3K_U + \alpha^*(-5K_X)\big) \cdot \bar{S} \cdot \bar{D} \geqslant 0,$$

because $D \neq S$. However we know that $\mu > n/5$. So we have a contradiction. $\square$

Thus, the divisor $-K_U$ is nef and big, because $-K_U^3 > 0$ by Lemmas 2.5 and 2.6.

Lemma 2.9. *The inequality $\mu/n - 1/r < 1$ holds.*

Proof. We only consider the case when $\daleth = 58$ and O is a singular point of type $\frac{1}{10}(1,3,7)$, because the proof is similar in all other cases (cf. Lemma 5.1). Then X can be given by

$$w^2z + wf_{14}(x,y,z,t) + f_{24}(x,y,z,t) = 0 \subset \mathbb{P}(1,3,4,7,10) \cong \mathrm{Proj}\big(\mathbb{C}[x,y,z,t,w]\big),$$

where $\mathrm{wt}(x)=1$, $\mathrm{wt}(y)=3$, $\mathrm{wt}(z)=4$, $\mathrm{wt}(t)=7$, $\mathrm{wt}(w)=10$, and f_i is a quasihomogeneous polynomial of degree i. Let R be the surface on X that is cut out by $z=0$, and $\bar{R}$ be the proper transform of the surface R on the threefold U. Then

$$\bar{R} \equiv \alpha^*(-4K_X) - \frac{7}{5}E,$$

and $(X, \frac{1}{4}R)$ is log canonical at O by Lemma 8.12 and Proposition 8.14 in [9]. Then $R \neq D$ and

$$0 \leqslant -K_U \cdot \bar{R} \cdot \bar{D} = 4n/35 - 2\mu/3,$$

because $-K_U$ is nef. Thus, we have $\mu \leqslant 6n/35$, which implies that $\mu/n - 1/10 < 1$. □

So the log pair $(U, \frac{1}{n}\bar{D} + (\mu/n - 1/r)E)$ is not log canonical at some point $P \in E$, because

$$K_U + \frac{1}{n}\bar{D} \equiv \alpha^*\left(K_X + \frac{1}{n}D\right) + (1/r - \mu/n)E.$$

Lemma 2.10. *The threefold U is smooth at the point P.*

Proof. See Section 4. □

Thus, the inequality $\mathrm{mult}_P(\bar{D}) > n + n/r - \mu$ holds. But it follows from [5] that

- either $d = 2r + a_j$ for some j, and there is a quadratic involution $\tau \in \mathrm{Bir}(X)$ induced by O,
- or $d = 3r + a_j$ for some j, and there is an elliptic involution $\tau \in \mathrm{Bir}(X)$ induced by O,

where $d = \sum_{i=1}^{4} a_i$.

Lemma 2.11. *The inequality $d \neq 2r + a_j$ holds for every $j \in \{1,2,3,4\}$.*

Proof. See Section 5. □

Thus, it follows from [5] that there is $j \in \{1,2,3,4\}$ such that $d = 3r + a_j$.

Remark 2.12. Let V be a threefold with isolated singularities, let $B \neq T$ be effective irreducible divisors on the threefold V, and let H be a nef divisor on the threefold V. Put

$$B \cdot T = \sum_{i=1}^{r} \epsilon_i L_i + \Delta,$$

where L_i is an irreducible curve, ϵ_i is a non-negative integer, and Δ is an effective one-cycle whose support does not contain the curves $L_1, \dots, L_r$. Then $\sum_{i=1}^{r} \epsilon_i H \cdot L_i \leqslant B \cdot T \cdot H$.

It follows from Lemma 2.11 that $\daleth \in \{7, 20, 23, 36, 40, 44, 61, 76\}$ (see [5]).

Lemma 2.13. *The case* $\daleth \in \{7, 20, 36\}$ *is impossible.*

Proof. Suppose that $\daleth \in \{7, 20, 36\}$. Then $a_1 = 1$, and it follows from Lemma 2.11 that O is a singular point of type $\frac{1}{a_2}(1, 1, a_2 - 1)$. Then $|-rK_U|$ induces a birational morphism $\sigma : U \to V$ such that σ contracts smooth rational curves $C_1, \dots, C_l$, and V is a hypersurface in $\mathbb{P}(1, 1, a_3, 2a_4, 3a_4)$ of degree $6a_4$, where $l = d(d - a_4)/a_3$. Let T be the surface in $|-K_U|$ that contains P.

Suppose that $P \notin \bigcup_{i=1}^{l} C_i$. Then it follows from the proof of Theorem 5.6.2 in [5] that there are natural number $s > 0$ and a surface $H \in |-s2a_4K_U|$ such that

$$s2a_4\big(-nK_X^3 - \mu/(a_2 - 1)\big) = \bar{D} \cdot T \cdot H \geqslant \mathrm{mult}_P(\bar{D})s > (n + n/a_2 - \mu)s,$$

which is impossible, because $\mu > n/a_2$. So we may assume that $P \in C_1$.

Put $\bar{D} \cdot T = mC_1 + \Delta$, where m is a non-negative integer, and Δ is an effective cycle such that the support of Δ does not contain the curve C_1. The curve C_1 is a smooth rational curve such that $\alpha^*(-K_X) \cdot C_1 = 2/a_2$ and $E \cdot C_1 = 2$.

It follows from [5] that there is a surface $R \in |-a_3K_U|$ such that R contains C_1, but R does not contain components of the cycle Δ passing through the point P. Then

$$a_3\big(-nK_X^3 - \mu/(a_2 - 1)\big) = R \cdot \Delta \geqslant \mathrm{mult}_P(\Delta) > n + n/a_2 - \mu - m,$$

which implies that $m > a_3n/a_4$, because $\mu > n/a_2$. Therefore, we have

$$a_3n/a_4 < m \leqslant \frac{-dnK_X \cdot \alpha(C_1)}{a_1a_2a_3a_4} = \frac{dn}{2a_1a_3a_4}$$

by Remark 2.12, because $-K_X \cdot \alpha(C_1) = 2/a_2$. The inequalities just obtained imply that $\daleth = 7$.

Let $\psi : X \dashrightarrow \mathbb{P}(1, a_1, a_2)$ be a natural projection. The fiber of ψ over $\psi(P)$ consists of two irreducible components, and one of them is C_1. Let Z be the other component of this fiber. Then

$$C_1^2 = -2, \qquad C_1 \cdot Z = 2, \qquad Z^2 = -4/3$$

on the surface T. Put $\Delta = \bar{m}Z + \Omega$, where $\bar{m}$ is a non-negative integer and Ω is an effective one-cycle whose support does not contain the curve Z. Then

$$4n/3 - 2\mu - 5\bar{m}/3 = (Z + C_1) \cdot \Omega > 3n/2 - \mu - m,$$

and $4\bar{m}/3 \geqslant 2m - 5n/6$, because $\Omega \cdot Z \geqslant 0$. The inequalities just obtained immediately imply that the inequality $\mu \leqslant n/2$ holds. So we have a contradiction, because $\mu > n/2$. $\square$

Hence, it follows from Lemmas 2.11 and 2.13 that $\gimel \in \{23, 40, 44, 61, 76\}$ and $d = 3r + a_j$, where $r = a_3 > 2a$ and $1 \leqslant j \leqslant 2$. Then X has a singular point Q of type $\frac{1}{\bar{r}}(1, \bar{a}, \bar{r} - \bar{a})$ such that

$$-K_X^3 = \frac{1}{ra(r-a)} + \frac{1}{\bar{r}\bar{a}(\bar{r}-\bar{a})},$$

where $\bar{r} = a_4 > 2\bar{a}$ and $\bar{a} \in \mathbb{N}$. It follows from [5] that there is a commutative diagram

$$\begin{tikzcd}
U \arrow[dd, "\sigma"'] \arrow[dr, "\alpha"] & & W \arrow[ll, "\gamma"'] \arrow[ddr, "\eta"] & \\
& X \arrow[ddrr, dashed, "\psi"] & & \\
V \arrow[r, hook] & \mathbb{P}(1, a_1, a_2, 2a_4, 3a_4) \arrow[r, dashed, "\chi"'] & \mathbb{P}(1, a_1, a_2, 2a_4) \arrow[r, dashed, "\xi"'] & \mathbb{P}(1, a_1, a_2),
\end{tikzcd}$$

where ξ, χ, ψ are projections, η is an elliptic fibration, γ is a weighted blow up of a point that dominates the point Q with weights $(1, \bar{a}, \bar{r} - \bar{a})$, and σ is a birational morphism that contracts smooth curves $C_1, \ldots, C_l$ such that V is a hypersurface in $\mathbb{P}(1, a_1, a_2, 2a_4, 3a_4)$ of degree $6a_4$, where $l = d(d - a_4)/(a_1 a_2)$. Let L be a curve in $|\mathcal{O}_{\mathbb{P}(1,a,r-a)}(1)|$, where $E \cong \mathbb{P}(1, a, r - a)$.

Lemma 2.14. *Suppose that $P \notin L$. Then $\mu > na(r + 1)/(r^2 + ar)$.*

Proof. There is a unique curve $C \in |\mathcal{O}_{\mathbb{P}(1,a,r-a)}(a)|$ such that $P \in C$. Put $\bar{D}|_E = \delta C + \Upsilon \equiv r\mu L$, where δ is a non-negative integer and Υ is an effective cycle such that $C \not\subset \operatorname{Supp}(\Upsilon)$. Then

$$(r\mu - a\delta)/(r - a) = (r\mu - a\delta)L \cdot C = C \cdot \Upsilon \geqslant \operatorname{mult}_P(\Upsilon) > n + n/r - \mu - \delta,$$

which implies that $\mu > na(r + 1)/(r^2 + ar)$, because $\delta \leqslant r\mu/a$. □

Let T be a surface in $|-K_U|$. Then $-K_U \cdot T \cdot \bar{D} \geqslant 0$, which implies that $\mu \leqslant -na(r - a)K_X^3$.

Lemma 2.15. *The point P is not contained in the surface T.*

Proof. Suppose that P is contained in the surface T. Then P is not contained in the base locus of the pencil $|-a_1 K_U|$, because the base locus of the pencil $|-a_1 K_U|$ does not contain smooth points of the surface E. The point P is not contained in the union $\bigcup_{i=1}^{l} C_i$, because $P \in T$.

The proof of Theorem 5.6.2 in [5] implies the existence of a surface $H \in |-s2a_1 a_4 K_U|$ such that

$$s2a_1a_4\bigl(-nK_X^3 - \mu/a_2\bigr) = \bar{D} \cdot H \cdot T \geqslant \operatorname{mult}_P(\bar{D})s > (n + n/r - \mu)s,$$

where s is a natural number, which is impossible, because $\mu > n/r$. □

We have $T|_E \sim \mathcal{O}_{\mathbb{P}(1,a,r-a)}(1)$. Taking into account that $\gimel \in \{7, 20, 23, 36, 40, 44, 61, 76\}$, we see that $\gimel \in \{23, 44\}$ by Lemmas 2.14 and 2.15, because $\mu \leqslant -na(r - a)K_X^3$.

Let S be a surface in $|-a_1K_U|$ that contains P. Then $\bar{D} \neq S$, because $\mu > n/r$.

Lemma 2.16. *The point P is contained in $\bigcup_{i=1}^{l} C_i$.*

Proof. Suppose that $P \notin \bigcup_{i=1}^{l} C_i$. Then the proof of Theorem 5.6.2 in [5] implies that

$$s2a_1a_4\left(-nK_X^3 - \mu/a_2\right) = \bar{D} \cdot H \cdot S \geqslant \mathrm{mult}_P(\bar{D})s > (n + n/r - \mu)s$$

for some $s \in \mathbb{N}$ and a surface $H \in |-s2a_4K_U|$, which is impossible, because $\mu > n/r$. □

We may assume that $P \in C_1$. Put $\bar{D} \cdot S = mC_1 + \Delta$, where m is a non-negative integer, and Δ is an effective cycle whose support does not contain C_1. Then it follows from Remark 2.12 that the inequality $m \leqslant nd/(a_2d - a_2a_3)$ holds, because $-K_X \cdot \alpha(C_1) = (d - a_3)/(a_3a_4)$.

It follows from the proof of Theorem 5.6.2 in [5] that there is $R \in |-s2a_4K_U|$ such that

$$s2a_1a_4\left(-nK_X^3 - \mu/a_2\right) = R \cdot \Delta \geqslant \mathrm{mult}_P(\Delta)s > s(n + n/r - \mu - m),$$

where $s \in \mathbb{N}$. However we have $m \leqslant nd/(a_2d - a_2a_3)$, which implies that $\daleth = 23$.

Therefore, we proved that X is a hypersurface $\mathbb{P}(1, 2, 3, 4, 5)$ of degree 14 and O is a singular point of type $\frac{1}{4}(1, 1, 3)$. Let M be a general surface in the linear system $|-3K_X|$ that passes through the point P. Then $S \cdot M = C_1 + Z_1$, where Z_1 is a curve such that $-K_U \cdot Z_1 = 1/5$. Put

$$\bar{D} \cdot S = mC_1 + \bar{m}Z_1 + \Upsilon,$$

where $\bar{m}$ is a non-negative integer, and Υ is an effective cycle, whose support does not contain the curves C_1 and Z_1. Then $m < 7n/15$ by Remark 2.12. But $\mu > n/4$ and

$$7n/10 - 6\mu/3 - 3\bar{m}/5 = M \cdot \Upsilon \geqslant \mathrm{mult}_P(\Upsilon) > 5n/4 - \mu - m,$$

because $P \notin Z_1$. The inequality obtained implies a contradiction.

Therefore, the assertion of Theorem 1.3 is completely proved.

3. Non-singular points

In this section we prove the assertion of Lemma 2.3. Let us use the assumptions and notation of Lemma 2.3. Take an arbitrary smooth point P of the threefold X.

Lemma 3.1. *Suppose that a_4 divides d, $a_1 \neq a_2$, and $-a_2a_3K_X^3 \leqslant 1$. Then* $\mathrm{mult}_P(D) \leqslant n$.

Proof. Suppose that $\mathrm{mult}_P(D) > n$. Let L be the base curve of $|-a_1K_X|$, and T be a surface in the linear system $|-K_X|$. Then $D \cdot T$ is an effective one-cycle, and $\mathrm{mult}_P(L) = 1$.

Suppose that $P \in L$. Let R be a general surface in $|-a_1K_X|$. Put $D \cdot T = mL + \Delta$, where m is non-negative integer, and Δ is an effective cycle whose support does not contain L. Then

$$-a_1(n - a_1m)K_X^3 = D \cdot T \cdot R - mR \cdot L = R \cdot \Delta \geqslant \mathrm{mult}_P(\Delta) > n - m,$$

which is impossible, because $-a_1K_X^3 \leqslant 1$. Thus, we see that $P \notin L$.

Suppose that $P \in T$. It follows from Theorem 5.6.2 in [5] that

$$ns \geqslant -sa_1a_3nK_X^3 = D \cdot S \cdot T \geqslant \mathrm{mult}_P(D)s > ns$$

for some $s \in \mathbb{N}$ and some surface $S \in |-sa_1a_3K_X|$. Hence, we see that $P \notin T$.

Let G be a general surface in $|-a_2K_X|$ that contains P. Then $G \cdot D$ is an effective cycle, but it follows from Theorem 5.6.2 in [5] that there are $s \in \mathbb{N}$ and $H \in |-sa_3K_X|$ such that

$$ns \geqslant -sa_2a_3nK_X^3 = D \cdot H \cdot G \geqslant \mathrm{mult}_P(D)s > ns,$$

because $-a_2a_3K_X^3 \leqslant 1$. So we have a contradiction. □

Lemma 3.2. *Suppose that a_4 divides d, $1 = a_1 \neq a_2$, and $-a_3K_X^3 \leqslant 1$. Then* $\mathrm{mult}_P(D) \leqslant n$.

Proof. Suppose that $\mathrm{mult}_P(D) > n$. Arguing as in the proof of Lemma 3.1, we see that P is not contained in the base curve of the pencil $|-K_X|$.

Let T be a general surface in $|-K_X|$ that contains P. Then Theorem 5.6.2 in [5] implies that there are $s \in \mathbb{N}$ and $S \in |-sa_3K_X|$ such that $ns \geqslant -sa_3nK_X^3 = D \cdot S \cdot T \geqslant \mathrm{mult}_P(D)s > ns$. □

Lemma 3.3. *Suppose that $a_1 \neq a_2$ and $-a_1a_4K_X^3 \leqslant 1$. Then* $\mathrm{mult}_P(D) \leqslant n$.

Proof. Suppose that $\mathrm{mult}_P(D) > n$. The proof of Lemma 3.2 implies that $a_1 \neq 1$. Arguing as in the proof of Lemma 3.1, we see that P is not contained in the unique surface of $|-K_X|$.

Let S be a surface in $|-a_1K_X|$ that contains P. Then we may assume that $\mathrm{mult}_P(S) \leqslant a_1$, because $P \notin T$ and X is sufficiently general. Thus, we have $S \neq D$.

It follows from Theorem 5.6.2 in [5] that there are $s \in \mathbb{N}$ and $H \in |-sa_4K_X|$ such that H has multiplicity at least $s > 0$ at P and contains no components of $D \cdot S$ passing through P. Then

$$ns \geqslant -sa_1a_4nK_X^3 = D \cdot S \cdot H \geqslant \mathrm{mult}_P(D)s > ns,$$

because $-a_1a_4K_X^3 \leqslant 1$. So we have a contradiction. □

Taking into account the possible values of (a_1, a_2, a_3, a_4), we see that $\mathrm{mult}_P(D) \leqslant n$ for

$$\daleth \notin \{6, 7, 8, 9, 10, 12, 13, 14, 16, 18, 19, 20, 22, 23, 24, 25, 32, 33, 38\}$$

by Lemmas 3.1–3.3. The log pair $(X, \frac{1}{n}D)$ is log canonical at P if $\mathrm{mult}_P(D) \leqslant n$ (see [9]).

Lemma 3.4. *Suppose that $\daleth = 18$. Then $(X, \frac{1}{n}D)$ is log canonical at P.*

Proof. Suppose that the log pair $(X, \frac{1}{n}D)$ is log canonical at the point P. Let us show that this assumption leads to a contradiction. Note that the inequality $\mathrm{mult}_P(D) > n$ holds.

The threefold X is a hypersurface in $\mathbb{P}(1,2,2,3,5)$ of degree 12, whose singularities consist of six points of type $\frac{1}{2}(1,1,1)$, and a point O of type $\frac{1}{5}(1,2,3)$. It follows from [2] that the diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & & W \arrow[ll, "\beta"'] \arrow[d, "\eta"] \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,2,2) \end{tikzcd}$$

commutes, where α is a weighted blow up of the point O with weights $(1,2,3)$, β is a weighted blow up with weights $(1,1,3)$ of a singular point of type $\frac{1}{3}(1,1,2)$, and η is an elliptic fibration.

Let C be a fiber of the projection ψ that passes through the point P, and L be its irreducible reduced component. We have $-K_X\cdot C=4/5$. But the number $-5K_X\cdot L$ is natural if L contains no points of type $\frac{1}{2}(1,1,1)$. Then $C=2L$ whenever C contains a point of type $\frac{1}{2}(1,1,1)$.

Let T be the surface in $|-K_X|$, and let S and $\grave{S}$ be general surfaces in $|-2K_X|$ that passes through the point P. Then S and $\grave{S}$ are irreducible and $S\supset L\subset \grave{S}$, but $S\neq D\neq \grave{S}$.

Suppose now that L is contained in T. Then $C=2L$ and $-K_X\cdot L=2/5$, but the singularities of the curve L consists of at most double points. Put $D|_T=mL+\Upsilon$, where m is a non-negative integer, and Υ is an effective cycle whose support does not contain L. Then

$$2n/5-4m/5=S\cdot\Upsilon\geqslant \mathrm{mult}_P(\Upsilon)\geqslant \mathrm{mult}_P(D)-\mathrm{mult}_P(L)>n-2m,$$

which implies that $m>n/2$. But $m\leqslant n/2$ by Remark 2.12, which implies that $L\not\subset T$.

Suppose that $C=L$. Then $\mathrm{mult}_P(L)\leqslant 2$. Put $D\cdot\grave{S}=\grave{m}C+\grave{\Upsilon}$, where $\grave{m}$ is a non-negative integer and $\grave{\Upsilon}$ is an effective cycle whose support does not contain C. Then

$$4n/5-8\grave{m}/5=S\cdot\grave{\Upsilon}\geqslant \mathrm{mult}_P(\grave{\Upsilon})>n-2m,$$

which implies that $m>n/2$. But $m\leqslant n/2$ by Remark 2.12, which implies that $C\neq L$.

The curve L does not pass through a point of type $\frac{1}{2}(1,1,1)$, and it follows from the generality of the threefold X that $C=L+Z$, where Z is an irreducible curve such that $Z\neq L$.

Put $D|_S=m_LL+m_ZZ+\Omega$, where m_L and m_Z are non-negative integers and Ω is an effective cycle whose support does not contain L and Z. We may assume that $-K_X\cdot L\leqslant -K_X\cdot Z$, which implies that either $-K_X\cdot L=1/5$ and $-K_X\cdot Z=3/5$, or $-K_X\cdot L=-K_X\cdot Z=2/5$.

Suppose that $-K_X\cdot L=2/5$. Then L and Z are smooth outside of O, and

$$4n/5-4m_L/5-4m_Z/5=\grave{S}|_S\cdot\Omega\geqslant \mathrm{mult}_P(\Omega)>n-m_L-m_C,$$

which implies that $m_L+m_C>n$. But $m_L+m_C\leqslant n$ by Remark 2.12.

Thus, we have $-K_X\cdot L=1/5$. The hypersurface X can be given by an equation

$$w^2z+wg(x,y,z,t)+h(x,y,z,t)=0\subset\mathbb{P}(1,2,2,3,5)\cong\mathrm{Proj}\big(\mathbb{C}[x,y,z,t,w]\big),$$

where $\mathrm{wt}(x)=1$, $\mathrm{wt}(y)=\mathrm{wt}(z)=2$, $\mathrm{wt}(t)=3$, $\mathrm{wt}(w)=5$, and g and h are quasihomogeneous polynomials of degree 7 and 12, respectively.

Let R be a surface on the threefold X that is cut out by $z = 0$, and let $\bar{R}$ and $\bar{L}$ be proper transforms of R and L on U, respectively. Then $\bar{R} \cdot \bar{L} < 0$, which implies that $L \subset R \supset Z$, and the curve L is contracted by the projection $X \dashrightarrow \mathbb{P}(1, 2, 2, 3)$ to a point.

Let $\bar{Z}$ be the proper transform of the curve Z on the threefold U, let $\pi : \bar{R} \to R$ be a birational morphism induced by α, and let $\bar{E}$ be the curve on the surface $\bar{R}$ that is contracted by π. Then

$$\bar{L}^2 = -1, \qquad \bar{L} \cdot \bar{Z} = \bar{L} \cdot \bar{E} = 1, \qquad \bar{Z}^2 = -1/3, \qquad \bar{E}^2 = -35/6, \qquad \bar{Z} \cdot \bar{E} = 4/3$$

on the surface $\bar{R}$, which implies that $L^2 = -29/35$, $L \cdot Z = 43/35$, $Z^2 = -1/35$ on the surface R.

Suppose that $P \in L \cap Z$. Then $m_L + 3m_C \leqslant 5n$ by Remark 2.12, but

$$n/5 + 29m_L/35 - 43m_Z/35 = \Omega \cdot L > n - m_L - m_Z,$$
$$2n/5 - 43m_L/35 + m_Z/35 = \Omega \cdot Z > n - m_L - m_Z,$$

which leads to a contradiction. Hence, either $L \ni P \notin Z$, or $Z \ni P \notin L$.

Suppose that $Z \ni P \notin L$. Then $\Omega \cdot Z > n - m_Z$ and $\Omega \cdot L \geqslant 0$, which implies a contradiction.

Thus, we see that $L \ni P \notin Z$. Then we have

$$n/5 + 29m_L/35 - 43m_Z/35 = \Omega \cdot L > n - m_L,$$
$$2n/5 - 43m_L/35 + m_Z/35 = \Omega \cdot Z \geqslant 0,$$

which implies that $m_L < n$. Now it follows from Theorem 7.5 in [9] that the log pair

$$\left(R, L + \frac{m_C}{n}C + \frac{1}{n}\Omega\right)$$

is not log canonical at the point P. Then $\mathrm{mult}_P(\Omega|_L) > n$ by Theorem 7.5 in [9], which implies that the inequality $\Omega \cdot L > n$ holds. The inequality $\Omega \cdot L > n$ leads to a contradiction. $\square$

Lemma 3.5. *Suppose that* $\gimel = 6$. *Then* $\mathrm{mult}_P(D) \leqslant n$.

Proof. Suppose that $\mathrm{mult}_P(D) > n$. It follows from [2] that the threefold X has two quotient singular points O_1 and O_2 of type $\frac{1}{2}(1, 1, 1)$ such that there is a commutative diagram

$$\begin{tikzcd}
 & & U \arrow[dll, "\sigma"'] \arrow[d, "\alpha"] & Y \arrow[l, "\gamma"'] \arrow[dd, "\eta"] \\
V \arrow[dr, "\omega"'] & & X \arrow[dl, dashed, "\xi"'] \arrow[dr, dashed, "\psi"] & \\
 & \mathbb{P}(1,1,1,4) \arrow[rr, dashed, "\chi"'] & & \mathbb{P}(1,1,1),
\end{tikzcd}$$

where ξ, ψ and χ are projections, α is a blow up of O_1 with weights $(1, 1, 1)$, γ is a blow up with weights $(1, 1, 1)$ of the point that dominates O_2, η is an elliptic fibration, ω is a double cover, and σ is a birational morphism that contracts 48 irreducible curves $C_1, \ldots, C_{48}$.

The threefold U contains 48 curves $Z_1, \dots, Z_{48}$ such that $\alpha(Z_i) \cup \alpha(C_i)$ is a fiber of ψ over the point $\psi(C_i)$. Put $\bar{Z}_i = \alpha(Z_i)$ and $\bar{C}_i = \alpha(C_i)$. Let L be a fiber of the projection ψ that passes through the point P, and let T_1 and T_2 be general surfaces in $|-K_X|$ that contain P.

Suppose that L is irreducible. Put $D \cdot T_1 = mL + \Upsilon$, where m is non-negative integer and Υ is an effective cycle whose support does not contain L. Then $m \leqslant n$ by Remark 2.12. But

$$n - m = D \cdot T_1 \cdot T_2 - mT_2 \cdot L = T_2 \cdot \Delta \geqslant \mathrm{mult}_P(\Delta) > n - m\,\mathrm{mult}_P(L),$$

which implies that L is singular at the point P. Then there is a surface $T \in |-K_X|$ that is singular at the point P. Let S is a general surface in $|-2K_X|$ that contains P. Then

$$2n = D \cdot T \cdot S \geqslant \mathrm{mult}_P(D \cdot T) > 2n,$$

which is a contradiction. Hence, the curve L is reducible.

We have $L = \bar{C}_i \cup \bar{Z}_i$. Put $D|_{T_1} = m_1\bar{C}_i + m_2\bar{Z}_i + \Delta$, where m_1 and m_2 are non-negative integers and Δ is an effective cycle whose support does not contain $\bar{C}_i$ and $\bar{Z}_i$.

In the case when $P \in \bar{C}_i \cap \bar{Z}_i$, there is $T \in |-K_X|$ such that T singular at P, and we can obtain a contradiction as above. So we may assume that $P \in \bar{C}_i$ and $P \notin \bar{Z}_i$. Arguing as in the proof of Lemma 3.4, we see that

$$n - 3m_1/2 + m_2 = \bar{C}_i \cdot \Delta > n,$$

because $\bar{C}_i$ is smooth. But we have

$$0 \leqslant \Delta \cdot \bar{Z}_i = n/2 - m_1\bar{C}_i \cdot \bar{Z}_i - m_2\bar{Z}_i^2 = n/2 - 2m_1 + 3m_2/2 < n/2 - m_1/2,$$

which easily leads to a contradiction. $\square$

Arguing as in the proofs of Lemmas 3.4 and 3.5, we see that $(X, \frac{1}{n}D)$ is log canonical at P for

$$\daleth \in \{7, 8, 9, 10, 12, 13, 14, 16, 19, 20, 22, 23, 24, 25, 32, 33, 38\},$$

which completes the proof Lemma 2.3.

4. Singular points

In this section we prove the assertion of Lemma 2.10. Let us use the assumptions and notation of Lemma 2.10. Suppose that P is a singular point of U. Let us derive a contradiction.

The point P is a singular point of type $\frac{1}{\bar{r}}(1, \bar{a}, \bar{r} - \bar{a})$, where $\bar{a}$ and $\bar{r}$ are coprime natural numbers such that $\bar{r} > 2\bar{a}$. Let $\beta : W \to U$ be a blow up of P with weights $(1, \bar{a}, \bar{r} - \bar{a})$. Then

$$-K_W^3 = -K_X^3 - \frac{1}{ra(r-a)} - \frac{1}{\bar{r}\bar{a}(\bar{r} - \bar{a})}.$$

Let $\breve{D}$ be the proper transform of D on W. There is a rational number ν such that

$$\breve{D} \equiv (\alpha \circ \beta)^*(-nK_X) - \mu\beta^*(E) - \nu G,$$

where G is the β-exceptional divisor. Then

$$K_W + \frac{1}{n}\breve{D} + (\mu/n - 1/r)\breve{E} \equiv \beta^*\left(K_U + \frac{1}{n}\bar{D} + (\mu/n - 1/r)E\right) - \epsilon G \equiv -\epsilon G,$$

where $\breve{E}$ is a proper transform of E on the threefold W, and $\epsilon \in \mathbb{Q}$. Then $\epsilon > 0$ due to [8].

Lemma 4.1. *The inequality $-K_W^3 \neq 0$ holds.*

Proof. Suppose that $-K_W^3 = 0$. Then it follows from [2] that the linear system $|-rK_W|$ induces an elliptic fibration $\eta: W \to Y$ for $r \gg 0$. Then $0 \leqslant \breve{D} \cdot C = -\epsilon G \cdot C < 0$, where C is a general fiber of the elliptic fibration η. So we have a contradiction. □

Thus, it follows from [2] that either $-K_W^3 < 0$, or $-K_W$ is nef and big.

Lemma 4.2. *Suppose that $-K_W^3 < 0$. Then $-K_W$ is not big.*

Proof. Suppose that $-K_W$ is big. Then it follows from [2] that we have the following possibilities:

- the equality $\daleth = 25$ holds, and O is a singular point of type $\frac{1}{7}(1,3,4)$;
- the equality $\daleth = 43$ holds, and O is a singular point of type $\frac{1}{9}(1,4,5)$;

but both cases are similar. So we assume that $\daleth = 43$. Then $-K_W - 4\beta^*(K_U)$ is nef (see [2]), and there is a surface H in the linear system $|-2K_X|$ such that

$$\breve{H} \equiv (\alpha \circ \beta)^*(-2K_X) - \frac{11}{9}\beta^*(E) - \frac{3}{2}G,$$

where $\breve{H}$ is a proper transform of the surface H on the threefold W. Thus, we have

$$0 \leqslant \breve{H} \cdot \breve{D} \cdot \big(-K_W - 4\beta^*(K_U)\big) = 5n/9 - 11\mu/4 - \nu,$$

which is impossible, because $\nu - n/3 + 3\mu/4 = n\epsilon > 0$ and $\mu > n/9$. □

Let T be a surface in $|-K_X|$, and $\mathcal{P}$ be the pencil generated by the divisors nT and D. Then

$$\mathcal{B} \equiv -nK_W \equiv (\alpha \circ \beta)^*(-nK_X) - \frac{n}{r}\beta^*(E) - \frac{n}{\bar{r}}G, \tag{4.3}$$

where $\mathcal{B}$ is the proper transforms of the pencil $\mathcal{P}$ on the threefold W.

Lemma 4.4. *The divisor $-K_W$ is nef and big.*

Proof. Suppose that the divisor $-K_W$ is not nef and big. Then $-K_W^3 < 0$, but $-K_W$ is not big by Lemma 4.2. Then the equivalence (4.3) almost uniquely determines[3] the pencil $\mathcal{P}$ due to [4].

[3] For example, it follows from [4] that the equivalence (4.3) implies that $n = 1$ in the case when $a_1 = 1$.

All possible cases are similar. So we assume that $\daleth \in \{45, 48, 58, 69, 74, 79\}$. Then O is a singular point of type $\frac{1}{a_4}(1, a_1, a_3)$, and X can be given by an equation

$$w^2 z + w f(x, y, z, t) + g(x, y, z, t) = 0 \subset \mathbb{P}(1, a_1, a_2, a_3, a_4) \cong \operatorname{Proj}\bigl(\mathbb{C}[x, y, z, t, w]\bigr),$$

where $\mathrm{wt}(x) = 1$, $\mathrm{wt}(y) = a_1$, $\mathrm{wt}(z) = a_2$, $\mathrm{wt}(t) = a_3$, $\mathrm{wt}(w) = a_4$, and f and g are polynomials.

Let S be a surface that is cut out on the threefold X by $z = 0$, and $\mathcal{M}$ be a pencil generated by the divisors $a_2 T$ and S. Then it follows from [4] that either $\mathcal{P} = \mathcal{M}$, or $\mathcal{P} = |-a_1 K_X|$.

Suppose that $\mathcal{P} = |-a_1 K_X|$. Then $\mu = n/a_1$, which is impossible, because $\mu > n/a_4$.

We see that $\mathcal{P} = \mathcal{M}$. Let M be a divisor in $\mathcal{M}$, and $\bar{M}$ be its proper transform on U. Then

$$\bar{M} \equiv \alpha^*(M) - \frac{a_3}{a_4} E$$

in the case when $M \neq S$, but $\mu > n/a_4$. Thus, we see that $D = S$, but $(X, \frac{1}{a_2} S)$ is log canonical at the point O by Lemma 8.12 and Proposition 8.14 in [9], which is a contradiction. $\square$

Taking into account the possible values of (a_1, a_2, a_3, a_4), we see that

$$\daleth \in \{8, 12, 13, 16, 20, 24, 25, 26, 31, 33, 36, 38, 46, 47, 48, 54, 56, 58, 65, 74, 79\}$$

by Lemmas 4.1, 4.2 and 4.4 (see [2]).

Lemma 4.5. *The case* $\daleth \notin \{12, 13, 20, 25, 31, 33, 38, 58\}$ *is impossible.*

Proof. Suppose that $\daleth \notin \{12, 13, 20, 25, 31, 33, 38, 58\}$. Then $r = a_4$, $r - a = a_3$, $\bar{r} = r - a$, $\bar{a} = a$, and $n\epsilon = \nu - (\bar{r} - \bar{a})(n/r - \mu)/\bar{r} - n/\bar{r}$. We may assume that $\daleth \neq 24$, because the case $\daleth = 24$ can be considered in a similar way. Then X can be given by the equation

$$w^2 z + w f(x, y, z, t) + g(x, y, z, t) = 0 \subset \mathbb{P}(1, a_1, a_2, a_3, a_4) \cong \operatorname{Proj}\bigl(\mathbb{C}[x, y, z, t, w]\bigr),$$

where $\mathrm{wt}(x) = 1$, $\mathrm{wt}(y) = a$, $\mathrm{wt}(z) = d - 2a_4$, $\mathrm{wt}(t) = a_3$, $\mathrm{wt}(w) = a_4$, the point O is given by the equations $x = y = z = t = 0$, and f and g are quasihomogeneous polynomials. Then

$$\breve{R} \equiv (\alpha \circ \beta)^*(-a_2 K_X) - \frac{d - r}{r} \beta^*(E) - \frac{\bar{r} - \bar{a}}{\bar{r}} G,$$

where $\breve{R}$ is a proper transform on W of the surface cut out on X by $z = 0$. Then $\breve{D} \neq \breve{R}$, and

$$n \frac{\sum_{i=1}^{4} a_i}{a_1 a_3 a_4} - \frac{\mu(d - r)}{a(r - a)} - \frac{\nu(\bar{r} - \bar{a})}{\bar{a}(\bar{r} - \bar{a})} = -K_W \cdot \breve{D} \cdot \breve{R} \geqslant 0,$$

which implies that $\mu < n/r$, because $\epsilon > 0$. However we know that $\mu > n/r$. $\square$

So, the divisor $-K_W$ is nef and big, and $\daleth \in \{12, 13, 20, 25, 31, 33, 38, 58\}$, which implies that

$$r = a_4, \qquad r - a = a_3, \qquad \bar{a} = a_1, \qquad \bar{r} - \bar{a} = a_2, \qquad a_2 \neq a_3,$$
$$n\epsilon = \nu + (r - 2a)\mu/(r - a) - 2n/r$$

due to [2]. Then W has a singular point $\bar{P} \neq P$ of type $\frac{1}{\bar{r}}(1, \bar{a}, \bar{r} - \bar{a})$ such that the diagram

$$\begin{tikzcd}
U & & W \arrow[ll, "\beta"'] & & V \arrow[ll, "\gamma"'] \arrow[d, "\eta"] \\
X \arrow[from=u, "\alpha"'] \arrow[rrrr, dashed, "\psi"'] & & & & \mathbb{P}(1, a_1, a_2)
\end{tikzcd}$$

commutes, where ψ is a natural projection, γ be a blow up of the point $\bar{P}$ with weights $(1, \bar{a}, \bar{r} - \bar{a})$, and η is an elliptic fibration. Let F be the exceptional divisor of γ, and $\bar{G}$ be the proper transform of the divisor G on the threefold V. Then F and $\bar{G}$ are sections of η, and $G \not\ni \bar{P} \notin \breve{E}$.

It follows from the inequality $-K_W \cdot \breve{D} \geqslant 0$ and the proof of Lemma 2.9 that $\epsilon < 1$, which implies that the log pair $(W, \frac{1}{n}\breve{D} + (\mu/n - 1/r)\breve{E} + \epsilon G)$ is not log canonical at some point $Q \in G$.

Lemma 4.6. *The threefold W is smooth at the point Q.*

Proof. Suppose that W is singular at the point Q. Then Q is a singular point of type $\frac{1}{\breve{r}}(1, 1, \breve{r} - 1)$, where either $\breve{r} = \bar{r} - \bar{a}$, or $\breve{r} = \bar{a} \neq 1$. Let $\omega : \breve{W} \to W$ be a blow up of Q with weights $(1, 1, \breve{r} - 1)$, and $\mathcal{H}$ be the proper transform of $\mathcal{P}$ on $\breve{W}$. Then it follows from [8] that $\mathcal{H} \equiv -nK_{\breve{W}}$, which implies that $n = r\mu = a_1$ due to [4]. However we know that $\mu > n/r$. □

Thus, it follows from Lemma 4.6 that $\mathrm{mult}_Q(\breve{D}) > n - n\epsilon - (\mu - n/r)\mathrm{mult}_Q(\breve{E})$.

Lemma 4.7. *There is a surface $T \in |-K_W|$ such that $Q \in T$.*

Proof. The existence of a surface $T \in |-K_W|$ that passes through the point Q is obvious in the case when $a_1 = 1$. Thus, we may assume that $a_1 \neq 1$. Then $\gimel \in \{33, 38, 58\}$, but we consider only the case $\gimel = 38$, because the cases $\gimel = 33$ and $\gimel = 58$ can be considered in a similar way.

Suppose that $\gimel = 38$. Then there is a unique surface $T \in |-K_W|$. Suppose that $Q \notin T$.

Arguing as in the proof of Lemma 2.14, we see that $\mathrm{mult}_Q(\breve{D}) \leqslant (a_1 + a_2)\nu/a_1$. Then

$$\nu \frac{a_1 + a_2}{a_1} > n - (\mu - n/7) - (\nu + 3\mu/5 - 2n/7),$$

but $\mathrm{mult}_Q(\breve{D}) > n + n/r - \mu - n\epsilon$. Thus, we have $\mu > 55n/56 - 5\nu/2$.

The inequality $-K_W \cdot \breve{D} \geqslant 0$ and the proof of Lemma 2.9 imply that $\nu \leqslant 10\mu/7$ and $\mu \leqslant 9n/40$, respectively. The hypersurface X can be given by the equation

$$w^2 y + w\big(t^2 + tf_5(x, y, z) + f_{10}(x, y, z)\big) + tf_{13}(x, y, z) + f_{18}(x, y, z)$$
$$= 0 \subset \mathrm{Proj}\big(\mathbb{C}[x, y, z, t, w]\big),$$

where $\mathrm{wt}(x)=1$, $\mathrm{wt}(y)=2$, $\mathrm{wt}(z)=3$, $\mathrm{wt}(t)=5$, $\mathrm{wt}(w)=8$, and $f_i(x,y,z)$ is a quasihomogeneous polynomial of degree i. Let $\breve{S}$ be the proper transform on the threefold W of the surface that is cut out on X by the equation $wy+(t^2+tf_5(x,y,z)+f_{10}(x,y,z))=0$. Then

$$\breve{S}\equiv(\alpha\circ\beta)^*(-10K_X)-\frac{18}{8}\beta^*(E)-\frac{13}{5}G,$$

but $\breve{S}\neq\breve{D}$. The divisor $-K_W$ is nef. Hence, we have

$$0\leqslant -K_W\cdot\breve{D}\cdot\breve{S}=3n/4-6\mu/5-13\nu/6,$$

but $\nu\leqslant 8\mu/5$, which implies $\nu\leqslant 9n/35$. Now we can easily obtain a contradiction. □

It follows from [2] that $|-rK_W|$ does not have base points for $r\gg 0$ and induces a birational morphism $\omega\colon W\to\bar{W}$ such that $\bar{W}$ is a hypersurface in $\mathbb{P}(1,a_1,a_2,2a_3,3a_3)$ of degree $6a_3$.

Lemma 4.8. *The morphism ω is not an isomorphism in a neighborhood of the point Q.*

Proof. Suppose that ω is an isomorphism in a neighborhood of the point Q. Then it follows from the proof of Theorem 5.6.2 in [5] that there is $R\in|-s2a_1a_3K_W|$ such that $\mathrm{mult}_Q(R)\geqslant s$, but R does not contain components of the cycle $\breve{D}\cdot S$ that pass through Q, where $s\in\mathbb{N}$. Then

$$\begin{aligned}s2a_3\big(-na_1K_X^3-\mu/a_3-\nu/a_2\big)=R\cdot\breve{D}\cdot T&\geqslant \mathrm{mult}_Q(\breve{D}\cdot T)s\\&>\big(n-\nu-\mu(a_3-a_1)/a_3+2n/a_4\big)s,\end{aligned}$$

because $Q\notin\breve{E}$. Now we can derive a contradiction using $n\epsilon=\nu+(a_3-a_1)\mu/a_3-2n/a_4>0$. □

It follows from Lemma 4.8 that there is a unique curve $C\subset W$ that contains Q such that

$$-K_W\cdot C=0,\qquad \beta^*(-K_U)\cdot C=1/a_4,\qquad C\cdot G=1,$$

which implies that $\daleth\notin\{33,38,58\}$ by Lemma 4.7. Hence, we have $\daleth\in\{12,13,20,25,31\}$.

Put $\breve{D}\cdot T=mC+\Omega$, where m is a non-negative integer, and Ω is an effective one-cycle, whose support does not contain the curve C. Then it follows from Remark 2.12 that

$$m\leqslant 5n/4-\mu,\qquad m\leqslant 11n/15-\mu/2,\qquad m\leqslant 13n/15-\mu,$$
$$m\leqslant 5n/7-\mu/3,\qquad m\leqslant 2n/3-\mu$$

in the case when $\daleth=12,13,20,25,31$, respectively. Recall that $\bar{G}$ is a section of η.

Let $\mathcal{H}$ be a pencil in $|-a_2K_W|$ of surfaces passing through the point Q, and H be a general surface in $\mathcal{H}$. Then C is the only curve in the base locus of $\mathcal{H}$ that passes through Q. Then

$$-na_2K_X^3-a_2\mu/(a_1a_3)-\nu/a_1=H\cdot\Omega\geqslant\mathrm{mult}_Q(\Omega)>n-\nu-\mu(a_3-a_1)/a_3+2n/a_4-m,$$

which immediately implies that either $\daleth=12$, or $\daleth=13$.

Lemma 4.9. *The inequality* $\daleth \neq 12$ *holds.*

Proof. Suppose that $\daleth = 12$. Let R be a sufficiently general surface in $|-2K_W|$ that contains the point Q. Then $R|_T = C + L + Z$, where $L = G|_T$, the curve Z is reduced, and $P \notin \beta(Z)$.

Suppose that Z is irreducible. Then $Z^2 = -4/3$, $C^2 = -2$, $L^2 = -3/2$ on the surface T. Put

$$\breve{D}|_T = m_C C + m_L L + m_Z Z + \Upsilon,$$

where m_C, m_L and m_Z are non-negative integers, and Υ is an effective one-cycle, whose support does not contain the curve C, L and Z.

Suppose that $Q \notin \breve{E}$. Then $m_C > 2n/3 - m_Z/3$, because

$$\begin{aligned} 5n/6 - 2\mu/3 - \nu = R \cdot \breve{D} \cdot T &= m_L + m_Z/3 + R \cdot \Upsilon \\ &> m_L + m_Z/3 + 3n/2 - \nu - 2\mu/3 - m_L - m_C, \end{aligned}$$

but $4m_Z/3 \geqslant 2m_C - n/3$, because $\Upsilon \cdot Z \geqslant 0$. Thus, we have

$$m_C > 2n/3 + m_Z/3 \geqslant 7n/12 + m_C/2,$$

which gives $m_C > 7n/6$, but $m_C \leqslant 5n/6$ by Remark 2.12, because $-K_X \cdot \alpha \circ \beta(C) = 5/6$.

Thus, we see that $Q \in \breve{E}$. Then $C \subset \breve{E}$ and $\beta(C) \in |\mathcal{O}_{\mathbb{P}(1,1,3)}(1)|$, but

$$\begin{aligned} 5n/6 - 2\mu/3 - \nu = R \cdot \breve{D} \cdot T &= m_L + m_Z/3 + R \cdot \Upsilon \\ &> m_L + m_Z/3 + 7n/4 - \nu - 5\mu/3 - m_L - m_C, \end{aligned}$$

which gives $m_C > 11n/12 - \mu + m_Z/3$. We have $-K_X \cdot \alpha \circ \beta(Z) = 5/6$ and $Z \cdot \breve{E} = 2$, but

$$4m_Z/3 \geqslant 2m_C + 2\mu - 5n/6,$$

because $Z \cdot \Upsilon \geqslant 0$. Thus, we have $m_Z > 3n/2$, but $m_Z \leqslant n/2$ by Remark 2.12.

Therefore, the curve Z is reducible. Then $Q \in \breve{E}$ and $Z = \acute{Z} + \grave{Z}$, where $\acute{Z}$ and $\grave{Z}$ are irreducible curves such that $G \cdot \acute{Z} = G \cdot \grave{Z} = -K_U \cdot \beta(\grave{Z}) = 0$ and $-K_X \cdot \alpha \circ \beta(\acute{Z}) = 7/12$. Then

$$\acute{Z}^2 = -4/3, \qquad \grave{Z}^2 = C^2 = -2, \qquad L^2 = -3/2,$$
$$L \cdot C = \acute{Z} \cdot C = \acute{Z} \cdot \grave{Z} = \grave{Z} \cdot C = 1, \qquad L \cdot \grave{Z} = L \cdot \acute{Z} = 0$$

on the surface T. Put $\breve{D}|_T = \bar{m}_C C + \bar{m}_L L + \bar{m}_Z \acute{Z} + \Phi$, where $\bar{m}_C$, $\bar{m}_L$, $\bar{m}_Z$ are non-negative integers, and Φ is an effective cycle, whose support does not contain C, L and $\acute{Z}$. Then

$$R|_T \cdot \Phi \geqslant \mathrm{mult}_Q(\Phi) > 7n/4 - \nu - 5\mu/3 - m_L - m_C,$$

and $\Phi \cdot \acute{Z} \geqslant 0$. We have $\beta^*(-K_U)|_T \cdot \Phi \geqslant 0$. Thus, we see that

$$\bar{m}_C > 11n/12 - \mu + \bar{m}_Z/3, \qquad 4\bar{m}_Z/3 \geqslant \bar{m}_C + \mu - 5n/6, \qquad \bar{m}_C + \mu \leqslant 5/4 - \bar{m}_Z,$$

but this system of linear inequalities is inconsistent, which completes the proof. $\square$

Thus, we see that $\beth = 13$. Then $C \subset \breve{E}$, because otherwise we have

$$2(11n/30 - \mu/6 - \nu/2) = H \cdot \Omega \geqslant \mathrm{mult}_Q(\Omega) > 7n/5 - \nu - \mu/3 - m,$$

which implies that $m > 2n/3$, which is impossible, because $m \leqslant 11n/15 - \mu/2$ and $\mu > n/5$. Put

$$\bar{D}|_{\breve{E}} = \bar{m}C + \Upsilon,$$

where $\bar{m}$ is a non-negative integer, and Υ is an effective cycle, whose support does not contain the curve C. Then $\bar{m} \leqslant 5\mu/2$, because we have $\beta(C) \in |\mathcal{O}_{\mathbb{P}(1,2,3)}(2)|$ and the curve C is reduced, where $E \cong \mathbb{P}(1,2,3)$. Then $\bar{m} \leqslant 11n/12$, because $\mu \leqslant 11n/30$.

The log pair $(W, \frac{1}{n}\breve{D} + \breve{E} + \epsilon G)$ is not log canonical at the point Q. Hence, the log pair

$$\left(\breve{E}, C + \frac{\nu + \mu/3 - 2n/5}{n} G|_{\breve{E}} + \frac{1}{n}\Upsilon\right)$$

is not log canonical at the point Q by Theorem 7.5 in [9]. Then

$$5\mu/3 - \nu = (\bar{m}C + \Upsilon) \cdot C = \Upsilon \cdot C > 7n/5 - \nu - \mu/3,$$

because $\mathrm{mult}_Q(\Upsilon|_C) > 7n/5 - \nu - \mu/3$ by Theorem 7.5 in [9]. Thus, we see that $\mu > 7n/10$, which is impossible, because $\mu \leqslant 11n/30 < 7n/10$. The assertion of Lemma 2.10 is proved.

5. Quadratic involutions

In this section we prove the assertion of Lemma 2.11. Let us use the assumptions and notation of Lemma 2.11. Suppose that $d = 2r + a_j$. To prove Lemma 2.11 we must derive a contradiction.

It follows from the equality $d = 2r + a_j$ that the threefold X can be given by the equation

$$x_i^2 x_j + x_i f(x_0, x_1, x_2, x_3, x_4) + g(x_0, x_1, x_2, x_3, x_4) = 0 \subset \mathrm{Proj}\big(\mathbb{C}[x_0, x_1, x_2, x_3, x_4]\big),$$

where $i \neq j$, $a_i = r$, $\mathrm{wt}(x_0) = 1$, $\mathrm{wt}(x_k) = a_k$, f and g are quasihomogeneous polynomials that do not depend on x_i. Put $\bar{a}_3 = a_{3+4-i}$, $\bar{a}_4 = a_i a_j$, $\bar{d} = 2\bar{a}_4$. Then there is a commutative diagram

$$\begin{tikzcd} U \arrow[rr, "\alpha"] \arrow[d, "\sigma"'] & & X \arrow[d, dashed, "\xi"] \\ V \arrow[r, hook] & \mathbb{P}(1, a_1, a_2, \bar{a}_3, \bar{a}_4) \arrow[r, dashed, "\chi"'] & \mathbb{P}(1, a_1, a_2, \bar{a}_3), \end{tikzcd}$$

where ξ and χ are projections, and σ is a birational morphism that contracts smooth irreducible rational curves $C_1, \ldots, C_l$ such that V is a hypersurface in $\mathbb{P}(1, a_1, a_2, \bar{a}_3, \bar{a}_4)$ of degree $\bar{d}$ with terminal singularities, where $l = a_i a_j (d - a_i) \sum_{i=1}^{4} a_i$. Then $-K_X \cdot \alpha(C_k) = 1/a_i$.

Let M be the surface that is cut out on the threefold X by $x_j = 0$, and $\bar{M}$ be the proper transform of M on the threefold U. Then $M \neq D$ by Lemma 8.12 and Proposition 8.14 in [9].

Lemma 5.1. *The inequalities* $\mu \leqslant -a_j n K_X^3(r-a)a/(d-r) \leqslant n(d-r)/(ra_j)$ *hold.*

Proof. The divisor $-K_U$ is nef. The inequality $\mu \leqslant -a_j n K_X^3(r-a)a/(d-r)$ follows from

$$0 \leqslant -K_U \cdot \bar{M} \cdot \bar{D} = -a_j n K_X^3 - \mu(d-r)/\big(ar-a^2\big),$$

and to conclude the proof we must show that $-a_j n K_X^3(r-a)a/(d-r) \leqslant n(d-r)/(ra_j)$. Suppose that $-a_j n K_X^3(r-a)a/(d-r) > n(d-r)/(ra_j)$. Then

$$\frac{d-r}{ra_j} < -a_j K_X^3 \frac{(r-a)a}{d-r} = \frac{da_j(r-a)a}{(d-r)a_1a_2a_3a_4},$$

but $a_1a_2a_3a_4 \geqslant a_j r(r-a)a$. Thus, we have $(d-r)^2 < d(d-2r)$, which is a contradiction. $\square$

We have $E \cong \mathbb{P}(1,a,r-a)$, and $|\mathcal{O}_{\mathbb{P}(1,a,r-a)}(1)|$ consists of a single curve when $a \neq 1$.

Lemma 5.2. *The inequality* $a \neq 1$ *holds.*

Proof. Suppose that $a = 1$. Taking into account the possible values of (a_1, a_2, a_3, a_4), we see that

$$\gimel \in \{6, 7, 8, 9, 12, 13, 16, 15, 17, 20, 25, 26, 30, 36, 31, 41, 47, 54\},$$

but we only consider the cases $\gimel = 7$ and $\gimel = 36$. The remaining 16 cases can be considered in a similar way. So the reader can easily obtain a contradiction in these cases by himself.

Suppose that $\gimel = 7$. Then X is a hypersurface in $\mathbb{P}(1,1,2,2,3)$ of degree 8, which implies that the point O is a singular point of type $\frac{1}{3}(1,1,2)$. Let S be the unique surface in $|-K_U|$ that contains the point P. Then S is an irreducible surface, which is smooth at the point P.

The singularities of U consists of singular points P_0, P_1, P_2, P_3 and P_4 of type $\frac{1}{2}(1,1,1)$ such that P_0 is a singular point of E. It follows from [2] that there is a commutative diagram

$$\begin{tikzcd}
U \arrow[d, "\alpha"'] & & Y_i \arrow[ll, "\beta_i"'] \arrow[dr, "\eta_i"] & \\
X \arrow[rrr, dashed, "\xi_i"'] & & & \mathbb{P}(1,1,2),
\end{tikzcd}$$

where ξ_i is a projection, β_i is a blow of P_i with weights $(1,1,1)$, and η_i is a morphism.

Suppose that $P \notin \bigcup_{i=1}^{l} C_i$. The proper transform of E on the threefold Y_i is a section of η_i in the case when $i \neq 0$. Hence, there is a surface $H \in |-2K_U|$ such that

$$2(2n/3 - \mu/2) = \bar{D} \cdot H \cdot S \geqslant \mathrm{mult}_P(\bar{D}) > 4n/3 - \mu,$$

which is a contradiction. So we may assume that $P \in C_1$. Then $-K_X \cdot \alpha(C_1) = 1/3$.

Let Z_1 be an irreducible curve such that $\bar{M}|_S = C_1 + Z_1$. Put $L = E|_S$. Then

$$C_1^2 = -2, \qquad Z_1^2 = L^2 = -3/2, \qquad C_1 \cdot Z_1 = L \cdot C_1 = 1, \qquad L \cdot Z_1 = 3/2$$

on the surface S. Put $\bar{D}|_S = m_C C_1 + m_Z Z_1 + m_L L + \Omega$, where m_C, m_Z and m_L are non-negative integers, and Ω is an effective cycle, whose support does not contain C_1, Z_1 and L. Then

$$n - 3\mu/2 + 3m_Z/2 - 3m_L/2 - m_C = Z_1 \cdot \Omega \geqslant 0,$$
$$3\mu/2 - 3m_Z/2 + 3m_L/2 - m_C = L \cdot \Omega \geqslant 0,$$

which implies that $3m_Z/2 \geqslant 3(\mu + m_L)/2 + m_C - n$ and $3(\mu + m_L)/2 \geqslant 3m_Z/2 + m_C$. Then

$$4n/3 - \mu - m_L - m_Z = (L + C_1 + Z_1) \cdot \Omega \geqslant \mathrm{mult}_P(\Omega) > 4n/3 - \mu - m_L - m_C,$$

which gives $m_C > m_Z$ and $4n/3 \geqslant \mu + m_L + m_Z$. So we see that $m_Z \leqslant n/2$ and $m_C \leqslant n/2$.

Then it follows from Theorem 7.5 in [9] that the log pair

$$\left(S, C_1 + \frac{\bar{m}_L + \mu - n/3}{n} L + \frac{m_Z}{n} Z + \frac{1}{n}\Omega\right)$$

is not log canonical at P, because $m_C \leqslant n$. So it follows from Theorem 7.5 in [9] that

$$C_1 \cdot \Omega \geqslant \mathrm{mult}_P(\Omega|_{C_1}) > n - m_L - \mu + n/3,$$

which implies that $m_C > m_Z/2 + n/2$, but $m_C \leqslant n/2$. So the case $\daleth = 7$ is impossible.

Suppose that $\daleth = 36$. Then X is a general hypersurface in $\mathbb{P}(1,1,4,6,7)$ of degree 18, and O is a singular point of type $\frac{1}{7}(1,1,6)$. Arguing as in the case $\daleth = 7$, we see that $P \notin \bigcup_{i=1}^{l} C_i$, which implies that we may assume that $P \in C_1$. Put $L = C_1$.

Let S be a surface in $|-K_U|$ such that $P \in S$. Then $\bar{M}|_S = L + Z$, where Z is an irreducible curve. Put $C = E|_S$. Then the intersection form of C, L, Z on S is given by

$$Z^2 = C^2 = -7/6, \qquad L^2 = -2, \qquad Z \cdot L = C \cdot L = 1, \qquad Z \cdot C = 5/6,$$

and P is the intersection point of the curves L and C. Put

$$\bar{D}|_S = m_L L + m_C C + m_Z Z + \Omega,$$

where m_L, m_C and m_Z are a non-negative integers, and Ω is an effective cycle, whose support does not contain the curves L, C, and Z.

It follows from the proof of Theorem 5.6.2 in [5] that we can find $H \in |-s6K_U|$ that has multiplicity at least $s > 0$ at the point P, but does not contain components of Ω that pass through the point P, where s is a natural number. Then

$$s6(3n/28 - \mu/6 - m_C/6 - m_Z/6) = H|_S \cdot \Omega \geqslant \mathrm{mult}_P(\Omega)s > (8n/7 - \mu - m_L - m_C)s,$$

which implies that $m_L > n/2 + m_Z$, but $m_L \leqslant 3n/4$ by Remark 2.12. We have

$$3n/28 - \mu/6 = -K_U|_S \cdot (m_L L + m_C C + m_Z Z + \Omega) \geqslant -K_U|_S \cdot (m_L L + m_C C + m_Z Z) = \frac{m_C + m_Z}{6},$$

which implies that $m_C + m_Z \leqslant 9n/14 - \mu$. On the surface S we have

$$7\mu/6 + 7m_C/6 - 5m_Z/6 - m_L = \Omega \cdot C > 8n/7 - \mu - m_L - m_C,$$

which implies that $13(\mu + m_C)/6 > 8n/7 + 5m_Z/6$. The inequality $\Omega \cdot Z \geqslant 0$ implies that

$$2n/7 - 5\mu/6 - m_L - 5m_C/6 + 7m_Z/6 \geqslant 0,$$

which implies that $7m_Z/6 \geqslant 5\mu/6 + m_L + 5m_C/6 - 2n/7$, but $m_Z \leqslant 3n/8$ by Remark 2.12.

It follows from Lemma 5.1 that $18n/77 \geqslant \mu > n/7$. The inequalities obtained

$$\begin{cases} 13(\mu + m_C)/6 > 8n/7 + 5m_Z/6, \\ 21n/48 \geqslant 7m_Z/6 \geqslant 5\mu/6 + m_L + 5m_C/6 - 2n/7, \\ m_C + m_Z \leqslant 9n/14 - \mu, \\ 3n/4 \geqslant m_L > n/2 + m_Z, \\ 18n/77 \geqslant \mu > n/7, \end{cases}$$

are inconsistent. So we have a contradiction. Thus, the case $\gimel = 36$ is impossible as well. $\square$

Taking into account the possible values of the quadruple (a_1, a_2, a_3, a_4), we see that

$$\gimel \in \{13, 18, 23, 24, 27, 32, 38, 40, 42, 43, 44, 45, 46, 48, 56, 58, 60, 61, 65, 68, 69, 74, 76, 79\}$$

by Lemmas 5.2. Let T be a general surface in $|-K_U|$. Then $T|_E \in |\mathcal{O}_{\mathbb{P}(1,a,r-a)}(1)|$.

Lemma 5.3. *The point P is contained in the surface T.*

Proof. It follows from Lemmas 2.14 and 5.1 that $P \in T$ unless $\gimel \in \{13, 24\}$. Therefore, we may assume that $\gimel \in \{13, 24\}$ and $P \notin T$. Let us derive a contradiction.

Let L be the curve in $|\mathcal{O}_{\mathbb{P}(1,a,r-a)}(1)|$. Then $P \notin L$, because $P \notin T$. Thus, there is a unique smooth irreducible curve C in the linear system $|\mathcal{O}_{\mathbb{P}(1,a,r-a)}(a)|$ that contains the point P. Put

$$\bar{D}|_E = \delta C + \Upsilon \equiv r\mu L,$$

where δ is a non-negative integer, and Υ is an effective cycle such that $C \not\subset \mathrm{Supp}(\Upsilon)$.

Arguing as in the proof of Lemma 2.14, we see that $\delta \leqslant r\mu/a$. Then $\delta < n$ by Lemma 5.1.

It follows from Theorem 7.5 in [9] that $(E, \frac{1}{n}\bar{D}|_E)$ is not log canonical at P, which implies that the log pair $(E, C + \frac{1}{n}\Upsilon)$ is not log canonical at P. It follows from Theorem 7.5 in [9] that

$$r\mu/(r-a) \geqslant (r\mu - a\delta)/(r-a) = C \cdot \Upsilon \geqslant \mathrm{mult}_P(\Upsilon|_C) > n,$$

which implies that $\mu \geqslant n(r-a)/r$, which is impossible by Lemma 5.1. $\square$

It follows from [5] that $T \cap E \cap \bigcup_{i=1}^{l} C_i \neq \emptyset \Leftrightarrow \gimel \in \{43, 46, 69, 74, 76, 79\}$.

Lemma 5.4. *The case* $\gimel \notin \{13, 24, 32, 43, 46\}$ *is impossible.*

Proof. Suppose that $\gimel \notin \{13, 24, 32, 43, 46, 56\}$. It follows from the proof of Theorem 5.6.2 in [5] that there are $s \in \mathbb{N}$ and $H \in |-sa_1\bar{a}_3K_U|$ such that $\mathrm{mult}_P(H) \geqslant s$, but H does not contain components of the cycle $\bar{D} \cdot T$ passing through P that are different from the curves $C_1, \ldots, C_l$.

We have $\gimel \in \{69, 74, 76, 79\}$ and $P \in \bigcup_{i=1}^{l} C_i$, because otherwise we get a contradiction using

$$sa_1\bar{a}_3\left(-nK_X^3 - \frac{\mu}{a(r-a)}\right) = \bar{D} \cdot H \cdot T \geqslant \mathrm{mult}_P(\bar{D})s > (n + n/r - \mu)s.$$

We may assume that $P \in C_1$. Put $\bar{D} \cdot T = mC_1 + \Delta$, where m is a non-negative integer number, and Δ is an effective cycle, whose support does not contain the curve C_1. Then it follows from the proof of Theorem 5.6.2 in [5] that there is a surface $R \sim -sa_1\bar{a}_3K_U$ such that

$$sa_1\bar{a}_3\left(-nK_X^3 - \frac{\mu}{a(r-a)}\right) = R \cdot \Delta \geqslant \mathrm{mult}_P(\Delta)s > (n + n/r - \mu - m)s,$$

where $s \in \mathbb{N}$. The inequality obtained is impossible, because $m \leqslant -a_i nK_X^3$ by Remark 2.12.

Suppose that $\gimel = 56$. As in the previous case, there is $H \in |-s24K_U|$ such that

$$s24(n/22 - \mu/24) = \bar{D} \cdot H \cdot T \geqslant \mathrm{mult}_P(\bar{D})s > (12n/11 - \mu)s,$$

where s is a natural number. Now we can easily obtain a contradiction with $\mu > n/r$. $\square$

Thus, to complete the proof of Lemma 2.11, we have to consider the cases $\gimel = 13, 24, 32, 43, 46$ one-by-one. For the sake of simplicity, we only consider the cases $\gimel = 13$ and $\gimel = 43$, because the remaining cases can be considered in a similar way.

Lemma 5.5. *The inequality* $\gimel \neq 43$ *holds.*

Proof. Suppose that $\gimel = 43$. Then X is a general hypersurface in $\mathbb{P}(1, 2, 3, 5, 9)$ of degree 20, and O is a singular point of type $\frac{1}{9}(1, 4, 5)$. The base locus of $|-2K_U|$ consists of two irreducible curves C and L such that $L = T \cdot E$, and C is the curve among $C_1, \ldots, C_l$ such that $C \cap L \neq \emptyset$.

Suppose that $P \notin C$. Then it follows from the proof of Theorem 5.6.2 in [5] that we can find a surface $H \in |-s20K_U|$ that has multiplicity at least $s > 0$ at the point P and does not contain components of $\bar{D} \cdot T$ that pass through P, where s is a natural number. Then

$$s20(n/18 - \mu/20) = \bar{D} \cdot H \cdot T \geqslant \mathrm{mult}_P(\bar{D})s > (10n/9 - \mu)s,$$

which implies that $\mu < n/9$, but $\mu > n/10$.

We see that $P \in C$. Then $\bar{M}$ contains C and L. Put

$$\bar{D}|_{\bar{M}} = m_1L + m_2C + \Delta,$$

where m_1 and m_2 are non-negative integers, and Δ is an effective cycle, whose support does not contain L and C. Then $m_2 \leqslant n$ by Remark 2.12, because $\alpha^*(-K_X) \cdot C = 1/9$.

The surface $\bar{M}$ is smooth at P. So it follows from Theorem 7.5 in [9] that the log pair

$$\left(\bar{M}, \frac{1}{n}\bar{D}|_{\bar{M}} + (\mu/n - 1/9)E|_{\bar{M}}\right)$$

is not log canonical in a neighborhood of the point P, but $E|_{\bar{M}} = L + Z$, where Z is an irreducible curve that does not pass through the point P. Therefore, the singularities of the log pair

$$\left(\bar{M}, (m_1/n + \mu/n - 1/9)L + C + \frac{1}{n}\Delta\right)$$

are not log canonical at the point P. So it follows from Theorem 7.5 in [9] that

$$n/9 - \mu - m_1 + m_2 = \Delta \cdot C \geqslant \mathrm{mult}_P(\Delta|_C) > n - m_1 - \mu + n/9,$$

because $C^2 = -1$ and $C \cdot L = 1$ on $\bar{M}$. Thus, we have $m_2 > n$, which is a contradiction. □

Suppose that $\daleth = 13$. Then $r = 5$ by Lemma 5.2. The base locus of the pencil $|-K_U|$ consists of two curves $\bar{C}$ and $\bar{L}$ such that $\bar{C} = E|_T$, and $\alpha(\bar{L})$ is the base curve of $|-K_X|$. Then

$$\bar{C}^2 = \bar{L}^2 = -5/6, \qquad \bar{L} \cdot \bar{C} = 1$$

on the surface T. Put $\bar{D}|_T = \bar{m}_L\bar{L} + \bar{m}_C\bar{C} + \Upsilon$, where $\bar{m}_L$ and $\bar{m}_C$ are non-negative integers, and Υ is an effective cycle, whose support does not contain $\bar{L}$ and $\bar{C}$. Then

$$11n/5 - 11\mu/6 = (6L + 5C) \cdot (\bar{m}_L\bar{L} + \bar{m}_C\bar{C} + \Upsilon) = 11\bar{m}_C/6 + (6L + 5C) \cdot \Upsilon \geqslant 11\bar{m}_C/6,$$

which implies that $\bar{m}_C \leqslant 6n/5 - \mu$. Thus, we have $\bar{m}_C < n$, because $\mu > n/5$.

Suppose that $P \notin \bar{L}$. Then it follows from Theorem 7.5 in [9] that the log pair

$$\left(S, \bar{C} + \frac{\bar{m}_L}{n}L + \frac{1}{n}\Upsilon\right)$$

is not log canonical in the neighborhood of the point P, because $\bar{m}_C + \mu - n/5 \leqslant n$, which implies that the inequality $\mathrm{mult}_P(\Upsilon|_{\bar{C}}) > n$ holds by Theorem 7.5 in [9]. Hence, we have

$$5\mu/6 + 5\bar{m}_C/6 \geqslant 5\mu/6 - \bar{m}_L + 5\bar{m}_C/6 = \Upsilon \cdot \bar{C} > n,$$

which is impossible, because $\bar{m}_C \leqslant 6/5 - \mu$. Thus, we see that $P = \bar{L} \cap \bar{C}$.

Put $\bar{D}|_{\bar{M}} = m\bar{L} + \Omega$, where m is a non-negative integer, and Ω is an effective cycle, whose support does not contain $\bar{L}$. Then $L^2 = 1/6$ on the surface $\bar{M}$. But $m \leqslant n$ by Remark 2.12.

Arguing as in the case $P \notin \bar{L}$ we see that $\mathrm{mult}_P(\Omega|_{\bar{L}}) > n$ by Theorem 7.5 in [9], and

$$11n/30 - \mu = \bar{D} \cdot \bar{L} = m/6 + \Omega \cdot \bar{L} > m/6 + n,$$

which implies that $m < 0$. So we have a contradiction, which completes the proof of Lemma 2.11.

6. Direct products

Let X be an arbitrary Fano variety with terminal $\mathbb{Q}$-factorial singularities of Picard rank one, and Γ be a subgroup of the group $\mathrm{Bir}(X)$.

Definition 6.1. The subgroup $\Gamma \subset \mathrm{Bir}(X)$ untwists all maximal singularities if for every linear system $\mathcal{M}$ on the variety X that has no fixed components there is $\xi \in \Gamma$ such that the singularities of the log pair $(X, \lambda\xi(\mathcal{M}))$ are canonical, where $\lambda \in \mathbb{Q}$ such that $K_X + \lambda\, \xi(\mathcal{M}) \equiv 0$.

It is well known that the group $\mathrm{Bir}(X)$ is generated by the subgroups Γ and $\mathrm{Aut}(X)$ in the case when the subgroup Γ untwists all maximal singularities (see [1]).

Definition 6.2. The variety X is birationally superrigid[4] (rigid, respectively) if the trivial subgroup (the whole group $\mathrm{Bir}(X)$, respectively) untwists all maximal singularities.

The birational rigidity of X implies that there is no dominant rational map $\rho : X \dashrightarrow Y$ such that $\dim(Y) \geqslant 1$, and sufficiently general fiber of the map ρ is rationally connected (see [1]).

Example 6.3. It follows from [12] that the variety X is birationally superrigid and $\mathrm{lct}(X) = 1$ in the case when X is one of the following smooth Fano varieties:

- a general hypersurface in $\mathbb{P}^r$ of degree $r \geqslant 6$;
- a general hypersurface in $\mathbb{P}(1^{m+1}, m)$ of degree $2m \geqslant 6$.

Definition 6.4. The subgroup Γ universally untwists all maximal singularities if for every variety U, and every linear system $\mathcal{M}$ on the variety $X \times U$ that does not have fixed components, there is a birational automorphism $\xi \in \Gamma$ such that the log pair

$$\big(F, \lambda\xi(\mathcal{M})|_F\big)$$

has at most canonical singularities, where F is a sufficiently general fiber of the natural projection $X \times U \to U$, and λ is a positive rational number such that $K_F + \lambda\, \xi(\mathcal{M})|_F \equiv 0$.

Let $X_1, \ldots, X_r$ be Fano varieties of Picard rank 1 with terminal $\mathbb{Q}$-factorial singularities. Put

$$U_i = X_1 \times \cdots \times X_{i-1} \times \widehat{X_i} \times X_{i+1} \times \cdots \times X_r$$

and $V = X_1 \times \cdots \times X_r$. Let $\pi_i : V \to U_i$ be a natural projection.

For every $i \in \{1, \ldots, r\}$, suppose that $\mathrm{lct}(X_i) \geqslant 1$, and there is a subgroup $\Gamma_i \subset \mathrm{Bir}(X_i)$ that universally untwists all maximal singularities. Then the following result holds.[5]

Theorem 6.5. *The variety $X_1 \times \cdots \times X_r$ is non-rational, and*

[4] There are several similar definitions of birational superrigidity and birational rigidity (see [1,5]).

[5] The assertion of Theorem 6.5 is proved in [12] for smooth birationally superrigid Fano varieties.

$$\mathrm{Bir}(X_1 \times \cdots \times X_r) = \left\langle \prod_{i=1}^{r} \Gamma_i,\ \mathrm{Aut}(X_1 \times \cdots \times X_r) \right\rangle,$$

for any dominant rational map $\rho : X_1 \times \cdots \times X_r \dashrightarrow Y$ whose general fiber is rationally connected, there is a commutative diagram

$$\begin{tikzcd} X_1 \times \cdots \times X_r \arrow[r, dashed, "\sigma"] \arrow[d, "\pi"'] & X_1 \times \cdots \times X_r \arrow[d, dashed, "\rho"] \\ X_{i_1} \times \cdots \times X_{i_k} \arrow[r, dashed, "\xi"'] & Y, \end{tikzcd}$$

where ξ and σ are birational maps, and π is a projection for some $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, r\}$.

It is well known that Theorem 6.5 is implied by the following technical result (see [12]).

Proposition 6.6. *For every linear system $\mathcal{M}$ on the variety V such that*

- *the linear system $\mathcal{M}$ does not have fixed components,*
- *the linear system $\mathcal{M}$ does not lie in the fibers of the projections $\pi_1, \ldots, \pi_r$,*

there are $k \in \{1, \ldots, r\}$, birational map $\xi \in \prod_{i=1}^{r} \Gamma_i$ and a positive rational number μ such that

- *the inequality $\kappa(V, \mu\xi(\mathcal{M})) \geqslant 0$ holds,*[6]
- *the equivalence $K_V + \mu\, \xi(\mathcal{M}) \equiv \pi_k^*(D)$ holds for some nef $\mathbb{Q}$-divisor D on U_k.*

Proof. Let F_i be a sufficiently general fiber of π_i. The subgroups $\Gamma_1, \ldots, \Gamma_r$ universally untwist all maximal singularities for every $i = 1, \ldots, r$. So there is $\xi \in \prod_{i=1}^{r} \Gamma_i$ such that the log pairs

$$\big(F_1, \mu_1\xi(\mathcal{M})|_{F_1}\big), \quad \ldots, \quad \big(F_r, \mu_r\xi(\mathcal{M})|_{F_r}\big)$$

are canonical, where μ_i is a rational number such that

$$K_V + \mu_i\, \xi(\mathcal{M}) \equiv \pi_i^*(D_i),$$

where D_i is a $\mathbb{Q}$-divisor on U_i. Then there is $m \in \{1, \ldots, r\}$ such that D_m is nef.

Now arguing as in the proof of Theorem 1 in [12], we see that $\kappa(V, \mu_k\xi(\mathcal{M})) \geqslant 0$. □

Let X be a general quasismooth hypersurface in $\mathbb{P}(1, a_1, a_2, a_3, a_4)$ of degree $\sum_{i=1}^{4} a_i$ with terminal singularities, where $a_1 \leqslant a_2 \leqslant a_3 \leqslant a_4$. Then X is a Fano threefold, whose divisor class group is generated by $-K_X$. The possible values of (a_1, a_2, a_3, a_4) are given in Table 5 in [7].

There are finitely many non-biregular birational involutions $\tau_1, \ldots, \tau_k \in \mathrm{Bir}(X)$ explicitly constructed in [5] such that the following result holds (see [5]).

[6] The number $\kappa(V, \mu\xi(\mathcal{M}))$ is a Kodaira dimension of the movable log pair $(V, \mu\xi(\mathcal{M}))$ (see [1]).

Theorem 6.7. *The subgroup $\langle\tau_1, \dots, \tau_k\rangle$ universally untwists all maximal singularities.*

Hence, the following two examples follow from [3,10,12] and Theorems 1.3, 6.5, 6.7.

Example 6.8. Let X be a general hypersurface in $\mathbb{P}(1, 1, 4, 5, 10)$ of degree 20, and U be a general hypersurface in $\mathbb{P}(1^{n+1}, n)$ of degree $2n \geqslant 6$. Then $\mathrm{Bir}(X \times U) \cong (\mathbb{Z}_2 * \mathbb{Z}_2) \oplus \mathbb{Z}_2$.

Example 6.9. Let X be a general hypersurface in $\mathbb{P}(1, 1, 2, 3, 3)$ of degree 9, and U be a general hypersurface in $\mathbb{P}^r$ of degree $r \geqslant 6$. Then

$$\mathrm{Bir}(X \times U) \cong \left\langle a, b, c \;\middle|\; a^2 = b^2 = c^2 = (abc)^2 = 1\right\rangle.$$

It follows from [5] that $\mathrm{Aut}(X) \neq \mathrm{Bir}(X)$ for exactly 45 values of (a_1, a_2, a_3, a_4).

Acknowledgments

The author thanks J. Howie, J. Kollár, L. O'Carroll, J. Park, A. Pukhlikov, V. Shokurov and the referees for useful comments. The author is grateful to the IHES for hospitality.

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