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Halphen pencils on weighted Fano threefold hypersurfaces

Research Article

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Abstract: On a general quasismooth well-formed weighted hypersurface of degree $\sum_{i=1}^{4} a_i$ in $\mathbb{P}(1, a_1, a_2, a_3, a_4)$, we classify all pencils whose general members are surfaces of Kodaira dimension zero.

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Throughout this article, all varieties are projective and defined over $\mathbb{C}$ and morphisms are proper unless otherwise stated.

1. Introduction

Let C be a smooth curve in $\mathbb{P}^2$ defined by a cubic homogeneous equation $f(x, y, z) = 0$. Suppose that we have nine distinct points $P_1, \cdots, P_9$ on C such that the divisor

$$\sum_{i=1}^{9} P_i - \mathcal{O}_{\mathbb{P}^2}(3)\Big|_C$$

is a torsion divisor of order $m \geq 1$ on the curve C. Then there is a curve $Z \subset \mathbb{P}^2$ of degree $3m$ such that $\mathrm{mult}_{P_i}(Z) = m$ for each point P_i. Let $\mathcal{P}$ be the pencil given by the equation

$$\lambda f^m(x, y, z) + \mu g(x, y, z) = 0 \subset \mathrm{Proj}\Big(\mathbb{C}[x, y, z]\Big) \cong \mathbb{P}^2,$$

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where $g(x, y, z) = 0$ is a homogeneous equation of the curve Z and $(\lambda : \mu) \in \mathbb{P}^1$. Then a general curve of the pencil $\mathcal{P}$ is birational to an elliptic curve. The pencil $\mathcal{P}$ is called a plane Halphen pencil and the construction of $\mathcal{P}$ can be generalized to the case when the curve C has ordinary double points and the points $P_1, \cdots, P_9$ are not necessarily distinct ([6]). In fact, every plane elliptic pencil is birational to a Halphen pencil. Namely, the following result is proved by Bertini but its rigorous proof is due to [6].

Theorem 1.1.

Let $\mathcal{M}$ be a pencil on $\mathbb{P}^2$ whose general curve is birational to an elliptic curve. Then there is a birational automorphism ρ of $\mathbb{P}^2$ such that $\rho(\mathcal{M})$ is a plane Halphen pencil.

A problem similar to Theorem 1.1 can be considered for Fano varieties whose groups of birational automorphisms are well understood. In particular, it is an interesting problem to classify pencils of K3 surfaces on three-dimensional weighted Fano hypersurfaces.

Definition 1.1.

A *Halphen pencil* is a one-dimensional linear system whose general element is birational to a smooth variety of Kodaira dimension zero.

Let X be a general quasismooth well-formed hypersurface of degree $d = \sum_{i=1}^{4} a_i$ in $\mathbb{P}(1, a_1, a_2, a_3, a_4)$ that has terminal singularities, where $a_1 \leq a_2 \leq a_3 \leq a_4$. Then

$$-K_X \sim_{\mathbb{Q}} \mathcal{O}_{\mathbb{P}(1,a_1,a_2,a_3,a_4)}(1)\Big|_X,$$

which implies that X is a Fano threefold. The divisor class group $\mathrm{Cl}(X)$ is generated by the anticanonical divisor $-K_X$ and there are exactly 95 possibilities for the quadruple (a_1, a_2, a_3, a_4), which were found by Iano-Fletcher. We use the notation $\gimel$ for the entry numbers of these famous 95 families. They are ordered in the same way as in [5], which is nowadays standard.
Birational geometry on such threefolds is extensively studied in [1], [2], [5] and [10]. The article [5] describes the generators of the group $\mathrm{Bir}(X)$ of birational automorphisms of X. Also the article [2] shows the relations among these generators. The former article proves the following result as well.

Theorem 1.2.

The threefold X cannot be rationally fibred by rational curves or surfaces.

As for birational maps into elliptic fibrations, the hypersurface X in each family except the families of $\gimel = 3, 60, 75, 84, 87$ and 93 is birational to an elliptic fibration ([2]). Furthermore, all birational transformations of the threefold X into elliptic fibrations are classified in [1].
It is known that Halphen pencils on the threefold X always exist, to be precise, the threefold X can be always rationally fibred by K3 surfaces ([2]). In this article, we will classify all Halphen pencils on hypersurfaces in the 95 families as is done for elliptic fibrations in [1].
Let us explain five examples of pencils on the threefold X. They exhaust all the possible Halphen pencils on X. It follows from [5] that the pencils constructed below are $\mathrm{Bir}(X)$-invariant (Proposition 4.1). We will show, throughout this article, that they are indeed Halphen pencils.

Example 1.1.

Suppose that $a_2 = 1$. Then every one-dimensional linear system in $|-K_X|$ is a Halphen pencil. It follows from adjunction that a general surface in $|-K_X|$ is birational to a smooth K3 surface.

Therefore in the cases in Example 1.1, or equivalently $\gimel = 1, 2, 3, 4, 5, 6, 8, 10, 14$, there are infinitely many Halphen pencils on the hypersurface X.

Example 1.2.

Suppose that $a_1 \neq a_2$. Then the linear system $|-a_1K_X|$ is a pencil. If $a_1 = 1$, then the linear system $|-K_X|$ is a Halphen pencil and its general surface belongs to Reid's 95 codimension 1 weighted K3 surfaces as in Example 1.1. In fact, it is a Halphen pencil if only if $a_1 \neq a_2$ ([2]). We will see that it is a unique Halphen pencil except the cases with $a_2 = 1$ and the cases in three Examples below.

Note that $a_1 = a_2 \neq 1$ exactly when $\daleth = 18$, 22 and 28.

Example 1.3.

Suppose that $\daleth = 18$, 22, or 28. In such cases, $a_1 = a_2 \neq 1$ and $a_3 = a_1 + 1$. The threefold X has singular points $O_1, \cdots, O_r$ of type $\frac{1}{a_1}(1, 1, a_1 - 1)$, where $r = \frac{3a_1+a_4+1}{a_1}$. There is a unique index $j \geq 3$ such that $a_2 + a_3 + a_4 = ma_j$, where m is a natural number. In particular, the threefold X is given by an equation

$$\sum_{k=0}^{m} x_j^k f_k\big(x_0, x_1, x_2, x_3, x_4\big) = 0 \subset \mathrm{Proj}\Big(\mathbb{C}[x_0, x_1, x_2, x_3, x_4]\Big),$$

where $\mathrm{wt}(x_0) = 1$, $\mathrm{wt}(x_l) = a_l$ and f_k is a quasihomogeneous polynomial of degree $a_1 + a_2 + a_3 + a_4 - ka_j$ that is independent of the variable x_j. Let $\mathcal{P}_i$ be the pencil of surfaces in $|-a_1K_X|$ that pass through the point O_i and $\mathcal{P}$ be the pencil on the threefold X that is cut out by the pencil $\lambda x_0^{a_1} + \mu f_m(x_0, x_1, x_2, x_3, x_4) = 0$, where $(\lambda : \mu) \in \mathbb{P}^1$. It will be proved that $\mathcal{P}$ and $\mathcal{P}_i$ are Halphen pencils in $|-a_1K_X|$.

The cases in Example 1.3 are the only cases that have more than two but finitely many Halphen pencils.

Example 1.4.

Suppose that $\daleth = 45$, 48, 55, 57, 58, 66, 69, 74, 76, 79, 80, 81, 84, 86, 91, 93 or 95. We then see $1 \neq a_1 \neq a_2$. Moreover there is a unique index $j \neq 2$ such that $a_1 + a_3 + a_4 = ma_j$, where m is a natural number. Therefore the threefold X is given by an equation

$$\sum_{k=0}^{m} x_j^k f_k\big(x_0, x_1, x_2, x_3, x_4\big) = 0 \subset \mathrm{Proj}\Big(\mathbb{C}[x_0, x_1, x_2, x_3, x_4]\Big),$$

where $\mathrm{wt}(x_0) = 1$, $\mathrm{wt}(x_i) = a_i$ and f_k is a quasihomogeneous polynomial of degree $a_1+a_2+a_3+a_4-ka_j$ that is independent of the variable x_j. Let $\mathcal{P}$ be the pencil on the threefold X that is cut out by the pencil $\lambda x_0^{a_2} + \mu f_m(x_0, x_1, x_2, x_3, x_4) = 0$, where $(\lambda : \mu) \in \mathbb{P}^1$. It will be shown that $\mathcal{P}$ is a Halphen pencil in $|-a_2K_X|$.

Example 1.5.

Suppose that $\daleth = 60$. Then X is a general hypersurface of degree 24 in $\mathbb{P}(1, 4, 5, 6, 9)$. Hence the threefold X is given by an equation

$$w^2 f_6\big(x, y, z, t\big) + w f_{15}\big(x, y, z, t\big) + f_{24}\big(x, y, z, t\big) = 0 \subset \mathrm{Proj}\Big(\mathbb{C}[x, y, z, t, w]\Big),$$

where $\mathrm{wt}(x) = 1$, $\mathrm{wt}(y) = 4$, $\mathrm{wt}(z) = 5$, $\mathrm{wt}(t) = 6$, $\mathrm{wt}(w) = 9$ and $f_k(x, y, z, t)$ is a general quasihomogeneous polynomial of degree k. Then the linear system on the threefold X cut out by the pencil $\lambda x^6 + \mu f_6\big(x, y, z, t\big) = 0$, where $(\lambda : \mu) \in \mathbb{P}^1$, is a Halphen pencil in $|-6K_X|$.

The cases in Examples 1.4 and 1.5 have at least two Halphen pencils because they also satisfy the condition for Example 1.2. Furthermore, we will see that these are the only Halphen pencils on the hypersurface X of each family in Examples 1.4 and 1.5.

The main purpose of this article is to prove the following[1]:

[1] *Theorem 1.3 is proved in [9] and [10] for the cases* $\daleth = 34$, 75, 88 *and* 90.

Theorem 1.3.

Let X be a general hypersurface in the 95 families. Then the pencils constructed in Examples 1.1, 1.2, 1.3, 1.4 and 1.5 exhaust all possibilities for Halphen pencils on the threefold X.

The following are immediate consequence of Theorem 1.3.

Corollary 1.1.

Let X be a general hypersurface in the 95 families with entry number $\gimel$.

(1) There are finitely many Halphen pencils on the threefold X if and only if $a_2 \neq 1$.

(2) There are at most two Halphen pencils on X in the case when $a_1 \neq a_2$.

(3) Every Halphen pencil on the threefold X is contained in $|-K_X|$ if $a_1 = 1$.

(4) The linear system $|-K_X|$ is the only Halphen pencil on X if $a_1 = 1$ and $a_2 \neq 1$.

(5) The linear system $|-a_1K_X|$ is the only Halphen pencil on the threefold X if and only if $\gimel \neq 1, 2, 3, 4, 5, 6, 8, 10, 14, 18, 22, 28, 45, 48, 55, 57, 58, 60, 66, 69, 74, 76, 79, 80, 81, 84, 86, 91, 93, 95$.

Furthermore, Theorem 1.3 with Proposition 4.1 forces us to conclude:

Corollary 1.2.

Let X be a general hypersurface in the 95 families. Then every Halphen pencil on the threefold X is invariant under the action of $\mathrm{Bir}(X)$.

The proof of Theorem 1.3 is based on Theorems 2.1, 2.2 and Lemmas 2.2, 2.3. In addition, we prove that general surfaces of the pencils constructed in Examples 1.1, 1.2, 1.3, 1.4, 1.5 are birational to smooth K3 surfaces.

Theorem 1.4.

Let X be a general hypersurface in the 95 families. Then a general surface of every Halphen pencil on X is birational to a smooth K3 surface.

When $a_1 = 1$, a general surface of a pencil contained in the linear system $|-K_X|$ is birational to a K3 surface. Furthermore, general surfaces in the pencils of Examples 1.1 and 1.2 are birational to K3 surfaces due to the following:

Proposition 1.1.

If $a_1 \neq a_2$, then a general surface of the pencil $|-a_1K_X|$ is birational to a K3 surface.

Proof. See [2]. □

To prove Theorem 1.4, we must check that general surfaces of Halphen pencils of Examples 1.3, 1.4 and 1.5 are also birational to smooth K3 surfaces. The proof of Theorem 1.4 is based on Corollaries 2.1 and 2.2. However, in order to save the space, we will verify Theorem 1.4 only by showing how to apply Corollaries 2.1 and 2.2 to some cases (see Propositions 6.7 and 6.11). Following this method, one can prove the other cases. For the proof of the other cases, the reader is referred to [3].

Theorems 1.3 and 1.4 tell us how a general hypersurface in the 95 families can be rationally fibred by smooth surfaces of Kodaira dimension zero.

Corollary 1.3.

Let X be a general hypersurface in the 95 families and let $\pi : Y \to Z$ be a morphism whose general fiber is birational to a smooth surface of Kodaira dimension zero. If there is a birational map $\alpha : X \dashrightarrow Y$, then there is an isomorphism $\phi : \mathbb{P}^1 \to Z$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, dashed, "\alpha"] \arrow[d, dashed, "\psi"'] & Y \arrow[d, "\pi"] \\ \mathbb{P}^1 \arrow[r, "\phi"'] & Z, \end{tikzcd}$$

where the rational map $\psi : X \dashrightarrow \mathbb{P}^1$ is induced by one of the pencils in Examples 1.1, 1.2, 1.3, 1.4 and 1.5. In particular, a general fiber of the morphism π is birational to a smooth K3 surface.

In what follows, we outline how to prove Theorem 1.3. Some of the 95 families then are fully studied for the theorem in order to show precisely how to carry out our scheme of the proof for the theorem. For details, the reader is referred to [3] where all the 95 families have been investigated. However, the families chosen to be studied in this article show all the essential methods to prove Theorems 1.3 and 1.4 completely.

2. Preliminaries

Let X be a threefold with $\mathbb{Q}$-factorial singularities and $\mathcal{M}$ be a linear system on the threefold X without fixed components. We consider the log pair $(X, \mu\mathcal{M})$ for some non-negative rational number μ.
Let $\alpha : Y \to X$ be a proper birational morphism such that Y is smooth and the proper transform $\mathcal{M}_Y$ of the linear system $\mathcal{M}$ by the birational morphism α is base-point-free. Then the rational equivalence

$$K_Y + \mu\mathcal{M}_Y \sim_{\mathbb{Q}} \alpha^*\Big(K_X + \mu\mathcal{M}\Big) + \sum_{i=1}^{k} a_i E_i$$

holds, where E_i is an exceptional divisor of the birational morphism α and a_i is a rational number.

Definition 2.1.

The singularities of the log pair $(X, \mu\mathcal{M})$ are *terminal* (*canonical*, *log-terminal*, respectively) if each rational number a_i is positive (non-negative, greater than -1, respectively). In this case we also say that the log pair $(X, \mu\mathcal{M})$ is *terminal* (*canonical*, *log-terminal*, respectively).

It is convenient to specify where the log pair $(X, \mu\mathcal{M})$ is not terminal.

Definition 2.2.

A proper irreducible subvariety $Z \subset X$ is called a *center of canonical singularities* of the log pair $(X, \mu\mathcal{M})$ if there is an exceptional divisor E_i such that $\alpha(E_i) = Z$ and $a_i \leq 0$. The set of all proper irreducible subvarieties of X that are centers of canonical singularities of the log pair $(X, \mu\mathcal{M})$ is denoted by $\mathbb{CS}(X, \mu\mathcal{M})$.

A curve not contained in the singular locus of the threefold X is a center of canonical singularities of the log pair $(X, \mu\mathcal{M})$ if and only if the multiplicity of a general surface of $\mathcal{M}$ along the curve is not smaller than $\frac{1}{\mu}$. Furthermore, we obtain

Lemma 2.1.

Let C be a curve on the threefold X that is not contained in the singular locus. Suppose that the curve C is a center of canonical singularities of the log pair $(X, \mu\mathcal{M})$ and the linear system $|-mK_X|$ is base-point-free for some natural number $m > 0$. If $-K_X \sim_{\mathbb{Q}} \mu\mathcal{M}$, then $-K_X \cdot C \leq -K_X^3$.

Proof. See Lemma 2.4 in [1]. □

The following result is a generalization of the so-called Noether–Fano inequality.

Theorem 2.1.

Suppose that the linear system $\mathcal{M}$ is a pencil whose general surface is birational to a smooth surface of Kodaira dimension zero, the linear system $|-mK_X|$ is base-point-free for some natural m and $-K_X \sim_{\mathbb{Q}} \mu\mathcal{M}$. If the linear system $|-mK_X|$ induces either a birational morphism or an elliptic fibration, then the log pair $(X, \mu\mathcal{M})$ is not terminal.

Proof. Let M be a general surface in $\mathcal{M}$. Suppose that the log pair $(X, \mu M)$ is terminal. Then for some positive rational number $\epsilon > \mu$, the log pair $(X, \epsilon M)$ is also terminal and the divisor $K_X + \epsilon M$ is nef. We have a resolution of indeterminacy of the rational map $\rho : X \dashrightarrow \mathbb{P}^1$ induced by the pencil $\mathcal{M}$ as follows:

$$\begin{tikzcd} & Y \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ X \arrow[rr, dashed, "\rho"'] & & \mathbb{P}^1, \end{tikzcd}$$

where Y is smooth and β is a morphism. We consider the linear equivalence

$$K_Y + \epsilon M_Y \sim_{\mathbb{Q}} \alpha^*\Big(K_X + \epsilon M\Big) + \sum_{i=1}^{k} c_i E_i,$$

where M_Y is the proper transform of the surface M and c_i is a rational number. Then each c_i is positive. Also we may assume that the proper transform $\mathcal{M}_Y$ of the pencil $\mathcal{M}$ by the birational morphism α is base-point-free. In particular, the surface M_Y is smooth.
Let l be a sufficiently big and divisible natural number. Then the negativity property of the exceptional locus of a birational morphism (Section 1.1 in [11]) implies that the linear system $|l(K_Y + \epsilon M_Y)|$ gives a dominant rational map $\xi : Y \dashrightarrow V$ with $\dim(V) \geq 2$. One the other hand, since the proper transform $\mathcal{M}_Y$ is a base-point-free pencil, the adjunction formula implies that

$$l\Big(K_Y + \epsilon M_Y\Big)\Big|_{M_Y} \sim lK_{M_Y}.$$

However, the surface M_Y has Kodaira dimension zero, which implies that $\dim(V) \leq 1$. This is a contradiction. □

Suppose that X has a quotient singular point P of type $\frac{1}{r}(1, a, r-a)$, where $r \geq 2$, $r > a$ and a is coprime to r. The weighted blow up $\pi : Y \to X$ at the point P with weights $(1, a, r-a)$ is called the Kawamata blow up at the point P with weights $(1, a, r-a)$ or simply the Kawamata blow up at the point P. One can easily check that the exceptional divisor E of the birational morphism π is isomorphic to $\mathbb{P}(1, a, r-a)$. Furthermore, we see

$$K_Y = \pi^*(K_X) + \frac{1}{r}E,\ K_Y^3 = K_X^3 + \frac{1}{ra(r-a)},\ E^3 = \frac{r^2}{a(r-a)}.$$

Unless otherwise mentioned, from this point throughout this section, we always assume that the linear system $\mathcal{M}$ is a pencil with $-K_X \sim_{\mathbb{Q}} \mu\mathcal{M}$. In addition, we always assume that a general surface of the pencil $\mathcal{M}$ is irreducible.

Lemma 2.2.

Let P be a singular point of a threefold X that is a quotient singularity of type $\frac{1}{r}(1, a, r-a)$, $r \geq 2$, $r > a$ and a is coprime to r. Suppose that the log pair $(X, \mu\mathcal{M})$ is canonical but the set $\mathbb{CS}(X, \mu\mathcal{M})$ contains either the point P or a curve passing through the point P. Let $\pi : Y \to X$ be the Kawamata blow up at the point P and let $\mathcal{M}_Y$ be the proper transform of $\mathcal{M}$ by the birational morphism π. Then $\mu\mathcal{M}_Y \sim_{\mathbb{Q}} -K_Y$, where E is the exceptional divisor of π.

Proof. See [7]. □

Lemma 2.2 can be generalized in the following way ([1]).

Lemma 2.3.

Under the assumptions and notations of Lemma 2.2, suppose that we have a proper subvariety $Z \subset E \cong \mathbb{P}(1, a, r-a)$ that belongs to $\mathbb{CS}(Y, \mu\mathcal{M}_Y)$. Then the following hold:

(1) The subvariety Z is not a smooth point of the surface E.

(2) If the subvariety Z is a curve, then it belongs to the linear system $|\mathcal{O}_{\mathbb{P}(1,a,r-a)}(1)|$ defined on the surface E and all singular points of the surface E are contained in the set $\mathbb{CS}(Y, \mu\mathcal{M}_Y)$.

Proof. For the convenience of the reader we consider only the case when $r = 5$ and $a = 2$. Thus we have $E \cong \mathbb{P}(1, 2, 3)$. Let Q_1 and Q_2 be the singular points of the surfaces E, and L the unique curve in $|\mathcal{O}_{\mathbb{P}(1,2,3)}(1)|$ on the surface E. Then L contains the singular points Q_1 and Q_2 but the equivalence $\mu\mathcal{M}_Y|_E \equiv L$ holds by Lemma 2.2. Also it follows from Lemma 2.2 that the set $\mathbb{CS}(Y, \mu\mathcal{M}_Y)$ contains both the points Q_1 and Q_2 if the curve L is contained in the set $\mathbb{CS}(Y, \mu\mathcal{M}_Y)$.

Suppose that the subvariety Z is different from L, Q_1 and Q_2. Let us show that this assumption gives us a contradiction. Suppose that Z is a point. Then Z is a smooth point of the threefold Y, which implies the inequality $\mathrm{mult}_Z(\mathcal{M}_Y) > \frac{1}{\mu}$. Let C be a general curve in $|\mathcal{O}_{\mathbb{P}(1,2,3)}(6)|$ on the surface E that passes through the point Z. Then C is not contained in the base locus of $\mathcal{M}_Y$, which implies the following contradictory inequality:

$$\frac{1}{\mu} = C \cdot \mathcal{M}_Y \geq \mathrm{mult}_Z\big(\mathcal{M}_Y\big) > \frac{1}{\mu}.$$

Therefore the subvariety Z must be a curve. Then $\mathrm{mult}_Z(\mathcal{M}_Y) \geq \frac{1}{\mu}$. Let C be a general curve in the linear system $|\mathcal{O}_{\mathbb{P}(1,2,3)}(6)|$ on the surface E. Then the curve C is not contained in the base locus of the pencil $\mathcal{M}_Y$. Therefore we have

$$\frac{1}{\mu} = C \cdot \mathcal{M}_Y \geq \mathrm{mult}_Z\big(\mathcal{M}_Y\big) C \cdot Z \geq \frac{1}{\mu} C \cdot Z,$$

which implies that $C \cdot Z = 1$ on the surface E. The equality $C \cdot Z = 1$ implies that the curve Z is contained in the linear system $|\mathcal{O}_{\mathbb{P}(1,2,3)}(1)|$ on the surface E, which is impossible due to our assumption. □

The following result is a generalization of Lemma A.20 in [1].

Theorem 2.2.

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be linear systems on a threefold X such that a general surface of each linear system $\mathcal{B}_i$ is irreducible. Then the linear system $\mathcal{B}_1$ coincides with the linear system $\mathcal{B}_2$ and they are pencils if one of the following holds:

(0) There is a Zariski closed proper subset $\Sigma \subset X$ such that for any general divisors $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$

$$\mathrm{Supp}\big(B_1\big) \cap \mathrm{Supp}\big(B_2\big) \subseteq \Sigma,$$

Note that the general divisors B_1 and B_2 are chosen independently of the proper subset Σ.

For the remainder of the theorem, let B_1 and B_2 be general surfaces of the linear systems $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively.

(1) There is a nef and big divisor D on the threefold X such that $D \cdot B_1 \cdot B_2 = 0$.

(2) The base locus of $\mathcal{B}_2$ consists of an irreducible curve C such that $B_1 \cdot B_2 \equiv \lambda C$ and $B_2 \cdot C < 0$ for some positive rational number λ.

(3) The equivalence $B_1 \equiv \lambda B_2$ holds for some positive rational number λ and the base locus of $\mathcal{B}_1$ consists of an irreducible curve C such that $B_1 \cdot C < 0$.

(4) *The surface B_1 is normal, the equivalence $B_2 \equiv \lambda B_1$ holds for some positive rational number λ and the base locus of $\mathcal{B}_1$ consists of irreducible curves $C_1, \cdots, C_r$ whose intersection form on the surface B_1 is negative-definite.*

Proof. It is easy to check. See [1] for instance. □

Theorem 2.3.

Suppose that a log pair $(X, \mu\mathcal{M})$ is canonical with $K_X + \mu\mathcal{M} \sim_{\mathbb{Q}} 0$. In addition, we suppose that one of the following holds:

(1) *The base locus of the pencil $\mathcal{M}$ consists of irreducible curves $C_1, \cdots, C_r$ and there is a nef and big divisor D on X such that $D \cdot C_i = 0$ for each i;*

(2) *the base locus of $\mathcal{M}$ consists of an irreducible curve C such that $\mathcal{M} \cdot C < 0$;*

(3) *a general surface of the pencil $\mathcal{M}$ is normal, the base locus of $\mathcal{M}$ consists of irreducible curves $C_1, \cdots, C_r$ whose intersection form is negative-definite on a general surface in $\mathcal{M}$.*

Then the linear system $\mathcal{M}$ is a Halphen pencil and there is an isomorphism $\xi : X \dashrightarrow X'$ in codimension 1 such that it is an isomorphism in the outside of the curves $C_1, \cdots, C_r$ (or C) and the proper transform $\mathcal{M}_{X'}$ of the pencil $\mathcal{M}$ by ξ is base-point-free.

Proof. The log pair $(X, \lambda\mathcal{M})$ is log-terminal for some rational number $\lambda > \mu$. Hence it follows from [11] that there is an isomorphism $\xi : X \dashrightarrow X'$ in codimension 1 such that it is an isomorphism in the outside of the curves $C_1, \cdots, C_r$ (or C), the log pair $(X', \lambda\mathcal{M}_{X'})$ is log-terminal and the divisor $K_{X'} + \lambda\mathcal{M}_{X'}$ is nef.
Let H be a general surface in the pencil $\mathcal{M}_{X'}$. Since

$$H \equiv \frac{1}{\lambda - \mu}\Big(K_{X'} + \lambda\mathcal{M}_{X'} - (K_X' + \mu\mathcal{M}_{X'})\Big),$$

the divisor H is nef. Hence it follows from the log abundance theorem ([8]) that the linear system $|mH|$ is base-point-free for some $m \gg 0$.
Let $\mathcal{B}$ be the proper transform of the linear system $|mH|$ on X. Also let B and M be general surfaces of the linear system $\mathcal{B}$ and the pencil $\mathcal{M}$, respectively. Then $B \equiv mM$ and one of the conditions in Theorem 2.2 is satisfied. Hence we have $\mathcal{M} = \mathcal{B}$, which implies that $m = 1$ and $\mathcal{M}_{X'} = |H|$ is base-point-free and induces a morphism $\pi' : X' \to \mathbb{P}^1$. Thus every member of the pencil $\mathcal{M}_{X'}$ is contracted to a point by the morphism π'.
The log pair $(X', \mu\mathcal{M}_{X'})$ is canonical because the map ξ is a log flop with respect to the log pair $(X, \mu\mathcal{M})$. In particular, the singularities of X' are canonical. Hence the surface H has at most Du Val singularities because the pencil $\mathcal{M}_{X'}$ is base-point-free. Moreover the equivalence $K_{X'} + \mu H \sim_{\mathbb{Q}} 0$ and the Adjunction formula imply that $K_H \sim 0$. Consequently, the linear system $\mathcal{M}$ is a Halphen pencil. □

Corollary 2.1.

Under the assumptions and with the notation of Theorem 2.3, additionally suppose that a general surface of the pencil $\mathcal{M}$ is linearly equivalent to $-nK_X$ for some natural number n. Then a general element of $\mathcal{M}$ is birational either to a smooth K3 surface or to an Abelian surface.

Proof. It immediately follows from the proof of Theorem 2.3 and the classification of smooth surfaces of Kodaira dimension zero. □

Corollary 2.2.

Under the assumptions and with the notation of Corollary 2.1, suppose that a general surface of the pencil $\mathcal{M}$ has a rational curve not contained in the base locus of the pencil $\mathcal{M}$. Then a general element of $\mathcal{M}$ is birational to a smooth K3 surface.

Proof. In the proof of Theorem 2.3, suppose that the surface M has a rational curve L not contained in the base locus of the pencil $\mathcal{M}$. Then the surface H contains a rational curve because the birational map ξ makes no change along the curve L. On the other hand, the surface H is birational either to a smooth K3 surface or to an Abelian surface. However, an Abelian surface cannot contain a rational curve. □

3. Scheme of the proof

Let us describe the notations we will use in the rest of the article. Unless otherwise mentioned, these notations are fixed from now until the end of the article.

- In the weighted projective space $\mathbb{P}(1, a_1, a_2, a_3, a_4)$, we assume that $a_1 \leq a_2 \leq a_3 \leq a_4$. For weighted homogeneous coordinates, we always use x, y, z, t and w with $\mathrm{wt}(x) = 1$, $\mathrm{wt}(y) = a_1$, $\mathrm{wt}(z) = a_2$, $\mathrm{wt}(t) = a_3$ and $\mathrm{wt}(w) = a_4$.
- The number $\daleth$ always means the entry number of each family of weighted Fano hypersurfaces in the Big Table of [5].
- In each family, we always let X be a general quasismooth hypersurface of degree d in the weighted projective space $\mathbb{P}(1, a_1, a_2, a_3, a_4)$, where $d = \sum_{i=1}^{4} a_i$.
- On the threefold X, a given Halphen pencil is denoted by $\mathcal{M}$.
- For a given Halphen pencil $\mathcal{M}$, we always assume that $\mathcal{M} \sim_{\mathbb{Q}} -nK_X$.
- When a morphism $f : V \to W$ is given, the proper transforms of a curve Z, a surface D and a linear system $\mathcal{D}$ on W by the morphism f will be always denoted by Z_V, D_V and $\mathcal{D}_V$, respectively, i.e., we use the ambient space V as their subscripts.
- S : the surface on X defined by the equation $x = 0$.
- S^y : the surface on X defined by the equation $y = 0$.
- S^z : the surface on X defined by the equation $z = 0$.
- S^t : the surface on X defined by the equation $t = 0$.
- S^w : the surface on X defined by the equation $w = 0$.
- C : the curve on X defined by the equations $x = y = 0$.
- $\bar{C}$: the curve on X defined by the equations $x = z = 0$.
- $\tilde{C}$: the curve on X defined by the equations $x = t = 0$.
- $\hat{C}$: the curve on X defined by the equations $x = w = 0$.

From now, we explain our scheme of the proof for Theorem 1.3.
In each case, for a given general hypersurface X and a given Halphen pencil $\mathcal{M}$, we consider the log pair $(X, \frac{1}{n}\mathcal{M})$. Note that the natural number n is given by $\mathcal{M} \sim_{\mathbb{Q}} -nK_X$. To prove Theorem 1.3, we must show that the Halphen pencil $\mathcal{M}$ is one of the pencils given in Examples 1.1, 1.2, 1.3, 1.4 and 1.5. To do so, we will do the following:

Step 1. We may always assume that the log pair $(X, \frac{1}{n}\mathcal{M})$ is canonical for the following reason:
Due to Theorem 4.1, there is a birational automorphism $\rho \in \mathrm{Bir}(X)$ such that the log pair $(X, \frac{1}{\bar{n}}\rho(\mathcal{M}))$ is canonical for the natural number $\bar{n}$ with $\rho(\mathcal{M}) \sim_{\mathbb{Q}} -\bar{n}K_X$. It will turn out that the pencil $\rho(\mathcal{M})$ is one of the pencils constructed in Examples 1.1, 1.2, 1.3, 1.4 and 1.5 that are $\mathrm{Bir}(X)$-invariant (Proposition 4.1). This implies that $\mathcal{M} = \rho(\mathcal{M})$.

Step 2. We observe that the set $\mathbb{CS}\left(X, \frac{1}{n}\mathcal{M}\right)$ is not empty by Theorem 2.1.

Step 3. We observe that the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains no smooth point of X. For the families with $\daleth \geq 3$ this follows from the proof of Theorem 5.3.1 in [5]. For the families with $\daleth = 1$ and 2 we will prove it directly.

Step 4. If $a_2 \neq 1$ then we may assume that the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains only singular points of X by Corollaries 4.3 and 4.4. If $a_2 = 1$ then we show that if the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains a curve, then $n = 1$. Therefore we may assume that the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains only singular points of X.

Step 5. If a singular point P of X satisfies the conditions of Lemmas 4.2 and 4.3 and the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains the point P, Lemma 4.3 implies that $\mathcal{M}$ is the pencil described in either Example 1.2 or Example 1.4. If a singular point Q of X satisfies the condition of Proposition 4.2 but the integer c in Proposition 4.2 is positive, then we can derive a contradiction from Lemma 4.2. Therefore the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ cannot contain the point Q.
Using the Big Table in [5], we may check whether a singular point satisfies the conditions or not. Therefore we may assume that the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ does not contain such singular points.

Step 6. We take a singular point P in the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ and then consider the Kawamata blow up $\pi_1 : X_1 \to X$ at the point P. Then Lemma 2.2 implies that $\mathcal{M}_{X_1} \sim_{\mathbb{Q}} -nK_{X_1}$. In particular, the image on X of every element of the set $\mathbb{CS}(X_1, \frac{1}{n}\mathcal{M}_{X_1})$ must belong to the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$.

Step 7. If $-K_{X_1}$ is nef then one can observe that the linear system $|-mK_{X_1}|$ is base-point-free for some natural m and the linear system $|-mK_X|$ induces either a birational morphism or an elliptic fibration. In particular, if $-K_{X_1}$ is nef, then the set $\mathbb{CS}\left(X_1, \frac{1}{n}\mathcal{M}_{X_1}\right)$ is not empty by Theorem 2.1.

Step 8. The set $\mathbb{CS}\left(X_1, \frac{1}{n}\mathcal{M}_{X_1}\right)$ does not contain smooth points of the π_1-exceptional divisor by Lemma 2.3. Moreover we show that the set $\mathbb{CS}\left(X_1, \frac{1}{n}\mathcal{M}_{X_1}\right)$ does not contain curves that are contained in the π_1-exceptional divisor (either by using Lemma 2.1 if $-K_{X_1}^3$ is small, or by some individual methods otherwise). Therefore the set $\mathbb{CS}(X_1, \frac{1}{n}\mathcal{M}_{X_1})$ contains only singular points of X_1.

Step 9. If $-K_{X_1}$ is nef, we apply Steps 2-8 to the log pair $(X_1, \frac{1}{n}\mathcal{M}_{X_1})$. We repeat this procedure to get a sequence of Kawamata blow ups

$$X_r \xrightarrow{\pi_r} X_{r-1} \xrightarrow{\pi_{r-1}} \cdots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X$$

such that $-K_{X_r}$ is not nef and $\mathcal{M}_{X_r} \sim_{\mathbb{Q}} -nK_{X_r}$. In fact, it turns out that r is at most 4.

Step 10. The linear system $|-mK_{X_r}|$ is not empty for a sufficiently large and divisible positive integer m. However it may contain only one divisor. In such a case, we derive a contradiction from the fact that $\mathcal{M}_{X_r} \sim_{\mathbb{Q}} -nK_{X_r}$.

Step 11. If the linear system $|-mK_{X_r}|$ for sufficiently large and divisible positive integer m is composed from a pencil, then we show that $\mathcal{M}$ is one of the pencils constructed in Examples I-V by using Theorem 2.2. Therefore we may assume that the linear system $|-mK_{X_r}|$ gives a rational map $\xi : X_r \dashrightarrow V$ such that $\dim V \geq 2$.

Step 12. We show the existence of a commutative diagram

$$\begin{tikzcd} & U \arrow[dr, "\psi"] & \\ X_r \arrow[ur, dashed, "\zeta"] \arrow[rr, dashed, "\xi"'] & & V \end{tikzcd}$$

such that the variety U has terminal $\mathbb{Q}$-factorial singularities, the rational map ζ is an isomorphism in codimension 1, $-K_U$ is nef, the linear system $|-mK_U|$ for sufficiently large and divisible positive integer m is base-point-free and the morphism ψ is given by $|-mK_U|$. We see that either $-K_U$ is nef and big or the morphism ψ is an elliptic fibration.

Step 13. Since the map ζ is an isomorphism in codimension 1, the equivalence $\mathcal{M}_U \sim_{\mathbb{Q}} -nK_U$ holds. Therefore the set $\mathbb{CS}\left(U, \frac{1}{n}\mathcal{M}_U\right)$ is not empty by Theorem 2.1.

Step 14. The set $\mathbb{CS}\left(X_r, \frac{1}{n}\mathcal{M}_{X_r}\right)$ is not empty because the map ζ is a log-flop with respect to the log pair $(X_r, \frac{1}{n}\mathcal{M}_{X_r})$.

Step 15. Now we can apply Steps 2-14 to the log pair $(X_r, \frac{1}{n}\mathcal{M}_{X_r})$ to get a sequence of Kawamata blow ups

$$X_q \xrightarrow{\pi_q} X_{q-1} \xrightarrow{\pi_{q-1}} \cdots \xrightarrow{\pi_{r+1}} X_{r+1} \xrightarrow{\pi_r} X_r$$

such that $-K_{X_q}$ is not nef, the equivalence $\mathcal{M}_{X_q} \sim_{\mathbb{Q}} -nK_{X_q}$ holds and the linear system $|-mK_{X_q}|$ either consists of a single divisor or is composed from a pencil, where m is a sufficiently large and divisible positive integer. In fact, it turns out that we need at most two Kawamata blow ups.

Step 16. If the linear system $|-mK_{X_q}|$ consists of a single divisor, we derive a contradiction from the fact that $\mathcal{M}_{X_q} \sim_{\mathbb{Q}} -nK_{X_q}$. If it is composed from a pencil, we show that $\mathcal{M}$ is one of the pencils constructed in Examples 1.1-1.5 using Theorem 2.2.

4. General results

Let $X \subset \mathbb{P}(1, a_1, a_2, a_3, a_4)$ be a general hypersurface in one of the 95 families with entry number $\daleth$. In addition, let $\mathcal{M}$ be a Halphen pencil on the threefold X. Then the log pair $(X, \frac{1}{n}\mathcal{M})$ is not terminal by Theorem 2.1, where n is the natural number such that $\mathcal{M} \sim_{\mathbb{Q}} -nK_X$. The following result is due to [5].

Theorem 4.1.

There is a birational automorphism $\tau \in \mathrm{Bir}(X)$ such that the log pair $(X, \frac{1}{m}\tau(\mathcal{M}))$ is canonical, where m is the natural number such that $\tau(\mathcal{M}) \sim_{\mathbb{Q}} -mK_X$.

To classify Halphen pencils on X up to the action of $\mathrm{Bir}(X)$, we may assume that the log pair $(X, \frac{1}{n}\mathcal{M})$ is canonical. However, it is not terminal by Theorem 2.1.

Proposition 4.1.

The pencils constructed in Examples 1.1, 1.2, 1.3, 1.4 and 1.5 are invariant under the action of the group $\mathrm{Bir}(X)$ of birational automorphisms of X.

Proof. Suppose that the hypersurface X is defined by the equation

$$f_d(x, y, z, t, w) = 0 \subset \mathrm{Proj}\big(\mathbb{C}[x, y, z, t, w]\big),$$

where $\mathrm{wt}(x) = 1$, $\mathrm{wt}(y) = a_1$, $\mathrm{wt}(z) = a_2$, $\mathrm{wt}(t) = a_3$, $\mathrm{wt}(w) = a_4$ and f_d is a general quasihomogeneous polynomial of degree $d = \sum a_i$.

Since the hypersurface X is general, it is not hard to see that the group $\mathrm{Aut}(X)$ of automorphisms of X is either trivial or isomorphic to the group of order 2. The latter case happens when $2a_4 = d$. In such a case, the hypersurface X can be defined by an equation of the form

$$w^2 = g_d(x, y, z, t),$$

where g_d is a general quasihomogeneous polynomial of degree d in variables x, y, z and t. The group $\mathrm{Aut}(X)$ is generated by the involution $[x : y : z : t : w] \mapsto [x : y : z : t : -w]$. Therefore in both cases, we can see that the pencils constructed in Examples 1.1, 1.2, 1.3, 1.4 and 1.5 are invariant under the action of the group $\mathrm{Aut}(X)$ of automorphisms of X.

Suppose that the hypersurface X is not superrigid, i.e., it has a birational automorphism that is not biregular. Then it is either a quadratic involution or an elliptic involution that are described in [5]. A quadratic involution has no effect on things defined with the variables x, y, z and t (see Theorem 4.9 in [5]). On the other hand, an elliptic involution has no effect on things defined with the variables x, y and z (see Theorem 4.13 in [5]). The pencils constructed in Examples 1.1, 1.2, 1.3 and 1.4 are defined by the variables x, y and z. Therefore such pencils are invariant under the action of the group $\mathrm{Bir}(X)$ of birational automorphisms of X. Meanwhile, the pencil constructed in Example 1.5 is contained in $|-a_3K_X|$. However, in the case $\daleth = 60$, the hypersurface X does not have an elliptic involution (see The Big Table in [5]) and hence the pencil is also $\mathrm{Bir}(X)$-invariant. ∎

To be precise, the poorf of Theorem 1.3 shows that for a given Halpen pencil $\mathcal{M}$, there is a birational automorphism τ of X such that the pencil $\tau(\mathcal{M})$ is exactly one of the Halpen pencils described in Examples 1.1-1.5. However, Proposition 4.1 proves that the pencils in Examples 1.1-1.5 are $\mathrm{Bir}(X)$-invariant. Therefore every Halphen pencil on X is $\mathrm{Bir}(X)$-invariant.

Lemma 4.1.
Suppose that $\gimel \geq 3$. Then the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains no smooth point of X.

Proof. This follows from the proof of Theorem 5.3.1 in [5]. □

Corollary 4.1.
Suppose that the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains a curve C. Then $-K_X \cdot C \leq -K_X^3$.

Proof. This is an immediate consequence of Lemma 2.1. □

Corollary 4.2.
The set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains a singular point of X whenever $\gimel \geq 7$.

Proof. If $\gimel \geq 7$, then $-K_X^3 < 1$. Therefore each curve C with $-K_X \cdot C \leq -K_X^3$ passes through a singular point of X. Then the result follows from Lemmas 2.2, 4.1 and Corollary 4.1. □

In fact, we have a stronger result as follows:

Theorem 4.2.
Suppose that $\gimel \geq 3$ and the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains a curve C. Then

$$\mathrm{Supp}\big(C\big) \subset \mathrm{Supp}\Big(S_1 \cdot S_2\Big),$$

where S_1 and S_2 are distinct surfaces of the linear system $|-K_X|$.

Proof. See Section 3.1 in [10]. □

Corollary 4.3.
The set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains no curves whenever $a_1 \neq 1$.

Corollary 4.4.
If the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains a curve and $a_2 \neq 1$, then $a_1 = 1$ and $\mathcal{M} = |-K_X|$.

Proof. This follows from Theorem 4.2 and Theorem 2.2. □

Suppose that $\gimel \geq 7$. Then the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains a singular point P of type $\frac{1}{r}(1, r-a, a)$, where $r \geq 2$, $r > a$ and a is coprime to r. Let $\pi : Y \to X$ be the Kawamata blow up at the singular point P and E be its exceptional divisor. Then $\mathcal{M}_Y \sim_{\mathbb{Q}} -nK_Y$ by Lemma 2.2. The cone of effective 1-cycles $\overline{\mathbb{NE}}(Y)$ of the threefold Y contains two extremal rays R_1 and R_2 such that π is a contraction of the extremal ray R_1. Moreover the following result holds:

Proposition 4.2.
Suppose that $-K_Y^3 \leq 0$ and $\gimel \neq 82$. Then the threefold Y contains irreducible surfaces $S \sim_{\mathbb{Q}} -K_Y$ and $T \sim_{\mathbb{Q}} -bK_Y + cE$ whose scheme-theoretic intersection is an irreducible reduced curve that generates R_2, where $b > 0$ and $c \geq 0$ are integer numbers.

Proof. See Lemma 5.4.3 in [5]. □

We can find the values of b and c for a given singular point in the Big Table of [5].

Lemma 4.2.
With the assumptions and the notation of Proposition 4.2, suppose that $-K_Y^3 < 0$. Then the number c is zero.

Proof. Let M_1 and M_2 be general surfaces of the pencil $\mathcal{M}_Y$. Then $M_1 \cdot M_2 \equiv n^2 K_Y^2$, which implies that $M_1 \cdot M_2 \notin \overline{\mathbb{NE}}(Y)$ in the case $c > 0$ by Proposition 4.2. □

Lemma 4.3.
With the assumptions and the notation of Proposition 4.2, suppose that the inequality $-K_Y^3 < 0$ holds. Then the pencil $\mathcal{M}_Y$ is generated by the divisors bS and T.

Proof. Let M_1 and M_2 be general surfaces in $\mathcal{M}_Y$. Then $M_1 \cdot M_2 \in \overline{\mathbb{NE}}(Y)$ and $M_1 \cdot M_2 \equiv n^2 K_Y^2$, which implies that $M_1 \cdot M_2 \in \mathbb{R}^+ R_2$ because $c = 0$ by Lemma 4.2. Moreover we have

$$\mathrm{Supp}(\Gamma) = \mathrm{Supp}\Big(M_1 \cdot M_2\Big)$$

since $T \cdot R_2 < 0$ and $S \cdot R_2 < 0$. Therefore the pencil $\mathcal{M}_Y$ coincides with the pencil generated by the divisors bS and T by Theorem 2.2-(0). This completes the proof. □

5. Easy cases

In this section, we apply our scheme to several easy cases in order to show how to prove Theorem 1.3.

Proposition 5.1.
If $\daleth = 3$, then every Halphen pencil is contained in $|-K_X|$.

Proof. The threefold X is a general hypersurface of degree 6 in $\mathbb{P}(1,1,1,1,3)$ with $-K_X^3 = 2$. It is smooth. It cannot be birationally transformed into an elliptic fibration ([2]).
It follows from Lemma 4.1 that the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ does not contain any point of X. Hence it must contain a curve Z; so the inequality

$$\mathrm{mult}_Z(\mathcal{M}) \geq n$$

holds.
For general surfaces M_1 and M_2 in $\mathcal{M}$ and a general surface D in $|-K_X|$, we have

$$2n^2 = M_1 \cdot M_2 \cdot D \geq \mathrm{mult}_Z^2(\mathcal{M})(-K_X \cdot Z) \geq n^2(-K_X \cdot Z).$$

This implies that $2 \geq -K_X \cdot Z$. Theorem 4.2 shows that there are different surfaces D_1 and D_2 in the linear system $|-K_X|$ such that the intersection $D_1 \cap D_2$ contains the curve Z.
Let $\mathcal{P}$ be the pencil in $|-K_X|$ consisting of surfaces passing through the curve Z.
Suppose that $-K_X \cdot Z = 2$. For a general surface D' in the pencil $\mathcal{P}$, the inequality

$$2n = M_1 \cdot D' \cdot D \geq \mathrm{mult}_Z(M_1)\mathrm{mult}_Z(D')(-K_X \cdot Z) \geq 2n$$

implies that $\mathrm{Supp}(M_1) \cap \mathrm{Supp}(D') \subset \mathrm{Supp}(Z)$. It follows from Theorem 2.2-(0) that the linear system $\mathcal{M}$ is the pencil in $|-K_X|$ consisting of surfaces that pass through Z.
Now we suppose that $-K_X \cdot Z = 1$. The generality of X implies that the general surface D in $|-K_X|$ is smooth and that $D_1 \cap D_2$ consists of the curve Z and an irreducible curve $\bar{Z}$ such that $Z \neq \bar{Z}$. Hence we have $Z^2 = \bar{Z}^2 = -2$ on the surface D and $\mathcal{M}|_D \equiv nZ + n\bar{Z}$. Therefore the inequality $\mathrm{mult}_Z(\mathcal{M}) \geq n$ shows that

$$\mathcal{M}|_D = n_1 Z + n_2 \bar{Z} + \Delta,$$

where $n_1 \geq n \geq n_2$ and Δ is an effective divisor whose support does not contain Z and $\bar{Z}$. Then we see

$$0 \leq (n_1 - n)Z \cdot \bar{Z} + \Delta \cdot \bar{Z} = (n - n_2)\bar{Z}^2 = -2(n - n_2),$$

which implies $n_2 \geq n$ and hence $n = n_1 = n_2$ and $\Delta = 0$. Consequently, we see that

$$\mathcal{M}|_D = nZ + n\bar{Z}.$$

Theorem 2.2-(0) then implies the identity $\mathcal{M} = \mathcal{P}$. □

Proposition 5.2.

If $\gimel = 11$, the linear system $|-K_X|$ is a unique Halphen pencil on X.

Proof. The threefold X is a general hypersurface of degree 10 in $\mathbb{P}(1, 1, 2, 2, 5)$ with $-K_X^3 = \frac{1}{2}$. Its singularities consist of five points $P_1, \cdots, P_5$ that are quotient singularities of type $\frac{1}{2}(1, 1, 1)$. For each singular point P_i, we have an elliptic fibration as follows:

$$\begin{tikzcd} & U_i \arrow[dl, "\pi_i"'] \arrow[dr, "\eta_i"] & \\ X \arrow[rr, dashed, "\xi_i"'] & & \mathbb{P}(1,1,2), \end{tikzcd}$$

where π_i is the Kawamata blow up at P_i with weights $(1, 1, 1)$ and η_i is an elliptic fibration.
If the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains a curve, then we obtain $\mathcal{M} = |-K_X|$ from Corollary 4.4. Thus we may assume that $\mathbb{CS}(X, \frac{1}{n}\mathcal{M}) \subset \{P_1, P_2, P_3, P_4, P_5\}$ by Lemma 4.1. Furthermore, it cannot consist of a single point by Lemmas 2.1 and 2.3. Indeed if it consists of a single singular point, say P_1, then Theorem 2.1 and Lemma 2.3 show that the set $\mathbb{CS}(U_1, \frac{1}{n}\mathcal{M}_{U_1})$ must contain a curve of degree 1 on the exceptional divisor of π_1. However, Lemma 2.1 gives us a contradiction.
Therefore it contains at least two, say P_i and P_j, of the five singular points. Let $\pi : U \to U_i$ be the Kawamata blow up at the singular point whose image on X is the point P_j. Then the pencil $|-K_U|$ is the proper transform of the pencil $|-K_X|$ and its base locus consists of the irreducible curve C_U. Since $-K_U \cdot C_U < 0$ and $\mathcal{M}_U \sim_{\mathbb{Q}} -nK_U$, Theorem 2.2-(3) completes the proof. □

Proposition 5.3.

If $\gimel = 14$, then every Halphen pencil on X is contained in $|-K_X|$.

Proof. Let X be a general hypersurface of degree 12 in $\mathbb{P}(1, 1, 1, 4, 6)$ with $-K_X^3 = \frac{1}{2}$. It has only one singular point P that is a quotient singularity of type $\frac{1}{2}(1, 1, 1)$.
We have an elliptic fibration as follows:

$$\begin{tikzcd} & Y \arrow[dl, "\pi"'] \arrow[dr, "\eta"] & \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}^2, \end{tikzcd}$$

where ψ is the natural projection, π is the Kawamata blow up at the point P with weights $(1, 1, 1)$ and η is an elliptic fibration.
The log pair $(X, \frac{1}{n}\mathcal{M})$ is not terminal by Theorem 2.1. However, it is terminal at a smooth point by Lemma 4.1.
Suppose that the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ consists of only the singular point P. Since $-K_Y^3 = 0$ and $\mathcal{M}_Y \sim_{\mathbb{Q}} -nK_Y$, every surface in the pencil $\mathcal{M}_Y$ is contracted to a curve by the morphism η. The log pair $(Y, \frac{1}{n}\mathcal{M}_Y)$ is not terminal along a curve Z' contained in the exceptional divisor of π by Theorem 2.1 and Lemma 2.3. The exceptional divisor of π is

a section of η, which implies $-K_Y \cdot Z'$ is positive because the elliptic fibration η is given by $|-K_Y|$. However, Lemma 2.1 shows that $-K_Y \cdot Z' \leq -K_Y^3 = 0$, which is a contradiction.

Consequently, the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ must contain a curve Z. Then the inequality $\mathrm{mult}_Z(\mathcal{M}) \geq n$ holds. Lemma 2.1 shows $-K_X \cdot Z \leq \frac{1}{2}$. But if the curve Z is not a fiber of the rational map ψ, then $-K_X \cdot Z \geq 1$. Therefore it must be a fiber of the rational map ψ.

For a general surface M in $\mathcal{M}$, a general surface D in $|-K_X|$ and a general surface D' in $|-K_X|$ that contains the curve Z we have

$$\frac{n}{2} = M \cdot D \cdot D' \geq \frac{n}{2},$$

which implies that $\mathrm{Supp}(M) \cap \mathrm{Supp}(D') \subset \mathrm{Supp}(Z)$. It follows from Theorem 2.2-(0) that the linear system $\mathcal{M}$ is the pencil in $|-K_X|$ consisting of surfaces that pass through Z. □

Proposition 5.4.

If $\gimel = 75$, then $|-a_1K_X|$ is a unique Halphen pencil on X.

Proof. The threefold X is a general hypersurface of degree 30 in $\mathbb{P}(1, 4, 5, 6, 15)$ with $-K_X^3 = \frac{1}{60}$. Its singularities consist of one quotient singular point of type $\frac{1}{4}(1, 1, 3)$, one quotient singular point of type $\frac{1}{3}(1, 1, 2)$, two quotient singular points of type $\frac{1}{2}(1, 1, 1)$ and two quotient singular points of type $\frac{1}{5}(1, 4, 1)$.

All these singular points except two of type $\frac{1}{5}(1, 4, 1)$ satisfy the conditions of Proposition 4.2 with positive c. However Lemma 4.2 shows that c must be zero. Therefore the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ cannot contain these singular points. Furthermore, it follows from Theorem 2.1 and Lemma 4.1 that the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ must contains a singular point P of type $\frac{1}{5}(1, 4, 1)$. The singular point P satisfies all the conditions of Lemma 4.3 and hence we can conclude that $\mathcal{M} = |-4K_X|$. □

6. Hard cases

In order show how to prove Theorem 1.3, in this section, we select nine out of the 95 families to study in detail. The selected families demonstrate how to apply our scheme of the proof. They have more complicated features for our scheme than those in the previous section. In addition, they show some individual methods for each case. Throughout studying the selected families, we will see all the techniques to prove Theorem 1.3.

$\gimel$=1: Hypersurface of degree 4 in $\mathbb{P}^4$.

Let X be a general quartic hypersurface in $\mathbb{P}^4$. It is smooth and the log pair $(X, \frac{1}{n}\mathcal{M})$ is canonical (Theorem 3.6 in [4]).

Proposition 6.1.

Every Halphen pencil is contained in $|-K_X|$.

Let us prove Proposition 6.1. Suppose that the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains a point P of the quartic X. We are to show this assumption leads to a contradiction.

Let M_1 and M_2 be two general surfaces in $\mathcal{M}$. Then the inequality $\mathrm{mult}_P(M_1 \cdot M_2) \geq 4n^2$ holds ([4]). On the other hand the degree of the cycle $M_1 \cdot M_2$ is $4n^2$, which implies that $\mathrm{mult}_P(M_1 \cdot M_2) = 4n^2$. In particular, the support of the cycle $M_1 \cdot M_2$ consists of the union of all lines passing through the point P. This implies that there are at most finitely many lines on the quartic X passing through the point P. Moreover the equality $\mathrm{mult}_P(\mathcal{M}) = 2n$ holds (see [1]).

Let $\pi : V \to X$ be the blow up at the point P and E be the exceptional divisor of π. In addition, let B_i be the proper transform of the divisor M_i by π. Then the equalities $\mathrm{mult}_P(M_1 \cdot M_2) = 4n^2$ and $\mathrm{mult}_P(\mathcal{M}) = 2n$ imply that

$$B_1 \cdot B_2 = \sum_{i=1}^{k} \mathrm{mult}_{\bar{L}_i}\big(B_1 \cdot B_2\big)\bar{L}_i,$$

where k is a number of lines on X that passes through the point P and $\bar{L}_i$ is an irreducible curve such that $\pi(\bar{L}_i)$ is a line on X that passes through the point P.

Lemma 6.1.
Let Z be an irreducible curve on X that is not a line passing through the point P. Then

$$\deg(Z) \geq 2\mathrm{mult}_P(Z),$$

where the equality holds only if the proper transform Z_V does not intersect the curve $\bar{L}_i$ for any i.

Proof. The proper transform Z_V is not contained in B_i because the base locus of the pencil $\mathcal{M}_V$ consists of the curves $\bar{L}_1, \cdots, \bar{L}_k$. Hence we have

$$0 \leq B_i \cdot Z_V \leq n\big(\deg(Z) - 2\mathrm{mult}_P(Z)\big),$$

which concludes the proof. □

Note that so far we have not used the generality of the quartic X beyond its smoothness. In the following we assume that there are at most 3 lines on X passing though a given point and every line on X has normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}$. The former condition is satisfied on a general quartic threefold. The latter condition is also satisfied on a general quartic. Moreover the latter condition is equivalent to the following: No two-dimensional linear subspace of $\mathbb{P}^4$ is tangent to the quartic X along a line. In particular, we see that no hyperplane section of X can be singular at three points that are contained in a single line.

Lemma 6.2.
For a line L in X passing through P, $\mathrm{mult}_L(\mathcal{M}) \geq \frac{n}{2}$.

Proof. Let $\alpha : W \to X$ be the blow up along the line L and F be the exceptional divisor of the blow up α. Then the surface F is the rational ruled surface $\mathbb{F}_1$.
Let Δ be the irreducible curve on the surface F such that $\Delta^2 = -1$ and Z be the fiber of the restricted morphism $\pi|_F : F \to L$ over the point P. Then $F|_F \equiv -(\Delta + Z)$, which implies that

$$\mathcal{M}_W\Big|_F \equiv nZ + \mathrm{mult}_L(\mathcal{M})(\Delta + Z).$$

Let $\beta : U \to W$ be the blow up along the curve Z and G be the exceptional divisor of β. Then the exceptional divisor E of π is the proper transform of the divisor G on the threefold V. Hence we have

$$\mathrm{mult}_Z(\mathcal{M}_W) = 2n - \mathrm{mult}_L(\mathcal{M}),$$

which implies that $\mathrm{mult}_Z(\mathcal{M}_W|_F) \geq 2n - \mathrm{mult}_L(\mathcal{M})$. Therefore we have

$$n + \mathrm{mult}_L(\mathcal{M}) \geq 2n - \mathrm{mult}_L(\mathcal{M}),$$

which gives $\mathrm{mult}_L(\mathcal{M}) \geq \frac{n}{2}$. □

Let T be a hyperplane section of X that is singular at the point P. Then T has only isolated singularities. Moreover we have $\mathrm{mult}_P(T \cdot M_i) = 4n$, which implies that the point P is an isolated double point of the surface T. Put $L_i = \pi(\bar{L}_i)$.

Lemma 6.3.
The point P is not an ordinary double point of the surface T.

Proof. Suppose that P is an ordinary double point of T. Let us show that this assumption leads us to a contradiction. Let H_i be a general hyperplane section of the quartic X that passes through the line L_i. Then

$$H_i \cdot T = L_i + Z_i,$$

where Z_i is a cubic curve. The cubic curve Z_i intersects the line L_i at the point P and at some smooth point of the surface T, because L_i does not contain three singular points of the surface T. Hence we have

$$L_i^2 = H_i \cdot L_i - Z_i \cdot L_i < -\frac{1}{2}.$$

The proper transform T_V has isolated singularities and is normal. Moreover the inequality $L_i^2 < -\frac{1}{2}$ implies that $\bar{L}_i^2 < -1$. Let M be a general surface in $\mathcal{M}$. The support of the cycle $T \cdot M$ consists of the union of all lines on X passing through the point P because $\mathrm{mult}_P(T \cdot M) = 4n$. Thus the equalities $\mathrm{mult}_P(T) = 2n$ and $\mathrm{mult}_P(M) = 2n$ imply that the support of the cycle $T_V \cdot M_V$ consists of the union of the curves $\bar{L}_1, \cdots, \bar{L}_k$. Hence we have

$$M_V\Big|_{T_V} = \sum_{i=1}^{k} m_i \bar{L}_i,$$

but $M_V \cdot \bar{L}_l = -n$ and $\bar{L}_i \cdot \bar{L}_j = 0$ for $i \neq j$. Therefore we see

$$-n = M_V \cdot \bar{L}_j = \sum_{i=1}^{k} m_i \bar{L}_i \cdot \bar{L}_j = m_j \bar{L}_j^2.$$

This implies that $m_j < n$.
Let H be the proper transform of a general hyperplane section of X on the threefold V. Then

$$4n = M_V \cdot T_V \cdot H = \sum_{i=1}^{k} m_i \bar{L}_i \cdot H = \sum_{i=1}^{k} m_i < kn,$$

which implies that $k > 4$. Thus the threefold X has at least five lines that pass through the point P, which is a contradiction. □

Thus the point P is not an ordinary double point on the surface T. Therefore there is a hyperplane section Z of the quartic surface T with $\mathrm{mult}_P(Z) \geq 3$. Hence the curve Z is reducible by Lemma 6.1. Moreover the curve Z is reduced and $\mathrm{mult}_P(Z) = 3$ by our generality assumption on X.

Lemma 6.4.

The curve Z is not a union of four lines.

Proof. Suppose that the curve Z is a union of four lines. Then one component of Z is a line L that does not pass through the point P. Also L intersects M_i in at least three points that are contained in the union of the lines $L_1, \cdots, L_k$. On the other hand, we have $M_i \cdot L = n$. This implies that L is contained in M_i by Lemma 6.2, which is impossible because the base locus of $\mathcal{M}$ is the union of the lines $L_1, \cdots, L_k$. □

The curve Z is not a union of an irreducible cubic curve and a line because of Lemma 6.1. Hence the curve Z is a union of two different lines passing through the point P and a conic that also passes through the point P, which is impossible by Lemma 6.1.
Therefore the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains a curve Z. So we have $\mathrm{mult}_Z(\mathcal{M}) = n$. It follows from Lemma 2.1 that $\deg(Z) \leq 4$.

Lemma 6.5.
The curve Z is contained in a two-dimensional linear subspace of $\mathbb{P}^4$.

Proof. Suppose that the curve Z is not contained in any plane in $\mathbb{P}^4$. Then the degree of the curve Z is either 3 or 4. If the degree is 3, then the curve is smooth. If the degree is 4, then the curve can be singular but the singularities consist of only one double point.
Suppose that Z is smooth. Let $\alpha : U \to X$ be the blow up along the curve Z and F be its exceptional divisor. Then the base locus of the linear system $|\alpha^*(-\deg(Z)K_X) - F|$ does not contain any curve but

$$\Big(\alpha^*(-\deg(Z)K_X) - F\Big) \cdot D_1 \cdot D_2 < 0,$$

where D_1 and D_2 are general surfaces of the linear system $\mathcal{M}_U$, which is a contradiction.
Suppose that the curve Z is a quartic curve with a double point P. Let $\beta : W \to X$ be the composition of the blow up at the point P with the blow up along the proper transform of the curve Z. Let G and E be the exceptional divisors of β such that $\beta(E) = Z$ and $\beta(G) = P$. Then the base locus of the linear system $|\beta^*(-4K_X) - E - 2G|$ does not contain any curve but

$$\Big(\beta^*(-4K_X) - E - 2G\Big) \cdot D_1 \cdot D_2 < 0$$

where D_1 and D_2 are general surfaces of the linear system $\mathcal{M}_W$, which is a contradiction. ☐

Lemma 6.6.
If the curve Z is a line, then the pencil $\mathcal{M}$ is contained in $|-K_X|$.

Proof. Let $\pi : V \to X$ be the blow up along the line Z. Then the linear system $|-K_V|$ is base-point-free and induces an elliptic fibration $\eta : V \to \mathbb{P}^2$. Therefore $\mathcal{M}_V$ is contained in fibers of η. In particular, the base locus of the pencil $\mathcal{M}_V$ does not contain curves not contracted by the morphism η.
The set $\mathbb{CS}(V, \frac{1}{n}\mathcal{M}_V)$ is not empty by Theorem 2.1. However, it does not contain any point because we assume that the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ does not contain points. Hence there is an irreducible curve $L \subset V$ such that $\mathrm{mult}_L(\mathcal{M}_V) = n$ and $\eta(L)$ is a point.
The pencil $\mathcal{M}_V$ is the pull-back via the morphism η of a pencil $\mathcal{P}$ on $\mathbb{P}^2$ with $\mathcal{P} \sim_{\mathbb{Q}} \mathcal{O}_{\mathbb{P}^2}(n)$. Hence the equality $\mathrm{mult}_L(\mathcal{M}_V) = n$ implies that the multiplicity of the pencil $\mathcal{P}$ at the point $\eta(L)$ is n, which implies that $n = 1$. ☐

Thus we may assume that the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ does not contain lines. Moreover the pencil $\mathcal{M}$ is contained in $|-K_X|$ if Z is a plane quartic curve by Theorem 2.2. Therefore we may assume further that Z is either a plane cubic curve or a conic.

Lemma 6.7.
If the curve Z is a cubic, then $\mathcal{M}$ is a pencil in $|-K_X|$.

Proof. Let $\mathcal{P}$ be the pencil in $|-K_X|$ that contains all surfaces passing through the cubic curve Z, and let D be a general surface in $\mathcal{P}$. Then D is a smooth K3 surface but the base locus of the pencil $\mathcal{P}$ consists of the curve Z and some line $L \subset X$. We have

$$\mathcal{M}\Big|_D = nZ + \mathrm{mult}_L(\mathcal{M})L + \mathcal{B} \equiv nZ + nL,$$

where $\mathcal{B}$ is a pencil on D without fixed components. On the other hand, we have $L^2 = -2$ on the surface D. This implies that $\mathrm{mult}_L(\mathcal{M}) = n$ and $\mathcal{B} = \varnothing$. Hence we have $\mathcal{M} = \mathcal{P}$ by Theorem 2.2. ☐

Therefore we may assume that the curve Z is a conic. Let Π be the plane in $\mathbb{P}^4$ that contains the conic Z.

Lemma 6.8.
If $\Pi \cap X = Z$, then $\mathcal{M}$ is a pencil in $|-K_X|$.

Proof. Let $\alpha : U \to X$ be the blow up along the curve Z and D be a general surface of the pencil $|-K_U|$. Then D is a smooth K3 surface but the base locus of the pencil $|-K_U|$ consists of the irreducible curve L such that $\alpha(L) = Z$ and $-K_U|_D \equiv L$. Therefore we have $\mathcal{M}_U\big|_D \equiv nL$, but $L^2 = -2$ on the surface D. Hence we have $\mathcal{M}_U = |-K_U|$ by Theorem 2.2. □

In the case when the set-theoretic intersection $\Pi \cap X$ contains a curve different from a conic Z, the arguments of the proof of Lemma 6.7 easily imply that $\mathcal{M}$ is a pencil in $|-K_X|$.
Thus we have completed the proof of Proposition 6.1.

$\gimel = 2$: **Hypersurface of degree** 5 **in** $\mathbb{P}(1,1,1,1,2)$.

The threefold X is a general hypersurface of degree 5 in $\mathbb{P}(1,1,1,1,2)$ with $-K_X^3 = \frac{5}{2}$. It has only one singular point O at $(0:0:0:0:1)$ which is a quotient singularity of type $\frac{1}{2}(1,1,1)$. The hypersurface X can be given by the equation

$$w^2 f_1\big(x,y,z,t\big) + w f_3\big(x,y,z,t\big) + f_5\big(x,y,z,t\big) = 0,$$

where f_i is a homogeneous polynomial of degree i.
There is a commutative diagram

$$\begin{tikzcd}
& Y \arrow[dl, "\pi"'] \arrow[dr, "\gamma"] & & W_i \arrow[ll, "\alpha_i"'] \arrow[dr, "w_i"] & \\
X \arrow[dr, dashed, "\psi"'] & & Y' \arrow[dl, "\omega"] & & U_i \arrow[ll, "\beta_i"'] \arrow[dl, "\eta_i"] \\
& \mathbb{P}^3 \arrow[rr, dashed, "\chi_i"'] & & \mathbb{P}^2 &
\end{tikzcd}$$

where

- ψ is the natural projection,
- π is the Kawamata blow up at the point O with weights $(1,1,1)$,
- γ is the birational morphism that contracts 15 smooth rational curves $L_1, \cdots, L_{15}$ to 15 isolated ordinary double points $P_1, \cdots, P_{15}$ of the variety Y', respectively,
- α_i is the blow up along the curve L_i,
- β_i is the blow up at the point P_i,
- w_i is a birational morphism,
- ω is a double cover of $\mathbb{P}^3$ branched over a sextic surface $R \subset \mathbb{P}^3$,
- χ_i is the projection from the point $\omega(P_i)$,
- η_i is an elliptic fibration.

The surface R is given by the equation

$$f_3\big(x,y,z,t\big)^2 - 4f_1\big(x,y,z,t\big)f_5\big(x,y,z,t\big) = 0 \subset \mathbb{P}^3 \cong \mathrm{Proj}\Big(\mathbb{C}[x,y,z,t]\Big).$$

It has 15 ordinary double points $\omega(P_1), \cdots, \omega(P_{15})$ that are given by the equations

$$f_3\big(x,y,z,t\big) = f_1\big(x,y,z,t\big) = f_5\big(x,y,z,t\big) = 0 \subset \mathbb{P}^3.$$

We may assume that the curves in $\mathbb{P}^3$ defined by $f_3 = f_1 = 0$ and $f_5 = f_1 = 0$ are irreducible.
For convenience, let M and M' be general surfaces in the pencil $\mathcal{M}$.

Lemma 6.9.

The set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ does not contain any smooth point of X.

Proof. Suppose that the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains a smooth point P of X. Let D be a general surface of the linear system $|-K_X|$ that passes through the point P. The surface D does not contain an irreducible component of the cycle $M \cdot M'$ if none of $\pi(L_i)$ passes through the point P. In particular, in such a case, we see

$$\mathrm{mult}_P\Big(M \cdot M'\Big) \leq M \cdot M' \cdot D = -n^2K_X^3 = \frac{5}{2}n^2.$$

This is impossible by Theorem 3.1 in [4]. Thus we may assume that the curve $\pi(L_1)$ passes through the point P.
Let us use the arguments of the article [5]. Put $L = \pi(L_1)$ and

$$\mathcal{M}\Big|_D = \mathcal{L} + \mathrm{mult}_L\big(\mathcal{M}\big)L,$$

where $\mathcal{L}$ is a pencil on the surface D without fixed curves. Then the point P is a center of log canonical singularities of the log pair $(D, \frac{1}{n}\mathcal{M}|_D)$ by the Shokurov connectedness principle ([4]). This implies that

$$\mathrm{mult}_P\Big(\Lambda_1 \cdot \Lambda_2\Big) \geq 4n\Big(n - \mathrm{mult}_L\big(\mathcal{M}\big)\Big)$$

by Theorem 3.1 in [4], where Λ_1 and Λ_2 are general curves in $\mathcal{L}$. The equality

$$\Lambda_1 \cdot \Lambda_2 = \frac{5}{2}n^2 - \mathrm{mult}_L\big(\mathcal{M}\big)n - \frac{3}{2}\mathrm{mult}_L^2\big(\mathcal{M}\big)$$

holds on the surface D because $L^2 = -\frac{3}{2}$ on the surface D. Hence we have

$$\frac{5}{2}n^2 - \mathrm{mult}_L\big(\mathcal{M}\big)n - \frac{3}{2}\mathrm{mult}_L^2\big(\mathcal{M}\big) \geq 4n\Big(n - \mathrm{mult}_L\big(\mathcal{M}\big)\Big),$$

which gives $\mathrm{mult}_L(\mathcal{M}) = n$. Thus the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains the curve $\pi(L_1)$.
The set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains the point O by Lemma 2.2. So $\mathcal{M}_{W_1} \sim_{\mathbb{Q}} -nK_{W_1}$ because $\mathrm{mult}_L(\mathcal{M}) = n$, which implies that each surface of $\mathcal{M}_{W_1}$ is contracted to a curve by the elliptic fibration $\eta_1 \circ w_1$. On the other hand, the set $\mathbb{CS}(W_1, \frac{1}{n}\mathcal{M}_{W_1})$ contains a subvariety of the threefold W_1 that dominates the point P.
Let E_1 be the exceptional divisor of α_1. Then $E_1 \cong \mathbb{P}^1 \times \mathbb{P}^1$ and the pencil $\mathcal{M}_{W_1}|_{E_1}$ does not have fixed components because E_1 is a section of the elliptic fibration $\eta_1 \circ \omega_1$ and the base locus of the pencil $\mathcal{M}_{W_1}$ can only contain curves contracted by the elliptic fibration $\eta_1 \circ w_1$. Thus the set $\mathbb{CS}(W_1, \frac{1}{n}\mathcal{M}_{W_1})$ contains a point Q of the surface E_1 such that $\pi \circ \alpha_1(Q) = P$.
The point Q is a center of log canonical singularities of the log pair $(E_1, \frac{1}{n}\mathcal{M}_{W_1}|_{E_1})$ by the Shokurov connectedness principle ([4]). Let Δ_1 and Δ_2 be general curves in $\mathcal{M}_{W_1}|_{E_1}$. Then the inequality

$$2n^2 = \mathrm{mult}_Q\Big(\Delta_1 \cdot \Delta_2\Big) \geq 4n^2$$

holds by Theorem 3.1 in [4], which is a contradiction. □

Lemma 6.10.

If the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains a curve Λ not passing through the singular point O, then the pencil $\mathcal{M}$ is contained in $|-K_X|$.

Proof. We have $\mathrm{mult}_{\Lambda}(\mathcal{M}) = n$ and $-K_X \cdot \Lambda \leq 2$ by Corollary 4.1.
Suppose that $-K_X \cdot \Lambda = 2$ and $\psi(\Lambda)$ is a line. Then the line $\psi(\Lambda)$ passes through a unique singular point of R. Hence we may assume that the curve Λ intersects $\pi(L_i)$ only for $i = 1$.
Let $\mathcal{D}$ be the pencil in the linear system $|-K_X|$ consisting of surfaces that pass through the curve Λ and D be a general surface of the pencil $\mathcal{D}$. Then the surface D is smooth in the outside of the singular point O, the point O is an ordinary double point of the surface D and the base locus of the pencil $\mathcal{D}$ consists of the curve Λ and the curve $\pi(L_1)$. Put $L = \pi(L_1)$. Then

$$\mathcal{M}\Big|_D = \mathrm{mult}_{\Lambda}\big(\mathcal{M}\big)\Lambda + \mathrm{mult}_L\big(\mathcal{M}\big)L + \mathcal{L} \equiv n\big(\Lambda + L\big),$$

where $\mathcal{L}$ is a pencil with no fixed curves. It gives $\mathcal{M} = \mathcal{D}$ by Theorem 2.2 since the inequality $L^2 < 0$ holds on the surface D.
We may assume that either the equality $-K_X \cdot \Lambda = 1$ holds or $\psi(\Lambda)$ is a conic, both of which imply that Λ is smooth. Let $\sigma : \check{X} \to X$ be the blow up along the curve Λ and G be its exceptional divisor.
Suppose that $-K_X \cdot \Lambda = 2$. Then Λ is cut out as a set by the surfaces of the linear system $|-2K_X|$ that pass through the curve Λ. Moreover the scheme-theoretic intersection of two general surfaces of the linear system $|-2K_X|$ passing through the curve Λ is reduced at a generic point of the curve Λ. This implies that the divisor $\sigma^*(-2K_X) - G$ is nef by Lemma 5.2.5 in [5]. However, we obtain an absurd inequality

$$-3n^2 = \Big(\sigma^*(-2K_X) - G\Big) \cdot M_{\check{X}} \cdot M'_{\check{X}} \geq 0.$$

Therefore the equality $-K_X \cdot \Lambda = 1$ holds, which implies that $|-K_{\check{X}}|$ is a pencil.
Suppose that $\psi(\Lambda)$ is not contained in the plane $f_1(x, y, z, t) = 0$. Then $\psi(\Lambda)$ contains a unique singular point of the surface $R \subset \mathbb{P}^3$. Hence we may assume that the curve Λ intersects $\pi(L_i)$ only for $i = 1$. This implies that the base locus of the linear system $|-K_{\check{X}}|$ consists of irreducible curves $\check{\Lambda}$ and $\check{L}_1$ such that $(\psi \circ \sigma)(\check{\Lambda}) = \psi(\Lambda)$ and $\sigma(\check{L}_1) = \pi(L_1)$. Let $\check{D}$ be a general surface in $|-K_{\check{X}}|$. Then we can consider the curves $\check{\Lambda}$ and $\check{L}_1$ as divisors on $\check{D}$. We have

$$\check{\Lambda}^2 = -2,\ \check{L}_1^2 = -\frac{3}{2},\ \check{\Lambda} \cdot \check{L}_1 = 1,$$

which implies that the intersection form of $\check{\Lambda}$ and $\check{L}_1$ is negative-definite. Since

$$\mathcal{M}_{\check{X}}\Big|_{\check{D}} \equiv -nK_{\check{X}}\Big|_{\check{D}} \equiv n\big(\check{\Lambda} + \check{L}_1\big),$$

it follows from Theorem 2.2 that $\mathcal{M}_{\check{X}} = |-K_{\check{X}}|$.
Finally, we suppose that the line $\psi(\Lambda)$ is contained in the plane $f_1(x, y, z, t) = 0$. In particular, the line $\psi(\Lambda)$ is not contained in the surface R because the curve $f_3 = f_1 = 0$ is irreducible. Moreover the line $\psi(\Lambda)$ contains exactly three singular points of the ramification surface[2]; otherwise the point O would belong to the curve Λ. Thus the curve Λ intersects exactly three curves among the curves $L_1, \cdots, L_{15}$; otherwise Λ would contain the point O.
We may assume that Λ intersects the curves $\pi(L_1)$, $\pi(L_2)$ and $\pi(L_3)$. This means that the points $\omega(P_1)$, $\omega(P_2)$, $\omega(P_3)$ are contained in $\psi(\Lambda)$. The base locus of $|-K_{\check{X}}|$ consists of the curves $\check{L}_1$, $\check{L}_2$, $\check{L}_3$ such that $\sigma(\check{L}_i) = \pi(L_i)$. The curves $\check{L}_1$, $\check{L}_2$, $\check{L}_3$ can be contracted on the surface $\check{D}$ to a singular point of type $\mathbb{D}_4$, which implies that their intersection form is negative-definite. Hence we have $\mathcal{M}_{\check{X}} = |-K_{\check{X}}|$ by Theorem 2.2 . $\square$

The equivalence $\mathcal{M}_Y \sim_{\mathbb{Q}} -nK_Y$ holds by Lemma 2.2. This implies that the set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ contains no point of Y due to Lemmas 2.3 and 6.9. Let $\mathcal{M}_{Y'}$ be the push-forward of the pencil $\mathcal{M}_Y$ by the birational morphism γ. Then $\mathcal{M}_{Y'} \sim_{\mathbb{Q}} -nK_{Y'}$, the log pair $(Y', \frac{1}{n}\mathcal{M}_{Y'})$ has canonical singularities but it follows from Theorem 2.1 that the singularities of the log pair $(Y', \frac{1}{n}\mathcal{M}_{Y'})$ are not terminal.

[2] *In fact, we may assume that no three points of the set* $\mathrm{Sing}(R)$ *are collinear.*

Lemma 6.11.
If the set $\mathbb{CS}(Y', \frac{1}{n}\mathcal{M}_{Y'})$ contains an irreducible curve Γ with $-K_{Y'} \cdot \Gamma \neq 1$, then the pencil $\mathcal{M}$ is contained in $|-K_X|$.

Proof. Let D be a general divisor in $|-K_{Y'}|$. In addition, let $M_{Y'} = \gamma(M_Y)$ and $M'_{Y'} = \gamma(M'_Y)$. Then

$$2n^2 = D \cdot M_{Y'} \cdot M'_{Y'} \geq \mathrm{mult}_\Gamma\Big(M_{Y'} \cdot M'_{Y'}\Big) D \cdot \Gamma \geq -n^2 K_{Y'} \cdot \Gamma,$$

because $\mathrm{mult}_\Gamma(\mathcal{M}_{Y'}) = n$. Therefore the inequality $-K_{Y'} \cdot \Gamma \leq 2$ holds.
Suppose that $-K_{Y'} \cdot \Gamma = 2$ but the curve $\omega(\Gamma)$ is a line. Let $\mathcal{T}$ be the linear subsystem of the linear system $|-K_{Y'}|$ consisting of surfaces passing through the curve Γ and T be a general surface in the pencil $\mathcal{T}$. Then the base locus of the pencil $\mathcal{T}$ consists of the curve Γ and the rational map induced by the pencil $\mathcal{T}$ is the composition of the double cover ω with the projection from the line $\omega(\Gamma)$. On the other hand, we have

$$2n = D \cdot T \cdot M_{Y'} \geq \mathrm{mult}_\Gamma\Big(T \cdot M_{Y'}\Big) D \cdot \Gamma \geq -nK_{Y'} \cdot \Gamma.$$

This implies that the support of the cycle $T \cdot M_{Y'}$ is contained in Γ. Thus we have $\mathcal{M}_{Y'} = \mathcal{T}$ by Theorem 2.2.
For now, we suppose that $-K_{Y'} \cdot \Gamma = 2$ but the curve $\omega(\Gamma)$ is a conic. Then Γ is smooth and $\omega|_\Gamma$ is an isomorphism. Moreover the curve Γ contains at most 2 singular points of the threefold Y' if the curve $\omega(\Gamma)$ is not contained in the plane $f_1(x, y, z, t) = 0$ and the curve Γ contains at most 6 singular points of the threefold Y' otherwise. We may assume that Γ passes through $P_1, \cdots, P_k$, where $0 \leq k \leq 6$. The equality $k = 0$ means that Γ lies in the smooth locus of the threefold Y'.
Let $\beta : V \to Y'$ be the blow up at the points $P_1, \cdots, P_k$ and E_i be the exceptional divisor of the blow up β with $\beta(E_i) = P_i$. The exceptional divisor E_i is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. The proper transform Γ_V intersects the surface E_i transversally at a single point, which we denote by Q_i.
Let $\upsilon : W \to V$ be the blow up along the curve Γ_V and G be the exceptional divisor of the birational morphism υ. In addition, let A_i and B_i be the fibers of the natural projections of the surface E_i that pass through the point Q_i, and $\bar{A}_i$ and $\bar{B}_i$ be the proper transforms of the curves A_i and B_i on the threefold W, respectively. Then we can flop the curves $\bar{A}_i$ and $\bar{B}_i$.
Let $\upsilon_1 : U \to W$ be the blow up along the curves $\bar{A}_1, \bar{B}_1, \cdots, \bar{A}_k, \bar{B}_k$. Also let F_i and H_i be the exceptional divisors of υ_1 such that $\upsilon_1(F_i) = \bar{A}_i$ and $\upsilon_1(H_i) = \bar{B}_i$. Then all the exceptional divisors are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. There is a birational morphism $\upsilon'_1 : U \to W'$ such that $\upsilon'_1(F_i)$ and $\upsilon'_1(H_i)$ are rational curves but $\upsilon'_1 \circ \upsilon_1^{-1}$ is not biregular in a neighborhood of $\bar{A}_i$ and $\bar{B}_i$. Let E'_i be the proper transform of E_i on the threefold W'. Then we can contract the surface E'_i to a singular point of type $\frac{1}{2}(1, 1, 1)$.
Let $\upsilon' : W' \to V'$ be the contraction of $E'_1, \cdots, E'_k$ and G' be the proper transform of the surface G on the threefold V'. Then there is a birational morphism $\beta' : V' \to Y'$ that contracts the divisor G' to the curve Γ. Hence we constructed the commutative diagram

$$\begin{tikzcd}
& W \arrow[dl, "\upsilon"'] & & U \arrow[ll, "\upsilon_1"'] \arrow[dr, "\upsilon'_1"] & \\
V \arrow[dr, "\beta"'] & & & & W' \arrow[dl, "\upsilon'"] \\
& Y' & & V'. \arrow[ll, "\beta'"'] &
\end{tikzcd}$$

The threefold V' is projective. Its singularities consist of $15 - k$ ordinary double points and k singular points of type $\frac{1}{2}(1, 1, 1)$. However, it is not $\mathbb{Q}$-factorial because the threefold Y' is not $\mathbb{Q}$-factorial.
The construction of the birational morphism β' implies that

$$\mathcal{M}_{V'} \sim_{\mathbb{Q}} -n\beta'^*(K_{Y'}) - nG' \sim_{\mathbb{Q}} -nK_{V'}.$$

Let D' be a general surface of the linear system $|\beta'^*(-4K_{Y'}) - G'|$. Then the divisor D' is nef by Lemma 5.2.5 in [5]. The construction of the birational morphism β' implies that

$$0 > \Big(-4 + \frac{k}{2}\Big)n^2 = \Big(\beta'^*(-4K_{Y'}) - G'\Big) \cdot \Big(\beta'^*(-nK_{Y'}) - nG'\Big)^2 = D' \cdot M_{V'} \cdot M'_{V'} \geq 0,$$

where $M_{V'}$ and $M'_{V'}$ are the proper transforms of $M_{Y'}$ and $M'_{Y'}$ by the birational morphism β'. We have obtained a contradiction. □

Lemma 6.12.

If the set $\mathbb{CS}(Y', \frac{1}{n}\mathcal{M}_{Y'})$ contains a curve Γ with $-K_Z \cdot \Gamma = 1$, then the pencil $\mathcal{M}$ is contained in $|-K_X|$.

Proof. The curve $\omega(\Gamma)$ is a line in $\mathbb{P}^3$. The restricted morphism $\omega|_\Gamma : \Gamma \to \omega(\Gamma)$ is an isomorphism. The curve Γ contains at most one singular point of Y' if $\omega(\Gamma)$ is not contained in the plane $f_1(x, y, z, t) = 0$ and the curve $\omega(\Gamma)$ contains at most three singular points of the threefold Y' otherwise. We may assume that Γ contains $P_1, \cdots, P_k$, where $0 \leq k \leq 3$. Here, the equality $k = 0$ means that Γ lies in the smooth locus of the threefold Y'.
Suppose that the line $\omega(\Gamma)$ is not contained in R. Let D be a general surface in $|-K_{Y'}|$ that passes through the curve Γ. Then

$$\mathcal{M}_{Y'}\Big|_D = \mathrm{mult}_\Gamma\big(\mathcal{M}_{Y'}\big)\Gamma + \mathrm{mult}_\Omega\big(\mathcal{M}_{Y'}\big)\Omega + \mathcal{L},$$

where $\mathcal{L}$ is a pencil without fixed curves and Ω is a smooth rational curve different from Γ such that $\omega(\Omega) = \omega(\Gamma)$. Moreover the surface D is smooth in the outside the points $P_1, \cdots, P_k$ but the points $P_1, \cdots, P_k$ are isolated ordinary double points of the surface D. We have $\Omega^2 = -2 + \frac{k}{2}$ on the surface D and

$$\Big(n - \mathrm{mult}_\Omega(\mathcal{M}_{Y'})\Big)\Omega^2 = \Big(\mathrm{mult}_\Gamma(\mathcal{M}_{Y'}) - n\Big)\Gamma \cdot \Omega + L \cdot \Omega = L \cdot \Omega \geq 0,$$

where L is a general curve in $\mathcal{L}$. Therefore the equality $\mathrm{mult}_\Omega(\mathcal{M}_{Y'}) = n$ holds. This easily implies that $\mathcal{M}_{Y'}$ is a pencil in $|-K_{Y'}|$ because of Theorem 2.2.
Finally, we suppose that $\omega(\Gamma)$ is contained in the ramification surface of ω. It implies that $\omega(\Gamma)$ is not contained in the plane $f_1(x, y, z, y) = 0$. The proof of Lemma 6.11 shows the existence of a birational morphism $\upsilon' : W' \to V'$ that contracts a single irreducible divisor G' to the curve Γ, the surface G' contains k singular points of the threefold V' of type $\frac{1}{2}(1, 1, 1)$ and υ' is the blow up of Γ at a generic point of Γ.
Let D' be a general surface in $|-K_{V'}|$. Then $\omega \circ \beta'(D')$ is a plane that passes through $\omega(\Gamma)$. It implies that the base locus of the pencil $|-K_{V'}|$ consists of an irreducible curve Γ' such that $\beta'(\Gamma') = \Gamma$ and

$$D' \cdot \Gamma' = -K_{V'}^3 = -2 + \frac{k}{2}.$$

Then one can easily see that $\mathcal{M}_{V'} = |-K_{V'}|$ by Theorem 2.2. Hence the linear system $\mathcal{M}$ is a pencil in $|-K_X|$. □

Proposition 6.2.

Every Halphen pencil is contained in $|-K_X|$.

Proof. Let $\mathcal{M}_{U_i}$ be the push-forward of the pencil $\mathcal{M}_{W_i}$ by the morphism w_i. Due to the previous arguments, we may assume that

$$P_1 \in \mathbb{CS}\Big(Y', \frac{1}{n}\mathcal{M}_{Y'}\Big) \subseteq \Big\{P_1, \cdots, P_{15}\Big\},$$

which implies that $\mathcal{M}_{U_1} \sim_{\mathbb{Q}} -nK_{U_1}$ by Theorem 3.10 in [4]. Therefore each member in the pencil $\mathcal{M}_{U_1}$ is contracted to a curve by the elliptic fibration η_1. Therefore the base locus of the pencil $\mathcal{M}_{U_1}$ does not contain curves that are not contracted by η_1. On the other hand, the singularities of the log pair $(U_1, \frac{1}{n}\mathcal{M}_{U_1})$ are not terminal by Theorem 2.1.

The proof of Lemma 6.9 implies that the set $\mathbb{CS}(U_1, \frac{1}{n}\mathcal{M}_{U_1})$ does not contain a smooth point of the exceptional divisor of β_1. Therefore the set $\mathbb{CS}(U_1, \frac{1}{n}\mathcal{M}_{U_1})$ contains a singular point of the threefold U_1, which implies that

$$\Big\{P_1, P_i\Big\} \subseteq \mathbb{CS}\Big(Y', \frac{1}{n}\mathcal{M}_{Y'}\Big) \subseteq \Big\{P_1, \cdots, P_{15}\Big\},$$

for some $i \neq 1$. Thus each member in the pencil $\mathcal{M}_{U_i}$ is contracted to a curve by the elliptic fibration η_i, which implies that $\mathcal{M}$ is a pencil in $|-K_X|$. $\square$

$\gimel = 4$: **Hypersurface of degree** 6 **in** $\mathbb{P}(1,1,1,2,2)$.

The weighted hypersurface X is defined by a general quasihomogeneous polynomial of degree 6 in $\mathbb{P}(1,1,1,2,2)$ with $-K_X^3 = \frac{3}{2}$. The singularities of the hypersurface X consist of points P_1, P_2, P_3 that are quotient singularities of types $\frac{1}{2}(1,1,1)$. The hypersurface X can be given by the equation

$$w^2t + \Big(t^2 + tf_2(x,y,z) + f_4(x,y,z)\Big)w + f_6(x,y,z,t) = 0$$

such that P_1 is given by the equations $x = y = z = t = 0$, where f_i is a general quasihomogeneous polynomial of degree i.

There is a commutative diagram

$$\begin{tikzcd}
 & U \arrow[ld, "\beta"'] \arrow[r, "\alpha"] & X \arrow[ddl, dashed, "\chi"'] \arrow[ddr, dashed, "\psi"] & Y \arrow[l, "\pi"'] \arrow[dd, "\eta"] \\
W \arrow[dr, "\omega"'] & & & \\
 & \mathbb{P}(1,1,1,2) \arrow[rr, dashed, "\xi"'] & & \mathbb{P}^2,
\end{tikzcd}$$

where

- ψ is the natural projection,
- π is the composition of the Kawamata blow ups at the points P_1, P_2 and P_3,
- η is an elliptic fibration
- α is the Kawamata blow up of the point P_1,
- ξ and χ are the natural projections,
- β is a birational morphism,
- ω is a double cover ramified along an octic surface $R \subset \mathbb{P}(1,1,1,2)$.

The surface R is given by the equation

$$\Big(t^2 + tf_2(x,y,z) + f_4(x,y,z)\Big)^2 - 4tf_6(x,y,z,t) = 0 \subset \mathbb{P}(1,1,1,2) \cong \mathrm{Proj}\Big(\mathbb{C}[x,y,z,t]\Big),$$

which implies that the surface R has exactly 24 isolated ordinary double points given by the equations

$$t = t^2 + tf_2(x,y,z) + f_4(x,y,z) = f_6(x,y,z,t) = 0.$$

The birational morphism β contracts 24 smooth rational curves $C_1, \cdots, C_{24}$ to isolated ordinary double points of the variety W that dominate the singular points of R.

It easily follows from Theorems 2.1, 4.2, Lemmas 2.1, 2.3 and 4.1 that either the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains an irreducible curve passing through a singular point of X or the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ consists of a single singular point of X. In particular, we may assume that the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains the point P_1.

Proposition 6.3.
Every Halphen pencil on X is contained in $|-K_X|$.

Proof. Suppose that the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains an irreducible curve Z that passes through P_1. Then it follows from Theorem 4.2 that the linear system $\mathcal{M}$ is a pencil in $|-K_X|$ in the case when $-K_X \cdot Z = \frac{3}{2}$. Therefore we may assume that the curve Z is contracted by the rational map ψ to a point. Also we may assume that either $-K_X \cdot Z = \frac{1}{2}$ or $-K_X \cdot Z = 1$.
Let $\mathcal{B}$ be the pencil in $|-K_X|$ consisting of surfaces passing through Z. In addition, let B and B' be general surfaces in $\mathcal{B}$. Then the cycle $B \cdot B'$ is reduced and contains the curve Z. Put $\tilde{Z} = B \cdot B'$ and let $\tilde{Z}_W$ be the image of the curve $\tilde{Z}_U$ by the birational morphism β. Then $\omega(\tilde{Z}_W)$ is a ruling of the cone $\mathbb{P}(1,1,1,2)$. In particular, the curve $\omega(\tilde{Z}_W)$ contains at most one singular point of the surface R.
There are exactly 24 rulings of the cone $\mathbb{P}(1,1,1,2)$ that pass through the singular points of the surface R. Thus we may assume that the curve $\tilde{Z}_W$ is irreducible in the case when the curve $\omega(\tilde{Z}_W)$ passes through a singular point of the surface R. Moreover the surface B_W that is the image of the surface B_U by β has an isolated ordinary double point at the point $\beta(C_i)$ in the case when $\omega \circ \beta(C_i) \in \omega(\tilde{Z}_W)$. Therefore the cycle $\tilde{Z}$ consists of two irreducible components.
Let $\bar{Z}$ be the irreducible component of $\tilde{Z}$ that is different from Z. Then the generality of the hypersurface X implies that $\bar{Z}^2 < 0$ on the surface B, but $M|_B \equiv nZ + n\bar{Z}$. On the other hand, we have

$$M\Big|_B = m_1 Z + m_2 \bar{Z} + F,$$

where and m_1 and m_2 are natural numbers and F is an effective divisor on B whose support contains neither the curve Z nor the curve $\bar{Z}$. We also have

$$m_1 \geq \mathrm{mult}_Z(\mathcal{M}) \geq n$$

and

$$(n - m_2)\bar{Z} \equiv F + (m_1 - n)Z.$$

Together they imply that $m_2 = m_1 = n$ and the support of the cycle $M \cdot B$ is contained in $Z \cup \bar{Z}$. Therefore the identity $\mathcal{M} = \mathcal{B}$ follows from Theorem 2.2.
For now, we suppose that the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ consists of the point P_1. It follows from Lemma 2.2 that $\mathcal{M}_U \sim_{\mathbb{Q}} -nK_U$. Therefore the set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ is not empty by Theorem 2.1. Let E be the exceptional divisor of α. Then $E \cong \mathbb{P}^2$ and the set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ contains a line L on the surface E by Lemma 2.3.
Let Z be the curve $S^t_U \cap E$. Then Z does not contain the curve L, the surface S^t_U contains every curve C_i and the curve Z is a smooth plane quartic curve. The hypersurface X is general by assumption. In particular, the surface S^t_U is smooth along the curve C_i, the morphism $\beta|_{S^t_U}$ contracts the curve C_i to a smooth point of the surface S^t_W which is the image of S^t_U by β. Moreover we may assume that the intersection $L \cap Z$ contains at least one point of the curve Z that is not contained in $\cup_{i=1}^{24} C_i$. Indeed, it is enough to assume that the set $\cup_{i=1}^{24}(C_i \cap Z)$ does not contain bi-tangent points of the plane quartic curve Z.
Let M' be a general surface in $\mathcal{M}$ and D be a general surface in $|-2K_U|$. Then

$$2n^2 = D \cdot M_U \cdot M'_U \geq 2\mathrm{mult}_L(M_U \cdot M'_U) \geq 2\mathrm{mult}_L(M_U)\mathrm{mult}_L(M'_U) \geq 2n^2,$$

which implies that the support of the cycle $M_U \cdot M'_U$ is contained in the union of the curve L and $\cup_{i=1}^{24} C_i$. Hence we have

$$\mathcal{M}_U\Big|_{S^t_U} = \mathcal{D} + \sum_{i=1}^{24} m_i C_i,$$

where m_i is a natural number and $\mathcal{D}$ is a pencil without fixed components. Let P be a point of $L \cap Z$ that is not contained in $\cup_{i=1}^{24} C_i$. For general curves D_1 and D_2 in $\mathcal{D}$,

$$n^2 - \sum_{i=1}^{24} m_i^2 = D_1 \cdot D_2 \geq \mathrm{mult}_P(D_1)\mathrm{mult}_P(D_2) \geq n^2,$$

which implies that $m_1 = m_2 = \cdots = m_{24} = 0$. Therefore we have $M_U \cdot M'_U = n^2 L$, which is impossible because the suppose of the cycle $M \cdot M'$ must contain a curve on X. $\square$

$\gimel = 16$: **Hypersurface of degree** 12 **in** $\mathbb{P}(1, 1, 2, 4, 5)$.

The threefold X is a general hypersurface of degree 12 in $\mathbb{P}(1, 1, 2, 4, 5)$ with $-K_X^3 = \frac{3}{10}$. Its singularities consist of three quotient singularities of type $\frac{1}{2}(1, 1, 1)$ and one point O that is a quotient singularity of type $\frac{1}{5}(1, 1, 4)$.
There is a commutative diagram

$$\begin{tikzcd}
U \arrow[d, "\alpha"'] & W \arrow[l, "\beta"'] & Y \arrow[l, "\gamma"'] \arrow[d, "\eta"] \\
X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1, 1, 2)
\end{tikzcd}$$

where

- ψ is the natural projection,
- α is the Kawamata blow up at the point O with weights $(1, 1, 4)$,
- β is the Kawamata blow up with weights $(1, 1, 3)$ at the singular point of the variety U that is contained in the exceptional divisor of α,
- γ is the Kawamata blow up with weights $(1, 1, 2)$ at the singular point of W that is contained in the exceptional divisor of β,
- η is an elliptic fibration.

The hypersurface X can be given by the equation

$$w^2 z + f_7(x, y, z, t)w + f_{12}(x, y, z, t) = 0,$$

where f_i is a quasihomogeneous polynomial of degree i. Moreover there is commutative diagram

$$\begin{tikzcd}
& U \arrow[dl, "\pi"'] \arrow[r, "\alpha"] & X \arrow[dl, dashed, "\chi"'] \arrow[dr, dashed, "\psi"] & \\
V \arrow[r, "\omega"'] & \mathbb{P}(1, 1, 2, 4) \arrow[rr, dashed, "\xi"'] & & \mathbb{P}(1, 1, 2),
\end{tikzcd}$$

where

- ξ and χ are the natural projections,
- π is a birational morphism,
- ω is a double cover of $\mathbb{P}(1, 1, 2, 4)$ ramified along a surface R of degree 12.

The surface R is given by the equation

$$f_7(x, y, z, t)^2 - 4zf_{12}(x, y, z, t) = 0 \subset \mathbb{P}(1, 1, 2, 4) \cong \mathrm{Proj}\big(\mathbb{C}[x, y, z, t]\big),$$

which implies that R has 21 isolated ordinary double points, given by the equations $z = f_7 = f_{12} = 0$. The morphism π contracts 21 smooth rational curves $C_1, C_2, \cdots, C_{21}$ to isolated ordinary double points of V which dominate the singular points of R.

Proposition 6.4.
The linear system $|-K_X|$ is a unique Halphen pencil on X.

First of all, Corollary 4.4 and Lemma 4.1 imply that the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ can contain only singular points of X. Furthermore, if it contains a singular point of type $\frac{1}{2}(1,1,1)$, then we see that the singular point satisfies all the conditions of Lemma 4.3 and hence $\mathcal{M} = |-K_X|$. Therefore to prove Proposition 6.4, we may assume that

$$\mathbb{CS}\Big(X, \frac{1}{n}\mathcal{M}\Big) = \Big\{O\Big\}.$$

Let E be the exceptional divisor of the birational morphism α. It contains one singular point P of U that is a quotient singularity of type $\frac{1}{4}(1,1,3)$. The surface E is isomorphic to $\mathbb{P}(1,1,4)$. The set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ contains the point P by Theorem 2.1 and Lemma 2.3. Furthermore, the following shows it consists of the point P.

Lemma 6.13.
The set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ cannot contain a curve.

Proof. Suppose that the set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ contains a curve Z. Then Z is contained in the surface E. Furthermore, it follows from Lemma 2.3 that Z is a curve in the linear system $|\mathcal{O}_{\mathbb{P}(1,1,4)}(1)|$. Therefore for a general surface M in $\mathcal{M}$, we have

$$\mathrm{Supp}\Big(M_U \cdot E\Big) = Z$$

because $M_U|_E \sim_{\mathbb{Q}} |\mathcal{O}_{\mathbb{P}(1,1,4)}(n)|$ and $\mathrm{mult}_Z(M_U) \geq n$.
Let M'_U be a general surface in $\mathcal{M}_U$ and D be a general surface in $|-4K_U|$. Then

$$n^2 = D \cdot M_U \cdot M'_U \geq \mathrm{mult}_Z(M_U \cdot M'_U) \geq \mathrm{mult}_Z(M_U)\mathrm{mult}_Z(M'_U) \geq n^2,$$

which implies that $\mathrm{mult}_L(M_U \cdot M'_U) = n^2$ and

$$\mathrm{Supp}\Big(M_U \cdot M'_U\Big) \subset Z \cup \bigcup_{i=1}^{21} C_i.$$

We consider the surface S^z. The image S^z_V of S^z_U to V is isomorphic to $\mathbb{P}(1,1,4)$. The surface S^z_U does not contain the curve Z due to the generality in the choice of X, but it contains every curve C_i. Moreover the surface S^z_U is smooth along the curves C_i and the morphism $\pi|_{S^z_U}$ contracts the curve C_i to a smooth point of S^z_V. Hence we have

$$\mathcal{M}_U\Big|_{S^z_U} = \mathcal{D} + \sum_{i=1}^{21} m_i C_i,$$

where m_i is a natural number and $\mathcal{D}$ is a pencil without fixed components. Therefore the inequality $m_i > 0$ implies that $C_i \cap Z \neq \varnothing$ and there is a point P' of the intersection $Z \cap S^z_U$ that is different from the singular point P. We may assume that $m_1 > 0$. Let D_1 and D_2 be general curves in $\mathcal{D}$. Then

$$\mathrm{mult}_{P'}(D_1) = \mathrm{mult}_{P'}(D_2) \geq \begin{cases} n & \text{in the case when } P' \notin \cup_{i=1}^{21} C_i \\ n - m_i & \text{in the case when } P' \in C_i \end{cases}$$

and the curves D_1 and D_2 pass through the point P because the point P is a base point of the pencil $\mathcal{M}_U$. Therefore we have

$$\frac{n^2}{4} - \sum_{i=1}^{21} m_i^2 = D_1 \cdot D_2 > \mathrm{mult}_P(D_1)\mathrm{mult}_P(D_2) \geq (n - m_1)^2 \geq \frac{n^2}{4} - m_1^2,$$

which is a contradiction. □

Let F be the exceptional divisor of the birational morphism β. It contains the singular point Q of W that is a quotient singularity of type $\frac{1}{3}(1,1,2)$. The set $\mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W)$ consists of the singular point Q by Theorem 2.1, Lemmas 2.3 and 2.1.

Let G be the exceptional divisor of γ and Q_1 be the unique singular point of G. The set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ must consist of the point Q_1 by Theorem 2.1 and Lemma 2.3 because every member in $\mathcal{M}_Y$ is contracted to a curve by the morphism η. Let $\sigma : V_1 \to Y$ be the Kawamata blow up at the point Q_1. Then $\mathcal{M}_{V_1} \sim_{\mathbb{Q}} -nK_{V_1}$ by Lemma 2.2, the linear system $|-K_{V_1}|$ is the proper transform of the pencil $|-K_X|$ and the base locus of the pencil $|-K_{V_1}|$ consist of the curve C_{V_1}. Therefore the inequality $-K_{V_1} \cdot C_{V_1} < 0$ implies $\mathcal{M} = |-K_X|$ by Theorem 2.2.

$\gimel = 18$: **Hypersurface of degree** 12 **in** $\mathbb{P}(1,2,2,3,5)$**.**

The threefold X is a general hypersurface of degree 12 in $\mathbb{P}(1,2,2,3,5)$ with $-K_X^3 = \frac{1}{5}$. The singularities of X consist of six points O_1, O_2, O_3, O_4, O_5 and O_6 that are quotient singularities of type $\frac{1}{2}(1,1,1)$ and one point P that is a quotient singularity of type $\frac{1}{5}(1,2,3)$.

There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & W \arrow[l, "\beta"'] \arrow[d, "\eta"] \\ X \arrow[r, dashed, "\psi"'] & \mathbb{P}(1,2,2), \end{tikzcd}$$

where

- ψ is the natural projection,
- α is the Kawamata blow up at the point P with weights $(1,2,3)$,
- β is the Kawamata blow up with weights $(1,2,1)$ of the singular point of the variety U that is a quotient singularity of type $\frac{1}{3}(1,2,1)$,
- η is an elliptic fibration.

The hypersurface X can be given by the equation

$$w^2z + wf_7(x,y,z,t) + f_{12}(x,y,z,t) = 0,$$

where $f_i(x,y,z,t)$ is a general quasihomogeneous polynomial of degree i. Let $\mathcal{P}$ be the pencil of surfaces that are cut out on the hypersurface X by the equations $\lambda x^2 + \mu z = 0$, where $(\lambda : \mu) \in \mathbb{P}^1$.

Proposition 6.5.

A general surface of the pencil $\mathcal{P}$ is birational to a K3 surface. In particular, the linear system $\mathcal{P}$ is a Halphen pencil.

Proof. A general surface of the pencil $\mathcal{P}$ is not ruled because X is birationally rigid ([5]). Hence a general surface of the pencil $\mathcal{P}$ is birational to a K3 surface because it is a compactification of a double cover of $\mathbb{C}^2$ branched over a sextic curve. $\square$

The hypersurface X can also be given by the equation

$$xg_{11}(x,y,z,t,w) + tg_9(x,y,z,t,w) + wg_7(x,y,z,t,w) + yg_5(y,z) = 0$$

such that the point O_1 is given by the equations $x = y = t = w = 0$, where g_i is a general quasihomogeneous polynomial of degree i. Let $\mathcal{P}_1$ be the pencil of surfaces that are cut out on the hypersurface X by the pencil $\lambda x^2 + \mu y = 0$, where $(\lambda : \mu) \in \mathbb{P}^1$. We will see that the linear system $\mathcal{P}_1$ is a Halphen pencil. The base locus of $\mathcal{P}_1$ does not contain the points O_2, O_3, O_4, O_5 and O_6. Similarly, we can construct a Halphen pencil $\mathcal{P}_i$ such that $\mathcal{P}_i \subset |-2K_X|$ and the base locus of the pencil $\mathcal{P}_i$ contains the point O_i.

Proposition 6.6.

The linear systems $\mathcal{P}$, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, $\mathcal{P}_4$, $\mathcal{P}_5$ and $\mathcal{P}_6$ are the only Halphen pencils on X.

We may assume that the singularities of the log pair $(X, \frac{1}{n}\mathcal{M})$ are canonical. Moreover it follows from Lemmas 4.1 and Corollary 4.3 that

$$\mathbb{CS}\Big(X, \frac{1}{n}\mathcal{M}\Big) \subset \Big\{O_1, O_2, O_3, O_4, O_5, O_6, P\Big\}.$$

Lemma 6.14.

If $O_i \in \mathbb{CS}(X, \frac{1}{n}\mathcal{M})$, then $\mathcal{M} = \mathcal{P}_i$.

Proof. Let $\pi_i : V_i \to X$ be the Kawamata blow up at the point O_i with weights $(1,1,1)$. Then $\mathcal{M}_{V_i} \sim_{\mathbb{Q}} -nK_{V_i}$ by Lemma 2.2.
The linear system $|-2K_{V_i}|$ is the proper transform of the pencil $\mathcal{P}_i$ and the base locus of $|-2K_{V_i}|$ consists of the irreducible curve C_{V_i} such that $\pi(C_{V_i})$ is the base curve of the pencil $\mathcal{P}_i$.
Let D be a general surface in $|-2K_{V_i}|$. Then the surface D is normal and $C_{V_i}^2 < 0$ on the surface D. On the other hand, we have $C_{V_i} \equiv -K_{V_i}|_D$, which implies that $\mathcal{M}_{V_i} = |-2K_{V_i}|$ by Theorem 2.2. □

Proposition 6.7.

A general surface of each pencil $\mathcal{P}_i$ is birational to a K3 surface. In particular, $\mathcal{P}_i$ is a Halphen pencil.

Proof. We use the same notations as in the proof of Lemma 6.14. The pencil $|-2K_{V_i}|$ satisfies the condition of Theorem 2.3. Therefore it is a Halphen pencil. The intersection of the surface D and the exceptional divisor $E_i \cong \mathbb{P}^2$ of the birational morphism π is a conic on E_i. An irreducible component of the intersection $D \cdot E_i$ is a rational curve not contained in the base locus of the pencil $|-2K_{V_i}|$. Therefore the surface D is birational to a K3 surface by Corollary 2.2. □

Let E be the exceptional divisor of the birational morphism α. It has two singular points Q and O that are quotient singularities of types $\frac{1}{3}(1,2,1)$ and $\frac{1}{2}(1,1,1)$, respectively.
Let C be the base curve of the pencil $\mathcal{P}$ and L be the unique curve in of the linear system $|\mathcal{O}_{\mathbb{P}(1,2,3)}(1)|$ on E.

Lemma 6.15.

If the set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ contains the point O, then $\mathcal{M} = \mathcal{P}$.

Proof. Let $\pi : V \to U$ be the Kawamata blow up at the point O with weights $(1,1,1)$ and F be the exceptional divisor of the birational morphism π. Let $\mathcal{L}$ be the proper transform of the linear system $|-3K_U|$ by the birational morphism π. We have $\mathcal{M}_V \sim_{\mathbb{Q}} -nK_V$ by Lemma 2.2, $\mathcal{P}_V \sim_{\mathbb{Q}} -2K_V$ and

$$\mathcal{L} \sim_{\mathbb{Q}} \pi^*(-3K_U) - \frac{1}{2}F.$$

The base locus of the linear system $\mathcal{L}$ consists of the irreducible curve $\tilde{C}_V$. Moreover for a general surface T of the linear system $\mathcal{L}$, the inequality $T \cdot \tilde{C}_V > 0$ holds, which implies that the divisor $\pi^*(-6K_U) - F$ is nef and big.
Let M and D be general surfaces of the pencils $\mathcal{M}_V$ and $\mathcal{P}_V$, respectively. Then

$$\Big(\pi^*(-6K_U) - F\Big) \cdot M \cdot D = \Big(\pi^*(-6K_U) - F\Big) \cdot \Big(\pi^*(-nK_U) - \frac{n}{2}F\Big) \cdot \Big(\pi^*(-2K_U) - F\Big) = 0,$$

which implies that $\mathcal{M}_V = \mathcal{P}_V$ by Theorem 2.2-(1). □

For now, to prove Proposition 6.6, we may assume that $\mathbb{CS}(X, \frac{1}{n}\mathcal{M}) = \{P\}$. Since $\mathcal{M}_U \sim_{\mathbb{Q}} -nK_U$ by Lemma 2.2, the set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M})_U$ is not empty by Theorem 2.1. Therefore Lemma 6.15 enables us to assume that the set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ consists of the point Q. The equivalence $\mathcal{M}_W \sim_{\mathbb{Q}} -nK_W$ by Lemma 2.2 implies that every surface in the pencil $\mathcal{M}_W$ is contracted to a curve by the elliptic fibration η. Moreover the set $\mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W)$ is not empty by Theorem 2.1.
Let G be the exceptional divisor of the birational morphism β and Q_1 be the singular point of the surface G. Then the point Q_1 is the quotient singularity of type $\frac{1}{2}(1,1,1)$ on the variety W. Moreover it follows from Lemma 2.3 that the set $\mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W)$ contains the point Q_1.
Let $\gamma: Y \to W$ be the Kawamata blow up at the point Q_1 with weights $(1,1,1)$. The base locus of the pencil $\mathcal{P}_Y$ consists of the irreducible curves C_Y and L_Y. Let D be a general surface of the pencil $\mathcal{P}_Y$. Then explicit local calculations show that $D \sim_{\mathbb{Q}} -2K_Y$. On the other hand, the surface D is normal and the intersection form of the curves C_Y and L_Y on the surface D is negative-definite. Hence we obtain the identity $\mathcal{M}_Y = \mathcal{P}_Y$ from Theorem 2.2 because $\mathcal{M}_Y|_D \equiv n(C_Y + L_Y)$. Therefore we see that $\mathcal{M} = \mathcal{P}$, which completes our proof of Proposition 6.6.

$\gimel = 25$: **Hypersurface of degree** 15 **in** $\mathbb{P}(1,1,3,4,7)$.

The threefold X is a general hypersurface of degree 15 in $\mathbb{P}(1,1,3,4,7)$ with $-K_X^3 = \frac{5}{28}$. It has two singular points. One is a quotient singularity P of type $\frac{1}{4}(1,1,3)$ and the other is a quotient singularity Q of type $\frac{1}{7}(1,3,4)$.
There is a commutative diagram

$$\begin{tikzcd}
 & & & Y \arrow[dl, "\gamma_O"'] \arrow[dr, "\gamma_P"] \arrow[drrr, "\eta"] & & & \\
 & & U_{PQ} \arrow[dl, "\beta_Q"'] \arrow[dr, "\beta_P"] & & U_{QO} \arrow[dl, "\beta_O"] & & \mathbb{P}(1,1,3), \\
 & U_P \arrow[dr, "\alpha_P"'] & & U_Q \arrow[dl, "\alpha_Q"] & & & \\
 & & X \arrow[uurrrr, dashed, "\psi"'] & & & &
\end{tikzcd}$$

where

- ψ is the natural projection,
- α_P is the Kawamata blow up at the point P with weights $(1,1,3)$,
- α_Q is the Kawamata blow up at the point Q with weights $(1,3,4)$,
- β_Q is the Kawamata blow up with weights $(1,3,4)$ at the point whose image to X is the point Q,
- β_P is the Kawamata blow up with weights $(1,1,3)$ at the point whose image to X is the point P,
- β_O is the Kawamata blow up with weights $(1,3,1)$ at the singular point O of type $\frac{1}{4}(1,3,1)$ contained in the exceptional divisor of the birational morphism α_Q,
- γ_P is the Kawamata blow up with weights $(1,1,3)$ at the point whose image to X is the point P,
- γ_O is the Kawamata blow up with weights $(1,3,1)$ at the singular point of type $\frac{1}{4}(1,3,1)$ contained in the exceptional divisor of the birational morphism β_Q,
- η is an elliptic fibration.

Proposition 6.8.
The linear system $|-K_X|$ is a unique Halphen pencil on X.

In what follows, we prove Proposition 6.8. For the convenience, let D be a general surface in $|-K_X|$.
It follows from [5] that $|-K_X|$ is invariant under the action of the group $\mathrm{Bir}(X)$. Therefore we may assume that the log pair $(X, \frac{1}{n}\mathcal{M})$ is canonical. In fact, we can assume that

$$\varnothing \neq \mathbb{CS}\Big(X, \frac{1}{n}\mathcal{M}\Big) \subseteq \Big\{P, Q\Big\}$$

by Lemma 4.1 and Corollary 4.4.

Lemma 6.16.
If the point Q is not contained in $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$, then $\mathcal{M} = |-K_X|$.

Proof. The set $\mathbb{CS}(U_P, \frac{1}{n}\mathcal{M}_{U_P})$ is not empty by Theorem 2.1 because $\mathcal{M}_{U_P} \sim_{\mathbb{Q}} -nK_{U_P}$ by Lemma 2.2. Let P_1 be the singular point of the variety U_P contained in the exceptional divisor of the birational morphism α_P. It is a quotient singularity of type $\frac{1}{3}(1,1,2)$. Lemma 2.3 implies that the set $\mathbb{CS}(U_P, \frac{1}{n}\mathcal{M}_{U_P})$ contains the point P_1.
Let $\pi_P : W_P \to U_P$ be the Kawamata blow up at the point P_1 with weights $(1,1,2)$. We can easily check that $|-K_{W_P}|$ is the proper transform of the pencil $|-K_X|$ and the base locus of the pencil $|-K_W|$ consists of the irreducible curve C_{W_P}. We can also see $D_{W_P} \cdot C_{W_P} = -K_{W_P}^3 = -\frac{1}{14} < 0$. Hence Theorem 2.2 implies the identity $\mathcal{M} = |-K_X|$ because $\mathcal{M}_{W_P} \sim_{\mathbb{Q}} nD_{W_P}$ by Lemma 2.2. ☐

The exceptional divisor $E \cong \mathbb{P}(1,3,4)$ of the birational morphism α_Q contains two singular points O and Q_1 that are quotient singularities of types $\frac{1}{4}(1,3,1)$ and $\frac{1}{3}(1,2,1)$. Let L be the unique curve of the linear system $|\mathcal{O}_{\mathbb{P}(1,3,4)}(1)|$ on the surface E.
Due to Lemma 6.16, we may assume that the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains the singular point Q. The proof of Lemma 6.16 also shows that the set $\mathbb{CS}(U_Q, \frac{1}{n}\mathcal{M}_{U_Q})$ cannot consist of the single point $\bar{P}$ whose image to X is the point P. It implies

$$\mathbb{CS}\Big(U_Q, \frac{1}{n}\mathcal{M}_{U_Q}\Big) \cap \Big\{O, Q_1\Big\} \neq \varnothing$$

by Theorem 2.1 and Lemma 2.3.

Lemma 6.17.
If the set $\mathbb{CS}(U_Q, \frac{1}{n}\mathcal{M}_{U_Q})$ contains both the point O and the point Q_1, then $\mathcal{M} = |-K_X|$.

Proof. Let $\gamma_Q : W_Q \to U_{QO}$ be the Kawamata blow up with weights $(1,2,1)$ at the point whose image to U_Q is the point Q_1.
The proper transform D_{W_Q} is irreducible and normal. The base locus of the pencil $|-K_{W_Q}|$ consists of the irreducible curves C_{W_Q} and L_{W_Q}. On the other hand, we have

$$\mathcal{M}_{W_Q}\Big|_{D_{W_Q}} \equiv -nK_{W_Q}\Big|_{D_{W_Q}} \equiv nC_{W_Q} + nL_{W_Q},$$

but the intersection form of the curves L_{W_Q} and C_{W_Q} on the normal surface D_{W_Q} is negative-definite. Then Theorem 2.2 completes the proof. ☐

It follows from Lemma 2.3 that we may assume the following possibilities:

- $\mathbb{CS}(U_Q, \frac{1}{n}\mathcal{M}_{U_Q}) = \{\bar{P}, O\}$;
- $\mathbb{CS}(U_Q, \frac{1}{n}\mathcal{M}_{U_Q}) = \{O\}$;
- $\mathbb{CS}(U_Q, \frac{1}{n}\mathcal{M}_{U_Q}) = \{\bar{P}, Q_1\}$;

- $\mathbb{CS}(U_Q, \frac{1}{n}\mathcal{M}_{U_Q}) = \{Q_1\}$.

The exceptional divisor $F \cong \mathbb{P}(1,3,1)$ of β_O contains one singular point Q_2 that is a quotient singularity of type $\frac{1}{3}(1,2,1)$.

Lemma 6.18.

If $\mathbb{CS}(U_Q, \frac{1}{n}\mathcal{M}_{U_Q}) = \{O\}$, then $\mathcal{M} = |-K_X|$.

Proof. The set $\mathbb{CS}(U_{QO}, \frac{1}{n}\mathcal{M}_{U_{QO}})$ contains the singular point Q_2 by Theorem 2.1 and Lemma 2.3.
Let $\gamma : W \to U_{QO}$ be the Kawamata blow up at the point Q_2 with weights $(1,2,1)$. Then $\mathcal{M}_W \sim_{\mathbb{Q}} -nK_W$ by Lemma 2.2 and the base locus of the pencil $|-K_W|$ consists of the curves C_W and L_W. The proper transform D_W is irreducible and normal, the equivalence $\mathcal{M}_W\Big|_{D_W} \equiv nC_W + nL_W$ holds, but the equalities

$$C_W^2 = -\frac{7}{12}, \quad L_W^2 = -\frac{5}{6}, \quad C_W \cdot L_W = \frac{2}{3}$$

hold on the surface D_W. So, the intersection form of the curves C_W and L_W on the normal surface D_W is negative-definite, which implies $\mathcal{M} = |-K_X|$ by Theorem 2.2. □

Lemma 6.19.

If $\mathbb{CS}(U_Q, \frac{1}{n}\mathcal{M}_{U_Q}) = \{\bar{P}, O\}$, then $\mathcal{M} = |-K_X|$.

Proof. We have $\mathcal{M}_Y \sim_{\mathbb{Q}} -nK_Y$, which implies that every surface of the pencil $\mathcal{M}_Y$ is contracted to a curve by the morphism η. In particular, the set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ does not contain curves because the exceptional divisors of $\beta_O \circ \gamma_P$ are sections of η.
Due to Theorem 2.1 and Lemmas 2.3, 6.18, we may assume that the set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ contains the singular point P_2 of Y contained in the exceptional divisor γ_P. Let $\sigma_P : Y_P \to Y$ be the Kawamata blow up at the point P_2 with weights $(1,2,1)$. Then $\mathcal{M}_{Y_P} \sim_{\mathbb{Q}} -nK_{Y_P}$ but the base locus of the pencil $|-K_{Y_P}|$ consists of the irreducible curves C_{Y_P} and L_{Y_P}. The proper transform D_{Y_P} is normal and $\mathcal{M}_{Y_P}|_{D_{Y_P}} \equiv nC_{Y_P} + nL_{Y_P}$. The intersection form of the curves C_{Y_P} and L_{Y_P} on the normal surface D_{Y_P} is negative-definite because the curves are contained in a fiber of $\eta \circ \sigma_P|_{D_{Y_P}}$ that consists of three irreducible components. Therefore we obtain the identity $\mathcal{M} = |-K_X|$ from Theorem 2.2. □

Thus to conclude the proof of Proposition 6.8, we may assume the following possibilities:

- $\mathbb{CS}(U_Q, \frac{1}{n}\mathcal{M}_{U_Q}) = \{\bar{P}, Q_1\}$;
- $\mathbb{CS}(U_Q, \frac{1}{n}\mathcal{M}_{U_Q}) = \{Q_1\}$.

The hypersurface X can be given by the equation

$$w^2y + wt^2 + wtf_4(x,y,z) + wf_8(x,y,z) + tf_{11}(x,y,z) + f_{15}(x,y,z) = 0,$$

where f_i is a general quasihomogeneous polynomial of degree i.

Lemma 6.20.

The case $\mathbb{CS}(U_Q, \frac{1}{n}\mathcal{M}_{U_Q}) = \{Q_1\}$ never happens.

Proof. Suppose that $\mathbb{CS}(U_Q, \frac{1}{n}\mathcal{M}_{U_Q}) = \{Q_1\}$. Let $\pi : V \to U_Q$ be the Kawamata blow up at the point Q_1 with weights $(1,2,1)$.
Let G be the exceptional divisor of the birational morphism π. The proof of Lemma 6.18 implies that the set $\mathbb{CS}(V, \frac{1}{n}\mathcal{M}_V)$ does not contain the singular point of V contained in the exceptional divisor G. So, the log pair $(V, \frac{1}{n}\mathcal{M}_V)$ is terminal by Lemma 2.3 and Corollary 4.1.

We have

$$\begin{cases}(\alpha_Q\circ\pi)^*\big(-K_X\big)\sim_{\mathbb{Q}} S_V+\dfrac{3}{7}G+\dfrac{1}{7}E_V,\\(\alpha_Q\circ\pi)^*\big(-K_X\big)\sim_{\mathbb{Q}} S_V^y+\dfrac{10}{7}G+\dfrac{8}{7}E_V,\\(\alpha_Q\circ\pi)^*\big(-3K_X\big)\sim_{\mathbb{Q}} S_V^z+\dfrac{2}{7}G+\dfrac{3}{7}E_V,\\(\alpha_Q\circ\pi)^*\big(-4K_X\big)\sim_{\mathbb{Q}} S_V^t+\dfrac{5}{7}G+\dfrac{4}{7}E_V,\\(\alpha_Q\circ\pi)^*\big(-7K_X\big)\sim_{\mathbb{Q}} S_V^w.\end{cases}$$

The equivalences imply

$$\begin{cases}(\alpha_Q\circ\pi)^*\Big(\dfrac{y}{x}\Big)\in|S_V|,\\(\alpha_Q\circ\pi)^*\Big(\dfrac{yz}{x^4}\Big)\in|4S_V|,\\(\alpha_Q\circ\pi)^*\Big(\dfrac{yt}{x^5}\Big)\in|5S_V|,\\(\alpha_Q\circ\pi)^*\Big(\dfrac{y^3w}{x^{10}}\Big)\in|10S_V|,\end{cases}$$

and hence the complete linear system $|-20K_V|$ induces a birational map $\chi_1 : V \dashrightarrow X'$ such that X' is a hypersurface in $\mathbb{P}(1,1,4,5,10)$, which implies that the divisor $-K_V$ is big.
The base locus of the pencil $|-K_V|$ consists of the irreducible curves C_V and L_V. It follows from [11] that there is an isomorphism $\zeta : V \dashrightarrow V'$ of codimension 1 such that ζ is regular in the outside of $C_V\cup L_V$ and the anticanonical divisor $-K_{V'}$ is nef and big. The singularities of the log pair $(V', \frac{1}{n}\mathcal{M}_{V'})$ are terminal because the rational map ζ is a log flop with respect to the log pair $(V, \frac{1}{n}\mathcal{M}_V)$, which contradicts Theorem 2.1. □

Now we suppose $\mathbb{CS}(U_Q, \frac{1}{n}\mathcal{M}_{U_Q}) = \{\bar{P}, Q_1\}$. Let $\sigma : U \to U_{PQ}$ be the Kawamata blow up with weights $(1,2,1)$ at the point $\bar{Q}_1$ whose image to U_Q is the point Q_1. Let $\bar{E}$ and $\tilde{E}$ be the exceptional divisors of α_P and σ, respectively. Then

$$\begin{cases}S_U\sim_{\mathbb{Q}}(\alpha_P\circ\beta_Q\circ\sigma)^*\big(-K_X\big)-\dfrac{3}{7}\tilde{E}-\dfrac{1}{7}E_U-\dfrac{1}{4}\bar{E}_U\sim_{\mathbb{Q}}-K_W,\\S_U^y\sim_{\mathbb{Q}}(\alpha_P\circ\beta_Q\circ\sigma)^*\big(-K_X\big)-\dfrac{10}{7}\tilde{E}-\dfrac{8}{7}E_U-\dfrac{1}{4}\bar{E}_U,\\S_U^z\sim_{\mathbb{Q}}(\alpha_P\circ\beta_Q\circ\sigma)^*\big(-3K_X\big)-\dfrac{2}{7}\tilde{E}-\dfrac{3}{7}E_U-\dfrac{3}{4}\bar{E}_U,\\S_U^t\sim_{\mathbb{Q}}(\alpha_P\circ\beta_Q\circ\sigma)^*\big(-4K_X\big)-\dfrac{5}{7}\tilde{E}-\dfrac{4}{7}E_U,\\S_U^w\sim_{\mathbb{Q}}(\alpha_P\circ\beta_Q\circ\sigma)^*\big(-7K_X\big)-\dfrac{11}{4}\bar{E}_U.\end{cases}$$

The equivalences imply that the pull-backs of rational functions

$$\frac{y}{x},\ \frac{yz}{x},\ \frac{y^3w}{x^{10}},\ \frac{y^4tw}{x^{15}}$$

are contained in the linear system $|aS_U|$, where $a = 1, 4, 10$ and 15, respectively. Therefore the linear system $|-60K_U|$ induces a birational map $\chi_2 : U \dashrightarrow X''$ such that the variety X'' is a hypersurface of degree 30 in $\mathbb{P}(1,1,4,10,15)$, which implies that the anticanonical divisor $-K_U$ is big. However, the proof of Lemma 6.18 shows that the singularities of $(U, \frac{1}{n}\mathcal{M}_U)$ are terminal. Then we can obtain a contradiction in the same way as in the proof of Lemma 6.20.

$\daleth = 32$: **Hypersurface of degree** 16 **in** $\mathbb{P}(1,2,3,4,7)$.

The hypersurface X is given by a general quasihomogeneous polynomial of degree 16 in $\mathbb{P}(1,2,3,4,7)$ with $-K_X^3 = \frac{2}{21}$. The singularities of the threefold X consist of four quotient singular points of type $\frac{1}{2}(1,1,1)$, one quotient singular point of type $\frac{1}{3}(1,2,1)$ and one quotient singular point P of type $\frac{1}{7}(1,3,4)$. There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & & Y \arrow[ll, "\beta"'] \arrow[d, "\eta"] \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,2,3), \end{tikzcd}$$

where

- ψ is the natural projection,
- α is the Kawamata blow up at the point P with weights $(1,3,4)$,
- β is the Kawamata blow up with weights $(1,1,3)$ at the singular point Q of the variety U that is a quotient singularity of type $\frac{1}{4}(1,1,3)$ contained in the exceptional divisor of α,
- η is an elliptic fibration.

The hypersurface X can be given by the quasihomogeneous equation

$$w^2y + wf_9\big(x,y,z,t\big) + f_{16}\big(x,y,z,t\big) = 0$$

where f_9 and f_{16} are quasihomogeneous polynomials of degrees 9 and 16, respectively. Let D be a general surface in $|-2K_X|$. It is cut out on the threefold X by the equation

$$\lambda x^2 + \mu y = 0,$$

where $(\lambda : \mu) \in \mathbb{P}^1$. The surface D is irreducible and normal. The base locus of the pencil $|-2K_X|$ consists of the curve C, which implies that $C = D \cdot S$.
The set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ cannot contain any singular point of type $\frac{1}{2}(1,1,1)$ because of Lemma 4.2 (see the Big Table in [5]). If it contains the singular point of type $\frac{1}{3}(1,2,1)$, we obtain $\mathcal{M} = |-2K_X|$ from Lemma 4.3. It then follows from Corollary 4.3 and Lemma 4.1 that we may assume

$$\mathbb{CS}\Big(X, \frac{1}{n}\mathcal{M}\Big) = \Big\{P\Big\}.$$

Furthermore, the set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ is not empty by Theorem 2.1 because $-K_U$ is nef and big.
The exceptional divisor $E \cong \mathbb{P}(1,3,4)$ of the birational morphism α contains two singular points O and Q that are quotient singularities of types $\frac{1}{3}(1,1,2)$ and $\frac{1}{4}(1,1,3)$, respectively. Let L be the unique curve contained in the linear system $|\mathcal{O}_{\mathbb{P}(1,3,4)}(1)|$ on the surface E. Let F be the exceptional divisor of β. It contains a singular point Q_1 that is quotient singularity of type $\frac{1}{3}(1,1,2)$.
Then it follows from Lemma 2.3 that either $Q \in \mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ or $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U) = \{O\}$.

Lemma 6.21.
If the set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ consists of the point Q, then $\mathcal{M} = |-2K_X|$.

Proof. It follows from Lemma 2.2 that $\mathcal{M}_Y \sim_{\mathbb{Q}} -nK_Y$, which implies that every surface in the pencil $\mathcal{M}_Y$ is contracted to a curve by the morphism η and the set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ contains the point Q_1.
Let $\pi : V \to Y$ be the Kawamata blow up at the point Q_1 with weights $(1,1,2)$. Then the transform D_V is normal but the base locus of the pencil $|-2K_V|$ consists of the irreducible curves C_V and L_V.
The intersection form of the curves C_V and L_V on the surface D_V is negative-definite because the curves C_V and L_V are components of a fiber of the elliptic fibration $\eta \circ \pi|_{D_V}$ that contains three irreducible components. On the other hand, we have

$$\mathcal{M}_V\Big|_{D_V} \equiv -nK_V\Big|_{D_V} \equiv nC_V + nL_V.$$

Therefore it follows from Theorem 2.2 that $\mathcal{M} = |-2K_X|$. ◻

From now on, we may assume that the set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ contains the point O due to Lemma 2.3. Let $\gamma : W \to U$ be the Kawamata blow up at the point O with weights $(1, 1, 2)$ and G be the exceptional divisor of the birational morphism γ. Then the surface $G \cong \mathbb{P}(1, 1, 2)$ and

$$\mathcal{M}_W \sim_{\mathbb{Q}} -nK_W \sim_{\mathbb{Q}} \gamma^*(-nK_U) - \frac{n}{3}G \sim_{\mathbb{Q}} (\alpha \circ \gamma)^*(-nK_X) - \frac{n}{7}\gamma^*(E) - \frac{n}{3}G.$$

In a neighborhood of the point P, the monomials x, z and t can be considered as weighted local coordinates on X such that $\mathrm{wt}(x) = 1$, $\mathrm{wt}(z) = 3$ and $\mathrm{wt}(z) = 4$. Then in a neighborhood of the singular point P, the surface D can be given by equation

$$\lambda x^2 + \mu\Big(\epsilon_1 x^9 + \epsilon_2 zx^6 + \epsilon_3 z^2x^3 + \epsilon_4 z^3 + \epsilon_5 t^2x + \epsilon_6 tx^5 + \epsilon_7 tzx^2 + h_{16}(x, z, t) + \text{higher terms}\Big) = 0,$$

where $\epsilon_i \in \mathbb{C}$ and h_{16} is a quasihomogeneous polynomial of degree 16. In a neighborhood of the singular point O, the birational morphism α can be given by the equations

$$x = \tilde{x}\tilde{z}^{\frac{1}{7}},\ z = \tilde{z}^{\frac{3}{7}},\ t = \tilde{t}\tilde{z}^{\frac{4}{7}},$$

where $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$ are weighted local coordinates on the variety U in a neighborhood of the singular point O such that $\mathrm{wt}(\tilde{x}) = 1$, $\mathrm{wt}(\tilde{z}) = 2$ and $\mathrm{wt}(\tilde{t}) = 1$.

In a neighborhood of the point O, the surface E is given by $\tilde{z} = 0$, the surface D_U is given by

$$\lambda\tilde{x}^2 + \mu\Big(\epsilon_1\tilde{x}^9\tilde{z} + \epsilon_2\tilde{z}\tilde{x}^6 + \epsilon_3\tilde{z}\tilde{x}^3 + \epsilon_4\tilde{z} + \epsilon_5\tilde{t}^2\tilde{x}\tilde{z} + \epsilon_6\tilde{t}\tilde{x}^5\tilde{z} + \epsilon_7\tilde{t}\tilde{z}\tilde{x}^2 + \text{higher terms}\Big) = 0,$$

and the surface S_U is given by the equation $\tilde{x} = 0$.

In a neighborhood of the singular point of G, the birational morphism γ can be given by

$$\tilde{x} = \bar{x}\bar{z}^{\frac{1}{3}},\ \tilde{z} = \bar{z}^{\frac{2}{3}},\ \tilde{t} = \bar{t}\bar{z}^{\frac{1}{3}},$$

where $\bar{x}$, $\bar{z}$ and $\bar{t}$ are weighted local coordinates on the variety W in a neighborhood of the singular point of G such that $\mathrm{wt}(\bar{x}) = \mathrm{wt}(\bar{z}) = \mathrm{wt}(\bar{t}) = 1$. The surface G is given by the equation $\bar{z} = 0$, the proper transform D_W is given by

$$\lambda\bar{x}^2 + \mu\Big(\epsilon_1\bar{x}^9\bar{z}^3 + \epsilon_2\bar{z}^2\bar{x}^6 + \epsilon_3\bar{z}\bar{x}^3 + \epsilon_4 + \epsilon_5\bar{t}^2\bar{x}\bar{z} + \epsilon_6\bar{t}\bar{x}^5\bar{z}^2 + \epsilon_7\bar{t}\bar{z}\bar{x}^2 + \text{higher terms}\Big) = 0,$$

the proper transform S_W is given by the equation $\bar{x} = 0$ and the proper transform E_W is given by the equation $\bar{z} = 0$.

Let $\mathcal{P}$ be the proper transforms on the variety W of the pencil $|-2K_X|$. The curves C_W and L_W are contained in the base locus of the pencil $\mathcal{P}$. Moreover easy calculations show that the base locus of the pencil $\mathcal{P}$ does not contain any other curve than C_W and L_W. We also have

$$\begin{cases} E_W \sim_{\mathbb{Q}} \gamma^*(E) - \dfrac{2}{3}F, \\ D_W \sim_{\mathbb{Q}} (\alpha \circ \gamma)^*\big(-2K_X\big) - \dfrac{2}{7}\gamma^*(E) - \dfrac{2}{3}G, \\ S_W \sim_{\mathbb{Q}} (\alpha \circ \gamma)^*\big(-K_X\big) - \dfrac{1}{7}\gamma^*(E) - \dfrac{1}{3}G, \end{cases}$$

Also we have $C_W + L_W = S_W \cdot D_W$ and $2L_W = D_W \cdot E_W$.

The curves C_W and L_W can be considered as irreducible effective divisors on the normal surface D_W. Then it follows from the equivalences above that

$$L_W^2 = -\frac{5}{8},\ C_W^2 = -\frac{7}{24},\ C_W \cdot L_W = \frac{3}{8},$$

which implies that the intersection form of C_W and L_W on D_W is negative-definite. Let M be a general surface of the linear system $\mathcal{M}_W$. Then

$$M\Big|_{D_W} \equiv -nK_W\Big|_{D_W} \equiv nS_W\Big|_{D_W} \equiv nC_W + nL_W,$$

which implies that $\mathcal{M} = |-2K_X|$ by Theorem 2.2.

Consequently, we have proved:

Proposition 6.9.

The linear system $|-2K_X|$ is the only Halphen pencil on X.

$\daleth = 56$**: Hypersurface of degree** 24 **in** $\mathbb{P}(1,2,3,8,11)$**.**

The threefold X is a general hypersurface of degree 24 in $\mathbb{P}(1,2,3,8,11)$ with $-K_X^3 = \frac{1}{22}$. Its singularities consist of three points that are quotient singularities of type $\frac{1}{2}(1,1,1)$ and the point $O = (0:0:0:0:1)$ that is a quotient singularity of type $\frac{1}{11}(1,3,8)$.

Before we proceed, let us first describe some birational transformations of the hypersurface X with elliptic fibrations, which are useful to explain the geometrical nature of our proof. There is a commutative diagram

$$\begin{tikzcd}
 & & Z \arrow[r, dashed, "\zeta"] \arrow[dl, "\nu"'] & Z' \arrow[r, "\sigma"] & \bar{Z}' \arrow[r, hookrightarrow] & \mathbb{P}(1,2,5,14,21) \arrow[ddl, dashed, "\phi"] \\
U \arrow[d, "\alpha"'] & W \arrow[l, "\beta"'] & Y \arrow[l, "\gamma"'] \arrow[d, "\eta"] & V \arrow[l, "\xi"'] \arrow[ul, "\omega"'] \arrow[r, dashed, "\chi"] & V' \arrow[d, "\upsilon"] & \\
X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,2,3) \arrow[rr, dashed, "\rho"'] & & \mathbb{P}(1,2,5) &
\end{tikzcd}$$

where

- ψ and ϕ are natural projections,
- α is the Kawamata blow up at the point O with weights $(1,3,8)$,
- β is the Kawamata blow up with weights $(1,3,5)$ at the singular point Q contained in the exceptional divisor E of α that is a quotient singularity of type $\frac{1}{8}(1,3,5)$,
- γ is the Kawamata blow up with weights $(1,3,2)$ at the singular point Q_1 of contained in the exceptional divisor F of β that is a quotient singularity of type $\frac{1}{5}(1,3,2)$,
- ν is the Kawamata blow up with weights $(1,1,2)$ at the singular point Q_2 of contained in the exceptional divisor of β that is a quotient singularity of type $\frac{1}{3}(1,1,2)$,
- ξ is the Kawamata blow up with weights $(1,1,2)$ at the point $\bar{Q}_2$ whose image to W is the point Q_2,
- ω is the Kawamata blow up with weights $(1,3,2)$ at the point $\bar{Q}_1$ whose image to W is the point Q_1,
- η and υ are elliptic fibrations,
- the maps ζ and χ are isomorphisms in codimension 1,
- the birational morphism σ is given by the plurianticanonical linear system of Z',
- the rational map ρ is a toric map,

The exceptional divisor E of the birational morphism α contains two singular points P and Q of U that are quotient singularities of types $\frac{1}{3}(1,1,2)$ and $\frac{1}{8}(1,3,5)$, respectively. Meanwhile, the exceptional divisor F of the birational morphism β also contains two singular points Q_1 and Q_2 of W that are quotient singularities of types $\frac{1}{5}(1,3,2)$ and $\frac{1}{3}(1,1,2)$, respectively.

Remark 6.1.

The divisors $-K_{Z'}$, $-K_U$ and $-K_W$ are nef and big. Thus the anticanonical models of the threefolds Z', U and W are Fano threefolds with canonical singularities. The anticanonical model of Z' is a hypersurface $\bar{Z}'$ of degree 42 in $\mathbb{P}(1,2,5,14,21)$. The anticanonical model of U is a hypersurface of degree 26 in $\mathbb{P}(1,2,3,8,13)$ and the anticanonical model of W is a hypersurface of degree 30 in $\mathbb{P}(1,2,3,10,15)$.

For the convenience, we denote the pencil $|-2K_X|$ by $\mathcal{B}$. In addition, a general surface in $\mathcal{B}$ is denoted by B and a general surface in $\mathcal{M}$ by M
It follows from Corollary 4.3 and Lemmas 4.1, 4.2 that we may assume that $\mathbb{CS}(X, \frac{1}{n}\mathcal{M}) = \{O\}$.

Lemma 6.22.
If the set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ contains the point P, then $\mathcal{M} = \mathcal{B}$.

Proof. Let $\beta_P : U_P \to U$ be the Kawamata blow up at the point P and E_P be its exceptional divisor. For a general surface D in $|-8K_X|$, we have

$$D_{U_P} \sim_{\mathbb{Q}} (\alpha \circ \beta_P)^*(-8K_X) - \frac{8}{11}\beta_P^*(E) - \frac{2}{3}E_P.$$

Since the base locus of the proper transform of the linear system $|-8K_X|$ on U_P does not contain any curve, the divisor D_{U_P} is nef and big.
Since $M_{U_P} \sim_{\mathbb{Q}} nS_{U_P}$ by Lemma 2.2 and $B_{U_P} \sim_{\mathbb{Q}} 2S_{U_P}$, we obtain

$$D_{U_P} \cdot B_{U_P} \cdot S_{U_P} = 2n\Big(\beta_P^*(-8K_U) - \frac{2}{3}E_P\Big) \cdot \Big(\beta_P^*(-K_U) - \frac{1}{3}E_P\Big)^2 = 0.$$

It implies $\mathcal{M} = \mathcal{B}$ by Theorem 2.2. □

Due to Theorem 2.1 and Lemma 2.3, we may assume that the set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ consists of the singular point Q. Thus it follows from Theorem 2.1, Lemmas 2.1, 2.3 that

$$\varnothing \neq \mathbb{CS}\Big(W, \frac{1}{n}\mathcal{M}_W\Big) \subseteq \Big\{Q_1,\ Q_2\Big\}.$$

Now we consider some local computation. We may assume that X is given by the equation

$$w^2y + wf_{13}(x, y, z, t) + f_{24}(x, y, z, t) = 0,$$

where $f_i(x, y, z, t)$ is a general quasihomogeneous polynomial of degree i. The surface B is given by the equation $\lambda x^2 + \mu y = 0$, where $(\lambda : \mu) \in \mathbb{P}^1$. The base locus of $\mathcal{B}$ consists of the irreducible curve C that is given by $x = y = 0$. We have $B \cdot S = C$.
In a neighborhood of O, the monomials x, z and t can be considered as weighted local coordinates on X such that $\mathrm{wt}(x) = 1$, $\mathrm{wt}(z) = 3$ and $\mathrm{wt}(z) = 8$. Then in a neighborhood of the singular point O, the surface B can be given by equation

$$\lambda x^2 + \mu\Big(\epsilon_1 x^{13} + \epsilon_2 zx^{10} + \epsilon_3 z^2x^7 + \epsilon_4 z^3x^4 + \epsilon_5 z^4x + \epsilon_6 tx^5 + \epsilon_7 tzx^2 + \epsilon_8 t^3 + \epsilon_9 z^8 + \text{other terms}\Big) = 0,$$

where $\epsilon_i \in \mathbb{C}$. In a neighborhood of the singular point Q, the birational morphism α can be given by the equations

$$x = \bar{x}\bar{t}^{\frac{1}{11}},\ z = \bar{z}\bar{t}^{\frac{3}{11}},\ t = \bar{t}^{\frac{8}{11}},$$

where $\bar{x}$, $\bar{z}$ and $\bar{t}$ are weighted local coordinates on U in a neighborhood of the singular point Q such that $\mathrm{wt}(\bar{x}) = 1$, $\mathrm{wt}(\bar{z}) = 3$ and $\mathrm{wt}(\bar{t}) = 8$. Thus in a neighborhood of the singular point Q, the divisor E is given by the equation $\bar{t} = 0$, the divisor S_U is given by $\bar{x} = 0$ and the divisor B_U is given by the equation

$$\lambda\bar{x}^2 + \mu\Big(\epsilon_1\bar{x}^{13}\bar{t} + \cdots + \epsilon_5\bar{z}^4\bar{x}\bar{t} + \epsilon_6\bar{t}\bar{x}^5 + \epsilon_7\bar{t}\bar{z}\bar{x}^2 + \epsilon_8\bar{t}^2 + \epsilon_9\bar{z}^8\bar{t}^2 + \text{other terms}\Big) = 0,$$

which implies that $B_U \sim_{\mathbb{Q}} 2S_U$ and the base locus of $\mathcal{B}_U$ is the union of C_U and the curve $L \subset E$ that is given by $\bar{x} = \bar{t} = 0$. We have $E \cong \mathbb{P}(1, 3, 8)$ and the curve L is the unique curve in $|\mathcal{O}_{\mathbb{P}(1,3,8)}(1)|$ on the surface E. The surface B_U is not normal. Indeed, B_U is singular at a generic point of L. We have $S_U \cdot B_U = C_U + 2L$ and $E \cdot B_U = 2L$, which implies that $S_U \cdot C_U = 0$ and $S_U \cdot L = \frac{1}{24}$.

Lemma 6.23.
If the set $\mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W)$ consists of the point Q_2, then $\mathcal{M} = \mathcal{B}$.

Proof. In a neighborhood of Q_2, the birational morphism β can be given by the equations

$$\bar{x} = \tilde{x}\tilde{z}^{\frac{1}{8}},\ \bar{z} = \tilde{z}^{\frac{3}{8}},\ \bar{t} = \tilde{t}\tilde{z}^{\frac{5}{8}},$$

where $\tilde{x}$, $\tilde{z}$ and $\tilde{t}$ are weighted local coordinates on W in a neighborhood of Q_2 such that $\mathrm{wt}(\tilde{x}) = 1$, $\mathrm{wt}(\tilde{z}) = 1$ and $\mathrm{wt}(\tilde{t}) = 2$. Thus in a neighborhood of the singular point Q_2, the divisor F is given by the equation $\tilde{z} = 0$, the divisor S_W is given by $\tilde{x} = 0$, the divisor E_W is given by $\tilde{t} = 0$ and the divisor B_W is given by the equation

$$\lambda\tilde{x}^2 + \mu\Big(\epsilon_7\tilde{t}\tilde{z}\tilde{x}^2 + \epsilon_8\tilde{t}^2\tilde{z} + \epsilon_9\tilde{z}^4\tilde{t}^2 + \text{other terms}\Big) = 0,$$

which implies that $B_W \sim_{\mathbb{Q}} 2S_W$, the base locus of $\mathcal{B}_W$ is the union of C_W, L_W and the curve $L' \subset F$ that is given by $\tilde{x} = \tilde{z} = 0$.
The surface F is isomorphic to $\mathbb{P}(1,3,5)$ and the curve L' is the unique curve of the linear system $|\mathcal{O}_{\mathcal{P}(1,3,5)}(1)|$ on the surface F. The surface B_W is smooth at a generic point of L'. We have

$$S_W \cdot B_W = C_W + 2L_W + L',\quad E_W \cdot B_W = 2L_W,\quad F \cdot B_W = 2L',$$

which implies that

$$S_W \cdot C_W = 0,\quad S_W \cdot L_W = 0,\quad S_W \cdot L' = \frac{1}{15}$$

because

$$\begin{cases} S_W \sim_{\mathbb{Q}} (\alpha\circ\beta)^*(-K_X) - \frac{1}{11}\beta^*(E) - \frac{1}{8}F, \\ B_W \sim_{\mathbb{Q}} (\alpha\circ\beta)^*(-2K_X) - \frac{2}{11}\beta^*(E) - \frac{2}{8}F, \\ E_W \sim_{\mathbb{Q}} \beta^*(E) - \frac{5}{8}F. \end{cases}$$

Let R be the exceptional divisor of ν. Let O_1 be the singular point of Z that is contained in R. Then $R \cong \mathbb{P}(1,1,2)$ and O_1 is a quotient singularity of type $\frac{1}{2}(1,1,1)$ on the threefold Z. In a neighborhood of O_1, the birational morphism ν can be given by the equations

$$\tilde{x} = \hat{x}\hat{t}^{\frac{1}{3}},\ \tilde{z} = \hat{z}\hat{t}^{\frac{1}{3}},\ \tilde{t} = \hat{t}^{\frac{2}{3}},$$

where $\hat{x}$, $\hat{z}$ and $\hat{t}$ are weighted local coordinates on Z in a neighborhood of O_1 with weight 1. Thus in a neighborhood of the singular point O_1, the divisor R is given by the equation $\hat{t} = 0$, the divisor S_Z is given by $\hat{x} = 0$, the divisor E_Z does not pass through the point O_1, the divisor F_Z is given by $\bar{z} = 0$ and the divisor B_Z is given by the equation

$$\lambda\hat{x}^2 + \mu\Big(\epsilon_8\hat{t}\hat{z} + \epsilon_9\hat{z}^4\hat{t}^2 + \text{other terms}\Big) = 0,$$

which implies that $B_Z \sim_{\mathbb{Q}} 2S_Z$, the base locus of $\mathcal{B}_Z$ consists of C_Z, L_Z, L'_Z and the curve L'' that is given by the equations $\hat{x} = \hat{t} = 0$. The curve L'' is the unique curve in $|\mathcal{O}_{\mathbb{P}(1,1,2)}(1)|$ on the surface R. The surface B_Z is smooth at a generic point of L''. Therefore we obtain

$$S_Z \cdot B_Z = C_Z + 2L_Z + L'_Z + L'',\ E_Z \cdot B_Z = 2L_Z,\ F_Z \cdot B_Z = 2L'_Z,\ R \cdot B_Z = 2L'',$$

which gives

$$S_Z \cdot C_Z = 0,\quad S_Z \cdot L_Z = -\frac{1}{3},\quad S_Z \cdot L'_Z = -\frac{1}{10},\quad S_Z \cdot L'' = \frac{1}{2}$$

because
$$\begin{cases} S_Z \sim_{\mathbb{Q}} (\alpha\circ\beta\circ\nu)^*(-K_X)-\dfrac{1}{11}(\beta\circ\nu)^*(E)-\dfrac{1}{8}\nu^*(F)-\dfrac{1}{3}R,\\ B_Z \sim_{\mathbb{Q}} (\alpha\circ\beta\circ\nu)^*(-2K_X)-\dfrac{2}{11}(\beta\circ\nu)^*(E)-\dfrac{2}{8}\nu^*(F)-\dfrac{2}{3}R,\\ E_Z \sim_{\mathbb{Q}} (\beta\circ\nu)^*(E)-\dfrac{5}{8}\nu^*(F)-\dfrac{2}{3}R,\\ F_Z \sim_{\mathbb{Q}} \nu^*(F)-\dfrac{1}{3}R. \end{cases}$$

In particular, the curves L_Z and L'_Z are the only curves on the variety Z that have negative intersection with the divisor $-K_Z$.

Due to Lemma 2.3, either the set $\mathbb{CS}(Z,\frac{1}{n}\mathcal{M}_Z)$ contains the point O_1 or the log pair $(Z,\frac{1}{n}\mathcal{M}_Z)$ is terminal.

We first suppose that the log pair $(Z,\frac{1}{n}\mathcal{M}_Z)$ is not terminal. Then the set $\mathbb{CS}(Z,\frac{1}{n}\mathcal{M}_Z)$ must contain the point O_1. Let $\pi_2 : Z_2 \to Z$ be the Kawamata blow up at the point O_1 and H be the exceptional divisor of π_2. Then our local calculations imply that $B_{Z_2} \sim_{\mathbb{Q}} 2S_{Z_2}$ and the base locus of $\mathcal{B}_{Z_2}$ consists of the curves C_{Z_2}, L_{Z_2}, L'_{Z_2} and L''_{Z_2}. Furthermore, we have
$$S_{Z_2}\cdot B_T = C_{Z_2}+2L_{Z_2}+L'_{Z_2}+L''_{Z_2},\quad E_{Z_2}\cdot B_{Z_2}=2L_{Z_2},$$
$$F_{Z_2}\cdot B_{Z_2}=2L'_{Z_2},\quad R_{Z_2}\cdot B_{Z_2}=2L''_{Z_2},$$
which implies that
$$S_{Z_2}\cdot C_{Z_2}=0,\quad S_{Z_2}\cdot L_{Z_2}=-\frac{1}{3},\quad S_{Z_2}\cdot L'_{Z_2}=-\frac{3}{5},\quad S_{Z_2}\cdot L''_{Z_2}=0,$$
because
$$\begin{cases} S_{Z_2} \sim_{\mathbb{Q}} (\alpha\circ\beta\circ\nu\circ\pi_2)^*(-K_X)-\dfrac{1}{11}(\beta\circ\nu\circ\pi_2)^*(E)-\dfrac{1}{8}(\nu\circ\pi_2)^*(F)-\dfrac{1}{3}\pi_2^*R-\dfrac{1}{2}H,\\ E_{Z_2} \sim_{\mathbb{Q}} (\beta\circ\nu\circ\pi_2)^*(E)-\dfrac{5}{8}(\nu\circ\pi_2)^*(F)-\dfrac{2}{3}\pi_2^*R,\\ F_{Z_2} \sim_{\mathbb{Q}} (\nu\circ\pi_2)^*(F)-\dfrac{1}{3}\pi_2^*R-\dfrac{1}{2}H,\\ R_{Z_2} \sim_{\mathbb{Q}} \pi_2^*R-\dfrac{1}{2}H, \end{cases}$$

The curves L_{Z_2} and L'_{Z_2} are the only curves on the variety Z_2 that have negative intersection with the divisor $-K_{Z_2}$. Moreover we see
$$\Big(B_{Z_2}+(\beta\circ\nu\circ\pi_2)^*(-16K_U)+(\nu\circ\pi_2)^*(-18K_W)\Big)\cdot L_{Z_2}=0,$$
$$\Big(B_{Z_2}+(\beta\circ\nu\circ\pi_2)^*(-16K_U)+(\nu\circ\pi_2)^*(-18K_W)\Big)\cdot L'_{Z_2}=0,$$
and hence the divisor $D_{Z_2} := B_{Z_2}+(\beta\circ\nu\circ\pi_2)^*(-16K_U)+(\nu\circ\pi_2)^*(-18K_W)$ is nef and big because $-K_U$ and $-K_W$ are nef and big. Therefore we obtain
$$D_{Z_2}\cdot B_{Z_2}\cdot M_{Z_2}=0,$$
and hence $\mathcal{M}=\mathcal{B}$ by Theorem 2.2.

For now, we suppose that the log pair $(Z,\frac{1}{n}\mathcal{M}_Z)$ is terminal. We will derive a contradiction from this assumption, so that the set $\mathbb{CS}(Z,\frac{1}{n}\mathcal{M}_Z)$ must contain the point O_1.

The log pair $(Z,\epsilon B_Z)$ is terminal for some rational number $\epsilon>\frac{1}{2}$ but the divisor $K_Z+\epsilon B_Z$ has non-negative intersection with all curves on the variety Z except the curves L_Z and L'_Z. It follows from [11] that there is an isomorphism $\zeta : Z \dashrightarrow Z'$ of codimension 1 and the divisor $-K_{Z'}$ is nef. Then the singularities of the log pair $(Z',\frac{1}{n}\mathcal{M}_{Z'})$ are terminal because the singularities of the log pair $(Z,\frac{1}{n}\mathcal{M}_Z)$ are terminal and the rational map ζ is a log flop with respect to the log pair $(Z,\frac{1}{n}\mathcal{M}_Z)$.

We obtain

$$\begin{cases} S_Z \sim_{\mathbb{Q}} (\alpha\circ\beta\circ\nu)^*(-K_X) - \frac{1}{11}(\beta\circ\nu)^*(E) - \frac{1}{8}\nu^*(F) - \frac{1}{3}R \\ \quad \sim_{\mathbb{Q}} (\alpha\circ\beta\circ\nu)^*(-K_X) - \frac{1}{11}E_Z - \frac{2}{11}F_Z - \frac{5}{11}R, \\ S_Z^y \sim_{\mathbb{Q}} (\alpha\circ\beta\circ\nu)^*(-2K_X) - \frac{13}{11}(\beta\circ\nu)^*(E) - \frac{5}{8}\nu^*(F) - \frac{2}{3}R \\ \quad \sim_{\mathbb{Q}} (\alpha\circ\beta\circ\nu)^*(-2K_X) - \frac{13}{11}E_Z - \frac{15}{11}F_Z - \frac{21}{11}R, \\ S_Z^z \sim_{\mathbb{Q}} (\alpha\circ\beta\circ\nu)^*(-3K_X) - \frac{3}{11}(\beta\circ\nu)^*(E) - \frac{3}{8}\nu^*(F) - \frac{1}{3}R \\ \quad \sim_{\mathbb{Q}} (\alpha\circ\beta\circ\nu)^*(-3K_X) - \frac{3}{11}E_Z - \frac{6}{11}F_Z - \frac{4}{11}R, \\ S_Z^t \sim_{\mathbb{Q}} (\alpha\circ\beta\circ\nu)^*(-8K_X) - \frac{8}{11}(\beta\circ\nu)^*(E) \\ \quad \sim_{\mathbb{Q}} (\alpha\circ\beta\circ\nu)^*(-8K_X) - \frac{8}{11}E_Z - \frac{5}{11}F_Z - \frac{7}{11}R. \end{cases}$$

from

$$F_Z \sim_{\mathbb{Q}} \nu^*(F) - \frac{1}{3}R, \quad E_Z \sim_{\mathbb{Q}} (\beta\circ\nu)^*(E) - \frac{5}{8}\nu^*(F) - \frac{2}{3}R.$$

Thus the pull-backs of the rational functions $\frac{y}{x^2}$, $\frac{zy}{x^5}$ and $\frac{ty^3}{x^{14}}$ are contained in the linear systems $|2S_Z|$, $|5S_Z|$ and $|14S_Z|$, respectively. In particular, the complete linear system $|-70K_Z|$ induces a dominant rational map $Z \dashrightarrow \mathbb{P}(1,2,5,14)$. Thus the anticanonical divisor $-K_{Z'}$ is nef and big. It contradicts Theorem 2.1 because the log pair $(Z', \frac{1}{n}\mathcal{M}_{Z'})$ is terminal. □

Due to the lemma above, we may assume that the set $\mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W)$ contains the point Q_1. In particular, the set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ is not empty and each member of the linear system $\mathcal{M}_Y$ is contracted to a curve by the morphism η.
Let G be the exceptional divisor of γ. Then G contains two singular points Q_1' and Q_2' of Y that are quotient singularities of types $\frac{1}{2}(1,1,1)$ and $\frac{1}{3}(1,1,2)$, respectively. Then

$$\mathbb{CS}\Big(Y, \frac{1}{n}\mathcal{M}_Y\Big) \subseteq \Big\{Q_1',\ Q_2',\ \bar{Q}_2\Big\},$$

where $\bar{Q}_2$ is the point on Y whose image to W by γ is the point Q_2.
In a neighborhood of Q_1, the birational morphism β can be given by the equations

$$\bar{x} = \tilde{x}\tilde{z}^{\frac{1}{8}},\ \bar{z} = \tilde{z}\tilde{t}^{\frac{3}{8}},\ \bar{t} = \tilde{t}^{\frac{5}{8}},$$

where $\tilde{x}$, $\tilde{z}$ and $\tilde{t}$ are weighted local coordinates on W in a neighborhood of Q_1 such that $\mathrm{wt}(\tilde{x}) = 1$, $\mathrm{wt}(\tilde{z}) = 3$ and $\mathrm{wt}(\tilde{t}) = 2$. Thus in a neighborhood of the singular point Q_1, the divisor F is given by the equation $\tilde{t} = 0$, the divisor S_W is given by $\tilde{x} = 0$, the divisor E_W does not pass though the point Q_1 and the divisor B_W is given by the equation

$$\lambda\tilde{x}^2 + \mu\Big(\epsilon_8\tilde{t} + \epsilon_9\tilde{z}^8\tilde{t}^4 + \text{other terms}\Big) = 0.$$

Therefore $B_W \sim_{\mathbb{Q}} 2S_W$ and the base locus of $\mathcal{B}_W$ is the union of C_W, L_W and the curve L' that is given by the equations $\tilde{x} = \tilde{t} = 0$. We have

$$S_W\cdot B_W = C_W + 2L_W + L',\ E_W\cdot B_W = 2L_W,\ F\cdot B_W = 2L',$$

which gives us

$$S_W\cdot C_W = S_W\cdot L_U = 0,\quad S_W\cdot L' = \frac{1}{15}.$$

In a neighborhood of Q_2, the birational morphism γ can be given by the equations

$$\tilde{x} = \hat{x}\hat{t}^{\frac{1}{5}},\ \tilde{z} = \hat{z}^{\frac{3}{5}},\ \tilde{t} = \hat{t}\hat{z}^{\frac{2}{5}},$$

where $\hat{x}$, $\hat{z}$ and $\hat{t}$ are weighted local coordinates on Y in the neighborhood of Q_2' such that $\mathrm{wt}(\hat{x}) = 1$, $\mathrm{wt}(\hat{z}) = 1$ and $\mathrm{wt}(\hat{t}) = 2$. Thus in a neighborhood of the singular point Q_2', the divisor G is given by the equation $\hat{z} = 0$, the divisor S_Y is given by $\hat{x} = 0$, the divisor F_Y is given by the equation $\bar{t} = 0$ and the divisor B_Y is given by the equation

$$\lambda\hat{x}^2 + \mu\Big(\epsilon_8\hat{t} + \epsilon_9\hat{z}^6\hat{t}^2 + \text{other terms}\Big) = 0.$$

Thus $B_Y \sim_{\mathbb{Q}} 2S_Y$ and that the base locus of $\mathcal{B}_Y$ is the union of the irreducible curves C_Y, L_Y and L_Y'. We have

$$\begin{cases} S_Y \sim_{\mathbb{Q}} (\alpha\circ\beta\circ\gamma)^*(-K_X) - \dfrac{1}{11}(\beta\circ\gamma)^*(E) - \dfrac{1}{8}\gamma^*(F) - \dfrac{1}{5}G, \\ E_Y \sim_{\mathbb{Q}} (\beta\circ\gamma)^*(E) - \dfrac{5}{8}\gamma^*(F), \\ F_Y \sim_{\mathbb{Q}} \gamma^*(F) - \dfrac{2}{5}G, \end{cases}$$

and

$$S_Y\cdot C_Y = S_Y\cdot L_Y = S_Y\cdot L_Y' = 0,$$

which simply means that C_Y, L_Y and L_Y' are components of a fiber of η.

Lemma 6.24.
If the set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ contains Q_2', then $\mathcal{M} = \mathcal{B}$.

Proof. Let $\sigma_2 : Y_2 \to Y$ be the Kawamata blow up at the point Q_2' and let H_2 be the exceptional divisor of σ_2. Then our local calculations imply that $B_{Y_2} \sim_{\mathbb{Q}} 2S_{Y_2}$ and the base locus of $\mathcal{B}_{Y_2}$ is the union of curves C_{Y_2}, L_{Y_2} and L_{Y_2}'. Thus we have

$$S_{Y_2}\cdot B_{Y_2} = C_{Y_2} + 2L_{Y_2} + L_{Y_2}',\quad E_{Y_2}\cdot B_{Y_2} = 2L_{Y_2},\quad F_{Y_2}\cdot B_{Y_2} = 2L_{Y_2}',$$

which implies that

$$S_{Y_2}\cdot C_{Y_2} = 0,\quad S_{Y_2}\cdot L_{Y_2} = 0,\quad S_{Y_2}\cdot L_{Y_2}' = -\frac{1}{3}$$

because

$$\begin{cases} S_{Y_2} \sim_{\mathbb{Q}} (\alpha\circ\beta\circ\gamma\circ\sigma_2)^*(-K_X) - \dfrac{1}{11}(\beta\circ\gamma\circ\sigma_2)^*(E) - \dfrac{1}{8}(\gamma\circ\sigma_2)^*(F) - \dfrac{1}{5}\sigma_2^*(G) - \dfrac{1}{3}H_2, \\ E_{Y_2} \sim_{\mathbb{Q}} (\beta\circ\gamma\circ\sigma_2)^*(E) - \dfrac{5}{8}(\gamma\circ\sigma_2)^*(F), \\ F_{Y_2} \sim_{\mathbb{Q}} (\gamma\circ\sigma_2)^*(F) - \dfrac{2}{5}\sigma_2^*(G) - \dfrac{2}{3}H_2, \\ G_{Y_2} \sim_{\mathbb{Q}} \sigma_2^*(G) - \dfrac{1}{3}H_2, \end{cases}$$

The curve L_{Y_2}' is the only curve on Y_2 that has negative intersection with $-K_{Y_2}$. Moreover we have $(S_{Y_2} + (\gamma\circ\sigma_2)^*(-5K_W))\cdot L_{Y_2}' = 0$, which implies that the divisor $S_{Y_2} + (\gamma\circ\sigma_2)^*(-5K_W)$ is nef and big because $-K_W$ is nef and big. Therefore

$$(S_{Y_2} + (\gamma\circ\sigma_2)^*(-5K_W))\cdot B_{Y_2}\cdot M_{Y_2} = 0$$

by Lemma 2.2 and hence $\mathcal{M} = \mathcal{B}$ by Theorem 2.2. □

Lemma 6.25.
The set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ cannot contain the point Q_1'.

Proof. Suppose that the set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ contains the point Q_1'. Let $\sigma_1 : Y_1 \to Y$ be the Kawamata blow up at the point Q_1'.
Let $\mathcal{D}$ be a general pencil in the linear system $|-3K_X|$. Then the base curve of $\mathcal{D}$ is the curve $\bar{C}$ given by $x = z = 0$. Moreover the base locus of $\mathcal{D}_{Y_1}$ consists of the curve $\bar{C}_{Y_1}$. Thus we see that $\bar{C}_{Y_1} = S_{Y_1} \cdot D_{Y_1}$ for a general surface D_{Y_1} in $\mathcal{D}_{Y_1}$. On the other hand, we have $D_{Y_1} \cdot C_{Y_1} < 0$, which implies that $n = 3$ and $\mathcal{M}_{Y_1} = \mathcal{D}_{Y_1}$ by Theorem 2.2. However, $D_{Y_1} \not\sim_{\mathbb{Q}} -3K_{Y_1}$. □

Consequently, we may assume that the set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ consists of the point $\bar{Q}_2$ whose image to W is the point Q_2. It implies that $\mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W) = \{Q_1, Q_2\}$. We have $\mathcal{M}_V \sim_{\mathbb{Q}} -nK_V$ by Lemma 2.2.
Let H be the exceptional divisor of ξ. Then it follows from the local computations made during the proof of Lemma 6.23 that $B_V \sim_{\mathbb{Q}} 2S_V$. The base locus of $\mathcal{B}_V$ is the union of the irreducible curves C_V, L_V, L_V' and the curve L'' such that $L'' = S_V \cdot H$. We have

$$S_V \cdot C_V = 0, \quad S_V \cdot L_V = -\frac{1}{3}, \quad S_V \cdot L_V' = -\frac{1}{6}, \quad S_V \cdot L'' = \frac{1}{2}.$$

Let $\bar{O}$ be the singular point of V that is contained in H. Then $\omega(\bar{O})$ is the singular point of Z contained in the exceptional divisor of ν. It follows from Lemma 2.3 that either the set $\mathbb{CS}(V, \frac{1}{n}\mathcal{M}_V)$ contains the point $\bar{O}$ or the log pair $(V, \frac{1}{n}\mathcal{M}_V)$ is terminal.
Suppose that the set $\mathbb{CS}(V, \frac{1}{n}\mathcal{M}_V)$ contains the point $\bar{O}$. Then the set $\mathbb{CS}(Z, \frac{1}{n}\mathcal{M}_Z)$ contains the point $\omega(\bar{O})$. The proof of Lemma 6.23 shows that $\mathcal{M} = \mathcal{B}$ if the set $\mathbb{CS}(Z, \frac{1}{n}\mathcal{M}_Z)$ contains the point $\omega(\bar{O}) = O_1$.
From now, we suppose that the singularities of the log pair $(V, \frac{1}{n}\mathcal{M}_V)$ are terminal. The singularities of the log pair $(V, \epsilon B_V)$ are log-terminal for some rational number $\epsilon > \frac{1}{2}$ but the divisor $K_V + \epsilon B_V$ has non-negative intersection with all curves on the variety V except the curves L_V and L_V'. Then there is an isomorphism $\chi : V \dashrightarrow V'$ of codimension 1 and the divisor $-K_{V'}$ is nef. Hence the linear system $|-rK_{V'}|$ is base-point-free for $r \gg 0$ by the log abundance theorem ([8]).
It follows from the proof of Lemma 6.23 that the pull-backs of the rational functions $\frac{y}{x^2}$ and $\frac{zy}{x^5}$ are contained in the linear systems $|2S_V|$ and $|5S_V|$, respectively. In particular, the complete linear system $|-10K_V|$ induces a dominant rational map $V \dashrightarrow \mathbb{P}(1, 2, 5)$, which implies that the linear system $|-rK_{V'}|$ induces a dominant morphism to a surface. In fact, the linear system $|-rK_{V'}|$ induces the morphism υ. The singularities of the log pair $(V', \frac{1}{n}\mathcal{M}_{V'})$ are terminal because the singularities of the log pair $(V, \frac{1}{n}\mathcal{M}_V)$ are terminal and the rational map χ is a log flop with respect to the log pair $(V, \frac{1}{n}\mathcal{M}_V)$. However, the singularities of the log pair $(V', \frac{1}{n}\mathcal{M}_{V'})$ cannot be terminal by Theorem 2.1. We have obtained a contradiction.
Summing up, we have proved:

Proposition 6.10.
The linear system $|-2K_X|$ is a unique Halphen pencil on X.

$\daleth = 79$: **Hypersurface of degree** 33 **in** $\mathbb{P}(1, 3, 5, 11, 14)$**.**

The threefold X is a general hypersurface of degree 33 in $\mathbb{P}(1, 3, 5, 11, 14)$ with $-K_X^3 = \frac{1}{70}$. It has two singular points. One is a quotient singularity of type $\frac{1}{5}(1, 1, 4)$ and the other is a quotient singularity O of type $\frac{1}{14}(1, 3, 11)$. The hypersurface X can be given by the equation

$$w^2 z + w f_{19}\big(x, y, z, t\big) + f_{33}\big(x, y, z, t\big) = 0,$$

where f_i is a quasihomogeneous polynomial of degree i. Let $\mathcal{P}$ be the pencil cut out on X by

$$\lambda x^5 + \mu z = 0,$$

where $(\lambda : \mu) \in \mathbb{P}^1$.

There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & W \arrow[l, "\beta"'] & Y \arrow[l, "\gamma"'] \arrow[d, "\eta"] \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,3,5), \end{tikzcd}$$

where

- ψ is the natural projection,
- α is the Kawamata blow up at the point O with weights $(1,3,11)$,
- β is the Kawamata blow up with weights $(1,3,8)$ at the singular point of type $\frac{1}{11}(1,3,8)$ contained in the exceptional divisor of the birational morphism α,
- γ is the Kawamata blow up with weights $(1,3,5)$ at the singular point of type $\frac{1}{8}(1,3,5)$ contained in the exceptional divisor of the birational morphism β,
- η is an elliptic fibration.

If the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M}_X)$ contains the singular point of type $\frac{1}{5}(1,1,4)$, then $\mathcal{M} = |-3K_X|$ by Lemma 4.3. Therefore we may assume that $\mathbb{CS}\left(X, \frac{1}{n}\mathcal{M}\right) = \{O\}$ due to Lemma 4.1 and Corollary 4.3.
The exceptional divisor E of α contains two quotient singular points P and Q of types $\frac{1}{3}(1,1,2)$ and $\frac{1}{11}(1,3,8)$, respectively.

Lemma 6.26.

The set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ consists of the point Q.

Proof. Suppose that $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U) \neq \{Q\}$. Then the set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ contains the point P. Let $\pi_P : U_P \to U$ be the Kawamata blow up at P with weights $(1,1,2)$. Then $\mathcal{M}_{U_P} \sim_{\mathbb{Q}} -nK_{U_P}$ by Lemma 2.2.
Let $\mathcal{D}$ be the proper transforms of $|-11K_X|$ on the threefold U_P and D be a general surface of the linear system $\mathcal{D}$. Then the base locus of the linear system $\mathcal{D}$ does not contain curves, which implies that the divisor D is nef. Thus we obtain an absurd inequality

$$0 \leq D \cdot M_1 \cdot M_2 = -\frac{n^2}{5},$$

where M_1 and M_2 are general surfaces of the pencil $\mathcal{M}_{U_P}$. □

The exceptional divisor F of the birational morphism β contains two singular points Q_1 and Q_2 that are quotient singularities of types $\frac{1}{3}(1,1,2)$ and $\frac{1}{8}(1,3,5)$ respectively.

Lemma 6.27.

If the set $\mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W)$ contains the point Q_1, then $\mathcal{M} = \mathcal{P}$.

Proof. Suppose that the set $\mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W)$ contains the point Q_1. Let $\pi : W_1 \to W$ be the Kawamata blow up of Q_1 with weights $(1,1,2)$ and G be its exceptional divisor. Then $\mathcal{M}_{W_1} \sim_{\mathbb{Q}} -nK_{W_1}$ by Lemma 2.2.
Let $\mathcal{L}$ be the linear system on the hypersurface X cut out by

$$\lambda_0 x^{30} + \lambda_1 y^{10} + \lambda_2 z^6 + \lambda_3 t^2 x^8 + \lambda_4 t^2 y^2 x^2 + \lambda_5 t y^6 x + \lambda_6 w t z = 0$$

where $(\lambda_0 : \cdots : \lambda_6) \in \mathbb{P}^6$. Then the base locus of $\mathcal{L}$ does not contain curves. Then it follows from simple calculations that the base locus of the linear system $\mathcal{L}_{W_1}$ does not contain any curve and for a general surface B in $\mathcal{L}$, we obtain

$$B_{W_1} \sim_{\mathbb{Q}} \left(\alpha\circ\beta\circ\pi\right)^*\Big(-30K_X\Big) - \frac{30}{14}(\beta\circ\pi)^*(E) - \frac{8}{11}\pi^*(F) - \frac{2}{3}G.$$

In particular, the divisor B_{W_1} is nef and big.

Let M be a general surface of the pencil $\mathcal{M}_{W_1}$ and D be a general surface of the linear system $|-5K_{W_1}|$. Then $B_{W_1}\cdot M\cdot D = 0$, which implies that $\mathcal{M} = \mathcal{P}$ by Theorem 2.2 because the linear system $|-5K_{W_1}|$ is the proper transform of the pencil $\mathcal{P}$. □

Proposition 6.11.
A general surface in the pencil $\mathcal{P}$ is birational to a smooth K3 surface.

Proof. We use the same notations in the proof of Lemma 6.27. The surface G is isomorphic to the projective space $\mathbb{P}(1,1,2)$. Let T be a general surface in the pencil $\mathcal{P}$ and let $\Delta = G\cdot T_{W_1}$. Then by simple calculation, we see that the curve Δ on G is defined by the equation

$$\epsilon_1\bar{x}^5 + \epsilon_2\bar{x}^2\bar{y}\bar{t} + \epsilon_3\bar{x}\bar{y}^2\bar{t} + \epsilon_4\bar{y}\bar{t}^2 = 0 \subset \mathrm{Proj}(\mathbb{C}[\bar{x},\bar{y},\bar{t}]) = \mathbb{P}(1,1,2),$$

where each ϵ_i is a general complex number. It has two nodes at the points $(1:0:0)$ and $(0:1:0)$. But it is smooth at the point $(0:0:1)$ which is a $\mathbb{A}_1$ singular point of the surface G. Let $\tilde{G}$ be the blow up of the surface G at these three points. The genus of the normalization $\tilde{\Delta}$ of the curve Δ is

$$p_g(\tilde{\Delta}) = \frac{(K_{\tilde{G}}+\tilde{\Delta})\cdot\tilde{\Delta}}{2} + 1 = \frac{(K_G+\Delta)\cdot\Delta - \frac{1}{2}}{2} + 1 - 2 = 0,$$

and hence the curve Δ is a rational curve not contained in the base locus of the pencil $\mathcal{P}_{W_1}$. Therefore Corollary 2.2 completes the proof. □

We may assume that $\mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W) = \{Q_2\}$ due to Theorem 2.1 and Lemma 2.3. Let O_1 and O_2 be the quotient singular points of the threefold Y contained in the exceptional divisor of γ that are of types $\frac{1}{3}(1,1,2)$ and $\frac{1}{5}(1,3,2)$, respectively. Then

$$\varnothing \neq \mathbb{CS}\Big(Y, \frac{1}{n}\mathcal{M}_Y\Big) \subset \Big\{O_1, O_2\Big\}$$

by Theorem 2.1, Lemmas 2.2 and 2.3. The proof of Lemma 6.25 implies that the $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ does not contain the point O_1. Now the proofs of Lemma 6.21 shows $\mathcal{M} = |-3K_X|$.

Therefore we have proved:

Proposition 6.12.
The linear systems $|-3K_X|$ and $\mathcal{P}$ are the only Halphen pencils on X.

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