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Cylinders in rational surfaces

I. A. Cheltsov

Abstract. We answer a question of Ciliberto's about cylinders in rational surfaces obtained by blowing up the plane at points in general position.
Bibliography: 13 titles.

Keywords: rational surfaces, del Pezzo surfaces, cylinders.

§ 1. Introduction

Let S be a smooth rational surface. A *cylinder* in S is an open subset $U \subset S$ such that $U \cong \mathbb{C}^1 \times Z$ for an affine curve Z. The surface S contains many cylinders, and it seems a hopeless task to describe all of them. Instead, we consider a similar problem for polarized surfaces (see [7]–[9], [2], [3] and [11]). To describe it, fix an ample $\mathbb{Q}$-divisor A on the surface S.

Definition 1.1. An *A-polar cylinder* in S is a Zariski open subset U in S such that

(C) $U \cong \mathbb{C}^1 \times Z$ for some affine curve Z, that is, U is a cylinder in S;
(P) there is an effective $\mathbb{Q}$-divisor D on S such that $D \sim_{\mathbb{Q}} A$ and $U = S \setminus \operatorname{Supp}(D)$.

An ample divisor A can always be chosen such that S contains an A-polar cylinder. This follows from Proposition 3.13 in [7]. On the other hand, we have the following.

Theorem 1.2 (see [9], [2] and [3]). *Let S_d be a smooth del Pezzo surface*[1] *of degree $d = K^2_{S_d}$. Then the following assertions hold*:

(1) *the surface S_d contains a $(-K_{S_d})$-polar cylinder if and only if $d \geqslant 4$;*
(2) *if $d \geqslant 4$, then S_d contains an H-polar cylinder for every ample $\mathbb{Q}$-divisor H on S_d;*
(3) *if $d = 3$, then S_d contains an H-polar cylinder for every ample $\mathbb{Q}$-divisor H on S_d such that $H \notin \mathbb{Q}_{>0}[-K_{S_d}]$.*

The paper [3] also contains one relevant result for del Pezzo surfaces of degree 1 and 2. To describe this result, let

$$\mu_A = \inf\bigl\{\lambda \in \mathbb{Q}_{>0} \mid \text{the } \mathbb{Q}\text{-divisor } K_S + \lambda A \text{ is pseudo-effective}\bigr\} \in \mathbb{Q}.$$

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[1]Unless explicitly stated otherwise, all varieties are assumed to be algebraic, projective and defined over $\mathbb{C}$.

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The number μ_A is known as the Fujita invariant, pseudo-effective threshold or spectral value of the divisor A (see [6] and [13]). Let Δ_A be the smallest extremal face of the Mori cone $\overline{\mathbb{NE}}(S)$ that contains $K_S + \mu_A A$. Denote the dimension of the face Δ_A by r_A. Observe that $r_A = 0$ if and only if S is a smooth del Pezzo surface and $\mu_A A \sim_{\mathbb{Q}} -K_S$. The number r_A is known as the Fujita rank of the divisor A (see [3]).

Theorem 1.3 (see [3]). *Let S_d be a smooth del Pezzo surface of degree $d = K_{S_d}^2$, let H be an ample $\mathbb{Q}$-divisor on S_d, and let r_H be the Fujita rank of the divisor H. Suppose that $r_H + d \leqslant 3$. Then S_d does not contain H-polar cylinders.*

At the conference "Complex affine geometry, hyperbolicity and complex analysis" held in Grenoble in October 2016, Ciro Ciliberto asked the following.

Question 1.4. Let S be a rational surface that is obtained from $\mathbb{P}^2$ by blowing up points in general position, and let A be an ample $\mathbb{Q}$-divisor on S such that $r_A + K_S^2 \leqslant 3$. Is it true that S does not contain A-polar cylinders?

Ciliberto also suggested that Question 1.4 be considered modulo Conjecture 2.3 in [4]. In this paper, we show that the answer to Question 1.4 is 'Yes'. To be precise, we prove the following.

Theorem 1.5. *Let S be a smooth rational surface that satisfies the following generality condition*:

($*$) *the self-intersection of every smooth rational curve in S is at least -1.*

Let A be an ample $\mathbb{Q}$-divisor on S, and let r_A be the Fujita rank of the divisor A. Suppose that $r_A + K_S^2 \leqslant 3$. Then S does not contain A-polar cylinders.

By Proposition 2.4 in [5], rational surfaces obtained by blowing up $\mathbb{P}^2$ at points in general position satisfy ($*$). Thus, the answer to Question 1.4 is 'Yes'.

Remark 1.6. Smooth del Pezzo surfaces satisfy ($*$). Moreover, if $K_S^2 \geqslant 1$, then the divisor $-K_S$ is ample if and only if S satisfies ($*$). This shows that Theorem 1.5 is a generalization of Theorem 1.3.

By Corollary 3.2 in [8], Theorem 1.5 implies the following.

Corollary 1.7. *Let S be a smooth rational surface that satisfies ($*$), let A be an ample $\mathbb{Z}$-divisor on S, let r_A be the Fujita rank of the divisor A, and let*

$$V = \operatorname{Spec}\biggl(\bigoplus_{n\geqslant 0} H^0(S, \mathscr{O}_S(nA))\biggr).$$

Suppose that $r_A + K_S^2 \leqslant 3$. Then V does not admit an effective action of the additive group $\mathbb{C}_+$.

The following example shows that the inequality $r_A + K_S^2 \leqslant 3$ in Theorem 1.5 is sharp.

Example 1.8. Let S be a rational surface that satisfies ($*$). Suppose that $K_S^2 \leqslant 3$. Then there exists a blow-down $f\colon S \to \mathbb{P}^2$ of $9 - K_S^2$ different points. Put $k = 4 - K_S^2 \geqslant 1$. Let $E_1, \dots, E_5$, $G_1, \dots, G_k$ be the exceptional curves of f, let $\mathscr{C}$ be the unique conic in $\mathbb{P}^2$ that passes through $f(E_1), \dots, f(E_5)$, let L be a general line in $\mathbb{P}^2$ tangent to $\mathscr{C}$, and let $\mathscr{P}$ be the pencil generated by $\mathscr{C}$ and $2L$. Denote the conic in $\mathscr{P}$ that contains $f(G_i)$ by C_i. Then

$$\mathbb{P}^2 \setminus (\mathscr{C} \cup L \cup C_1 \cup \dots \cup C)$$

is a cylinder. Denote the proper transforms of $\mathscr{C}$ and L on S by $\widetilde{\mathscr{C}}$ and $\widetilde{L}$, respectively. Similarly, denote the proper transform of the conic C_i on the surface S by $\widetilde{C}_i$. Then

$$S \setminus (\widetilde{\mathscr{C}} \cup \widetilde{L} \cup E_1 \cup \dots \cup E_5 \cup \widetilde{C}_1 \cup \dots \cup \widetilde{C}_k \cup G_1 \cup \dots \cup G_k) \cong \mathbb{P}^2 \setminus (\mathscr{C} \cup L \cup C_1 \cup \dots \cup C_k).$$

Let ε_1, ε_2 and x be rational numbers such that $1/2 > \varepsilon_1 > \varepsilon_2/2 > 0$ и $1 > x > 1-(1-2\varepsilon_1)/(2k)$. Let $A = -K_S + x(G_1 + \dots + G_k)$. Then A is ample and $r_A = k$, since

$$A \sim_{\mathbb{Q}} \left(1 + \varepsilon_1 - \frac{\varepsilon_2}{2}\right)\widetilde{\mathscr{C}} + \varepsilon_2 \widetilde{L} + \left(\varepsilon_1 - \frac{\varepsilon_2}{2}\right)\sum_{i=1}^{5} E_i + \frac{1-2\varepsilon_1}{2k}\sum_{i=1}^{k}\widetilde{C}_i + \left(x + \frac{1-2\varepsilon_1}{2k} - 1\right)\sum_{i=1}^{k} G_i.$$

Thus, the surface S contains an A-polar cylinder, and $r_A + K_S^2 = 4$.

The following example shows that the inequality $r_A + K_S^2 \geqslant 4$ does not always imply the existence of A-polar cylinders in S.

Example 1.9. Let $f\colon S \to \mathbb{P}^2$ be a blow-up of nine points such that $|-K_S|$ is a base point free pencil. Suppose that all curves in the pencil $|-K_S|$ are irreducible. Then S satisfies $(*)$. Suppose, in addition, that all singular curves in the pencil $|-K_S|$ do not have cusps. Let $E_1, \dots, E_4$ be any four f-exceptional curves. Fix $x \in \mathbb{Q}$ such that $0 < x < 1$. Let

$$A = -K_S + x(E_1 + \dots + E_4).$$

Then A is ample. Moreover, we have $r_A = 4$. Furthermore, if $x > 7/8$, then it follows from Example 1.8 that S contains an A-polar cylinder. On the other hand, the surface S does not contain A-polar cylinders for $x \leqslant 1/4$ by Lemmas 2.4, 2.6 and 2.7.

The following examples shows that we cannot omit $(*)$ in Theorem 1.5.

Example 1.10. Let L_1 and L_2 be two distinct lines in $\mathbb{P}^2$. Then

$$\mathbb{P}^2 \setminus (L_1 \cup L_2) \cong \mathbb{C}^1 \times \mathbb{C}^*.$$

Let P_1 be a point in $L_1 \setminus L_2$. Let $P_2, \dots, P_7$ be general points in $L_2 \setminus L_1$. Let $f\colon \widehat{S} \to \mathbb{P}^2$ be the blow-up of these seven points $P_1, \dots, P_7$. Denote the f-exceptional curves such that $f(F_i) = P_i$ by $F_1, \dots, F_7$. Let $g\colon \widetilde{S} \to \widehat{S}$ be the blow-up of the point in F_1 contained in the proper transform of L_1. Denote the g-exceptional curve by G. Let $\widetilde{F}_1$ be the proper transform on $\widetilde{S}$ of the curve F_1. Let $h\colon \overline{S} \to \widetilde{S}$ be the blow-up of the point $\widetilde{F}_1 \cap G$. Denote the h-exceptional curve by H. Let $e\colon \mathscr{S} \to \widetilde{S}$ be the blow-up of a general point in H. Denote the e-exceptional curve by $\mathscr{E}$. Denote the proper transforms of the curves $H, G, F_1, \dots, F_7, L_1, L_2$ on the surface $\mathscr{S}$ by $\mathscr{H}, \mathscr{G}, \mathscr{F}_1, \dots, \mathscr{F}_7, \mathscr{L}_1, \mathscr{L}_2$, respectively. Fix a positive rational number ε such that $\varepsilon < 1/3$. Then

$$-K_{\mathscr{S}} \sim_{\mathbb{Q}} (2-\varepsilon)\mathscr{L}_1 + (1+\varepsilon)\mathscr{L}_2 + (1-\varepsilon)\mathscr{F}_1 + \varepsilon\sum_{i=2}^{7}\mathscr{F}_i + (2-2\varepsilon)\mathscr{G} + (2-3\varepsilon)\mathscr{H} + (1-3\varepsilon)\mathscr{E}.$$

We also have

$$\mathscr{S} \setminus (\mathscr{L}_1 \cup \mathscr{L}_2 \cup \mathscr{F}_1 \cup \dots \cup \mathscr{F}_7 \cup \mathscr{G} \cup \mathscr{H} \cup \mathscr{E}) \cong \mathbb{P}^2 \setminus (L_1 \cup L_2).$$

Let $\pi\colon \mathscr{S} \to S$ be the contraction of the curves $\mathscr{L}_1$, $\mathscr{G}$ and $\mathscr{H}$. Then S is a smooth surface. We have $K_S^2 = 2$, the divisor $-K_S$ is nef, but

$$\pi(\mathscr{F}_1)\cdot\pi(\mathscr{F}_1) = \pi(\mathscr{L}_2)\cdot\pi(\mathscr{L}_2) = -2.$$

In particular, the surface S does not satisfy $(*)$. Let L_{12} be the line in $\mathbb{P}^2$ that contains P_1 and P_2, and let $\mathscr{L}_{12}$ be its proper transform on $\mathscr{S}$. Fix a positive rational number x such that $\varepsilon > x > 3\varepsilon - 1$. Then

$$-K_{\mathscr{S}} + x\mathscr{L}_{12} \sim_{\mathbb{Q}} (2-\varepsilon)\mathscr{L}_1 + (1+\varepsilon)\mathscr{L}_2 + (1-\varepsilon)\mathscr{F}_1 + (\varepsilon - x)\mathscr{F}_2 + \varepsilon(\mathscr{F}_3 + \dots + \mathscr{F}_7) + (2+x-2\varepsilon)\mathscr{G} + (2+x-3\varepsilon)\mathscr{H} + (1+x-3\varepsilon)\mathscr{E}.$$

Let $A = -K_S + x\pi(\mathscr{L}_{12})$. Then the divisor A is ample and $r_A = 1$, so that $r_A + K_S^2 = 3$. On the other hand, the surface S contains an A-polar cylinder, since

$$A \sim_{\mathbb{Q}} (1+\varepsilon)\pi(\mathscr{L}_2) + (1-\varepsilon)\pi(\mathscr{F}_1) + (\varepsilon - x)\pi(\mathscr{F}_2) + \varepsilon\sum_{i=3}^{7}\pi(\mathscr{F}_i) + (1+x-3\varepsilon)\pi(\mathscr{E})$$

and

$$S \setminus (\pi(\mathscr{L}_2) \cup \pi(\mathscr{F}_1) \cup \dots \cup \pi(\mathscr{F}_7) \cup \pi(\mathscr{E})) \cong \mathbb{C}^1 \times \mathbb{C}^*.$$

Now we describe the structure of this paper. In § 2 we present results that are used in the proof of Theorem 1.5. In § 3 we prove three lemmas that constitute the main part of the proof of Theorem 1.5. In § 4 we finish the proof of Theorem 1.5.

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§ 2. Preliminaries

Let S be a smooth rational surface, and let $C_1, \dots, C_n$ be irreducible curves on S. Fix nonnegative rational numbers $\lambda_1, \dots, \lambda_n$. Let $D = \lambda_1 C_1 + \dots + \lambda_n C_n$. For consistency, we will use this notation throughout the paper. In this section, we present a few well-known (local and global) results about S and D that will be used in the proof of Theorem 1.5. We start with the following.

Lemma 2.1 (see [10], Theorem 4.57, (2)). *Let P be a point in S. Suppose the singularities of the log pair (S, D) are not log canonical at P. Then* $\operatorname{mult}_P(D) > 1$.

The following lemma is a special case of a much more general result, known as the inversion of adjunction (see [10], Theorem 5.50).

Lemma 2.2 (see [10], Corollary 5.57). *Let P be a smooth point of the curve C_1. Suppose that $\lambda_1 \leqslant 1$ and the log pair (S, D) is not log canonical at P. Let*

$$\Delta = \lambda_2 C_2 + \cdots + \lambda_n C_n.$$

Then $C_1 \cdot \Delta \geqslant (C_1 \cdot \Delta)_P > 1$.

We will also use the following (local) result.

Lemma 2.3 (see [1], Theorem 13). *Let P be a point in $C_1 \cap C_2$. Suppose that $\lambda_1 \leqslant 1$ and $\lambda_2 \leqslant 1$. Suppose further that at P the curves C_1 and C_2 are smooth and intersect transversally, and that the log pair (S, D) is not log canonical at P. Let*

$$\Delta = \lambda_3 C_3 + \cdots + \lambda_n C_n.$$

If $\operatorname{mult}_P(\Delta) \leqslant 1$, *then $(C_1 \cdot \Delta)_P > 1 - \lambda_2$ or $(C_2 \cdot \Delta)_P > 1 - \lambda_1$.*

The following result was used in Example 1.9.

Lemma 2.4. *Using the assumptions and notation from Example 1.9, suppose that $D \sim_{\mathbb{Q}} A$ and $x \leqslant 1/4$. Then the log pair (S, D) is log canonical.*

Proof. Suppose that (S, D) is not log canonical at some point $P \in S$. Let $\mathscr{C}$ be the curve in the pencil $|-K_S|$ that contains P. By assumption, the curve $\mathscr{C}$ is irreducible. Moreover, its arithmetic genus is 1, so that it is either smooth or has one simple node, because we assume that curves in the pencil $|-K_S|$ do not have cusps.

If $\mathscr{C}$ is not contained in $\operatorname{Supp}(D)$, then $1 \geqslant 4x = C_1 \cdot \Delta \geqslant \operatorname{mult}_P(D) > 1$ by Lemma 2.1. This shows that $\mathscr{C}$ is contained in the support of the divisor D. Without loss of generality we can assume that $\mathscr{C} = C_1$ and $\lambda_1 > 1$. Let $\Delta = \lambda_2 C_2 + \cdots + \lambda_n C_n$.

We claim that $\lambda_1 < 1$. Indeed, we have

$$C_1 + x(E_1 + \cdots + E_4) \sim_{\mathbb{Q}} \lambda_1 C_1 + \Delta,$$

and the intersection form of the curves $E_1, \ldots, E_4$ is negative definite. Thus, if $\lambda_1 \geqslant 1$, then $\lambda_1 = 1$ and $\Delta = x(E_1 + \cdots + E_4)$, which is impossible, because the singularities of the log pair $(S, C_1 + x(E_1 + \cdots + E_4))$ are log canonical, since C_1 is either smooth or has one simple node (by assumption).

If C_1 is smooth at P, then $1 \geqslant 4x = C_1 \cdot \Delta \geqslant (C_1 \cdot \Delta)_P > 1$ by Lemma 2.2, so that the curve C_1 has a simple node at the point P. This implies that $P \notin E_1 \cup \cdots \cup E_4$, because $\mathscr{C} \cdot E_i = -K_S \cdot E_i = 1$ for every i.

We can assume that one of the curves $E_1, \ldots, E_4$ is not contained in $\operatorname{Supp}(\Delta)$, since otherwise we can swap D with the divisor

$$(1 + \mu)D - \mu(C_1 + x(E_1 + \cdots + E_4)r)$$

for an appropriate positive rational number μ. Without loss of generality, we can assume that $E_4 \not\subset \mathrm{Supp}(\Delta)$. Then

$$1 - x = E_4 \cdot (\lambda_1 C_1 + \Delta) = \lambda_1 + E_4 \cdot \Delta \geqslant \lambda_1.$$

Let $m = \mathrm{mult}_P(\Delta)$. Then $4x = C_1 \cdot \Delta \geqslant 2m$, so that $m \leqslant 2x$.

Let $f\colon \widetilde{S} \to S$ be the blow-up of the point P. Denote the f-exceptional curve by F and the proper transforms on $\widetilde{S}$ of the divisors C_1 and Δ by $\widetilde{C}_1$ and $\widetilde{\Delta}$, respectively. Then $(\widetilde{S}, \lambda_1\widetilde{C}_1 + \widetilde{\Delta} + (2\lambda_1 + m - 1)F)$ is not log canonical at some point $Q \in F$, since

$$K_{\widetilde{S}} + \lambda_1\widetilde{C}_1 + \widetilde{\Delta} + (2\lambda_1 + m - 1)F \sim_{\mathbb{Q}} f^*(K_S + D).$$

Moreover, $2\lambda_1 + m - 1 \leqslant 1$, since we have already proved that $\lambda_1 \leqslant 1 - x$ and $m \leqslant 2x$.

If $Q \notin \widetilde{C}_1$, then $(\widetilde{S}, \widetilde{\Delta} + F)$ is not log canonical at Q, so that $1/2 \geqslant 2x \geqslant m = F \cdot \widetilde{\Delta} > 1$ by Lemma 2.2. This shows that $Q \in \widetilde{C}_1$.

The curve $\widetilde{C}_1$ is smooth and intersects F transversally at Q. We know that $m \leqslant 2x \leqslant 1$. Thus, we can apply Lemma 2.3 to the log pair $(\widetilde{S}, \lambda_1\widetilde{C}_1 + \widetilde{\Delta} + (2\lambda_1 + m - 1)F)$. Then

$$4x - 2m = \widetilde{\Delta} \cdot \widetilde{C}_1 > 2(1 - (2\lambda_1 + m - 1)),$$

or $m = \widetilde{\Delta} \cdot F > 2(1 - \lambda_1)$. This leads to a contradiction, since $m \leqslant 2x$ and $\lambda_1 \leqslant 1 - x$. The lemma is proved.

In the proof of Theorem 1.5 we will use the following (global) result.

Theorem 2.5 (see [2], Theorem 1.12). *Suppose that S is a smooth del Pezzo surface such that $K_S^2 \leqslant 3$, and let*

$$D \sim_{\mathbb{Q}} -K_S.$$

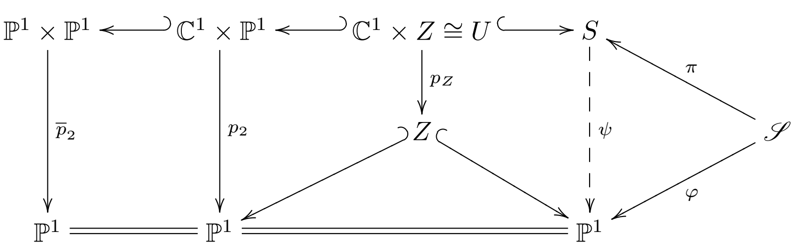
Let P be a point in S. Suppose that (S, D) is not log canonical at the point P. Then the linear system $|-K_S|$ contains a unique curve T such that (S, T) is not log canonical at P. Moreover, the support of the divisor D contains all the irreducible components of the curve T.

Let $U = S \setminus (C_1 \cup \dots \cup C_n)$. Suppose that $U \cong \mathbb{C}^1 \times Z$ for an affine curve Z.

Lemma 2.6. *The inequality $n \geqslant 10 - K_S^2$ holds.*

This follows from the proof of Lemma 4.11 in [7].

The embeddings $Z \hookrightarrow \mathbb{P}^1$ and $\mathbb{C}^1 \hookrightarrow \mathbb{P}^1$ induce the commutative diagram



where p_Z, p_2 and $\overline{p}_2$ are the projections onto the second factors, ψ is the map induced by p_Z, the map π is a birational morphism resolving the indeterminacy of ψ and φ is a morphism. Let $\mathscr{E}_1, \dots, \mathscr{E}_m$ be the π-exceptional curves (if π is an isomorphism, we let $m = 0$). Let C be the section of the projection $\overline{p}_2$ that is the complement of $\mathbb{C}^1 \times \mathbb{P}^1$ in $\mathbb{P}^1 \times \mathbb{P}^1$. Denote the proper transforms on $\mathscr{S}$ of the curves $C_1, \dots, C_n$ by $\mathscr{C}_1, \dots, \mathscr{C}_n$, respectively. Similarly, denote the proper transform of the curve C on the surface $\mathscr{S}$ by $\mathscr{C}$.

Lemma 2.7. *Suppose that $K_S + D$ is pseudo-effective, and $\lambda_i < 2$ for every i. Then $\pi(\mathscr{C})$ is a point, and (S, D) is not log canonical at $\pi(\mathscr{C})$.*

Proof. By construction, a general fibre of the morphism φ is a smooth rational curve, and the curve $\mathscr{C}$ is its section. Then $\mathscr{C}$ is either one of the curves $\mathscr{C}_1, \dots, \mathscr{C}_n$ or one of the curves $\mathscr{E}_1, \dots, \mathscr{E}_m$. All the other curves among $\mathscr{C}_1, \dots, \mathscr{C}_n$ and $\mathscr{E}_1, \dots, \mathscr{E}_m$ are mapped by φ to points in $\mathbb{P}^1$. Thus, without loss of generality we can assume that $\mathscr{C} = \mathscr{C}_1$ or $\mathscr{C} = \mathscr{E}_m$.

There are rational numbers $\mu_1, \dots, \mu_m$ such that

$$K_{\mathscr{S}} + \sum_{i=1}^{n} \lambda_i \mathscr{C}_i + \sum_{i=1}^{m} \mu_i \mathscr{E}_i = \pi^*(K_S + D).$$

Let $\mathscr{F}$ be a general fibre of the morphism φ. If $\mathscr{C} = \mathscr{C}_1$, then

$$\begin{aligned} -2 + \lambda_1 &= \left(K_{\mathscr{S}} + \sum_{i=1}^{n} \lambda_i \mathscr{C}_i + \sum_{i=1}^{m} \mu_i \mathscr{E}_i\right) \cdot \mathscr{F} \\ &= \pi^*(K_S + D) \cdot \mathscr{F} = (K_S + D) \cdot \pi(\mathscr{F}) \geqslant 0, \end{aligned}$$

because $K_S + D$ is pseudo-effective. Thus, in this case $\lambda_1 > 2$, which is impossible by assumption. Hence we conclude that $\mathscr{C} = E_m$, so that $\pi(\mathscr{C})$ is a point. Then

$$\begin{aligned} -2 + \mu_m &= \left(K_{\mathscr{S}} + \sum_{i=1}^{n} \lambda_i \mathscr{C}_i + \sum_{i=1}^{m} \mu_i \mathscr{E}_i\right) \cdot \mathscr{F} \\ &= \pi^*(K_S + D) \cdot \mathscr{F} = (K_S + D) \cdot \pi(\mathscr{F}) \geqslant 0, \end{aligned}$$

because the divisor $K_S + D$ is pseudo-effective. This shows that the singularities of the log pair (S, D) are not log canonical at the point $\pi(\mathscr{C})$. The lemma is proved.

§ 3. Three main lemmas

In this section, we prove three results which will be used later, in the proof of Theorem 1.5 in § 4, namely Lemmas 3.4–3.6 below.

Let S be a smooth rational surface that satisfies $(*)$, let $C_1, \dots, C_n$ be irreducible curves on S, let

$$U = S \setminus (C_1 \cup \dots \cup C_n)$$

and let $D = \sum_{i=1}^{n} \lambda_i C_i$ for some non-negative rational numbers $\lambda_1, \dots, \lambda_n$. Suppose also that S contains disjoint smooth rational curves $E_1, \dots, E_r$ such that $E_i^2 = -1$ for every i, and

$$D \sim_{\mathbb{Q}} -K_S + \sum_{i=1}^{r} a_i E_i$$

for some nonnegative rational numbers $a_1, \dots, a_r$.

Remark 3.1. If D is ample, then r is the Fujita rank of the divisor D. However, in this section, we deliberately do not assume that D is ample. We hope that this will not cause much confusion. We have to consider nonample divisors here, because Lemmas 3.4–3.6 can also be applied to nonample divisors, and this is also used in proving them.

Let $g\colon S \to \overline{S}$ be a blow-down of the curves $E_1, \dots, E_r$, let $\overline{C}_1 = g(C_1), \dots, \overline{C}_n = g(C_n)$, and let $\overline{D} = \lambda_1 \overline{C}_1 + \dots + \lambda_n \overline{C}_n$. Then $K_{\overline{S}}^2 = r + K_S^2$ and $\overline{D} \sim_{\mathbb{Q}} -K_{\overline{S}}$.

Remark 3.2. Since S satisfies $(*)$ by assumption, the surface $\overline{S}$ also satisfies $(*)$. In particular, if $r + K_S^2 \geqslant 1$, then $\overline{S}$ is a smooth del Pezzo surface by Remark 1.6.

First we prove an auxiliary result.

Lemma 3.3. *Suppose that $C_i \neq E_j$ for all i and j. Then the log pair (S, D) is log canonical along $E_1 \cup \dots \cup E_r$.*

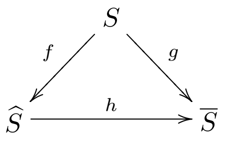
Proof. Suppose that the log pair (S, D) is not log canonical at some point $P \in E_1 \cup \dots \cup E_r$. Then $\mathrm{mult}_P(D) > 1$ by Lemma 2.1. Thus, if $P \in E_1$, then $1 \geqslant 1 - a_1 = D \cdot E_1 > 1$, which is absurd. Similarly, we see that $P \notin E_2 \cup \dots \cup E_r$. The lemma is proved.

Recall that $U = S \setminus (C_1 \cup \dots \cup C_n)$, and r is the number of g-exceptional curves.

Lemma 3.4. *Suppose that $r + K_S^2 = 1$, and $\lambda_i > 0$ for every i. Then U is not a cylinder.*

Proof. We have $U = S \setminus \mathrm{Supp}(D)$, and $\overline{S}$ is a smooth del Pezzo surface by Remark 3.2. If $K_{\overline{S}}^2 = 1$, then $r = 0$, so that $S \cong \overline{S}$ and $D \sim_{\mathbb{Q}} -K_S$. In this case, if U is a cylinder, then U is a $(-K_S)$-polar cylinder, which is impossible by Theorem 1.2. Therefore, we can assume that $K_S^2 \leqslant 0$. We prove the required assertion by induction on K_S^2.

Suppose first that $C_1 = E_1$. Then there exists a commutative diagram



where $f\colon S \to \widehat{S}$ is a contraction of the curve $C_1 = E_1$, and h is a birational morphism. Denote the proper transforms on $\widehat{S}$ of the curves $E_2, \dots, E_r$ by $\widehat{E}_2, \dots, \widehat{E}_r$, respectively and the proper transforms on $\widehat{S}$ of the curves $C_2, \dots, C_n$ by $\widehat{C}_2, \dots, \widehat{C}_n$, respectively. Then $K_{\widehat{S}}^2 = K_S^2 + 1$ and

$$-K_{\widehat{S}} + \sum_{i=2}^{r} a_i \widehat{E}_i \sim_{\mathbb{Q}} \sum_{i=2}^{n} \lambda_i \widehat{C}_i.$$

By induction, the subset $\widehat{S}\setminus(\widehat{C}_2\cup\cdots\cup\widehat{C}_n)\cong U$ is not a cylinder. Thus, we can assume that $C_1\neq E_1$. Similarly, we can assume that $C_i\neq E_j$ for all possible i and j, which means that none of the curves $E_1,\ldots,E_r$ is contained in $\mathrm{Supp}(D)$.

Suppose that U is a cylinder. Then $n\geqslant 10-K_S^2\geqslant 10$ by Lemma 2.6, and

$$1=-K_{\overline{S}}\cdot\overline{D}=-K_{\overline{S}}\cdot(\lambda_1\overline{C}_1+\cdots+\lambda_n\overline{C}_n)\geqslant\sum_{i=1}^n\lambda_i$$

because the divisor $-K_{\overline{S}}$ is ample. Thus, we see that $\lambda_i<1$ for every i.

By Lemma 2.7, the surface S contains a point P such that the log pair (S,D) is not log canonical at P. In the notation of § 2, P is the point $\pi(\mathscr{C})$. Let $\overline{P}=g(P)$. Then $(\overline{S},\overline{D})$ is not log canonical at $\overline{P}$ because $P\notin E_1\cup\cdots\cup E_r$ by Lemma 3.3.

By Theorem 2.5 there is a unique curve $\overline{T}\in|-K_{\overline{S}}|$ such that $(\overline{S},\overline{T})$ is not log canonical at the point $\overline{P}$. Note that $\overline{T}$ is irreducible. Thus, Theorem 2.5 also implies that $\overline{T}$ is one of the curves $\overline{C}_1,\ldots,\overline{C}_n$. Without loss of generality we can assume that $\overline{T}=\overline{C}_1$.

The curve $\overline{T}=\overline{C}_1$ is singular at $\overline{P}$. In fact, we can say more: this curve has a cuspidal singularity at $\overline{P}$, and it is smooth away from this point. For every $i\in\{1,\ldots,r\}$, we let

$$m_i=\begin{cases}0 & \text{if } g(E_i)\notin\overline{T},\\ 1 & \text{if } g(E_i)\in\overline{T}.\end{cases}$$

Then

$$C_1\sim g^*(\overline{C}_1)-\sum_{i=1}^r m_iE_i\sim -K_S+\sum_{i=1}^r(1-m_i)E_i.$$

We replace D by a divisor $(1+\mu)D-\mu C_1$ for an appropriate rational number $\mu>0$ such that the new divisor is effective and its support does not contain the curve C_1. Let

$$D'=\frac{1}{1-\lambda_1}D-\frac{\lambda_1}{1-\lambda_1}C_1=\sum_{i=2}^n\frac{\lambda_i}{1-\lambda_1}C_i.$$

Then D' is an effective divisor whose support does not contain the curve C_1. On the other hand, we have

$$D'\sim_{\mathbb{Q}}-K_S+\sum_{i=1}^r\frac{a_i+(m_i-1)\lambda_1}{1-\lambda_1}E_i.$$

Thus, if $(a_i+(m_i-1)\lambda_1)/(1-\lambda_1)\geqslant 0$ for every i, then (S,D') is not log canonical at P by Lemma 2.7. In this case the singularities of the log pair

$$\left(\overline{S},\sum_{i=2}^n\frac{\lambda_i}{1-\lambda_1}\overline{C}_i\right)$$

are not log canonical at the point $\overline{P}$, because $P\notin E_1\cup\cdots\cup E_r$. The latter is impossible by Theorem 2.5. Therefore, at least one rational number among

$$\frac{a_1+(m_1-1)\lambda_1}{1-\lambda_1},\ \frac{a_2+(m_2-1)\lambda_1}{1-\lambda_1},\ \ldots,\ \frac{a_r+(m_r-1)\lambda_1}{1-\lambda_1}$$

must be negative. Without loss of generality we can assume that there exists $k \leqslant r$ such that

$$\frac{a_i + (m_i - 1)\lambda_1}{1 - \lambda_1} < 0$$

for every $i \leqslant k$, and $(a_i + (m_i - 1)\lambda_1)/(1 - \lambda_1) \geqslant 0$ for every $i > k$ (if $k < r$). Then $m_1 = \cdots = m_k = 0$. We can also assume that $a_1 \leqslant \cdots \leqslant a_k$. Let

$$D'' = \frac{1}{1 - a_1} D - \frac{a_1}{1 - a_1} C_1 = \frac{\lambda_1 - a_1}{1 - a_1} C_1 + \sum_{i=2}^{n} \frac{\lambda_i}{1 - a_1} C_i.$$

Then D'' is an effective $\mathbb{Q}$-divisor such that

$$\begin{aligned} D'' &\sim_{\mathbb{Q}} -K_S + \sum_{i=2}^{r} \frac{a_i - a_1(1 - m_i)}{1 - a_1} E_i \\ &= -K_S + \sum_{i=2}^{k} \frac{a_i - a_1}{1 - a_1} E_i + \sum_{i=k+1}^{r} \frac{a_i - a_1(1 - m_i)}{1 - a_1} E_i. \end{aligned}$$

Note that $(a_i - a_1(1 - m_i))/(1 - a_1) \geqslant 0$ for every possible $i > k$, because $a_1 < \lambda_1$.

Let $e\colon \widetilde{S} \to \overline{S}$ be the blow-up of the point $g(E_1)$, and let $\widetilde{E}_1$ be its exceptional curve. Denote the proper transforms on $\widetilde{S}$ of the curves $C_1, \ldots, C_n$ by $\widetilde{C}_1, \ldots, \widetilde{C}_n$, respectively. Likewise, let $\widetilde{D}''$ denote the proper transform of the divisor D'' on the surface $\widetilde{S}$. Then

$$\widetilde{D}'' = \frac{\lambda_1 - a_1}{1 - a_1} \widetilde{C}_1 + \sum_{i=2}^{n} \frac{\lambda_i}{1 - a_1} \widetilde{C}_i \sim_{\mathbb{Q}} -K_{\widetilde{S}}.$$

Since $g(E_1) \notin \overline{T}$, the point $g(E_1)$ is not the base point of the pencil $|-K_{\overline{S}}|$. Thus, $|-K_{\overline{S}}|$ contains a unique irreducible curve that passes through $g(E_1)$. Denote this curve by $\overline{R}$, and let $\widetilde{R}$ and R be the proper transforms of this curve on the surfaces $\widetilde{S}$ and S, respectively. If $\overline{R}$ is singular at $g(E_1)$, then R is a smooth rational curve such that

$$R^2 \leqslant \widetilde{R}^2 = -3,$$

which is impossible, because S satisfies $(*)$. Thus, we see that the curve R is smooth at the point $g(E_1)$. Then $\widetilde{R} \sim -K_{\widetilde{S}}$ and $\widetilde{R}^2 = 0$. In particular, $\widetilde{R}$ is a nef divisor. On the other hand, we have $\widetilde{C}_1 \cdot \widetilde{R} = 1$, because $\overline{C}_1 = \overline{T}$ does not contain $g(E_1)$ since $m_1 = 0$. Then

$$0 = K_{\widetilde{S}}^2 = \widetilde{D}'' \cdot \widetilde{R} = \frac{\lambda_1 - a_1}{1 - a_1} \widetilde{C}_1 \cdot \widetilde{R} + \sum_{i=2}^{n} \frac{\lambda_i}{1 - a_1} \widetilde{C}_i \cdot \widetilde{R} \geqslant \frac{\lambda_1 - a_1}{1 - a_1} \widetilde{C}_1 \cdot \widetilde{R} = \frac{\lambda_1 - a_1}{1 - a_1},$$

so that $a_1 \geqslant \lambda_1$. This is a contradiction, since we have already proved that $a_1 < \lambda_1$. The lemma is proved.

Lemma 3.5. *Suppose that $r + K_S^2 = 2$ and $\lambda_i > 0$ for every i. Then U is not a cylinder.*

Proof. We have $K_{\overline{S}}^2 = 2$, so that $\overline{S}$ is a smooth del Pezzo surface by Remark 3.2. If $K_S^2 = 2$, then $r = 0$ and $S \cong \overline{S}$. In this case the required assertion follows from Theorem 1.2. Thus, we can assume that $K_S^2 \leqslant 1$. Moreover, arguing as in the proof of Lemma 3.4 we can assume that $C_i \neq E_j$ for all i and j. Then, applying Lemma 3.1 from [2] to the log pair $(\overline{S}, \overline{D})$, we conclude that $\lambda_i \leqslant 1$ for each i.

Suppose that $U = S \setminus \mathrm{Supp}(D)$ is a cylinder. Then $n \geqslant 9$ by Lemma 2.6. Moreover, by Lemma 2.7 the surface S contains a point P such that the log pair (S, D) is not log canonical at P. In the notation of § 2, the point P is the point $\pi(\mathscr{C})$. Let $\overline{P} = g(P)$. Then $(\overline{S}, \overline{D})$ is not log canonical at $\overline{P}$ because $P \notin E_1 \cup \cdots \cup E_r$ by Lemma 3.3.

By Theorem 2.5 the linear system $|-K_{\overline{S}}|$ contains a curve $\overline{T}$ such that $(\overline{S}, \overline{T})$ is not log canonical at the point $\overline{P}$, and irreducible components of the curve $\overline{T}$ are among the curves $\overline{C}_1, \ldots, \overline{C}_n$. In particular, $\overline{T}$ is singular at $\overline{P}$. Note that this property determines the curve $\overline{T}$ uniquely. Moreover, since $\overline{S}$ is a smooth del Pezzo surface of degree $K_{\overline{S}}^2 = 2$, $\overline{T}$ has at most two irreducible components. Thus, without loss of generality, we can assume that either $\overline{T} = \overline{C}_1$, or $\overline{T} = \overline{C}_1 + \overline{C}_2$ and $\lambda_1 \leqslant \lambda_2$.

If $\overline{T} = \overline{C}_1$, then $\overline{T}$ has a cuspidal singularity at $\overline{P}$. Likewise, if $\overline{T} = \overline{C}_1 + \overline{C}_2$, then $\overline{T}$ has a tacknodal singularity at $\overline{P}$. In both cases $\overline{P}$ is the unique singular point of the curve $\overline{T}$. As in the proof of Lemma 3.4, for every $i \in \{1, \ldots, r\}$, let

$$m_i = \begin{cases} 0 & \text{if } g(E_i) \notin \overline{T}, \\ 1 & \text{if } g(E_i) \in \overline{T}. \end{cases}$$

Let T be the proper transform of the curve $\overline{T}$ on the surface S. Then

$$T \sim g^*(\overline{T}) - \sum_{i=1}^{r} m_i E_i \sim -K_S + \sum_{i=1}^{r} (1 - m_i) E_i.$$

If $\overline{T} = \overline{C}_1$, then $\lambda_1 < 1$, because

$$2 = -K_{\overline{S}} \cdot \overline{D} = \sum_{i=1}^{n} \lambda_i (-K_{\overline{S}} \cdot \overline{C}_i) = 2\lambda_1 + \sum_{i=2}^{n} \lambda_i (-K_{\overline{S}} \cdot \overline{C}_i) \geqslant 2\lambda_1 + \sum_{i=2}^{n} \lambda_i > 2\lambda_1.$$

Similarly, if $\overline{T} = \overline{C}_1 + \overline{C}_2$, then $\lambda_1 < 1$, because

$$2 = -K_{\overline{S}} \cdot \overline{D} = \lambda_1 + \lambda_2 + \sum_{i=3}^{n} \lambda_i (-K_{\overline{S}} \cdot \overline{C}_i) > \lambda_1 + \lambda_2 \geqslant 2\lambda_1.$$

Let $D' = \frac{1}{1-\lambda_1} D - \frac{\lambda_1}{1-\lambda_1} T$ and $\overline{D}' = \frac{1}{1-\lambda_1} \overline{D} - \frac{\lambda_1}{1-\lambda_1} \overline{T}$. If $\overline{T} = \overline{C}_1$, then

$$D' = \sum_{i=2}^{n} \frac{\lambda_i}{1 - \lambda_1} C_i.$$

Similarly, if $\overline{T} = \overline{C}_1 + \overline{C}_2$, then

$$D' = \frac{\lambda_2 - \lambda_1}{1 - \lambda_1} C_2 + \sum_{i=3}^{n} \frac{\lambda_i}{1 - \lambda_1} C_i.$$

In both cases the divisor D' is effective, and its support does not contain C_1. On the other hand we have

$$D' \sim_{\mathbb{Q}} -K_S + \sum_{i=1}^{r} \frac{a_i + (m_i - 1)\lambda_1}{1-\lambda_1} E_i.$$

Thus, if $(a_i + (m_i-1)\lambda_1)/(1-\lambda_1) \geqslant 0$ for every i, then the log pair (S, D') is not log canonical at the point P by Lemma 2.7. Then $\overline{D}' \sim_{\mathbb{Q}} -K_{\overline{S}}$ and $(\overline{S}, \overline{D}')$ is not log canonical at $\overline{P}$, which contradicts Theorem 2.5. Hence at least one of the numbers $(a_1 + (m_1-1)\lambda_1)/(1-\lambda_1), \dots, (a_r + (m_r-1)\lambda_1)/(1-\lambda_1)$ must be negative. Without loss of generality we can assume that

$$\frac{a_i + (m_i-1)\lambda_1}{1-\lambda_1} < 0 \quad \Longleftrightarrow \quad i \leqslant k$$

for some $k \leqslant r$, and $a_1 \leqslant \dots \leqslant a_k$. Then $m_i = 0$ and $a_i < \lambda_1$ for every $i \leqslant k$.

Let $D'' = \frac{1}{1-a_1} D - \frac{a_1}{1-a_1} T$. Then D'' is effective. Indeed, if $\overline{T} = \overline{C}_1$, then

$$D'' = \frac{\lambda_1 - a_1}{1-a_1} C_1 + \sum_{i=2}^{n} \frac{\lambda_i}{1-a_1} C_i.$$

Similarly, if $\overline{T} = \overline{C}_1 + \overline{C}_2$, then

$$D'' = \frac{\lambda_1 - a_1}{1-a_1} C_1 + \frac{\lambda_2 - a_1}{1-a_1} C_2 + \sum_{i=3}^{n} \frac{\lambda_i}{1-a_1} C_i.$$

Note that $\mathrm{Supp}(D'') = \mathrm{Supp}(D)$. On the other hand we have

$$D'' \sim_{\mathbb{Q}} -K_S + \sum_{i=2}^{r} \frac{a_i - a_1(1-m_i)}{1-a_1} E_i.$$

Applying Lemma 3.4 to D'' we see that U is not a cylinder. This is a contradiction. The lemma is proved.

Lemma 3.6. *Suppose that $r + K_S^2 = 3$, and $\lambda_i > 0$ for every i. Then U is not a cylinder.*

Proof. Since $K_{\overline{S}}^2 = 3$, we see that $\overline{S}$ is a smooth cubic surface in $\mathbb{P}^3$ by Remark 3.2. Thus, if $K_S^2 = 3$, then $r = 0$ and $S \cong \overline{S}$ and $D \sim_{\mathbb{Q}} -K_S$, so that $U = S \setminus \mathrm{Supp}(D)$ is not a cylinder by Theorem 1.2. Therefore, we can assume that $K_S^2 \leqslant 2$. Moreover, arguing as in the proof of Lemma 3.4 we can assume that $C_i \neq E_j$ for all possible i and j. Then, applying Lemma 4.1 from [2] to the log pair $(\overline{S}, \overline{D})$, we conclude that $\lambda_i \leqslant 1$ for each i.

Suppose that U is a cylinder. Let us seek for a contradiction. By Lemma 2.6, we have $n \geqslant 8$. By Lemma 2.7, the surface S contains a point P such that (S, D) is not log canonical at P. In the notations of § 2, the point P is the point $\pi(\mathscr{C})$. Let $\overline{P} = g(P)$. Then $(\overline{S}, \overline{D})$ is not log canonical at $\overline{P}$ because $P \notin E_1 \cup \dots \cup E_r$ by Lemma 3.3.

Let $\overline{T}$ be the hyperplane section of $\overline{S}$ that is singular at $\overline{P}$. By Theorem 2.5, the pair $(\overline{S},\overline{T})$ is not log canonical at $\overline{P}$, and all irreducible components of the curve $\overline{T}$ are among the irreducible curves $\overline{C}_1,\ldots,\overline{C}_n$. Thus, we can assume that either

- $\overline{T}=\overline{C}_1$, or
- $\overline{T}=\overline{C}_1+\overline{C}_2$ and $\lambda_1\leqslant\lambda_2$, or
- $\overline{T}=\overline{C}_1+\overline{C}_2+\overline{C}_3$ and $\lambda_1\leqslant\lambda_2\leqslant\lambda_3$.

If $\overline{T}=\overline{C}_1$, then $\overline{T}$ has a cuspidal singularity at $\overline{P}$. Likewise, if $\overline{T}=\overline{C}_1+\overline{C}_2$, then $\overline{T}$ has a tacknodal singularity at $\overline{P}$. Finally, if $\overline{T}=\overline{C}_1+\overline{C}_2+\overline{C}_3$, then the curves $\overline{C}_1$, $\overline{C}_2$ and $\overline{C}_3$ are lines passing through the point $\overline{P}$. Therefore, in all possible cases $\overline{P}$ is the unique singular point of the curve $\overline{T}$. As in the proofs of Lemmas 3.4 and 3.5 for every $i\in\{1,\ldots,r\}$ we let

$$m_i=\begin{cases}0 & \text{if } g(E_i)\notin\overline{T},\\ 1 & \text{if } g(E_i)\in\overline{T}.\end{cases}$$

Let T be the proper transform of $\overline{T}$ on the surface S. Then

$$T\sim g^*(\overline{T})-\sum_{i=1}^{r}m_iE_i\sim -K_S+\sum_{i=1}^{r}(1-m_i)E_i.$$

We claim that $\lambda_1<1$. Indeed, if $\overline{T}=\overline{C}_1$, then

$$3=-K_{\overline{S}}\cdot\overline{D}=\sum_{i=1}^{n}\lambda_i(-K_{\overline{S}}\cdot\overline{C}_i)=3\lambda_1+\sum_{i=2}^{n}\lambda_i(-K_{\overline{S}}\cdot\overline{C}_i)\geqslant 3\lambda_1+\sum_{i=2}^{n}\lambda_i>3\lambda_1,$$

so that $\lambda_1<1$. Similarly, if $\overline{T}=\overline{C}_1+\overline{C}_2$, then $\lambda_1<1$ because

$$3=\lambda_1\deg(\overline{C}_1)+\lambda_2\deg(\overline{C}_2)+\sum_{i=3}^{n}\lambda_i(-K_{\overline{S}}\cdot\overline{C}_i)>\lambda_1\big(\deg(\overline{C}_1)+\deg(\overline{C}_2)\big)=3\lambda_1.$$

Finally, if $\overline{T}=\overline{C}_1+\overline{C}_2+\overline{C}_3$, then we also have $\lambda_1<1$ because

$$3=-K_{\overline{S}}\cdot\overline{D}=\lambda_1+\lambda_2+\lambda_3+\sum_{i=4}^{n}\lambda_i(-K_{\overline{S}}\cdot\overline{C}_i)>\lambda_1+\lambda_2+\lambda_3\geqslant 3\lambda_1.$$

Let $D'=\frac{1}{1-\lambda_1}D-\frac{\lambda_1}{1-\lambda_1}T$ and $\overline{D}'=\frac{1}{1-\lambda_1}\overline{D}-\frac{\lambda_1}{1-\lambda_1}\overline{T}$. If $\overline{T}=\overline{C}_1$, then

$$D'=\sum_{i=2}^{n}\frac{\lambda_i}{1-\lambda_1}C_i.$$

Similarly, if $\overline{T}=\overline{C}_1+\overline{C}_2$, then

$$D'=\frac{\lambda_2-\lambda_1}{1-\lambda_1}C_2+\sum_{i=3}^{n}\frac{\lambda_i}{1-\lambda_1}C_i.$$

Finally, if $\overline{T} = \overline{C}_1 + \overline{C}_2 + \overline{C}_3$, then

$$D' = \frac{\lambda_2 - \lambda_1}{1 - \lambda_1} C_2 + \frac{\lambda_3 - \lambda_1}{1 - \lambda_1} C_3 + \sum_{i=4}^{n} \frac{\lambda_i}{1 - \lambda_1} C_i.$$

Therefore, in all cases, the divisor D' is effective, and its support does not contain the curve C_1. On the other hand, we have

$$D' \sim_{\mathbb{Q}} -K_S + \sum_{i=1}^{r} \frac{a_i + (m_i - 1)\lambda_1}{1 - \lambda_1} E_i.$$

Thus, if $(a_i + (m_i - 1)\lambda_1)/(1 - \lambda_1) \geqslant 0$ for every i, then (S, D') is not log canonical at P by Lemma 2.7, so that the log pair $(\overline{S}, \overline{D}')$ is not log canonical at $\overline{P}$, which contradicts Theorem 2.5, because $\overline{D}' \sim_{\mathbb{Q}} -K_{\overline{S}}$ and the support of the divisor $\overline{D}'$ does not contain the curve $\overline{C}_1$. Hence at least one number among $(a_1 + (m_1 - 1)\lambda_1)/(1 - \lambda_1), \ldots, (a_r + (m_r - 1)\lambda_1)/(1 - \lambda_1)$ is negative.

Without loss of generality we can assume that

$$\frac{a_i + (m_i - 1)\lambda_1}{1 - \lambda_1} < 0 \quad \Longleftrightarrow \quad i \leqslant k$$

for some $k \leqslant r$, and $a_1 \leqslant \cdots \leqslant a_k$. Then $m_i = 0$ and $a_i < \lambda_1$ for every $i = 1, \ldots, k$.

Put $D'' = \frac{1}{1-a_1} D - \frac{a_1}{1-a_1} T$. Then D'' is an effective divisor. Indeed, if $\overline{T} = \overline{C}_1$, then

$$D'' = \frac{\lambda_1 - a_1}{1 - a_1} C_1 + \sum_{i=2}^{n} \frac{\lambda_i}{1 - a_1} C_i.$$

Similarly, if $\overline{T} = \overline{C}_1 + \overline{C}_2$, then

$$D'' = \frac{\lambda_1 - a_1}{1 - a_1} C_1 + \frac{\lambda_2 - a_1}{1 - a_1} C_2 + \sum_{i=3}^{n} \frac{\lambda_i}{1 - a_1} C_i.$$

Finally, if $\overline{T} = \overline{C}_1 + \overline{C}_2 + \overline{C}_3$, then

$$D'' = \frac{\lambda_1 - a_1}{1 - a_1} C_1 + \frac{\lambda_2 - a_1}{1 - a_1} C_2 + \frac{\lambda_3 - a_1}{1 - a_1} C_3 + \sum_{i=4}^{n} \frac{a_i}{1 - a_1} C_i.$$

In all cases $\mathrm{Supp}(D'') = \mathrm{Supp}(D)$. On the other hand, we have

$$D'' \sim_{\mathbb{Q}} -K_S + \sum_{i=2}^{r} \frac{a_i - a_1(1 - m_i)}{1 - a_1} E_i.$$

Applying Lemma 3.5 to D'' we see that U is not a cylinder. This is a contradiction. The lemma is proved.

§ 4. The proof

In this section we prove Theorem 1.5 using Lemmas 3.4–3.6.

Let S be a smooth rational surface, let A be an ample $\mathbb{Q}$-divisor on S, and let μ_A be its Fujita invariant. Then

$$K_S + \mu_A A \in \partial\overline{\mathbb{NE}(S)}.$$

Thus, the divisor $K_S + \mu_A A$ is pseudo-effective, and it is not big. Let Δ_A be the smallest extremal face of the cone $\overline{\mathbb{NE}(S)}$ that contains $K_S + \mu_A A$, and let r_A be the dimension of this face, that is, r_A is the Fujita rank of the divisor A. To prove Theorem 1.5 we have to show that S does not contain A-polar cylinders if S satisfies $(*)$, and $r_A + K_S^2 \leqslant 3$.

First, we describe the Zariski decomposition of the divisor $K_S + \mu_A A$, which follows from Theorem 1 in [13] or [12]. To be precise, we have the following.

Lemma 4.1. *There is a birational morphism $g\colon S \to \overline{S}$ such that $\overline{S}$ is smooth, and*

$$K_S + \mu_A A \sim_{\mathbb{Q}} g^*(K_{\overline{S}} + \mu_A \overline{A}) + \sum_{i=1}^{r} a_i E_i,$$

where $E_1, \dots, E_r$ are all g-exceptional curves, $a_1, \dots, a_r$ are positive rational numbers, $\overline{A} = g_*(A)$, the divisor $K_{\overline{S}} + \mu_A \overline{A}$ is nef, and

$$(K_{\overline{S}} + \mu_A \overline{A})^2 = 0.$$

Moreover, one of the following two cases holds:

(1) *$\overline{S}$ is a smooth del Pezzo surface, $K_{\overline{S}} + \mu_A \overline{A} \sim_{\mathbb{Q}} 0$, and $r = r_A$;*

(2) *there exists a conic bundle $h\colon \overline{S} \to \mathbb{P}^1$ such that $K_{\overline{S}} + \mu_A \overline{A} \sim_{\mathbb{Q}} qF$ for a positive rational number q, where F is a fibre of h, and $r_A = \operatorname{rk}\operatorname{Pic}(S) - 1$.*

Proof. The surface S contains an irreducible curve C such that $\mu_A A \sim_{\mathbb{Q}} aC$ for some positive rational number a, and the singularities of the log pair (S, aC) are log terminal. Thus, we can apply the Log Minimal Model Program to this log pair (see [10]).

If $K_S + aC \sim_{\mathbb{Q}} 0$, the required assertion is obvious. Likewise, if $K_S + aC \not\sim_{\mathbb{Q}} 0$ and the divisor $K_S + aC$ is nef, then $(K_S + aC)^2 = 0$, because $K_S + aC$ is not big by assumption. In this case the required assertion follows from [10], Theorem 3.3, because C is ample. Thus, we can assume that $K_S + aC$ is not nef.

If $\operatorname{rk}\operatorname{Pic}(S) = 1$, then $S = \mathbb{P}^2$. If $\operatorname{rk}\operatorname{Pic}(S) = 2$, then S is one of Hirzebruch surfaces. In both cases the required assertion is obvious. Thus, we can assume that $\operatorname{rk}\operatorname{Pic}(S) \geqslant 3$.

Then, since $K_S + aC$ is not nef, there exists a birational map $g_1\colon S \to S_1$ that contracts an irreducible curve E_1 such that $(K_S + aC) \cdot E_1 < 0$. Since C is ample, we see that $E_1 \neq C$ and $K_S \cdot E_1 < 0$, which implies that E_1 is a smooth rational curve, and $E_1^2 = -1$. In particular, the surface S_1 is smooth.

Let $C_1 = g(C)$. Then

$$K_S + aC \sim_{\mathbb{Q}} g_1^*(K_{S_1} + aC_1) + b_1 E_i$$

for some rational number $b_1 > 0$. Then (S_1, aC_1) is log terminal, the divisor aC_1 is ample, and the divisor $K_{S_1} + aC_1$ is contained in the boundary of the Mori cone $\overline{\mathbb{NE}}(S_1)$. Hence we can apply the same arguments to $K_{S_1} + aC_1$ and iterate the whole process. Eventually, after finitely many steps this gives us the required assertions. The lemma is proved.

Now we suppose that $r_A + K_S^2 \leqslant 3$. Since $\operatorname{rk}\operatorname{Pic}(S) = 10 - K_S^2$, the face Δ_A has large codimension in $\overline{\mathbb{NE}(S)}$. Thus, by Lemma 4.1 the nef part of the Zariski decomposition of the divisor $K_S + \mu_A A$ is trivial, and there exists a birational morphism $g\colon S \to \overline{S}$ such that $\overline{S}$ is a smooth del Pezzo surface, g contracts r_A smooth rational curves, and

$$\mu_A A \sim_{\mathbb{Q}} -K_S + \sum_{i=1}^{r_A} a_i E_i$$

where $E_1, \dots, E_{r_A}$ are g-exceptional curves, and $a_1, \dots, a_{r_A}$ are positive rational numbers. Observe also that $K_{\overline{S}}^2 = r_A + K_S^2$, so that $r_A + K_S^2 \geqslant 1$.

Finally, we suppose that S satisfies $(*)$. Then the curves $E_1, \dots, E_{r_A}$ must be disjoint, so that $E_1^2 = E_2^2 = \dots = E_{r_A}^2 = -1$.

To prove Theorem 1.5, we have to show that S does not contain A-polar cylinders. Suppose that this is not the case. Then there is an effective $\mathbb{Q}$-divisor D on the surface S such that $S \setminus \operatorname{Supp}(D)$ is a cylinder, and $D \sim_{\mathbb{Q}} A$. This contradicts Lemmas 3.4–3.6, because $r_A + K_S^2 \in \{1, 2, 3\}$.

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