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KÄHLER–EINSTEIN FANO THREEFOLDS OF DEGREE 22

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Abstract

We study the problem of existence of Kähler–Einstein metrics on smooth Fano threefolds of Picard rank one and anticanonical degree 22 that admit a faithful action of the multiplicative group $\mathbb{C}^*$. We prove that, with the possible exception of two explicitly described cases, all such smooth Fano threefolds are Kähler–Einstein.

All varieties are assumed to be projective and are defined over the field of complex numbers.

1. Introduction

Smooth Fano threefolds of Picard rank 1 have been classified by Iskovskikh in [I77,I78]. Among them, he found a family missing in the original works by Fano. Threefolds in this family have the same cohomology groups as $\mathbb{P}^3$ does. Their anticanonical degree is 22, hence they are called threefolds of type V_{22}. In fact, Iskovskikh himself missed one threefold in this family, which was later recovered by Mukai and Umemura in [MU83]. This threefold, usually called the Mukai–Umemura threefold, is an equivariant compactification of $\mathrm{SL}_2(\mathbb{C})/\mathbf{I}$, where $\mathbf{I}$ denotes the icosahedral group. Its automorphism group is isomorphic to the group $\mathrm{PGL}_2(\mathbb{C})$.

The automorphism groups of threefolds of type V_{22} have been studied by Prokhorov in [P90]. He proved that this group is finite except for a unique threefold for which the connected component of identity of the automorphism

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group is isomorphic to the additive group $\mathbb{C}^+$; and a one-parameter family of threefolds that admit a faithful action of the multiplicative group $\mathbb{C}^*$, which includes the Mukai–Umemura threefold as a special member. We refer to the latter varieties as threefolds of type V_{22}^*.

In [Ti97], Tian showed that there are threefolds of type V_{22} with trivial automorphism group that do not admit Kähler–Einstein metrics, which disproved a folklore conjecture that all smooth Fano varieties without holomorphic vector fields are Kähler–Einstein. On the other hand, Donaldson proved

Theorem 1.1 ([D08, Theorem 3])**.** *Let X be the Mukai–Umemura threefold, and G be its automorphism group. Then*

$$\alpha_G(X) = \frac{5}{6}.$$

Here $\alpha_G(X)$ is the G-equivariant α-invariant defined by Tian in [Ti87]. If X is a smooth Fano variety, and G is a reductive subgroup in $\mathrm{Aut}(X)$, then Demailly's [CS08, Theorem A.3] gives

$$\alpha_G(X) = \sup\left\{ \epsilon \in \mathbb{Q} \;\middle|\; \begin{array}{l} \text{the log pair } \Big(X, \dfrac{\epsilon}{n}\mathcal{D}\Big) \text{ is log canonical} \\ \text{for any } n \in \mathbb{Z}_{>0} \text{ and every} \\ G\text{-invariant linear system } \mathcal{D} \subset \big| -nK_X \big| \end{array} \right\}. \tag{1.2}$$

Donaldson's Theorem 1.1 implies the existence of a Kähler–Einstein metric on the Mukai–Umemura threefold by famous Tian's criterion:

Theorem 1.3 ([Ti87])**.** *Let X be a smooth Fano variety of dimension n, and G be a reductive subgroup in* $\mathrm{Aut}(X)$*. Suppose that*

$$\alpha_G(X) > \frac{n}{n+1}.$$

Then X admits a Kähler–Einstein metric.

An example of a Kähler–Einstein threefold of type V_{22} with finite automorphism group has been constructed in [CS12]. On the other hand, there exist threefolds of this type that are not Kähler–Einstein.

Example 1.4. Let X^{a} be the unique threefold of type V_{22} such that the connected component of identity of its automorphism group is isomorphic to the additive group $\mathbb{C}^+$. By the Matsushima obstruction, the variety X^{a} is not Kähler–Einstein. It is interesting to point out that X^{a} is K-semistable. Indeed, it follows from [KPS18, Proposition 5.4.4] and the Mukai construction of varieties of type V_{22} (cf. [KPS18, Remark 5.4.8]) that the Mukai–Umemura threefold is a degeneration of X^{a}. Since the Mukai–Umemura threefold is Kähler–Einstein, it is K-polystable by [CDS15], so that in particular it is

K-semistable. On the other hand, K-semistability is an open condition, see [X19, Theorem 1.4] or [BLX19, Corollary 1.2]. Hence X^{a} is K-semistable.

The problem of existence of Kähler–Einstein metrics on threefolds of type V_{22}^* was addressed by Donaldson in [D08, D18], by Rollin, Simanca and Tipler in [RST13], and by Dinew, Kapustka and Kapustka in [DKK17]. In particular, they proved that the set of such threefolds that are Kähler–Einstein is open in moduli in the Euclidean topology. Donaldson suggested that in fact all threefolds of type V_{22}^* are Kähler–Einstein. In [D08], he wrote

> The Mukai–Umemura manifold has $\tau = 1$. When τ is close to 1 we have seen that the corresponding manifold admits a Kähler–Einstein metric. It seems likely that this true for all τ but, as far as the author is aware, this is not known. It seems an interesting test case for future developments in the existence theory.

Here τ is a parameter in the moduli space of threefolds of type V_{22}^* that is used in [D08]. The Mukai–Umemura threefold corresponds to $\tau = 1$.

In [D18, §4.1], Donaldson made a more precise suggestion about which threefolds of type V_{22} are Kähler–Einstein metric and which are not. It also predicts that each threefold of type V_{22}^* must admit a Kähler–Einstein metric.

To verify Donaldson's suggestion, Dinew, Kapustka and Kapustka estimated the $\alpha_{\mathbb{C}^*}$-invariants of threefolds of type V_{22}^*. It appeared that they do not exceed $\frac{1}{2}$, so that Tian's Theorem 1.3 cannot be applied. However, the automorphism groups of all threefolds of type V_{22}^* are actually larger than $\mathbb{C}^*$. It was pointed out in [RST13, DKK17] that there exists an additional involution that anticommutes with the $\mathbb{C}^*$-action, so that together they generate a subgroup isomorphic to $\mathbb{C}^* \rtimes \boldsymbol{\mu}_2$. Here $\boldsymbol{\mu}_2$ denotes the group of order 2. In fact, by [KP17, Theorem 1.3], one has

$$\mathrm{Aut}\big(X\big) \cong \mathbb{C}^* \rtimes \boldsymbol{\mu}_2$$

for every threefold X of type V_{22}^* that is not the Mukai–Umemura threefold.

Dinew, Kapustka and Kapustka posed

Problem 1.5 ([DKK17, Problem 7.1])**.** Let X be a smooth Fano threefold of type V_{22}^*, and let G be a subgroup in $\mathrm{Aut}(X)$ that is isomorphic to $\mathbb{C}^* \rtimes \boldsymbol{\mu}_2$. Compute $\alpha_G(X)$.

In this paper we completely solve this problem using the description of smooth Fano threefolds of type V_{22}^* obtained recently by Kuznetsov and Prokhorov in [KP17].

Kuznetsov and Prokhorov proved that the isomorphisms classes of Fano threefolds of type V_{22}^* are naturally parameterized by $u \in \mathbb{C} \setminus \{0, 1\}$. In §2, we present their construction in details. Note that the parameter u used by

Kuznetsov and Prokhorov in [KP17] differs from the parameter τ used by Donaldson in [D08].

To state our main result, we denote by V_u the smooth Fano threefold of type V_{22}^* that corresponds to the parameter u in the construction of [KP17]. Then the Mukai–Umemura threefold is V_u for $u=-\frac{1}{4}$ by [KP17, Theorem 1.3]. Let G be a subgroup in $\mathrm{Aut}(V_u)$ such that

$$G\cong\mathbb{C}^*\rtimes\boldsymbol{\mu}_2.$$

The main result of our paper is

Theorem 1.6. *One has*

$$\alpha_G(V_u)=\begin{cases}\dfrac{4}{5} & \textit{if } u\neq\dfrac{3}{4} \textit{ and } u\neq 2,\\ \dfrac{3}{4} & \textit{if } u=\dfrac{3}{4},\\ \dfrac{2}{3} & \textit{if } u=2.\end{cases}$$

Applying Tian's Theorem 1.3, we obtain

Corollary 1.7. *If $u\neq\frac{3}{4}$ and $u\neq 2$, then V_u is Kähler–Einstein.*

Remark 1.8. If $u=\frac{3}{4}$ or $u=2$, then V_u is also Kähler–Einstein. This has been recently proved by Fujita in [Fu21]. Note also that Theorem 1.6 and [ACCF$^+$, Theorem 1.4.10] imply that V_u is Kähler–Einstein for $u=\frac{3}{4}$.

Let us describe the scheme of the proof of Theorem 1.6. To estimate $\alpha_G(V_u)$, one has to describe irreducible G-invariant subvarieties of small degree in V_u. Since G acts on V_u without fixed points, we have to deal with irreducible G-invariant curves of small degree, and G-invariant anticanonical surfaces in V_u. However, the geometry of the threefold V_u is rather complicated, and it is hard to complete these tasks in a straightforward way. Instead, we use a construction of the threefold V_u as a G-equivariant birational image of a smooth quadric hypersurface in $\mathbb{P}^4$ found recently by Kuznetsov and Prokhorov in [KP17], see the diagram (2.5) for more details. This allows to describe irreducible G-invariant curves of small degree in V_u and G-invariant surfaces in $|-K_{V_u}|$ in terms of the quadric, whose G-equivariant geometry is much easier to control. In particular, this description gives us an upper bound on $\alpha_G(V_u)$. To show that the latter bound is sharp, we have to study G-equivariant birational geometry of the threefold V_u. We do this using three explicit G-equivariant Sarkisov links that start from V_u. As a result, we obtain the formula for $\alpha_G(V_u)$ in Theorem 1.6.

Let us describe the structure of this paper. In §2, we recall from [KP17] the explicit construction of the threefold V_u using a birational map from a three-dimensional quadric. In this section, we also describe this birational map explicitly in coordinates. In §3, we start an explicit classification of irreducible G-invariant curves of small degree in the threefold V_u, which will be used in the proof of Theorem 1.6. In §4, we complete this classification, see Proposition 4.12. In §5, we study the pencil in the linear system $|-K_{V_u}|$ that consists of all G-invariant surfaces and describe singularities of surfaces in this pencil. In §6, we describe one Sarkisov link that plays a crucial role in the proof of Theorem 1.6. In this section, we also describe two special birational transformations of the threefold V_u, which are known as *bad Sarkisov links*. They are also used in the proof of our Theorem 1.6. Finally, in §7, we prove Theorem 1.6.

2. Kuznetsov–Prokhorov construction

Consider the projective space $\mathbb{P}^4$ with homogeneous coordinates x, y, z, t, and w. Suppose that the group $\mathbb{C}^*$ acts on $\mathbb{P}^4$ by

$$\lambda\colon (x:y:z:t:w) \mapsto (x:\lambda y:\lambda^3 z:\lambda^5 t:\lambda^6 w). \tag{2.1}$$

Furthermore, consider the involution ι acting on $\mathbb{P}^4$ by

$$\iota\colon (x:y:z:t:w) \mapsto (w:t:z:y:x). \tag{2.2}$$

This defines the action of the group $G \cong \mathbb{C}^* \rtimes \boldsymbol{\mu}_2$ on $\mathbb{P}^4$.

Let the quadric Q_u, where $u \in \mathbb{C}$, be given by equation

$$u(xw - z^2) + (z^2 - yt) = 0. \tag{2.3}$$

Then the quadric Q_u is G-invariant. Note that Q_u is smooth provided that $u \notin \{0,1\}$. Therefore, until the end of the paper (with the only exception of Remark 2.12), we will always assume that neither $u = 0$ nor $u = 1$.

Let Γ be the image of $\mathbb{P}^1$ with homogeneous coordinates $(s_0 : s_1)$ embedded into $\mathbb{P}^4$ by

$$(s_0 : s_1) \mapsto (s_0^6 : s_0^5 s_1 : s_0^3 s_1^3 : s_0 s_1^5 : s_1^6).$$

Then Γ is a G-invariant curve contained in the quadric Q_u. It is the closure of the G-orbit of the point $(1:1:1:1:1)$. One easily checks that $\deg(\Gamma) = 6$, cf. Lemma 3.1.

Let $\mathcal{S}$ be the complete intersection in $\mathbb{P}^4$ that is given by

$$\begin{cases} xw - z^2 = 0, \\ z^2 - yt = 0. \end{cases}$$

Then the surface $\mathcal{S}$ is G-invariant, and $\Gamma \subset \mathcal{S} \subset Q_u$.

Remark 2.4. The surface $\mathcal{S}$ is a toric singular del Pezzo surface of degree 4 that has 4 ordinary double points. These points are $(1:0:0:0:0)$,

$(0:0:0:0:1)$, $(0:1:0:0:0)$ and $(0:0:0:1:0)$. The first two of them are contained in the curve Γ.

It was proved in [KP17, Theorem 2.2] (cf. [Ta89, (2.13.2)]) that there exists the following G-equivariant commutative diagram

$$
\begin{tikzcd}
 & \widetilde{Q}_u \arrow[rr, dashed, "\chi"] \arrow[dr, "\alpha"] \arrow[dl, "\pi"'] & & \widetilde{V}_u \arrow[dl, "\beta"'] \arrow[dr, "\phi"] & \\
 & & Y_u & & \\
Q_u \arrow[urr, dashed, "\gamma"] \arrow[rrrr, dashed, "\zeta"] & & & & V_u \arrow[ull, dashed, "\omega"']
\end{tikzcd}
\tag{2.5}
$$

Here V_u is a smooth Fano threefold of type V_{22}^*, the morphism π is the blowup of the quadric Q_u along the curve Γ, the morphism ϕ is the blowup of the threefold V_u along a (unique) G-invariant smooth rational curve $\mathcal{C}_2$ with $-K_{V_u} \cdot \mathcal{C}_2 = 2$, the map χ is a flop in two smooth rational curves, which we will describe later in Remark 2.11. The morphisms α and β in (2.5) are small birational morphisms that are given by the linear systems $|-nK_{\widetilde{Q}_u}|$ and $|-nK_{\widetilde{V}_u}|$ for $n \gg 0$, respectively. By construction, the threefold Y_u is a non-$\mathbb{Q}$-factorial Fano threefold with terminal singularities such that $-K_{Y_u}^3 = 16$.

Remark 2.6. Kuznetsov and Prokhorov showed in [KP17] that every smooth Fano threefold of type V_{22}^* can be obtained via diagram (2.5) for some $u \in \mathbb{C} \setminus \{0, 1\}$. Moreover, they proved that for distinct u the resulting varieties V_u are not isomorphic. Furthermore, if $u = -\frac{1}{4}$, then V_u is the Mukai–Umemura threefold by [KP17, Theorem 1.3]. For other descriptions of threefolds of type V_{22}^*, see [D08, §5.3], [DKK17, §2.2] and [KPS18, §5.3].

Recall from [IP99, Proposition 4.1.11] that the divisor $-K_{V_u}$ is very ample, and the linear system $|-K_{V_u}|$ gives an embedding $V_u \hookrightarrow \mathbb{P}^{13}$. In particular, the curve $\mathcal{C}_2$ is a conic in this embedding. Let us identify V_u with its anticalonical image in $\mathbb{P}^{13}$ and fix the following notation.

- We denote by H_{Q_u} a hyperplane section of the quadric Q_u in $\mathbb{P}^4$.
- We denote by H_{V_u} a hyperplane section of the threefold V_u in $\mathbb{P}^{13}$.
- We denote by $\widetilde{\mathcal{S}}$ the proper transform of the surface $\mathcal{S}$ on the threefold $\widetilde{Q}_u$.
- We denote by E_{Q_u} the exceptional surface of the blowup π.
- We denote by E_{V_u} the exceptional surface of the blowup ϕ.

Then $\widetilde{\mathcal{S}}$ is the proper transform of E_{V_u} on $\widetilde{Q}_u$, which is the unique divisor in the linear system $|2\pi^*(H_{Q_u}) - E_{Q_u}|$. Similarly, the proper transform of E_{Q_u}

on $\widetilde{V}_u$ is the unique surface in the linear system $|2\phi^*(H_{V_u}) - 5E_{V_u}|$. Thus, we also fix the following notation.

- We denote by $\widetilde{\mathcal{R}}$ the unique surface in the linear system $|2\phi^*(H_{V_u}) - 5E_{V_u}|$.
- We denote by $\mathcal{R}$ the proper transform of the surface $\widetilde{\mathcal{R}}$ on the threefold V_u.

Corollary 2.7. *One has* $\alpha_G(V_u) \leqslant \frac{4}{5}$.

Proof. Let $D = \frac{1}{2}\mathcal{R}$. Then $D \sim_{\mathbb{Q}} -K_{V_u}$. Moreover, since $\mathcal{R} \sim -2K_{V_u}$ and $\mathrm{mult}_{\mathcal{C}_2}(\mathcal{R}) = 5$, the log pair $(V_u, \frac{4}{5}D)$ is not Kawamata log terminal. Indeed, we have

$$K_{\widetilde{V}_u} + \frac{4}{5}\widetilde{D} + E_{V_u} \sim_{\mathbb{Q}} \phi^*\Big(K_{V_u} + \frac{4}{5}D\Big).$$

This shows that $\alpha_G(V_u) \leqslant \frac{4}{5}$. $\square$

Using the information about the classes of the exceptional divisors E_{Q_u} and E_{V_u}, one can easily check that the rational map $\phi \circ \chi \colon \widetilde{Q}_u \dashrightarrow V_u$ is given by the linear system $|5\pi^*(H_{Q_u}) - 2E_{Q_u}|$, and the rational map $\pi \circ \chi^{-1} \colon \widetilde{V}_u \dashrightarrow Q_u$ is given by the linear system $|\phi^*(H_{V_u}) - 2E_{V_u}|$.

Remark 2.8. By [IP99, Proposition 4.1.12(iii)], the threefold V_u is a scheme-theoretic intersection of quadrics in $\mathbb{P}^{13}$. Thus since $-K_{\widetilde{V}_u} \sim \phi^*(H_{V_u}) - E_{V_u}$ and $h^0(\mathcal{O}_{\widetilde{V}_u}(-K_{\widetilde{V}_u})) = 11$, the linear system $|-K_{\widetilde{V}_u}|$ gives a morphism $V_u \to \mathbb{P}^{10}$ that is birational on its image. Hence, there is a commutative diagram

$$\begin{tikzcd} & \widetilde{V}_u \arrow[dl] \arrow[dr, "\phi"] & \\ \mathbb{P}^{10} & & V_u \arrow[ll, dashed] \end{tikzcd}$$

such that the dashed arrow is a linear projection from the conic $\mathcal{C}_2$. This implies that we can assume that the morphism β in (2.5) is given by the linear system $|-K_{\widetilde{V}_u}|$. Hence, we can also assume that the morphism α is given by the linear system $|-K_{\widetilde{Q}_u}|$. Thus, the threefold Y_u is a (singular) Fano threefold anticanonically embedded into $\mathbb{P}^{10}$.

Let L_1 and L_2 be the tangent lines in $\mathbb{P}^4$ to the curve Γ at the points $(1:0:0:0:0)$ and $(0:0:0:0:1)$, respectively. Then L_1 is given by

$$z = t = w = 0, \tag{2.9}$$

and the line L_2 is given by

$$x = y = z = 0. \tag{2.10}$$

Thus, both lines L_1 and L_2 are contained in the surface $\mathcal{S}$. Denote by $\widetilde{L}_1$ and $\widetilde{L}_2$ the proper transforms of the lines L_1 and L_2 on the threefold $\widetilde{Q}_u$, respectively.

Remark 2.11. By [KP17, Remark 5.3], the curves $\widetilde{L}_1$ and $\widetilde{L}_2$ are the flopping curves of the map χ. The flopping curves of χ^{-1} are described in [KP17, Proposition 5.1]. Namely, the threefold V_u contains exactly two lines that intersect the conic $\mathcal{C}_2$. Denote them by ℓ_1 and ℓ_2, and denote their proper transforms on $\widetilde{V}_u$ by $\widetilde{\ell}_1$ and $\widetilde{\ell}_2$, respectively. The lines ℓ_1 and ℓ_2 intersect the conic $\mathcal{C}_2$ transversally, because V_u is an intersection of quadrics. Moreover, the lines ℓ_1 and ℓ_2 are contained in the surface $\mathcal{R}$, since $\mathcal{R} \sim -2K_{V_u}$ and $\mathrm{mult}_{\mathcal{C}_2}(\mathcal{R}) = 5$. By [KP17, Remark 5.3], the curves $\widetilde{\ell}_1$ and $\widetilde{\ell}_2$ are exactly the flopping curves of the map χ^{-1}. Thus, the birational map ζ in (2.5) induces an isomorphism

$$Q_v \setminus \mathcal{S} \cong V_u \setminus \mathcal{R}.$$

Without loss of generality, we may assume that $\beta(\widetilde{\ell}_1) = \alpha(\widetilde{L}_1)$ and $\beta(\widetilde{\ell}_2) = \alpha(\widetilde{L}_2)$. Note that the lines ℓ_1 and ℓ_2 on the Fano threefold V_u are *special*, i.e., their normal bundles in V_u are isomorphic to $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$; see the proof of [KP17, Proposition 5.1]. This implies that the normal bundles of the curves $\widetilde{\ell}_1$ and $\widetilde{\ell}_2$ in $\widetilde{V}_u$ are isomorphic to $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$, so that the flop χ^{-1} is given by Reid's pagoda [R83, §5].

Remark 2.12. It follows from Theorem 1.6 and Remark 1.8 that V_u is K-polystable for every $u \notin \{0, 1\}$. It would be interesting to find the K-polystable limits of the threefolds V_u when $u \to 0$, $u \to 1$ and $u \to \infty$. In fact, we have a candidate for the limit in the case when $u \to 1$. Namely, if $u = 1$, then the quadric threefold Q_u is singular at the point $(0 : 0 : 1 : 0 : 0)$. This point is not contained in the surface $\mathcal{S}$, and it is not contained in the curve Γ. Thus, the commutative diagram (2.5) still makes sense in this case. The threefold V_1 is a Fano threefold with one ordinary double point such that $-K_{V_1}^3 = 22$. By [KP17, Proposition 5.4], one has $\mathrm{Pic}(V_1) \cong \mathbb{Z}$ and $\mathrm{Cl}(V_1) \cong \mathbb{Z}^2$, so that V_1 is one of the threefolds described in [P16, Theorem 1.2]. Note also that $\mathrm{Cl}(V_1)^G \cong \mathbb{Z}^2$. We expect that V_1 is K-polystable, so that it is the K-polystable limit of our threefolds V_u when $u \to 1$.

The commutative diagram (2.5) is a Sarkisov link (that starts at Q_u and ends at V_u). It plays a crucial role in the proof of our Theorem 1.6. In §6, we describe another G-equivariant Sarkisov link that starts at V_u and ends at another threefold of type V_{22}^* (possibly isomorphic to V_u). This link also helps to prove Theorem 1.6.

Remark 2.13 (cf. [CS12, CS14, CS15, CS16, CS19])**.** It would be interesting to study other G-Sarkisov links that start at the threefold V_u or the quadric Q_u. Such links usually arise from G-irreducible curves of small degree or G-orbits of small length. For example, the inverse of the link (2.5) arises from the conic $\mathcal{C}_2$, which is irreducible and G-invariant. The curve $\ell_1 + \ell_2$ from Remark 2.11 also gives rise to a G-Sarkisov link. Namely, one can show that there exists a G-equivariant commutative diagram

(2.14)
$$\begin{tikzcd}
 & \overline{V}_u \arrow[dl, "\upsilon"'] \arrow[dr, "\varsigma"] \arrow[rr, dashed, "\varrho"] & & \overline{W} \arrow[dl, "\varphi"'] \arrow[dr, "\nu"] & \\
V_u & & U & & W
\end{tikzcd}$$

Here υ is a blowup of the lines ℓ_1 and ℓ_2, the morphisms ς and φ are small and birational, the map ϱ flops the curves contracted by ς, the threefold U is a Fano threefold with terminal singularities such that $-K_U^3 = 14$, the threefold W is a smooth Fano threefold such that $\mathrm{Pic}(W) \cong \mathbb{Z}^2$ and $-K_W^3 = 28$, and ν is a birational morphism that contracts the proper transform of the unique surface in $|-K_{V_u}|$ which is singular along the lines ℓ_1 and ℓ_2 to a smooth rational curve of (anticanonical) degree 6. Note that $\mathrm{Pic}(W)^G \cong \mathbb{Z}$, and W is the threefold No. (1.2.3) in [P13, Theorem 1.2]. It can be realized as the blow-up of a smooth quadric in $\mathbb{P}^4$ along a twisted quartic curve. Note that unlike (2.5) the diagram (2.14) is not a Sarkisov link in the usual sense [C95], because the curve $\ell_1 + \ell_2$ is reducible.

Now we describe the birational maps γ and ζ in the diagram (2.5) explicitly using coordinates on $\mathbb{P}^4$. To describe the map γ, recall that this map is given by the restriction of the linear system of all cubic hypersurfaces in $\mathbb{P}^4$ that pass through the curve Γ to the quadric Q_u. Since γ is G-equivariant and, in particular, $\mathbb{C}^*$-equivariant, we are in position to choose $\mathbb{C}^*$-invariant generators of this linear system. To start with, set

$$f = xw - yt,$$

so that the equation $f = 0$ cuts out the surface $\mathcal{S}$ on the quadric Q_u. Then we set

$$\begin{aligned}
(2.15)\quad & h_3 = y^3 - x^2 z, \quad h_5 = x^2 t - y^2 z, \quad h_6 = xf, \quad h_7 = yf, \\
& h_8 = y^2 w - xzt, \quad h_9 = zf, \quad h_{10} = xt^2 - yzw, \quad h_{11} = tf, \\
& h_{12} = wf, \quad h_{13} = yw^2 - zt^2, \quad h_{15} = t^3 - zw^2.
\end{aligned}$$

Then the involution ι swaps the polynomials h_i and h_{18-i} for $3 \leqslant i \leqslant 8$, and it preserves the polynomial h_9. Observe also that these 11 cubic polynomials all vanish on the curve Γ. Moreover, the corresponding surfaces in Q_u cut out by $h_i = 0$ are smooth at a general point of the curve Γ, so that their proper transforms on $\widetilde{Q}_u$ are all contained in the linear system $|-K_{\widetilde{Q}_u}| = |3\pi^*(H_{Q_u}) - E_{Q_u}|$.

Every polynomial h_i is semi-invariant with respect to the $\mathbb{C}^*$-action (2.1). Moreover, the weight of the polynomial h_i equals i. This implies, in particular, that they define linearly independent sections in $H^0(\mathcal{O}_{Q_u}(3H_{Q_u}))$. Since $h^0(\mathcal{O}_{\widetilde{Q}_u}(-K_{\widetilde{Q}_u}))=11$ by the Riemann–Roch formula and Kawamata–Viehweg vanishing theorem, we conclude that the birational map γ in (2.5) is given by

(2.16)
$$(x:y:z:t:w)\mapsto \big(h_3:h_5:h_6:h_7:h_8:h_9:h_{10}:h_{11}:h_{12}:h_{13}:h_{15}\big).$$

Thus, using (2.9) and (2.10), we see that $\gamma(L_1)=(1:0:0:0:0:0:0:0:0:0:0)$ and $\gamma(L_2)=(0:0:0:0:0:0:0:0:0:0:1)$.

Now let us describe the map ζ in (2.5). To do this, we set

$$g_{i+6}=f\cdot h_i \tag{2.17}$$

for $i\in\{3,5,6,7,8,9,10,11,12,13,15\}$. Let

$$\begin{aligned} g_{10}&=(u-1)x^2yzw-3xy^2zt+(2-u)xyz^3+y^4w+x^3t^2,\\ g_{20}&=(u-1)xztw^2-3yzt^2w+(2-u)z^3tw+xt^4+y^2w^3,\\ g'_{15}&=(u-1)x^2t^3+(u-1)y^3w^2-(u+4)y^2zt^2+(3u+2)xyztw+(4-4u)yz^3t.\end{aligned} \tag{2.18}$$

Note that the involution ι swaps the polynomials g_i and g_{30-i} for $9\leqslant i\leqslant 14$, and it preserves both polynomials g_{15} and g'_{15}. Observe that all polynomials g_i and the polynomial g'_{15} are semi-invariant with respect to the $\mathbb{C}^*$-action (2.1). Moreover, the weight of the polynomial g_i equals i, and the weight of the polynomial g'_{15} equals 15. Also observe that

$$g'_{15}(0,1,0,0,1)=1\neq 0=g_{15}(0,1,0,0,1),$$

and the point $(0:1:0:0:1)$ is contained in the quadric Q_u. This implies, in particular, that these 14 quintic polynomials define linearly independent sections in $H^0(\mathcal{O}_{Q_u}(5H_{Q_u}))$.

For every $i\in\{9,\ldots,21\}$, denote by M_i the surface in the quadric Q_u that is cut out by the equation $g_i=0$. Similarly, denote by M'_{15} the surface in Q_u that is cut out by the equation $g'_{15}=0$. It is easy to see that all these surfaces pass through the curve Γ.

Lemma 2.19. *The surfaces M_i and M'_{15} are singular along* Γ.

Proof. For $i\in\{3,5,6,7,8,9,10,11,12,13,15\}$ this follows from the fact that the polynomials h_i and f vanish along Γ. To check the assertion for the surfaces M_{10}, M_{20} and M'_{15}, one can just write down the partial derivatives of g_{10}, g_{20} and g'_{15} at the point $(1:1:1:1:1)$, compare them with the partial derivatives of the left hand side of (2.3), and then use the fact that Γ is the closure of the orbit of the latter point. ∎

One can check that the multiplicities of the surfaces M_i and M'_{15} along the curve Γ equal 2. This also follows from the fact that the surfaces E_{Q_u} and $\widetilde{\mathcal{S}}$ generate the cone of effective divisors of the threefold $\widetilde{Q}_u$. We conclude that the proper transforms of the surfaces M_i and M'_{15} on the threefold $\widetilde{Q}_u$ generate the linear system $|5H_{Q_u} - 2E_{Q_u}|$. Hence, the birational map ζ in (2.5) is given by

$$\begin{aligned}(x:y:z:t:w) \mapsto \big(&g_9 : g_{10} : g_{11} : g_{12} : g_{13} : g_{14} : g_{15} \\ &: g'_{15} : g_{16} : g_{17} : g_{18} : g_{19} : g_{20} : g_{21}\big).\end{aligned} \tag{2.20}$$

In particular, this reproves [DKK17, Proposition 4.1].

Denote by T_i and T'_{15} the proper transforms of the surfaces M_i and M'_{15} on the threefold V_u, respectively. Then

$$T_i \sim T'_{15} \sim -K_{V_u} \sim H_{V_u}.$$

This implies that all surfaces T_i and T'_{15} are irreducible, because the group $\mathrm{Pic}(V_u)$ is generated by the divisor H_{V_u}. This implies that the surface M'_{15} is irreducible, since the surface T'_{15} is irreducible and M'_{15} does not contain the surface $\mathcal{S}$. Similarly, the surfaces M_{10} and M_{20} are also irreducible. However, the remaining surfaces M_i are reducible. Namely, let N_3, N_5, N_8, N_{10}, N_{13} and N_{15} be the surfaces in Q_u that are cut out by the equations $h_3 = 0$, $h_5 = 0$, $h_8 = 0$, $h_{10} = 0$ and $h_{15} = 0$, respectively. Similarly, let H_x, H_y, H_z, H_t and H_w be the hyperplane sections of the quadric Q_u that are cut out by $x = 0$, $y = 0$, $z = 0$, $t = 0$ and $w = 0$, respectively. Then we see from (2.15) that

$$\begin{gathered} M_9 = N_3 + \mathcal{S}, \quad M_{11} = N_5 + \mathcal{S}, \quad M_{12} = H_x + 2\mathcal{S}, \quad M_{13} = H_y + 2\mathcal{S}, \\ M_{14} = N_8 + \mathcal{S}, \quad M_{15} = H_z + 2\mathcal{S}, \quad M_{16} = N_{10} + \mathcal{S}, \quad M_{17} = H_t + 2\mathcal{S}, \\ M_{18} = H_w + 2\mathcal{S}, \quad M_{19} = N_{13} + \mathcal{S}, \quad M_{21} = N_{15} + \mathcal{S}. \end{gathered}$$

Thus, the surfaces T_9, T_{11}, T_{14}, T_{16}, T_{19} and T_{21} are actually the proper transforms on the threefold V_u of the surfaces N_3, N_5, N_8, N_{10}, N_{13} and N_{15}, respectively. Similarly, the surfaces T_{12}, T_{13}, T_{15}, T_{17} and T_{18} are the proper transforms on the threefold V_u of the surfaces H_x, H_y, H_z, H_t and H_w, respectively.

Remark 2.21. It follows from (2.20) that the conic $\mathcal{C}_2$ is contained in the surfaces T_9, T_{11}, T_{12}, T_{13}, T_{14}, T_{15}, T_{16}, T_{17}, T_{18}, T_{19} and T_{21}, and it is not contained in the surfaces T_{10}, T_{20} and T'_{15}.

Lemma 2.22. *The line ℓ_1 is contained in the surfaces T_{11}, T_{12}, T_{13}, T_{14}, T_{15}, T'_{15}, T_{16}, T_{17}, T_{18}, T_{19}, T_{20}, T_{21}, and it is not contained in the surfaces T_9 and T_{10}. Similarly, the line ℓ_2 is contained in the surfaces T_9, T_{10}, T_{11},*

T_{12}, T_{13}, T_{14}, T_{15}, T'_{15}, T_{16}, T_{17}, T_{18}, T_{19}, *and it is not contained in the surfaces* T_{20} *and* T_{21}.

Proof. Let $P_\lambda \in \mathbb{P}^4$ be the point

$$\Big(\frac{\lambda(u\lambda-\lambda+1)}{u} : \lambda : \lambda : 1 : 1\Big),$$

where $\lambda \in \mathbb{C}$. Let C be the (closure of the) curve swept out by P_λ. Then C is contained in the quadric Q_u, and

$$C \cap L_2 = P_0 = \big(0 : 0 : 0 : 1 : 1\big).$$

Note that the point P_0 is not contained in the curve Γ, so that the proper transforms of the curves C and L_2 on the threefold $\widetilde{Q}_u$ still meet at the preimage of the point P_0. This implies that the proper transform C_{V_u} of the curve C on the threefold V_u intersects the line ℓ_2. Substitute the coordinates of the point P_λ into (2.20), multiply the coordinates of the resulting point by $\frac{u}{\lambda}$, and let $\lambda = 0$. This gives the point

$$C_{V_u} \cap \ell_2 = (0:0:0:0:0:0:0:0:0:0:0:0:1:1-u).$$

Using the $\mathbb{C}^*$-action on $\mathbb{P}^{13}$, we immediately obtain the equations of the line ℓ_2. The equations for the line ℓ_1 are obtained in a similar way. Now the required assertion follows from (2.20). $\square$

Let us conclude this section by Lemma 2.23.

Lemma 2.23. *There are no G-fixed points in Q_u and V_u.*

Proof. It follows from (2.1) that the only $\mathbb{C}^*$-fixed points in the quadric Q_u are the points $(1:0:0:0:0)$, $(0:0:0:0:1)$, $(0:1:0:0:0)$ and $(0:0:0:1:0)$. Note that ι swaps the points $(1:0:0:0:0)$ and $(0:0:0:0:1)$, and it also swaps the remaining two $\mathbb{C}^*$-fixed points, so that there are no G-fixed points in Q_u. This also implies that there are no G-fixed points in $\widetilde{Q}_u$.

By Remark 2.11, the flopping curves of χ are disjoint and swapped by the involution ι. Hence, there are no G-fixed points in $\widetilde{V}_u$. Thus, if V_u contains a G-fixed point, then it must be contained in the conic $\mathcal{C}_2$.

Let $\Pi \cong \mathbb{P}^2$ be the linear span of the conic $\mathcal{C}_2$ in $\mathbb{P}^{13}$. Then Π is G-invariant. The action of G on Π is not faithful (indeed, it contains all elements of order 5 in $\mathbb{C}^*$). However, the kernel is finite, and the automorphism ι acts faithfully on Π. This implies that there is a faithful action of a quotient of G that is isomorphic to G on Π and thus on $\mathcal{C}_2$. Therefore, the conic $\mathcal{C}_2$ does not contain G-fixed points, so that there are no G-fixed points in V_u. $\square$

3. Invariant curves

In this section, we make the first steps needed for a description of irreducible G-invariant curves in Q_u and V_u. We start with

Lemma 3.1. *Fix a point $(a_0 : \ldots : a_n) \in \mathbb{P}^n$, and fix positive integers $r_0 \leqslant \ldots \leqslant r_n$. Let Z be the curve in $\mathbb{P}^n$ that is the closure of the subset*

$$\Big\{(\lambda^{r_0}a_0 : \ldots : \lambda^{r_n}a_n) \mid \lambda \in \mathbb{C}^*\Big\} \subset \mathbb{P}^n.$$

Denote by Σ the set of indices i such that $a_i \neq 0$. Set

$$r_k = \min\{r_i \mid i \in \Sigma\}, \quad r_K = \max\{r_i \mid i \in \Sigma\}.$$

Denote by d the greatest common divisor of the numbers $r_i - r_k$ for $i \in \Sigma$. Then

$$\deg(Z) = \frac{r_K - r_k}{d}.$$

Furthermore, let s be the maximal number of indices i in Σ with distinct r_i. Then Z is a rational normal curve if and only if $\deg(Z) = s$.

Proof. Cancelling a common factor in the homogeneous coordinates if necessary, we may assume that $r_k = 0$. To compute the degree of Z, note that the intersection points of Z with a general hyperplane Λ in $\mathbb{P}^n$ correspond to the roots of a polynomial $P_\Lambda(\lambda)$ of degree r_K in λ. Since P_Λ is actually a polynomial of degree r_K/d in λ^d, the r_K roots of P_Λ produce r_K/d points of $\Lambda \cap Z$. Thus, the degree of Z equals r_K/d. It remains to notice that the linear span of Z has dimension s, so that Z is a rational normal curve if and only if $\deg(Z) = s$. $\square$

There are no G-fixed points in Q_u by Lemma 2.23. This implies, in particular, that every irreducible G-invariant curve in Q_u is rational and contains at least one ι-fixed point. Hence, every irreducible G-invariant curve is a closure of the $\mathbb{C}^*$-orbit of any of its ι-fixed points.

Lemma 3.2. *All ι-fixed points in Q_u are the points*

$$P_\pm = (1 : \pm\sqrt{u} : 0 : \mp\sqrt{u} : -1)$$

and the points

$$\begin{aligned}(3.3)\quad \Big(b^2 - (1-u)(a-b)^2 &: u(a^2-b^2) - a^2 : \\ &a^2 - u(a-b)^2 : u(a^2-b^2) - a^2 : b^2 - (1-u)(a-b)^2\Big),\end{aligned}$$

where $(a : b) \in \mathbb{P}^1$.

Proof. Using (2.2), one can see that the ι-fixed points in $\mathbb{P}^4$ are the points of the line

$$\begin{cases} x+w=0, \\ y+t=0, \\ z=0, \end{cases}$$

and the points of the plane

$$\begin{cases} x-w=0, \\ y-t=0. \end{cases}$$

Intersecting the line with Q_u, we obtain the points $P_{\pm}$. Similarly, intersecting the plane with the quadric Q_u, we obtain the conic parameterized by (3.3). □

Observe that the $\mathbb{C}^*$-orbit of the point P_+ is the same as the $\mathbb{C}^*$-orbit of the point P_-. We denote its closure by $\Theta_{\pm}$. Similarly, we denote the closure of the $\mathbb{C}^*$-orbit of the point (3.3) by $\Theta_{a,b}$. By construction, the curves $\Theta_{\pm}$ and $\Theta_{a,b}$ are all irreducible G-invariant curves contained in the quadric Q_u.

Lemma 3.4. *The only irreducible G-invariant curves in $\mathcal{S}$ are*

$$\Gamma = \Theta_{0,1} = \Theta_{u,u-1}$$

and $\Theta_{1,0} = \Theta_{1,1}$. The degree of the curve $\gamma(\Theta_{1,0})$ in $\mathbb{P}^{10}$ is 12.

Proof. Recall from §2 that the surface $\mathcal{S}$ is cut out on the quadric Q_u by the equation $f=0$, where $f = xw - yt$. Substituting $x=1$, $y=\pm\sqrt{u}$, $z=0$, $t=\mp\sqrt{u}$ and $w=-1$ into the polynomial f, we get $u-1$, so that the curve $\Theta_{\pm}$ is not contained in $\mathcal{S}$. Similarly, substituting the coordinates of the point (3.3) into f, we obtain

$$4(1-u)ab(a-b)(u(a-b)-a),$$

and the first assertion follows.

The curve $\Theta_{1,0}$ is the closure of the $\mathbb{C}^*$-orbit of the point $P=(1:1:-1:1:1)$. Thus, by (2.16), the curve $\gamma(\Theta_{1,0})$ is the closure of the $\mathbb{C}^*$-orbit of the point

$$\gamma(P) = (1:1:0:0:1:0:1:0:0:1:1),$$

so that the degree of the curve $\gamma(\Theta_{0,1})$ is 12 by Lemma 3.1. □

Let Δ be the conic in Q_u that is cut out by

$$y=t=0. \tag{3.5}$$

Then Δ is G-invariant. One can check that

$$\Delta = \Theta_{\sqrt{u},\sqrt{u-1}} = \Theta_{-\sqrt{u},\sqrt{u-1}}.$$

Similarly, let Υ be the conic in Q_u that is cut out by

$$x=w=0. \tag{3.6}$$

Then Υ is G-invariant. One can check that

$$\Upsilon = \Theta_{\sqrt{1-u}+1,\sqrt{1-u}} = \Theta_{\sqrt{1-u}-1,\sqrt{1-u}}.$$

Lemma 3.7. *The following assertions hold.*

(i) *The curve $\zeta(\Theta_\pm)$ is a curve of degree 12. One has $\zeta(\Theta_\pm) \subset T_{15} \cap T'_{15}$.*

(ii) *The curve $\zeta(\Delta)$ is a rational normal curve of degree 4. One has $\zeta(\Delta) \subset T_{10} \cap T_{20}$.*

(iii) *The curve $\zeta(\Upsilon)$ is a rational normal curve of degree 6. One has $\zeta(\Upsilon) \subset T_{10} \cap T_{20}$.*

(iv) *For every curve $\Theta_{a,b}$ not contained in the surface $\mathcal{S}$ and different from Δ and Υ, the degree of $\zeta(\Theta_{a,b})$ is either 10 or 12.*

(v) *If $\Theta_{a,b}$ is not contained in the surface $\mathcal{S}$, then the degree of the curve $\zeta(\Theta_{a,b})$ equals 10 if and only if the curve $\Theta_{a,b}$ is contained in $N_3 \cap N_{15}$.*

Proof. By (2.20), the curve $\zeta(\Theta_\pm)$ is the closure of the $\mathbb{C}^*$-orbit of the point $\zeta(P_+)$ that is

$$\Big(u\sqrt{u} : -u : -\sqrt{u} : u-1 : \sqrt{u}(u-1) : \\ -u : 0 : 0 : u : -\sqrt{u}(u-1) : -u+1 : \sqrt{u} : u : -u\sqrt{u}\Big),$$

which is contained in $T_{15} \cap T'_{15}$. Then $\zeta(\Theta_\pm)$ is a curve of degree 12 by Lemma 3.1, and it is contained in $T_{15} \cap T'_{15}$. This proves assertion (i).

To prove assertions (ii), (iii) and (iv), we need some auxiliary computations. Define the polynomial

$$q_0 = (u-1)^2a^4 - 2(u-1)^2a^3b + 2(u-1)(u-2)a^2b^2 - 6u(u-1)ab^3 + u(3u-2)b^4.$$

Furthermore, define the polynomials

$$\begin{aligned}
q_1 &= (u-1)a^2 - ub^2,\\
q_2 &= (u-1)a^2 - (2u-2)ab + ub^2,\\
q_3 &= (u-1)a^2 + 2ab - (u+2)b^2,\\
q_4 &= (u-1)a^2 - (2u-2)ab + (u-2)b^2,\\
q_5 &= (u-1)a^2 - 2uab + ub^2,\\
q_6 &= (u-1)a^2 - (2u-4)ab + (u-4)b^2.
\end{aligned}$$

Recall that $u \neq 0$ and $u \neq 1$. Observe that q_i is coprime to q_j for $0 \leqslant i < j \leqslant 6$ with the following exceptions:

- q_0 is divisible by q_6 provided that $u^2 - 2u + 2 = 0$;
- $q_1 = q_6$ provided that $u = 2$;
- $q_3 = q_5$ provided that $u = -1$;
- q_2 and q_3 have a common linear factor provided that $u = \frac{-1\pm\sqrt{5}}{2}$.

Substituting the coordinates of the point (3.3) into the polynomials g_i and g'_{15}, we obtain the polynomials p_i and p'_{15} (in a and b), respectively. We compute

$$
\begin{aligned}
p_9 = p_{21} &= -8(u-1)a^2b(a-b)((u-1)a-ub)^2q_0,\\
p_{10} = p_{20} &= 4a^2((u-1)a-ub)^2q_1q_2q_3,\\
p_{11} = p_{19} &= -8(u-1)a^2b(a-b)((u-1)a-ub)^2q_1q_4,\\
p_{12} = p_{18} &= 16(u-1)^2a^2b^2(a-b)^2((u-1)a-ub)^2q_2,\\
p_{13} = p_{17} &= 16(u-1)^2a^2b^2(a-b)^2((u-1)a-ub)^2q_1,\\
p_{14} = p_{16} &= -8(u-1)a^2b(a-b)((u-1)a-ub)^2q_1q_2,\\
p_{15} &= -16(u-1)^2a^2b^2(a-b)^2((u-1)a-ub)^2q_5,\\
p'_{15} &= 4(u-1)a^2((u-1)a-ub)^2q_1^2q_6.
\end{aligned}
$$

Let us consider the curve $\Theta_{a,b}$ not contained in the surface $\mathcal{S}$. By Lemma 3.4 this means that $a \neq 0$, $b \neq 0$, $a-b \neq 0$ and $(u-1)a-ub \neq 0$. These conditions imply that

- the polynomials p_9 and p_{21} vanish if and only if q_0 does,
- the polynomials p_{10} and p_{20} vanish if and only if one of q_1, q_2, or q_3 does,
- the polynomials p_{11} and p_{19} vanish if and only if either q_1 or q_4 does,
- the polynomials p_{12} and p_{18} vanish if and only if q_2 does,
- the polynomials p_{13} and p_{17} vanish if and only if q_1 does,
- the polynomials p_{14} and p_{16} vanish if and only if either q_1 or q_2 does,
- the polynomial p_{15} vanishes if and only if q_5 does,
- the polynomial p'_{15} vanishes if and only if either q_1 or q_6 does.

Note that $q_1 = 0$ if and only if $\Theta_{a,b} = \Delta$, and $q_2 = 0$ if and only if $\Theta_{a,b} = \Upsilon$.

Suppose that $\Theta_{a,b} = \Delta$. Then $q_1 = 0$, so that

$$
p_{10} = p_{11} = p_{13} = p_{14} = p'_{15} = p_{16} = p_{17} = p_{19} = p_{20} = 0. \tag{3.8}
$$

The coprimeness properties of the polynomials q_i imply that p_9, p_{12}, p_{15}, p_{18} and p_{21} do not vanish. Therefore, $\zeta(\Delta)$ is a rational normal curve of degree 4 by (2.20) and Lemma 3.1, which proves assertion (ii).

Suppose that $\Theta_{a,b} = \Upsilon$. Then $q_2 = 0$, so that

$$
p_{10} = p_{12} = p_{14} = p_{16} = p_{18} = p_{20} = 0. \tag{3.9}
$$

The coprimeness properties of the polynomials q_i imply that p_9, p_{11}, p_{13}, p_{15}, p_{17}, p_{19} and p_{21} do not vanish. Therefore, we see that $\zeta(\Upsilon)$ is a rational normal curve of degree 6 by (2.20) and Lemma 3.1, which proves assertion (iii).

Now suppose that $\Theta_{a,b}$ is different from Δ and Υ. This means that $q_1 \neq 0$ and $q_2 \neq 0$, so that in particular p_{12} and p_{13} do not vanish. If $q_0 \neq 0$, then p_9 and p_{21} do not vanish as well, so that the degree of the curve $\zeta(\Theta_{a,b})$ is 12 by (2.20) and Lemma 3.1. Thus, we may assume that $q_0 = 0$, so that

$$p_9 = p_{21} = 0.$$

The coprimeness properties of the polynomials q_i imply that p_{10}, p_{11} and p_{20} do not vanish, so that the degree of the curve $\zeta(\Theta_{a,b})$ is 10 by (2.20) and Lemma 3.1. This proves assertion (iv). The condition $p_9 = p_{21} = 0$ means that the curve $\Theta_{a,b}$ is contained in M_9 and M_{21}. Since $M_9 = N_3 + \mathcal{S}$ and $M_{21} = N_{15} + \mathcal{S}$, we see that $\Theta_{a,b}$ is contained in N_3 and N_{15}, because we assume that $\Theta_{a,b}$ is not contained in $\mathcal{S}$. This proves assertion (v) and completes the proof of the lemma. ☐

Taking a more careful look at the proof of Lemma 3.7, one can deduce that there are only a finite number of curves among $\zeta(\Theta_{a,b})$ that are *not* rational normal curves of degree 12. Moreover, one can explicitly describe all such curves for any given u.

Remark 3.10. By Lemma 3.7(i), the intersection $T_{15} \cap T'_{15}$ contains the curve $\zeta(\Theta_{\pm})$, which is a curve of degree 12. Moreover, it follows from Lemma 2.22 that $T_{15} \cap T'_{15}$ contains both lines ℓ_1 and ℓ_2. Thus, the intersection $T_{15} \cap T'_{15}$ does not contain irreducible G-invariant curves of degree greater than 8 that are different from the curve $\zeta(\Theta_{\pm})$. Note that $T_{15} \cap T'_{15}$ does not contain the conic $\mathcal{C}_2$ by Remark 2.21. Using (3.5), we see that $T_{15} \cap T'_{15}$ does not contain the curve $\mathcal{C}_4$. Similarly, using (3.6), we see that $T_{15} \cap T'_{15}$ does not contain the curve $\mathcal{C}_6$.

Let us describe explicitly the curves $\Theta_{a,b}$ in the case when $\zeta(\Theta_{a,b})$ is a curve of degree 10. If $u \neq -\frac{1}{3}$, let ϑ be one of the roots $\sqrt{(3u+1)(1-u)}$. If $u = -\frac{1}{3}$, let $\vartheta = 0$. If $u = \frac{2}{3}$, then

$$(3u+1)(1-u) = 1.$$

In this case, we assume that $\vartheta = 1$. Observe that the quadric Q_u contains the point

$$\Big(1 : 1 : 1 : \frac{(u-1)(\vartheta - u - 1)}{2u^2} : \frac{(u-1)(2u^2 + \vartheta - u - 1)}{2u^3}\Big). \tag{3.11}$$

Similarly, the quadric Q_u contains the point

$$\Big(1 : 1 : 1 : \frac{(u-1)(-\vartheta - u - 1)}{2u^2} : \frac{(u-1)(2u^2 - \vartheta - u - 1)}{2u^3}\Big). \tag{3.12}$$

Let Ψ be the closure of the $\mathbb{C}^*$-orbit of the point (3.11), and let Ψ' be the closure of the $\mathbb{C}^*$-orbit of the point (3.12). Then the curve Ψ is G-invariant, since the $\mathbb{C}^*$-orbit of the point (3.11) contains the image of this point via the involution ι, because

$$\Big(1:\lambda:\lambda^3:\lambda^5\frac{(u-1)(\vartheta-u-1)}{2u^2}:\lambda^6\frac{(u-1)(2u^2+\vartheta-u-1)}{2u^3}\Big)$$
$$=\Big(\frac{(u-1)(2u^2+\vartheta-u-1)}{2u^3}:\frac{(u-1)(\vartheta-u-1)}{2u^2}:1:1:1\Big)$$

for $\lambda=\frac{u(\vartheta-u-1)}{(2u^2+\vartheta-u-1)}\in\mathbb{C}^*$. Similarly, we see that the curve Ψ' is G-invariant. Of course, the curves Ψ and Ψ' are of the form $\Theta_{a,b}$ for certain a and b, but we will never use the values of these parameters.

It is straightforward to check that $\Psi=\Psi'$ if and only if $u=-\frac{1}{3}$. Moreover, if $u=\frac{2}{3}$, then $\Psi\neq\Gamma$ and $\Psi'=\Gamma$. This explains why we let $\vartheta=1$ in this case.

Lemma 3.13. *The following assertions hold.*

- (i) *Both curves Ψ and Ψ' are contained in the intersection $N_3\cap N_{15}$.*
- (ii) *The curve Ψ is not contained in $\mathcal{S}$. If $u\neq\frac{2}{3}$, then Ψ' is not contained in $\mathcal{S}$.*
- (iii) *The curve $\zeta(\Psi)$ is a curve of degree* 10.
- (iv) *If $u\neq\frac{2}{3}$, then $\zeta(\Psi')$ is a curve of degree* 10.
- (v) *If $\Theta_{a,b}\not\subset\mathcal{S}$ and $\zeta(\Theta_{a,b})$ is a curve of degree* 10*, then $\Theta_{a,b}=\Psi$ or $\Theta_{a,b}=\Psi'$.*
- (vi) *The surfaces N_3 and N_{15} are tangent along Γ if and only if $u=\frac{2}{3}$.*
- (vii) *If $u=\frac{2}{3}$, then N_3 and N_{15} do not tangent $\mathcal{S}$ at a general point of the curve Γ.*
- (viii) *If $u=-\frac{1}{3}$, then N_3 and N_{15} are tangent along $\Psi=\Psi'$.*

Proof. Using (2.3), we see that the intersection $N_3\cap N_{15}$ is given in $\mathbb{P}^4$ by

$$\begin{cases} y^3-x^2z=0,\\ t^3-zw^2=0,\\ u(xw-z^2)+(z^2-yt)=0. \end{cases}\tag{3.14}$$

In fact, this system of equation defines an effective one-cycle in Q_u of degree 18, which contains the curve Γ.

Let us show that $N_3\cap N_{15}$ contains the curves Ψ and Ψ'. To do this, we may consider the subset where $x\neq 0$, so that we let $x=1$. Substituting $z=y^3$ and

$$w=\frac{yt}{u}+\frac{u-1}{u}z^2$$

into $t^3-zw^2=0$, we obtain the equation

$$\big(t-y^5\big)\Big(t^2u^2+(u^2-1)ty^5+(u-1)^2y^{10}\Big)=0.$$

If $t = y^5$, we get the curve Γ. Thus, the remaining part of the subset (3.14) consists of the $\mathbb{C}^*$-orbits of the points

$$\Big(1 : 1 : 1 : t : \frac{t+u-1}{u}\Big),$$

where t is a solution of the quadratic equation

$$u^2t^2 + (u^2-1)t + (u-1)^2 = 0.$$

Solving this equation, we obtain exactly the points (3.11) and (3.12). This shows that (3.14) contains the curves Ψ and Ψ'. This proves assertion (i).

Observe that the intersection $\mathcal{S} \cap N_3$ consists of the curve Γ, the line L_2, and the line $y = z = w = 0$. Similarly, the intersection $\mathcal{S} \cap N_{15}$ consists of the curve Γ, the line L_1, and the line $x = z = t = 0$. Thus, the curve Ψ is contained in $\mathcal{S}$ if and only if $\Psi = \Gamma$. Since $\mathcal{S}$ is cut out on Q_u by the equation $xw = yt$, we see that if Ψ is contained in $\mathcal{S}$, then

$$\frac{(u-1)(\vartheta-u-1)}{2u^2} = \frac{(u-1)(2u^2+\vartheta-u-1)}{2u^3}.$$

Simplifying this equation, we get $\vartheta = \frac{3u^2-1}{u-1}$, which implies that $u = \frac{2}{3}$, so that $\vartheta = 1$ by assumption, which implies that the point (3.11) is not contained in $\mathcal{S}$. Hence, we see that Ψ is not contained in $\mathcal{S}$. Similarly, we see that Ψ' is contained in $\mathcal{S}$ if and only if $u = \frac{2}{3}$. This proves assertion (ii).

Since Ψ is not contained in $\mathcal{S}$, we see that $\zeta(\Psi)$ is a curve of degree 10 by Lemma 3.7(v). Similarly, if $u \neq \frac{2}{3}$, then Ψ' is not contained in $\mathcal{S}$, so that $\zeta(\Psi')$ is a curve of degree 10 by Lemma 3.7(v) as well. This proves assertions (iii) and (iv).

If $\Theta_{a,b}$ is not contained in the surface $\mathcal{S}$ and $\zeta(\Theta_{a,b})$ is a curve of degree 10, then $\Theta_{a,b}$ is contained in $N_3 \cap N_{15}$ by Lemma 3.7(v). On the other hand, the intersection $N_3 \cap N_{15}$ is given by (3.14). We just proved that this system of equation defines the union $\Gamma \cup \Psi \cup \Psi'$, so that either $\Theta_{a,b} = \Psi$ or $\Theta_{a,b} = \Psi'$. This proves assertion (v).

To prove assertions (vi) and (vii), let us find the local equations of the surfaces N_3, N_{15} and $\mathcal{S}$ at the point $(1 : 1 : 1 : 1 : 1)$. We may work in a chart $x \neq 0$, so that we let $x = 1$. Substituting $w = \frac{yt}{u} + \frac{u-1}{u}z^2$ into the equation $t^3 - w^2z = 0$ and multiplying the resulting equation by u^2, we obtain the equation

$$t^3u^2 - t^2y^2z + 2(1-u)tyz^3 - (u-1)^2z^5 = 0.$$

Similarly, the surface $\mathcal{S}$ is given by $ty = z^2$, and the surface N_3 is given by $z = y^3$. Now introducing new coordinates $\bar{y} = y - 1$, $\bar{z} = z - 1$ and $\bar{t} = t - 1$, we see that N_{15} is given by

$$2\bar{y} + (5u-4)\bar{z} + (2-3u)\bar{t} + \text{higher order terms} = 0.$$

Similarly, the surface $\mathcal{S}$ is given by

$$\bar{y} - 2\bar{z} + \bar{t} + \text{higher order terms} = 0, \tag{3.15}$$

while the linear term of the defining equation of the surface N_3 is $3\bar{y}-\bar{z}$. Hence, the surface N_3 is not tangent to $\mathcal{S}$ at the point $(1:1:1:1:1)$. Similarly, we see that the surface N_3 is tangent to N_{15} at the point $(1:1:1:1:1)$ if and only if $u = \frac{2}{3}$. This proves assertions (vi) and (vii).

To prove assertion (viii), we assume that $u = -\frac{1}{3}$. Then $\Psi = \Psi'$, and the point (3.11) is the point $(1:1:1:4:-8)$. Arguing as above, we see that the local equations of the surfaces N_3 and N_{15} at the point $(1:1:1:4:-8)$ have the same linear part (in coordinates $\bar{y} = y-1$, $\bar{z} = z-1$ and $\bar{t} = t-4$). Hence, the surface N_3 is tangent to N_{15} at the point $(1:1:1:4:-8)$. This proves assertion (viii) and completes the proof of the lemma. $\square$

Recall from Remark 2.11 that the birational map ζ in (2.5) induces an isomorphism

$$Q_v \setminus \mathcal{S} \cong V_u \setminus \mathcal{R}.$$

Therefore, from (2.20) and Lemmas 3.7 and 3.13, we obtain an explicit description of all irreducible G-invariant curves in the Fano threefold V_u that are not contained in the surface $\mathcal{R}$. Thus, to classify all such curves in V_u, we need to describe those of them that are contained in $\mathcal{R}$. This will be done in the next section.

4. Invariant curves in the surface $\mathcal{R}$

In this section we describe irreducible G-invariant curves in the surface $\mathcal{R}$, and complete the classification of irreducible G-invariant curves in the threefold V_u (see Proposition 4.12). We will show that $\mathcal{R}$ contains exactly two irreducible G-invariant curves, one of which is the conic $\mathcal{C}_2$. To describe the other curve, we analyze all irreducible G-invariant curves in surface E_{Q_u}. We start with

Remark 4.1. Recall from Remark 2.4 that the surface $\mathcal{S}$ is smooth at every point of the curve Γ except for the points $(1:0:0:0:0)$ and $(0:0:0:0:1)$, which are isolated ordinary double singularities. This implies that

$$\widetilde{\mathcal{S}}\big|_{E_{Q_u}} = \widetilde{\Gamma} + \mathbf{l}_1 + \mathbf{l}_2$$

for some section $\widetilde{\Gamma}$ of the projection $E_{Q_u} \to \Gamma$, where $\mathbf{l}_1$ and $\mathbf{l}_2$ are the fibers of this projection over the points $(1:0:0:0:0)$ and $(0:0:0:0:1)$, respectively. The curve $\widetilde{\Gamma}$ is irreducible and G-invariant. Since $\widetilde{\Gamma}$ is contained in $\widetilde{\mathcal{S}}$, its image in V_u is the conic $\mathcal{C}_2$.

Now let us show that E_{Q_u} contains exactly two irreducible G-invariant curves.

Lemma 4.2. *The surface E_{Q_u} contains exactly two irreducible G-invariant curves. One of them is the curve $\widetilde{\Gamma}$ from Remark* 4.1*. The second one is also a section of the projection $E_{Q_u} \to \Gamma$.*

Proof. Let $\mathbf{l}$ be the fiber of the natural projection $E_{Q_u} \to \Gamma$ over the point $(1:1:1:1:1)$. Then $\mathbf{l} \cong \mathbb{P}^1$ and the curve $\mathbf{l}$ is ι-invariant. Thus, either ι fixes every point in $\mathbf{l}$ or ι fixes exactly two points in $\mathbf{l}$. Let us show that the former case is impossible. To do this, recall from §2 that

$$\Gamma \subset N_3 \cap N_5 \cap N_8 \cap N_{10} \cap N_{13} \cap N_{15},$$

and the surfaces N_3, N_5, N_8, N_{10}, N_{13}, N_{15} are smooth at a general point of the curve Γ. Denote by $\widetilde{N}_3$, $\widetilde{N}_5$, $\widetilde{N}_8$, $\widetilde{N}_{10}$, $\widetilde{N}_{13}$ and $\widetilde{N}_{15}$ the proper transforms of the surfaces N_3, N_5, N_8, N_{10}, N_{13} and N_{15} on the threefold $\widetilde{Q}_u$, respectively. Then each intersection

$$\widetilde{N}_3 \cap \mathbf{l}, \quad \widetilde{N}_5 \cap \mathbf{l}, \quad \widetilde{N}_8 \cap \mathbf{l}, \quad \widetilde{N}_{10} \cap \mathbf{l}, \quad \widetilde{N}_{13} \cap \mathbf{l}, \quad \widetilde{N}_{15} \cap \mathbf{l}$$

consists of a single point. Moreover, if $u \neq \frac{2}{3}$, then N_3 is not tangent to N_{15} at a general point of Γ by Lemma 3.13(vi). Hence, in this case, we have

$$\widetilde{N}_3 \cap \mathbf{l} \neq \widetilde{N}_{15} \cap \mathbf{l},$$

so that the involution ι swaps these two points, since $\iota(N_3) = N_{15}$. Thus, if $u \neq \frac{2}{3}$, then the involution ι acts on the curve $\mathbf{l}$ nontrivially.

Recall that $\iota(N_5) = N_{13}$, the surface N_5 is cut out on Q_u by $x^2t - y^2z = 0$, and the surface N_5 is cut out on Q_u by $yw^2 - zt^2 = 0$. Let us find out when N_5 is tangent to N_{13} at a general point of Γ. To do this, let us describe the local equations of the surfaces N_5 and N_{13} at the point $(1:1:1:1:1)$. We may work in a chart $x \neq 0$, so that we let $x = 1$. Substituting

$$w = \frac{yt}{u} + \frac{u-1}{u} z^2$$

into $yw^2 - zt^2 = 0$ and multiplying the resulting equation by u^2, we obtain the equation

$$t^2y^3 - u^2t^2z + 2(u-1)ty^2z^2 + (u-1)^2yz^4 = 0.$$

This is the equation of N_{13}. The equation of the surface N_5 is simply $t = y^2z$.

Now introducing new coordinates $\bar{y} = y - 1$, $\bar{z} = z - 1$ and $\bar{t} = t - 1$, we see that N_{13} is given by

$$(u+2)\bar{y} + (3u-4)\bar{z} + 2(1-u)\bar{t} + \text{higher order terms} = 0.$$

Similarly, the surface N_{13} is given by

$$2\bar{y} + \bar{z} - \bar{t} + \text{higher order terms} = 0.$$

This implies that N_5 is tangent to N_{13} at the point $(1:1:1:1:1)$ if and only if $u = 2$.

Recall from Lemma 3.13(vi) that N_3 is tangent to N_{15} at a general point of the curve Γ if and only if $u = \frac{2}{3}$. We see that N_5 is tangent to the surface N_{13} at a general point of the curve Γ if and only if $u = 2$. The same arguments imply that N_8 is never tangent to N_{10} at a general point of the curve Γ. Arguing as above, we see that ι acts on $\mathbf{l}$ nontrivially as claimed.

Since ι acts nontrivially on the fiber $\mathbf{l}$, it fixes two points in $\mathbf{l}$. One of them is the point $\mathbf{l} \cap \widetilde{\mathcal{S}}$. It is contained in $\widetilde{\Gamma}$, so that $\widetilde{\Gamma}$ is the closure of the $\mathbb{C}^*$-orbit of the point $\mathbf{l} \cap \widetilde{\mathcal{S}}$. Similarly, the closure of the $\mathbb{C}^*$-orbit of the second fixed point of the involution ι is another irreducible G-invariant curve in E_{Q_u}. Then every irreducible G-invariant curve in E_{Q_u} must be one of these two curves. Indeed, an irreducible G-invariant curve in E_{Q_u} cannot be contracted by π, since Q_u does not have G-fixed points. Moreover, since all $\mathbb{C}^*$-orbits in E_{Q_u} that are not contained in the fibers of the projection $E_{Q_u} \to \Gamma$ are its sections, we conclude that an intersection of any irreducible G-invariant curve in E_{Q_u} with $\mathbf{l}$ must consist of a ι-invariant point, which in turn uniquely determines this curve. Since we proved that $\mathbf{l}$ contains exactly two ι-fixed points, an irreducible G-invariant curve in E_{Q_u} must be the closure of the $\mathbb{C}^*$-orbit of one of these two points. This completes the proof of the lemma. $\square$

Thus, the surface E_{Q_u} contains exactly two irreducible G-invariant curves. One of them is the curve $\widetilde{\Gamma}$ from Remark 4.1. The second curve can be described rather explicitly.

Remark 4.3. Let us use the notation of the proof of Lemma 4.2. Recall from this proof that ι fixes exactly two points in $\mathbf{l}$. One of them is the point $\mathbf{l} \cap \widetilde{\mathcal{S}}$. To describe the second ι-fixed point in $\mathbf{l}$, denote by M_{15}^{μ} the surface in Q_u that is cut out by the equation

$$g_{15}' + \mu g_{15} = 0,$$

where $\mu \in \mathbb{C}$. Denote by $\widetilde{M_{15}^{\mu}}$ the proper transform of the surface M_{15}^{μ} on the threefold $\widetilde{Q}_u$. Then M_{15}^{μ} is singular along Γ by Lemma 2.19. Moreover, it has a double point at a general point of Γ. To determine its type, let us describe the local equation of the surface M_{15}^{μ} at the point $(1:1:1:1:1)$. We may work in the chart $x \neq 0$, so that we let $x = 1$. Substituting $x = 1$ and

$w = \frac{yt}{u} + \frac{u-1}{u}z^2$ into $g'_{15} + \mu g_{15}$ and multiplying the result by u^2, we obtain the polynomial

$$
\begin{aligned}
u^2t^3 + t^2y^5 + (u^2\mu - 2u\mu + \mu + u - 4)t^2y^2z \\
+ 2(u-1)ty^4z^2 + (8 - 2u^2\mu + 4u\mu - 3u^2 - 2\mu - 4u)tyz^3 \\
+ (u-1)^2y^3z^4 + (u^2\mu - 2u\mu + u^2 + \mu + 3u - 4)z^5.
\end{aligned}
$$

Then introducing new coordinates $\bar{y} = y - 1$, $\bar{z} = z - 1$ and $\bar{t} = t - 1$, we rewrite this polynomial as

$$
\begin{aligned}
(4.4)\quad & (\mu u^2 - 2\mu u + 3u^2 + \mu + u - 3)\bar{t}^2 \\
&+ (2\mu u^2 - 4\mu u - 3u^2 + 2\mu + 8u - 6)\bar{t}\bar{y} + (12 - 4\mu u^2 + 8\mu u - 9u^2 - 4\mu - 6u)\bar{t}\bar{z} \\
&+ (\mu u^2 - 2\mu u + 3u^2 + \mu + 7u - 3)\bar{y}^2 + (12 - 4\mu u^2 + 8\mu u + 3u^2 - 4\mu - 18u)\bar{y}\bar{z} \\
&+ (4\mu u^2 - 8\mu u + 7u^2 + 4\mu + 8u - 12)\bar{z}^2 + \text{higher order terms.}
\end{aligned}
$$

If $\mu \neq -\frac{3u^2+16u-16}{4(u-1)^2}$, then the surface M_{15}^{μ} has a nonisolated ordinary double point at a general point of Γ. Vice versa, if $\mu = -\frac{3u^2+16u-16}{4(u-1)^2}$, then the quadratic part of the polynomial (4.4) simplifies as

$$\frac{1}{4}\Big((2+3u)\bar{y} + 4(u-1)\bar{z} + (2-3u)\bar{t}\Big)^2.$$

Comparing it with (3.15), we see that the intersection $\widetilde{M}_{15}^{\mu} \cap \mathbf{l}$ consists of a single point that is not contained in $\widetilde{\mathcal{S}}$. This is the second point fixed in $\mathbf{l}$ by the involution ι.

Remark 4.5. Suppose that $u = \frac{2}{3}$. Let $\widetilde{Z}$ be an irreducible G-invariant curve contained in the surface E_{Q_u} that is different from the curve $\widetilde{\Gamma}$. Denote by $\widetilde{\Psi}$ the proper transform of the curve Ψ on the threefold $\widetilde{Q}_u$. Let us use the notation from the proof of Lemma 4.2 and Remark 4.3. Then

$$\widetilde{N}_3 \cap \widetilde{N}_{15} = \widetilde{Z} \cup \widetilde{\Psi}$$

by Lemma 3.13(vi), because N_3 is smooth at the point $(1:0:0:0:0)$, and N_{15} is smooth at the point $(0:0:0:0:1)$. Observe also that the curve $\widetilde{L}_1$ is contained in $\widetilde{N}_3$, and it is not contained in $\widetilde{N}_{15}$. Similarly, the curve $\widetilde{L}_2$ is contained in $\widetilde{N}_{15}$, and it is not contained in $\widetilde{N}_3$. Thus, since $\widetilde{N}_{15} \cdot \widetilde{L}_1 = 0$ and $\widetilde{N}_3 \cdot \widetilde{L}_2 = 0$, we see that $\widetilde{L}_1$ is disjoint from $\widetilde{N}_{15}$, and $\widetilde{L}_2$ is disjoint from $\widetilde{N}_3$. Using (2.5) and (2.20), we see that

$$T_9 \cap T_{21} = \mathcal{C}_2 \cup \zeta\big(\Psi\big) \cup \phi \circ \chi\big(\widetilde{Z}\big).$$

Moreover, the surfaces T_9 and T_{21} intersect transversally at a general point of the conic $\mathcal{C}_2$, since the surface $\widetilde{\mathcal{S}}$ does not contain the curves $\widetilde{Z}$ and $\widetilde{\Psi}$.

Furthermore, the curve $\zeta(\Psi)$ has degree 10 by Lemma 3.13(iii). Thus $\phi\circ\chi(\widetilde{Z})$ is also a curve of degree 10.

Remark 4.6. Suppose that $u=2$. Let $\widetilde{Z}$ be an irreducible G-invariant curve contained in the surface E_{Q_u} that is different from the curve $\widetilde{\Gamma}$. Let us use the notation from the proof of Lemma 4.2 and Remark 4.3. In the proof of Lemma 4.2, we showed that both surfaces $\widetilde{N}_5$ and $\widetilde{N}_{13}$ contain the curve $\widetilde{Z}$. On the other hand, we have

$$N_5\cap N_{13}=\Gamma\cup\Delta\cup L_1\cup L_2.$$

Moreover, the surfaces N_5 and N_{13} are not tangent at a general point of the conic Δ. This can be checked, for example, using local equations of the surfaces N_5 and N_{13} at the point $(1:0:2:0:2)$. Observe also that the surface N_5 is smooth at the point $(0:0:0:0:1)$, and the surface N_{13} is smooth at the point $(1:0:0:0:0)$. Hence, we deduce that

$$\widetilde{N}_5\cap\widetilde{N}_{13}=\widetilde{Z}\cup\widetilde{\Delta}\cup\widetilde{L}_1\cup\widetilde{L}_2,$$

where $\widetilde{\Delta}$ is the proper transform of the conic Δ. Moreover, the surfaces $\widetilde{N}_5$ and $\widetilde{N}_{13}$ intersect transversally at a general point of the curve $\widetilde{Z}$. Indeed, otherwise the curve Γ would be contained in the one-cycle $N_5\cdot N_{13}$ with multiplicity at least 3, which is impossible, since $H_{Q_u}\cdot N_5\cdot N_{13}=18$, and the one-cycle $N_5\cdot N_{13}$ also contains the conic Δ and the lines L_1 and L_2. Thus, keeping in mind that the curves $\widetilde{L}_1$ and $\widetilde{L}_2$ are contracted by α, we conclude that

$$\alpha\big(\widetilde{N}_5\big)\cap\alpha\big(\widetilde{N}_{13}\big)=\alpha(\widetilde{Z})\cup\gamma(\Delta).$$

On the other hand, the degree of the curve $\gamma(\Delta)$ is 4, one has $-K_{Y_u}^3=16$ and

$$\alpha\big(\widetilde{N}_5\big)\sim\alpha\big(\widetilde{N}_{13}\big)\sim -K_{Y_u}.$$

This implies that $\alpha(\widetilde{Z})$ is a curve of degree 12, because $\alpha(\widetilde{N}_5)$ and $\alpha(\widetilde{N}_{13})$ intersect transversally at general points of the curves $\alpha(\widetilde{Z})$ and $\gamma(\Delta)$. Denote by $\widetilde{C}$ the proper transform of the curve $\widetilde{Z}$ on the threefold $\widetilde{V}_u$. Then

$$\begin{aligned}12=\deg\big(\alpha(\widetilde{Z})\big)&=-K_{\widetilde{Q}_u}\cdot\widetilde{Z}=-K_{Y_u}\cdot\alpha(\widetilde{Z})=-K_{Y_u}\cdot\beta(\widetilde{C})=-K_{\widetilde{V}_u}\cdot\widetilde{C}\\&=\Big(\phi^*\big(H_{V_u}\big)-E_{V_u}\Big)\cdot\widetilde{C}\leqslant\phi^*\big(H_{V_u}\big)\cdot\widetilde{C}=H_{V_u}\cdot\widetilde{C}=\deg\big(\phi(\widetilde{C})\big).\end{aligned}$$

We conclude our investigation of irreducible G-invariant curves in E_{Q_u} by the following result, which also completes the description of irreducible G-invariant curves in V_u of degree 10 started in Lemma 3.13 and Remark 4.5.

Lemma 4.7. *Let $\widetilde{Z}$ be an irreducible G-invariant curve contained in the surface E_{Q_u}. Then one of the following two possibilities holds.*

- *The curve $\widetilde{Z}$ is the curve $\widetilde{\Gamma}$ from Remark* 4.1. *The curve $\phi\circ\chi(\widetilde{Z})$ is the conic $\mathcal{C}_2$. The degree of the curve $\alpha(\widetilde{Z})$ is at least* 12.

- *The curve $\widetilde{Z}$ is the unique irreducible G-invariant curve in E_{Q_u} not contained in $\widetilde{\mathcal{S}}$. If $u \neq \frac{2}{3}$, then $\deg(\phi \circ \chi(\widetilde{Z})) \geqslant 12$. If $u = \frac{2}{3}$, then $\deg(\phi \circ \chi(\widetilde{Z})) = 10$, and the curve $\phi \circ \chi(\widetilde{Z})$ is contained in $T_9 \cap T_{21}$.*

Proof. The normal bundle of the smooth rational curve Γ in Q_u is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(p) \oplus \mathcal{O}_{\mathbb{P}^1}(q)$ for some integers p and q such that $p \geqslant q$ and $p + q = 16$. Thus, the exceptional surface E_{Q_u} is a Hirzebruch surface $\mathbb{F}_n$ for $n = p - q \geqslant 0$. Denote by $\mathbf{s}$ the section of the natural projection $E_{Q_u} \to \Gamma$ such that $\mathbf{s}^2 = -n$. Then $-E_{Q_u}|_{E_{Q_u}} \sim \mathbf{s} + \kappa\mathbf{l}$ for some integer κ. One has

$$-16 = E_{Q_u}^3 = \Big(\mathbf{s} + \kappa\mathbf{l}\Big)^2 = -n + 2\kappa,$$

so that $\kappa = \frac{n-16}{2}$. This implies that $\widetilde{\mathcal{S}}|_{E_{Q_u}} \sim \mathbf{s} + \frac{n+8}{2}\mathbf{l}$. On the other hand, it follows from Remark 4.1 that $\widetilde{\mathcal{S}}|_{E_{Q_u}} = \widetilde{\Gamma} + \mathbf{l}_1 + \mathbf{l}_2$, where $\mathbf{l}_1$ and $\mathbf{l}_2$ are the fibers of the natural projection $E_{Q_u} \to \Gamma$ over the points $(1:0:0:0:0)$ and $(0:0:0:0:1)$, respectively. This gives $\widetilde{\Gamma} \sim \mathbf{s} + \frac{n+4}{2}\mathbf{l}$, which implies, in particular, that $\widetilde{\Gamma} \neq \mathbf{s}$. Hence, we have

$$0 \leqslant \widetilde{\Gamma} \cdot \mathbf{s} = \Big(\mathbf{s} + \frac{n+4}{2}\mathbf{l}\Big) \cdot \mathbf{s} = \frac{4-n}{2},$$

which implies that $n \leqslant 4$. Thus, we compute

$$\deg\big(\alpha(\widetilde{Z})\big) = -K_{\widetilde{Q}_u} \cdot \widetilde{Z} = \Big(3\pi^*\big(H_{Q_u}\big) - E_{Q_u}\Big) \cdot \widetilde{Z} = \Big(\mathbf{s} + \frac{n+20}{2}\mathbf{l}\Big) \cdot \widetilde{Z}. \tag{4.8}$$

In particular, if $\widetilde{Z} = \widetilde{\Gamma}$, then (4.8) gives

$$\deg\big(\alpha(\widetilde{Z})\big) = \Big(\mathbf{s} + \frac{n+20}{2}\mathbf{l}\Big) \cdot \Big(\mathbf{s} + \frac{n+4}{2}\mathbf{l}\Big) = 12.$$

Let $\widetilde{C}$ be the proper transform of the curve $\widetilde{Z}$ on the threefold $\widetilde{V}_u$, and let $C = \phi(\widetilde{C})$. If $\widetilde{Z} \neq \widetilde{\Gamma}$, then

(4.9)
$$\begin{aligned}\deg\big(\alpha(\widetilde{Z})\big) &= -K_{\widetilde{Q}_u} \cdot \widetilde{Z} = -K_{Y_u} \cdot \alpha(\widetilde{Z}) = -K_{Y_u} \cdot \beta(\widetilde{C}) = -K_{\widetilde{V}_u} \cdot \widetilde{C} \\ &= \Big(\phi^*\big(H_{V_u}\big) - E_{V_u}\Big) \cdot \widetilde{C} \leqslant \phi^*\big(H_{V_u}\big) \cdot \widetilde{C} = H_{V_u} \cdot \widetilde{C} = \deg(C).\end{aligned}$$

Now let us use the notation from the proof of Lemma 4.2 and Remark 4.3. To complete the proof, we may assume that $\widetilde{Z}$ is the closure of the $\mathbb{C}^*$-orbit of the point $\widetilde{M}_{15}^{\mu} \cap \mathbf{l}$. Then $\widetilde{Z}$ is contained in $\widetilde{M}_{15}^{\mu}$, it is a section of the natural projection $E_{Q_u} \to \Gamma$, and it is not contained in $\widetilde{\mathcal{S}}$. In particular, we have $\widetilde{Z} \neq \widetilde{\Gamma}$.

By Remarks 4.5 and 4.6, we may assume that $u \neq \frac{2}{3}$ and $u \neq 2$. This implies that $n = 0$, cf. Remark 4.10. Indeed, suppose that $n > 0$. Then

$\widetilde{Z} = \mathbf{s}$ by Lemma 4.2, because the curve $\mathbf{s}$ is clearly G-invariant. Then it follows from (4.8) that

$$\deg\big(\alpha(\widetilde{Z})\big) = -K_{\widetilde{Q}_u} \cdot \widetilde{Z} = \frac{20-n}{2} < 10.$$

Hence, at least one surface among $\widetilde{N}_3$, $\widetilde{N}_5$, $\widetilde{N}_8$, $\widetilde{N}_{10}$, $\widetilde{N}_{13}$ and $\widetilde{N}_{15}$ contains the curve $\widetilde{Z}$. Since $\iota(\widetilde{N}_3) = \widetilde{N}_{15}$, $\iota(\widetilde{N}_5) = \widetilde{N}_{13}$ and $\iota(\widetilde{N}_8) = \widetilde{N}_{10}$, this implies that $\widetilde{Z}$ is contained in at least one of the intersections $\widetilde{N}_3 \cap \widetilde{N}_{15}$, $\widetilde{N}_5 \cap \widetilde{N}_{13}$, $\widetilde{N}_8 \cap \widetilde{N}_{10}$. On the other hand, it follows from Lemma 3.13(vi) that N_3 is tangent to N_{15} at a general point of the curve Γ if and only if $u = \frac{2}{3}$. Since we assumed that $u \neq \frac{2}{3}$, we see that

$$\widetilde{Z} \not\subset \widetilde{N}_3 \cap \widetilde{N}_{15}.$$

Likewise, the surface N_5 is tangent to the surface N_{13} at a general point of the curve Γ if and only if $u = 2$. We showed this in the proof of Lemma 4.2. Similar computations imply that the surface N_8 is not tangent to N_{10} at a general point of the curve Γ. Therefore, the curve $\widetilde{Z}$ is contained neither in $\widetilde{N}_5 \cap \widetilde{N}_{13}$ nor in $\widetilde{N}_8 \cap \widetilde{N}_{10}$. The obtained contradiction shows that the case $n > 0$ is impossible, so that $n = 0$.

Since $n = 0$, one has $E_{Q_u} \cong \mathbb{P}^1 \times \mathbb{P}^1$. By (4.8), we have

$$-K_{\widetilde{Q}_u} \cdot \widetilde{Z} = \Big(\mathbf{s} + 10\mathbf{l}\Big) \cdot \widetilde{Z} \geqslant \Big(\mathbf{s} + 10\mathbf{l}\Big) \cdot \mathbf{s} = 10.$$

This also shows that $-K_{\widetilde{Q}_u} \cdot \widetilde{Z} = 10$ if and only if $\widetilde{Z} \sim \mathbf{s}$. However, this case is impossible. Indeed, if $\widetilde{Z} \sim \mathbf{s}$, then the linear system $|\mathbf{s}|$ contains at least two irreducible G-invariant curves. On the other hand, we already know from Lemma 4.2 that $\widetilde{Z}$ and $\widetilde{\Gamma} \sim \mathbf{s} + 2\mathbf{l}$ are the only irreducible G-invariant curves in the surface E_{Q_u}. Hence, using (4.9) we conclude that $\deg(C) \geqslant -K_{\widetilde{Q}_u} \cdot \widetilde{Z} \geqslant 11$.

Using Lemma 3.7, we see that V_u does not contain irreducible G-invariant curves of degrees 1, 3, 5, 7, 8 and 9. In particular, the threefold V_u does not contain G-invariant lines, which also follows from [KP17, Lemma 4.1(i)].

By Remark 3.10, there exists a unique surface in the pencil generated by T_{15} and T'_{15} that contains C. In fact, we know this surface from Remark 4.3. It is the image of the surface $\widetilde{M}^{\mu}_{15}$ from Remark 4.3, where $\mu = -\frac{3u^2+16u-16}{4(u-1)^2}$. Thus, if $\deg(C) = 11$, there should be at least one surface among T_9, T_{10}, T_{11}, T_{12}, T_{13}, T_{14}, T_{16}, T_{17}, T_{18}, T_{19}, T_{20}, T_{21} that also contains C. But we proved above that none of the surfaces $\widetilde{N}_3$, $\widetilde{N}_5$, $\widetilde{N}_8$, $\widetilde{N}_{10}$, $\widetilde{N}_{13}$, $\widetilde{N}_{15}$ contains the curve $\widetilde{Z}$, so that the surfaces T_9, T_{11}, T_{14}, T_{16}, T_{19} and T_{21} do not contain C either. Similarly, the surfaces T_{12}, T_{13}, T_{17} and T_{18} do not contain the curve C, because the surfaces H_x, H_y, H_z, H_t and H_w do not contain the curve Γ. Thus, to complete the proof, we may assume that either T_{10} or T_{20} contains the curve C. Actually, this assumption implies that both surfaces T_{10} and T_{20} contain the curve C, since $\iota(T_{10}) = T_{20}$. Note that this case is indeed possible when $u = -2$ by Remark 4.11.

By Lemma 3.7, both surfaces T_{10} and T_{20} contain the curves $\zeta(\Delta)$ and $\zeta(\Upsilon)$, the degree of the curve $\zeta(\Delta)$ is 4, and the degree of the curve $\zeta(\Upsilon)$ is 6. Since we already know that $\deg(C) \geqslant 11$, we see that the G-invariant one-cycle $T_{10} \cdot T_{20}$ consists of the curves $\zeta(\Delta)$, $\zeta(\Upsilon)$, C and a G-invariant curve of degree $12 - \deg(C)$. Since V_u does not contain G-invariant lines, we see that

$$T_{10} \cdot T_{20} = \zeta(\Delta) + \zeta(\Upsilon) + C,$$

so that $\deg(C) = 12$. This completes the proof of the lemma. $\square$

Remark 4.10. If $u \neq \frac{2}{3}$ and $u \neq 2$, then $E_{Q_u} \cong \mathbb{P}^1 \times \mathbb{P}^1$, so that the normal bundle of the curve Γ in the quadric Q_u is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(8) \oplus \mathcal{O}_{\mathbb{P}^1}(8)$. We showed this in the proof of Lemma 4.7. Vice versa, if $u = \frac{2}{3}$ or $u = 2$, then, arguing as in the proof of Lemma 4.7, one can show that $E_{Q_u} \cong \mathbb{F}_4$, so that the normal bundle of the curve Γ is $\mathcal{O}_{\mathbb{P}^1}(6) \oplus \mathcal{O}_{\mathbb{P}^1}(10)$ in this case. However, we will not use this information in the sequel.

Remark 4.11. Denote by $\widetilde{M}_{10}$ and $\widetilde{M}_{20}$ the proper transform of the surfaces M_{10} and M_{20} on the threefold $\widetilde{Q}_u$, respectively. Recall that both M_{10} and M_{20} have quadratic singularity at the point $(1:1:1:1:1)$. Substituting $x = 1$ and $w = \frac{yt}{u} + \frac{u-1}{u}z^2$ into the polynomial ug_{10}, we obtain the polynomial $ut^2 + ty^5 - (2u+1)y^2zt + (u-1)y^4z^2 + yz^3$. The quadratic part of its local expansion at the point $(1:1:1:1:1)$ is

$$u\bar{t}^2 + (3-4u)\bar{y}\bar{t} - (2u+1)\bar{t}\bar{z} + (4u+3)\bar{y}^2 + (4u-7)\bar{y}\bar{z} + (u+2)\bar{z}^2,$$

where $\bar{y} = y - 1$, $\bar{z} = z - 1$ and $\bar{t} = t - 1$. Similarly, substituting $x = 1$ and $w = \frac{yt}{u} + \frac{u-1}{u}z^2$ into the polynomial u^3g_{20}, we obtain the polynomial

$$\begin{aligned} u^3t^4 + t^3y^5 - (2u^2+u)t^3y^2z + (3u-3)t^2y^4z^2 + (-2u^3+u^2+2u)t^2yz^3 \\ + (3u^2-6u+3)ty^3z^4 + (u^2-u)tz^5 + (u^3-3u^2+3u-1)y^2z^6. \end{aligned}$$

Then the quadratic part of the local expansion of the polynomial u^2g_{20} is

$$\begin{aligned} (4u^2-5u+2)\bar{t}^2 + (4-4u^2-u)\bar{y}\bar{t} - (12u^2-17u+8)\bar{t}\bar{z} \\ + (u^2+4u+2)\bar{y}^2 + (6u^2-u-8)\bar{y}\bar{z} + (9u^2-14u+8)\bar{z}^2. \end{aligned}$$

Both these quadric forms are degenerate, so that they define reducible conics in $\mathbb{P}^2_{\bar{y},\bar{z},\bar{t}}$. If $u \neq -2$, then these conics do not have common components. However, if $u = -2$, then the former quadratic form is $(\bar{t} - 5\bar{y})(\bar{y} + 3\bar{z} - 2\bar{t})$, and the latter quadratic form is $4(\bar{y} - 12\bar{z} + 7\bar{t})(\bar{y} + 3\bar{z} - 2\bar{t})$. Note that the quadratic part of the polynomial (4.4) is a multiple of $(\bar{y} + 3\bar{z} - 2\bar{t})^2$. Thus, if $u=-2$, then $\widetilde{M}_{10}\cap\widetilde{M}_{20}$ contains the irreducible G-invariant curve in E_{Q_u} that is different from the curve $\widetilde{\Gamma}$, see Remark 4.1.

Recall that $\zeta(\mathcal{S})=\mathcal{C}_2$. Denote the curves $\zeta(\Delta)$ and $\zeta(\Upsilon)$ by $\mathcal{C}_4$ and $\mathcal{C}_6$, respectively. Similarly, if $u\neq\frac{2}{3}$, let $\mathcal{C}_{10}=\zeta(\Psi)$ and $\mathcal{C}'_{10}=\zeta(\Psi')$. Finally, if $u=\frac{2}{3}$, let $\mathcal{C}_{10}=\zeta(\Psi)$ and let $\mathcal{C}'_{10}=\phi\circ\chi(\widetilde{Z})$, where $\widetilde{Z}$ is the irreducible G-invariant curve in E_{Q_u} that is different from the curve $\widetilde{\Gamma}$.

Proposition 4.12. *Let C be an irreducible G-invariant curve in V_u with $\deg(C)<12$. Then one of the following holds: $C=\mathcal{C}_2$, $C=\mathcal{C}_4$, $C=\mathcal{C}_6$, $C=\mathcal{C}_{10}$, or $C=\mathcal{C}'_{10}$.*

Proof. We may assume that $C\neq\mathcal{C}_2$. Denote by $\widetilde{C}$ the proper transform of the curve C on the threefold $\widetilde{V}_u$. By Remark 2.11, the curve $\widetilde{C}$ is not flopped by χ^{-1}. Denote by $\widetilde{Z}$ the proper transform of the curve $\widetilde{C}$ on the threefold $\widetilde{Q}_u$. Then $\widetilde{Z}$ is not contracted by π, since Q_u does not have G-fixed points by Lemma 2.23.

Let $Z=\pi(\widetilde{Z})$. Then Z is an irreducible G-invariant curve. Hence, the curve Z is either the curve $\Theta_\pm$ or the curve $\Theta_{a,b}$ for some $(a:b)\in\mathbb{P}^1$. Therefore, if Z is not contained in $\mathcal{S}$, the required assertion follows from Lemmas 3.7 and 3.13. Thus, we may assume that $Z\subset\mathcal{S}$, which implies that $Z=\Gamma$, because $C\neq\mathcal{C}_2$ by assumption. This simply means that $\widetilde{Z}$ is contained in the exceptional surface E_{Q_u}. Then $u=\frac{2}{3}$ and $Z=\mathcal{C}'_{10}$ by Lemma 4.7. $\square$

Using Remark 2.21 and Lemmas 3.13 and 4.7, we see that

$$T_9\cdot T_{21}=\mathcal{C}_{10}+\mathcal{C}'_{10}+\mathcal{C}_2. \tag{4.13}$$

5. Anticanonical pencil

Let $\mathcal{P}_{Q_u}$ be the pencil of surfaces in $|5H_{Q_u}|$ that are cut out on Q_u by

$$\mu_0 g_{15}+\mu_1 g'_{15}=0,$$

where $(\mu_0:\mu_1)\in\mathbb{P}^1$. Here g_{15} is the polynomial of weight 15 in (2.17), and g'_{15} is the polynomial of weight 15 in (2.18). Then the pencil $\mathcal{P}_{Q_u}$ is free from base components.

Denote by $\mathcal{P}_{V_u}$ the proper transform of the pencil $\mathcal{P}_{Q_u}$ on the threefold V_u. Then $\mathcal{P}_{V_u}$ is generated by the irreducible surfaces T_{15} and T'_{15}, and it contains all G-invariant surfaces in the linear system $|-K_{V_u}|$. This follows from (2.20).

By Lemma 2.22, the base locus of the pencil $\mathcal{P}_{V_u}$ contains the lines ℓ_1 and ℓ_2 from Remark 2.11. Similarly, we know from Lemma 3.7(i) that the base locus of the pencil $\mathcal{P}_{V_u}$ contains the curve $\zeta(\Theta_\pm)$. Thus, using Remark 3.10 and Proposition 4.12, we obtain

Corollary 5.1. *The curve $\zeta(\Theta_\pm)$ is the only irreducible G-invariant curve in V_u which is contained in the base locus of the pencil $\mathcal{P}_{V_u}$.*

Therefore, for every irreducible G-invariant curve in V_u that is different from $\zeta(\Theta_\pm)$, there exists a unique surface in the pencil $\mathcal{P}_{V_u}$ that contains this curve. In particular, the pencil $\mathcal{P}_{V_u}$ contains a unique surface that passes through $\mathcal{C}_4$, and it contains a unique surface that passes through $\mathcal{C}_6$. Below we describe both of them.

Lemma 5.2. *The curve $\mathcal{C}_6$ is not contained in T'_{15}. On the other hand, the curve $\mathcal{C}_4$ is contained in T'_{15}. Moreover, the surface T'_{15} is singular along the curve $\mathcal{C}_4$. If $u \neq 2$, then T'_{15} has a nonisolated ordinary double point at a general point of the curve $\mathcal{C}_4$. If $u = 2$, then T'_{15} has a nonisolated ordinary triple point at general point of the curve $\mathcal{C}_4$.*

Proof. Recall from (2.18) that

$$g'_{15} = (u-1)x^2t^3 + (u-1)y^3w^2 - (u+4)y^2zt^2 + (3u+2)xyztw + (4-4u)yz^3t.$$

Substituting (3.6) into g'_{15}, we see that Υ is not contained in M'_{15}, so that $\mathcal{C}_6$ is not contained in T'_{15}. Similarly, substituting (3.5) into g'_{15}, we see that Δ is contained in M'_{15}, so that $\mathcal{C}_4$ is contained in T'_{15}.

To describe the singularity of the surface T'_{15} at a general point of the curve $\mathcal{C}_4$, it is enough to describe the singularity of the surface M'_{15} at a general point of the curve Δ. The latter point has the form $(\frac{u-1}{u}\tau^2 : 0 : \tau : 0 : 1)$ with $\tau \in \mathbb{C}^*$. Substituting $w = 1$ and $x = z^2 + \frac{ty-z^2}{u}$ into $g'_{15} = 0$ and multiplying the resulting equation by $\frac{u^2}{u-1}$, we obtain

$$-u(u-2)tyz^3 + u^2y^3 + (u-1)^2t^3z^4 - u(u+2)t^2y^2z + 2(u-1)t^4yz^2 + t^5y^2 = 0. \tag{5.3}$$

Thus, at a general point of the curve $\mathcal{C}_4$, the surface M'_{15} has singularity locally isomorphic to the product of $\mathbb{C}$ and the germ of the curve singularity given by

$$-u(u-2)ty + u^2y^3 + (u-1)^2t^3 - u(u+2)t^2y^2 + 2(u-1)t^4y + t^5y^2 = 0.$$

If $u \neq 2$, the quadratic part $-u(u-2)ty$ of the left hand side is nondegenerate, so that M'_{15} has a nonisolated ordinary double point at P. If $u = 2$, the above equation becomes $t^3 + 4y^3 - 8t^2y^2 + 2t^4y + t^5y^2 = 0$, which defines an ordinary triple point (also known as curve singularity of type $\mathbf{D}_4$), and the assertion follows. □

Corollary 5.4. *If $u = 2$, then $\alpha_G(V_u) \leqslant \frac{2}{3}$.*

Let $g''_{15} = ug_{15} + g'_{15}$. Then

$$g''_{15} = (u-1)x^2t^3 + (u-1)y^3w^2 - 4y^2zt^2 + (u+2)xyztw - 4(u-1)yz^3t + ux^2zw^2.$$

Denote by M''_{15} the surface in the quadric Q_u that is cut out by $g''_{15} = 0$. Let T''_{15} be its proper transform on the threefold V_u. Then T''_{15} is an irreducible surface in $\mathcal{P}_{V_u}$.

Lemma 5.5. *The curve $\mathcal{C}_4$ is not contained in T''_{15}. On the other hand, the curve $\mathcal{C}_6$ is contained in T''_{15}. Moreover, the surface T''_{15} is singular along the curve $\mathcal{C}_6$. If $u \neq \frac{3}{4}$, then T''_{15} has a nonisolated ordinary double point at a general point of the curve $\mathcal{C}_6$. If $u = \frac{3}{4}$, then T''_{15} has a nonisolated tacnodal singularity at a general point of the curve $\mathcal{C}_6$.*

Proof. Substituting (3.5) into g''_{15}, we see that $\Delta \not\subset M''_{15}$, so that $\mathcal{C}_4 \not\subset T''_{15}$. Similarly, substituting (3.6) into g''_{15}, we see that $\Upsilon \subset M''_{15}$, so that $\mathcal{C}_6 \subset T''_{15}$.

To describe the singularity of the surface T''_{15} at a general point of the curve $\mathcal{C}_6$, it is enough to describe the singularity of the surface M''_{15} at a general point of the curve Υ. The latter point has the form $P = (0 : (1-u)\tau^2 : \tau : 1 : 0)$ with $\tau \in \mathbb{C}^*$.

Substituting $t = 1$ and $y = z^2 + u(wx - z^2)$ into $g''_{15} = 0$ and dividing the resulting equation by $(u-1)$, we obtain

$$x^2 + (3u-2)z^3xw - (u-1)^3w^2z^6 + 3u(u-1)^2z^4xw^3 \\ - 3uw^2x^2z - 3u^2(u-1)z^2x^2w^4 + u^3w^5x^3 = 0.$$

Thus, at a general point of the curve $\mathcal{C}_6$, the surface M''_{15} has singularity locally isomorphic to the product of $\mathbb{C}$ and the germ of the curve singularity given by

$$x^2 + (3u-2)xw - (u-1)^3w^2 + 3u(u-1)^2xw^3 \\ - 3uw^2x^2 - 3u^2(u-1)x^2w^4 + u^3w^5x^3 = 0.$$

If $u \neq \frac{3}{4}$, the quadratic part $x^2 + (3u-2)xw - (u-1)^3w^2$ of the left hand side is nondegenerate, so that M''_{15} has a nonisolated ordinary double point at P. If $u = \frac{3}{4}$, the above equation becomes $w^2 + 16wx + 64x^2 + 9w^3x - 144w^2x^2 + 27w^4x^2 + 27w^5x^3 = 0$. So, introducing new auxiliary coordinates $w = v - 8x$, we get

$$v^2 - 13824x^4 + 4032vx^3 + 110592x^6 - 360v^2x^2 \\ + 9v^3x - 55296vx^5 + 10368v^2x^4 - 884736x^8 + 552960vx^7 - 864v^3x^3 \\ + 27v^4x^2 - 138240v^2x^6 + 17280v^3x^5 - 1080v^4x^4 + 27v^5x^3 = 0.$$

This equation defines a tacnodal point (also known as curve singularity of type $\mathbf{A}_3$), and the assertion follows. $\square$

Corollary 5.6. *If $u = \frac{3}{4}$, then $\alpha_G(V_u) \leqslant \frac{3}{4}$.*

Proof. Suppose that $u = \frac{3}{4}$. Recall that $T''_{15} \sim -K_{V_u}$. Since T''_{15} has a tacnodal singularity at a general point of the curve $\mathcal{C}_6$ by Lemma 5.5, the log pair $(V_u, \frac{3}{4}T''_{15})$ is not Kawamata log terminal. Hence $\alpha_G(V_u) \leqslant \frac{3}{4}$. $\square$

6. Sarkisov links

Let $\mathcal{C}$ be one of the irreducible G-invariant curves $\mathcal{C}_4$ or $\mathcal{C}_6$ in the threefold V_u, let $\sigma\colon \widehat{V}_u \to V_u$ be the blowup of the curve $\mathcal{C}$, and let E_σ be the exceptional surface of σ. Denote by $\widehat{T}_i$, $\widehat{T}'_{15}$, $\widehat{T}''_{15}$ the proper transforms on $\widehat{V}_u$ of the surfaces T_i, T'_{15}, T''_{15}, respectively.

Remark 6.1. Suppose that $\mathcal{C} = \mathcal{C}_4$. Then $\widehat{T}'_{15} \sim \sigma^*(H_{V_u}) - m'E_\sigma$, where $m' = \mathrm{mult}_{\mathcal{C}}(T'_{15})$. By Lemma 5.2, one has

$$m' = \begin{cases} 2 & \text{if } u \neq 2, \\ 3 & \text{if } u = 2. \end{cases}$$

Moreover, if $u \neq 2$, then T'_{15} has a nonisolated ordinary double point at a general point of the curve $\mathcal{C}$. In this case, one has

$$\widehat{T}'_{15}\big|_{E_\sigma} = \widehat{\mathcal{C}} + \varkappa\big(\mathbf{l}_1 + \mathbf{l}_2\big),$$

where $\widehat{\mathcal{C}}$ is a 2-section of the natural projection $E_\sigma \to \mathcal{C}_4$, the curves $\mathbf{l}_1$ and $\mathbf{l}_2$ are the fibers of this projection over two $\mathbb{C}^*$-fixed points in $\mathcal{C}_4$, respectively, and $\varkappa$ is a nonnegative integer. Moreover, it can be seen from (5.3) that the curve $\widehat{\mathcal{C}}$ is reducible, so that it consists of two sections of the projection $E_\sigma \to \mathcal{C}$. However, the curve $\widehat{\mathcal{C}}$ is G-irreducible. This follows from (2.2) and (5.3).

Let us show that the divisor $-K_{\widehat{V}_u} \sim \sigma^*(H_{V_u}) - E_\sigma$ is nef.

Lemma 6.2. *Suppose that $\mathcal{C} = \mathcal{C}_4$. Then $\sigma^*(H_{V_u}) - E_\sigma$ is nef.*

Proof. Recall from (3.5) that the conic Δ is the scheme-theoretic intersection of the surfaces H_y and H_t. Moreover, it follows from (3.8) that $\mathcal{C}_4$ is contained in the intersection

$$T_{10} \cap T_{11} \cap T_{13} \cap T_{14} \cap T'_{15} \cap T_{16} \cap T_{17} \cap T_{19} \cap T_{20}. \tag{6.3}$$

Recall also that T_{13} is the proper transform on V_u of the surface H_y, and the surface T_{17} is the proper transform on V_u of the surface H_t. Thus, using Remark 2.21 and Lemma 2.22, we see that the intersection $T_{13} \cap T_{17}$ consists of the curve $\mathcal{C}_4$, the conic $\mathcal{C}_2$, the lines ℓ_1 and ℓ_2 from Remark 2.11, and the proper transform on V_u of the fibers of π over the points $(1:0:0:0:0)$ and $(0:0:0:0:1)$.

Recall that T_{11} is the proper transform on V_u of the surface N_5, and the surface T_{19} is the proper transform on V_u of the surface N_{13}. Since N_5 contains Γ and is smooth at the point $(1:0:0:0:0)$, the surface $\widetilde{N}_5$ does not contain the fiber of π over this point. Similarly, the surface $\widetilde{N}_{13}$ does not contain the fiber of π over the point $(0:0:0:0:1)$. Hence, using Remark 2.21 again, we see that the only curves contained in the intersection $T_{11} \cap T_{13} \cap T_{17} \cap T_{19}$ are the conic $\mathcal{C}_2$, the curve $\mathcal{C}_4$, and the lines ℓ_1 and ℓ_2.

By Remark 2.21, the surface T'_{15} does not contain the conic $\mathcal{C}_2$. Similarly, it follows from Lemma 2.22 that the intersection $T_{10} \cap T_{20}$ contains neither ℓ_1 nor ℓ_2. Thus, we see that $\mathcal{C}_4$ is the only curve contained in the intersection (6.3).

The base locus of the linear system $|\sigma^*(H_{V_u}) - E_\sigma|$ does not contain any curves outside the exceptional surface E_σ. Moreover, the surfaces T_{13} and T_{17} intersect transversally at a general point of the curve $\mathcal{C}_4$, because the surfaces H_y and H_t intersect transversally at every point of the conic Δ. Hence, the base locus of the linear system $|\sigma^*(H_{V_u}) - E_\sigma|$ does not contain curves, with the only possible exception of finitely many fibers of the projection $E_\sigma \to \mathcal{C}_4$. This implies the required assertion. $\square$

Lemma 6.4. *Suppose that $\mathcal{C} = \mathcal{C}_6$. Then $\sigma^*(H_{V_u}) - E_\sigma$ is nef.*

Proof. Recall from (3.6) that the conic Υ is the scheme-theoretic intersection of the surfaces H_x and H_w. Moreover, it follows from (3.9) that $\mathcal{C}_6$ is contained in the intersection

$$T_{10} \cap T_{12} \cap T_{14} \cap T''_{15} \cap T_{16} \cap T_{18} \cap T_{20}. \tag{6.5}$$

Recall also that T_{12} is the proper transform on V_u of the surface H_x, and the surface T_{18} is the proper transform on V_u of the surface H_w. Moreover, the surface H_x does not contain the point $(1:0:0:0:0)$, and the surface H_w does not contain the point $(0:0:0:0:1)$. Thus, using Remark 2.21 and Lemma 2.22, we see that the intersection $T_{12} \cap T_{18}$ consists of the curve $\mathcal{C}_6$, the conic $\mathcal{C}_2$, and the lines ℓ_1 and ℓ_2 from Remark 2.11.

By Remark 2.21, the surface T''_{15} does not contain the conic $\mathcal{C}_2$. Similarly, it follows from Lemma 2.22 that the intersection $T_{10} \cap T_{20}$ contains neither ℓ_1 nor ℓ_2. Thus, the curve $\mathcal{C}_6$ is the only curve contained in the intersection (6.5).

The base locus of the linear system $|\sigma^*(H_{V_u}) - E_\sigma|$ does not contain any curves outside the exceptional surface E_σ. Moreover, the surfaces T_{13} and T_{18} intersect transversally at a general point of the curve $\mathcal{C}_6$, because the surfaces H_x and H_w intersect transversally at every point of the conic Υ. Therefore, the base locus of the linear system $|\sigma^*(H_{V_u}) - E_\sigma|$ does not contain curves with the only possible exception of finitely many fibers of the projection $E_\sigma \to \mathcal{C}_6$. This implies the required assertion. $\square$

We see that $-K_{\widehat{V}_u}$ is nef. Since $E_\sigma^3 = -\deg(\mathcal{C}) + 2$ and $\sigma^*(H_{V_u}) \cdot E^2 = -\deg(\mathcal{C})$, we compute

$$-K_{\widehat{V}_u}^3 = \begin{cases} 12 \text{ if } \mathcal{C} = \mathcal{C}_4, \\ 8 \text{ if } \mathcal{C} = \mathcal{C}_6. \end{cases}$$

Therefore, the divisor $-K_{\widehat{V}_u}$ is also big. Thus, it follows from Basepoint-free Theorem that the linear system $|-nK_{\widehat{V}_u}|$ is free from base points for $n \gg 0$, see [KM98, Theorem 3.3]. This linear system gives a crepant birational morphism $\eta\colon \widehat{V}_u \to Y$, so that Y is a Fano threefold with at most canonical singularities such that $-K_Y^3 = -K_{\widehat{V}_u}^3$. Observe that according to the classification of smooth Fano threefolds with Picard rank 2, the threefold $\widehat{V}_u$ is not Fano. In other words, η is not an isomorphism, and Y is indeed singular.

Lemma 6.6. *Suppose that $\mathcal{C} = \mathcal{C}_4$. Then η is small if and only if $u \neq 2$.*

Proof. If $u = 2$, then $\mathrm{mult}_{\mathcal{C}}(T'_{15}) = 3$ by Lemma 5.2, so that

$$\begin{aligned} 0 \leqslant -K_{\widehat{V}_u}^2 \cdot \widehat{T}'_{15} &= \Big(\sigma^*(H_{V_u}) - E_\sigma\Big)^2 \cdot \Big(\sigma^*(H_{V_u}) - 3E_\sigma\Big) \\ &= 22 + 3\sigma^*(H_{V_u}) \cdot E_\sigma^2 + 4\sigma^*(H_{V_u}) \cdot E_\sigma^2 - 3E_\sigma^3 = 0, \end{aligned}$$

which implies that $\widehat{T}'_{15}$ is contracted by η.

We may assume that $u \neq 2$. Then $\mathrm{mult}_{\mathcal{C}}(T'_{15}) = 2$ by Lemma 5.2. Let F be an irreducible surface in $\widehat{V}_u$. Then $F \sim \sigma^*(nH_{V_u}) - mE_\sigma$ for some integers n and m. We compute

$$\begin{aligned} -K_{\widehat{V}_u}^2 \cdot F &= \Big(\sigma^*(H_{V_u}) - E_\sigma\Big)^2 \cdot \Big(\sigma^*(nH_{V_u}) - mE_\sigma\Big) \\ &= 22n + n\sigma^*(H_{V_u}) \cdot E_\sigma^2 + 2m\sigma^*(H_{V_u}) \cdot E_\sigma^2 - mE_\sigma^3 = 18n - 6m, \end{aligned}$$

so that F is contracted by η if and only if $m = 3n$. In particular, the surface $\widehat{T}'_{15}$ is not contracted by η. On the other hand, if $F \neq \widehat{T}'_{15}$, then

$$\begin{aligned} 0 &\leqslant \Big(\sigma^*(H_{V_u}) - E_\sigma\Big) \cdot F \cdot \widehat{T}'_{15} \\ &= \Big(\sigma^*(H_{V_u}) - E_\sigma\Big) \cdot \Big(\sigma^*(nH_{V_u}) - mE_\sigma\Big) \cdot \Big(\sigma^*(H_{V_u}) - 2E_\sigma\Big) \\ &= 22n + 2n\sigma^*(H_{V_u}) \cdot E_\sigma^2 + 3m\sigma^*(H_{V_u}) \cdot E_\sigma^2 - 2mE_\sigma^3 = 14n - 8m, \end{aligned}$$

so that $m \neq 3n$, which implies that F is also not contracted by η. $\square$

Therefore, if $\mathcal{C} = \mathcal{C}_4$ and $u \neq 2$, then it follows from standard computations as in [IP99, §4.1] or [Ta89, ACM17, CM13] that there exists a G-equivariant commutative diagram

$$
\begin{tikzcd}
 & \widehat{V}_u \arrow[dl, "\sigma"'] \arrow[dr, "\eta"] \arrow[rr, dashed, "\rho"] & & \widehat{V}_{u'} \arrow[dl, "\eta'"'] \arrow[dr, "\sigma'"] & \\
V_u & & Y & & V_{u'}
\end{tikzcd}
\tag{6.7}
$$

where ρ is the flop in the curves contracted by η, and the variety $V_{u'}$ is a smooth Fano threefold of type V_{22}^* that corresponds to (some) parameter u', which is possibly different from u. Here the map σ' is a birational morphism that contracts the proper transform of the surface $\widehat{T}'_{15}$ to a unique irreducible G-invariant (rational normal) curve $\mathcal{C}'_4$ of degree 4 in $V_{u'}$. The diagram (6.7) is Sarkisov link No. 104 in [CM13].

Remark 6.8. It would be interesting to know whether the threefold $V_{u'}$ in (6.7) is isomorphic to the threefold V_u or not, that is, whether $u = u'$ or not.

Lemma 6.9. *Suppose that $\mathcal{C} = \mathcal{C}_4$ and $u \neq 2$. Then η does not contract curves in E_σ.*

Proof. The normal bundle of the curve $\mathcal{C}_4$ in V_u is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(p) \oplus \mathcal{O}_{\mathbb{P}^1}(q)$ for some integers p and q such that $p \geqslant q$ and $p + q = 2$. Thus, the exceptional surface E_σ is a Hirzebruch surface $\mathbb{F}_n$ for $n = p - q \geqslant 0$. Denote by $\mathbf{s}$ a section of the natural projection $E_\sigma \to \mathcal{C}_4$ such that $\mathbf{s}^2 = -n$, and denote by $\mathbf{l}$ a fiber of this projection. Then $-E_\sigma|_{E_\sigma} \sim \mathbf{s} + \kappa\mathbf{l}$ for some integer κ. One has

$$-2 = E_\sigma^3 = \Big(\mathbf{s} + \kappa\mathbf{l}\Big)^2 = -n + 2\kappa,$$

so that $\kappa = \frac{n-2}{2}$. By Remark 6.1, one has

$$\widehat{T}'_{15}\big|_{E_\sigma} = \widehat{\mathcal{C}} + \varkappa\big(\mathbf{l}_1 + \mathbf{l}_2\big),$$

where $\widehat{\mathcal{C}}$ is a reducible G-irreducible 2-section of the projection $E_\sigma \to \mathcal{C}_4$, the curves $\mathbf{l}_1$ and $\mathbf{l}_2$ are the fibers of this projection over two $\mathbb{C}^*$-fixed points in $\mathcal{C}_4$, respectively, and $\varkappa$ is a nonnegative integer. This gives

$$\widehat{\mathcal{C}} \sim 2\mathbf{s} + (n + 2 - 2\varkappa)\mathbf{l}.$$

Since $\widehat{\mathcal{C}} \neq \mathbf{s}$, we have $0 \leqslant \widehat{\mathcal{C}} \cdot \mathbf{s} = 2 - n - 2\varkappa$, which gives $n \leqslant 2$. This implies that the divisor

$$-K_{\widehat{V}_u}|_{E_\sigma} \sim \mathbf{s} + \frac{n+6}{2}\mathbf{l}$$

is ample, and the assertion follows. $\square$

If $\mathcal{C} = \mathcal{C}_6$, then the morphism η is never small, since it contracts the surface $\widehat{T}''_{15}$. Indeed, in this case, we have $\widehat{T}''_{15} \sim \sigma^*(H_{V_u}) - 2E_\sigma$ by Lemma 5.5, which implies that

$$K^2_{\widehat{V}_u} \cdot \widehat{T}''_{15} = \Big(\sigma^*(H_{V_u}) - E_\sigma\Big)^2 \cdot \Big(\sigma^*(H_{V_u}) - 2E_\sigma\Big) = 22 + 5\sigma^*(H_{V_u}) \cdot E_\sigma^2 - 2E_\sigma^3 = 0.$$

This is a so-called *bad link* (cf. Sarkisov link No. 93 in [ACM17]).

7. The proof

In this section, we prove Theorem 1.6. Let

$$\varepsilon(u) = \begin{cases} \dfrac{4}{5} & \text{if } u \neq \dfrac{3}{4} \text{ and } u \neq 2, \\ \dfrac{3}{4} & \text{if } u = \dfrac{3}{4}, \\ \dfrac{2}{3} & \text{if } u = 2. \end{cases}$$

By Corollaries 2.7, 5.4 and 5.6, we know that $\alpha_G(V_u) \leqslant \varepsilon(u)$. Thus, by (1.2), to prove Theorem 1.6, we have to show that the log pair $(V_u, \frac{\varepsilon(u)}{n}\mathcal{D})$ has log canonical singularities for every G-invariant linear system $\mathcal{D} \subset |-nK_{V_u}|$ and for every positive integer n. For basic properties of singularities of such log pairs, we refer the reader to [Ko97, Theorem 4.8].

Remark 7.1. Let $\mathcal{D}$ be a nonempty G-invariant linear subsystem in $|-nK_{V_u}|$ for some $n \in \mathbb{Z}_{>0}$. Fix a positive rational number ϵ. Suppose that the log pair $(V_u, \frac{\epsilon}{n}\mathcal{D})$ is strictly log canonical, i.e., log canonical but not Kawamata log terminal. Let Z be a center of log canonical singularities of the log pair $(V_u, \frac{\epsilon}{n}\mathcal{D})$ (see [Ka97, Definition 1.3]). Then Z is $\mathbb{C}^*$-invariant. This follows from the existence of an equivariant strong resolution of singularities (see [RY02, Ko07]).

Remark 7.2. In the assumptions of Remark 7.1, let $\mathcal{F}$ be the fixed part of the linear system $\mathcal{D}$, and let $\mathcal{M}$ be its mobile part, so that

$$\mathcal{D} = \mathcal{F} + \mathcal{M}.$$

Since $\mathrm{Pic}(V_u) = \mathbb{Z}[-K_{V_u}]$, one has $\mathcal{F} \sim -n_1K_{V_u}$ and $\mathcal{M} \sim -n_2K_{V_u}$ for some nonnegative integers n_1 and n_2 such that $n_1 + n_2 = n$. Then Z is a center of log canonical singularities of either $(V_u, \frac{\epsilon}{n_1}\mathcal{F})$ or $(V_u, \frac{\epsilon}{n_2}\mathcal{M})$, see [CS09, Remark 2.9] and the proof of [CS09, Lemma 2.10].

Remark 7.3. In the assumptions of Remark 7.2, there is a $\mathbb{C}^*$-invariant divisor $D \in \mathcal{D}$. Then Z is a center of log canonical singularities of the log pair $(V_u, \frac{\epsilon}{2n}(D + \iota(D)))$.

Hence, to prove Theorem 1.6, it is enough to show that the log pair $(V_u, \varepsilon(u)D)$ is log canonical for every G-invariant effective $\mathbb{Q}$-divisor D on the threefold V_u such that

$$D \sim_{\mathbb{Q}} -K_{V_u}.$$

Moreover, if necessary, we may assume that $D = \frac{1}{n}S$ for some irreducible surface S in the linear system $|-nK_{V_u}|$. This follows from

Remark 7.4. Let D be a G-invariant effective $\mathbb{Q}$-divisor D on the threefold V_u such that $D \sim_{\mathbb{Q}} -K_{V_u}$, and let Z be an irreducible subvariety in V_u such that Z is a center of log canonical singularities of the log pair $(V_u, \epsilon D)$, where ϵ is a positive rational number. Suppose that

$$D = D_1 + D_2$$

for two nonzero effective G-invariant $\mathbb{Q}$-divisors $D_1 \sim_{\mathbb{Q}} -\epsilon_1 K_{V_u}$ and $D_2 \sim_{\mathbb{Q}} -\epsilon_2 K_{V_u}$. Here ϵ_1 and ϵ_2 are positive rational numbers such that $\epsilon_1 + \epsilon_2 = 1$. Then either Z is a center of log canonical singularities of the log pair $(V_u, \frac{\epsilon}{\epsilon_1}D_1)$ or Z is a center of log canonical singularities of the log pair $(V_u, \frac{\epsilon}{\epsilon_2}D_2)$ (or both). This is well known and easy to prove. See, for instance, [CS08, Remark 2.22] or [CP16, Lemma 2.2].

The key point in the proof of Theorem 1.6 is the following

Proposition 7.5. *Let D be a G-invariant effective $\mathbb{Q}$-divisor on V_u such that $D \sim_{\mathbb{Q}} -K_{V_u}$. Suppose that $(V_u, \varepsilon(u)D)$ is not log canonical. Then $(V_u, \varepsilon(u)D)$ is not log canonical at a general point of one of the curves $\mathcal{C}_2$, $\mathcal{C}_4$ or $\mathcal{C}_6$.*

Proof. Let ϵ be a positive rational number such that $(V_u, \epsilon D)$ is strictly log canonical. Then $\epsilon < \varepsilon(u)$. Let Z be a minimal center of log canonical singularities of the log pair $(V_u, \epsilon D)$. Since $\mathrm{Pic}(V_u)$ is generated by $-K_{V_u}$ and $\epsilon < 1$, the center Z is either a point or a curve. Recall from Remark 7.1 that Z is $\mathbb{C}^*$-invariant. Observe that $\iota(Z)$ is also a minimal center of log canonical singularities of the log pair $(V_u, \frac{\epsilon}{n}D)$.

Now we will use the so-called *perturbation trick.* For details, see [CS16, Lemma 2.4.10], and the proofs of [Ka97, Theorem 1.10] and [Ka98, Theorem 1]. Observe that there exists a mobile G-invariant linear system $\mathcal{B}$ on the threefold V_u, and there are rational numbers $1 \gg \epsilon_1 \geqslant 0$ and $1 \gg \epsilon_2 \geqslant 0$ such that

$$\big(\epsilon - \epsilon_1\big)D + \epsilon_2\mathcal{B} \sim_{\mathbb{Q}} -\theta K_{V_u},$$

for some positive rational number $\theta < \varepsilon(u)$, the log pair

$$\Big(V_u, \big(\epsilon - \epsilon_1\big)D + \epsilon_2\mathcal{B}\Big) \tag{7.6}$$

has strictly log canonical singularities, and the only centers of log canonical singularities of the log pair (7.6) are Z and $\iota(Z)$.

Observe that the divisor $-(K_{V_u}+(\epsilon-\epsilon_1)D+\epsilon_2\mathcal{B})$ is ample, since $\theta<\varepsilon(u)<1$. Thus, the locus of log canonical singularities of the pair (7.6) is connected by the Kollár–Shokurov connectedness principle [KM98, Corollary 5.49]. Since there are no G-fixed points on V_u by Lemma 2.23, the center Z is not a point, so that Z is a curve.

By [Ka97, Proposition 1.5], either $Z=\iota(Z)$ or the centers Z and $\iota(Z)$ are disjoint. Using the Kollár–Shokurov connectedness, we see that $Z=\iota(Z)$, so that Z is G-invariant.

Since $(\theta-\varepsilon(u))K_{V_u}$ is an ample $\mathbb{Q}$-divisor, using Kawamata subadjunction theorem [Ka98, Theorem 1], we see that Z is smooth and

$$(1-\varepsilon(u))K_{V_u}\Big|_Z\sim_{\mathbb{Q}}\Big(K_{V_u}+(\epsilon-\epsilon_1)D+\epsilon_2\mathcal{B}+(\theta-\varepsilon(u))K_{V_u}\Big)\Big|_Z\sim_{\mathbb{Q}}K_Z+D_Z$$

for some ample divisor D_Z on the curve Z. In particular, we see that Z is rational and

$$(\varepsilon(u)-1)\deg\big(Z\big)>-2,$$

which implies that $\deg(Z)<\frac{2}{1-\varepsilon(u)}\leqslant 10$, so that $\deg(Z)\leqslant 9$. Thus, by Proposition 4.12, the curve Z is one of the curves $\mathcal{C}_2$, $\mathcal{C}_4$ or $\mathcal{C}_6$, which is exactly what we need. □

In the remaining part of this section, we will show that $(V_u,\varepsilon(u)D)$ is log canonical at general points of the curves $\mathcal{C}_2$, $\mathcal{C}_4$ or $\mathcal{C}_6$ for every G-invariant effective $\mathbb{Q}$-divisor D on the threefold V_u such that $D\sim_{\mathbb{Q}}-K_{V_u}$. We start with the conic $\mathcal{C}_2$.

Lemma 7.7. *Let D be a G-invariant effective $\mathbb{Q}$-divisor on V_u such that $D\sim_{\mathbb{Q}}-K_{V_u}$. Then the log pair $(V_u,\frac{4}{5}D)$ is log canonical at a general point of the curve $\mathcal{C}_2$.*

Proof. The normal bundle of the conic $\mathcal{C}_2$ in V_u is either isomorphic to $\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}$ or isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}^1}(1)$. Thus, the exceptional surface E_{V_u} is either $\mathbb{P}^1\times\mathbb{P}^1$ or the Hirzebruch surface $\mathbb{F}_2$.

If $E_{V_u}\cong\mathbb{P}^1\times\mathbb{P}^1$, we denote by $\mathbf{s}$ the section of the natural projection $E_{V_u}\to\mathcal{C}_2$ such that $\mathbf{s}^2=0$. Similarly, if $E_{V_u}\cong\mathbb{F}_2$, we denote by $\mathbf{s}$ the section of the projection $E_{V_u}\to\mathcal{C}_2$ such that $\mathbf{s}^2=-2$. If $E_{V_u}\cong\mathbb{P}^1\times\mathbb{P}^1$, then $-E_{V_u}|_{E_{V_u}}\sim\mathbf{s}$. Similarly, if $E_{V_u}\cong\mathbb{F}_2$, then

$$-E_{V_u}\Big|_{E_{V_u}}\sim\mathbf{s}+\mathbf{l},$$

where $\mathbf{l}$ is the fiber of the natural projection $E_{V_u}\to\mathcal{C}_2$.

Denote by $\widetilde{D}$ the proper transform of the divisor D on the threefold $\widetilde{V}_u$. Then

$$\widetilde{D}\sim_{\mathbb{Q}}\phi^*\big(H_{V_u}\big)-mE_{V_u},$$

where $m = \mathrm{mult}_{\mathcal{C}_2}(D)$. One the other hand, we know that $\mathcal{R} \sim 2\phi^*(H_{V_u}) - 5E_{V_u}$, so that

$$\widetilde{D} \sim_{\mathbb{Q}} \frac{1}{2}\mathcal{R} + \Big(\frac{5}{2} - m\Big)E_{V_u},$$

which implies that $m \leqslant \frac{5}{2}$, because E_{Q_u} is the proper transform of the surface $\mathcal{R}$ on the threefold $\widetilde{Q}_u$.

Suppose that the log pair $(V_u, \frac{4}{5}D)$ is not log canonical at a general point of the curve $\mathcal{C}_2$. Then $m > \frac{5}{4}$. Moreover, the surface E_{V_u} contains a G-irreducible curve $\widetilde{C}$ such that $\phi(\widetilde{C}) = \mathcal{C}_2$, and the log pair

$$\left(\widetilde{V}_u, \frac{4}{5}\widetilde{D} + \Big(\frac{4m}{5} - 1\Big)E_{V_u}\right) \tag{7.8}$$

is not log canonical at a general point of the curve $\widetilde{C}$. Furthermore, since we know that $m \leqslant \frac{5}{2}$, the curve $\widetilde{C}$ must be a section of the natural projection $E_{V_u} \to \mathcal{C}_2$. This fact is well-known. See for instance [CP16, Remark 2.5]. Thus, the curve $\widetilde{C}$ is irreducible.

When we apply [KM98, Theorem 5.50] to (7.8), we see that the log pair $(E_{V_u}, \frac{4}{5}\widetilde{D}|_{E_{V_u}})$ is also not log canonical at a general point of the curve $\widetilde{C}$. This simply means that

$$\frac{4}{5}\widetilde{D}\big|_{E_{V_u}} = \theta\widetilde{C} + \Omega$$

for some rational number $\theta > 1$ and some effective $\mathbb{Q}$-divisor Ω on the surface E_{V_u}.

One has $\widetilde{C} \sim \mathbf{s} + \kappa\mathbf{l}$ for some nonnegative integer κ. If $E_{V_u} \cong \mathbb{P}^1 \times \mathbb{P}^1$, then

$$\theta\mathbf{s} + \theta\kappa\mathbf{l} + \Omega \sim_{\mathbb{Q}} \theta\widetilde{C} + \Omega = \frac{4}{5}\widetilde{D}\big|_{E_{V_u}} \sim_{\mathbb{Q}} \frac{4m}{5}\mathbf{s} + \frac{8}{5}\mathbf{l},$$

so that either $\kappa = 0$ or $\kappa = 1$. Thus, in this case we have

$$-K_{\widetilde{V}_u} \cdot \widetilde{C} = -K_{\widetilde{V}_u}\big|_{E_{V_u}} \cdot \widetilde{C} = \big(\mathbf{s} + 2\mathbf{l}\big) \cdot \big(\mathbf{s} + \kappa\mathbf{l}\big) = 2 + \kappa \leqslant 3.$$

Similarly, if $E_{V_u} \cong \mathbb{F}_2$, then

$$\theta\mathbf{s} + \theta\kappa\mathbf{l} + \Omega \sim_{\mathbb{Q}} \theta\widetilde{C} + \Omega = \frac{4}{5}\widetilde{D}\big|_{E_{V_u}} \sim_{\mathbb{Q}} \frac{4m}{5}\mathbf{s} + \frac{8+4m}{5}\mathbf{l},$$

so that $\kappa \leqslant 3$, which gives

$$-K_{\widetilde{V}_u} \cdot \widetilde{C} = -K_{\widetilde{V}_u}\big|_{E_{V_u}} \cdot \widetilde{C} = \big(\mathbf{s} + 3\mathbf{l}\big) \cdot \big(\mathbf{s} + \kappa\mathbf{l}\big) = 1 + \kappa \leqslant 4.$$

We proved that $-K_{\widetilde{V}_u} \cdot \widetilde{C} \leqslant 4$. Then the degree of the curve $\beta(\widetilde{C})$ is $-K_{\widetilde{V}_u} \cdot \widetilde{C} \leqslant 4$. This is impossible by Lemmas 3.4 and 4.7. $\square$

Now we deal with the curve $\mathcal{C}_6$.

Lemma 7.9. *Let D be an effective $\mathbb{Q}$-divisor on the threefold V_u such that $D \sim_{\mathbb{Q}} -K_{V_u}$. Suppose that* $\mathrm{Supp}(D)$ *does not contain T''_{15}. Then the log pair (V_u, D) is log canonical at a general point of the curve $\mathcal{C}_6$.*

Proof. Let us use the notation of §6 with $\mathcal{C} = \mathcal{C}_6$. Denote by $\widehat{T}''_{15}$ the proper transform of the surface T''_{15} on the threefold $\widehat{V}_u$. Then

$$\widehat{T}''_{15} \sim \sigma^*(H_{V_u}) - 2E_\sigma$$

by Lemma 5.5.

Denote by $\widehat{D}$ the proper transform on $\widehat{V}_u$ of the divisor D. We also let $m = \mathrm{mult}_{\mathcal{C}_6}(D)$. Using $E_\sigma^3 = -4$ and $\sigma^*(H_{V_u}) \cdot E^2 = -6$, we compute

$$\begin{aligned}
&\Big(\sigma^*(H_{V_u}) - E_\sigma\Big) \cdot \widehat{D} \cdot \widehat{T}''_{15} \\
&\quad = \Big(\sigma^*(H_{V_u}) - E_\sigma\Big) \cdot \Big(\sigma^*(H_{V_u}) - mE_\sigma\Big) \cdot \Big(\sigma^*(H_{V_u}) - 2E_\sigma\Big) \\
&\quad = 22 + 2\sigma^*(H_{V_u}) \cdot E_\sigma^2 + 3m\sigma^*(H_{V_u}) \cdot E_\sigma^2 - 2mE_\sigma^3 = 10 - 10m.
\end{aligned}$$

On the other hand, the divisor $\sigma^*(H_{V_u}) - E_\sigma$ is nef by Lemma 6.4. Thus, we have $m \leqslant 1$, and the assertion follows. □

Corollary 7.10. *Let D be an effective $\mathbb{Q}$-divisor on V_u such that $D \sim_{\mathbb{Q}} -K_{V_u}$. If $u = \frac{3}{4}$, then the log pair $(V_u, \frac{3}{4}D)$ is log canonical at a general point of the curve $\mathcal{C}_6$. If $u \neq \frac{3}{4}$, then the log pair (V_u, D) is log canonical at a general point of the curve $\mathcal{C}_6$.*

Proof. If $u = \frac{3}{4}$, then $(V_u, \frac{3}{4}T''_{15})$ is log canonical at a general point of $\mathcal{C}_6$ by Lemma 5.5. Likewise, if $u \neq \frac{3}{4}$, then the pair (V_u, T''_{15}) is log canonical at a general point of the curve $\mathcal{C}_6$. Thus, by Remark 7.4, we may assume that $\mathrm{Supp}(D)$ does not contain the surface T''_{15}. Now the assertion follows from Lemma 7.9. □

Combining Proposition 7.5, Lemma 7.7 and Corollary 7.10, we obtain

Corollary 7.11. *Let D be a G-invariant effective $\mathbb{Q}$-divisor on V_u such that $D \sim_{\mathbb{Q}} -K_{V_u}$. Suppose that the log pair $(V_u, \varepsilon(u)D)$ is log canonical at a general point of the curve $\mathcal{C}_4$. Then the log pair $(V_u, \varepsilon(u)D)$ is log canonical.*

Finally, we deal with the curve $\mathcal{C}_4$ using Corollary 7.11.

Lemma 7.12. *Let D be a G-invariant effective $\mathbb{Q}$-divisor on V_u such that $D \sim_{\mathbb{Q}} -K_{V_u}$. Suppose that* $\mathrm{Supp}(D)$ *does not contain T'_{15}. Then the log pair $(V_u, \frac{5}{6}D)$ is log canonical at a general point of the curve $\mathcal{C}_4$.*

Proof. Let us use the notation of §6 with $\mathcal{C} = \mathcal{C}_4$. Then $\sigma^*(H_{V_u}) - E_\sigma$ is nef by Lemma 6.2. Denote by $\widehat{D}$ the proper transform on $\widehat{V}_u$ of the divisor D. We also let $m = \mathrm{mult}_{\mathcal{C}_4}(D)$. If $u = 2$, then $\mathrm{mult}_{\mathcal{C}_4}(T'_{15}) = 3$ by Remark 6.1, so that

$$\begin{aligned}0 &\leqslant \Big(\sigma^*(H_{V_u}) - E_\sigma\Big)\cdot \widehat{D}\cdot \widehat{T}'_{15}\\ &= \Big(\sigma^*(H_{V_u}) - E_\sigma\Big)\cdot\Big(\sigma^*(H_{V_u}) - mE_\sigma\Big)\cdot\Big(\sigma^*(H_{V_u}) - 3E_\sigma\Big)\\ &= 22 + 3\sigma^*(H_{V_u})\cdot E_\sigma^2 + 4m\sigma^*(H_{V_u})\cdot E_\sigma^2 - 3mE_\sigma^3 = 10 - 10m,\end{aligned}$$

so that $m \leqslant 1$, which implies that the log pair (V_u, D) is log canonical at a general point of the curve $\mathcal{C}_4$.

Hence, we may assume that $u \neq 2$, so that $\mathrm{mult}_{\mathcal{C}_4}(T'_{15}) = 2$ by Remark 6.1. Then

$$\begin{aligned}0 &\leqslant \Big(\sigma^*(H_{V_u}) - E_\sigma\Big)\cdot \widehat{D}\cdot \widehat{T}'_{15}\\ &= \Big(\sigma^*(H_{V_u}) - E_\sigma\Big)\cdot\Big(\sigma^*(H_{V_u}) - mE_\sigma\Big)\cdot\Big(\sigma^*(H_{V_u}) - 2E_\sigma\Big)\\ &= 22 + 2\sigma^*(H_{V_u})\cdot E_\sigma^2 + 3m\sigma^*(H_{V_u})\cdot E_\sigma^2 - 2mE_\sigma^3 = 14 - 8m,\end{aligned}$$

which gives $m \leqslant \frac{7}{4}$. Let us show that this implies that $(V_u, \frac{5}{6}D)$ is log canonical at a general point of the curve $\mathcal{C}_4$.

Let $\epsilon = \frac{5}{6}$. Suppose that $(V_u, \epsilon D)$ is not log canonical at a general point of the curve $\mathcal{C}_4$. Then the surface E_σ contains a G-irreducible curve $\widehat{Z}$ such that $\sigma(\widehat{Z}) = \mathcal{C}_4$, and the log pair

$$\left(\widehat{V}_u, \epsilon\widehat{D} + \Big(\epsilon m - 1\Big)E_\sigma\right) \tag{7.13}$$

is not log canonical at a general point of the curve $\widehat{Z}$. Moreover, since $\epsilon m = \frac{5m}{6} \leqslant \frac{35}{24} < 2$, the curve $\widehat{Z}$ must be a section of the natural projection $E_\sigma \to \mathcal{C}_4$. This is well-known. See for instance [CP16, Remark 2.5].

We see that $\widehat{Z}$ is irreducible. Thus, the curve $\widehat{Z}$ is not contained in $\widehat{T}'_{15}$ by Remark 6.1. Moreover, it follows from Lemma 6.9 that the curve $\widehat{Z}$ is not contracted by η, so that $\widehat{Z}$ is not flopped by ρ.

Denote by D' the proper transform of the divisor D on the threefold $V_{u'}$, and denote by T' the proper transform of the exceptional surface E_σ on the threefold $V_{u'}$. Then the log pair

$$\Big(V_{u'}, \epsilon D' + \big(\epsilon m - 1\big)T'\Big) \tag{7.14}$$

is not log canonical, because the log pair (7.13) is not log canonical at a general point of the curve $\widehat{Z}$.

Let us compute the class of the divisor D' in the group $\mathrm{Pic}(V_{u'})$, and the multiplicity of the divisor D' at a general point of the curve $\mathcal{C}'_4$. Recall from

(6.7) that $\mathcal{C}_4'$ is the unique irreducible G-invariant curve of degree 4 in the threefold $V_{u'}$. We have

$$\widehat{D}+\big(m-1\big)E_\sigma\sim_{\mathbb{Q}}-K_{\widehat{V}_u}.$$

This implies that $D'+\big(m-1\big)T'\sim_{\mathbb{Q}}-K_{V_{u'}}$, where T' is the unique surface in the linear system $|-K_{V_{u'}}|$ that is singular along the curve $\mathcal{C}_4'$. Thus, we have

$$D'\sim_{\mathbb{Q}}-\big(2-m\big)K_{V_{u'}}.$$

Similar arguments applied to the divisor $\frac{1}{2-m}D'$ give

$$-\frac{1}{2-m}K_V\sim_{\mathbb{Q}}\frac{1}{2-m}D\sim_{\mathbb{Q}}-\left(2-\frac{\mathrm{mult}_{\mathcal{C}_4'}(D')}{2-m}\right)K_V,$$

so that $\mathrm{mult}_{\mathcal{C}_4'}(D')=3-2m$.

Observe that $\mathrm{mult}_{\mathcal{C}_4'}(T')=2$. Thus, we have

$$\mathrm{mult}_{\mathcal{C}_4'}\Big(\epsilon D'+\big(\epsilon m-1\big)T'\Big)=3\epsilon-2<1,$$

so that (7.14) is log canonical at a general point of the curve $\mathcal{C}_4'$. On the other hand, we have

$$\epsilon D'+\big(\epsilon m-1\big)T'\sim_{\mathbb{Q}}-\big(2\epsilon-1\big)K_{V_{u'}}$$

and $2\epsilon-1=\frac{2}{3}\leqslant\varepsilon(u)$. Thus, the log pair (7.14) must be log canonical by Corollary 7.11 applied to $V_{u'}$. The obtained contradiction completes the proof of the lemma. □

Corollary 7.15. *Let D be an effective $\mathbb{Q}$-divisor on V_u such that $D\sim_{\mathbb{Q}}-K_{V_u}$. If $u=2$, then the log pair $(V_u,\frac{2}{3}D)$ is log canonical at a general point of the curve $\mathcal{C}_4$. If $u\neq 2$, then the log pair $(V_u,\frac{5}{6}D)$ is log canonical at a general point of the curve $\mathcal{C}_4$.*

Proof. If $u=2$, then $(V_u,\frac{2}{3}T_{15}')$ is log canonical at a general point of $\mathcal{C}_4$ by Lemma 5.2. Similarly, if $u\neq 2$, then the pair (V_u,T_{15}') is log canonical at a general point of the curve $\mathcal{C}_4$. Thus, by Remark 7.4, we may assume that $\mathrm{Supp}(D)$ does not contain the surface T_{15}'. Now the assertion follows from Lemma 7.12. □

Combining Corollaries 7.11 and 7.15, we obtain the assertion of Theorem 1.6. Indeed, let D be an effective $\mathbb{Q}$-divisor on the threefold V_u such that $D\sim_{\mathbb{Q}}-K_{V_u}$. As we already mentioned, we have to show that the log pair $(V_u,\varepsilon(u)D)$ is log canonical. But the log pair $(V_u,\varepsilon(u)D)$ is log canonical at a general point of the curve $\mathcal{C}_4$ by Corollary 7.15, so that it is log canonical everywhere by Corollary 7.11.

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