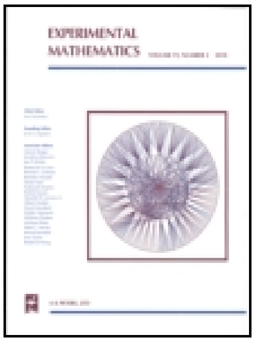

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Del Pezzo Zoo

Ivan Cheltsov and Constantin Shramov

CONTENTS

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We study del Pezzo surfaces that are quasismooth and well-formed weighted hypersurfaces. In particular, we find all such surfaces whose α-invariant of Tian is greater than 2/3.

1. INTRODUCTION

In this paper, all varieties are assumed to be complex, projective and normal. Let X be a hypersurface in $\mathbb{P}(a_0,\ldots,a_n)$ of degree d, where $a_0 \leq \cdots \leq a_n$. Then X is given by

$$\phi(x_0,\ldots,x_n) = 0 \subset \mathbb{P}(a_0,\ldots,a_n) \cong \operatorname{Proj}(\mathbb{C}[x_0,\ldots,x_n]),$$

where $\mathrm{wt}(x_i) = a_i$, and ϕ is a quasihomogeneous polynomial of degree d. The equation

$$\phi(x_0,\ldots,x_n) = 0 \subset \mathbb{C}^{n+1} \cong \operatorname{Spec}(\mathbb{C}[x_0,\ldots,x_n])$$

defines a quasihomogeneous singularity (V, O), where O is the origin of $\mathbb{C}^{n+1}$.

Definition 1.1. The hypersurface X is quasismooth if the singularity (V, O) is isolated.

Remark 1.2. Suppose that X is quasismooth. It follows from [Kollár 97, Theorem 7.9], [Kollár 97, Proposition 8.13], and [Kollár 97, Remark 8.14.1] that $\sum_{i=0}^{n} a_i > d$ if and only if the singularity (V, O) is canonical. Moreover, since (V, O) is Gorenstein, it is canonical if and only if it is rational (see [Kollár 97, Theorem 11.1]).

Definition 1.3. The hypersurface $X \subset \mathbb{P}(a_0,\ldots,a_n)$ is *well formed* if

$$\gcd(a_0,\ldots,\widehat{a}_i,\ldots,\widehat{a}_j,\ldots,a_n) \mid d$$

and

$$\gcd(a_0,\ldots,\widehat{a}_i,\ldots,a_n) = 1$$

for every $i \neq j$.

Suppose that X is well formed. Then $\sum_{i=0}^{n} a_i > d$ if and only if X is a Fano variety. Put

$$I = \sum_{i=0}^{n} a_i - d,$$

and suppose that $\sum_{i=0}^{n} a_i > d$. We call I the index of the Fano variety X. Note that I should not be confused with the Fano index of X (see Remark 1.8).

Definition 1.4. The *global log canonical threshold* of the Fano variety X is the real number

$$\begin{aligned}\mathrm{lct}(X) &= \sup\{\lambda \in \mathbb{Q} \mid \text{the log pair } (X, \lambda D) \text{ is log canonical} \\ &\qquad \text{for every effective } \mathbb{Q}\text{-divisor } D \equiv -K_X\}.\end{aligned}$$

The number $\mathrm{lct}(X)$ is an algebraic counterpart of the α-invariant introduced in [Tian 87]. In particular, the global log canonical threshold and the α-invariant are known to coincide in the nonsingular case (see, e.g., [Cheltsov and Shramov 08, Theorem A.3]). One of the important applications of (either of) these invariants is the problem of existence of an orbifold Kähler–Einstein metric on the variety X.

Theorem 1.5. [Tian 87, Demailly and Kollár, 01] *The variety X admits an orbifold Kähler–Einstein metric if*

$$\mathrm{lct}(X) > \frac{\dim(X)}{\dim(X) + 1}.$$

There are Fano orbifolds that do not admit orbifold Kähler–Einstein metrics [Matsushima 57, Futaki 83, Gauntlett et al. 07].

Theorem 1.6. [Gauntlett et al. 07] *The variety X admits no Kähler–Einstein metrics if either $I > na_0$ or*

$$dI^n > n^n \prod_{i=0}^{n} a_i.$$

The two inequalities mentioned in Theorem 1.6 are known as Lichnerowicz and Bishop obstructions, respectively. A remarkable fact is that in our case they are not independent. Namely, we prove the following result in Section 3.

Theorem 1.7. *Let $\bar{a}_0 \leq \bar{a}_1 \leq \cdots \leq \bar{a}_n$ and $\bar{d}$ be positive real numbers such that*

$$\bar{d}\Bigg(\sum_{i=0}^{n} \bar{a}_i - \bar{d}\Bigg)^n > n^n \prod_{i=0}^{n} \bar{a}_i$$

and $\bar{d} < \sum_{i=0}^{n} \bar{a}_i$. Then $\sum_{i=0}^{n} \bar{a}_i - \bar{d} > n\bar{a}_0$.

It is well known that $I \leq n = \dim(X) + 1$ if X is smooth. On the other hand, we know that

$$dI^n > n^n \prod_{i=0}^{n} a_i \iff I(-K_X)^{n-1} > (\dim(X) + 1)^n.$$

Remark 1.8. Let U be a smooth Fano variety of dimension m. Define the Fano index $\beth$ of U to be the maximal integer such that $-K_U \sim \beth H$ for some $H \in \mathrm{Pic}(U)$. Then the inequality

$$\beth(-K_U)^m \leq (\dim(U) + 1)^{m+1}$$

fails in general if $m \gg 1$ [Debarre 01, Proposition 5.22]. But we always have $\beth \leq m + 1$.

Suppose that $n = 3$. Then X is a del Pezzo surface with at most quotient singularities, which is an interesting object of study, in particular from the point of view of the question of existence of orbifold Kähler–Einstein metrics and Sasakian–Einstein structures (see, e.g., [Johnson and Kollár 01, Araujo 02, Boyer et al. 02, Boyer et al. 03]) and some others (see, e.g., [Elagin 07]). The classification of such surfaces X with $I = 1$ is known due to [Johnson and Kollár 01].

Theorem 1.9. [Johnson and Kollár 01, Theorem 8] *Suppose that $I = 1$. Then either $(a_0, a_1, a_2, a_3, d) = (2, 2m+1, 2m+1, 4m+1, 8m+4)$, where m is a positive integer, or the quintuple (a_0, a_1, a_2, a_3, d) lies in the sporadic set*

$$\begin{aligned}&\{(1,1,1,1,3),\ (1,1,1,2,4),\ (1,1,2,3,6),\ (1,2,3,5,10),\\ &(1,3,5,7,15),\ (1,3,5,8,16),\ (2,3,5,9,18),\ (3,3,5,5,15),\\ &(3,5,7,11,25),\ (3,5,7,14,28),\ (3,5,11,18,36),\\ &(5,14,17,21,56),\ (5,19,27,31,81),\ (5,19,27,50,100),\\ &(7,11,27,37,81),\ (7,11,27,44,88),\ (9,15,17,20,60),\\ &(9,15,23,23,69),\ (11,29,39,49,127),\\ &(11,49,69,128,256), (13,23,35,57,127),\\ &(13,35,81,128,256)\}.\end{aligned}$$

Note that we cannot apply Theorem 1.5 to the surface X if $I \geq 3a_0/2$, because $\mathrm{lct}(X) \leq a_0/I$.

The authors of [Boyer et al. 03] went further to classify the cases with $2 \leq I \leq 10$ and to suggest that I cannot attain larger values.

Theorem 1.10. [Boyer et al. 03, Theorem 4.5] *Suppose that* $2 \leq I \leq 10$ *and* $I < 3a_0/2$. *Then one of the following holds:*

1. *There exist a nonnegative integer* $k < I$ *and a positive integer* $a \geq I + k$ *such that*

$$(a_0, a_1, a_2, a_3, d) = (I - k, I + k, a, a + k, 2a + k + I).$$

2. *The quintuple* (a_0, a_1, a_2, a_3, d) *belongs to one of the following infinite series:*

$$\begin{gathered}(3, 3m, 3m + 1, 3m + 1, 9m + 3),\\ (3, 3m + 1, 3m + 2, 3m + 2, 9m + 6),\\ (3, 3m + 1, 3m + 2, 6m + 1, 12m + 5),\\ (3, 3m + 1, 6m + 1, 9m, 18m + 3),\\ (3, 3m + 1, 6m + 1, 9m + 3, 18m + 6),\\ (4, 2m + 3, 2m + 3, 4m + 4, 8m + 12),\\ (4, 2m + 3, 4m + 6, 6m + 7, 12m + 18),\\ (6, 6m + 3, 6m + 5, 6m + 5, 18m + 15),\\ (6, 6m + 5, 12m + 8, 18m + 9, 36m + 24),\\ (6, 6m + 5, 12m + 8, 18m + 15, 36m + 30),\\ (8, 4m + 5, 4m + 7, 4m + 9, 12m + 23),\\ (9, 3m + 8, 3m + 11, 6m + 13, 12m + 35),\end{gathered}$$

where m *is a positive integer.*

3. *The quintuple* (a_0, a_1, a_2, a_3, d) *lies in the sporadic set shown in Figure 1.*

Remark 1.11. Note that Theorem 1.10 differs from [Boyer et al. 03, Theorem 4.5] in the following ways:

1. The series $(3, 3m + 1, 3m + 2, 6m + 1, 12m + 5)$ is omitted in [Boyer et al. 03, Theorem 4.5].
2. We have removed the quintuple $(5, 7, 8, 9, 23)$ from the list of sporadic cases, since $(5, 7, 8, 9, 23) = (I - k, I + k, a, a + k, 2a + k + I)$ for $I = 6$, $k = 1$ and $a = 8$.
3. The infinite series in [Boyer et al. 03, Theorem 4.5] corresponding to our series $(4, 2m + 3, 4m + 6, 6m + 7, 12m + 18)$ starts from $m = 0$; we have shifted it and extracted the sporadic case $(3, 4, 6, 7, 18)$ corresponding to $m = 0$.
4. The infinite series in [Boyer et al. 03, Theorem 4.5] corresponding to our series $(8, 4m + 5, 4m + 7, 4m + 9, 12m + 23)$ in [Boyer et al. 03, Theorem 4.5] starts with $m = 0$; we have shifted it and extracted the sporadic case $(5, 7, 8, 9, 23)$ corresponding to $m = 0$.
5. The infinite series in [Boyer et al. 03, Theorem 4.5] corresponding to our series $(9, 3m + 8, 3m + 11, 6m + 13, 12m + 35)$ starts with $m = -1$; we have shifted it and extracted the sporadic case $(8, 9, 11, 13, 35)$ corresponding to $m = 0$ (note that the quintuple $(5, 7, 8, 9, 23)$ corresponding to $m = -1$ has already appeared from the previous series).

Remark 1.12. Arguing as in the proof of [Boyer et al. 03, Lemma 5.2], one can show that

$$\mathrm{lct}(X) \geq 2/3 \iff (a_0, a_1, a_2, a_3, d) \in \{(1, 1, 1, 1, 3), (1, 1, 2, 3, 6)\}$$

in the case that $(a_0, a_1, a_2, a_3, d) = (I - k, I + k, a, a + k, 2a + k + I)$ for some nonnegative integer $k < I$

$$\left\{\begin{matrix}(2, 3, 4, 7, 14), & (3, 4, 5, 10, 20), & (3, 4, 6, 7, 18), & (3, 4, 10, 15, 30), & (5, 13, 19, 22, 57),\\ (5, 13, 19, 35, 70), & (6, 9, 10, 13, 36), & (7, 8, 19, 25, 57), & (7, 8, 19, 32, 64), & \\ (9, 12, 13, 16, 48), & (9, 12, 19, 19, 57), & (9, 19, 24, 31, 81), & (10, 19, 35, 43, 105), & \\ (11, 21, 28, 47, 105), & (11, 25, 32, 41, 107), & (11, 25, 34, 43, 111), & (11, 43, 61, 113, 226), & \\ (13, 18, 45, 61, 135), & (13, 20, 29, 47, 107), & (13, 20, 31, 49, 111), & (13, 31, 71, 113, 226), & \\ (14, 17, 29, 41, 99), & (5, 7, 11, 13, 33), & (5, 7, 11, 20, 40), & (11, 21, 29, 37, 95), & \\ (11, 37, 53, 98, 196), & (13, 17, 27, 41, 95), & (13, 27, 61, 98, 196), & (15, 19, 43, 74, 148), & \\ (9, 11, 12, 17, 45), & (10, 13, 25, 31, 75), & (11, 17, 20, 27, 71), & (11, 17, 24, 31, 79), & \\ (11, 31, 45, 83, 166), & (13, 14, 19, 29, 71), & (13, 14, 23, 33, 79), & (13, 23, 51, 83, 166), & \\ (11, 13, 19, 25, 63), & (11, 25, 37, 68, 136), & (13, 19, 41, 68, 136), & (11, 19, 29, 53, 106), & \\ (13, 15, 31, 53, 106), & (11, 13, 21, 38, 76), & (3, 7, 8, 13, 29), & (3, 10, 11, 19, 41), & \\ (5, 6, 8, 9, 24), & (5, 6, 8, 15, 30), & (2, 3, 4, 5, 12), & (7, 10, 15, 19, 45), & \\ (7, 18, 27, 37, 81), & (7, 15, 19, 32, 64), & (7, 19, 25, 41, 82), & (7, 26, 39, 55, 117). & \end{matrix}\right\}.$$

FIGURE 1. Sporadic quintuples appearing in Theorem 1.10.

$$\left\{\begin{array}{l}(1,1,2,2,4),\ (1,4,5,7,15),\ (1,4,5,8,16),\ (1,5,7,11,22),\ (1,6,9,13,27),\\ (1,7,12,18,36),\ (1,8,13,20,40),\ (1,9,15,22,45),\ (1,3,4,6,12),\ (1,4,6,9,18),\\ (1,6,10,15,30),\ (2,3,4,5,12),\ (2,3,4,7,14),\ (3,4,5,10,20),\ (3,4,6,7,18),\ (3,4,10,15,30),\\ (3,4,6,7,18),\ (5,13,19,22,57),\ (5,13,19,35,70),\ (6,9,10,13,36),\ (7,8,19,25,57),\\ (7,8,19,32,64),\ (9,12,13,16,48),\ (9,12,19,19,57),\ (9,19,24,31,81),\\ (10,19,35,43,105),\ (11,21,28,47,105),\ (11,25,32,41,107),\ (11,25,34,43,111),\\ (11,43,61,113,226),\ (13,18,45,61,135),\ (13,20,29,47,107),\\ (13,20,31,49,111),\ (13,31,71,113,226),\ (14,17,29,41,99)\end{array}\right\}.$$

FIGURE 2. Sporadic quintuples appearing in Corollary 1.14.

and some positive integer $a \geq I + k$ (cf. [Cheltsov 08, Theorem-1.7]). These two cases are exactly those in which X is smooth.

The main purpose of this paper is to prove a technical result, Theorem 2.2, which we derive from the classification of isolated quasihomogeneous rational three-dimensional hypersurface singularities obtained in [Yau and Yu 03]. While not very attractive on its own, Theorem 2.2 easily implies the following.

Theorem 1.13. *The assertion of Theorem 1.10 holds without the assumption $I \leq 10$.*

Therefore, we obtain a proof of the (corrected version of the) half-experimental result of [Boyer et al. 03] (i.e., Theorem 1.10) modulo [Yau and Yu 03].

As our second application of Theorem 2.2, we derive from it a classificatory result in the style of [Johnson and Kollár 01] that is more explicit than the corresponding result of [Boyer et al. 03]. Namely, we list the cases with $I = 2$. Note that obtaining the list of the cases with any bounded index requires just a bit of elementary computation modulo Theorem 2.2.

Corollary 1.14. *Suppose that $I = 2$. Then one of the following holds* (1) $(a_0, a_1, a_2, a_3, d) = (1, 1, s, r, s + r)$, *where $s \leqslant r$ are positive integers,*

(2) *The quintuple (a_0, a_1, a_2, a_3, d) belongs to one of the following infinite series:*

- $(1, 2, m + 1, m + 2, 2m + 4)$,
- $(1, 3, 3m, 3m + 1, 6m + 3)$,
- $(1, 3, 3m + 1, 3m + 2, 6m + 5)$,
- $(3, 3m, 3m + 1, 3m + 1, 9m + 3)$,
- $(3, 3m + 1, 3m + 2, 3m + 2, 9m + 6)$,
- $(3, 3m + 4, 3m + 5, 6m + 7, 12m + 17)$,
- $(3, 3m + 1, 6m + 1, 9m, 18m + 3)$,
- $(3, 3m + 1, 6m + 1, 9m + 3, 18m + 6)$,
- $(4, 2m + 3, 2m + 3, 4m + 4, 8m + 12)$,
- $(4, 2m + 3, 4m + 6, 6m + 7, 12m + 18)$,

where m is a positive integer,

(3) *The quintuple (a_0, a_1, a_2, a_3, d) lies in the sporadic set shown in Figure 2.*

As was already mentioned above, an interesting question about a surface X is whether it admits an orbifold Kähler–Einstein metric. Some obstructions are provided by Theorem 1.6, and the main instrument to prove the existence is the sufficient condition given by Theorem 1.5. Most of the examples mentioned in Theorems 1.9 and 1.10 have already been studied from this point of view. As for the series omitted in [Boyer et al. 03], we have the following.

Theorem 1.15. *Suppose that*

$$(a_0, a_1, a_2, a_3, d) = (3, 3m + 1, 3m + 2, 6m + 1, 12m + 5),$$

where $m \in \mathbb{Z}_{>0}$. Then $\mathrm{lct}(X) = 1$.

Theorem 1.15 can be proved along the same lines as the results of [Cheltsov et al. 10].

The results of [Tian 90, Johnson and Kollár 01, Araujo 02, Boyer et al. 02, Boyer et al. 03, Cheltsov et al. 10] together with Theorem 1.15 imply the following result concerning orbifold Kähler–Einstein metrics on the del Pezzo hypersurfaces X.

Corollary 1.16. *Suppose that $I < 3a_0/2$. Then either X admits an orbifold Kähler–Einstein metric, or one of the following possible exceptions occur:*

- *there exist a nonnegative integer $k < I$ and a positive integer $a \geq I + k$ such that*

$$(a_0, a_1, a_2, a_3, d) = (I-k, I+k, a, a+k, 2a+k+I),$$

- *the quintuple (a_0, a_1, a_2, a_3, d) lies in the set*

$$\left\{\begin{array}{l}(2,3,4,7,14),(7,10,15,19,45),\\ \quad(7,18,27,37,81),\\ (7,15,19,32,64),(7,19,25,41,82),\\ \quad(7,26,39,55,117)\end{array}\right\},$$

- $(a_0, a_1, a_2, a_3, d) = (1, 3, 5, 7, 15)$ *and* $\phi(x_0, x_1, x_2, x_3)$ *does not contain* $x_1x_2x_3$,
- $(a_0, a_1, a_2, a_3, d) = (2, 3, 4, 5, 12)$ *and* $\phi(x_0, x_1, x_2, x_3)$ *does not contain* $x_1x_2x_3$.

Remark 1.17. One can show that there are infinitely many quintuples

$$(i-k, i+k, a, a+k, 2a+k+i)$$

such that there exists a quasismooth well-formed hypersurface in $\mathbb{P}(i-k, i+k, a, a+k)$ of degree $2a+k+i$, where k, a, i are nonnegative integers such that $0 \leq k < i$ and $a \geq i+k$.

Example 1.18. A general hypersurface in $\mathbb{P}(1, 2n-1, 2n-1, 3n-2)$ of degree $6n-3$ is a quasismooth well-formed del Pezzo surface for every positive integer n. This series corresponds to the values $k = n-1$, $a = 2n-1$, and $i = n$ of Remark 1.17.

2. TECHNICAL RESULTS

Let X be a quasismooth hypersurface in $\mathbb{P}(a_0, a_1, a_2, a_3)$ of degree d (throughout this section we will not assume that the numbers a_i are ordered). The hypersurface X is given by

$$\phi(x, y, z, w) = 0 \subset \mathbb{P}(a_0, a_1, a_2, a_3) \cong \mathrm{Proj}(\mathbb{C}[x, y, z, w]),$$

where $\mathrm{wt}(x) = a_0$, $\mathrm{wt}(y) = a_1$, $\mathrm{wt}(z) = a_2$, $\mathrm{wt}(w) = a_3$, and $\phi(x, y, z, w)$ is a quasihomogeneous polynomial of degree d.

Definition 2.1. We say that X is *degenerate* if $d = a_i$ for some i (cf. [Iano-Fletcher 00, Definition 6.5]).

The purpose of this section is to prove the following result.

Theorem 2.2. *Suppose that $a_0 \leqslant \cdots \leqslant a_3$ and the hypersurface $X \subset \mathbb{P}(a_0, a_1, a_2, a_3)$ is a well-formed nondegenerate del Pezzo surface. Then one of the following holds:*

1. *There exist a nonnegative integer $k < I$ and a positive integer $a \geq I + k$ such that*
$$(a_0, a_1, a_2, a_3, d) = (I-k, I+k, a, a+k, 2a+k+I),$$
2. *$I = a_i + a_j$ for some distinct i and j;*
3. *$I = a_i + \frac{a_j}{2}$ for some distinct i and j;*
4. *$(a_0, a_1, a_2, a_3, d, I)$ belongs to one of the infinite series listed in Table 1;*
5. *$(a_0, a_1, a_2, a_3, d, I)$ lies in the sporadic set listed in Table 2.*

Remark 2.3. Note that the first three cases of Theorem 2.2 are not mutually exclusive. On the other hand, since the most interesting cases (say, from the point of view of Kähler–Einstein metrics) appear in the last two cases of Theorem 2.2, we designed Tables 1 and 2 so that the cases listed there are mutually exclusive, and none of them is contained in any of the first three cases of Theorem 2.2. One can check that for each sextuple $(a_0, a_1, a_2, a_3, d, I)$ listed in Tables 1 and 2, there exists a well-formed quasismooth hypersurface in $\mathbb{P}(a_0, a_1, a_2, a_3)$ of degree d (apparently, this is not the case with the first three cases of Theorem 2.2).

Remark 2.4. If $I = a_i + a_j$ or $I = a_i + a_j/2$ for some i and j, then $\mathrm{lct}(X) \leq 2/3$. Unfortunately, we do not know how to handle the problem of existence of Kähler–Einstein metrics in these cases. Nor do we know this for the first case of Theorem 2.2. Note that the Bishop and Lichnerowicz obstructions (see Theorem 1.6) are not enough to settle this question.

The proof of Theorem 2.2 is based on the classification of isolated three-dimensional quasihomogeneous rational hypersurface singularities. Consider a singularity (V, O) defined by the equation

$$\phi(x, y, z, w) = 0 \subset \mathbb{C}^4 \cong \mathrm{Spec}(\mathbb{C}[x, y, z, w]),$$

where O is the origin of $\mathbb{C}^4$. Suppose that (V, O) is an isolated singularity (this happens if and only if the corresponding hypersurface $X \subset \mathbb{P}(a_0, a_1, a_2, a_3)$ is quasismooth). Suppose also that V is indeed singular at the point O, i.e., $\mathrm{mult}_O(V) \geqslant 2$ (this happens if and only if the corresponding hypersurface $X \subset \mathbb{P}(a_0, a_1, a_2, a_3)$ is nondegenerate). The following classificatory result may

(a_0, a_1, a_2, a_3)	d	I	Source
$(1, 3n-2, 4n-3, 6n-5)$	$12n-9$	n	VII.2(3)
$(1, 3n-2, 4n-3, 6n-4)$	$12n-8$	n	II.2(2)
$(1, 4n-3, 6n-5, 9n-7)$	$18n-14$	n	VII.3(1)
$(1, 6n-5, 10n-8, 15n-12)$	$30n-24$	n	III.1(4)
$(1, 6n-4, 10n-7, 15n-10)$	$30n-20$	n	III.2(2)
$(1, 6n-3, 10n-5, 15n-8)$	$30n-15$	n	III.2(4)
$(1, 8n-2, 12n-3, 18n-5)$	$36n-9$	$2n$	IV.3(3)
$(2, 6n-3, 8n-4, 12n-7)$	$24n-12$	$2n$	II.2(4)
$(2, 6n+1, 8n+2, 12n+3)$	$24n+6$	$2n+2$	II.2(1)
$(3, 6n+1, 6n+2, 9n+3)$	$18n+6$	$3n+3$	II.2(1)
$(7, 28n-18, 42n-27, 63n-44)$	$126n-81$	$7n-1$	XI.3(14)
$(7, 28n-17, 42n-29, 63n-40)$	$126n-80$	$7n+1$	X.3(1)
$(7, 28n-13, 42n-23, 63n-31)$	$126n-62$	$7n+2$	X.3(1)
$(7, 28n-10, 42n-15, 63n-26)$	$126n-45$	$7n+1$	XI.3(14)
$(7, 28n-9, 42n-17, 63n-22)$	$126n-44$	$7n+3$	X.3(1)
$(7, 28n-6, 42n-9, 63n-17)$	$126n-27$	$7n+2$	XI.3(14)
$(7, 28n-5, 42n-11, 63n-13)$	$126n-26$	$7n+4$	X.3(1)
$(7, 28n-2, 42n-3, 63n-8)$	$126n-9$	$7n+3$	XI.3(14)
$(7, 28n-1, 42n-5, 63n-4)$	$126n-8$	$7n+5$	X.3(1)
$(7, 28n+2, 42n+3, 63n+1)$	$126n+9$	$7n+4$	XI.3(14)
$(7, 28n+3, 42n+1, 63n+5)$	$126n+10$	$7n+6$	X.3(1)
$(7, 28n+6, 42n+9, 63n+10)$	$126n+27$	$7n+5$	XI.3(14)
$(2, 2n+1, 2n+1, 4n+1)$	$8n+4$	1	II.3(4)
$(3, 3n, 3n+1, 3n+1)$	$9n+3$	2	III.5(1)
$(3, 3n+1, 3n+2, 3n+2)$	$9n+6$	2	II.5(1)
$(3, 3n+1, 3n+2, 6n+1)$	$12n+5$	2	XVIII.2(2)
$(3, 3n+1, 6n+1, 9n)$	$18n+3$	2	VII.3(2)
$(3, 3n+1, 6n+1, 9n+3)$	$18n+6$	2	II.2(2)
$(4, 2n+3, 2n+3, 4n+4)$	$8n+12$	2	V.3(4)
$(4, 2n+3, 4n+6, 6n+7)$	$12n+18$	2	XII.3(17)
$(6, 6n+3, 6n+5, 6n+5)$	$18n+15$	4	III.5(1)
$(6, 6n+5, 12n+8, 18n+9)$	$36n+24$	4	VII.3(2)
$(6, 6n+5, 12n+8, 18n+15)$	$36n+30$	4	IV.3(1)
$(8, 4n+5, 4n+7, 4n+9)$	$12n+23$	6	XIX.2(2)
$(9, 3n+8, 3n+11, 6n+13)$	$12n+35$	6	XIX.2(2)

TABLE 1. One-parameter infinite series of values of $(a_0, a_1, a_2, a_3, d, I)$ in Theorem 2.2. We always assume that $a_0 \leqslant \cdots \leqslant a_3$. The last columns represent the cases in [Yau and Yu 03] from which the sixtuples $(a_0, a_1, a_2, a_3, d, I)$ originate. Note that sometimes a sixtuple $(a_0, a_1, a_2, a_3, d, I)$ originates from several cases in [Yau and Yu 03]. The parameter n is any positive integer.

be obtained by studying Newton diagrams of the corresponding polynomials.

Theorem 2.5. [Yau and Yu 03, Theorem 2.1] *One has*

$$\phi(x,y,z,w) = \xi(x,y,z,w) + \chi(x,y,z,w),$$

where $\xi(x,y,z,w)$ and $\chi(x,y,z,w)$ are quasihomogeneous polynomials of degree d with respect to the weights $\mathrm{wt}(x) = a_0$, $\mathrm{wt}(y) = a_1$, $\mathrm{wt}(z) = a_2$, $\mathrm{wt}(t) = a_3$ *such that the quasihomogeneous polynomials $\xi(x,y,z,w)$ and $\chi(x,y,z,w)$ do not have common monomials, the equation*

$$\xi(x,y,z,w) = 0 \subset \mathbb{C}^4 \cong \mathrm{Spec}(\mathbb{C}[x,y,z,w])$$

defines an isolated singularity, and $\xi(x,y,z,w)$ is one of the following polynomials:

I. $Ax^\alpha + By^\beta + Cz^\gamma + Dw^\delta$,

II. $Ax^\alpha + By^\beta + Cz^\gamma + Dzw^\delta$,

III. $Ax^\alpha + By^\beta + Cz^\gamma w + Dzw^\delta$,

IV. $Ax^\alpha + Bxy^\beta + Cz^\gamma + Dzw^d$,

V. $Ax^\alpha y + Bxy^\beta + Cz^\gamma + Dzw^\delta$,

VI. $Ax^\alpha y + Bxy^\beta + Cz^\gamma w + Dzw^\delta$,

VII. $Ax^\alpha + By^\beta + Cyz^\gamma + Dzw^\delta$,

VIII. $Ax^\alpha + By^\beta + Cyz^\gamma + Dyw^\delta + Ez^\epsilon w^\zeta$,

(a_0, a_1, a_2, a_3)	d	I	Source	(a_0, a_1, a_2, a_3)	d	I	Source
$(1,3,5,8)$	16	1	VIII.3(5)	$(2,3,5,9)$	18	1	II.2(3)
$(3,3,5,5)$	15	1	I.19	$(3,5,7,11)$	25	1	X.2(3)
$(3,5,7,14)$	28	1	VII.4(4)	$(3,5,11,18)$	36	1	VII.3(1)
$(5,14,17,21)$	56	1	XI.3(8)	$(5,19,27,31)$	81	1	X.3(3)
$(5,19,27,50)$	100	1	VII.3(3)	$(7,11,27,37)$	81	1	X.3(4)
$(7,11,27,44)$	88	1	VII.3(5)	$(9,15,17,20)$	60	1	VII.6(3)
$(9,15,23,23)$	69	1	III.5(1)	$(11,29,39,49)$	127	1	XIX.2(2)
$(11,49,69,128)$	256	1	X.3(1)	$(13,23,35,57)$	127	1	XIX.2(2)
$(13,35,81,128)$	256	1	X.3(2)	$(1,3,4,6)$	12	2	I.3
$(1,4,6,9)$	18	2	IV.3(3)	$(1,6,10,15)$	30	2	I.4
$(2,3,4,7)$	14	2	IX.3(1)	$(3,3,4,4)$	12	2	V.3(4)
$(3,4,5,10)$	20	2	II.3(2)	$(3,4,6,7)$	18	2	VII.3(10)
$(3,4,10,15)$	30	2	II.2(3)	$(5,13,19,22)$	57	2	X.3(3)
$(5,13,19,35)$	70	2	VII.3(3)	$(6,9,10,13)$	36	2	VII.3(8)
$(7,8,19,25)$	57	2	X.3(4)	$(7,8,19,32)$	64	2	VII.3(3)
$(9,12,13,16)$	48	2	VII.6(2)	$(9,12,19,19)$	57	2	III.5(1)
$(9,19,24,31)$	81	2	XI.3(20)	$(10,19,35,43)$	105	2	XI.3(18)
$(11,21,28,47)$	105	2	XI.3(16)	$(11,25,32,41)$	107	2	XIX.3(1)
$(11,25,34,43)$	111	2	XIX.2(2)	$(11,43,61,113)$	226	2	X.3(1)
$(13,18,45,61)$	135	2	XI.3(14)	$(13,20,29,47)$	107	2	XIX.3(1)
$(13,20,31,49)$	111	2	XIX.2(2)	$(13,31,71,113)$	226	2	X.3(2)
$(14,17,29,41)$	99	2	XIX.2(3)	$(5,7,11,13)$	33	3	X.3(3)
$(5,7,11,20)$	40	3	VII.3(3)	$(11,21,29,37)$	95	3	XIX.2(2)
$(11,37,53,98)$	196	3	X.3(1)	$(13,17,27,41)$	95	3	XIX.2(2)
$(13,27,61,98)$	196	3	X.3(2)	$(15,19,43,74)$	148	3	X.3(1)
$(5,6,8,9)$	24	4	VII.3(2)	$(5,6,8,15)$	30	4	IV.3(1)
$(9,11,12,17)$	45	4	XI.3(20)	$(10,13,25,31)$	75	4	XI.3(14)
$(11,17,20,27)$	71	4	XIX.3(1)	$(11,17,24,31)$	79	4	XIX.2(2)
$(11,31,45,83)$	166	4	X.3(1)	$(13,14,19,29)$	71	4	XIX.3(1)
$(13,14,23,33)$	79	4	XIX.2(2)	$(13,23,51,83)$	166	4	X.3(2)
$(6,7,9,10)$	27	5	XI.3(14)	$(11,13,19,25)$	63	5	XIX.2(2)
$(11,25,37,68)$	136	5	X.3(1)	$(13,19,41,68)$	136	5	X.3(2)
$(11,19,29,53)$	106	6	X.3(1)	$(13,15,31,53)$	106	6	X.3(2)
$(11,13,21,38)$	76	7	X.3(1)				

TABLE 2. Sporadic cases of values of $(a_0, a_1, a_2, a_3, d, I)$ in Theorem 2.2. We always assume that $a_0 \leqslant \cdots \leqslant a_3$. The last columns represent the cases in [Yau and Yu 03] from which the sixtuples $(a_0, a_1, a_2, a_3, d, I)$ originate. Note that sometimes a sixtuple $(a_0, a_1, a_2, a_3, d, I)$ originates from several cases in [Yau and Yu 03]. The parameter n is any positive integer.

IX. $Ax^\alpha + By^\beta w + Cz^\gamma w + Dyw^\delta + Ey^\epsilon z^\zeta$,

X. $Ax^\alpha + By^\beta z + Cz^\gamma w + Dyw^\delta$,

XI. $Ax^\alpha + Bxy^\beta + Cyz^\gamma + Dzw^\delta$,

XII. $Ax^\alpha + Bxy^\beta + Cxz^\gamma + Dyw^\delta + Ey^\epsilon z^\zeta$,

XIII. $Ax^\alpha + Bxy^\beta + Cyz^\gamma + Dyw^\delta + Ez^\epsilon w^\zeta$,

XIV. $Ax^\alpha + Bxy^\beta + Cxz^\gamma + Dxw^\delta + Ey^\epsilon z^\zeta + Fz^\eta w^\theta$,

XV. $Ax^\alpha y + Bxy^\beta + Cxz^\gamma + Dzw^\delta + Ey^\epsilon z^\zeta$,

XVI. $Ax^\alpha y + Bxy^\beta + Cxz^\gamma + Dxw^\delta + Ey^\epsilon z^\zeta + Fz^\eta w^\theta$,

XVII. $Ax^\alpha y + Bxy^\beta + Cyz^\gamma + Dxw^\delta + Ey^\epsilon w^\zeta + Fx^\eta z^\theta$,

XVIII. $Ax^\alpha z + Bxy^\beta + Cyz^\gamma + Dyw^\delta + Ez^\epsilon w^\zeta$,

XIX. $Ax^\alpha z + Bxy^\beta + Cz^\gamma w + Dyw^\delta$,

where $\alpha, \beta, \gamma, \delta$ are positive integers, $\epsilon, \zeta, \eta, \theta$ are non-negative integers, and A, B, C, D, E, F are complex numbers.

We will refer to the latter polynomials according to case labeling in Theorem 2.5. For simplicity of notation, we suppose that $A = B = C = D = E = F = 1$ in the rest of the paper.[1]

In order to prove Theorem 2.2, we will suppose that $d < \sum_{i=0}^{3} a_i$ (this happens if and only if X is a del Pezzo surface, provided that X is well formed). Then the singularity (V, O) is canonical (see Remark 1.2), and thus $\mathrm{mult}_O(V) \leqslant 3$. Moreover, the singularity (V, O) is rational (see Remark 1.2).

The main result of [Yau and Yu 03] is a classification of (the deformation families of) the quasihomogeneous polynomials that define isolated three-dimensional quasihomogeneous *rational* hypersurface singularities up to an analytical change of coordinates (in some sense, it is a refinement of Theorem 2.5).

To give a classification of quasismooth del Pezzo hypersurfaces in weighted projective spaces, we actually need the classification of such polynomials up to a change of coordinates that is compatible with the corresponding $\mathbb{C}^*$-action (i.e., a change of coordinates that respects the weights).

Remark 2.6. Note that these two classifications indeed differ. If, say, one denotes by $\upsilon(x, y, z, w)$ the $(\alpha, \beta, \gamma, \delta)$-part of the polynomial $\xi(x, y, z, w)$, one sees that the cases in which υ has fewer than four different monomials are absent from the list of [Yau and Yu 03]. These are $\xi(x, y, z, w) = x^\alpha + y^\beta + z^\gamma w + zw^\delta$ with $\gamma = \delta = 1$ (cf. [Yau and Yu 03, Case III]), $\xi(x, y, z, w) = x^\alpha y + xy^\beta + z^\gamma + zw^\delta$ with $\alpha = \beta = 1$ (cf. [Yau and Yu 03, Case V]), and $\xi(x, y, z, w) = x^\alpha y + xy^\beta + z^\gamma w + zw^\delta$ with $\alpha = \beta = 1$ or/and $\gamma = \delta = 1$ (cf. [Yau and Yu 03, Case VI]). It is easy to check that the listed cases are equivalent up to an analytical change of coordinates to some other cases that are present in the list of [Yau and Yu 03], but one can choose the weights of variables so that there exists no such change of coordinates that respects the weights.

Indeed, while the weights of the variables are not fixed even if one fixes a polynomial $\xi(x, y, z, w)$ from Theorem 2.5 that is homogeneous with respect to these weights (since one can multiply all of them by some constant), the corresponding *well-formed* weighted projective space and thus the family of the corresponding well-formed hypersurfaces becomes fixed in this case. Fortunately, these two classifications are not very far from each other. To recover the latter from the former is not a difficult task, but still it requires some additional work.

Luckily, to prove Theorem 2.2 we do not need to do this in full generality, since we can disregard polynomials whose degree d (and thus the index I as well) equals a sum of two of the weights. The latter are included in one of the types of our resulting classification (see Theorem 2.2). If there is a unique choice of weights $\mathrm{wt}(x)$, $\mathrm{wt}(y)$, $\mathrm{wt}(z)$, $\mathrm{wt}(w)$ that makes some of the polynomials obtained from the polynomial $\xi(x, y, z, w)$ by an analytical change of coordinates quasihomogeneous, then one trivially obtains that every change of coordinates that turns ξ into another quasihomogeneous polynomial must agree with the corresponding $\mathbb{C}^*$-action.

Furthermore, this is the case if we restrict ourselves to the weights that are at most $d/2$, where d is the total weight of a corresponding polynomial (see [Saito 71, Lemma 4.3]). Therefore, the polynomials that we need to recover must be homogeneous with respect to the weights such that one of the weights, say $\mathrm{wt}(x)$, is strictly larger than $d/2$.

In this case, we have $\xi = xg + h$, where g and h are polynomials that do not depend on x. By quasismoothness, at least one other variable occurs linearly in g, so by a $\mathbb{C}^*$-equivariant coordinate transformation, we may assume that g is a coordinate, say y. Now collect all terms divisible by y and absorb them in xy by a ($\mathbb{C}^*$-equivariant) coordinate change in x. We still have to take care of all polynomials that are obtained from ξ by an analytical change of coordinates (note that these may not contain a monomial that is a product of two variables even if ξ does). The rank of the hypersurface singularity in question is at least 2. The latter is preserved under the analytical change of coordinates, so it is enough for our purposes to describe all possible quasihomogeneous polynomials f (say in variables x_0, x_1, x_2, and x_4) giving a singularity of rank r equal to 2, 3, or 4, and not containing monomials $x_i x_j$ for $i \neq j$.

The latter condition implies that (up to a $\mathbb{C}^*$-equivariant coordinate change)

$$f = x_0^2 + \cdots + x_r^2 + g(x_{r+1}, \ldots, x_4),$$

where g is a polynomial in $4 - r$ variables of rank 0 (i.e., corank $0 \leqslant 4 - r \leqslant 2$). If $r = 4$, then $g = 0$, and if $r = 3$, then $g = x_3^n$, so that in both of these cases, f is found in [Yau and Yu 03, Case I]. If $r = 2$, applying [Arnold et al. 85, Section 13.1] (and keeping in mind [Saito 71, Lemma 4.3]), we again see that f is

[1] The singularity defined by $\xi(x, y, z, w)$ is not necessarily isolated if $A = \cdots = F = 1$ (this happens, for instance, in case XIX if $\alpha = \beta = \gamma = \delta = 1$). We hope that such abuse of notation will not lead to any confusion.

contained in the list of [Yau and Yu 03] (cases I.1, II.1, and III.1).[2]

To summarize, for every $\xi(x,y,z,w)$, the possible values (up to a $\mathbb{C}^*$-equivariant change of coordinates) of the quadruple $(\alpha,\beta,\gamma,\delta)$ are listed in [Yau and Yu 03] up to the polynomials that contain a monomial that is a product of two variables. Unfortunately, as sometimes happens with long lists, there are some omissions in the list of [Yau and Yu 03]. Namely, apart from minor misprints (see Examples 2.9 and 2.15 below), the following cases are omitted:[3]

XI. $\xi(x,y,z,w) = x^\alpha + xy^\beta + yz^\gamma + zw^\delta$ and $(\alpha,\beta,\gamma,\delta) = (2,4,13,3)$,

XII. $\xi(x,y,z,w) = x^\alpha + xy^\beta + xz^\gamma + yw^\delta + y^\epsilon z^\zeta$ and

$$(\alpha,\beta,\gamma,\delta,\epsilon,\zeta) \in \big\{(5,4,3,2,1,3),(7,4,3,2,2,2),(6,5,3,2,1,3)\big\}.$$

Remark 2.7. Note that different cases in the list of [Yau and Yu 03] are not mutually exclusive. For example, for Case I.1 with $r = s = 2$ and Case XIII.1(7) with $r = 2$, there are a $\mathbb{C}^*$-action and a change of coordinates equivariant with respect to this action such that the two (deformation families of) singularities are the same (in fact, such coincidences are numerous in the list). A side effect of this is that sometimes one has to make a ($\mathbb{C}^*$-equivariant) coordinate change to find a given polynomial in the list. For example, the polynomial $\xi = x^4 + xy^4 + xz^3 + yw^2 + y^4 z$ is not found in Case XII as one could possibly expect, but in the new coordinates $x' = x$, $y' = z - x$, $z' = y$, $w' = w$, it gives the same deformation family as Case XI.3(16) for $r = s = 4$.

Therefore, given the list from [Yau and Yu 03], to prove Theorem 2.2, we must find all singularities in this list that correspond to the well-formed hypersurfaces $X \subset \mathbb{P}(a_0,a_1,a_2,a_3)$. This means that we need to find all possible values of the quadruple $(\alpha,\beta,\gamma,\delta)$ such that

$$\gcd(a_i,a_j,a_k) = 1$$

and d is divisible by $\gcd(a_i,a_j)$ for all $i \neq j \neq k \neq i$. Let us show how to do this in a few typical cases.

[2]We are grateful to J. Stevens, who explained this argument to us.
[3]We are grateful to L. Morris, who checked the computations of [Yau and Yu 03] and found these omissions.

Example 2.8. Suppose that the hypersurface X is well formed, and suppose that the quasihomogeneous polynomial $\xi(x,y,z,w)$ is found in the third part of [Yau and Yu 03, Case X.3(1)]. Then

$$\xi(x,y,z,w) = x^2 + y^3 z + z^5 w + yw^u,$$

where $5 \le u \le 18$. Hence $2a_0 = 3a_1 + a_2 = 5a_2 + a_3 = a_1 + ua_3$. Put $a_3 = a$. Then

$$(a_0,a_1,a_2,a_3,d) = \left(\frac{(15u+1)a}{22}, \frac{(4u+1)a}{11}, \frac{(3u-2)a}{11}, a, \frac{(15u+1)a}{11}\right),$$

where either $a = 1$ or $a = 11$, because $\gcd(a_1,a_2,a_3) = 1$.

Suppose that $a = 1$. Then $3u - 2$ and $4u + 1$ are divisible by 11. We see that $u = 8$. Then

$$a_0 = \frac{(15u+1)a}{22} = \frac{121}{22} \notin \mathbb{Z},$$

which is a contradiction.

We see that $a = 11$. Then u must be odd for a_0 to be an integer. Thus, we obtain the following solutions:

$$(a_0,a_1,a_2,a_3,d,I) = (38,21,13,11,76,7),$$
$$(a_0,a_1,a_2,a_3,d,I) = (53,29,19,11,106,6),$$
$$(a_0,a_1,a_2,a_3,d,I) = (68,37,25,11,136,5),$$
$$(a_0,a_1,a_2,a_3,d,I) = (83,45,31,11,166,4),$$
$$(a_0,a_1,a_2,a_3,d,I) = (98,53,37,11,196,3),$$
$$(a_0,a_1,a_2,a_3,d,I) = (113,61,43,11,226,2),$$
$$(a_0,a_1,a_2,a_3,d,I) = (128,69,49,11,256,1).$$

Example 2.9. Suppose that the hypersurface X is well formed, and suppose that the quasihomogeneous polynomial $\xi(x,y,z,w)$ is found in the second part of [Yau and Yu 03, Case XII.3(16)]. Then[4]

$$\xi(x,y,z,w) = x^3 + xy^5 + xz^2 + yw^4 + y^\epsilon z^\zeta,$$

which gives $3a_0 = a_0 + 5a_1 = a_0 + 2a_2 = a_1 + 4a_3$, which contradicts the well-formedness of X.

Example 2.10. Suppose that the hypersurface X is well formed, and suppose that the quasihomogeneous polynomial $\xi(x,y,z,w)$ is found in [Yau and Yu 03, Case I.2]. Then

$$\xi(x,y,z,w) = x^2 + y^3 + z^3 + w^r,$$

where $r \in \mathbb{Z}_{\ge 3}$. Hence $2a_0 = 3a_1 = 3a_2 = ra_3$. Thus $a_0 = 3$ and $a_1 = a_2 = 2$, because

$$\gcd(a_0,a_1,a_2) = 1.$$

[4]Note that there is a misprint in [Yau and Yu 03, Case XII.3(16)], and one should read $(5,4)$ instead of $(4,5)$.

We see that $a_3 = 6/r$. Since $r \geq 3$, we have $a_3 = 1$, because $\gcd(a_1, a_2, a_3) = 1$.

Example 2.11. Suppose that the hypersurface X is well formed, and suppose that the quasihomogeneous polynomial $\xi(x, y, z, w)$ is found in the fourth part of [Yau and Yu 03, Case IX.3(3)]. Then

$$\xi(x, y, z, w) = x^3 + y^2 w + z^6 w + yw^s + y^\epsilon z^\zeta,$$

where $s \in \mathbb{Z}_{\geq 6}$. Hence $3a_0 = 2a_1 + a_3 = 6a_2 + a_3 = a_1 + sa_3$. Put $a_3 = a$. Then

$$(a_0, a_1, a_2, a_3, d) = \left(\frac{(2s-1)a}{3}, (s-1)a, \frac{(s-1)a}{3}, a, (2s-1)a\right),$$

where d is divisible by $\gcd(a_1, a_2) = (s-1)a/3$. Thus, we have

$$(s-1) \mid 3(2s-1),$$

which is possible only if 3 is divisible by $s-1$, which contradicts the assumption $s \geq 6$.

Example 2.12. Suppose that the hypersurface X is well formed, and suppose that the quasihomogeneous polynomial $\xi(x, y, z, w)$ is found in the first part of [Yau and Yu 03, Case VIII.3(5)]. Then

$$\xi(x, y, z, w) = x^2 + y^s + yz^3 + yw^3 + z^\epsilon w^\zeta,$$

where $s \in \mathbb{Z}_{\geq 4}$. Hence $2a_0 = sa_1 = a_1 + 3a_2 = a_1 + 3a_3$. Put $a_1 = a$. Then

$$(a_0, a_1, a_2, a_3, d) = \left(\frac{sa}{2}, a, \frac{(s-1)a}{3}, \frac{(s-1)a}{3}, sa\right),$$

where $d = sa$ is divisible by $\gcd(a_2, a_3) = (s-1)a/3$, because X is well formed. Thus

$$(s-1) \mid 3s,$$

which implies that $s = 4$, because $s \geq 4$. Hence, we have

$$(a_0, a_1, a_2, a_3, d) = (2a, a, a, a, 4a),$$

which gives $a = 1$. Then X is a smooth del Pezzo surface X such that $K_X^2 = 2$.

Example 2.13. Suppose that the hypersurface X is well formed, and suppose that the quasihomogeneous polynomial $\xi(x, y, z, w)$ is found in the second part of [Yau and Yu 03, Case XVIII.2(2)]. Then

$$\xi(x, y, z, w) = x^2 z + xy^2 + yz^s + yw^3 + z^\epsilon w^\zeta,$$

where $s \in \mathbb{Z}_{\geq 4}$. Hence $2a_0 + a_2 = a_0 + 2a_1 = a_1 + sa_2 = a_1 + 3a_3$. Put $a_2 = a$. Then

$$(a_0, a_1, a_2, a_3, d) = \left(\frac{(2s-1)a}{3}, \frac{(s+1)a}{3}, a, \frac{sa}{3}, \frac{(4s+1)a}{3}\right).$$

Since either s or $s+1$ is not divisible by 3, we see that a is divisible by 3. But

$$\gcd(a_0, a_1, a_2) = 1,$$

because X is well formed. Then $a = 3$. Thus, we have

$$(a_0, a_1, a_2, a_3, d) = (2s-1, s+1, 3, s, 4s+1),$$

where $s \in \mathbb{Z}_{\geq 4}$. Note that if $s \equiv 2 \bmod 3$, then

$$\gcd(a_0, a_1, a_2) = 3,$$

which is impossible. Then either $s \equiv 0 \bmod 3$ or $s \equiv 1 \bmod 3$.

Suppose that $s \equiv 0 \bmod 3$. Then $s = 3n$ for some $n \in \mathbb{Z}_{\geq 2}$. We have

$$(a_0, a_1, a_2, a_3, d) = (6n-1, 3n+1, 3, 3n, 12n+1),$$

and d is not divisible by $\gcd(a_2, a_3) = 3$, which contradicts the well-formedness of X.

We see that $s \equiv 1 \bmod 3$. Then $s = 3n+1$ for some $n \in \mathbb{Z}_{\geq 2}$. We have

$$(a_0, a_1, a_2, a_3, d, I) = (6n+1, 3n+2, 3, 3n+1, 12n+5, 2).$$

Example 2.14. Suppose that the hypersurface X is well formed, and suppose that the quasihomogeneous polynomial $\xi(x, y, z, w)$ is found in [Yau and Yu 03, Case IX.2(1)]. Then

$$\xi(x, y, z, w) = x^2 + y^2 w + z^r w + yw^s + y^\epsilon z^\zeta,$$

where $r \in \mathbb{Z}_{\geq 2} \ni s$. Hence $2a_0 = 2a_1 + a_3 = ra_2 + a_3 = a_1 + sa_3$. Put $a_2 = a$. Then

$$(a_0, a_1, a_2, a_3, d) = \left(\frac{(2s-1)ra}{4(s-1)}, \frac{ra}{2}, a, \frac{ra}{2(s-1)}, \frac{(2s-1)ra}{2(s-1)}\right).$$

Note that $\gcd(2s-1, 4(s-1)) = 1$. Thus ra is divisible by $4(s-1)$. But

$$\gcd(a_0, a_1, a_3) = 1,$$

because the hypersurface X is well formed. Then $ra = 4(s-1)$. Hence, we have

$$(a_0, a_1, a_2, a_3, d) = \left(2s-1, 2s-2, \frac{4s-4}{r}, 2, 4s-2\right),$$

where d is divisible by $\gcd(a_1, a_2)$. Hence $r(4s-2)$ is divisible by $s-1$. Then

$$r = k(s-1)$$

for some $k \in \mathbb{Z}_{\geq 1}$. Since $4/k = a_2 \in \mathbb{Z}_{>0}$, one obtains that $k \in \{1, 2, 4\}$.

If $k \in \{1, 2\}$, then $\gcd(a_1, a_2, a_3) = 2$, which is impossible. We see that $k = 4$. Then

$$(a_0, a_1, a_2, a_3, d) = (2s - 1, 2s - 2, 1, 2, 4s - 2).$$

Example 2.15. Suppose that the hypersurface X is well formed, and suppose that the quasihomogeneous polynomial $\xi(x, y, z, w)$ is found in the first part of [Yau and Yu 03, Case V.3(4)]. Then[5]

$$\xi(x, y, z, w) \in \{yx^3 + xy^3 + z^2 + zw^s,\ yx^3 + xy^3 + z^s + zw^2\},$$

where $s \in \mathbb{Z}_{\geq 3}$. If $\xi(x, y, z, w) = yx^3 + xy^3 + z^2 + zw^s$, then

$$3a_0 + a_1 = a_0 + 3a_1 = 2a_2 = a_2 + sa_3,$$

which contradicts the well-formedness of the hypersurface X.

We have $\xi(x, y, z, w) = yx^3 + xy^3 + z^s + zw^2$. Then $3a_0 + a_1 = a_0 + 3a_1 = sa_2 = a_2 + 2a_3$, and

$$(a_0, a_1, a_2, a_3, d) = \left(\frac{sa}{4}, \frac{sa}{4}, a, \frac{(s-1)a}{2}, sa\right),$$

where $a_2 = a$. Since $\gcd(a_0, a_1, a_2) = 1$, we see that $4 \mid a$. Then $a \in \{2, 4\}$.

Suppose that $a = 2$. Then

$$(a_0, a_1, a_2, a_3, d) = \left(\frac{s}{2}, \frac{s}{2}, 2, s - 1, 2s\right),$$

where s is divisible by 2 and not divisible by 4. Then $s = 4n + 2$, where $n \in \mathbb{Z}_{\geq 1}$. We have

$$(a_0, a_1, a_2, a_3, d, I) = (2n + 1, 2n + 1, 2, 4n + 1, 8n + 4, 1).$$

Suppose that $a = 4$. Then $(a_0, a_1, a_2, a_3, d) = (s, s, 4, 2s - 2, 4s)$. Then

$$(a_0, a_1, a_2, a_3, d, I) = (2n + 1, 2n + 1, 4, 4n, 8n + 4, 2)$$

for some $n \in \mathbb{Z}_{\geq 1}$, because s must be odd.

Example 2.16. Suppose that the hypersurface X is well formed, and suppose that the quasihomogeneous polynomial $\xi(x, y, z, w)$ is found in the first part of [Yau and Yu 03, Case XI.3(14)]. Then

$$\xi(x, y, z, w) = x^3 + xy^3 + yz^s + zw^2,$$

where $s \in \mathbb{Z}_{\geq 3}$. Hence $3a_0 = a_0 + 3a_1 = a_1 + sa_2 = a_2 + 2a_3$. Put $a_2 = a$. Then

$$(a_0, a_1, a_2, a_3, d) = \left(\frac{3as}{7}, \frac{2as}{7}, a, \frac{a(9s-7)}{14}, \frac{9as}{7}\right),$$

and $\gcd(a_0, a_1, a_2) = 1$, because X is well formed. Thus either $a = 1$ or $a = 7$.

Suppose that $a = 1$. Then

$$(a_0, a_1, a_2, a_3, d) = \left(\frac{3s}{7}, \frac{2s}{7}, 1, \frac{9s-7}{14}, \frac{9s}{7}\right),$$

which implies that $s = 7k$ for some $k \in \mathbb{Z}_{\geq 1}$. Hence, we have

$$(a_0, a_1, a_2, a_3, d) = \left(3k, 2k, 1, \frac{9k-1}{2}, 9k\right),$$

which implies that $k = 2n - 1$ for some $n \in \mathbb{Z}_{\geq 1}$. We have

$$(a_0, a_1, a_2, a_3, d, I) = (6n-3, 4n-2, 1, 9n-5, 18n-9, n).$$

Suppose that $a = 7$. Then

$$(a_0, a_1, a_2, a_3, d) = \left(3s, 2s, 7, \frac{9s-7}{2}, 9s\right),$$

which implies that $s = 2k + 1$ for some $k \in \mathbb{Z}_{\geq 1}$. Hence, we have

$$(a_0, a_1, a_2, a_3, d) = (6k + 3, 4k + 2, 7, 9k + 1, 18k + 9),$$

but $\gcd(a_0, a_1, a_2) = 1$. Then $k \not\equiv 3 \bmod 7$. Thus, we have the following solutions:[6]

$$(a_0, a_1, a_2, a_3, d, I) = (28n - 22, 42n - 33, 7, 63n - 53, 126n - 99, 7n - 2),$$
$$(a_0, a_1, a_2, a_3, d, I) = (28n - 18, 42n - 27, 7, 63n - 44, 126n - 81, 7n - 1),$$
$$(a_0, a_1, a_2, a_3, d, I) = (28n - 10, 42n - 15, 7, 63n - 26, 126n - 45, 7n + 1),$$
$$(a_0, a_1, a_2, a_3, d, I) = (28n - 6, 42n - 9, 7, 63n - 17, 126n - 27, 7n + 2),$$
$$(a_0, a_1, a_2, a_3, d, I) = (28n - 2, 42n - 3, 7, 63n - 8, 126n - 9, 7n + 3),$$
$$(a_0, a_1, a_2, a_3, d, I) = (28n + 2, 42n + 3, 7, 63n + 1, 126n + 9, 7n + 4),$$

where $n \in \mathbb{Z}_{\geq 1}$.

Example 2.17. Suppose that the hypersurface X is well formed, and suppose that the quasihomogeneous polynomial $\xi(x, y, z, w)$ is found in [Yau and Yu 03, Case

[5] Note that there is a misprint in [Yau and Yu 03, Case V.3(4)], and one should read $(r, s) = (3, s)$ instead of $(s, r) = (3, s)$, and the same correction should be made in the second and the third parts of this subcase.

[6] Note that in Tables 1 and 2, we split the first of the obtained series into a sporadic case corresponding to $n = 1$ and a shifted series starting from $n = 2$. This is done to ensure that $a_0 \leqslant \cdots \leqslant a_3$.

VIII.2(1)]. Then

$$\xi(x, y, z, w) = x^2 + y^2 + yz^r + yw^s + z^\epsilon w^\zeta,$$

where $r \in \mathbb{Z}_{\geq 2} \ni s$. Hence $2a_0 = 2a_1 = a_1 + ra_2 = a_1 + sa_3$. Put $a_2 = a$. Then

$$(a_0, a_1, a_2, a_3, d) = \left(ra, ra, a, \frac{ra}{s}, 2ra\right),$$

where $a = \gcd(a_0, a_1, a_2) = 1$, because X is well formed. Thus, we have

$$(a_0, a_1, a_2, a_3, d) = \left(r, r, 1, \frac{r}{s}, 2r\right),$$

where

$$\frac{r}{s} = \gcd(a_0, a_1, a_3) = 1.$$

Then $(a_0, a_1, a_2, a_3, d) = (r, r, 1, 1, 2r)$.

Example 2.18. Suppose that the hypersurface X is well formed, and suppose that the quasihomogeneous polynomial $\xi(x, y, z, w)$ is found in [Yau and Yu 03, Case XIV.1(1)]. Then

$$\xi(x, y, z, w) = x^r + xy + xz^s + xw^t + y^\epsilon z^\zeta + z^\eta w^\theta,$$

where $r, s, t \in \mathbb{Z}_{\geq 2}$. Hence $ra_0 = a_0 + a_1 = a_0 + sa_2 = a_0 + ta_3$. Put $a_0 = a$. Then

$$(a_0, a_1, a_2, a_3, d) = \left(a, (r-1)a, \frac{(r-1)a}{s}, \frac{(r-1)a}{t}, ra\right).$$

It follows from the well-formedness of the hypersurface X that

$$\gcd(a_0, a_1, a_2) = \gcd(a_0, a_1, a_3) = 1,$$

so that a divides s and t. Put $s = ap$ and $t = aq$ for some $q \in \mathbb{Z}_{\geq 1} \ni p$. Then

$$\gcd\left(\frac{r-1}{p}, \frac{r-1}{q}\right) = 1,$$

because $\gcd(a_1, a_2, a_3) = 1$, where $r - 1$ is divisible by p and q. Thus, we see that

$$p = mk, \quad q = ml, \quad r - 1 = mkl,$$

where m, k, and l are positive integers such that $\gcd(k, l) = 1$. Then

$$(a_0, a_1, a_2, a_3, d) = (a, mkla, l, k, (mkl + 1)a).$$

By well-formedness, one obtains that d is divisible by $\gcd(a_1, a_2) = l$. Then $l \mid a$ and

$$l \mid \gcd(a_0, a_1, a_2),$$

so that by well-formedness, $l = 1$. In a similar way, we get $k = 1$. Then

$$(a_0, a_1, a_2, a_3, d, I) = \big(a, ma, 1, 1, (m + 1)a, 2\big),$$

where m and a are arbitrary positive integers.

The proof of Theorem 2.2 is similar in the remaining cases.

3. BISHOP VERSUS LICHNEROWICZ

In this section, we prove Theorem 1.7. Let $\bar{a}_0, \dots, \bar{a}_n, \bar{d}$ be positive real numbers such that

$$0 < \sum_{i=0}^{n} \bar{a}_i - \bar{d} \leq n\bar{a}_0$$

and $\bar{a}_0 \leq \bar{a}_1 \leq \cdots \leq \bar{a}_n$, where $n \geq 1$. To prove Theorem 1.7, we must show that

$$\bar{d}\left(\sum_{i=0}^{n} \bar{a}_i - \bar{d}\right)^n \leq n^n \prod_{i=0}^{n} \bar{a}_i.$$

Put $\bar{I} = \sum_{i=0}^{n} \bar{a}_i - \bar{d}$. Then $I = \alpha n \bar{a}_0$, where $\alpha \in \mathbb{R}$ such that $0 < \alpha \leq 1$. We must prove that

$$\left(\sum_{i=1}^{n} \bar{a}_i + (1 - \alpha n)\bar{a}_0\right)\bar{a}_0^{n-1}\alpha^n - \prod_{i=1}^{n} \bar{a}_i \leq 0. \qquad (3–1)$$

Put $a_i = \bar{a}_i/\bar{a}_0$ for every $i \in \{1, \dots, n\}$. Then (3–1) is equivalent to

$$\left(\sum_{i=1}^{n} a_i + 1 - \alpha n\right)\alpha^n - \prod_{i=1}^{n} a_i \leq 0, \qquad (3–2)$$

where $a_1 \geq 1, a_2 \geq 1, \dots, a_n \geq 1$. But to prove (3–2), it is enough to prove that

$$\sum_{i=1}^{n} a_i + 1 - n - \prod_{i=1}^{n} a_i \leq 0, \qquad (3–3)$$

because the derivative of the left-hand side of (3–2) with respect to α equals

$$n\alpha^{n-1}\left(\sum_{i=1}^{n} a_i + 1 - \alpha(n+1)\right) \geq n\alpha^{n-1}\left(\sum_{i=1}^{n} a_i - n\right) \geq 0,$$

since $\alpha \leq 1$ and $a_i \geq 1$ for every $i \in \{1, \dots, n\}$. Let us prove (3–3) by induction on n.

We may assume that $n \geq 2$, and $a_i \neq 1$ for every $i \in \{1, \dots, n\}$ by the induction assumption.

Lemma 3.1. *Suppose that $a_i \geq n$ for some $i \in \{1, \dots, n\}$. Then the inequality* (3–3) *holds.*

Proof. Without loss of generality, we may assume that $a_n \geq n$. Then

$$\sum_{i=1}^{n} a_i + 1 - n - \prod_{i=1}^{n} a_i = \sum_{i=1}^{n-1} \left(a_i - \prod_{i=1}^{n-1} a_i \right) + (a_n - n + 1)\left(1 - \prod_{i=1}^{n-1} a_i\right) \geq 0,$$

which completes the proof. □

Put $F(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i + 1 - n - \prod_{i=1}^{n} x_i$. Let $U \subset \mathbb{R}^n$ be an open set given by

$$1 < a_1 < n, \quad 1 < a_2 < n, \ldots, 1 < a_n < n,$$

and suppose that (3–3) fails. Then $F(a_1, \ldots, a_n) > 0$. But

$$(x_1, \ldots, x_n) \in \overline{U} \setminus U \Longrightarrow F(x_1, \ldots, x_n) \leq 0,$$

which implies that F attains its maximum at some point $(A_1, \ldots, A_n) \in U$. Thus, we have

$$A_k = \prod_{i=1}^{n} A_i$$

for every $k \in \{1, \ldots, n\}$ by the first derivative test. The latter implies $A_1 = A_2 = \cdots = A_n$. Then

$$nA_1 + 1 - n - A_1^n > 0,$$

which is impossible, because $nA_1 + 1 - n - A_1^n$ is a decreasing function of A_1 vanishing at $A_1 = 1$.

The assertion of Theorem 1.7 is proved.

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