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Log canonical thresholds of certain Fano hypersurfaces

Ivan Cheltsov · Jihun Park · Joonyeong Won

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Abstract We study log canonical thresholds on quartic threefolds, quintic fourfolds, and double spaces. As an important application, we show that they have Kähler–Einstein metrics if they are general.

Keywords Anticanonical linear system · Fano variety · Kähler–Einstein metric · Log canonical threshold

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1 Introduction

All varieties are defined over $\mathbb{C}$.

1.1 Introduction

The multiplicity of a nonzero polynomial $f \in \mathbb{C}[z_1, \dots, z_n]$ at a point $P \in \mathbb{C}^n$ is the nonnegative integer m such that $f \in \mathfrak{m}_P^m \backslash \mathfrak{m}_P^{m+1}$, where $\mathfrak{m}_P$ is the maximal ideal of polynomials vanishing at the point P in $\mathbb{C}[z_1, \dots, z_n]$. It can be also defined by derivatives. The multiplicity

I. Cheltsov (✉)
School of Mathematics, The University of Edinburgh, Edinburgh EH9 3JZ, UK
e-mail: I.Cheltsov@ed.ac.uk

J. Park
IBS Center for Geometry and Physics, Institute for Basic Science and Department of Mathematics, POSTECH, Pohang 790-784, Gyeongbuk, Korea
e-mail: wlog@postech.ac.kr

J. Won
KIAS, 85 Hoegiro, Dongdaemun-gu, Seoul 130-722, Korea
e-mail: leonwon@kias.re.kr

of f at the point P is the number

$$\mathrm{mult}_P(f) = \min\left\{m \,\middle|\, \frac{\partial^m f}{\partial^{m_1} z_1 \partial^{m_2} z_2 \dots \partial^{m_n} z_n}(P) \neq 0\right\}.$$

On the other hand, we have a similar invariant that is defined by integrations. This invariant, which is called the log canonical threshold of f at the point P, is given by

$$c_P(f) = \sup\left\{c \,\middle|\, |f|^{-c} \text{ is locally } L^2 \text{ near the point } P \in \mathbb{C}^n\right\}.$$

This number appears in many places. For instance, the log canonical threshold of the polynomial f at the origin is the same as the absolute value of the largest root of the Bernstein-Sato polynomial of f.

Even though log canonical threshold was implicitly known and extensively studied under different names by J. H. M. Steenbrink, A. Varchenko and so forth, it was formally introduced to birational geometry by Shokurov in [31] as follows. Let X be a $\mathbb{Q}$-factorial variety with at worst log canonical singularities, $Z \subset X$ a closed subvariety, and D an effective $\mathbb{Q}$-divisor on X. The log canonical threshold of D along Z is the number

$$c_Z(X, D) = \sup\left\{c \,\middle|\, \text{the log pair } (X, cD) \text{ is log canonical along } Z\right\}.$$

For the case $Z = X$ we use the notation $c(X, D)$ instead of $c_X(X, D)$. Because log canonicity is a local property, we see that

$$c_Z(X, D) = \inf_{P \in Z} \{c_P(X, D)\}.$$

If $X = \mathbb{C}^n$ and $D = (f = 0)$, then we also use the notation $c_0(f)$ for the log canonical threshold of D at the origin.

Even though several methods have been invented in order to compute log canonical thresholds, it is not easy to compute them in general. However, many problems in birational geometry are related to log canonical thresholds. The log canonical thresholds play a significant role in the study on birational geometry. They show many interesting properties (see [10–12, 17, 19–21, 23–26]).

We occasionally find it useful to consider the smallest value of log canonical thresholds of effective divisors linearly equivalent to a given divisor, in particular, an anticanonical divisor (for instance, see [23]).

Definition 1.1 Let X be a $\mathbb{Q}$-factorial Fano variety with at worst log terminal singularities. For a natural number $m > 0$, we define the m-th global log canonical threshold of X by the number

$$\mathrm{lct}_m(X) = \inf\left\{c\left(X, \frac{1}{m}H\right) \,\middle|\, H \in \left|-mK_X\right|\right\}.$$

Note that the number $\mathrm{lct}_m(X)$ is defined to be ∞ if the linear system $|-mK_X|$ is empty. Also, we define the global log canonical threshold of X by the number

$$\mathrm{lct}(X) = \inf_{n \in \mathbb{N}} \left\{\mathrm{lct}_m(X)\right\}.$$

We can immediately see

$$\mathrm{lct}(X) = \sup\left\{c \,\middle|\, \begin{array}{l}\text{the log pair } (X, cD) \text{ is log canonical for}\\ \text{every effective } \mathbb{Q}\text{-divisor } D \text{ with } D \equiv -K_X\end{array}\right\}.$$

To see the simplest case, let S be a smooth del Pezzo surface. It follows from [5, Theorem 1.7] and [23, Section 3] that

$$\mathrm{lct}(S) = \mathrm{lct}_1(S) = \begin{cases} 1/3 & \text{when } S \cong \mathbb{F}_1 \text{ or } K_S^2 \in \{7, 9\}, \\ 1/2 & \text{when } S \cong \mathbb{P}^1 \times \mathbb{P}^1 \text{ or } K_S^2 \in \{5, 6\}, \\ 2/3 & \text{when } K_S^2 = 4, \\ 2/3 & \text{when } S \text{ is a cubic in } \mathbb{P}^3 \text{ with an Eckardt point}, \\ 3/4 & \text{when } S \text{ is a cubic in } \mathbb{P}^3 \text{ without Eckardt points}, \\ 3/4 & \text{when } K_S^2 = 2 \text{ and } |-K_S| \text{ has a tacnodal curve}, \\ 5/6 & \text{when } K_S^2 = 2 \text{ and } |-K_S| \text{ has no tacnodal curves}, \\ 5/6 & \text{when } K_S^2 = 1 \text{ and } |-K_S| \text{ has a cuspidal curve}, \\ 1 & \text{when } K_S^2 = 1 \text{ and } |-K_S| \text{ has no cuspidal curves}. \end{cases} \tag{1.1}$$

For a quasismooth hypersurface X in $\mathbb{P}(a_0, \dots, a_4)$ of degree $\sum_{i=0}^4 a_i - 1$, where $a_0 \le \cdots \le a_4$, one can find $\mathrm{lct}(X) > \frac{3}{4}$ for 1,936 values of $(a_0, a_1, a_2, a_3, a_4)$ (see [16, Corollary 3.4]). Moreover, for a quasismooth hypersurface X in $\mathbb{P}(1, a_1, \dots, a_4)$ of degree $\sum_{i=1}^4 a_i$ having terminal singularities, there are exactly 95 possible quadruples (a_1, a_2, a_3, a_4) found in [14] and [16]. It follows from [4, Theorem 1.3] that $\mathrm{lct}(X) = 1$ if

$$(a_1, a_2, a_3, a_4) \notin \Big\{ (1, 1, 1, 1), (1, 1, 1, 2), (1, 1, 2, 2), (1, 1, 2, 3) \Big\}$$

and the hypersurface X is sufficiently general.

It is proved that the global log canonical threshold of a rational homogeneous space of Picard rank 1 and Fano index r is $\frac{1}{r}$ (see [13, Theorem 2]).

Example 1.2 Let X be a smooth hypersurface of degree $n \ge 3$ in $\mathbb{P}^n$. Then

$$\mathrm{lct}_m(X) \ge \frac{n-1}{n}$$

due to [3, Theorem 1.3] and [6, Theorem 3.3]. Furthermore, $\mathrm{lct}_m(X) = \frac{n-1}{n}$ if and only if X contains a cone of dimension $n-2$ (see [3, Conjecture 1.5], [6, Corollary 4.10], and [11, Theorem 0.2]). The inequality obviously implies that

$$\mathrm{lct}(X) \ge \frac{n-1}{n}.$$

However, it is shown that $\mathrm{lct}(X) = 1$ if X is general and $n \ge 6$ (see [30, Theorem 2]).

From the results of [30], it is natural to expect the following:

Conjecture 1.3 *The global log canonical thresholds of a general quartic threefold and a general quintic fourfold are 1.*

This conjecture has been proposed for canonical thresholds in [30, Conjecture 2].

For an evidence of the conjecture, we can consider the first global log canonical threshold of a general hypersurface. It is not hard to show that the first global log canonical threshold of a general hypersurface of degree $n \ge 4$ in $\mathbb{P}^n$ is one (see Proposition 2.1). In the case of smooth quartic threefolds, we can find all the first global log canonical thresholds (see Proposition 2.2).

For the global log canonical thresholds, we prove the following:

Theorem 1.4 *Let X be a general hypersurface of degree $n = 4$ or 5 in $\mathbb{P}^n$. Then*

$$\mathrm{lct}\,(X) \geq \begin{cases} \frac{7}{9} & \textit{for } n = 4; \\ \\ \frac{5}{6} & \textit{for } n = 5. \end{cases}$$

The global log canonical threshold of a Fano variety is an algebraic counterpart of the α-invariant introduced in [32]. One of the most interesting applications of the global log canonical thresholds of Fano varieties is the following result proved in [9, p. 549] (see also [22] and [32]).

Theorem 1.5 *Let X be an d-dimensional Fano variety with at most quotient singularities. The variety X has an orbifold Kähler–Einstein metric if the inequality*

$$\mathrm{lct}\,(X) > \frac{d}{d+1}$$

holds.

The inequality in Example 1.2 is not strong enough to apply Theorem 1.5 to a smooth hypersurface of degree n in $\mathbb{P}^n$. However, we see that (1.1) enables Theorem 1.5 to imply the existence of a Kähler–Einstein metric on a general cubic surface and that [30, Theorem 2] enables Theorem 1.5 to imply the existence of a Kähler–Einstein metric on a general hypersurface of degree $n \geq 6$ in $\mathbb{P}^n$. Even though Theorem 1.4 is much weaker than Conjecture 1.3, they are strong enough to imply the existence of a Kähler–Einstein metric. Consequently, we can obtain the following:

Corollary 1.6 *A general hypersurface of degree $n \geq 2$ in $\mathbb{P}^n$ has a Kähler–Einstein metric.*

In fact, a smooth conic in $\mathbb{P}^2$ has a Kähler–Einstein metric because it is isomorphic to $\mathbb{P}^1$ and the Fubini–Study metric of a projective space is Kähler–Einstein. Furthermore, a smooth cubic surface always admits a Kähler–Einstein metric (see [33, Section 2]). Meanwhile, it is proved that a Kähler–Einstein metric exists on a smooth hypersurface in $\mathbb{P}^n$ defined by a homogeneous polynomial equation of the form $z_0^n + z_1^n + f_n(z_2, \ldots, z_n) = 0$, where $n \geq 4$ and f_n is a homogeneous polynomial of degree n in variables $z_2, \ldots, z_n$ (see [1, Proposition 3.1]).

Also, in this paper, we will study log canonical thresholds on double spaces, *i.e.*, double covers of $\mathbb{P}^n$, and obtain similar results as what we have on Fano hypersurfaces in $\mathbb{P}^n$. For instance, we will prove that the first global log canonical threshold of a smooth double space is equal to its global log canonical threshold (see Proposition 3.2) and that every smooth double cover of $\mathbb{P}^n$ ramified along a hypersurface of degree $2n$ admits a Kähler–Einstein metric.

Let us close this section by a conjecture inspired by [34, Question 1].

Conjecture 1.7 *For a smooth Fano variety X, $lct(X) = lct_m(X)$ for some natural number $m \geq 1$.*

2 Log canonical threshold of a Fano hypersurface

2.1 Hypersurface of degree n in $\mathbb{P}^n$

As we mentioned, one can consider the first global log canonical threshold of a general hypersurface of degree $n \geq 4$ in $\mathbb{P}^n$ in behalf of Conjecture 1.3.

Proposition 2.1 *Let X be a general hypersurface of degree $n \geq 4$ in $\mathbb{P}^n$. Then* $\mathrm{lct}_1(X) = 1$.

Proof Consider the space $\mathcal{S}_n = \mathbb{P}^n \times \mathbb{P}\left(\mathrm{H}^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(n)\right)\right)$ with the natural projections $p\colon \mathcal{S}_n \to \mathbb{P}^n$ and $q\colon \mathcal{S}_n \to \mathbb{P}\left(\mathrm{H}^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(n))\right)$. Put $\mathcal{I}_n = \left\{(O, F) \in \mathcal{S}_n \,\middle|\, F(O) = 0 \text{ and } F = 0 \text{ is smooth.}\right\}$.

Let (O, F) be a pair in $\mathcal{I}_n$. Suppose that $O = [1 : 0 : \cdots : 0]$. Then F can be given by a polynomial of the form $z_0^{n-1}z_n + z_0^{n-2}q_2(z_1, \ldots, z_n) + \cdots + z_0 q_{n-1}(z_1, \ldots, z_n) + q_n(z_1, \ldots, z_n) \in \mathbb{C}[z_0, \ldots, z_n]$, where q_i is a homogeneous polynomial of degree i.

We say that the point O is *bad* on the hypersurface $F = 0$ if one of the following condition holds:

(1) $q_2(z_1, \ldots, z_{n-1}, 0) \equiv 0$.
(2) $q_2(z_1, \ldots, z_{n-1}, 0) = \{l(z_1, \ldots, z_{n-1})\}^2$ for some linear form $l(z_1, \ldots, z_{n-1})$ and if we assume $l(z_1, \ldots, z_{n-1}) = z_{n-1}$, either $q_3(z_1, \ldots, z_{n-2}, 0, 0) \equiv 0$ or $q_3(z_1, \ldots, z_{n-2}, 0, 0) = \{m(z_1, \ldots, z_{n-2})\}^3$ for some linear form $m(z_1, \ldots, z_{n-2})$.

Then consider a subset of $\mathcal{I}_n$,

$$\mathcal{Y}_n = \left\{(O, F) \in \mathcal{I}_n \,\middle|\, \text{the point } O \text{ is bad on the quartic } F = 0\right\}.$$

One can see that for a given point P on $\mathbb{P}^n$, the dimension of $p^{-1}(P) \cap \mathcal{Y}_n$ is strictly smaller than $h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(n)) - (n + 1)$, and hence the dimension of the space $\mathcal{Y}_n$ is smaller than the dimension of $\mathbb{P}\left(\mathrm{H}^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(n)\right)\right)$. Therefore, the image of the regular map $q|_{\mathcal{Y}_n}\colon \mathcal{Y}_n \to \mathbb{P}\left(\mathrm{H}^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(n)\right)\right)$ is a proper closed subset of $\mathbb{P}\left(\mathrm{H}^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(n)\right)\right)$. So, a general hypersurface of degree n in $\mathbb{P}^n$ has no bad point.

Let X be a general hypersurface of degree n in $\mathbb{P}^n$ and H be a divisor in $|-K_X|$. We claim that the pair (X, H) is log canonical at every point P on X. By a suitable coordinate change we may assume that the point $P = [1 : 0 : \cdots : 0]$. We may also assume that the hypersurface X is defined by the equation

$$z_0^{n-1}z_n + z_0^{n-2}q_2(z_1, \ldots, z_n) + \cdots + z_0 q_{n-1}(z_1, \ldots, z_n) + q_n(z_1, \ldots, z_n) = 0,$$

where q_i is a homogeneous polynomial of degree i. Unless the hyperplane section H is given by the tangent hyperplane at the point P, the divisor H is smooth at the point P, and hence the pair (X, H) is log canonical at the point P. Now we suppose that H is given by the tangent hyperplane T at the point P. The hyperplane T in $\mathbb{P}^n$ is defined by $z_n = 0$ in our case. Since both X and T are smooth and $H = T \cap X$, we obtain $c_P(X, H) = c_P(T, H)$ from [11, Theorem 3.1]. Furthermore, $c_P(T, H) = c_0(f)$, where $f = q_2(z_1, \ldots, z_{n-1}, 0) + \cdots + q_{n-1}(z_1, \ldots, z_{n-1}, 0) + q_n(z_1, \ldots, z_{n-1}, 0)$. Since P is not a bad point on X, the polynomial $q_2(z_1, \ldots, z_{n-1}, 0)$ is not zero polynomial. If the rank of the quadratic polynomial $q_2(z_1, \ldots, z_{n-1}, 0)$ is at least 2, then $c_0(f) = 1$ by [17, Lemma 8.10 (8.10.3)]. If the rank of the quadratic polynomial $q_2(z_1, \ldots, z_{n-1}, 0)$ is 1, we may assume that $q_2(z_1, \ldots, z_{n-1}, 0) = z_{n-1}^2$. Consider the polynomial f with weights $\mathrm{wt}(z_1) = \cdots = \mathrm{wt}(z_{n-2}) = 2$, $\mathrm{wt}(z_{n-1}) = 3$. The leading term of f with respect to the weights is $f_w = z_{n-1}^2 + q_3(z_1, \ldots, z_{n-2}, 0, 0)$. Since the polynomial $\widetilde{q_3} = q_3(z_1, \ldots, z_{n-2}, 0, 0)$ is neither zero polynomial nor a cube of a linear polynomial, we obtain $c_0(\widetilde{q_3}) \geq \frac{1}{2}$, and hence $c_0(f_w) = \max\{\frac{1}{2} + c_0(\widetilde{q_3}), 1\}$=1. By [19, Proposition 2.1], we have $c_0(f) \geq c_0(f_w) = 1$. Therefore, the pair (X, H) is log canonical at every point on X. Consequently, $\mathrm{lct}_1(X) = 1$. ◻

For smooth quartic threefolds, one can compute all the possible first global log canonical thresholds by studying normal quartic surfaces. Here we only list them and the brief idea to compute them as follows:

Proposition 2.2 *Let X be a smooth quartic threefold in $\mathbb{P}^4$. The first global log canonical threshold* $\mathrm{lct}_1(X)$ *is one of the following:*

$$\left\{\frac{3}{4}, \frac{29}{36}, \frac{22}{27}, \frac{5}{6}, \frac{16}{19}, \frac{17}{20}, \frac{6}{7}, \frac{13}{15}, \frac{37}{42}, \frac{7}{8}, \frac{8}{9}, \frac{9}{10}, \frac{23}{26}, \frac{11}{12}, \frac{12}{13}, \frac{13}{14}, \frac{14}{15}, \frac{15}{16}, \frac{31}{34}, \frac{17}{18}, \frac{21}{22}, \frac{23}{24}, \frac{29}{30}, \frac{41}{42}, 1\right\}.$$

Furthermore, for each number μ in the set above, there is a smooth quartic threefold X with $\mathrm{lct}_1(X) = \mu$.

Its proof goes as follows. A divisor $S \in |-K_X|$ is given by the intersection of X and a hyperplane H in $\mathbb{P}^4$. Because the log canonical threshold $c(X, S)$ is equal to the log canonical threshold $c(H, S)$ (see [11, Theorem 3.1]), the result above can be obtained by investigating log canonical thresholds of normal quartic surfaces H in $\mathbb{P}^3$. Note that a hyperplane section of a smooth hypersurface in $\mathbb{P}^n$, $n \geq 4$ is normal and that a normal hypersurface in $\mathbb{P}^{n-1}$ can be attained by a hyperplane section of a smooth hypersurface in $\mathbb{P}^n$ (see [15]). Let S be a normal surface in $\mathbb{P}^3$ defined by a homogeneous quartic polynomial F. We suppose that S has a singular point at [0:0:0:1]. We then consider the log pair $(\mathbb{C}^3, D)$, where D is the fourth affine piece of S that is defined by the polynomial $f(x, y, z) = F(x, y, z, 1)$. Since log canonical thresholds can be computed locally, it is enough to study the log pair $(\mathbb{C}^3, D)$ instead of (X, S). For the detail of the computation, see [35].

Before we prove Theorem 1.4, let us explain our generality condition. Let X_d be a hypersurface of degree d in $\mathbb{P}^n$, $d \geq n \geq 4$. Let P be an arbitrary point on X_d. By suitable coordinate changes, we assume that $P = [1 : 0 : \cdots : 0]$. Then the hypersurface X_d is defined by

$$z_0^{n-1}q_1(z_1, \ldots, z_n) + z_0^{n-2}q_2(z_1, \ldots, z_n) + \cdots + z_0 q_{n-1}(z_1, \ldots, z_n) + q_d(z_1, \ldots, z_n) = 0,$$

where q_i are homogeneous polynomials of degrees i in variables $z_1, \ldots, z_n$.

Definition 2.3 The hypersurface X_d is said to be k-regular at the point P, where $0 \leq k \leq d$, if the homogenous polynomials

$$q_1, q_2, \ldots, q_k$$

form a regular sequence in $\mathbb{C}[z_1, \ldots, z_n]$. The hypersurface X_d is said to be k-regular if it is k-regular everywhere.

Proposition 2.4 *A general hypersurface of degree n in $\mathbb{P}^n$ is $(n-1)$-regular.*

Proof See [28, Proposition 1]. □

To prove Theorem 1.4 we need a linear system on X_d that has a big multiplicity at a given point but a small base locus. Put

$$f_i(z_0, \ldots, z_n) = \sum_{j=1}^{i} z_0^{i-j} q_j(z_1, \ldots, z_n)$$

for each $1 \leq i \leq d$.

Definition 2.5 The m-th hypertangent linear system $\mathcal{M}$ at the point P is the linear subsystem of $|\mathcal{O}_{X_d}(m)|$ consisting of the divisors cut by hypersurfaces

$$\sum_{i=1}^{m} f_i(z_0, \ldots, z_n) p_{m-i}(z_1, \ldots, z_n) = 0,$$

where $p_j(z_1, \ldots, z_n)$ is a homogeneous polynomial of degree j.

Note that $mult_P(M) \geq m + 1$ for each divisor M in the m-th hypertangent linear system on X_d.

Lemma 2.6 *Suppose that the hypersurface X_d is $(n-1)$-regular at a point P. Then the following hold.*

1. *There are finitely many lines (possibly none) on X_d passing through the point P.*
2. *The base locus of the $(n-1)$-th hypertangent linear system $\mathcal{M}$ at the point P consists of lines passing through the point P on X_d.*

Proof There is a one-to-one correspondence between the set of lines passing through the point P and the zero locus of the polynomials $q_1 = \cdots = q_d = 0$ in $\mathbb{P}^{n-1}$. Since the homogeneous polynomials $q_1, \ldots, q_{n-1}$ form a regular sequence in $\mathbb{C}[z_1, \ldots, z_n]$, they defines a finite set in $\mathbb{P}^{n-1}$. This proves the first assertion.

The base locus of the linear system $\mathcal{M}$ is defined by the equations $f_1 = \cdots = f_{n-1} = 0$. Therefore, it is cut out by the equations $q_1 = q_2 = \cdots = q_{n-1} = 0$. This shows the second assertion. □

We close this section by the following useful lemma.

Lemma 2.7 *Let X be a smooth hypersurface of degree n in $\mathbb{P}^n$ and D be an effective $\mathbb{Q}$-divisor numerically equivalent to $-K_X$. For a non-negative number $\lambda \leq 1$, there is a point $P \in X$ such that $(X, \lambda D)$ is log canonical on $X \setminus P$.*

Proof The log pair $(X, \lambda D)$ is log canonical in the outside of finitely many points of the smooth hypersurface X (see [27, Theorem 2] or [28, Section 3]). Suppose that there are two points at which the log pair $(X, \lambda D)$ is not log canonical. Then for sufficiently small $\epsilon > 0$ the log pair $(X, (\lambda - \epsilon)D)$ is not log canonical at the two points either. Since the divisor $-(K_X + (\lambda - \epsilon)D)$ is nef and big, it follows from from the connectedness principle of Shokurov (see [18, Theorem 17.4]) that the locus of non-Kawamata log terminal singularities of the log pair $(X, (\lambda - \epsilon)D)$ is connected. This is a contradiction. □

2.2 General quartic

Let X be a smooth quartic hypersurface in $\mathbb{P}^4$ such that the following general conditions hold:

- the threefold X is 3-regular;
- every line on the hypersurface X has normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}$;
- the intersection of X with a two-dimensional linear subspace of $\mathbb{P}^4$ cannot be a double conic curve.

Remark 2.8 A line on the quartic X has normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}$ if and only if no two-dimensional linear subspace of $\mathbb{P}^4$ is tangent to the quartic X along the line (see [7, Theorem 1.9]).

Remark 2.9 It follows from Proposition 2.2 and [6] that $\mathrm{lct}_1(X) \geq \frac{7}{9}$. To avoid the long proof of Proposition 2.2, we can use instead Proposition 2.1 by adding extra generality conditions.

Remark 2.10 Let B and B' be effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors on a variety V. Then

$$\left(V,\ \alpha B + (1-\alpha)\, B'\right)$$

is log canonical if both (V, B) and (V, B') are log canonical, where $0 \leq \alpha \leq 1$.

Let us prove Theorem 1.4 for the case $n = 4$. Put $\lambda = \frac{7}{9}$. Let D be an effective $\mathbb{Q}$-divisor on X such that $D \equiv -K_X$. To prove Theorem 1.4, we have to show that $(X, \lambda D)$ is log canonical.

Suppose that $(X, \lambda D)$ is not log canonical. Due to Remarks 2.9 and 2.10, we may assume that $D = \frac{1}{n}R$ where R is an irreducible divisor with $R \sim -nK_X$ for some natural number $n > 1$. By Lemma 2.7, the log pair $(X, \lambda D)$ is not log canonical only at a single point P.

The threefold X can be given by

$$v^3x + v^2q_2(x, y, z, u) + vq_3(x, y, z, u) + q_4(x, y, z, u) = 0 \subset \mathbb{P}^4 \cong \mathrm{Proj}\left(\mathbb{C}\left[x, y, z, u, v\right]\right),$$

where $q_i(x, y, z, u)$ is a homogeneous polynomial of degree i. Furthermore, we may assume that the point P is located at $[0 : 0 : 0 : 0 : 1]$. Let T be the surface on X cut out by $x = 0$.

Lemma 2.11 *The multiplicity of D at the point P is at most 2.*

Proof The statement immediately follows from the inequalities

$$4 = H \cdot T \cdot D \geq \mathrm{mult}_P(T \cap D) \geq \mathrm{mult}_P(T)\,\mathrm{mult}_P(D) \geq 2\mathrm{mult}_P(D),$$

where H is a general hyperplane section of X passing through the point P. □

Let $\pi : U \to X$ be the blow up at the point P with the exceptional divisor E. Then

$$\bar{D} \equiv \pi^*(D) - \mathrm{mult}_P(D)\, E,$$

where $\bar{D}$ is the proper transform of the divisor D via the morphism π.

It follows from [8, Corollary 3.5] or [30, Proposition 3] that there is a line $L \subset E$ such that

$$\mathrm{mult}_P(D) + \mathrm{mult}_L\left(\bar{D}\right) > \frac{2}{\lambda}.$$

Recall that E is isomorphic to $\mathbb{P}^2$.

Let $\mathcal{L}$ be the linear system of hyperplane sections of X such that

$$S \in \mathcal{L} \iff \text{either } L \subset \bar{S} \text{ or } S = T,$$

where $\bar{S}$ is the proper transform of S via the birational morphism π. There is a two-dimensional linear subspace $\Pi \subset \mathbb{P}^4$ such that the base locus of $\mathcal{L}$ consists of the intersection $\Pi \cap X$.

Let S be a general surface in $\mathcal{L}$. Then S is a smooth K3 surface. Put

$$T_S = T\big|_S = \sum_{i=1}^{r} Z_i,$$

where each Z_i is an irreducible curve. The generality conditions imply that the curve T_S is reduced (see Remark 2.8). Then $\sum_{i=1}^{r} \deg(Z_i) = 4$. It follows that

$$\mathrm{mult}_P(S \cap D) \geq \mathrm{mult}_P(S)\,\mathrm{mult}_P(D) + \mathrm{mult}_L\left(\bar{S} \cap \bar{D}\right) \geq \mathrm{mult}_P(D) + \mathrm{mult}_L\left(\bar{D}\right) > \frac{2}{\lambda}.$$

Put

$$D_S = D\big|_S = \sum_{i=1}^{r} m_i Z_i + \Delta,$$

where m_i is a non-negative rational number and Δ is an effective one-cycle on S whose support does not contain the curves $Z_1, \ldots, Z_r$. Then

$$\sum_{i=1}^{r} m_i \mathrm{mult}_P\,(Z_i) + \mathrm{mult}_P\,(\Delta) = \mathrm{mult}_P\,(D_S) > \frac{2}{\lambda},$$

and the support of the cycle Δ does not contain any component of the cycle T_S. We have

$$\begin{aligned} 4 = T_S \cdot D_S &= \sum_{i=1}^{r} m_i \deg\,(Z_i) + T_S \cdot \Delta \\ &\geq \sum_{i=1}^{r} m_i \deg\,(Z_i) + \mathrm{mult}_P(T_S)\left(\frac{2}{\lambda} - \sum_{i=1}^{r} m_i \mathrm{mult}_P\,(Z_i)\right). \end{aligned}$$

Remark 2.12 The equality $m_i = \mathrm{mult}_{Z_i}(D)$ holds for every i because $X|_\Pi$ is reduced.

It follows from the 3-regularity of X that $\mathrm{mult}_P(T_S) \leq 3$.

Lemma 2.13 *Suppose that* $\mathrm{mult}_P(T_S) = 3$. *Then*

$$16 > \frac{12}{\lambda} + \deg\,(Z_k)\, m_k,$$

for Z_k that is not a line passing through the point P.

Proof Let $\bar{T}$ be the proper transform of the surface T via the birational morphism π. Then

$$3 = \mathrm{mult}_P(T_S) = \mathrm{mult}_P\,(T \cap S) = \mathrm{mult}_P\,(T)\,\mathrm{mult}_P\,(S) + \mathrm{mult}_L\left(\bar{T} \cap \bar{S}\right).$$

Hence, we see that $L \subset \bar{T}$. Since $\mathrm{mult}_P(D) > \frac{1}{\lambda}$ and $\mathrm{mult}_P(T) = 2$, it follows that

$$\mathrm{mult}_P\,(T \cap D) \geq \mathrm{mult}_P\,(T)\,\mathrm{mult}_P\,(D) + \mathrm{mult}_L\left(\bar{T} \cap \bar{D}\right) \geq 2\mathrm{mult}_P\,(D) + \mathrm{mult}_L\left(\bar{D}\right) > \frac{3}{\lambda}.$$

Let $L_1, \ldots, L_m$ be all the lines on X that pass through the point P. Put

$$T \cap D = \sum_{i=1}^{m} \epsilon_i L_i + \bar{m}_k Z_k + \Upsilon,$$

where ϵ_i and $\bar{m}_k$ are non-negative rational numbers, and Υ is an effective one-cycle on X whose support does not contain the lines $L_1, \ldots, L_m$. Then $\bar{m}_k \geq m_k$ by Remark 2.12.

Taking the intersection with a general hyperplane section of X, we see that

$$4 \geq \sum_{i=1}^{r} \epsilon_i + \bar{m}_k \deg\,(Z_k),$$

but $\bar{m}_k \mathrm{mult}_P(Z_k) + \mathrm{mult}_P(\Upsilon) > \frac{3}{\lambda} - \sum_{i=1}^{r} \epsilon_i$.

Take a general member M in the third hypertangent linear system $\mathcal{M}$ at the point P. Note that the base locus of $\mathcal{M}$ consists of the lines $L_1, \dots, L_m$ by Lemma 2.6. Hence, we have

$$12 = M \cdot T \cdot D \geq 3 \sum_{i=1}^{r} \epsilon_i + M \cdot (\bar{m}_k Z_k + \Upsilon)$$
$$> 3 \sum_{i=1}^{r} \epsilon_i + 4 \left(\frac{3}{\lambda} - \sum_{i=1}^{r} \epsilon_i \right) = \frac{12}{\lambda} - \sum_{i=1}^{r} \epsilon_i.$$

This implies $16 > 12/\lambda + \deg(Z_k) m_k$ since $4 \geq \sum_{i=1}^{r} \epsilon_i + \bar{m}_k \deg(Z_k)$ and $\bar{m}_k \geq m_k$. □

From now on, in order to describe the reduced curve T_S, we will use the following notations:

- C : an irreducible cubic not passing through the point P.
- $\widetilde{C}$: an irreducible cubic that is smooth at the point P.
- $\widehat{C}$: an irreducible cubic that is singular at the point P.

For $i = 1, 2$

- Q_i : an irreducible quadric not passing through the point P.
- $\widetilde{Q}_i$: an irreducible quadric passing through the point P.

For $i = 1, 2, 3, 4$

- L_i : a line not passing through the point P.
- $\widetilde{L}_i$: a line passing through the point P.

Then, the following are all the possible configuration of T_S. In each case, we derive a contradictory inequality from our assumptions so that the log pair $(X, \lambda D)$ should be log canonical. To obtain a contradictory inequality for each case, we start from the inequality

$$4 = T_S \cdot D_S = \sum m_i Z_i \cdot T_S + T_S \cdot \Delta \geq \sum m_i \deg(Z_i) + \mathrm{mult}_P(T_S)\mathrm{mult}_P(\Delta)$$
$$> \sum m_i \deg(Z_i) + \mathrm{mult}_P(T_S) \left(\frac{2}{\lambda} - \sum m_i \mathrm{mult}_P(Z_i) \right),$$

and then we show that the number

$$A := \sum m_i \deg(Z_i) + \mathrm{mult}_P(T_S) \left(\frac{2}{\lambda} - \sum m_i \mathrm{mult}_P(Z_i) \right)$$

is greater than 4.

CASE A The curve T_S is an irreducible quartic curve.

1. $\mathrm{mult}_P(T_S) = 2$.
   $D_S = mT_S + \Delta$.
   A contradictory inequality:
   $$A = 4m + 2 \left(\frac{2}{\lambda} - 2m \right) = \frac{4}{\lambda} > 4.$$
2. $\mathrm{mult}_P(T_S) = 3$.
   $D_S = mT_S + \Delta$.
   An auxiliary inequality:
   $$16 > \frac{12}{\lambda} + 4m \quad \text{by Lemma 2.13.}$$

A contradictory inequality:

$$A = 4m + 3\left(\frac{2}{\lambda} - 3m\right) = \frac{6}{\lambda} - 5m > \frac{6}{\lambda} - 5\left(4 - \frac{3}{\lambda}\right) = \frac{21}{\lambda} - 20 > 4.$$

CASE B The curve T_S is reducible and contains no line passing through the point P.

1. $T_S = \widehat{C} + L_1$.
   $D_S = m\widehat{C} + m_1 L_1 + \Delta$.
   An auxiliary inequality :

$$1 = L_1 \cdot D_S \geq 3m - 2m_1.$$

   A contradictory inequality:

$$A = 3m + m_1 + 2\left(\frac{2}{\lambda} - 2m\right) = \frac{4}{\lambda} + m_1 - m \geq \frac{4}{\lambda} + m_1 - \frac{1 + 2m_1}{3} > 4.$$

2. $T_S = \widetilde{Q_1} + \widetilde{Q_2}$.
   $D_S = m_1\widetilde{Q_1} + m_2\widetilde{Q_2} + \Delta$.
   A contradictory inequality:

$$A = 2m_1 + 2m_2 + 2\left(\frac{2}{\lambda} - m_1 - m_2\right) = \frac{4}{\lambda} > 4.$$

CASE C The curve T_S contains a unique line passing through the point P.

1. $T_S = \widetilde{Q_1} + \widetilde{L_1} + L_2$.
   $D_S = m\widetilde{Q_1} + m_1\widetilde{L_1} + m_2 L_2 + \Delta$.
   Auxiliary inequalities:

$$\left.\begin{array}{l} 2 = Q_1 \cdot D_S \geq -2m + 2m_1 + 2m_2 \\ 1 = L_2 \cdot D_S \geq 2m + m_1 - 2m_2 \end{array}\right\} \Rightarrow 1 \geq m_1$$

   A contradictory inequality:

$$A = 2m + m_1 + m_2 + 2\left(\frac{2}{\lambda} - m - m_1\right) = \frac{4}{\lambda} + m_2 - m_1 \geq \frac{4}{\lambda} + m_2 - 1 > 4.$$

2. $T_S = \widetilde{C} + \widetilde{L_1}$.
   $D_S = m\widetilde{C} + m_1\widetilde{L_1} + \Delta$.
   An auxiliary inequality:

$$3 = \widetilde{C} \cdot D_S \geq 3m_1.$$

   A contradictory inequality:

$$A = 3m + m_1 + 2\left(\frac{2}{\lambda} - m - m_1\right) = \frac{4}{\lambda} + m - m_1 \geq \frac{4}{\lambda} + m - 1 > 4.$$

3. $T_S = \widehat{C} + \widetilde{L_1}$.
   $D_S = m\widehat{C} + m_1\widetilde{L_1} + \Delta$.
   Auxiliary inequalities:

$$\begin{array}{l} 3 = \widehat{C} \cdot D_S \geq 3m_1 \\ 16 > \frac{12}{\lambda} + 3m \quad \text{by Lemma 2.13} \end{array}$$

A contradictory inequality:

$$A = \frac{6}{\lambda} - 3m - 2m_1 \geq \frac{6}{\lambda} - 2 - 3m > \frac{6}{\lambda} - 2 - \left(16 - \frac{12}{\lambda}\right) = \frac{18}{\lambda} - 18 > 4.$$

CASE D The curve T_S contains two lines passing through the point P.

1. $T_S = \widetilde{Q_1} + \widetilde{L_1} + \widetilde{L_2}$.
   $D_S = m\widetilde{Q_1} + m_1\widetilde{L_1} + m_2\widetilde{L_2} + \Delta$,
   Auxiliary inequalities:

$$\begin{aligned} &2 = \widetilde{Q_1} \cdot D_S \geq -2m + 2m_1 + 2m_2 \\ &16 > \tfrac{12}{\lambda} + 2m \quad \text{by Lemma 2.13.} \end{aligned}$$

   A contradictory inequality:

$$A = \frac{6}{\lambda} - m - 2m_1 - 2m_2 \geq \frac{6}{\lambda} - m - 2(1+m) > \frac{6}{\lambda} - 2 - 3\left(8 - \frac{6}{\lambda}\right) = \frac{24}{\lambda} - 26 > 4.$$

2. $T_S = \widetilde{L_1} + \widetilde{L_2} + L_3 + L_4$.
   $D_S = m_1\widetilde{L_1} + m_2\widetilde{L_2} + m_3L_3 + m_4L_4 + \Delta$, where we may assume that $m_3 \geq m_4$.
   An auxiliary inequality:

$$1 = L_4 \cdot D_S \geq m_1 + m_2 + m_3 - 2m_4$$

   A contradictory inequality:

$$\begin{aligned} A = \frac{4}{\lambda} + m_3 + m_4 - (m_1 + m_2) &\geq \frac{4}{\lambda} + m_3 + m_4 - m_2 - (1 - m_2 - m_3 + 2m_4) \\ &\geq \frac{4}{\lambda} + 2m_3 - m_4 - 1 > 4. \end{aligned}$$

3. $T_S = Q_1 + \widetilde{L_1} + \widetilde{L_2}$.
   $D_S = mQ_1 + m_1\widetilde{L_1} + m_2\widetilde{L_2} + \Delta$.
   An auxiliary inequality:

$$2 = Q_1 \cdot D_S \geq -2m + 2m_1 + 2m_2 \Rightarrow 1 + m \geq m_1 + m_2.$$

   A contradictory inequality:

$$A = \frac{4}{\lambda} + 2m - m_1 - m_2 \geq \frac{4}{\lambda} + m - 1 > 4.$$

CASE E The curve T_S contains three lines passing through the point P.

1. $T_S = \widetilde{L_1} + \widetilde{L_2} + \widetilde{L_3} + L_4$.
   $D_S = m_1\widetilde{L_1} + m_2\widetilde{L_2} + m_3\widetilde{L_3} + mL_4 + \Delta$.
   Auxiliary inequalities:

$$\begin{aligned} &1 = L_4 \cdot D_S \geq -2m + m_1 + m_2 + m_3 \\ &16 > \tfrac{12}{\lambda} + m \quad \text{by Lemma 2.13.} \end{aligned}$$

   A contradictory inequality:

$$\begin{aligned} A = \frac{6}{\lambda} + m - 2(m_1 + m_2 + m_3) \geq \frac{6}{\lambda} + m - 2(1 + 2m) &> \frac{6}{\lambda} - 2 - 3\left(16 - \frac{12}{\lambda}\right) \\ &= \frac{42}{\lambda} - 50 = 4. \end{aligned}$$

Therefore, Theorem 1.4 for $n = 4$ has been proved.

2.3 General quintic

In this section, we prove Theorem 1.4 for $n = 5$.

Let X be a quintic hypersurface in $\mathbb{P}^5$ such that the following generality conditions hold:

G 1. The hypersurface X is 4-regular;
G 2. For every 3-dimensional linear space Π in $\mathbb{P}^5$, the intersection $X \cap \Pi$ is irreducible and reduced;
G 3. For each point $P \in X$ and each 3-dimensional linear space Π contained in the tangent hyperplane at P and containing the point P, if the surface $Z := X \cap \Pi$ has multiplicity two at the point P, then it satisfies the following:

G 3.0. The surface Z
- G 3.0.1. cannot be singular along a line passing through the point P ;
- G 3.0.2. cannot contain four lines passing through the point P.

G 3.1. If Z contains only one line L passing through the point P,
- G 3.1.1. then the line L meets its residual curve by a general hyperplane section in Π either at at least one smooth point or at at least two ordinary double points.

G 3.2. If Z contains only two lines, L_1 and L_2, passing through the point P, then
- G 3.2.1. it has at most four singular points on $L_1 \cup L_2$;
- G 3.2.2. if it has four singular points on L_i, then all of them are ordinary double points;
- G 3.2.3. if it has three singular points on L_i, then two of them are ordinary double points;
- G 3.2.4. if it has exactly three singular points on the line L_i, then the line L_i meets its residual curve by a general hyperplane section in Π at one smooth point;
- G 3.2.5. if it has exactly two singular points on the line L_i, then either P is a non-ordinary double point and the line L_i meets its residual curve by a general hyperplane section in Π at two smooth points, or the point P is an ordinary double point and the line L_i meets its residual curve by a general hyperplane section in Π at at least one smooth point.
- G 3.2.6. if it has no singular point other than P on the line L_i, then the line L_i meets its residual curve by a general hyperplane section in Π at at least two smooth points.

G 3.3. If Z contains three lines, L_1, L_2 and L_3, passing through the point P,
- G 3.3.1. if the three lines are coplanar, then it is smooth on $(L_1 \cup L_2 \cup L_3) \setminus \{P\}$ and each line L_i meets its residual curve by a general hyperplane section in Π at four points;
- G 3.3.2. if the three lines are not coplanar, then either P is a non-ordinary double point, the surface Z is smooth at every point of $(L_1 \cup L_2 \cup L_3) \setminus \{P\}$ and each line L_i meets its residual curve by a general hyperplane section in Π at four points, or P is an ordinary double point and each line L_i meets its residual curve by a general hyperplane section in Π at two smooth points.

Lemma 2.14 *A general quintic hypersurface X in $\mathbb{P}^5$ satisfies the condition G 2.*

Proof This follows directly from [2, Theorem 5.1]. □

Lemma 2.15 *A general quintic hypersurface X in $\mathbb{P}^5$ satisfies the condition G 3.*

Proof See Appendix. □

Put $\lambda = \frac{5}{6}$. Let D be an effective $\mathbb{Q}$-divisor on X such that $D \equiv -K_X$. We claim that the log pair $(X, \lambda D)$ is log canonical.

Suppose that the log pair $(X, \lambda D)$ is not log canonical. As in the case of quartic threefolds, we may assume that $D = \frac{1}{n}R$ where R is an irreducible divisor with $R \sim -nK_X$ for some natural number n. Furthermore, the following lemma enables us to assume $n > 1$. We may use Proposition 2.1 with extra generality conditions in order to assume $n > 1$ without the aid of the lemma.

Lemma 2.16 *If a quintic hypersurface Y in $\mathbb{P}^5$ is 4-regular, then* $\mathrm{lct}_1(Y) = 1$.

Proof The main idea of the proof is the same as that of Proposition 2.2. It is enough to prove $c_0(f) = 1$ for a quintic polynomial $f(x, y, z, u) \in \mathbb{C}[x, y, z, u]$ obtained from the quintic polynomial defining the quintic Y. Using the 4-regular condition we can derive enough monomials from the polynomial f to have $c_0(f) = 1$. We omit the detailed computation. For the detail, see [35]. □

It follows from Lemma 2.7 that there is a point $P \in X$ such that the log pair $(X, \lambda D)$ is log canonical on $X \setminus P$. Therefore, the log pair $(X, \lambda D)$ is not log canonical only at the point P.

By suitable coordinate changes, we may assume that $P = [0 : 0 : 0 : 0 : 0 : 1]$ and that the fourfold X is given by an equation

$$w^4 x + \sum_{i=2}^{5} w^{5-i} q_i\left(x, y, z, u, v\right) = 0 \subset \mathbb{P}^5 \cong \mathrm{Proj}\left(\mathbb{C}\big[x, y, z, u, v, w\big]\right),$$

where $q_i(x, y, z, u, v)$ is a homogeneous polynomial of degree i. Let T be the threefold on X cut by $x = 0$.

Let $\pi : U \to X$ be the blow up at the point P with the exceptional divisor E. Then

$$\bar{D} \equiv \pi^*\left(D\right) - \mathrm{mult}_P\left(D\right) E,$$

where $\bar{D}$ is the proper transform of the divisor D via the morphism π. Note that $\mathrm{mult}_P(D) > \frac{1}{\lambda}$. It follows from [30, Proposition 3] that either $\mathrm{mult}_P(D) > \frac{2}{\lambda}$ or there is a plane $\Omega \subset E \cong \mathbb{P}^3$ such that

$$\mathrm{mult}_P\left(D\right) + \mathrm{mult}_\Omega\left(\bar{D}\right) > \frac{2}{\lambda}.$$

In the case when $\mathrm{mult}_P(D) > \frac{2}{\lambda}$, let $\mathcal{L}$ be a sufficiently general pencil of hyperplane sections of X that pass through the point P. In the case when $\mathrm{mult}_P(D) \leq \frac{2}{\lambda}$, let $\mathcal{L}$ be the pencil of hyperplane sections of X such that

$$S \in \mathcal{L} \iff \text{either}\, \Omega \subset \bar{S} \,\text{or}\, S = T,$$

where $\bar{S}$ is the proper transform of S via the birational morphism π. In both the cases, there is a three-dimensional linear subspace $\Pi \subset \mathbb{P}^5$ such that the base locus of $\mathcal{L}$ consists of the intersection $\Pi \cap X$.

Let S be a general threefold in $\mathcal{L}$. Then $S \neq T$ and $\mathrm{mult}_P(S \cap D) > \frac{2}{\lambda}$.

Put $Z = X|_\Pi$. The surface Z is reduced and irreducible because X contains neither quadric surfaces nor planes by our initial assumption. The 4-regularity of X implies that $\mathrm{mult}_P(Z) \leq 3$.

Lemma 2.17 *The multiplicity of Z at the point P is 3.*

Proof Suppose that $\mathrm{mult}_P(Z) \leq 2$. Let $\mathcal{M}$ be the 4th hypertangent linear system at the point P and let M be a general member in $\mathcal{M}$. The base locus of $\mathcal{M}$ consists of finitely many lines on X that pass through the point P.

Put $D \cap S = mZ + \Upsilon$, where m is a non-negative rational number and Υ is a 2-cycle whose support does not contain the surface Z. Then $\mathrm{mult}_P(\Upsilon) > \frac{2}{\lambda} - 2m$ but T does not contain components of Υ. We therefore have

$$\mathrm{mult}_P(T \cap \Upsilon) > \frac{4}{\lambda} - 4m.$$

We then consider the one cycle $T \cap \Upsilon$. We may write

$$T \cap \Upsilon = \sum_{i=1}^{k} \alpha_i L_i + \Delta,$$

where L_i is a line contained in Z and passing through the point P and the support of Δ contain none of the lines L_i's. We have

$$M \cdot \Delta = M \cdot \left(T \cdot D \cdot S - mT \cdot Z - \sum_{i=1}^{k} \alpha_i L_i \right) = 20 - 20m - 4 \sum_{1=1}^{k} \alpha_i$$

and

$$M \cdot \Delta \geq \mathrm{mult}_P(M)\, \mathrm{mult}_P(\Delta) > \frac{20}{\lambda} - 20m - 5 \sum_{i=1}^{k} \alpha_i,$$

and hence

$$4 = \frac{20}{\lambda} - 20 < \sum_{i=1}^{k} \alpha_i.$$

On the other hand, using our generality condition G3, we obtain the opposite inequality $\sum_{i=1}^{k} \alpha_i \leq 4$ case by case as follows, so that we could conclude that $\mathrm{mult}_P(Z)$=3.

By our generality condition, we have $k \leq 3$. Note that we may regard $T \cap \Upsilon$ as a divisor in $|\mathcal{O}_Z(1-m)|$ on the quintic surface $Z \subset \Pi \cong \mathbb{P}^3$ since $T \cap S = Z$. For each line L_j we consider the hyperplane section A_j of Z by a general hyperplane in Π passing through the line L_j. The divisor A_j on the surface Z consists of the line L_j and the residual curve C_j. On the surface Z, we have

$$\sum_{i=1}^{k} \alpha_i C_j \cdot L_i \leq C_j \cdot \left(\sum_{i=1}^{k} \alpha_i L_i + \Delta \right) = 4(1-m) \leq 4. \tag{2.1}$$

On the surface Z, the local intersection number of C_i and L_j at an ordinary double point of Z is well-defined and it is at least $\frac{1}{2}$ if these two curves intersect there. The local intersection number of C_i and L_j at a smooth point of Z is at least 1 if these two curves intersect there.

CASE $k = 1$.
G3.1.1 implies $1 \leq C_1 \cdot L_1$. Then the inequality (2.1) implies $\alpha_1 \leq 4$.

CASE $k = 2$.
First we suppose that neither L_1 nor L_2 contains exactly two singular points of Z. Then it follows from the conditions G 3.2.1, 3.2.2, 3.2.3, 3.2.4, 3.2.6 that $2 \leq C_j \cdot L_j$. This implies that $\alpha_j \leq 2$, and hence $\alpha_1 + \alpha_2 \leq 4$.

Now we suppose that L_1 contains exactly two singular points of Z. One of them are the point P.

Suppose that P is a non-ordinary double point. Then

$$2\alpha_1 \leq C_1 \cdot (\alpha_1 L_1 + \alpha_2 L_2) \leq 4$$

by G 3.2.5. Thus, we have $\alpha_1 \leq 2$. On the other hand, it follows from G 3.2.3, G 3.2.4, G 3.2.5 and G 3.2.6 that

$$2\alpha_2 \leq C_2 \cdot (\alpha_1 L_1 + \alpha_2 L_2) \leq 4,$$

which implies that $\alpha_2 \leq 2$. Then $\alpha_1 + \alpha_2 \leq 4$.

Suppose now that the point P is an ordinary double point. We obtain from G 3.2.5 that

$$\frac{3}{2}\alpha_1 + \frac{1}{2}\alpha_2 \leq C_1 \cdot (\alpha_1 L_1 + \alpha_2 L_2) \leq 4.$$

On the other hand, regardless of the number of the singular points on L_2, we see

$$\frac{1}{2}\alpha_1 + \frac{3}{2}\alpha_2 \leq C_2 \cdot (\alpha_1 L_1 + \alpha_2 L_2) \leq 4$$

by G 3.2.3, G 3.2.4, G 3.2.5 and G 3.2.6, since the point P is an ordinary double point. These imply that $\alpha_1 + \alpha_2 \leq 4$.

CASE $k = 3$.
Suppose that the three lines are coplanar. Then G 3.3.1 shows that for each $j = 1, 2, 3$ we have $3 \leq C_j \cdot L_j$, and hence $\alpha_j \leq \frac{4}{3}$. Therefore, we obtain $\alpha_1 + \alpha_2 + \alpha_3 \leq 4$.

Suppose that the three lines are not coplanar. We have to consider two cases: when the point P is an ordinary double point, and when the point P is not an ordinary double point.

Suppose that P is not an ordinary double point. Then G 3.3.2 shows that for each $j = 1, 2, 3$ we have $3 \leq C_j \cdot L_j$, and hence $\alpha_j \leq \frac{4}{3}$. Therefore, we obtain $\alpha_1 + \alpha_2 + \alpha_3 \leq 4$.

Suppose that P is an ordinary double point. Then G 3.3.2 shows that for each i and j,

$$C_j \cdot L_i \geq \frac{1}{2} + 2\delta_{ij},$$

where δ_{ij} is the Kronecker-delta function, i.e., $\delta_{ij} = 1$ if $i = j$; $\delta_{ij} = 0$ if $i \neq j$. Then the inequality (2.1) implies that for each $1 \leq j \leq 3$, we have

$$\frac{1}{2}(\alpha_1 + \alpha_2 + \alpha_3) + 2\alpha_j \leq \sum_{i=1}^{3} \alpha_i C_j \cdot L_i \leq 4,$$

and hence

$$3(\alpha_1 + \alpha_2 + \alpha_3) \leq \frac{7}{2}(\alpha_1 + \alpha_2 + \alpha_3) \leq \sum_{j=1}^{3}\sum_{i=1}^{3} \alpha_i C_j \cdot L_i \leq 12,$$

which implies that $\alpha_1 + \alpha_2 \leq 4$. This completes the proof. □

Let $\bar{T}$ be the proper transform of T via the birational morphism π. Then

$$3 = \mathrm{mult}_P(Z) = \mathrm{mult}_P\,(T \cap S) = \mathrm{mult}_P\,(T)\,\mathrm{mult}_P\,(S) + \mathrm{mult}_\Omega\left(\bar{T} \cap \bar{S}\right),$$

which implies that $\Omega \subset \bar{T}$. Since $\mathrm{mult}_P(D) > \frac{1}{\lambda}$ and $\mathrm{mult}_P(T) = 2$, it follows that

$$\mathrm{mult}_P\,(T \cap D) \geq \mathrm{mult}_P\,(T)\,\mathrm{mult}_P\,(D) + \mathrm{mult}_\Omega\left(\bar{T} \cap \bar{D}\right) \geq 2\mathrm{mult}_P\,(D) + \mathrm{mult}_\Omega\left(\bar{D}\right) > \frac{3}{\lambda}.$$

Now we restrict everything to a general hyperplane section of the fourfold X. Let H be a general hyperplane in $\mathbb{P}^5$ passing through the point P. Put

$$\tilde{X} = H \cap X, \quad \tilde{T} = H \cap T, \quad \tilde{S} = H \cap S, \quad \tilde{D} = H \cap D, \quad \tilde{Z} = H \cap Z, \quad \tilde{\Upsilon} = H \cap \Upsilon.$$

Let $\tilde{P} = [0 : 0 : 0 : 0 : 1]$. The threefold $\tilde{X}$ is 4-regular at the point $\tilde{P}$. The divisor $\tilde{D}$ is equivalent to $\mathcal{O}_{\mathbb{P}^4}(1)|_{\tilde{X}}$.

We have

$$\mathrm{mult}_{\tilde{P}}\left(\tilde{T}\right) = 2, \quad \mathrm{mult}_{\tilde{P}}(\tilde{Z}) = 3, \quad \mathrm{mult}_{\tilde{P}}\left(\tilde{T} \cap \tilde{D}\right) > \frac{3}{\lambda}, \quad \mathrm{mult}_{\tilde{P}}\left(\tilde{S} \cap \tilde{D}\right) > \frac{2}{\lambda}.$$

The intersection $\tilde{T} \cap \tilde{S}$ consists of the irreducible reduced curve $\tilde{Z}$. Put

$$\tilde{T} \cap \tilde{D} = \bar{m}\tilde{Z} + \Delta,$$

where $\bar{m}$ is a non-negative rational number and Δ is an effective one-cycle on $\tilde{X}$ whose support does not contain the curve $\tilde{Z}$. Then, $\mathrm{mult}_{\tilde{P}}(\Delta) > \frac{3}{\lambda} - 3\bar{m}$.

Let $\mathcal{N}$ be the third hypertangent linear system at the point $\tilde{P}$. Lemma 2.6 shows that the base locus of $\mathcal{N}$ does not contain any curves because the threefold $\tilde{X}$ contains no lines passing through the point $\tilde{P}$. Hence, for a general member N in $\mathcal{N}$ we have

$$15 = N \cdot \tilde{T} \cdot \tilde{D} \geq 15\bar{m} + N \cdot \Delta > 15\bar{m} + 4\left(\frac{3}{\lambda} - 3\bar{m}\right),$$

which implies $15 > \frac{12}{\lambda} + 3\bar{m}$. Since $\tilde{D} \cap \tilde{S} = m\tilde{Z} + \tilde{\Upsilon}$ and $\mathrm{mult}_{\tilde{P}}(\tilde{\Upsilon}) > \frac{2}{\lambda} - 3m$, on the surface $\tilde{S}$ we have

$$5 - 5m = \tilde{Z} \cap \tilde{\Upsilon} > \mathrm{mult}_{\tilde{P}}\left(\tilde{Z}\right)\mathrm{mult}_{\tilde{P}}\left(\tilde{\Upsilon}\right) > 3\left(\frac{2}{\lambda} - 3m\right).$$

Thus, we see that $4m > \frac{6}{\lambda} - 5$.

The curve $\tilde{Z}$ is reduced and $\tilde{S}$ is a sufficiently general hyperplane section of $\tilde{X}$ that contains the curve $\tilde{Z}$. Thus, we have

$$m = \mathrm{mult}_{\tilde{Z}}(\tilde{D}) \leq \mathrm{mult}_{\tilde{Z}}(\tilde{T} \cap \tilde{D}) = \bar{m},$$

which implies

$$15 \geq \frac{12}{\lambda} + 3m > \frac{12}{\lambda} + \frac{3}{4}\left(\frac{6}{\lambda} - 5\right).$$

It contradicts $\lambda = \frac{5}{6}$.

The obtained contradiction completes the proof of Theorem 1.4.

3 Log canonical threshold on a double space

3.1 Generalized global log canonical threshold

The Picard group of a smooth Fano hypersurface of degree $n \geq 4$ of $\mathbb{P}^n$ is generated by an anticanonical divisor. Therefore, it is natural that we consider only plurianticanonical divisors when we define its global log canonical threshold. However, in other varieties, it may not be enough. Therefore, we generalizes the global log canonical threshold as follows:

Definition 3.1 Let X be a $\mathbb{Q}$-factorial variety with at worst log canonical singularities. For an integral divisor D on the variety X and a natural number $m > 0$, we define the m-th global log canonical threshold of the divisor D by the number

$$\mathrm{lct}_m\left(X, D\right) = \inf\left\{c\left(X, \frac{1}{m}H\right) \;\Big|\; H \in \left|mD\right|\right\},$$

where the number $\mathrm{lct}_m(X, D)$ is defined to be ∞ if the linear system $|mD|$ is empty. Also, we define the global log canonical threshold of D by the number

$$\mathrm{lct}(X, D) = \inf_{n\in\mathbb{N}}\left\{\mathrm{lct}_n(X, D)\right\}.$$

3.2 Double spaces

Let $\pi : V \to \mathbb{P}^n$ be a smooth double cover ramified along a hypersurface S of degree $2m$ in $\mathbb{P}^n$, $n \geq 3$. In addition, let H be the pull-back of a hyperplane in $\mathbb{P}^n$ by the covering map π. We can consider the double cover V as a smooth hypersurface of degree $2m$ in $\mathbb{P}(1^{n+1}, m)$.

Proposition 3.2 *The global log canonical threshold* $\mathrm{lct}(V, H)$ *is equal to the first global log canonical threshold* $\mathrm{lct}_1(V, H)$.

Proof Let us use the arguments in the proof of [30, Proposition 5].

Suppose that there is a divisor D in the linear system $|\mu H|$ for some integer $\mu \geq 2$ such that

$$c\left(V, \frac{1}{\mu}D\right) < \mathrm{lct}_1(V, H) \leq 1.$$

It follows from Remark 2.10 that we may assume that the support of the divisor D does not contain divisors of the linear system $|H|$.

Choose a number λ such that $c\left(V, \frac{1}{\mu}D\right) < \lambda < \mathrm{lct}_1(V, H)$. Then the log pair $\left(V, \frac{\lambda}{\mu}D\right)$ is not log canonical. By [29, Proposition 4.3] we have the center of a non-log-canonical singularity of the log pair $\left(V, \frac{\lambda}{\mu}D\right)$ at a point P on V.

Suppose that $\pi(P) \in S$. Let T be the unique divisor in the linear system $|H|$ that is singular at the point P. Since we have $\mathrm{mult}_P(D) > \mu$, we obtain an absurd inequality

$$2\mu = D \cdot T^{n-1} \geq \mathrm{mult}_P\left(D \cap T\right) > 2\mu.$$

Now, we suppose that $\pi(P) \notin S$. Let $\xi\colon W \to V$ be the blow up at the point P and $E \cong \mathbb{P}^{n-1}$ be the exceptional divisor of the birational morphism ξ. Then, it follows from [30, Proposition 3] that there is a hyperplane $\Lambda \subset E$ such that

$$\mathrm{mult}_P\left(D\right) + \mathrm{mult}_\Lambda\left(\bar{D}\right) > 2\mu,$$

where $\bar{D}$ is the proper transform of D on the variety W.

Let G be a general divisor in $|H|$ such that $\Lambda \subset \mathrm{Supp}(\bar{G})$, where $\bar{G}$ is the proper transform of G on the variety W. Then, we also obtain a contradictory inequality

$$2\mu = D \cdot G^{n-1} \geq \mathrm{mult}_P\,(D \cap G) > 2\mu.$$

□

Now we are ready to prove the following result.

Proposition 3.3 *The following inequality holds:*

$$\mathrm{lct}\,(V, H) \geq \min\left(1, \frac{m+n-1}{2m}\right).$$

Proof By Proposition 3.2, it is enough to consider the first global log canonical threshold $\mathrm{lct}_1(V, H)$ instead of $\mathrm{lct}(V, H)$. Let D be a divisor in $|H|$.

The double space V can be defined by a quasi-homogenous equation $w^2 = f\,(x_0, \ldots, x_n)$ in the weighted projective space $\mathbb{P}\left(1^{n+1}, m\right) \cong \mathrm{Proj}\,(\mathbb{C}[x_0, \ldots, x_n, w])$, where $\mathrm{wt}(x_i) = 1$, $\mathrm{wt}(w) = m$, and f is a homogeneous polynomial of degree $2m$. Note that the homogenous polynomial f defines the smooth hypersurface S in $\mathbb{P}^n$ since V is smooth. We may assume that the divisor D is cut out on V by the equation $x_0 = 0$. The divisor D is a hypersurface in $\mathbb{P}\,(1^n, m) \cong \mathrm{Proj}\,(\mathbb{C}[x_1, \ldots, x_n, w])$ defined by the equation $w^2 = f\,(0, x_1, \ldots, x_n)$. It has isolated singularities since the hypersurface

$$D_S := \{f\,(0, x_1, \ldots, x_n) = 0\} \subset \mathbb{P}^{n-1} \cong \mathrm{Proj}\,(\mathbb{C}[x_1, \ldots, x_n])\,,$$

has isolated singularities (see [15]).

It follows from [11, Theorem 3.1] that the log pair $(V, \lambda D)$ is log terminal if and only if $(\mathbb{P}(1^n, m), \lambda D)$ is log terminal because V is smooth and the divisor D is contained in the smooth locus of $\mathbb{P}(1^n, m)$. It then follows from [17, Proposition 8.21] that

$$c(V, D) = c\left(\mathbb{P}\left(1^n, m\right), D\right) = \frac{1}{2} + c\left(\mathbb{P}^{n-1}, D_S\right).$$

We then see that [11, Theorem 3.1] and [6, Theorem 3.3] imply

$$c\left(\mathbb{P}^{n-1}, D_S\right) = c\,(S, D_S) \geq \frac{n-1}{2m}.$$

This completes the proof. □

Let $\pi : V \to \mathbb{P}^n$ be a double cover ramified along a smooth hypersurface of degree $2n \geq 4$. It is a Fano variety of Fano index 1 and the pull-back of a hyperplane in $\mathbb{P}^n$ is an anticanonical divisor of V. It follows from Proposition 3.3 (for $n \geq 3$) and [5, Theorem 1.7] (for $n = 2$) that

$$\mathrm{lct}\,(V) \geq \frac{2n-1}{2n},$$

while [30, Theorem 2] shows that $\mathrm{lct}(V) = 1$ if V is general and $n \geq 3$. Therefore, we immediately obtain the following result that has been proved by [1] in a different way.

Corollary 3.4 *A smooth double cover of $\mathbb{P}^n$ ramified along a hypersurface of degree $2n \geq 4$ admits a Kähler–Einstein metric.*

Remark 3.5 Combining the results of [3] and the proof of Proposition 3.3, we can easily obtain the following. Let V be the smooth hypersurface in $\mathbb{P}(1^{n+1}, m)$ of degree $2m \geq 2n \geq 6$ given by an equation

$$w^2 = f(x_0, \ldots, x_n) \subset \mathbb{P}\left(1^{n+1}, m\right) \cong \mathrm{Proj}\left(\mathbb{C}[x_0, \ldots, x_n, w]\right),$$

where $\mathrm{wt}(x_i) = 1,\ \mathrm{wt}(w) = m$, and f is a homogeneous polynomial of degree $2m$. Suppose that

$$c(V, D) = \frac{m+n-1}{2m\mu},$$

where $D \in |\mu H|$ and $\mu \in \mathbb{N}$. Then $D = \mu T$, where T is a divisor that is cut out on the hypersurface V by an equation $\sum_{i=0}^{n} \lambda_i x_i = 0$ such that the hypersurface

$$f(x_0, \ldots, x_n) = \sum_{i=0}^{n} \lambda_i x_i = 0 \subset \mathbb{P}^{n-1} \cong \mathrm{Proj}\left(\mathbb{C}[x_0, \ldots, x_n] \Big/ \left(\sum_{i=0}^{n} \lambda_i x_i\right)\right)$$

is a cone over a smooth hypersurface in $\mathbb{P}^{n-2}$ of degree $2m$.

We can also give an easy proof of the following result that is a corollary of [30, Theorem 2].

Proposition 3.6 *Let V be the double cover of $\mathbb{P}^n$, $n \geq 3$, ramified along a general hypersurface S of degree $2n$ in $\mathbb{P}^n$. Then* $\mathrm{lct}(V, H) = 1$.

Proof We assume that for every hyperplane $M \subset \mathbb{P}^n$, the intersection $S \cap M$ has at most isolated double points. This generality condition is obviously satisfied for a general hypersurface S because $n \geq 3$.

Let D be a divisor in the linear system $|H|$. It follows from [17, Lemma 8.12] that the singularities of the log pair (V, D) are log canonical if and only if the singularities of the log pair

$$\left(\mathbb{P}^n, \pi(D) + \frac{1}{2}S\right)$$

are log canonical. Put $M = \pi(D)$. It follows from [17, Theorem 7.5] that the singularities of the log pair (V, D) are log canonical if and only if the log pair $(M, \frac{1}{2}S|_M)$ is log canonical. But the log pair $(M, \frac{1}{2}S|_M)$ is log canonical because $S|_M$ has at most isolated double points. $\square$

The generality assumption in Proposition 3.6 is weaker than that of [30, Theorem 2].

Let V be the double cover of $\mathbb{P}^3$ ramified along a smooth sextic $S \subset \mathbb{P}^3$. Note that the pull-back of a hyperplane in $\mathbb{P}^3$ is an anticanonical divisor. As we did for quartic threefolds, we are also able to find all the possible first global log canonical thresholds of V.

Proposition 3.7 *Let V be the smooth double cover of $\mathbb{P}^3$ ramified along a sextic. Then, the first global log canonical threshold of the Fano variety V is one of the following:*

$$\left\{\frac{5}{6}, \frac{43}{50}, \frac{13}{15}, \frac{33}{38}, \frac{7}{8}, \frac{33}{38}, \frac{8}{9}, \frac{9}{10}, \frac{11}{12}, \frac{13}{14}, \frac{15}{16}, \frac{17}{18}, \frac{19}{20}, \frac{21}{22}, \frac{29}{30}, 1\right\}.$$

Furthermore, for each number μ in the set above, there is a smooth double cover V of $\mathbb{P}^3$ ramified along a sextic with $\mathrm{lct}_1(V) = \mu$.

Proof For the proof, see [35]. Its brief idea is as follows. For a hyperplane H in $\mathbb{P}^3$, we see that

$$c\left(V, \pi^*(H)\right) = \min\left\{1, \frac{1}{2} + c\left(H, H \cap S\right)\right\}.$$

The intersection $H \cap S$ is a reduced sextic plane curve on $H \cong \mathbb{P}^2$. Therefore, for the first statement of Proposition 3.7, it is enough to consider all the possible values of $c(\mathbb{P}^2, C)$ for reduced sextic plane curves. Furthermore, we can consider only the values for $c_0(f)$, where f is a reduced sextic polynomial vanishing at the origin. □

Because the first global log canonical thresholds coincide with the global log canonical thresholds on double spaces, Proposition 3.7 implies a stronger result as follows.

Corollary 3.8 *Let V be a smooth double cover of $\mathbb{P}^3$ ramified along a sextic. Then, the global log canonical threshold of the Fano variety V is one of the numbers in Proposition 3.7. Furthermore, for each number μ in Proposition 3.7, there is a smooth double cover V of $\mathbb{P}^3$ ramified along a sextic with lct$(V) = \mu$.*

Let us finish the paper by an example of a smooth double cover of $\mathbb{P}^3$ ramified along a sextic surface with the global log canonical threshold 1.

Example 3.9 Let V be the smooth double cover of $\mathbb{P}^3$ ramified along the sextic surface $S \subset \mathbb{P}^3$ defined by the equation

$$x_0^6 + x_1^6 + x_2^6 + x_3^6 + x_0^2x_1^2x_2x_3 = 0.$$

Let $C \subset \mathbb{P}^3$ be the curve defined by the intersection of the surface S and the Hessian surface Hess(S) of S. For the tangent hyperplane T_P at a point $P \in S$, if the multiplicity of the curve $T_P \cap S$ at the point P is at least 3, then the curve C is singular at the point P. Using the computer program, *Singular*, one can check that the curve C is smooth in the outside of the curves $x_i = x_j = 0$ with $i \neq j$. Furthermore, for a point P in S that belongs to the curves $x_i = x_j = 0$ with $i \neq j$, one can easily check that the log pair $(S, \frac{1}{2}H_P)$ is log canonical, where H_P is the hyperplane section of S by the tangent hyperplane to S at the point P. Consequently, $\mathrm{lct}(V) = \mathrm{lct}_1(V) = 1$. The variety V is an explicit example of smooth Fano variety with the following properties (We do not know any other explicit example of such a smooth Fano variety). For each $i = 1, 2, \ldots, r$, let $V_i = V$. Then, the paper [30] implies that the product $V_1 \times \cdots \times V_r$ is not rational and

$$\mathrm{Bir}(V_1 \times \cdots \times V_r) = \mathrm{Aut}(V_1 \times \cdots \times V_r).$$

Moreover, for each dominant rational map $\rho\colon V_1 \times \cdots \times V_r \dashrightarrow Y$ whose general fiber is rationally connected, there is a subset $\{i_1, \ldots, i_k\} \subset \{1, v, r\}$ such that the diagram

$$\begin{tikzcd} V_1 \times \cdots \times V_r \arrow[d, "\pi"'] \arrow[dr, dashed, "\rho"] & \\ V_{i_1} \times \cdots \times V_{i_k} \arrow[r, dashed, "\bar{\rho}"'] & Y \end{tikzcd}$$

commutes, where π is the natural projection and $\bar{\rho}$ is a birational map.

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4 Appendix

Let X_F be a smooth quintic hypersurface in $\mathbb{P}^5$ that is given by zeroes of a section $F \in H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(5))$. It follows from Proposition 2.4 that there exists a non-empty Zariski open subset $U_{G1} \in H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(5))$ such that X_F is 4-regular whenever $F \in U_{G1}$. Similarly, it follows from Lemma 2.14 that there exists a non-empty Zariski open subset $U_{G2} \in H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(5))$ such that for every 3-dimensional linear space Π in $\mathbb{P}^5$, the intersection $X_F \cap \Pi$ is irreducible and reduced if $F \in U_{G2}$.

The purpose of this Appendix is to prove Lemma 2.15, i.e., to prove the existence of a non-empty Zariski open subset $U_{G3} \in H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(5))$ such that for each $F \in U_{G3}$ the hypersurface X_F satisfies the condition G3 (see Sect. 2.3). Indeed, we prove the statement as follows:

> For each $a(= 0, 1, 2, 3)$ and $b(= 1, 2, \ldots, 6)$, there exists a non-empty Zariski open subset U in $H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(5))$ such that if $F \in U$, then for each point $P \in X$ and each 3-dimensional linear space Π_3 contained in the tangent hyperplane at P and containing the point P, the surface $Z := X \cap \Pi_3$ satisfies the condition G3.a.b.

Since we use the same method in order to prove the statement for each a and b, we first explain how the proof goes and then show the required computations in each case G3.a.b.

The proof goes as follows.

First we consider the space

$$\mathcal{S} = \mathcal{F} \times H^0\left(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(5)\right)$$

with the natural projections $p\colon \mathcal{S} \to H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(5))$ and $q\colon \mathcal{S} \to \mathcal{F}$. Here, $\mathcal{F}$ is a suitable flag variety in $\mathbb{P}^5$. Depending on the case, the flag $\mathcal{F}$ will be $Flag(0, 1, 2, 3, 4)$, $Flag(0, 2, 3, 4)$, $Flag(0, 1, 3, 4)$ or $Flag(0, 3, 4)$, where $Flag(n_1, \ldots, n_k)$ is the flag variety that parametrizes k-tuples $(\Pi_{n_1}, \ldots, \Pi_{n_k})$ of n_i-dimensional linear spaces with $\Pi_{n_1} \subset \cdots \subset \Pi_{n_k} \subset \mathbb{P}^5$. A 0-dimensional linear space will be denoted by P and a four dimensional linear space will be denoted by T.

We then put

$$\mathcal{I} = \left\{ \left((P, \Pi_{n_2}, \ldots, \Pi_{n_{k-1}}, T), F\right) \in \mathcal{S} \;\middle|\; \begin{array}{l} F(P) = 0; \\ T \text{ is the tangent hyperplane to } X_F \text{ at } P; \\ X_F \text{ satisfies the properties } \mathcal{P}_{G3.a.b}. \end{array} \right\},$$

where the properties $\mathcal{P}_{G3.a.b}$ will be specified in the individual proofs. Then in each case, we will see that it is easy to check that the morphism $q|_{\mathcal{I}} : \mathcal{I} \to \mathcal{F}$ is surjective.

With this set up, we compute the codimension c of $q|_{\mathcal{I}}^{-1}(P, \Pi_{n_2}, \ldots, \Pi_{n_{k-1}}, T)$ in $H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(5))$ for a point $(P, \Pi_{n_2}, \ldots, \Pi_{n_{k-1}}, T) \in \mathcal{F}$. We may always assume that T

is defined by $x = 0$, Π_3 is defined by $x = y = 0$, Π_2 by $x = y = z = 0$, Π_1 by $x = y = z = u = 0$ and $P = [0:0:0:0:0:1]$. We write the quintic polynomial F as

$$w^5 q_0 + w^4 q_1(x,y,z,u,v) + w^3 q_2(x,y,z,u,v) + w^2 q_3(x,y,z,u,v)$$
$$+ w q_4(x,y,z,u,v) + q_5(x,y,z,u,v),$$

where q_i is a homogeneous polynomial of degree i.

The condition $F(P) = 0$ is equivalent to $q_0 = 0$. The condition that T is the tangent hyperplane to X_F at P is equivalent to $q_1 = \lambda x$ for some $\lambda \in \mathbb{C}^*$. These two conditions contribute to the codimension c by 5. For each a and b, we will show that that the properties $\mathcal{P}_{G3.a.b}$ makes another contribution to the codimension c by more than $\dim \mathcal{F} - 5$.

These altogether show that the codimension of $q|_{\mathcal{I}}^{-1}(P, \Pi_{n_2}, \dots, \Pi_{n_{k-1}}, T)$ in $H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(5))$ is more than $\dim \mathcal{F}$. These implies that the morphism $p|_{\mathcal{I}}$ cannot be surjective. Taking the properties $\mathcal{P}_{G3.a.b}$ into consideration, we can immediately notice that this non-surjectivity implies the statement.

Therefore, to prove the statement for each case, it is enough to

- specify the flag $\mathcal{F}$ with its dimension;
- specify the property $\mathcal{P}_{G3.a.b}$;
- show that the properties $\mathcal{P}_{G3.a.b}$ makes another contribution to the codimension c by more than $\dim \mathcal{F} - 5$.

Now we do these jobs for each case.

Lemma 1 *The statement holds for $G3.0.1$.*

Proof The flag $\mathcal{F}$ is $Flag(0,1,3,4)$. It is of dimension 14. Put

$$\mathcal{P}_{G3.0.1} = \{X_F \cap \Pi_3 \text{ is singular along } \Pi_2.\}.$$

The condition that $X_F \cap \Pi_3$ contains the line Π_1 is equivalent to the condition that for each $i = 2,3,4,5$, the polynomial q_i contains no v^i. For $X_F \cap \Pi_3$ in order to be singular along L, for each $i = 2,3,4,5$, the polynomial q_i must not contain the monomials $zu^r v^{i-r-1}$, $r = 0,1,\dots,i-1$. These altogether show that the properties $\mathcal{P}_{G3.0.1}$ is of codimension > 9. □

Lemma 2 *The statement holds for $G3.0.2$.*

Proof The flag $\mathcal{F}$ is $Flag(0,3,4)$. It is of dimension 12. Put

$$\mathcal{P}_{G3.0.2} = \{X_F \cap \Pi_3 \text{ contains four lines.}\}.$$

Since we may assume that q_1, q_2, q_3, q_4 forms a regular sequence, $X_F \cap \Pi_3$ containing four lines is equivalent to $q_5(x,y,z,u,v)$ vanishing at four given points in $q_1(x,y,z,u,v) = q_2(x,y,z,u,v) = q_3(x,y,z,u,v) = q_4(x,y,z,u,v) = 0$ in $\mathbb{P}^4$ and $q_4(0,0,z,u,v)$ vanishing at four given points in $q_2(0,0,z,u,v) = q_3(0,0,z,u,v) = 0$ in $\mathbb{P}^2$. These altogether show that the properties $\mathcal{P}_{G3.0.2}$ is of codimension 8. □

Lemma 3 *The statement holds for $G3.1.1$.*

Proof The flag $\mathcal{F}$ is $Flag(0,1,3,4)$. It is of dimension 14.

Put

$$\mathcal{P}_{G3.1.1} = \left\{\begin{array}{l} X_F \cap \Pi_3 \text{ contains } \Pi_1; \\ \Pi_1 \text{ meets its residual curve by a general hyperplane} \\ \text{section of } X_F \cap \Pi_3 \text{ in } \Pi_3 \text{ only at singular points;} \\ X_F \cap \Pi_3 \text{ has at most one ordinary double point on } \Pi_1. \end{array}\right\}.$$

We write

$$q_i(0,0,z,u,v) = \sum_{r+s+t=i} A_{rst} z^r u^s v^t,$$

where A_{rst}'s are constants.

The condition that $X_F \cap \Pi_3$ contains the line Π_1 is equivalent to $A_{00t} = 0$ for $t = 2,\ 3,\ 4$ and 5 since the line Π_1 is defined by $x = y = z = u = 0$.

The surface $X_F \cap \Pi_3$ has singular points on the line Π_1 exactly where the polynomials $A_{101}vw^3 + A_{102}v^2w^2 + A_{103}v^3w + A_{104}v^4$ and $A_{011}vw^3 + A_{012}v^2w^2 + A_{013}v^3w + A_{014}v^4$ have common zeros in $\mathbb{P}^1$. The zero given by $v = 0$ corresponds to the singular point P. To see this, put $\bar{F}(z,u,v,w) = F(0,0,z,u,v,w)$. Since $A_{00t} = 0$ for $t = 2,\ 3,\ 4$ and 5, we always have $\frac{\partial \bar{F}}{\partial v}(0,0,v,w) = \frac{\partial \bar{F}}{\partial w}(0,0,v,w) = 0$. The common zeros of

$$\frac{\partial \bar{F}}{\partial z}(0,0,v,w) = v(A_{101}w^3 + A_{102}vw^2 + A_{103}v^2w + A_{104}v^3),$$
$$\frac{\partial \bar{F}}{\partial u}(0,0,v,w) = v(A_{011}w^3 + A_{012}vw^2 + A_{013}v^2w + A_{014}v^3)$$

are the singular points of $X_F \cap \Pi_3$ on the line Π_1. Note that Π_1 and its residual curve by a general hyperplane meet at every singular point of $X_F \cap \Pi_3$ on the line Π_1. Therefore, the second condition is equivalent to the condition that the polynomials $A_{101}vw^3 + A_{102}v^2w^2 + A_{103}v^3w + A_{104}v^4$ and $A_{011}vw^3 + A_{012}v^2w^2 + A_{013}v^3w + A_{014}v^4$ have four common zeros in $\mathbb{P}^1$ with counting multiplicity, i.e., these two polynomials are proportional. This imposes three additional independent conditions on the coefficients of F.

The condition that the polynomial $A_{101}vw^3 + A_{102}v^2w^2 + A_{103}v^3w + A_{104}v^4$ has k zeros without counting multiplicity imposes $4-k$ additional independent conditions on the coefficients of F. Note that $1 \le k \le 4$.

We claim that the last condition imposes $k-1$ independent conditions on the coefficients of F. Here we verify the claim only for the case with $k = 4$. The other cases with $k = 3$ and 2 can be verified in the same way.

We write the homogenized Hessian matrix of the polynomial $q_2(0,0,z,u,v) + q_3(0,0,z,u,v) + q_4(0,0,z,u,v) + q_5(0,0,z,u,v)$ along the line Π_1 as follows:

$$\begin{pmatrix} 2(A_{200}w^3{+}A_{201}vw^2{+}A_{202}v^2w{+}A_{203}v^3) & A_{110}w^3{+}A_{111}vw^2{+}A_{112}v^2w{+}A_{113}v^3 & A_{101}w^3{+}2A_{102}vw^2{+}3A_{103}v^2w{+}4A_{104}v^3 \\ A_{110}w^3{+}A_{111}vw^2{+}A_{112}v^2w{+}A_{113}v^3 & 2(A_{020}w^3{+}A_{021}vw^2{+}A_{022}v^2w{+}A_{023}v^3) & A_{011}w^3{+}2A_{012}vw^2{+}3A_{013}v^2w{+}4A_{014}v^3 \\ A_{101}w^3{+}2A_{102}vw^2{+}3A_{103}v^2w{+}4A_{104}v^3 & A_{011}w^3{+}2A_{012}vw^2{+}3A_{013}v^2w{+}4A_{014}v^3 & 0 \end{pmatrix}.$$

Let $H(v,w)$ be the determinant of the homogenized Hessian matrix. The condition that three of the four singular points on Π_1 is not ordinary double points is equivalent to the condition that $H(v,w)$ vanishes at three points out of the four points defined by $A_{101}w^3v + A_{102}v^2w^2 + A_{103}v^3w + A_{104}v^4 = 0$ and $A_{011}vw^3 + A_{012}v^2w^2 + A_{013}v^3w + A_{014}v^4 = 0$ in $\mathbb{P}^1$. We claim that it imposes three additional independent conditions on the coefficients of F. To verify the claim, we put

$$A_{110} = 0, \quad A_{111} = 0, \quad A_{112} = 0, \quad A_{113} = 0, \quad A_{102} = 0, \quad A_{103} = 0$$
$$A_{012} = 0, \quad A_{013} = 0 \quad A_{201} = 0, \quad A_{202} = 0, \quad A_{021} = 0, \quad A_{022} = 0.$$

Since $A_{101}w^3v + A_{104}v^4 = 0$ and $A_{011}vw^3 + A_{014}v^4 = 0$ defines four points in $\mathbb{P}^1$, we have $[\lambda : \mu] \in \mathbb{P}^1$ with $\lambda(A_{101}, A_{104}) = \mu(A_{011}, A_{014})$. We then see that in our restricted situation, the condition is equivalent to the condition that $A_{101}w^3v + A_{104}v^4 = 0$ has three common points with

$$\left(A_{101}w^3 + 4A_{104}v^3\right)^2 \left\{\lambda^2 \left(A_{200}w^3 + A_{203}v^3\right) + \mu^2 \left(A_{020}w^3 + A_{023}v^3\right)\right\} = 0$$

in $\mathbb{P}^1$. Since this is a condition of codimension 3 in the restricted situation, it verifies the claim.

These altogether show that the properties $\mathcal{P}_{G3.1.1}$ is of codimension > 9. □

Lemma 4 *The statement holds for G3.2.1.*

Proof The flag $\mathcal{F}$ is $Flag(0, 1, 2, 3, 4)$. It is of dimension 15. Put

$$\mathcal{P}_{G3.2.1} = \left\{ \begin{array}{l} X_F \text{ contains } \Pi_1; \quad X_F \cap \Pi_2 \text{ contains a line other than } \Pi_1 \text{ passing through } P; \\ X_F \cap \Pi_3 \text{ contains four singular points other than } P \text{ on the two lines on } X \cap \Pi_2 \\ \text{passing through the point } P. \end{array} \right\}.$$

The condition that X_F contains the line Π_1 is equivalent to the condition that for each $i = 2, 3, 4, 5$, the polynomial q_i contains no v^i. The condition that $X_F \cap \Pi_2$ contains a line other than Π_1 passing through P is equivalent to the condition that $q_3(0, 0, 0, u, v)$, $q_4(0, 0, 0, u, v)$ and $q_5(0, 0, 0, u, v)$ vanish at the point other than the point given by $u = 0$ in $\mathbb{P}^1$ where $q_2(0, 0, 0, u, v)$ vanishes. For $X_F \cap \Pi_3$ in order to have four singular points other than P on the two lines on $X_F \cap \Pi_2$ passing through the point P is a condition of codimension 4. These altogether show that the properties $\mathcal{P}_{G3.2.1}$ is of codimension 11. □

Lemma 5 *The statement holds for G3.2.2.*

Proof The flag $\mathcal{F}$ is $Flag(0, 1, 2, 3, 4)$. It is of dimension 15. Put

$$\mathcal{P}_{G3.2.2} = \left\{ \begin{array}{l} X_F \text{ contains } \Pi_1; \quad X_F \cap \Pi_2 \text{ contains two lines passing through } P; \\ X_F \cap \Pi_3 \text{ has three singular points other than } P \text{ on } \Pi_1; \\ X_F \cap \Pi_3 \text{ has at least one singular point on } \Pi_1 \text{ that is not an ordinary double point.} \end{array} \right\}.$$

The condition that $\Pi_1 \subset X_F$ is equivalent to the fact that each $q_i(x, y, z, u, v)$ does not have v^i monomial, which is condition of codimension 4. The condition that $X_F \cap \Pi_2$ contains another line passing through the point P is equivalent to the condition that either $q_3(0, 0, 0, u, v)$, $q_4(0, 0, 0, u, v)$ and $q_5(0, 0, 0, u, v)$ vanish at the points in $\mathbb{P}^1$ where $q_2(0, 0, 0, u, v)/u$ vanishes, or $q_2(0, 0, 0, u, v)$ is a zero polynomial and $q_3(0, 0, 0, u, v)$, $q_4(0, 0, 0, u, v)$ and $q_5(0, 0, 0, u, v)$ have common root in $\mathbb{P}^1$. Thus, the condition that $X_F \cap \Pi_2$ contains another line passing through the point P is a condition of codimension 3. For the surface $X_F \cap \Pi_3$ to have three singular points on Π_1 other than P is a condition of codimension 3. Arguing as in the proof of Lemma 3, we can see that the condition that one of the singular points of $X_F \cap \Pi_3$ on Π_1 is not an ordinary double point is a condition of codimension 1. These altogether show that the properties $\mathcal{P}_{G3.2.2}$ is of codimension > 10. □

Lemma 6 *The statement holds for G3.2.3.*

Proof The flag $\mathcal{F}$ is $Flag(0, 1, 2, 3, 4)$. It is of dimension 15. Put

$$\mathcal{P}_{G3.2.3} = \left\{ \begin{array}{l} X_F \text{ contains } \Pi_1; \\ X_F \cap \Pi_2 \text{ contains a line other than } \Pi_1 \text{ passing through } P; \\ X_F \cap \Pi_3 \text{ contains two non-ordinary singular points on } \Pi_1. \end{array} \right\}.$$

The condition that X_F contains the line Π_1 is equivalent to the condition that for each $i = 2, 3, 4, 5$, the polynomial q_i contains no v^i. The condition that $X_F \cap \Pi_2$ contains a line other than Π_1 passing through P is equivalent to the condition that $q_3(0, 0, 0, u, v)$, $q_4(0, 0, 0, u, v)$ and $q_5(0, 0, 0, u, v)$ vanish at the point other than the point given by $u = 0$ in $\mathbb{P}^1$ where $q_2(0, 0, 0, u, v)$ vanishes. As in the proof of Lemma 3, we can see that for $X_F \cap \Pi_3$ to have two non-ordinary singular points on Π_1 is a condition of codimension 4. These altogether show that the properties $\mathcal{P}_{G3.2.3}$ is of codimension > 10. □

Lemma 7 *The statement holds for G3.2.4.*

Proof The flag $\mathcal{F}$ is $Flag(0, 1, 2, 3, 4)$. It is of dimension 15. Put

$$\mathcal{P}_{G3.2.4} = \left\{ \begin{array}{l} X_F \text{ contains } \Pi_1; \quad X_F \cap \Pi_2 \text{ contains a line other than } \Pi_1 \text{ passing through } P; \\ X_F \cap \Pi_3 \text{ contains two singular points other than } P \text{ on the line } \Pi_1; \\ \Pi_1 \text{ meets its residual curve by a general hyperplane section of } X_F \cap \Pi_3 \text{ in } \Pi_3 \\ \text{only at three points.} \end{array} \right\}.$$

We write

$$q_i(0, 0, z, u, v) = \sum_{r+s+t=i} A_{rst} z^r u^s v^t,$$

where A_{rst}'s are constants.

The condition that $X_F \cap \Pi_3$ contains the line Π_1 is equivalent to $A_{00t} = 0$ for $t = 2,\ 3,\ 4$ and 5. The condition that $X_F \cap \Pi_2$ contains a line other than Π_1 passing through P is equivalent to the condition that $q_3(0, 0, 0, u, v)$, $q_4(0, 0, 0, u, v)$ and $q_5(0, 0, 0, u, v)$ vanish at the point other than the point given by $u = 0$ in $\mathbb{P}^1$ where $q_2(0, 0, 0, u, v)$ vanishes. For $X_F \cap \Pi_3$ in order to have two singular points other than P on the line Π_1 and for Π_1 to meet its residual curve by a general hyperplane section of $X_F \cap \Pi_3$ in Π_3 only at three point are equivalent to the condition that the polynomials $A_{101}vw^3 + A_{102}v^2w^2 + A_{103}v^3w + A_{104}v^4$ and $A_{011}vw^3 + A_{012}v^2w^2 + A_{013}v^3w + A_{014}v^4$ have four common zeros in $\mathbb{P}^1$ with counting multiplicity and the polynomial $A_{101}vw^3 + A_{102}v^2w^2 + A_{103}v^3w + A_{104}v^4$ has three zeros without counting multiplicity. This condition is of codimention 4. These altogether show that the properties $\mathcal{P}_{G3.2.4}$ is of codimension 11. □

Lemma 8 *The statement holds for G3.2.5.*

Proof The flag $\mathcal{F}$ is $Flag(0, 1, 2, 3, 4)$. It is of dimension 15. Put

$$\mathcal{P}_{G3.2.5} = \left\{ \begin{array}{l} X_F \text{ contains } \Pi_1; \quad X_F \cap \Pi_2 \text{ contains a line other than } \Pi_1 \text{ passing through } P; \\ X_F \cap \Pi_3 \text{ contains one singular points other than } P \text{ on the line } \Pi_1; \\ \text{either } \Pi_1 \text{ meets its residual curve by a general hyperplane section in } \Pi_3 \\ \text{only at singular points or} \\ \Pi_1 \text{ meets its residual curve by a general hyperplane section in } \Pi_3 \\ \text{only at three points and } X_F \cap \Pi_3 \text{ has a non-ordinary singular point } P. \end{array} \right\}.$$

The first two conditions imposes seven independent conditions on the coefficients of F as before. For the surface $X_F \cap \Pi_3$ to have a singular point on Π_1 other than P and plus for Π_1 to meet its residual curve by a general hyperplane section in Π_3 only at singular points impose at least four independent conditions on the coefficients of F. Meanwhile, for the surface $X_F \cap \Pi_3$ to have a singular point on Π_1 other than P and plus for Π_1 to meet its residual curve by a general hyperplane section in Π_3 only at three points impose at least four independent conditions on the coefficients of F. However, the condition that $X_F \cap \Pi_3$ has a non-ordinary singular point P is of codimension 1. Therefore, the properties $\mathcal{P}_{G3.2.5}$ is of codimension 11.

These altogether show that the properties $\mathcal{P}_{G3.2.5}$ is of codimension 11. □

Lemma 9 *The statement holds for G3.2.6.*

Proof The flag $\mathcal{F}$ is $Flag(0, 1, 2, 3, 4)$. It is of dimension 15. Put

$$\mathcal{P}_{G3.2.6} = \left\{ \begin{array}{l} X_F \text{ contains } \Pi_1; \quad X_F \cap \Pi_2 \text{ contains a line other than } \Pi_1 \text{ passing through } P; \\ \Pi_1 \text{ meets its residual curve by a general hyperplane section of } X_F \cap \Pi_3 \text{ in } \Pi_3 \\ \text{at at most two points.} \end{array} \right\}.$$

The first two conditions imposes seven independent conditions on the coefficients of F as before.

For the last condition, we write

$$q_i(0, 0, z, u, v) = \sum_{r+s+t=i} A_{rst} z^r u^s v^t,$$

where A_{rst}'s are constants.

The last condition is equivalent to the condition that either the polynomials $A_{101}vw^3 + A_{102}v^2w^2 + A_{103}v^3w + A_{104}v^4$ and $A_{011}vw^3 + A_{012}v^2w^2 + A_{013}v^3w + A_{014}v^4$ have a common zero at $v = 0$ with multiplicity at least 3 or they are proportional and have only two zeros (without counting multiplicities). The former and the latter are both a condition of codimension at least 4.

Therefore, the properties $\mathcal{P}_{G3.2.6}$ is of codimension at least 11. □

Lemma 10 *The statement holds for G3.3.1.*

Proof The flag $\mathcal{F}$ is $Flag(0, 2, 3, 4)$. It is of dimension 14. Put

$$\mathcal{P}_{G3.3.1} = \left\{ \begin{array}{l} X_F \cap \Pi_2 \text{ contains three lines passing through } P; \\ \text{either } X_F \cap \Pi_3 \text{ has a singular point on } \Pi_2 \text{ other than } P \text{ or one of the lines } L_i \text{ meets} \\ \text{its residual curve by a general hyperplane section in } \Pi_3 \text{ at at most three points.} \end{array} \right\}.$$

The condition that $X_F \cap \Pi_2$ contains three lines passing through the point P is equivalent to the condition that $q_2(0, 0, 0, u, v)$ is identically zero; $q_4(0, 0, 0, u, v)$ and $q_5(0, 0, 0, u, v)$ vanish at the three points in $\mathbb{P}^1$ where $q_3(0, 0, 0, u, v)$ vanishes. For the surface $X_F \cap \Pi_3$ to have a singular point on Π_2 other than P is a condition of codimension 1. For one of the lines L_i to meet its residual curve by a general hyperplane section in Π_3 at at most three points is also a condition of codimension 1. These altogether show that the properties $\mathcal{P}_{G3.3.1}$ is of codimension > 9. □

Lemma 11 *The statement holds for G3.3.2.*

Proof The flag $\mathcal{F}$ is $Flag(0, 3, 4)$. It is of dimension 12. Put

$$\mathcal{P}_{G3.3.2} = \left\{ \begin{array}{l} X_F \cap \Pi_3 \text{ contains three lines passing through } P; \\ \text{either one of the lines } L_i \text{ meets its residual curve by a general hyperplane section} \\ \text{in } \Pi_3 \text{ at at most one smooth point or} \\ \text{one of the lines } L_i \text{ meets its residual curve by a general hyperplane section in } \Pi_3 \\ \text{at at most two smooth points and } X_F \cap \Pi_3 \text{ has a non-ordinary singular point at } P. \end{array} \right\}.$$

The condition that $X_F \cap \Pi_3$ contains three lines passing through the point P is equivalent to the condition that $q_4(0, 0, z, u, v)$ and $q_5(0, 0, z, u, v)$ vanish at three points in $\mathbb{P}^2$ where both $q_2(0, 0, z, u, v)$ and $q_3(0, 0, z, u, v)$ vanish.

For one of the lines L_i to meet its residual curve by a general hyperplane section in Π_3 at at most one smooth is also a condition of codimension at least 2. For one of the lines L_i to meet its residual curve by a general hyperplane section in Π_3 at at most two smooth is also a condition of codimension at least 1. The condition that $X_F \cap \Pi_3$ has a non-ordinary singular point at P is equivalent to the condition that the quadratic polynomial $q_2(0, 0, z, u, v)$ is singular in variables z, u, v. This is a condition of codimension 1. These altogether show that the properties $\mathcal{P}_{G3.3.2}$ is of codimension > 7. □

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