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Katzarkov–Kontsevich–Pantev Conjecture for Fano Threefolds

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Abstract—We verify the Katzarkov–Kontsevich–Pantev conjecture for Landau–Ginzburg models of smooth Fano threefolds.

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INTRODUCTION

For a smooth Fano variety X, its Landau–Ginzburg model is a smooth quasiprojective variety Y equipped with a regular function $\mathsf{w}\colon Y \to \mathbb{C}$. The homological mirror symmetry conjecture predicts the equivalences between the derived category of coherent sheaves on X (the derived category of singularities of (Y, w), respectively) and the Fukaya–Seidel category of the pair (Y, w) (the Fukaya category of X, respectively).

In [9], Katzarkov, Kontsevich, and Pantev considered a *tame compactification* of the Landau–Ginzburg model (see [9, Definition 2.4]), that is, a commutative diagram

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[d, "\mathsf{w}"'] & Z \arrow[d, "\mathsf{f}"] \\ \mathbb{C} \arrow[r, hook] & \mathbb{P}^1 \end{tikzcd}$$

such that Z is a smooth compact variety that satisfies certain natural geometric conditions and f is a morphism such that $\mathsf{f}^{-1}(\infty) = -K_Z$. The compactification Z (if it exists) is unique up to flops in the fibers of the morphism f. The pair (Z, f) is usually called the compactified Landau–Ginzburg model of the Fano variety X.

Denote by D_Z the fiber over infinity $\mathsf{f}^{-1}(\infty)$. Assume that D_Z is a normal crossing divisor. Holomorphic forms with simple poles along components of D_Z form the logarithmic de Rham complex $\Omega^\bullet_Z(\log D_Z)$. For each $a \geq 0$ define a *sheaf* $\Omega^a_Z(\log D_Z, \mathsf{f})$ *of* f*-adapted logarithmic forms* as a subsheaf of $\Omega^a_Z(\log D_Z)$ consisting of forms which remain logarithmic after multiplication by $d\mathsf{f}$. More precisely,

$$\Omega^a_Z(\log D_Z, \mathsf{f}) = \big\{\alpha \in \Omega^a_Z(\log D_Z) \bigm| d\mathsf{f} \wedge \alpha \in \Omega^{a+1}_Z(\log D_Z)\big\},$$

where one considers f as a meromorphic function on Z and $d\mathsf{f}$ is viewed as a meromorphic 1-form. Katzarkov, Kontsevich, and Pantev in [9] defined the Hodge-type numbers $f^{p,q}(Y, \mathsf{w})$ of the Landau–Ginzburg model (Y, w).

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Definition. The *Landau–Ginzburg Hodge numbers* $f^{p,q}(Y,\mathsf{w})$ are defined as follows:

$$f^{p,q}(Y,\mathsf{w}) = \dim H^p\big(Z, \Omega_Z^q(\log D_Z, \mathsf{f})\big).$$

Note that according to [6] one has

$$f^{p,q}(Y,\mathsf{w}) = \dim \mathrm{Gr}_q^F H^{p+q}(Y,V),$$

where V is a smooth fiber of w and $H^{p+q}(Y,V)$ is equipped with the natural mixed Hodge structure on the relative cohomology with Hodge filtration $F^\bullet$.

Conjecture (Katzarkov, Kontsevich, Pantev)**.** *Let* (Y,w) *be a Landau–Ginzburg model of a smooth Fano variety* X *with* $\dim X = \dim Y = d$*. Suppose that it admits a tame compactification. Then*

$$h^{p,q}(X) = f^{p,d-q}(Y,\mathsf{w}).$$

In [11], this conjecture was proved for del Pezzo surfaces and their Landau–Ginzburg models constructed by Auroux, Katzarkov, and Orlov in [3]. In this paper, we verify the Katzarkov–Kontsevich–Pantev conjecture for smooth Fano threefolds and their *toric Landau–Ginzburg models* constructed in [14, 15, 1, 5], which satisfy all hypotheses of the Katzarkov–Kontsevich–Pantev conjecture by [16, Theorem 1].

From now on and until the end of this paper, we assume that X is a smooth Fano threefold. Its compactified Landau–Ginzburg model is given by the following commutative diagram:

$$\begin{tikzcd} (\mathbb{C}^*)^3 \arrow[r, hook] \arrow[d, "\mathsf{p}"'] & Y \arrow[r, hook] \arrow[d, "\mathsf{w}"] & Z \arrow[d, "\mathsf{f}"] \\ \mathbb{C} \arrow[r, equal] & \mathbb{C} \arrow[r, hook] & \mathbb{P}^1 \end{tikzcd} \qquad (\maltese)$$

where p is a surjective morphism that is given by one of the Laurent polynomials explicitly described in [1, 16, 5], the variety Y is a smooth threefold with $K_Y \sim 0$, and Z is a smooth compact threefold such that

$$-K_Z \sim \mathsf{f}^{-1}(\infty).$$

Moreover, the threefold Z in each case is obtained from $\mathbb{P}^3$ (or, rarely, from $\mathbb{P}^1 \times \mathbb{P}^2$) by blowing up rational curves. This shows that one gets $h^{1,2}(Z) = 0$. Moreover, in each case this gives the normal crossing fiber over infinity D_Z.

In [6], Harder showed how to compute the numbers $f^{p,q}(Y,\mathsf{w})$ using the global geometry of the compactification Z. He showed that under some natural conditions one has $f^{3,0}(Y,\mathsf{w}) = f^{0,3}(Y,\mathsf{w}) = 1$ and

$$f^{1,1}(Y,\mathsf{w}) = f^{2,2}(Y,\mathsf{w}) = \sum_{P \in \mathbb{C}^1} (\rho_P - 1), \qquad (\clubsuit)$$

where ρ_P is the number of irreducible components of the fiber $\mathsf{w}^{-1}(P)$. Moreover, he proved that

$$f^{1,2}(Y,\mathsf{w}) = f^{2,1}(Y,\mathsf{w}) = \dim \mathrm{coker}\big(H^2(Z,\mathbb{R}) \to H^2(F,\mathbb{R})\big) - 2 + h^{1,2}(Z), \qquad (\spadesuit)$$

where F is a general fiber of the morphism w. Finally, he proved that the remaining $f^{p,q}$ numbers of the Landau–Ginzburg model (Y,w) vanish.

Thus, to prove the Katzarkov–Kontsevich–Pantev conjecture for smooth Fano threefolds, one needs to compute the right-hand sides in ($\clubsuit$) and ($\spadesuit$) and compare them with the well-known Hodge numbers of smooth Fano threefolds. For smooth Fano threefolds of Picard rank 1, this was done in [15, 7]. The goal of this paper is to do the same for smooth Fano threefolds whose Picard rank is greater than 1.

To be more precise, we prove the following result.

Main Theorem. *Let X be a smooth Fano threefold, and let $\mathsf{f}\colon Z \to \mathbb{P}^1$ be its compactified Landau–Ginzburg model given by (✠), where p is a surjective morphism given by one of the Laurent polynomials described in [1, 5]. Then*

$$h^{1,2}(X) = \sum_{P\in\mathbb{C}^1} (\rho_P - 1), \tag{$\heartsuit$}$$

where ρ_P is the number of irreducible components of the fiber $\mathsf{w}^{-1}(P)$. Moreover, one has

$$\operatorname{rk}\operatorname{Pic}(X) = \dim\operatorname{coker}\big(H^2(Z,\mathbb{R}) \to H^2(F,\mathbb{R})\big) - 2, \tag{$\diamondsuit$}$$

where F is a general fiber of the morphism f.

Using (♣) and (♠), we obtain the following corollary.

Corollary. *Let X be a smooth Fano threefold. Then the Katzarkov–Kontsevich–Pantev conjecture holds for its compactified Landau–Ginzburg model (✠), where p is a morphism given by one of the Laurent polynomials described in [1, 16, 5].*

The proof of the Main Theorem gives an explicit description of the fiber $\mathsf{f}^{-1}(\infty)$ in (✠), which shows that the conditions of Harder's result are satisfied. This has already been verified in [16, Corollary 35] for smooth Fano threefolds with very ample anticanonical divisor. The proof of the Main Theorem also gives an explicit description of (isolated and non-isolated) singularities of the fibers of the morphism w in (✠) in the case when p is given by one of the Laurent polynomials from [1, 16, 5]. It seems possible to use this description to check that the Jacobian rings of these Landau–Ginzburg models are isomorphic to the quantum cohomology rings of the corresponding smooth Fano threefolds, which reflects homological mirror symmetry on the Hochschild cohomology level. Perhaps, one can also use the proof of the Main Theorem to compute the derived categories of singularities of our compactified Landau–Ginzburg models.

This paper is organized as follows. In Section 1 we give a detailed scheme of the proof of our Main Theorem. We illustrate each step of the scheme by an appropriate example (see Examples 1.7.1, 1.8.6, 1.10.11, 1.12.3, and 1.13.2). In Sections 2–10 we prove the Main Theorem for smooth Fano threefolds of Picard rank $2,\dots,10$, respectively. These sections are split into subsections, in each of which we consider one family of smooth Fano threefolds from the list of families given in [8]. For instance, if X is a blow-up of a smooth quadric threefold in a disjoint union of two lines, this is family 3.20, and we prove the Main Theorem in this case in the subsection entitled "Family 3.20" in Section 3. Similarly, in the subsection "Family 2.24" of Section 2, we prove the Main Theorem for family 2.24, which consists of divisors of bidegree $(1,2)$ in $\mathbb{P}^2\times\mathbb{P}^2$. Finally, in the Appendix, we review the basic intersection theory for smooth curves on surfaces with du Val singularities, which is probably well known to experts.

Notation and conventions. We assume that all varieties are defined over the field of complex numbers $\mathbb{C}$ unless otherwise stated. For a (not necessary reduced) variety V, we denote the number of its irreducible components by $[V]$. To denote Laurent polynomials from the database [4], we use the notation $P.N$, where P is the number of the Newton polytope of the polynomial and N is the number of the polynomial for the polytope. If the polynomial for a polytope is unique, we just say that it is polynomial P.

1. SCHEME OF THE PROOF OF THE MAIN THEOREM

To prove the Main Theorem, we fix a smooth Fano threefold X. Then X is contained in one of the 105 deformation families described in Iskovskikh and Prokhorov's book [8]. We add the variety found in [12] to the end of the list of Picard rank 4 threefolds. We always assume that X is a general threefold in its deformation family.

For each family, we have the commutative diagram ($\maltese$), where p is given by a Laurent polynomial, which we identify with p. Then we proceed as follows.

1.1. Mirror partners. The polynomial p is not uniquely determined by X. However, Akhtar, Coates, Galkin, and Kasprzyk proved in [1] that all of them are related by birational transformations, called mutations. Mutations preserve the right-hand sides of ($\heartsuit$) and ($\diamondsuit$) in the Main Theorem. Thus, to prove the Main Theorem, we may choose any Laurent polynomial p from [4] among mirror partners for X.

1.2. Rank of the Picard group. If X is a smooth Fano threefold such that $\operatorname{rk}\operatorname{Pic}(X) = 1$, then ($\heartsuit$) in the Main Theorem is already established in [15, 17], and ($\diamondsuit$) in the Main Theorem follows from the proof of [7, Theorem 4.1]. Thus, we will always assume that $\operatorname{rk}\operatorname{Pic}(X) \geq 2$. This leaves us with 87 deformation families described in [8] and one family described in [12] (which was missed in [8]).

1.3. Minkowski polynomials. If $-K_X$ is very ample, then X admits a Gorenstein toric degeneration. In this case, the Newton polytope of the Laurent polynomial p is a reflexive lattice polytope which is a *fan polytope* of the toric degeneration, and the coefficients of p correspond to expansions of its facets into Minkowski sums of elementary polygons. Because of this, the Laurent polynomial p is usually called a *Minkowski polynomial* (see [1]).

The divisor $-K_X$ is very ample except for five special families. These are the deformation families 2.1–2.3, 9.1, and 10.1 in [8]. To prove the Main Theorem, we deal with these cases separately. Thus, in the remaining part of this section, we assume that $-K_X$ is very ample and p is one of the corresponding Minkowski polynomials.

1.4. Pencil of quartic surfaces. For every smooth Fano threefold X such that its anticanonical divisor $-K_X$ is very ample, we can always choose the corresponding Minkowski polynomial p in [1] such that there is a pencil $\mathcal{S}$ of quartic surfaces on $\mathbb{P}^3$ given by

$$f_4(x,y,z,t) = \lambda xyzt \tag{1.4.1}$$

that expands ($\maltese$) to the following commutative diagram:

$$\begin{tikzcd}
(\mathbb{C}^*)^3 \arrow[r, hook] \arrow[d, "\mathsf{p}"] & Y \arrow[r, hook] \arrow[d, "\mathsf{w}"] & Z \arrow[d, "\mathsf{f}"] & V \arrow[l, dashed, "\chi"'] \arrow[r, "\pi"] \arrow[d, "\mathsf{g}"] & \mathbb{P}^3 \arrow[dl, dashed, "\phi"] \\
\mathbb{C} \arrow[r, equal] & \mathbb{C} \arrow[r, hook] & \mathbb{P}^1 \arrow[r, equal] & \mathbb{P}^1 &
\end{tikzcd} \tag{1.4.2}$$

where ϕ is a rational map given by the pencil $\mathcal{S}$, the map π is a birational morphism to be explicitly constructed later in this section, the threefold V is smooth, and χ is a composition of flops. Here $f_4(x,y,z,t)$ is a quartic homogeneous polynomial and $\lambda \in \mathbb{C} \cup \{\infty\}$, where $\lambda = \infty$ corresponds to the fiber $\mathsf{f}^{-1}(\infty)$.

1.5. Fibers of the Landau–Ginzburg model. By [16, Corollary 35], we have

$$\left[\mathsf{f}^{-1}(\infty)\right] = \frac{4 - K_X^3}{2}.$$

To verify ($\heartsuit$) in the Main Theorem, we must find $[\mathsf{f}^{-1}(\lambda)]$ for every $\lambda \neq \infty$. This can be done by checking the basic properties of the pencil $\mathcal{S}$. Let us show how to do this in simple cases.

Let S_λ be the quartic surface given by (1.4.1), let $\widetilde{S}_\lambda$ be its proper transform on the threefold V, and let $E_1, \ldots, E_n$ be the π-exceptional divisors. Then

$$K_V + \widetilde{S}_\lambda + \sum_{i=1}^{n} \mathbf{a}_i^\lambda E_i \sim \pi^*(K_{\mathbb{P}^3} + S_\lambda) \sim 0$$

for some nonnegative integers $\mathbf{a}_1^\lambda, \ldots, \mathbf{a}_n^\lambda$. Hence, since $-K_V \sim \mathsf{g}^{-1}(\infty)$, we conclude that

$$\mathsf{g}^{-1}(\lambda) = \widetilde{S}_\lambda + \sum_{i=1}^{n} \mathbf{a}_i^\lambda E_i. \tag{1.5.1}$$

Since χ in (1.4.2) is a composition of flops, it follows from (1.5.1) that

$$\big[\mathsf{f}^{-1}(\lambda)\big] = [S_\lambda] + \#\big\{i \in \{1, \ldots, n\} \colon \mathbf{a}_i^\lambda > 0\big\}. \tag{1.5.2}$$

The number $[S_\lambda]$ is easy to compute. How to determine the correction term in (1.5.2)? One way to do this is to explicitly describe the birational morphism π in (1.4.2) and then compute the numbers $\mathbf{a}_1^\lambda, \ldots, \mathbf{a}_n^\lambda$. However, this method is usually very time consuming. Our main goal is to show how to do the same with less effort. We start with the following.

Lemma 1.5.3. *Let P be a point in the base locus of the pencil $\mathcal{S}$. Suppose that the quartic surface S_λ has at most a du Val singularity at P. If $P \in \pi(E_i)$, then $\mathbf{a}_i^\lambda = 0$.*

Proof. By [10, Theorem 7.9], the log pair $(\mathbb{P}^3, S_\lambda)$ has canonical singularities at P, so that $\mathbf{a}_i^\lambda = 0$ for every E_i such that $P \in \pi(E_i)$. $\square$

Corollary 1.5.4. *Suppose that S_λ has du Val singularities at every point of the base locus of the pencil $\mathcal{S}$. Then $\mathsf{f}^{-1}(\lambda)$ is irreducible.*

Proof. The surface S_λ is irreducible, because S_λ has du Val singularities at every point of the base locus of the pencil $\mathcal{S}$. This follows from the fact that irreducible components of the surface S_λ are hypersurfaces in $\mathbb{P}^3$. By Lemma 1.5.3, we have

$$\mathbf{a}_1^\lambda = \mathbf{a}_2^\lambda = \ldots = \mathbf{a}_n^\lambda = 0,$$

so that the fiber $\mathsf{f}^{-1}(\lambda)$ is irreducible by (1.5.2). $\square$

Let us show how to apply this result to prove ($\heartsuit$) in the Main Theorem in one simple case. Before doing this, let us fix handy notation that will be used throughout the whole paper.

1.6. Handy notation. We will use $[x : y : z : t]$ as homogeneous coordinates on $\mathbb{P}^3$. For a nonempty subset I in $\{x, y, z, t\}$, we will write H_I for the plane in $\mathbb{P}^3$ on which the sum of coordinates in I is equal to zero. For instance, we denote by $H_{\{x\}}$ the plane in $\mathbb{P}^3$ given by $x = 0$. Similarly, we denote by $H_{\{y,t\}}$ the plane in $\mathbb{P}^3$ given by

$$y + t = 0.$$

For nonempty distinct subsets I, J, and K, we will also write $L_{I,J} = H_I \cap H_J$ and similarly $P_{I,J,K} = H_I \cap H_J \cap H_K$. For instance, by $L_{\{x\},\{y,z,t\}}$ we denote the line in $\mathbb{P}^3$ given by

$$\begin{cases} x = 0, \\ y + z + t = 0. \end{cases}$$

Similarly, we have $P_{\{x\},\{y\},\{z\}} = [0:0:0:1]$ and $P_{\{x\},\{y\},\{z,t\}} = [0:0:1:-1]$.

If the quartic surface S_λ has du Val singularities, we always denote by H_λ its general hyperplane section or its class in $\mathrm{Pic}(S_\lambda)$. We will often use this to compute the intersection form of some curves on S_λ in the proof of ($\diamondsuit$) in the Main Theorem.

1.7. Apéry–Fermi pencil. Let us use Corollary 1.5.4 in the case when $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. In this case, the pencil $\mathcal{S}$ was studied by Peters and Stienstra in [13].

Example 1.7.1. Suppose that $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. This is family 3.27. One of its mirror partners is given by the Laurent polynomial

$$x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z}.$$

This is the Minkowski polynomial 30. The corresponding pencil $\mathcal{S}$ is given by

$$x^2yz + y^2xz + z^2xy + t^2xy + t^2xz + t^2yz = \lambda xyzt.$$

Its base locus consists of the lines $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, and $L_{\{t\},\{x,y,z\}}$. If $\lambda \neq \infty$, then the singular points of S_λ contained in one of these lines are the points $P_{\{y\},\{z\},\{t\}}$, $P_{\{x\},\{z\},\{t\}}$, and $P_{\{x\},\{y\},\{t\}}$, which are du Val singular points of type $\mathbb{A}_3$, and the points $P_{\{x\},\{y\},\{z\}}$, $P_{\{x\},\{t\},\{y,z\}}$, $P_{\{y\},\{t\},\{x,z\}}$, and $P_{\{z\},\{t\},\{x,y\}}$, which are isolated ordinary double points. Then $[\mathsf{f}^{-1}(\lambda)] = 1$ for every $\lambda \neq \infty$ by Corollary 1.5.4. Since $h^{1,2}(X) = 0$, this proves (♡) in the Main Theorem in this case.

This approach works for 55 deformation families of smooth Fano threefolds.

1.8. Base points and base curves. In many cases, we cannot apply Corollary 1.5.4 to prove (♡) in the Main Theorem simply because the pencil $\mathcal{S}$ contains surfaces that have non-du Val singularities in its base locus. In fact, quite often, the pencil $\mathcal{S}$ contains reducible surfaces, so that they have non-isolated singularities. To deal with these cases, we have to refine formula (1.5.2). Let us make the first step in this direction.

Let $C_1, \ldots, C_r$ be irreducible curves contained in the base locus of the pencil $\mathcal{S}$. With very few exceptions (see families 3.8, 3.22, 3.24, 3.29, 7.1, and 8.1), these curves are either lines or conics. For every base curve C_j, we let

$$\mathbf{C}_j^\lambda = \#\bigl\{i \in \{1, \ldots, n\}\colon\ \mathbf{a}_i^\lambda > 0 \text{ and } \pi(E_i) = C_j\bigr\}. \tag{1.8.1}$$

Let Σ be the (finite) subset of the base locus of the pencil $\mathcal{S}$ such that for every $P \in \Sigma$ there is an index $i \in \{1, \ldots, n\}$ such that $\pi(E_i) = P$. For every $P \in \Sigma$, we let

$$\mathbf{D}_P^\lambda = \#\bigl\{i \in \{1, \ldots, n\}\colon\ \mathbf{a}_i^\lambda > 0 \text{ and } \pi(E_i) = P\bigr\}. \tag{1.8.2}$$

We say that $\mathbf{D}_P^\lambda$ is the *defect* of the *fixed* singular point P.

Using (1.5.2), we see that

$$\bigl[\mathsf{f}^{-1}(\lambda)\bigr] = [S_\lambda] + \sum_{i=1}^{r} \mathbf{C}_j^\lambda + \sum_{P \in \Sigma} \mathbf{D}_P^\lambda. \tag{1.8.3}$$

If P is a point in Σ such that the quartic surface S_λ has a du Val singularity at P, then its defect vanishes by Lemma 1.5.3. However, the defect may also vanish if S_λ has a singularity worse than a du Val singularity at the point P.

Remark 1.8.4. For a general $\lambda \in \mathbb{C}$, the singular points of the surface S_λ are all du Val. Moreover, they are of two kinds: those whose coordinates depend on λ, and those whose coordinates do not depend on λ. The latter will be called *fixed* singular points, and the former, *floating* singular points. The set Σ consists of fixed singular points.

For every point $P \in \Sigma$, the number $\mathbf{D}_P^\lambda$ can be computed locally near P. We will show how to do this later (see formula (1.10.9) below). Now let us show how to compute the number $\mathbf{C}_i^\lambda$ defined in (1.8.1). For every $\lambda \in \mathbb{C} \cup \{\infty\}$ and every $i \in \{1, \ldots, r\}$, we let

$$\mathbf{M}_i^\lambda = \operatorname{mult}_{C_i}(S_\lambda).$$

For any two distinct quartic surfaces S_{λ_1} and S_{λ_2} in the pencil $\mathcal{S}$, we have

$$S_{\lambda_1} \cdot S_{\lambda_2} = \sum_{i=1}^{r} \mathbf{m}_i C_i$$

for some positive numbers $\mathbf{m}_1, \ldots, \mathbf{m}_r$. Then $\mathbf{m}_i \geq \mathbf{M}_i^{\lambda}$ for every $\lambda \in \mathbb{C} \cup \{\infty\}$.

Lemma 1.8.5. *Fix $\lambda \in \mathbb{C} \cup \{\infty\}$ and $a \in \{1, \ldots, r\}$. Then*

$$\mathbf{C}_a^{\lambda} = \begin{cases} 0 & \text{if } \mathbf{M}_a^{\lambda} = 1, \\ \mathbf{m}_a - 1 & \text{if } \mathbf{M}_a^{\lambda} \geq 2. \end{cases}$$

Proof. The required assertion can be checked at a general point of the curve C_a. Because of this, we may assume that C_a is smooth. To resolve the base locus of the pencil $\mathcal{S}$ at a general point of the curve C_a, we observe that general surfaces in this pencil are smooth at a general point of the curve C_a. This implies that there exists a composition of $\mathbf{m}_a \geq 1$ blow-ups of smooth curves

$$V_{\mathbf{m}_a} \xrightarrow{\gamma_{\mathbf{m}_a}} V_{\mathbf{m}_a - 1} \xrightarrow{\gamma_{\mathbf{m}_a - 1}} \ldots \xrightarrow{\gamma_2} V_1 \xrightarrow{\gamma_1} \mathbb{P}^3$$

such that γ_1 is the blow-up of the curve C_a, for $i > 1$ the morphism γ_i is a blow-up of a smooth curve $C_a^{i-1} \subset V_{i-1}$ such that

$$\gamma_{i-1}\big(C_a^{i-1}\big) = C_a^{i-2} \subset V_{i-2},$$

and the curve C_a^{i-1} is contained in the proper transform of a general surface in $\mathcal{S}$ on the threefold V_{i-1}. Here, we have $V_0 = \mathbb{P}^3$ and $C_a^0 = C_a$.

For each index $i \in \{1, \ldots, \mathbf{m}_a\}$, let F_i be the exceptional surface of the morphism γ_i. Then $C_a^i \subset F_i$, and the curve C_a^i is a section of the $\mathbb{P}^1$-bundle $G_i \to C_a^{i-1}$ induced by γ_i. Note that C_a^i is not contained in the proper transform of the surface F_{i-1}.

For each $i \in \{0, 1, \ldots, \mathbf{m}_a\}$, denote by S_λ^i the proper transform of the surface S_λ on the threefold V_i. Then

$$\sum_{i=0}^{\mathbf{m}_a - 1} \mathrm{mult}_{C_a^i}(S_\lambda^i) = \mathbf{m}_a.$$

Moreover, for each $b \in \{1, \ldots, n\}$ such that $\beta(E_b) = C_a$, there is $j \in \{1, \ldots, \mathbf{m}_a - 1\}$ such that E_b is the proper transform of the divisor F_j on the threefold V in the diagram (1.4.2). Conversely, for each $j \in \{1, \ldots, \mathbf{m}_a - 1\}$, there is $b \in \{1, \ldots, n\}$ such that $\beta(E_b) = C_a$ and E_b is the proper transform of the divisor F_j on the threefold V, which implies

$$\mathbf{a}_b^{\lambda} = \sum_{i=0}^{j-1} \big(\mathrm{mult}_{C_a^i}(S_\lambda^i) - 1\big).$$

On the other hand, we also have

$$\mathbf{M}_a^{\lambda} = \mathrm{mult}_{C_a}(S_\lambda) \geq \mathrm{mult}_{C_a^1}(S_\lambda^1) \geq \mathrm{mult}_{C_a^2}(S_\lambda^2) \geq \ldots \geq \mathrm{mult}_{C_a^{j-1}}\big(S_\lambda^{j-1}\big) \geq 0.$$

Using this, we obtain a dichotomy:

- either $\mathbf{M}_a^{\lambda} = 1$ and $\mathbf{a}_b^{\lambda} = 0$ for every $b \in \{1, \ldots, n\}$ such that $\beta(E_b) = C_a$,
- or $\mathbf{M}_a^{\lambda} \geq 2$ and $\mathbf{a}_b^{\lambda} > 0$ for every $b \in \{1, \ldots, n\}$ such that $\beta(E_b) = C_a$, with a single exception in the case when E_b is a proper transform of the divisor $F_{\mathbf{m}_a}$ on the threefold V.

This immediately implies the required assertion. □

Let us show how to apply Lemma 1.8.5 to prove (♡) in the Main Theorem in one case.

Example 1.8.6. Suppose that X is a smooth Fano threefold in family 3.2. Then one of its mirror partners is given by the Laurent polynomial

$$\frac{z^2}{xy}+z+\frac{3z}{y}+\frac{3z}{x}+x+y+\frac{z}{xy}+\frac{3x}{y}+\frac{3y}{x}+\frac{1}{y}+\frac{1}{x}+\frac{x^2}{yz}+\frac{3x}{z}+\frac{3y}{z}+\frac{y^2}{xz}.$$

This is the Minkowski polynomial 2569. The pencil $\mathcal{S}$ is given by

$$z^3t+xyz^2+3z^2xt+3z^2yt+x^2yz+y^2xz+z^2t^2+3x^2tz+3y^2tz \\ +t^2xz+t^2yz+x^3t+3x^2yt+3y^2xt+y^3t=\lambda xyzt.$$

Suppose that $\lambda\neq\infty$. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be conics given by $x=y^2+2yz+z^2+tz=0$ and $y=x^2+2xz+z^2+tz=0$, respectively. Then

$$S_\infty\cdot S_\lambda=2L_{\{x\},\{t\}}+2L_{\{y\},\{t\}}+2L_{\{z\},\{t\}}+L_{\{x\},\{y,z\}}+L_{\{y\},\{x,z\}} \\ +3L_{\{z\},\{x,y\}}+L_{\{t\},\{x,y,z\}}+\mathcal{C}_1+\mathcal{C}_2.$$

Thus, we have $r=9$, and we may assume that $C_1=\mathcal{C}_1$, $C_2=\mathcal{C}_2$, $C_3=L_{\{x\},\{t\}}$, $C_4=L_{\{y\},\{t\}}$, $C_5=2L_{\{z\},\{t\}}$, $C_6=L_{\{x\},\{y,z\}}$, $C_7=L_{\{y\},\{x,z\}}$, $C_8=L_{\{z\},\{x,y\}}$, and $C_9=L_{\{t\},\{x,y,z\}}$. Then $\mathbf{m}_1=\mathbf{m}_2=\mathbf{m}_6=\mathbf{m}_7=\mathbf{m}_9=1$, $\mathbf{m}_3=\mathbf{m}_4=\mathbf{m}_5=2$, and $\mathbf{m}_8=3$. We have

$$\Sigma=\big\{P_{\{x\},\{y\},\{z\}},P_{\{x\},\{t\},\{y,z\}},P_{\{y\},\{t\},\{x,z\}},P_{\{z\},\{t\},\{x,y\}}\big\}.$$

If $\lambda\neq-6$, then S_λ is irreducible and has isolated singularities. In this case, the surface S_λ has du Val singularities at $P_{\{x\},\{y\},\{z\}}$, $P_{\{x\},\{t\},\{y,z\}}$, $P_{\{y\},\{t\},\{x,z\}}$, and $P_{\{z\},\{t\},\{x,y\}}$, and it does not have other singular points in the base locus of the pencil $\mathcal{S}$. Then $[\mathsf{f}^{-1}(\lambda)]=1$ for every $\lambda\neq-6$ by Corollary 1.5.4. On the other hand, we have

$$S_{-6}=H_{\{x,y,z\}}+\mathbf{S},$$

where $\mathbf{S}$ is a cubic surface given by

$$zt^2+x^2t+xyt+2xzt+y^2t+2yzt+z^2t+xyz=0.$$

We have $\mathbf{M}_1^{-6}=\ldots=\mathbf{M}_7^{-6}=\mathbf{M}_9^{-6}=1$ and $\mathbf{M}_8^{-6}=2$. Thus, it follows from Lemma 1.8.5 that

$$\mathbf{C}_8^{-6}=2\qquad\text{and}\qquad\mathbf{C}_1^{-6}=\ldots=\mathbf{C}_7^{-6}=\mathbf{C}_9^{-6}=0.$$

Note that S_{-6} has du Val singularities of type $\mathbb{A}$ or non-isolated ordinary double singularities at the points of the set Σ. We will see in Lemma 1.12.1 that this gives $\mathbf{D}_P^{-6}=0$ for each $P\in\Sigma$. Then $[\mathsf{f}^{-1}(-6)]=4$ by (1.8.3), which gives (♡) in the Main Theorem.

Unlike what we have just seen in Example 1.8.6, the numbers $\mathbf{D}_P^{\lambda}$ in (1.8.3) do not always vanish for every $P\in\Sigma$. Thus, we have to provide an algorithm to compute them. To this end, we should choose a suitable birational morphism π in (1.4.2).

1.9. Blowing up fixed singular points. We can (partially) resolve all fixed singular points of the surfaces in the pencil $\mathcal{S}$ by consecutive blow-ups of $\mathbb{P}^3$ at finitely many points. This gives a birational map $\alpha\colon U\to\mathbb{P}^3$ such that the proper transform of the pencil $\mathcal{S}$ on the threefold U does not have fixed singular points. Let $\widehat{\mathcal{S}}$ be the proper transform of the pencil $\mathcal{S}$ on the threefold U. Then we can (uniquely) choose α such that $\widehat{\mathcal{S}}\sim-K_U$.

Remark 1.9.1. By construction, for every point P in the base locus of the pencil $\widehat{\mathcal{S}}$, there exists a surface in $\widehat{\mathcal{S}}$ that is smooth at P. Note that a general surface in $\widehat{\mathcal{S}}$ is not necessarily smooth. However, in most cases it is smooth. In the remaining cases, it has du Val singular points of type $\mathbb{A}$ by [10, Theorem 4.4].

Denote by $\widehat{C}_1, \ldots, \widehat{C}_r$ proper transforms of the curves $C_1, \ldots, C_r$ on the threefold U, respectively. Then these curves are contained in the base locus of the pencil $\widehat{\mathcal{S}}$. However, the pencil $\widehat{\mathcal{S}}$ always has other base curves. Denote them by $\widehat{C}_{r+1}, \ldots, \widehat{C}_s$, where $s > r$. A posteriori, all base curves of the pencil $\widehat{\mathcal{S}}$ are smooth rational curves.

For any two distinct surfaces $\widehat{S}_{\lambda_1}$ and $\widehat{S}_{\lambda_2}$ in the pencil $\widehat{\mathcal{S}}$, we have

$$\widehat{S}_{\lambda_1} \cdot \widehat{S}_{\lambda_2} = \sum_{i=1}^{s} \mathbf{m}_i \widehat{C}_i \tag{1.9.2}$$

for some positive numbers $\mathbf{m}_1, \ldots, \mathbf{m}_s$. Since general surfaces in $\widehat{\mathcal{S}}$ are smooth at general points of the curves $\widehat{C}_1, \ldots, \widehat{C}_s$, we can resolve the base locus of the pencil $\widehat{\mathcal{S}}$ by

$$\mathbf{m}_1 + \mathbf{m}_2 + \mathbf{m}_3 + \ldots + \mathbf{m}_s$$

consecutive blow-ups of smooth rational curves (cf. Remarks 2.1.5 and 10.1.4). This gives a birational morphism $\beta\colon V' \to U$ such that there exists a commutative diagram

$$\begin{tikzcd}
U \arrow[d, "\alpha"'] & V' \arrow[l, "\beta"'] \arrow[d, "\mathsf{g}'"] \\
\mathbb{P}^3 \arrow[r, dashed, "\phi"'] & \mathbb{P}^1
\end{tikzcd}$$

where g' is a morphism whose general fibers are smooth K3 surfaces.

By construction, the threefold V' is smooth, and the anticanonical divisor $-K_{V'}$ is rationally equivalent to a scheme fiber of the fibration g'. This immediately implies that there exists a composition of flops $\eta\colon V \dashrightarrow V'$ that makes the following diagram commutative:

$$\begin{tikzcd}
V \arrow[rr, dashed, "\eta"] \arrow[dr, "\mathsf{g}"'] \arrow[dd, "\pi"'] & & V' \arrow[dl, "\mathsf{g}'"] \arrow[dd, "\beta"] \\
& \mathbb{P}^1 & \\
\mathbb{P}^3 \arrow[ur, dashed, "\phi"'] & & U \arrow[ll, "\alpha"]
\end{tikzcd}$$

Hence, in what follows, we will always assume that $V = V'$, $\pi = \alpha \circ \beta$, $\eta = \mathrm{Id}$, and $\mathsf{g}' = \mathsf{g}$. This gives us the commutative diagram

$$\begin{tikzcd}
U \arrow[d, "\alpha"'] & V \arrow[l, "\beta"'] \arrow[dl, "\pi"] \arrow[d, "\mathsf{g}"'] \arrow[r, dashed, "\chi"] & Z \arrow[d, "\mathsf{f}"] \\
\mathbb{P}^3 \arrow[r, dashed, "\phi"'] & \mathbb{P}^1 \arrow[r, equal] & \mathbb{P}^1
\end{tikzcd} \tag{1.9.3}$$

Let $k = \operatorname{rk}\operatorname{Pic}(U) - 1$. For simplicity, we assume that $\beta(E_1), \ldots, \beta(E_k)$ are exceptional surfaces of the morphism α, while the surfaces $E_{k+1}, \ldots, E_n$ are contracted by β.

1.10. Counting multiplicities. Let us show how to explicitly compute $\mathbf{D}_P^\lambda$ in (1.8.3) for every point $P \in \Sigma$.

To this end, we denote by $\widehat{E}_1, \ldots, \widehat{E}_k$ the proper transforms of the surfaces $E_1, \ldots, E_k$ on the threefold U, respectively. For every $\lambda \in \mathbb{C} \cup \{\infty\}$, we let

$$\widehat{D}_\lambda = \widehat{S}_\lambda + \sum_{i=1}^{k} \mathbf{a}_i^\lambda \widehat{E}_i. \tag{1.10.1}$$

Then $\widehat{D}_\lambda \sim -K_U$, and the numbers $\mathbf{a}_1^\lambda, \ldots, \mathbf{a}_k^\lambda$ are uniquely determined by this rational equivalence. Furthermore, we have $\widehat{D}_\lambda \in \widehat{\mathcal{S}}$ by construction.

Lemma 1.10.2. *Let P be a point in the set Σ. If $\mathrm{mult}_P(S_\lambda)=2$, then*

$$\mathbf{a}_i^\lambda = 0$$

for every $i\in\{1,\dots,k\}$ such that $\alpha(\widehat{E}_i)=P$.

Proof. Straightforward. $\square$

For every fixed singular point $P\in\Sigma$, we let

$$\mathbf{A}_P^\lambda = \#\big\{i\in\{1,\dots,k\}\colon\ \mathbf{a}_i^\lambda>0 \text{ and } \alpha(\widehat{E}_i)=P\big\}. \tag{1.10.3}$$

Then Lemma 1.10.2 can be reformulated as follows.

Corollary 1.10.4. *If $\mathrm{mult}_P(S_\lambda)=2$ for $P\in\Sigma$, then $\mathbf{A}_P^\lambda=0$.*

For every $\lambda\in\mathbb{C}\cup\{\infty\}$ and every $a\in\{r+1,\dots,s\}$, we let

$$\mathbf{C}_a^\lambda = \#\big\{i\in\{1,\dots,n\}\colon\ \mathbf{a}_i^\lambda>0 \text{ and } \beta(E_i)=\widehat{C}_a\big\}. \tag{1.10.5}$$

For every λ and every $a\in\{1,\dots,s\}$, we let

$$\mathbf{M}_a^\lambda = \mathrm{mult}_{\widehat{C}_a}(\widehat{D}_\lambda). \tag{1.10.6}$$

Lemma 1.10.7. *Fix $\lambda\in\mathbb{C}\cup\{\infty\}$ and $a\in\{1,\dots,s\}$. Then*

$$\mathbf{C}_a^\lambda = \begin{cases} 0 & \text{if } \mathbf{M}_a^\lambda=1,\\ \mathbf{m}_a-1 & \text{if } \mathbf{M}_a^\lambda\geq 2.\end{cases}$$

Proof. See the proof of Lemma 1.8.5. $\square$

On the other hand, it follows from (1.5.2) that

$$\big[\mathsf{f}^{-1}(\lambda)\big] = [\widehat{D}_\lambda] + \sum_{i=1}^{s}\mathbf{C}_i^\lambda = [S_\lambda] + \sum_{P\in\Sigma}\mathbf{A}_P^\lambda + \sum_{i=1}^{s}\mathbf{C}_i^\lambda. \tag{1.10.8}$$

Comparing formulas (1.8.3) and (1.10.8), we obtain the formula for the *defect*

$$\mathbf{D}_P^\lambda = \mathbf{A}_P^\lambda + \sum_{\substack{i=r+1\\ \alpha(\widehat{C}_i)=P}}^{s}\mathbf{C}_i^\lambda \tag{1.10.9}$$

for every point $P\in\Sigma$. Similarly, using (1.10.8) and Lemma 1.10.7, we get

Corollary 1.10.10. *If $\mathbf{M}_i^\lambda=1$ for each $i\in\{1,\dots,s\}$, then $[\mathsf{f}^{-1}(\lambda)]=[\widehat{D}_\lambda]$.*

Let us show how to apply this handy result.

Example 1.10.11. Suppose that $X=\mathbb{P}^1\times\mathbf{S}_3$, where $\mathbf{S}_3$ is a smooth cubic surface in $\mathbb{P}^3$. This is family 8.1 in [8]. One of its mirror partners is given by the Minkowski polynomial 768, which is the Laurent polynomial

$$\frac{1}{yz}+\frac{3}{y}+\frac{3z}{y}+x+\frac{z^2}{y}+\frac{3}{z}+3z+\frac{1}{x}+\frac{3y}{z}+3y+\frac{y^2}{z}.$$

Then the corresponding quartic pencil $\mathcal{S}$ is given by

$$t^3x+3t^2xz+3z^2xt+x^2zy+z^3x+3t^2xy+3z^2xy+t^2zy+3y^2xt+3y^2xz+y^3x=\lambda xyzt,$$

and it has six base curves: $C_1=L_{\{x\},\{y\}}$, $C_2=L_{\{x\},\{z\}}$, $C_3=L_{\{x\},\{t\}}$, $C_4=L_{\{y\},\{t,z\}}$, $C_5=L_{\{z\},\{t,y\}}$, and C_6 that is the singular cubic curve $t=xyz+y^3+3y^2z+3yz^2+z^3=0$. Suppose that $\lambda \neq \infty$. Then

$$S_\lambda \cdot S_\infty = 2C_1 + 2C_2 + 3C_3 + 3C_4 + 3C_5 + C_6,$$

and the surface S_λ is irreducible. Moreover, if $\lambda \neq -4, -8$, then the singularities of the surface S_λ are du Val, so that $[\mathsf{f}^{-1}(\lambda)] = 1$ by Corollary 1.5.4. However, the singular locus of the surface S_{-4} consists of the point $P_{\{x\},\{y\},\{z\}}$ and the line $x - t = y + z + t = 0$. Similarly, the singular locus of the surface S_{-8} consists of the point $P_{\{x\},\{y\},\{z\}}$ and the line $x + t = y + z + t = 0$. Thus, we cannot apply Corollary 1.5.4 when $\lambda = -4$ or $\lambda = -8$. Nevertheless, we have $[\mathsf{f}^{-1}(-4)] = 1$ and $[\mathsf{f}^{-1}(-8)] = 1$. To show this, observe that

$$\Sigma = \big\{P_{\{y\},\{z\},\{t\}}, P_{\{x\},\{t\},\{y,z\}}\big\}.$$

Moreover, if $\lambda \neq -4, -8$, then $P_{\{y\},\{z\},\{t\}}$ is a singular point of S_λ of type $\mathbb{A}_2$, and the point $P_{\{x\},\{t\},\{y,z\}}$ is a singular point of S_λ of type $\mathbb{A}_5$. The birational morphism $\alpha\colon U \to \mathbb{P}^3$ can be decomposed as follows:

$$\begin{tikzcd}
& U_2 \arrow[dl, "\alpha_2"'] & & U_3 \arrow[ll, "\alpha_3"'] & \\
U_1 \arrow[drr, "\alpha_1"'] & & & & U \arrow[ul, "\alpha_4"'] \arrow[dll, "\alpha"] \\
& & \mathbb{P}^3 & &
\end{tikzcd}$$

Here α_1 is the blow-up of the point $P_{\{y\},\{z\},\{t\}}$, the morphism α_2 is the blow-up of the preimage of the point $P_{\{x\},\{t\},\{y,z\}}$, the morphism α_3 is the blow-up of a point in the α_2-exceptional surface, and α_4 is the blow-up of a point in the α_3-exceptional surface. We may assume that $\widehat{E}_4$ is the α_4-exceptional surface. Similarly, we may assume that $\widehat{E}_1$, $\widehat{E}_2$, and $\widehat{E}_3$ are the proper transforms on U of the exceptional surfaces of the morphisms α_1, α_2, and α_3, respectively. Then

$$\widehat{D}_\infty = \widehat{S}_\infty + \widehat{E}_1 \sim \widehat{S}_\lambda \sim -K_U.$$

One can show that $\widehat{E}_2$, $\widehat{E}_3$, and $\widehat{E}_4$ do not contain base curves of the pencil $\widehat{\mathcal{S}}$, and the surface $\widehat{E}_1$ contains two base curves of the pencil $\widehat{\mathcal{S}}$. They are cut out on $\widehat{E}_1$ by the proper transforms on U of the planes $H_{\{y\}}$ and $H_{\{z\}}$. Let us denote them by $\widehat{C}_7$ and $\widehat{C}_8$, respectively. Then $\widehat{S}_\lambda$ and $\widehat{S}_\infty + \widehat{E}_1$ generate the pencil $\widehat{\mathcal{S}}$ and

$$\widehat{S}_\lambda \cdot \big(\widehat{S}_\infty + \widehat{E}_1\big) = 2\widehat{C}_1 + 2\widehat{C}_2 + 3\widehat{C}_3 + 3\widehat{C}_4 + 3\widehat{C}_5 + \widehat{C}_6 + 2\widehat{C}_7 + 2\widehat{C}_8.$$

Note that $\mathbf{M}_1^\lambda = \ldots = \mathbf{M}_8^\lambda = 1$ for every $\lambda \in \mathbb{C}$. Therefore, using Corollary 1.10.10, we conclude that $[\mathsf{f}^{-1}(\lambda)] = 1$ for every $\lambda \in \mathbb{C}$. Thus, we see that ($\heartsuit$) in the Main Theorem holds in this case, since $h^{1,2}(X) = 0$.

1.11. Extra notation. In Example 1.10.11, we explicitly decomposed the birational morphism α in (1.9.3) as a sequence of blow-ups. To verify ($\heartsuit$) in the Main Theorem, we have to do the same many times. To save space, let us introduce common notation that will be used in all these decompositions.

Recall that α is a composition of $k \geq 1$ blow-ups of points. Suppose we have the following commutative diagram:

$$\begin{tikzcd}
& U_2 \arrow[dl, "\alpha_2"'] & U_3 \arrow[l, "\alpha_3"'] & \ldots \arrow[l, "\alpha_4"'] & U_a \arrow[l, "\alpha_a"'] \\
U_1 \arrow[drr, "\alpha_1"'] & & & & \\
& & \mathbb{P}^3 & U \arrow[l, "\alpha"] \arrow[uur, "\gamma"'] &
\end{tikzcd}$$

where $a \leq k$, each α_i is a blow-up of a point, and γ is a (possibly biregular) birational morphism. Then we denote the exceptional divisor of α_i by $\mathbf{E}_i$. Moreover, for every $j \geq i$, we denote by $\mathbf{E}_i^j$ the proper transform of the divisor $\mathbf{E}_i$ on U_j. Furthermore, we will always assume that the proper transform of the surface $\mathbf{E}_i$ on U is the divisor $\widehat{E}_i$.

For every $\lambda \in \mathbb{C} \cup \{\infty\}$ and $i \leq a$, we denote by S_λ^i the proper transform of the quartic surface S_λ on the threefold U_i. Similarly, we denote by $\mathcal{S}^i$ the proper transform on U_i of the pencil $\mathcal{S}$, and we denote by D_λ^i the divisor in the pencil $\mathcal{S}^i$ that contains the surface S_λ^i. Then D_λ^i is just the image of the divisor $\widehat{D}_\lambda$ on the threefold U_i.

We denote by $C_1^i, \ldots, C_r^i$ the proper transforms on U_i of the curves $C_1, \ldots, C_r$, respectively. Similarly, if the surface $\mathbf{E}_i$ contains a base curve of the pencil $\mathcal{S}^i$, then we denote this curve by C_j^i for an appropriate $j > r$. We will always assume that its proper transform on the threefold U is the base curve $\widehat{C}_j$, which we introduced earlier.

1.12. Good double points. As we already saw in Example 1.8.6, in some cases all defects $\mathbf{D}_P^\lambda$ in (1.8.3) vanish, so that we do not need to blow up $\mathbb{P}^3$ to compute $[\mathsf{f}^{-1}(\lambda)]$. A handy observation is that

$$\mathbf{D}_P^\lambda = 0$$

for $P \in \Sigma$ if the rank of the quadratic form of the (local) defining equation of the quartic surface S_λ at the point P is at least 2. We will call such points *good* double points. This unifies du Val singular points of type $\mathbb{A}$ and non-isolated ordinary double points.

Lemma 1.12.1. *Let P be a fixed singular point in Σ. Suppose that P is a good double point of the surface S_λ. Then $\mathbf{D}_P^\lambda = 0$.*

Proof. By Corollary 1.10.4, we have $\mathbf{A}_P^\lambda = 0$. Therefore, it follows from (1.10.9) that we have to show that $\mathbf{C}_j^\lambda = 0$ for every $j > r$ such that $\alpha(\widehat{C}_j) = P$. Let $\widehat{E}_i$ be an α-exceptional surface such that $\alpha(\widehat{E}_i) = P$, and let $\widehat{C}_j$ be a base curve of $\widehat{\mathcal{S}}$ that is contained in $\widehat{E}_i$. By Lemma 1.10.7, it suffices to show that

$$\mathbf{M}_j^\lambda = \operatorname{mult}_{\widehat{C}_j}(\widehat{D}_\lambda) = 1.$$

To this end, we may assume that $\alpha\colon U \to \mathbb{P}^3$ is the blow-up of the point P and $\widehat{E}_i$ is the exceptional divisor of this blow-up. Then the restriction $\widehat{D}_\lambda|_{\widehat{E}_i}$ is a union of two distinct lines in $\widehat{E}_i \cong \mathbb{P}^2$. In particular, the surface $\widehat{D}_\lambda = \widehat{S}_\lambda$ is smooth at general points of any of these lines and the assertion follows. $\square$

Corollary 1.12.2. *Suppose that every fixed singular point of the pencil $\mathcal{S}$ is a good double point of the surface S_λ. Then*

$$\left[\mathsf{f}^{-1}(\lambda)\right] = [S_\lambda] + \sum_{i=1}^{r} \mathbf{C}_i^\lambda.$$

Let us show how to apply this corollary in one simple example.

Example 1.12.3. Suppose that X is contained in family 3.11 in [8]. Then its mirror partner is given by the Minkowski polynomial 1518, which is the Laurent polynomial

$$x + y + z + \frac{z}{x} + \frac{z}{y} + \frac{y}{x} + \frac{z}{xy} + \frac{y}{z} + \frac{1}{x} + \frac{1}{y} + \frac{y}{xz} + \frac{1}{xy}.$$

Thus, the pencil $\mathcal{S}$ is given by the equation

$$xyz^2 + x^2yz + xy^2z + xz^2t + yz^2t + y^2zt + z^2t^2 + xy^2t + xzt^2 + yzt^2 + y^2t^2 + zt^3 = \lambda xyzt,$$

and its base locus consists of the lines $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{z,t\}}$, $L_{\{y\},\{x,t\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{z\},\{x,t\}}$, $L_{\{t\},\{x,y,z\}}$, and the conic $\{x = y^2 + yz + zt = 0\}$. If $\lambda \neq -2, \infty$, then the surface S_λ has at worst du Val singularities, so that $[\mathsf{f}^{-1}(\lambda)] = 1$ by Corollary 1.5.4. On the other hand, we have $S_{-2} = H_{\{x,t\}} + \mathbf{S}$, where $\mathbf{S}$ is an irreducible cubic surface given by the equation $xyz + yz^2 + z^2t + zt^2 + y^2z + y^2t + yzt = 0$. Note also that S_{-2} is smooth at a general point of every base curve of the pencil $\mathcal{S}$. Thus, it follows from (1.8.3) that

$$\left[\mathsf{f}^{-1}(-2)\right] = 2 + \sum_{P \in \Sigma} \mathbf{D}_P^{-2}.$$

Furthermore, the set Σ consists of the points $P_{\{y\},\{z\},\{t\}}$, $P_{\{x\},\{z\},\{t\}}$, $P_{\{x\},\{y\},\{t\}}$, $P_{\{x\},\{t\},\{y,z\}}$, and $P_{\{y\},\{z\},\{x,t\}}$, and the quadratic terms of the Taylor expansions of the surface S_{-2} at these points can be described as follows:

$P_{\{y\},\{z\},\{t\}}$: quadratic term yz;

$P_{\{x\},\{z\},\{t\}}$: quadratic term $(x+t)(z+t)$;

$P_{\{x\},\{y\},\{t\}}$: quadratic term $(x+t)(y+t)$;

$P_{\{x\},\{t\},\{y,z\}}$: quadratic term $(x+t)(x+y+z-t) - (\lambda+2)xt$;

$P_{\{y\},\{z\},\{x,t\}}$: quadratic term $z(x+(\lambda+2)y+t)$.

By Corollary 1.12.2, we have $\mathbf{D}_P^{-2} = 0$ for every $P \in \Sigma$, so that $[\mathsf{f}^{-1}(-2)] = 2$. Thus, we see that ($\heartsuit$) in the Main Theorem holds in this case, since $h^{1,2}(X) = 1$.

1.13. Curves on singular quartic surfaces. We will prove ($\diamondsuit$) in the Main Theorem by computing the intersection form of the curves $C_1, \ldots, C_r$ on a general surface in the pencil $\mathcal{S}$. To this end, let $\Bbbk = \mathbb{C}(\lambda)$, let $S_\Bbbk$ be the quartic surface in $\mathbb{P}^3_\Bbbk$ given by (1.4.1), and let $\nu\colon \widetilde{S}_\Bbbk \to S_\Bbbk$ be the minimal resolution of singularities of the surface $S_\Bbbk$.

Lemma 1.13.1. *Suppose that λ is a general element of $\mathbb{C}$. Then the surface S_λ is singular, and it has du Val singularities. Let M be the $r \times r$ matrix with entries $M_{ij} \in \mathbb{Q}$ given by $M_{ij} = C_i \cdot C_j$, where $C_i \cdot C_j$ is the intersection of the curves C_i and C_j on the surface S_λ. Then the right-hand side of ($\diamondsuit$) is equal to*

$$20 - \operatorname{rk}\operatorname{Pic}(\widetilde{S}_\Bbbk) + \operatorname{rk}\operatorname{Pic}(S_\Bbbk) - \operatorname{rk} M.$$

Proof. The fact that the surface S_λ is singular and has du Val singularities can be verified by case-by-case analysis using the defining equation of the quartic surface S_λ. To prove the formula for the right-hand side of ($\diamondsuit$), we may assume that Z in ($\diamondsuit$) is the smooth threefold V constructed earlier in this section. Recall from Subsection 1.5 that $\widetilde{S}_\lambda$ is the proper transform of the quartic surface S_λ on the threefold V and $E_1, \ldots, E_n$ are the exceptional divisors of the constructed birational morphism $\pi\colon V \to \mathbb{P}^3$. Since $\widetilde{S}_\lambda$ is smooth, the morphism π induces the minimal resolution of singularities $\varpi\colon \widetilde{S}_\lambda \to S_\lambda$. Then

$$K_{\widetilde{S}_\lambda} \sim \varpi^*(K_{S_\lambda}) \sim 0,$$

because S_λ has du Val singularities. Thus, we see that $\widetilde{S}_\lambda$ is a smooth K3 surface, which implies $H^2(\widetilde{S}_\lambda, \mathbb{R}) \cong \mathbb{R}^{22}$. Therefore, the right-hand side of ($\diamondsuit$) is equal to

$$\dim\operatorname{coker}\bigl(H^2(V,\mathbb{R}) \to H^2(\widetilde{S}_\lambda,\mathbb{R})\bigr) - 2 = 20 - \dim\operatorname{im}\bigl(H^2\bigl(V,\mathbb{R}\bigr) \to H^2(\widetilde{S}_\lambda,\mathbb{R})\bigr).$$

But the cohomology group $H^2(V,\mathbb{R})$ is generated by the class of the divisor $\pi^*(\mathcal{O}_{\mathbb{P}^3}(1))$ and the classes of the π-exceptional surfaces $E_1, \ldots, E_n$. So, to complete the proof, we have to show that $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_\Bbbk/S_\Bbbk) + \operatorname{rk} M$ is the dimension of the vector subspace

$$\bigl\langle E_1|_{\widetilde{S}_\lambda}, \ldots, E_n|_{\widetilde{S}_\lambda}, \pi^*(\mathcal{O}_{\mathbb{P}^3}(1))|_{\widetilde{S}_\lambda} \bigr\rangle \subset H^2(\widetilde{S}_\lambda, \mathbb{R}).$$

Note that the class of the restriction $\pi^*(\mathcal{O}_{\mathbb{P}^3}(1))|_{\widetilde{S}_\lambda}$ is contained in the vector subspace in $H^2(\widetilde{S}_\lambda,\mathbb{R})$ spanned by the classes of the restrictions $E_1|_{\widetilde{S}_\lambda},\dots,E_n|_{\widetilde{S}_\lambda}$, because

$$\sum_{i=1}^r \mathbf{m}_i C_i \sim \pi^*(\mathcal{O}_{\mathbb{P}^3}(4))\big|_{S_\lambda}.$$

This shows that

$$\big\langle E_1|_{\widetilde{S}_\lambda},\dots,E_n|_{\widetilde{S}_\lambda},\pi^*(\mathcal{O}_{\mathbb{P}^3}(1))|_{\widetilde{S}_\lambda}\big\rangle=\big\langle E_1|_{\widetilde{S}_\lambda},\dots,E_n|_{\widetilde{S}_\lambda}\big\rangle\subset H^2(\widetilde{S}_\lambda,\mathbb{R}).$$

Furthermore, for every π-exceptional divisor E_i, the restriction $E_i|_{\widetilde{S}_\lambda}$ (if not empty) is either an effective divisor whose support consists of ϖ-exceptional curves or the strict transform on $\widetilde{S}_\lambda$ of one of the curves $C_1,\dots,C_r$ via the birational morphism ϖ. Moreover, we have

$$\dim\big\langle E_1|_{\widetilde{S}_\lambda},\dots,E_n|_{\widetilde{S}_\lambda}\big\rangle=\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk}/S_{\Bbbk})+\operatorname{rk}M,$$

because $\widetilde{S}_{\Bbbk}$ is a generic scheme fiber of the morphism $\mathsf{f}\colon V\to\mathbb{P}^1$ and $\widetilde{S}_\lambda$ is a sufficiently general fiber of this morphism. This gives the required equality. $\square$

Thus, to verify ($\diamondsuit$) in the Main Theorem, it suffices to show that

$$\operatorname{rk}\operatorname{Pic}(X)+\operatorname{rk}M+\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk})-\operatorname{rk}\operatorname{Pic}(S_{\Bbbk})=20,\tag{$\bigstar$}$$

where M is the intersection matrix defined in Lemma 1.13.1. For basic properties of the intersection of curves on surfaces with du Val singularities, see the Appendix.

Let us show how to check ($\bigstar$) in one case.

Example 1.13.2. Suppose that $X=\mathbb{P}^1\times\mathbb{P}^2$. This is family 2.34 in [8]. One of its mirror partners is given by the Minkowski polynomial 4, which is the Laurent polynomial

$$x+y+z+\frac{1}{x}+\frac{1}{yz}.$$

Then the pencil $\mathcal{S}$ is given by

$$x^2yz+y^2xz+z^2xy+t^2yz+t^3x=\lambda xyzt,$$

and its base locus consists of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, and $L_{\{t\},\{x,y,z\}}$. Suppose that $\lambda\neq\infty$. Then the singular points of S_λ contained in one of these lines are the points $P_{\{x\},\{y\},\{t\}}$ and $P_{\{x\},\{z\},\{t\}}$, which are singular points of type $\mathbb{A}_4$, the points $P_{\{y\},\{z\},\{t\}}$, $P_{\{y\},\{t\},\{x,z\}}$, and $P_{\{z\},\{t\},\{x,y\}}$, which are singular points of type $\mathbb{A}_2$, and the point $P_{\{x\},\{t\},\{y,z\}}$, which is an isolated ordinary double point of the surface S_λ. In particular, we see that ($\heartsuit$) in the Main Theorem holds by Corollary 1.5.4. Resolving the singularities of the quartic surface $S_{\Bbbk}$, we also see that

$$\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk})=\operatorname{rk}\operatorname{Pic}(S_{\Bbbk})+15.$$

Thus, to verify ($\bigstar$), we have to compute the rank of the intersection matrix of the lines $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, and $L_{\{t\},\{x,y,z\}}$ on the surface S_λ. This matrix has the same rank as the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, and H_λ, since

$$\begin{aligned}L_{\{x\},\{y\}}+L_{\{x\},\{z\}}+2L_{\{x\},\{t\}}&\sim L_{\{x\},\{y\}}+3L_{\{y\},\{t\}}\sim L_{\{x\},\{z\}}+3L_{\{z\},\{t\}}\\&\sim L_{\{x\},\{t\}}+L_{\{y\},\{t\}}+L_{\{z\},\{t\}}+L_{\{t\},\{x,y,z\}}\sim H_\lambda.\end{aligned}$$

These rational equivalences follow from the equalities

$$\begin{aligned}
H_{\{x\}} \cdot S_\lambda &= L_{\{x\},\{y\}} + L_{\{x\},\{z\}} + 2L_{\{x\},\{t\}},\\
H_{\{y\}} \cdot S_\lambda &= L_{\{x\},\{y\}} + 3L_{\{y\},\{t\}},\\
H_{\{z\}} \cdot S_\lambda &= L_{\{x\},\{z\}} + 3L_{\{z\},\{t\}},\\
H_{\{t\}} \cdot S_\lambda &= L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y,z\}}.
\end{aligned}$$

On the other hand, using Propositions A.1.2 and A.1.3 in the Appendix, we see that the intersection form of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, and H_λ on the surface S_λ is given by

$$\begin{array}{c} \\ L_{\{x\},\{y\}} \\ L_{\{x\},\{z\}} \\ H_\lambda \end{array}
\begin{array}{c} \begin{array}{ccc} L_{\{x\},\{y\}} & L_{\{x\},\{z\}} & H_\lambda \end{array} \\
\begin{pmatrix} -4/5 & 1 & 1 \\ 1 & -4/5 & 1 \\ 1 & 1 & 4 \end{pmatrix} \end{array}.$$

This matrix has rank 3, so that (★) holds in this case.

1.14. Scheme of the proof. In the remaining part of the paper, we prove (♡) and (♢) in the Main Theorem for every deformation family of smooth Fano threefolds, similar to what we did in Examples 1.7.1, 1.8.6, 1.10.11, 1.12.3, and 1.13.2. We will do this case by case, reserving one subsection per deformation family. The numbering of the families is taken from [8], and we group families with the same Picard rank in one section. For example, the subsection entitled "Family 4.1" contains the proof of the Main Theorem for family 4.1 in [8], which consists of smooth divisors of multidegree $(1,1,1,1)$ on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

In every case when $-K_X$ is very ample, we proceed as follows. First, we choose an appropriate toric Landau–Ginzburg model for the threefold X such that (1.4.2) exists for some pencil $\mathcal{S}$, which is given by equation (1.4.1). Second, we describe the base locus of this pencil. Third, we describe the singularities of every surface S_λ in the pencil $\mathcal{S}$ that are contained in the base locus of this pencil. This also gives an explicit construction of the birational map α in (1.9.3), which can be used to describe the minimal resolution of singularities $\nu\colon \widetilde{S}_{\Bbbk} \to S_{\Bbbk}$. Using it, we compute $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk}) - \operatorname{rk}\operatorname{Pic}(S_{\Bbbk})$, and verify (★) using the intersection theory on S_λ for general $\lambda \in \mathbb{C}$. To do this more efficiently, we use basic results on intersection of curves on singular surfaces, which we present in the Appendix.

If the singular points of S_λ contained in the base locus of the pencil $\mathcal{S}$ are all du Val for every $\lambda \neq \infty$, then we apply Corollary 1.5.4 to deduce (♡) in the Main Theorem. Similarly, if every fixed singular point is a good double point of every non-du Val surface S_λ in the pencil $\mathcal{S}$, then we can apply Corollary 1.12.2 together with Lemma 1.8.5 to compute the right-hand side of (♡) in the Main Theorem.

If the pencil $\mathcal{S}$ contains a non-du Val quartic surface S_λ that has a *bad* singularity at some fixed singular point $P \in \Sigma$, then we can compute the number of irreducible components of the fiber $\mathsf{f}^{-1}(\lambda)$ using (1.8.3). This gives us

$$\big[\mathsf{f}^{-1}(\lambda)\big] = [S_\lambda] + \sum_{i=1}^{r} \mathbf{C}_j^\lambda + \sum_{P\in\Sigma} \mathbf{D}_P^\lambda.$$

Here, the term $[S_\lambda]$ is easy to compute. Similarly, the second term in this formula can be computed using Lemma 1.8.5. Therefore, for every fixed singular point $P \in \Sigma$ that is neither du Val nor a good double point of the surface S_λ, we must compute its defect $\mathbf{D}_P^\lambda$.

To compute the defect $\mathbf{D}^{\lambda}_{P}$, we describe the birational morphism $\alpha\colon U \to \mathbb{P}^3$ in (1.9.3). This can be done locally in a neighborhood of the point P. Then we describe the divisor

$$\widehat{D}_{\lambda} = \widehat{S}_{\lambda} + \sum_{i=1}^{k} \mathbf{a}_i^{\lambda} \widehat{E}_i$$

in (1.10.1). In many cases, we can use Lemma 1.10.2 to show that some (or all) of the numbers $\mathbf{a}_1^{\lambda}, \dots, \mathbf{a}_k^{\lambda}$ vanish. But it is not hard to compute them in general.

Then we describe the base curves of the pencil $\widehat{\mathcal{S}}$, and compute the intersection multiplicities $\mathbf{m}_1, \dots, \mathbf{m}_s$ in (1.9.2) and the multiplicities $\mathbf{M}_1^{\lambda}, \dots, \mathbf{M}_s^{\lambda}$ in (1.10.6). For the proper transforms of the base curves of the pencil $\mathcal{S}$, these computations should have already been done at the previous steps. For the remaining base curves of the pencil $\widehat{\mathcal{S}}$, we can compute these numbers locally near every point in Σ. For each such point $P \in \Sigma$, we can compute its defect $\mathbf{D}^{\lambda}_{P}$ arguing as in Subsection 1.10. If the surface S_{λ} has a du Val singularity or a non-isolated ordinary double singularity at P, we can use Lemma 1.12.1 to deduce that its defect $\mathbf{D}^{\lambda}_{P}$ vanishes. This allows us to skip many local computations.

Finally, we use (1.8.3) to compute $[\mathsf{f}^{-1}(\lambda)]$ for every $\lambda \neq \infty$. This gives (♡) in the Main Theorem and completes the proof of the Main Theorem in the case when $-K_X$ is very ample.

Example 1.14.1. Suppose that the threefold X is contained in family 3.6 in [8]. Then X can be obtained by blowing up $\mathbb{P}^3$ in a disjoint union of a line and a smooth elliptic curve of degree 4, so that $h^{1,2}(X) = 1$. A toric Landau–Ginzburg model of the threefold X is given by the Minkowski polynomial 1899, which is

$$x + z + \frac{x}{z} + \frac{1}{xy} + \frac{z}{x} + \frac{1}{y} + \frac{1}{z} + 2y + \frac{3}{x} + \frac{yz}{x} + \frac{y}{z} + \frac{3y}{x} + \frac{y^2}{x}.$$

Then the corresponding pencil $\mathcal{S}$ is given by

$$x^2yz + xzt^2 + xyz^2 + x^2yt + zt^3 + yz^2t + xyt^2 + 2xy^2z + 3yzt^2 + y^2z^2 + xy^2t + 3y^2zt + y^3z = \lambda xyzt.$$

Suppose that $\lambda \neq \infty$. Let $\mathcal{C}$ be the conic $x = yz + (y+t)^2 = 0$. Then

$$\begin{aligned}
H_{\{x\}} \cdot S_{\lambda} &= L_{\{x\},\{z\}} + L_{\{x\},\{y,t\}} + \mathcal{C},\\
H_{\{y\}} \cdot S_{\lambda} &= L_{\{y\},\{z\}} + 2L_{\{y\},\{t\}} + L_{\{y\},\{x,t\}},\\
H_{\{z\}} \cdot S_{\lambda} &= L_{\{x\},\{z\}} + L_{\{y\},\{z\}} + L_{\{z\},\{t\}} + L_{\{z\},\{x,y,t\}},\\
H_{\{t\}} \cdot S_{\lambda} &= L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y\}} + L_{\{t\},\{x,y,z\}}.
\end{aligned} \tag{1.14.2}$$

Let $\mathbf{S}$ be an irreducible cubic surface given by $zt^2 + 2yzt + xyt + yz^2 + xyz + y^2z = 0$. Then $S_{-3} = H_{\{x,y,t\}} + \mathbf{S}$. If $\lambda \neq -3$, then S_{λ} is irreducible and its singularities contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

$P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_2$ with quadratic term $y(z+t)$;

$P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term $y(x+y+t)$;

$P_{\{x\},\{z\},\{y,t\}}$: type $\mathbb{A}_2$ with quadratic term $x(x+y+t-3z-\lambda z)$;

$P_{\{y\},\{z\},\{x,t\}}$: type $\mathbb{A}_1$ with quadratic term $4zy - (x+t)(y+z) - y^2 + \lambda yz$;

$P_{\{y\},\{t\},\{x,z\}}$: type $\mathbb{A}_1$ with quadratic term $2yt - t^2 - (x+z)y - y^2 + \lambda ty$;

$P_{\{z\},\{t\},\{x,y\}}$: type $\mathbb{A}_2$ with quadratic term $t(x+y+t-3z-\lambda z)$ for $\lambda \neq -4$, and type $\mathbb{A}_3$ for $\lambda = -4$.

These are the fixed singular points of the pencil $\mathcal{S}$. All of them are good double points of the surface S_{-3}. Now, using Corollaries 1.5.4 and 1.12.2, we obtain ($\heartsuit$) in the Main Theorem. To verify ($\bigstar$), we observe that $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk}) = \operatorname{rk}\operatorname{Pic}(S_{\Bbbk}) + 11$. Now we must compute the rank of the intersection matrix M in Lemma 1.13.1. We may assume that $\lambda \notin \{-4,-3\}$. Using (1.14.2), we see that M has the same rank as the intersection matrix of the curves $L_{\{x\},\{z\}}$, $L_{\{x\},\{y,t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{x,t\}}$, $L_{\{z\},\{x,y,t\}}$, $L_{\{t\},\{x,y,z\}}$, and H_λ, which is given by

$$\begin{array}{c} \\ L_{\{x\},\{z\}} \\ L_{\{x\},\{y,t\}} \\ L_{\{y\},\{z\}} \\ L_{\{y\},\{x,t\}} \\ L_{\{z\},\{x,y,t\}} \\ L_{\{t\},\{x,y,z\}} \\ H_\lambda \end{array}
\begin{array}{c} \begin{array}{ccccccc} L_{\{x\},\{z\}} & L_{\{x\},\{y,t\}} & L_{\{y\},\{z\}} & L_{\{y\},\{x,t\}} & L_{\{z\},\{x,y,t\}} & L_{\{t\},\{x,y,z\}} & H_\lambda \end{array} \\
\begin{pmatrix}
-4/3 & 2/3 & 1 & 0 & 1/3 & 0 & 1\\
2/3 & -7/12 & 0 & 1/2 & 1/3 & 0 & 1\\
1 & 0 & -5/6 & 1/2 & 1/2 & 0 & 1\\
0 & 1/2 & 1/2 & -1/2 & 1/2 & 0 & 1\\
1/3 & 1/3 & 1/2 & 1/2 & -1/6 & 1/3 & 1\\
0 & 0 & 0 & 0 & 1/3 & -5/6 & 1\\
1 & 1 & 1 & 1 & 1 & 1 & 4
\end{pmatrix} \end{array}.$$

It has rank 6, so that ($\bigstar$) holds, which gives ($\diamondsuit$) in the Main Theorem by Lemma 1.13.1.

In the remaining part of this paper, we will always use the notation of this section except for five families of smooth Fano threefolds whose anticanonical divisors are not very ample. These are families 2.1–2.3, 9.1, and 10.1 in [8]. We will deal with them in the corresponding subsections of Sections 2, 9, and 10. The proof of the Main Theorem in these cases is similar to that in the case when $-K_X$ is very ample. For instance, if $X = \mathbb{P}^1 \times \mathsf{S}_2$, where S_2 is a smooth del Pezzo surface of degree 2, the commutative diagram (1.4.2) also exists. But now by [16, Proposition 29] the pencil $\mathcal{S}$ is given by

$$x^3y = (\lambda yz - y^2 - z^2)(xt - xz - t^2),$$

where $\lambda \in \mathbb{C} \cup \{\infty\}$. In this case, which is family 9.1, we can still apply all the steps described above to prove the Main Theorem.

2. FANO THREEFOLDS OF PICARD RANK 2

Family 2.1. In this case, the threefold X can be obtained as a blow-up of a smooth sextic hypersurface in $\mathbb{P}(1,1,1,2,3)$ along a smooth elliptic curve. This implies that $h^{1,2}(X) = 22$. Note that $-K_X$ is not very ample. Because of this, there exists no Laurent polynomial with reflexive Newton polytope that gives the toric Landau–Ginzburg model of this deformation family. However, there are Laurent polynomials with non-reflexive Newton polytopes that give the commutative diagram (✠). One of them is

$$\frac{(r+s+1)^6(t+1)}{rs^2} + \frac{1}{t},$$

which we also denote by p.

Let $\gamma\colon \mathbb{C}^3 \dashrightarrow \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$ be a birational transformation given by the change of coordinates

$$r = \frac{1}{b} - \frac{1}{b^2c} - 1, \qquad s = \frac{1}{b^2c}, \qquad t = -\frac{1}{y} - 1.$$

Arguing as in Subsection 1.9, we can expand (✠) to the commutative diagram

$$\begin{tikzcd}
 & \mathbb{P}^2\times\mathbb{P}^1 \arrow[d, dashed, "\phi"] & \mathbb{C}^3 \arrow[l, hook'] \arrow[r, dashed, "\gamma"] \arrow[d, "\mathsf{q}"] & \mathbb{C}^*\times\mathbb{C}^*\times\mathbb{C}^* \arrow[r, hook] \arrow[d, "\mathsf{p}"] & Y \arrow[r, hook] \arrow[d, "\mathsf{w}"] & Z \arrow[d, "\mathsf{f}"] \\
V \arrow[ur, "\pi"] \arrow[r, "\mathsf{g}"'] & \mathbb{P}^1 & \mathbb{C}^1 \arrow[l, hook'] \arrow[r, equal] & \mathbb{C}^1 \arrow[r, equal] & \mathbb{C}^1 \arrow[r, hook] & \mathbb{P}^1
\end{tikzcd} \tag{2.1.1}$$

where q is a surjective morphism, π is a birational morphism, the threefold V is smooth, the map g is a surjective morphism such that $-K_V \sim \mathsf{g}^{-1}(\infty)$, and ϕ is a rational map given by the pencil

$$x(x+y)c^3 = y\big((x+y)\lambda + y\big)\big(abc - b^2c - a^3\big), \tag{2.1.2}$$

where $([x:y],[a:b:c])$ is a point in $\mathbb{P}^1 \times \mathbb{P}^2$ and $\lambda \in \mathbb{C} \cup \{\infty\}$.

The commutative diagram (2.1.1) is similar to the commutative diagram (1.4.2) presented in Subsection 1.4. As in (1.4.2), there exists a composition of flops $\chi\colon V \dashrightarrow Z$ that makes the following diagram commute:

$$\begin{tikzcd} V \arrow[r, dashed, "\chi"] \arrow[d, "\mathsf{g}"'] & Z \arrow[d, "\mathsf{f}"] \\ \mathbb{P}^1 \arrow[r, equal] & \mathbb{P}^1 \end{tikzcd}$$

So, to prove the Main Theorem in this case, we will follow the scheme described in Section 1. Moreover, we will use the same assumptions and notation as in the case when $-K_X$ is very ample. The only difference is that $\mathbb{P}^3$ is now replaced by $\mathbb{P}^1 \times \mathbb{P}^2$. For instance, we denote by $\mathcal{S}$ the pencil (2.1.2) and by S_λ the surface in $\mathcal{S}$ given by (2.1.2), where $\lambda \in \mathbb{C} \cup \{\infty\}$. Similarly, we extend the handy notation in Subsection 1.6 to bilinear sections of $\mathbb{P}^1 \times \mathbb{P}^2$. Note that the curve H_λ is not defined in this case.

Let S be the surface in $\mathbb{P}^1 \times \mathbb{P}^2$ given by $abc - b^2c - a^3 = 0$. Then S is irreducible and

$$S_\infty = H_{\{y\}} + H_{\{x,y\}} + \mathsf{S}.$$

Let $\mathbf{S}$ be the surface in $\mathbb{P}^1 \times \mathbb{P}^2$ given by the equation $xc^3 + yc^3 - yabc + yb^2c + ya^3 = 0$. Then $\mathbf{S}$ is irreducible and $S_{-1} = H_{\{x\}} + \mathbf{S}$. These are all reducible surfaces in $\mathcal{S}$.

To describe the base locus of the pencil $\mathcal{S}$, we observe that

$$H_{\{x,y\}} \cdot S_{-1} = \mathcal{C}_1, \qquad H_{\{y\}} \cdot S_{-1} = 3L_{\{y\},\{c\}}, \qquad \mathsf{S} \cdot S_{-1} = \mathcal{C}_1 + \mathcal{C}_2 + 9L_{\{a\},\{c\}}, \tag{2.1.3}$$

where $\mathcal{C}_1$ is the curve in $\mathbb{P}^1 \times \mathbb{P}^2$ given by $x + y = abc - b^2c - a^3 = 0$ and $\mathcal{C}_2$ is the curve in $\mathbb{P}^1 \times \mathbb{P}^2$ given by $x = abc - b^2c - a^3 = 0$. Thus, we have

$$S_{-1} \cdot S_\infty = 2\mathcal{C}_1 + \mathcal{C}_2 + 3L_{\{y\},\{c\}} + 9L_{\{a\},\{c\}},$$

so that the base locus of the pencil $\mathcal{S}$ consists of the curves $\mathcal{C}_1$, $\mathcal{C}_2$, $L_{\{y\},\{c\}}$, and $L_{\{a\},\{c\}}$.

To match the notation used in Subsection 1.8, we let $C_1 = \mathcal{C}_1$, $C_2 = \mathcal{C}_2$, $C_3 = L_{\{y\},\{c\}}$, and $C_4 = L_{\{a\},\{c\}}$. Then $\mathbf{m}_1 = 2$, $\mathbf{m}_2 = 2$, $\mathbf{m}_3 = 3$, and $\mathbf{m}_4 = 9$.

Observe that S_0 is singular along the curve $L_{\{y\},\{c\}}$. Moreover, if $\lambda \notin \{0, -1, \infty\}$, then the surface S_λ has isolated singularities. In this case the singular points of S_λ contained in the base locus of the pencil $\mathcal{S}$ are du Val and can be described as follows:

$$\begin{aligned} P_{\{y\},\{a\},\{c\}}&\colon\ \text{type } \mathbb{A}_8;\\ [\lambda+1:-\lambda]\times[0:1:0]&\colon\ \text{type } \mathbb{A}_8. \end{aligned}$$

Applying Corollary 1.5.4, we obtain the following.

Corollary 2.1.4. *The fiber* $\mathsf{f}^{-1}(\lambda)$ *is irreducible for every* $\lambda \notin \{0, -1, \infty\}$.

Observe that the point $P_{\{y\},\{a\},\{c\}}$ is the only fixed singular point of the pencil $\mathcal{S}$.

Remark 2.1.5. The base curve C_1 is singular at the point $P_{\{x,y\},\{a\},\{b\}}$. Similarly, the base curve C_2 is singular at the point $P_{\{x\},\{a\},\{b\}}$. Thus, in the notation of Subsection 1.9, both curves $\widehat{C}_1$ and $\widehat{C}_2$ are singular. This implies that the threefold V in (2.1.1) is singular: it has isolated ordinary double points. But this is not important for the proof of the Main Theorem in this case, because these singular points are contained in the fiber $\mathbf{g}^{-1}(\infty)$. Note that we can resolve them by composing the birational morphism π in (2.1.1) with a small resolution of these double points. However, the resulting smooth threefold would not be projective (cf. the proof of [16, Proposition 29]).

First, let us prove ($\diamondsuit$) in the Main Theorem. By Lemma 1.13.1, it follows from

Lemma 2.1.6. *Equality* ($\bigstar$) *holds.*

Proof. Suppose that $\lambda \notin \{0, -1, \infty\}$. Let H_λ be the intersection of the surface S_λ with a general surface in $\mathbb{P}^1 \times \mathbb{P}^2$ of bidegree $(0, 1)$. Then it follows from (2.1.3) that

$$C_1 + C_2 + 9C_4 \sim 3H_\lambda$$

and $C_1 \sim C_2 \sim 3C_3$ on the surface S_λ. Thus, the intersection matrix of the curves C_1, C_2, C_3, and C_4 on the surface S_λ has the same rank as the intersection matrix

$$\begin{pmatrix} C_1^2 & H_\lambda \cdot C_1 \\ H_\lambda \cdot C_1 & H_\lambda^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}.$$

On the other hand, we have $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk}) = \operatorname{rk}\operatorname{Pic}(S_{\Bbbk}) + 16$. This shows that ($\bigstar$) holds. $\square$

In the remaining part of this subsection, we will show that ($\heartsuit$) in the Main Theorem also holds in this case. To this end, we have to compute $[\mathsf{f}^{-1}(-1)]$ and $[\mathsf{f}^{-1}(0)]$. We start with

Lemma 2.1.7. *One has* $[\mathsf{f}^{-1}(-1)] = 2$.

Proof. As we have already mentioned, the point $P_{\{y\},\{a\},\{c\}}$ is the only fixed singular point of the pencil $\mathcal{S}$. The surface S_{-1} has a du Val singularity of type $\mathbb{A}_8$ at it. Since

$$\mathbf{M}_1^{-1} = \mathbf{M}_2^{-1} = \mathbf{M}_3^{-1} = \mathbf{M}_4^{-1} = 1,$$

we use Corollary 1.12.2 to deduce that $[\mathsf{f}^{-1}(-1)] = [S_{-1}] = 2$. $\square$

To compute $[\mathsf{f}^{-1}(0)]$, observe that $\mathbf{M}_1^0 = 1$, $\mathbf{M}_2^0 = 1$, $\mathbf{M}_3^0 = 2$, and $\mathbf{M}_4^0 = 1$. Thus, it follows from (1.8.3) and Lemma 1.8.5 that

$$[\mathsf{f}^{-1}(0)] = 3 + \mathbf{D}^0_{P_{\{y\},\{a\},\{c\}}}, \tag{2.1.8}$$

where $\mathbf{D}^0_{P_{\{y\},\{a\},\{c\}}}$ is the defect of the singular point $P_{\{y\},\{a\},\{c\}}$ defined in Subsection 1.8. The defect $\mathbf{D}^0_{P_{\{y\},\{a\},\{c\}}}$ can be computed locally near the point $P_{\{y\},\{a\},\{c\}}$. The recipe to compute it is given in Subsection 1.10. Let us use it.

Suppose that $\lambda \neq \infty$. Consider a local chart $x = b = 1$. Then the surface S_λ in this chart is given by

$$-\lambda yc + c\big(c^2 + \lambda ya - \lambda y^2 - y^2\big) + y\big(c^3 - \lambda a^3 + \lambda yac + yac\big) - (\lambda + 1)(y^2a^3) = 0.$$

Let $\alpha_1\colon U_1 \to \mathbb{P}^1 \times \mathbb{P}^2$ be the blow-up of the point $P_{\{y\},\{a\},\{c\}}$. A chart of the blow-up α_1 is given by the coordinate change $a_1 = a$, $y_1 = y/a$, $c_1 = c/a$. In this chart, the surface D^1_λ is given by the equation

$$-\lambda y_1c_1 + \lambda y_1a_1(c_1 - a_1) + a_1c_1\big(c_1^2 - \lambda y_1^2 - y_1^2\big) + y_1^2a_1^2(\lambda + 1)(c_1 - a_1) + a_1^2c_1^3y_1 = 0,$$

where $a_1 = 0$ defines the exceptional surface $\mathbf{E}_1$. Then $\mathbf{E}_1$ contains two base curves of the pencil $\mathcal{S}^1$. One of them is given by $a_1 = y_1 = 0$, and the other, by $a_1 = c_1 = 0$. Denote the former curve by C_5^1 and the latter by C_6^1.

If $\lambda \neq 0$, then the point $(a_1, y_1, c_1) = (0, 0, 0)$ is the only singular point of the surface D^1_λ that is contained in $\mathbf{E}_1$. Let $\alpha_2\colon U_2 \to U_1$ be the blow-up of this point. A chart of the blow-up α_2 is given by the coordinate change $a_2 = a_1$, $y_2 = y_1/a_1$, $c_2 = c_1/a_1$. Let $\widehat{y}_2 = y_2$, $\widehat{a}_2 = a_2$, and $\widehat{c}_2 = a_2 + c_2$. Then D^2_λ is given by

$$-\lambda\widehat{y}_2\widehat{c}_2 + \lambda\widehat{y}_2\widehat{a}_2(\widehat{c}_2 - \widehat{a}_2) + \widehat{a}_2^2\big(\widehat{c}_2^3 - \widehat{a}_2^3 + 3\widehat{a}_2^2\widehat{c}_2 - 3\widehat{c}_2^2\widehat{a}_2 - \lambda\widehat{y}_2^2\widehat{c}_2 - \widehat{y}_2^2\widehat{c}_2\big) \\ + (\lambda+1)\widehat{y}_2^2\widehat{a}_2^3(\widehat{c}_2 - \widehat{a}_2) + \widehat{a}_2^4\widehat{y}_2(\widehat{c}_2 - \widehat{a}_2)^3 = 0,$$

and $\mathbf{E}_2$ is given by $\widehat{a}_2 = 0$. Then $\mathbf{E}_2$ contains two base curves of the pencil $\mathcal{S}^2$. One of them is given by $\widehat{a}_2 = \widehat{y}_2 = 0$, and the other, by $\widehat{a}_2 = \widehat{c}_2 = 0$. Denote the former curve by C^2_7 and the latter by C^2_8.

If $\lambda \neq 0$, then $(\widehat{a}_2, \widehat{y}_2, \widehat{c}_2) = (0,0,0)$ is the only singular point of the surface D^2_λ that is contained in $\mathbf{E}_2$. Let $\alpha_3 \colon U_3 \to U_2$ be the blow-up of this point. A chart of this blow-up is given by the coordinate change $\widehat{a}_3 = \widehat{a}_2$, $\widehat{y}_3 = \widehat{y}_2/\widehat{a}_2$, $\widehat{c}_3 = \widehat{c}_2/\widehat{a}_2$. Let $\overline{y}_3 = \widehat{y}_3$, $\overline{a}_3 = \widehat{a}_3$, and $\overline{c}_3 = \widehat{a}_3 + \widehat{c}_3$. Denote by $\mathbf{E}_2$ the exceptional surface of the blow-up α_2. Then D^3_λ is given by

$$-\lambda\overline{y}_3\overline{c}_3 + \lambda\overline{a}_3\overline{y}_3\overline{c}_3 - \overline{a}_3^3 - \lambda\overline{y}_3\overline{a}_3^2 + 3\overline{a}_3^3(\overline{c}_3 - \overline{a}_3) - 3\overline{a}_3^3(\overline{c}_3 - \overline{a}_3)^2 \\ + \overline{a}_3^3\big(\overline{c}_3^3 - \overline{a}_3^3 + 3\overline{a}_3^2\overline{c}_3 - 3\overline{a}_3\overline{c}_3^2 - \lambda\overline{y}_3^2\overline{c}_3 - \overline{y}_3^2\overline{c}_3\big) + \overline{y}_3\overline{a}_3^4\big(\lambda\overline{y}_3\overline{c}_3 + \overline{y}_3\overline{c}_3 - \overline{a}_3^2 - \lambda\overline{y}_3\overline{a}_3 - \overline{y}_3\overline{a}_3\big) \\ + 3\overline{y}_3\overline{a}_3^6(\overline{c}_3 - \overline{a}_3) - 3\overline{y}_3\overline{a}_3^6(\overline{c}_3 - \overline{a}_3)^2 + \overline{y}_3\overline{a}_3^6(\overline{c}_3 - \overline{a}_3)^3 = 0,$$

and $\mathbf{E}_3$ is given by $\overline{a}_3 = 0$. Then $\mathbf{E}_3$ contains two base curves of the pencil $\mathcal{S}^3$. One of them is given by $\overline{a}_3 = \overline{y}_3 = 0$, and the other, by $\overline{a}_3 = \overline{c}_3 = 0$. Denote the former curve by C^3_9 and the latter by C^3_{10}.

There exists a commutative diagram

$$\begin{tikzcd}
 & U_2 \arrow[dl, "\alpha_2"'] & & U_3 \arrow[ll, "\alpha_3"'] & \\
U_1 \arrow[drr, "\alpha_1"'] & & & & U \arrow[ul, "\alpha_4"'] \arrow[dll, "\alpha"] \\
 & & \mathbb{P}^3 & &
\end{tikzcd}$$

where α_4 is the blow-up of the point $(\overline{a}_3, \overline{y}_3, \overline{c}_3) = (0,0,0)$. Note that $\widehat{E}_4$ contains two base curves of the pencil $\widehat{\mathcal{S}}$. Denote them by $\widehat{C}_{11}$ and $\widehat{C}_{12}$. Then $\widehat{C}_1, \ldots, \widehat{C}_{12}$ are all base curves of the pencil $\widehat{\mathcal{S}}$, because

$$\widehat{S}_{\lambda_1} \cdot \widehat{S}_{\lambda_2} = 2\widehat{C}_1 + \widehat{C}_2 + 3\widehat{C}_3 + 9\widehat{C}_4 + \widehat{C}_5 + 7\widehat{C}_6 + 2\widehat{C}_7 + 5\widehat{C}_8 + 3\widehat{C}_9 + 3\widehat{C}_{10} + \widehat{C}_{11} + \widehat{C}_{12}$$

for two general λ_1 and λ_2 in $\mathbb{C}$. This also shows that $\mathbf{m}_5 = 1$, $\mathbf{m}_6 = 7$, $\mathbf{m}_7 = 2$, $\mathbf{m}_8 = 5$, $\mathbf{m}_9 = 3$, $\mathbf{m}_{10} = 3$, $\mathbf{m}_{11} = 1$, and $\mathbf{m}_{12} = 1$.

Let us compute the term $\mathbf{A}^0_{P_{\{y\},\{a\},\{c\}}}$ in (1.10.9). We have $\widehat{D}_0 = \widehat{S}_0 + \widehat{E}_1 + 2\widehat{E}_2 + 3\widehat{E}_3 + \widehat{E}_4$. This gives $\mathbf{A}^0_{P_{\{y\},\{a\},\{c\}}} = 4$. Note also that $\mathbf{M}^0_5 = 1$, $\mathbf{M}^0_6 = 2$, $\mathbf{M}^0_7 = 2$, $\mathbf{M}^0_8 = 3$, $\mathbf{M}^0_9 = 3$, $\mathbf{M}^0_{10} = 3$, $\mathbf{M}^0_{11} = 1$, and $\mathbf{M}^0_{12} = 1$. Thus, it follows from (1.10.9) that

$$\mathbf{D}^0_P = 4 + \sum_{i=1}^{12} \mathbf{C}^0_i,$$

where $\mathbf{C}^0_i$ is the number defined in (1.10.5). By Lemma 1.10.7, we have

$$\mathbf{C}^0_i = \begin{cases} 0 & \text{if } \ \mathbf{M}^0_i = 1, \\ \mathbf{m}_i - 1 & \text{if } \ \mathbf{M}^0_i \geq 2. \end{cases}$$

Therefore, we have $\mathbf{D}^0_P = 19$. Now, using (2.1.8), we deduce that $[\mathsf{f}^{-1}(0)] = 22$. Keeping in mind that $h^{1,2}(X) = 22$ and $[\mathsf{f}^{-1}(-1)] = 2$, we see that ($\heartsuit$) in the Main Theorem holds.

Family 2.2. In this case, the threefold X is a double cover of $\mathbb{P}^1 \times \mathbb{P}^2$ ramified in a surface of bidegree $(2,4)$. This implies that $h^{1,2}(X) = 20$. As in the previous case, the divisor $-K_X$ is not very ample, and there are no toric Landau–Ginzburg models with reflexive Newton polytope in this case. However, we can find a Laurent polynomial p with non-reflexive Newton polytope that gives the commutative diagram ($\maltese$). For instance, we can choose p to be the Laurent polynomial

$$\frac{(a+b+c+1)^2}{a} + \frac{(a+b+c+1)^4}{bc}.$$

Let $\gamma\colon \mathbb{C}^3 \dashrightarrow \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$ be a birational transformation given by the change of coordinates

$$a = xy, \qquad b = yz, \qquad c = z - xy - yz - 1.$$

By [16, Proposition 16], we can expand ($\maltese$) to the commutative diagram

$$\begin{tikzcd}
& \mathbb{P}^3 \arrow[d, dashed, "\phi"] & \mathbb{C}^3 \arrow[l, hook'] \arrow[r, dashed, "\gamma"] \arrow[d, "\mathsf{q}"] & \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^* \arrow[r, hook] \arrow[d, "\mathsf{p}"'] & Y \arrow[r, hook] \arrow[d, "\mathsf{w}"] & Z \arrow[d, "\mathsf{f}"] \\
V \arrow[ur, "\pi"] \arrow[r, "\mathsf{g}"'] & \mathbb{P}^1 & \mathbb{C}^1 \arrow[l, hook'] \arrow[r, equal] & \mathbb{C}^1 \arrow[r, equal] & \mathbb{C}^1 \arrow[r, hook] & \mathbb{P}^1
\end{tikzcd} \tag{2.2.1}$$

where q is a surjective morphism, π is a birational morphism, the threefold V is smooth, the map g is a surjective morphism such that $-K_V \sim \mathsf{g}^{-1}(\infty)$, and ϕ is a rational map defined by a pencil of quartic surfaces $\mathcal{S}$ given by

$$xz^3 = \big(zt - xy - yz - t^2\big)(\lambda xy - z^2), \tag{2.2.2}$$

where $\lambda \in \mathbb{C} \cup \{\infty\}$. Note that a general fiber of the morphism g is a smooth K3 surface. Thus, a general surface in the pencil (2.2.2) has at worst du Val singularities.

The diagram (2.2.1) is very similar to the diagram (1.4.2) presented in Subsection 1.4. The only difference is that the pencil $\mathcal{S}$ is now given by equation (2.2.2). Because of this, we will follow the scheme described in Section 1 and use the assumptions and the notation introduced in this section.

As in Section 1, we denote by S_λ the surface in $\mathcal{S}$ given by (2.2.2). Then

$$S_\infty = H_{\{x\}} + H_{\{y\}} + \mathbf{Q},$$

where $\mathbf{Q}$ is the quadric in $\mathbb{P}^3$ given by $zt - xy - yz - t^2 = 0$. If $\lambda \neq \infty$, then

$$H_{\{x\}} \cdot S_\lambda = 2L_{\{x\},\{z\}} + \mathcal{C}_1, \qquad H_{\{y\}} \cdot S_\lambda = 2L_{\{y\},\{z\}} + \mathcal{C}_2, \qquad \mathbf{Q} \cdot S_\lambda = \mathcal{C}_1 + 3\mathcal{C}_3, \tag{2.2.3}$$

where $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ are conics in $\mathbb{P}^3$ that are given by the equations $x = zt - yz - t^2 = 0$, $y = xz - zt + t^2 = 0$, and $z = xy - t^2 = 0$, respectively. It follows from (2.2.3) that the base locus of the pencil $\mathcal{S}$ consists of the curves $L_{\{x\},\{z\}}$, $L_{\{y\},\{z\}}$, $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$.

We already know that the surface S_∞ is reducible. The surface S_0 is also reducible. Indeed, we have $S_0 = 2H_{\{z\}} + \mathsf{Q}$, where Q is a quadric surface given by $xz - yz + zt - t^2 - xy = 0$. On the other hand, if $\lambda \neq \infty, 0$, then the surface S_λ has isolated singularities, which implies that it is irreducible.

If $\lambda \neq \infty, 0$, then the singular points of S_λ contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

$P_{\{x\},\{y\},\{z\}}$: type $\mathbb{A}_1$ with quadratic term $\lambda xy - z^2$;

$P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_9$ (see the proof of Lemma 2.2.7 below) with quadratic term $\lambda x(x+z)$;

$P_{\{y\},\{z\},\{t\}}$: type $\mathbb{E}_6$ (see the proof of Lemma 2.2.8 below) with quadratic term λy^2.

If $\lambda \neq 0, \infty$, then the intersection matrix of the curves $L_{\{x\},\{z\}}$, $L_{\{y\},\{z\}}$, $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ on the surface S_λ has the same rank as the intersection matrix of the curves $L_{\{x\},\{z\}}$, $L_{\{y\},\{z\}}$, and H_λ, because

$$H_\lambda \sim 2L_{\{x\},\{z\}} + \mathcal{C}_1 \sim 2L_{\{y\},\{z\}} + \mathcal{C}_2 \sim_{\mathbb{Q}} \frac{1}{2}\mathcal{C}_1 + \frac{3}{2}\mathcal{C}_3$$

on the surface S_λ. This follows from (2.2.3). On the other hand, the following holds.

Lemma 2.2.4. *Suppose that $\lambda \neq 0, \infty$. Then the intersection matrix of the curves $L_{\{x\},\{z\}}$, $L_{\{y\},\{z\}}$, and H_λ on the surface S_λ is given by*

$$\begin{array}{c} \\ L_{\{x\},\{z\}} \\ L_{\{y\},\{z\}} \\ H_\lambda \end{array} \begin{array}{c} \begin{array}{ccc} L_{\{x\},\{z\}} & L_{\{y\},\{z\}} & H_\lambda \end{array} \\ \begin{pmatrix} 1/10 & 1/2 & 1 \\ 1/2 & -1/6 & 1 \\ 1 & 1 & 4 \end{pmatrix} \end{array}.$$

Proof. The equalities $H_\lambda^2 = 4$ and $H_\lambda \cdot L_{\{x\},\{z\}} = H_\lambda \cdot L_{\{y\},\{z\}} = 1$ are obvious. Note that

$$H_{\{z\}} \cdot S_\lambda = L_{\{x\},\{z\}} + L_{\{y\},\{z\}} + \mathcal{C}_3.$$

Thus, on the surface S_λ, we have

$$H_\lambda \sim L_{\{x\},\{z\}} + L_{\{y\},\{z\}} + \mathcal{C}_3 \sim_{\mathbb{Q}} L_{\{x\},\{z\}} + L_{\{y\},\{z\}} + \frac{1}{3}\big(2H_\lambda - \mathcal{C}_1\big)$$
$$\sim_{\mathbb{Q}} L_{\{x\},\{z\}} + L_{\{y\},\{z\}} + \frac{1}{3}\big(H_\lambda + 2L_{\{x\},\{z\}}\big) \sim_{\mathbb{Q}} \frac{5}{3}L_{\{x\},\{z\}} + L_{\{y\},\{z\}} + \frac{1}{3}H_\lambda,$$

so that

$$L_{\{x\},\{z\}} \sim_{\mathbb{Q}} \frac{2}{5}H_\lambda - \frac{3}{5}L_{\{y\},\{z\}}.$$

Therefore, to complete the proof, it suffices to compute the numbers $L_{\{y\},\{z\}}^2$ and $L_{\{y\},\{z\}} \cdot L_{\{x\},\{z\}}$.

Observe that $P_{\{y\},\{z\},\{t\}}$ and $P_{\{x\},\{y\},\{z\}}$ are the only singular points of S_λ contained in the line $L_{\{y\},\{z\}}$. So, by Proposition A.1.3 (see the Appendix), we get $L_{\{y\},\{z\}}^2 = -2 + 4/3 + 1/2 = -1/6$.

Since $L_{\{y\},\{z\}} \cap L_{\{x\},\{z\}} = P_{\{x\},\{y\},\{z\}}$, Proposition A.1.2 gives $L_{\{y\},\{z\}} \cdot L_{\{x\},\{z\}} = 1/2$. $\square$

The matrix in Lemma 2.2.4 has rank 2. Moreover, it follows from the proofs of Lemmas 2.2.7 and 2.2.8 below that $\mathrm{rk}\,\mathrm{Pic}(\widetilde{S}_{\Bbbk}) = \mathrm{rk}\,\mathrm{Pic}(S_{\Bbbk}) + 16$. Thus, we see that ($\bigstar$) holds. Therefore, by Lemma 1.13.1, we see that ($\diamondsuit$) in the Main Theorem also holds.

To prove ($\heartsuit$) in the Main Theorem, we observe that $[\mathsf{f}^{-1}(\lambda)] = 1$ for every $\lambda \notin \{0, \infty\}$. This follows from Corollary 1.5.4. Therefore, to verify ($\heartsuit$) in the Main Theorem, we have to show that $[\mathsf{f}^{-1}(0)] = 21$. We will do this in the remaining part of this subsection.

To match the notation introduced in Subsection 1.8, we let $C_1 = \mathcal{C}_1$, $C_2 = \mathcal{C}_2$, $C_3 = \mathcal{C}_3$, $C_4 = L_{\{x\},\{z\}}$, and $C_5 = L_{\{y\},\{z\}}$. Then (2.2.3) gives

$$S_0 \cdot S_\infty = 2C_1 + C_2 + 3C_3 + 2C_4 + 2C_5,$$

so that $\mathbf{m}_1 = 2$, $\mathbf{m}_2 = 1$, $\mathbf{m}_3 = 3$, and $\mathbf{m}_4 = \mathbf{m}_5 = 2$. Moreover, one has $\mathbf{M}_1^0 = \mathbf{M}_2^0 = 1$ and $\mathbf{M}_3^0 = \mathbf{M}_4^0 = \mathbf{M}_5^0 = 2$. Then $\mathbf{C}_1^0 = \mathbf{C}_1^0 = 0$, $\mathbf{C}_3^0 = 2$, and $\mathbf{C}_4^0 = \mathbf{C}_5^0 = 1$ by Lemma 1.8.5. Thus, using $[S_0] = 2$ and (1.8.3), we see that

$$[\mathsf{f}^{-1}(0)] = 6 + \mathbf{D}^0_{P_{\{x\},\{y\},\{z\}}} + \mathbf{D}^0_{P_{\{x\},\{z\},\{t\}}} + \mathbf{D}^0_{P_{\{y\},\{z\},\{t\}}}, \tag{2.2.5}$$

where $\mathbf{D}^0_{P_{\{x\},\{y\},\{z\}}}$, $\mathbf{D}^0_{P_{\{x\},\{z\},\{t\}}}$, and $\mathbf{D}^0_{P_{\{y\},\{z\},\{t\}}}$ are the defects of the singular points $P_{\{x\},\{y\},\{z\}}$, $P_{\{x\},\{z\},\{t\}}$, and $P_{\{y\},\{z\},\{t\}}$, respectively. For a precise definition of defects, see (1.8.2).

Lemma 2.2.6. *One has $\mathbf{D}^0_{P_{\{x\},\{y\},\{z\}}} = 0$.*

Proof. The required assertion follows from (1.10.9), because $P_{\{x\},\{y\},\{z\}}$ is a double point of the surface S_0 and the quadratic term of the surface S_λ at this point is $\lambda xy - z^2$. $\square$

Lemma 2.2.7. *One has* $\mathbf{D}^0_{P_{\{x\},\{z\},\{t\}}} = 10$.

Proof. In the chart $y = 1$, the surface S_λ is given by the equation

$$\lambda x(x+z) - \big(xz^2 + z^3 + \lambda xzt - \lambda xt^2\big) + z^2(xz + zt - t^2) = 0,$$

where $P_{\{x\},\{z\},\{t\}} = (0,0,0)$. We can rewrite this equation as

$$\lambda\widehat{x}\widehat{z} + \big(\lambda\widehat{x}\widehat{t}^{\,2} - \lambda\widehat{x}\widehat{z}\widehat{t} + \lambda\widehat{x}^2\widehat{t} + 2\widehat{x}\widehat{z}^2 - \widehat{x}^2\widehat{z} - \widehat{z}^3\big) + (\widehat{x} - \widehat{z})^2\big(\widehat{x}\widehat{z} + \widehat{z}\widehat{t} - \widehat{x}^2 - \widehat{x}\widehat{t} - \widehat{t}^{\,2}\big) = 0,$$

where $\widehat{x} = x$, $\widehat{z} = x + z$, and $\widehat{t} = t$.

Let $\alpha_1\colon U_1 \to \mathbb{P}^3$ be the blow-up of the point $P_{\{x\},\{z\},\{t\}}$. A chart of the blow-up α_1 is given by the coordinate change $\widehat{x}_1 = \widehat{x}/\widehat{t}$, $\widehat{z}_1 = \widehat{z}/\widehat{t}$, $\widehat{t}_1 = \widehat{t}$. Let $\overline{x}_1 = \widehat{x}_1$, $\overline{z}_1 = \widehat{z}_1 + \widehat{t}_1$, and $\bar{t}_1 = \widehat{t}_1$. Then S^1_λ is given by the equation

$$\lambda\overline{x}_1\overline{z}_1 + \lambda\overline{x}_1\bar{t}_1(\overline{x}_1 - \overline{z}_1 + \bar{t}_1) - \overline{z}_1\bar{t}_1(\overline{x}_1 - \overline{z}_1 + \bar{t}_1)^2 - \bar{t}_1^2(\overline{x}_1 - \overline{z}_1 + \bar{t}_1)^3 - \overline{x}_1\bar{t}_1^2(\overline{x}_1 - \overline{z}_1 + \bar{t}_1)^3 = 0$$

for every $\lambda \neq 0$. If $\lambda = 0$, this equation defines $D^1_0 = S^1_0 + \mathbf{E}_1$. By (1.10.3) and (1.10.9), this contributes 1 to the defect $\mathbf{D}^0_{P_{\{x\},\{z\},\{t\}}}$. Here and below we circle each contribution for the reader's convenience.

Note that $\mathbf{E}_1$ is given by $\bar{t}_1 = 0$. This shows that $\mathbf{E}_1$ contains two base curves of the pencil $\mathcal{S}^1$. One of them is given by $\overline{x}_1 = \bar{t}_1 = 0$, and the other, by $\overline{z}_1 = \bar{t}_1 = 0$. We denote the former curve by C^1_6 and the latter by C^1_7. Then $S^1_0 + \mathbf{E}_1$ is smooth at a general point of the curve C_6, so that this base curve does not give an extra contribution to the defect by Lemma 1.10.7 and (1.10.9). On the other hand, we have

$$\mathrm{mult}_{C^1_7}(S^1_0 + \mathbf{E}_1) = \mathbf{M}^0_7 = \mathbf{m}_7 = \mathrm{mult}_{C^1_7}\big((S^1_0 + \mathbf{E}_1)\cdot S^1_\lambda\big) = 2,$$

where $\lambda \neq 0$. By Lemma 1.10.7 and (1.10.9), the curve C^1_7 contributes 1 to the defect.

Let $\alpha_2\colon U_2 \to U_1$ be the blow-up of the point $C^1_6 \cap C^1_7$. Then $D^2_0 = S^2_0 + \mathbf{E}^2_1 + 2\mathbf{E}_2$. By (1.10.3) and (1.10.9), this contributes 1 to $\mathbf{D}^0_{P_{\{x\},\{z\},\{t\}}}$.

A chart of the blow-up α_2 is given by the coordinate change $\overline{x}_2 = \overline{x}_1/\bar{t}_1$, $\overline{z}_2 = \overline{z}_1/\bar{t}_1$, $\bar{t}_2 = \bar{t}_1$. Let $\breve{x}_2 = \overline{x}_2$, $\breve{z}_2 = \overline{z}_2 + \bar{t}_2$, and $\breve{t}_2 = \bar{t}_2$. Then $\mathbf{E}_2$ is given by $\bar{t}_2 = 0$, and D^2_λ is given by

$$\begin{aligned}\lambda\breve{x}_2\breve{z}_2 + \breve{t}_2\big(\lambda\breve{x}_2\breve{t}_2 - \breve{z}_2\breve{t}_2 + \lambda\breve{x}_2^2 - \lambda\breve{x}_2\breve{z}_2\big) - \breve{t}_2^2(\breve{t}_2 + 2\breve{z}_2)(\breve{x}_2 - \breve{z}_2 + \breve{t}_2)&\\ - \breve{t}_2^2\big(\breve{x}_2^2\breve{z}_2 - 3\breve{z}_2\breve{t}_2^2 + 2\breve{t}_2^3 - 2\breve{x}_2\breve{z}_2^2 + \breve{z}_2^3 - 2\breve{x}_2\breve{z}_2\breve{t}_2 + 2\breve{x}_2^2\breve{t}_2 + 5\breve{x}_2\breve{t}_2^2\big)&\\ - \breve{t}_2^3(\breve{x}_2 - \breve{z}_2 + \breve{t}_2)\big(\breve{z}_2^2 - 2\breve{x}_2\breve{z}_2 - 2\breve{z}_2\breve{t}_2 + \breve{t}_2^2 + 5\breve{x}_2\breve{t}_2 + \breve{x}_2^2\big)&\\ - 3\breve{x}_2\breve{t}_2^4(\breve{x}_2 - \breve{z}_2 + \breve{t}_2)^2 - \breve{x}_2\breve{t}_2^4(\breve{x}_2 - \breve{z}_2 + \breve{t}_2)^3 = 0.&\end{aligned}$$

The pencil $\mathcal{S}^2$ has two base curves contained in the surface $\mathbf{E}_2$. One of them is given by the equation $\overline{x}_2 = \bar{t}_2 = 0$, and the other, by the equation $\bar{t}_2 + \overline{z}_2 = \bar{t}_2 = 0$. Denote the former curve by C^2_8 and the latter by C^2_9. Then

$$\mathrm{mult}_{C^2_8}(S^2_0 + \mathbf{E}^2_1 + 2\mathbf{E}_2) = \mathbf{M}^0_8 = \mathbf{m}_8 = \mathrm{mult}_{C^2_8}\big((S^2_0 + \mathbf{E}^2_1 + 2\mathbf{E}_2)\cdot S^2_\lambda\big) = 2,$$

where $\lambda \neq 0$. Thus, this curve contributes 1 to the defect by Lemma 1.10.7 and (1.10.9). On the other hand, we have $\mathbf{M}^0_9 = 3$ and $\mathbf{m}_9 = 4$, because $S^2_0 + \mathbf{E}^2_1 + 2\mathbf{E}_2$ is given by

$$\breve{t}_2^4\breve{x}_2\big(\breve{x}_2 + \breve{t}_2 - \breve{z}_2 + 1\big)^3 + \breve{t}_2^2\big(\breve{t}_2^2 + \breve{t}_2\breve{x}_2 - \breve{t}_2\breve{z}_2 + \breve{z}_2\big)\big(\breve{x}_2 + \breve{t}_2 - \breve{z}_2 + 1\big)^2 = 0,$$

and S^2_∞ is given by $\breve{x}_2(\breve{t}_2^2+\breve{t}_2\breve{x}_2-\breve{t}_2\breve{z}_2+\breve{z}_2)=0$. Thus, by Lemma 1.10.7 and (1.10.9), the curve C^2_9 contributes ③ to the defect $\mathbf{D}^0_{P_{\{x\},\{z\},\{t\}}}$.

Let $\alpha_3\colon U_3\to U_2$ be the blow-up of the point $C^2_8\cap C^2_9$. Then $D^3_0=S^3_0+\mathbf{E}^3_1+2\mathbf{E}^3_2+\mathbf{E}_3$. By (1.10.3) and (1.10.9), this contributes ① to $\mathbf{D}^0_{P_{\{x\},\{z\},\{t\}}}$.

A chart of the blow-up α_3 is given by the coordinate change $\breve{x}_3=\breve{x}_2/\breve{t}_2$, $\breve{z}_3=\breve{z}_2/\bar{t}_2$, $\breve{t}_3=\breve{t}_2$. In this chart, the surface $\mathbf{E}_3$ is given by $\breve{t}_3=0$, and the surface S^3_λ is given by

$$
\begin{aligned}
&(\lambda\breve{x}_3-\breve{t}_3)(\breve{t}_3+\breve{z}_3)+\lambda\breve{t}_3\breve{x}_3^2-\lambda\breve{t}_3\breve{x}_3\breve{z}_3-2\breve{t}_3^3-\breve{t}_3^2\breve{x}_3-\breve{t}_3^2\breve{z}_3+3\breve{t}_3^3\breve{z}_3-\breve{t}_3^4-5\breve{t}_3^3\breve{x}_3\\
&\quad-2\breve{t}_3^2\breve{x}_3\breve{z}_3+2\breve{t}_3^2\breve{z}_3^2+3\breve{t}_3^4\breve{z}_3-6\breve{t}_3^4\breve{x}_3-2\breve{t}_3^3\breve{x}_3^2+2\breve{t}_3^3\breve{x}_3\breve{z}_3+9\breve{t}_3^4\breve{x}_3\breve{z}_3-3\breve{t}_3^5\breve{x}_3-6\breve{t}_3^4\breve{x}_3^2\\
&\quad-3\breve{t}_3^4\breve{z}_3^2-\breve{t}_3^3\breve{x}_3^2\breve{z}_3+2\breve{t}_3^3\breve{x}_3\breve{z}_3^2-\breve{t}_3^3\breve{z}_3^3+6\breve{t}_3^5\breve{x}_3\breve{z}_3-\breve{t}_3^6\breve{x}_3-6\breve{t}_3^5\breve{x}_3^2-\breve{t}_3^4\breve{x}_3^3+3\breve{t}_3^4\breve{x}_3^2\breve{z}_3\\
&\quad-3\breve{t}_3^4\breve{x}_3\breve{z}_3^2+\breve{t}_3^4\breve{z}_3^3+3\breve{t}_3^6\breve{x}_3\breve{z}_3-3\breve{t}_3^6\breve{x}_3^2-3\breve{t}_3^5\breve{x}_3^3+6\breve{t}_3^5\breve{x}_3^2\breve{z}_3-3\breve{t}_3^5\breve{x}_3\breve{z}_3^2+6\breve{t}_3^6\breve{x}_3^2\breve{z}_3\\
&\quad-3\breve{t}_3^6\breve{x}_3^3-3\breve{t}_3^6\breve{x}_3\breve{z}_3^2+3\breve{t}_3^6\breve{x}_3^3\breve{z}_3-\breve{t}_3^6\breve{x}_3^4-3\breve{t}_3^6\breve{x}_3^2\breve{z}_3^2+\breve{t}_3^6\breve{x}_3\breve{z}_3^3=0
\end{aligned}
$$

for $\lambda\neq 0$. If $\lambda=0$, then this equation defines $D^3_0=S^3_0+\mathbf{E}^3_1+2\mathbf{E}^3_2+\mathbf{E}_3$.

The pencil $\mathcal{S}^3$ has two base curves contained in the surface $\mathbf{E}_3$. One of them is given by the equation $\breve{t}_3=\breve{z}_3=0$, and the other, by the equation $\breve{t}_3=\breve{x}_3=0$. Denote the former curve by C^3_{10} and the latter by C^2_{11}. Then $\mathbf{M}^0_{10}=2$. Similarly, we have $\mathbf{m}_{10}=3$, because (at a general point of the curve C^3_{10}) the surface $S^3_0+\mathbf{E}^3_1+2\mathbf{E}_3+\mathbf{E}_3$ is given by

$$\breve{x}_3\breve{t}_3^3\big(\breve{t}_3\breve{x}_3-\breve{t}_3\breve{z}_3+\breve{t}_3+1\big)+\breve{t}_3\big(\breve{t}_3\breve{x}_3-\breve{t}_3\breve{z}_3+\breve{t}_3+\breve{z}_3\big)=0,$$

and S^3_∞ is given by $\breve{t}_3\breve{x}_3-\breve{t}_3\breve{z}_3+\breve{t}_3+\breve{z}_3=0$. Thus, the curve C^3_{10} contributes ② to the defect by Lemma 1.10.7 and (1.10.9). On the other hand, we have $\mathbf{M}^0_{11}=1$. Thus, by Lemma 1.10.7 and (1.10.9), the curve C^3_{11} does not contribute to the defect.

Let $\alpha_4\colon U_4\to U_3$ be the blow-up of the intersection point $C^3_{10}\cap C^3_{11}$. Then the birational map $\alpha\colon U\to\mathbb{P}^3$ in (1.9.3) can be decomposed via the following commutative diagram:

$$
\begin{tikzcd}
 & U_2 \arrow[dl, "\alpha_2"'] & & U_3 \arrow[ll, "\alpha_3"'] & \\
U_1 \arrow[dr, "\alpha_1"'] & & & & U_4 \arrow[ul, "\alpha_4"'] \\
 & \mathbb{P}^3 & & U \arrow[ll, "\alpha"] \arrow[ur, "\gamma"'] &
\end{tikzcd}
$$

where γ is a birational morphism that is an isomorphism along the exceptional locus of the composition $\alpha_1\circ\alpha_2\circ\alpha_3\circ\alpha_4$.

The surface $\mathbf{E}_4$ contains one base curve of the pencil $\mathcal{S}^4$. Denote this curve by C^4_{12}. Simple computations show that neither $\mathbf{E}_4$ nor the curve C^4_{12} contributes to the defect. Thus, summarizing, we see that $\mathbf{D}^0_{P_{\{x\},\{z\},\{t\}}}=10$. $\square$

Lemma 2.2.8. *One has* $\mathbf{D}^0_{P_{\{y\},\{z\},\{t\}}}=5$.

Proof. Let us use the notation of the proof of Lemma 2.2.7. In a neighborhood of the preimage of the point $P_{\{y\},\{z\},\{t\}}$ on the threefold U_4, we can identify the threefold U_4 with the chart of $\mathbb{P}^3$ given by $x=1$. In this chart, the surface S^4_λ is given by

$$\lambda y^2+z^3+z^3t-yz^2-\lambda yzt+\lambda y^2z+\lambda yt^2-yz^3-z^2t^2=0,$$

and $(0,0,0)$ is the preimage of the point $P_{\{y\},\{z\},\{t\}}$.

Let $\alpha_5\colon U_5\to U_4$ be the blow-up of the point $(0,0,0)$. Then $D^5_0=S^5_0+\mathbf{E}_5$. By (1.10.3) and (1.10.9), this contributes ① to the defect $\mathbf{D}^0_{P_{\{y\},\{z\},\{t\}}}$.

A chart of the blow-up α_5 is given by the coordinate change $y_5 = y/t$, $z_5 = z/t$, $t_5 = t$. In this chart, the surface D^5_λ is given by the equation

$$\lambda y_5(t_5 + y_5) - \lambda t_5 y_5 z_5 + \big(\lambda t_5 y_5^2 z_5 - t_5^2 z_5^2 - t_5 y_5 z_5^2 + t_5 z_5^3\big) + t_5^2 z_5^3 - t_5^2 y_5 z_5^3 = 0.$$

We can rewrite this equation as

$$\lambda\widehat{y}_5\widehat{t}_5 + \lambda\widehat{y}_5\widehat{z}_5(\widehat{y}_5 - \widehat{t}_5) - \widehat{z}_5(\widehat{y}_5 - \widehat{t}_5)\big(\lambda\widehat{y}_5^2 + \widehat{z}_5^2 - \widehat{z}_5\widehat{t}_5\big) + \widehat{z}_5^3(\widehat{y}_5 - \widehat{t}_5)^2 - \widehat{y}_5\widehat{z}_5^3(\widehat{y}_5 - \widehat{t}_5)^2 = 0,$$

where $\widehat{y}_5 = y_5$, $\widehat{z}_5 = z_5$, and $\widehat{t}_5 = y_5 + t_5$. Then $\mathbf{E}_5$ is given by $\widehat{y}_5 = \widehat{t}_5$.

The surface $\mathbf{E}_5$ contains one base curve of the pencil $\mathcal{S}^5$. Denote it by C^5_{13}. Then C^5_{13} is given by $\widehat{y}_5 = \widehat{t}_5 = 0$. One has $\mathbf{M}^0_{13} = 1$. By Lemma 1.10.7 and (1.10.9), the curve C^5_{13} does not contribute to the defect of the singular point $P_{\{y\},\{z\},\{t\}}$.

Let $\alpha_6\colon U_6 \to U_5$ be the blow-up of the point $(\widehat{y}_5, \widehat{z}_5, \widehat{t}_5) = (0, 0, 0)$. Then $D^6_0 = S^6_0 + \mathbf{E}^6_5 + 2\mathbf{E}_6$. Thus, by (1.10.3) and (1.10.9), this contributes ① to the defect $\mathbf{D}^0_{P_{\{y\},\{z\},\{t\}}}$.

One (local) chart of the blow-up α_6 is given by $\widehat{y}_6 = \widehat{y}_5/\widehat{z}_5$, $\widehat{z}_6 = \widehat{z}_5$, $\widehat{t}_6 = \widehat{t}_5/\widehat{z}_5$. Thus, if $\lambda \neq 0$, then S^6_λ is given by the equation

$$\begin{aligned}\lambda\widehat{y}_6\widehat{t}_6 + \widehat{z}_6(\widehat{y}_6 - \widehat{t}_6)(\lambda\widehat{y}_6 - \widehat{z}_6) + \widehat{z}_6^2\widehat{t}_6(\widehat{y}_6 - \widehat{t}_6)& \\ - \widehat{z}_6^2(\widehat{y}_6 - \widehat{t}_6)\big(\lambda\widehat{y}_6^2 - \widehat{y}_6\widehat{z}_6 + \widehat{z}_6\widehat{t}_6\big) - \widehat{y}_6\widehat{z}_6^4(\widehat{y}_6 - \widehat{t}_6)^2 &= 0.\end{aligned}$$

The surface $\mathbf{E}_6$ is given by $\widehat{z}_6 = 0$. It contains two base curves of the pencil $\mathcal{S}^6$. One of them is given by $\widehat{y}_6 = \widehat{z}_6 = 0$, and the other, by $\widehat{t}_6 = \widehat{z}_6 = 0$. Denote the former curve by C^6_{14} and the latter by C^6_{15}. If $\lambda \neq 0$, then

$$\mathrm{mult}_{C^6_{14}}(S^6_0 + \mathbf{E}^6_5 + 2\mathbf{E}_6) = \mathbf{M}^0_{14} = \mathbf{m}_{14} = \mathrm{mult}_{C^6_{14}}\big((S^6_0 + \mathbf{E}^6_5 + 2\mathbf{E}_6)\cdot S^6_\lambda\big) = 2.$$

Thus, the curve C^6_{14} contributes ① to the defect by Lemma 1.10.7 and (1.10.9). Similarly, we see that the curve C^6_{15} contributes ① to the defect of the singular point $P_{\{y\},\{z\},\{t\}}$.

Let $\alpha_7\colon U_7 \to U_6$ be the blow-up of the point $C^6_{14} \cap C^6_{15}$. Then $D^7_0 = S^7_0 + \mathbf{E}^7_5 + 2\mathbf{E}^7_6 + \mathbf{E}_7$. By (1.10.3) and (1.10.9), this contributes ① to the defect $\mathbf{D}^0_{P_{\{y\},\{z\},\{t\}}}$.

One (local) chart of the blow-up α_7 is given by $\widehat{y}_7 = \widehat{y}_6/\widehat{z}_6$, $\widehat{t}_7 = \widehat{t}_6/\widehat{z}_6$, $\widehat{z}_7 = \widehat{z}_6$. If $\lambda \neq 0$, then the surface S^7_λ is given by the equation

$$\begin{aligned}\widehat{z}_7\widehat{t}_7 - \widehat{y}_7\widehat{z}_7 + \lambda\widehat{y}_7\widehat{t}_7 + \lambda\widehat{y}_7\widehat{z}_7(\widehat{y}_7 - \widehat{t}_7) + \widehat{z}_7^2\widehat{t}_7(\widehat{y}_7 - \widehat{t}_7)& \\ + \widehat{z}_7^3(\widehat{y}_7 - \widehat{t}_7)^2 - \lambda\widehat{y}_7^2\widehat{z}_7^3(\widehat{y}_7 - \widehat{t}_7) - \widehat{y}_7\widehat{z}_7^5(\widehat{y}_7 - \widehat{t}_7)^2 &= 0.\end{aligned}$$

The surface $\mathbf{E}_7$ is given by $\widehat{z}_7 = 0$. It contains two base curves of the pencil $\mathcal{S}^7$. One of them is given by $\widehat{y}_7 = \widehat{z}_7 = 0$, and the other, by $\widehat{t}_7 = \widehat{z}_7 = 0$. Denote the former curve by C^7_{16} and the latter by C^7_{17}. Then $\mathbf{M}^0_{16} = \mathbf{M}^0_{17} = 1$, so that C^7_{16} and C^7_{17} do not contribute anything to $\mathbf{D}^0_{P_{\{y\},\{z\},\{t\}}}$ by Lemma 1.10.7 and (1.10.9).

Let $\alpha_8\colon U_8 \to U_8$ be the blow-up of the intersection point $C^7_{16} \cap C^7_{17}$. Then the birational map $\alpha\colon U \to \mathbb{P}^3$ in (1.9.3) can be decomposed via the following commutative diagram:

$$\begin{tikzcd}
& U_3 \arrow[dl, "\alpha_3"'] & U_4 \arrow[l, "\alpha_4"'] & U_5 \arrow[l, "\alpha_5"'] & U_6 \arrow[l, "\alpha_6"'] & \\
U_2 \arrow[dr, "\alpha_2"'] & & & & & U_7 \arrow[ul, "\alpha_7"'] \\
& U_1 \arrow[r, "\alpha_1"'] & \mathbb{P}^3 & U \arrow[l, "\alpha"'] \arrow[r, "\delta"'] & U_8 \arrow[ur, "\alpha_8"'] &
\end{tikzcd}$$

where δ is a birational morphism that is an isomorphism along the exceptional locus of the composition $\alpha_5 \circ \alpha_6 \circ \alpha_7 \circ \alpha_8$.

Arguing as above, we see that $\mathbf{E}_8$ does not contribute anything to the computation of the defect. Moreover, the surface $\mathbf{E}_8$ does not contain base curves of the pencil $\mathcal{S}^8$. Thus, summarizing, we see that $\mathbf{D}^0_{P_{\{x\},\{z\},\{t\}}} = 5$. $\square$

Using (2.2.5) and Lemmas 2.2.6–2.2.8, we conclude that $[\mathsf{f}^{-1}(0)] = 21$, so that (♡) in the Main Theorem holds in this case.

Family 2.3. In this case, the threefold X can be obtained from a smooth quartic hypersurface in $\mathbb{P}(1,1,1,1,2)$ by blowing up a smooth elliptic curve. In particular, we have $h^{1,2}(X) = 11$. Let p be the Laurent polynomial

$$\frac{(a+b+1)^4(c+1)}{abc} + c + 1.$$

Then p gives the commutative diagram (✠) by [16, Proposition 16].

Let $\gamma\colon \mathbb{C}^3 \dashrightarrow \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$ be a birational transformation given by the change of coordinates

$$a = -xz, \qquad b = x + xz - 1, \qquad c = -\frac{y}{z} - 1.$$

As for family 2.2, we can use γ to expand (✠) to the commutative diagram (2.2.1). The only difference is that now the pencil $\mathcal{S}$ is given by the equation

$$x^3y + (\lambda z + y)(y + z)(xz + xt - t^2) = 0, \tag{2.3.1}$$

where $\lambda \in \mathbb{C} \cup \{\infty\}$. As for family 2.2, we will follow the scheme described in Section 1 and use the assumptions and notation introduced in that section. But now S_λ denotes the quartic surface in $\mathbb{P}^3$ given by (2.3.1).

Let $\mathbf{Q}$ be the quadric given by $xz + xt - t^2 = 0$. Then $S_\infty = H_{\{z\}} + H_{\{y,z\}} + \mathbf{Q}$. Similarly, let $\mathbf{S}$ be the cubic surface in $\mathbb{P}^3$ given by the equation

$$x^3 + xyz + xyt - yt^2 + xz^2 + xzt - zt^2 = 0.$$

Then $S_0 = H_{\{y\}} + \mathbf{S}$. Thus, we see that both S_∞ and S_0 are reducible. In fact, these are the only reducible surfaces in $\mathcal{S}$. Indeed, if $\lambda \neq \infty, 0, 1$, then S_λ has isolated singularities, which implies that it is irreducible. Moreover, the surface S_1 is also irreducible, but it is singular along the line $L_{\{x\},\{y,z\}}$.

If $\lambda \neq \infty$, then

$$\begin{gathered} H_{\{z\}} \cdot S_\lambda = L_{\{y\},\{z\}} + \mathcal{C}_1, \\ H_{\{y,z\}} \cdot S_\lambda = L_{\{y\},\{z\}} + 3L_{\{x\},\{y,z\}}, \\ \mathbf{Q} \cdot S_\lambda = 6L_{\{x\},\{t\}} + \mathcal{C}_2, \end{gathered} \tag{2.3.2}$$

where $\mathcal{C}_1$ and $\mathcal{C}_2$ are the curves in $\mathbb{P}^3$ given by the equations $z = x^3 + xyt - yt^2 = 0$ and $y = xz + xt - t^2 = 0$, respectively. Thus, if $\lambda \neq \infty$, then

$$S_\infty \cdot S_\lambda = 6L_{\{x\},\{t\}} + 2L_{\{y\},\{z\}} + 3L_{\{x\},\{y,z\}} + \mathcal{C}_1 + \mathcal{C}_2.$$

Hence, the base curves of the pencil $\mathcal{S}$ are $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{x\},\{y,z\}}$, $\mathcal{C}_1$, and $\mathcal{C}_2$.

If $\lambda \neq 0, 1$, then the singular points of S_λ contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

$P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_1$ with quadratic term $xz + xt - t^2$;

$P_{\{x\},\{y\},\{z\}}$: type $\mathbb{A}_5$ with quadratic term $(y + \lambda z)(y + z)$;

$P_{\{x\},\{t\},\{y,z\}}$: type $\mathbb{A}_5$ with quadratic term $(\lambda - 1)x(y + z)$;

$[0 : \lambda : -1 : 0]$: type $\mathbb{A}_5$ with quadratic term $(\lambda - 1)x(y + \lambda z)$.

If $\lambda \notin \{\infty, 0, 1\}$, then it follows from (2.3.2) that

$$H_\lambda \sim L_{\{y\},\{z\}} + \mathcal{C}_1 \sim L_{\{y\},\{z\}} + 3L_{\{x\},\{y,z\}} \sim_{\mathbb{Q}} 3L_{\{x\},\{t\}} + \frac{1}{2}\mathcal{C}_2$$

on the (singular) quartic surface S_λ. Therefore, if $\lambda \notin \{\infty, 0, 1\}$, then the intersection matrix of the curves $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{x\},\{y,z\}}$, $\mathcal{C}_1$, and $\mathcal{C}_2$ on the surface S_λ has the same rank as the intersection matrix of the curves $L_{\{y\},\{z\}}$, $L_{\{x\},\{t\}}$, and H_λ. In this case, we also have

$$H_{\{y\}} \cdot S_\lambda = 2L_{\{y\},\{z\}} + \mathcal{C}_2,$$

so that $2L_{\{y\},\{z\}} + \mathcal{C}_2 \sim H_\lambda$, which gives $2L_{\{y\},\{z\}} + H_\lambda \sim 6L_{\{x\},\{t\}}$.

If $\lambda \notin \{\infty, 0, 1\}$, then the intersection matrix of the curves $L_{\{y\},\{z\}}$, $L_{\{x\},\{t\}}$, and H_λ on the surface S_λ is given by

$$\begin{array}{c c} & \begin{array}{ccc} L_{\{y\},\{z\}} & L_{\{x\},\{t\}} & H_\lambda \end{array} \\ \begin{array}{c} L_{\{y\},\{z\}} \\ L_{\{x\},\{t\}} \\ H_\lambda \end{array} & \begin{pmatrix} -1/2 & 0 & 1 \\ 0 & 1/6 & 1 \\ 1 & 1 & 4 \end{pmatrix} \end{array}.$$

Its rank is 2. On the other hand, the description of singular points of the surface S_λ easily implies that $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk}) = \operatorname{rk}\operatorname{Pic}(S_{\Bbbk}) + 16$, so that ($\bigstar$) holds. Thus, by Lemma 1.13.1, we see that ($\diamondsuit$) in the Main Theorem holds.

Let us prove ($\heartsuit$) in the Main Theorem. Observe that $[\mathsf{f}^{-1}(\lambda)] = 1$ for every $\lambda \notin \{\infty, 0, 1\}$. This follows from Corollary 1.5.4. Thus, to verify ($\heartsuit$) in the Main Theorem, we have to show that $[\mathsf{f}^{-1}(0)] + [\mathsf{f}^{-1}(1)] = 13$. We start with

Lemma 2.3.3. *One has* $[\mathsf{f}^{-1}(0)] = 2$.

Proof. Note that $[S_0] = 2$, and S_0 is smooth at general points of the curves $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{x\},\{y,z\}}$, $\mathcal{C}_1$, and $\mathcal{C}_2$. Furthermore, the points $P_{\{x\},\{z\},\{t\}}$, $P_{\{x\},\{y\},\{z\}}$, and $P_{\{x\},\{t\},\{y,z\}}$ are good double points of the surface S_0. Then $[\mathsf{f}^{-1}(0)] = 2$ by Corollary 1.12.2. □

Let us show that $[\mathsf{f}^{-1}(1)] = 11$. Let $C_1 = \mathcal{C}_1$, $C_2 = \mathcal{C}_2$, $C_3 = L_{\{x\},\{t\}}$, $C_4 = L_{\{y\},\{z\}}$, and $C_5 = L_{\{x\},\{y,z\}}$. Then $\mathbf{m}_1 = \mathbf{m}_2 = 1$, $\mathbf{m}_3 = 6$, $\mathbf{m}_4 = 2$, and $\mathbf{m}_5 = 3$. Moreover, one has $\mathbf{M}_1^1 = \mathbf{M}_2^1 = \mathbf{M}_3^1 = \mathbf{M}_4^1 = 1$ and $\mathbf{M}_5^0 = 2$. Then $\mathbf{C}_1^1 = \mathbf{C}_2^1 = \mathbf{C}_3^1 = \mathbf{C}_4^1 = 0$ and $\mathbf{C}_5^1 = 2$ by Lemma 1.8.5. Thus, using (1.8.3), we see that

$$\left[\mathsf{f}^{-1}(1)\right] = 3 + \mathbf{D}^1_{P_{\{x\},\{z\},\{t\}}} + \mathbf{D}^1_{P_{\{x\},\{y\},\{z\}}} + \mathbf{D}^1_{P_{\{x\},\{t\},\{y,z\}}}. \tag{2.3.4}$$

Lemma 2.3.5. *One has* $\mathbf{D}^1_{P_{\{x\},\{z\},\{t\}}} = 0$.

Proof. Observe that $P_{\{x\},\{z\},\{t\}}$ is an isolated ordinary double point of the surface S_1. Thus, we have $\mathbf{D}^1_{P_{\{x\},\{z\},\{t\}}} = 0$ by Lemma 1.12.1. □

Lemma 2.3.6. *One has* $\mathbf{D}^1_{P_{\{x\},\{y\},\{z\}}} = 1$.

Proof. In the chart $t = 1$, one has $P_{\{x\},\{y\},\{z\}} = (0,0,0)$, and the surface S_λ is given by

$$(y + \lambda z)(y + z) - x(y + \lambda z)(y + z) - x\big(x^2y + \lambda yz^2 + \lambda z^3 + y^2z + yz^2\big) = 0.$$

Let $\alpha_1\colon U_1 \to \mathbb{P}^3$ be the blow-up of the point $P_{\{x\},\{y\},\{z\}}$. Then $S^1_\lambda \sim -K_{U_1}$ for every $\lambda \in \mathbb{C}$. A chart of the blow-up α_1 is given by the coordinate change $x_1 = x$, $y_1 = y/x$, $z_1 = z/x$. In this chart, the surface $\mathbf{E}_1$ is given by $x_1 = 0$ and the surface S^1_λ is given by

$$(y_1 + \lambda z_1)(y_1 + z_1) - x_1\big(x_1y_1 + (y_1 + \lambda z_1)(y_1 + z_1)\big) - x_1^2z_1(y_1 + z_1)(y_1 + \lambda z_1) = 0.$$

This shows that $\mathbf{E}_1$ contains one base curve of the pencil $\mathcal{S}^1$. It is given by $x_1 = y_1 + z_1 = 0$. Denote this curve by C^1_6. Then $\mathbf{M}^1_6 = \mathbf{m}_6 = 2$. But surfaces in the pencil $\mathcal{S}^1$ do not have fixed singular points in $\mathbf{E}_1$. Thus, keeping in mind the construction of the birational morphism α, we see that $\mathbf{D}^1_{P_{\{x\},\{y\},\{z\}}} = 1$ by (1.10.9), (1.10.3), and Lemma 1.10.7. □

Lemma 2.3.7. *One has* $\mathbf{D}^1_{P_{\{x\},\{t\},\{y,z\}}} = 7$.

Proof. Let us use the notation of the proof of Lemma 2.3.6. In a neighborhood of the preimage of the point $P_{\{x\},\{t\},\{y,z\}}$, we can identify U_1 with the chart of $\mathbb{P}^3$ given by $z=1$. In this chart, the surface S^1_λ is given by the equation

$$(\lambda-1)\widehat{x}\widehat{y}+\big((\lambda-1)(\widehat{x}\widehat{y}\widehat{t}-\widehat{y}\widehat{t}^{\,2})+\widehat{x}\widehat{y}^2-\widehat{x}^3\big)+\widehat{y}(\widehat{x}^3+\widehat{x}\widehat{y}\widehat{t}-\widehat{y}\widehat{t}^{\,2})=0,$$

where $\widehat{x}=x$, $\widehat{t}=t$, and $\widehat{y}=y+z$. In these coordinates, the point $(0,0,0)$ is the preimage of the point $P_{\{x\},\{t\},\{y,z\}}$.

Let $\alpha_2\colon U_2\to U_1$ be the blow-up of the point $(0,0,0)$. Then $D^2_1=S^2_1+\mathbf{E}_2$. Thus, by (1.10.3) and (1.10.9), the surface $\mathbf{E}_2$ contributes ① to $\mathbf{D}^1_{P_{\{x\},\{t\},\{y,z\}}}$.

One chart of the blow-up α_2 is given by the coordinate change $\widehat{x}_2=\widehat{x}/\widehat{t}$, $\widehat{y}_2=\widehat{y}/\widehat{t}$, $\widehat{t}_2=\widehat{t}$. In this chart, the surface S^2_λ is given by

$$(\lambda-1)\widehat{y}_2(\widehat{x}_2-\widehat{t}_2)+\big(\lambda\widehat{t}_2\widehat{x}_2\widehat{y}_2-\widehat{t}_2\widehat{x}_2\widehat{y}_2\big)+\big(\widehat{t}_2\widehat{x}_2\widehat{y}_2^2-\widehat{t}_2^2\widehat{y}_2^2-\widehat{t}_2\widehat{x}_2^3\big)+\widehat{t}_2^2\widehat{x}_2\widehat{y}_2^2+\widehat{t}_2^2\widehat{x}_2^3\widehat{y}_2=0$$

for $\lambda\neq1$. Let $\overline{x}_2=\widehat{x}_2-\widehat{t}_2$, $\overline{y}_2=\widehat{z}_2$, and $\overline{t}_2=\widehat{t}_2$. We can rewrite the latter equation as

$$\begin{aligned}(\lambda-1)&\big(\overline{x}_2\overline{y}_2+\overline{y}_2\overline{t}_2(\overline{x}_2+\overline{t}_2)\big)\\&=\overline{x}_2^3\overline{t}_2+3\overline{x}_2^2\overline{t}_2^2+3\overline{x}_2\overline{t}_2^3+\overline{t}_2^4-\overline{x}_2\overline{y}_2^2\overline{t}_2-\overline{y}_2^2\overline{t}_2^2(\overline{x}_2+\overline{t}_2)-\overline{y}_2\overline{t}_2^2(\overline{x}_2+\overline{t}_2)^3.\end{aligned}$$

For $\lambda=1$, this equation defines $D^2_2=S^2_1+\mathbf{E}_2$.

The surface $\mathbf{E}_2$ is given by $\overline{t}_2=0$. It contains two base curves of the pencil $\mathcal{S}^2$. One of them is given by $\overline{x}_2=\overline{t}_2=0$, and the other, by $\overline{z}_2=\overline{t}_2=0$. Denote the former curve by C^2_7 and the latter by C^2_8. Then $\mathbf{M}^1_7=2$. Note that $\mathbf{m}_7=4$, because S_∞ is given by $\overline{y}_2(\overline{t}_2^2+t\overline{x}_2+\overline{x}_2)=0$ and $S^2_1+\mathbf{E}_2$ is given by

$$\big(\overline{t}_2^4+3\overline{t}_2^3\overline{x}_2+3\overline{t}_2^2\overline{x}_2^2+\overline{t}_2\overline{x}_2^3\big)(\overline{t}_2\overline{y}_2-1)+\overline{y}_2\big(\overline{t}_2^2+\overline{t}_2\overline{x}_2+\overline{x}_2\big)(\overline{t}_2\overline{y}_2-1)=0.$$

Thus, the curve C^2_7 contributes ③ to the defect by Lemma 1.10.7 and (1.10.9). On the other hand, one has $\mathbf{M}^1_8=1$, so that C^2_8 does not contribute to the defect.

Let $\alpha_3\colon U_3\to U_2$ be the blow-up of the point $C^2_7\cap C^2_8$. Then $D^3_0=S^3_0+\mathbf{E}^3_2+2\mathbf{E}_3$. By (1.10.3) and (1.10.9), the surface $\mathbf{E}_3$ contributes ① to the defect $\mathbf{D}^1_{P_{\{x\},\{t\},\{y,z\}}}$.

A chart of the blow-up α_3 is given by the coordinate change $\overline{x}_3=\overline{x}_2/\overline{t}_2$, $\overline{y}_3=\overline{y}_2/\overline{t}_2$, $\overline{t}_3=\overline{t}_2$. In this chart, the surface $\mathbf{E}_3$ is given by $\overline{t}_3=0$. Similarly, if $\lambda\neq1$, then S^3_λ is given by

$$\begin{aligned}(\lambda-1)&\overline{y}_3(\overline{x}_3+\overline{t}_3)-t_2^2+\overline{x}_3\overline{t}_3\big((\lambda-1)\overline{y}_3-3\overline{t}_3\big)-3\overline{x}_3^2\overline{t}_3^2\\&-\overline{t}_3^2\big(\overline{x}_3^3-\overline{y}_3\overline{t}_3^2-\overline{x}_3\overline{y}_3^2-\overline{y}_3^2\overline{t}_3\big)+\overline{x}_3\overline{y}_3t_2^3(3\overline{t}_3+\overline{y}_3)+3\overline{x}_3^2\overline{y}_3\overline{t}_3^4+\overline{x}_3^3\overline{y}_3\overline{t}_3^4=0.\end{aligned}$$

Then $\mathbf{E}_3$ contains two base curves of the pencil $\mathcal{S}^3$. One of them is given by $\overline{t}_3=\overline{x}_3=0$, and the other, by $\overline{t}_3=\overline{y}_3=0$. Denote the former curve by C^3_9 and the latter by C^2_{10}. Then $\mathbf{M}^1_9=\mathbf{M}^1_{10}=2$ and $\mathbf{m}_9=\mathbf{m}_{10}=2$. Thus, by Lemma 1.10.7 and (1.10.9), the curves C^3_9 and C^3_{10} contribute ② to the defect $\mathbf{D}^1_{P_{\{x\},\{t\},\{y,z\}}}$.

Summarizing, we see that $\mathbf{D}^1_{P_{\{x\},\{t\},\{y,z\}}}\geq7$. Looking at the defining equation of the surface S^3_λ, one can easily see that $\mathbf{D}^1_{P_{\{x\},\{t\},\{y,z\}}}=7$. $\square$

Using (2.3.4) and Lemmas 2.3.5–2.3.7, we see that (♡) in the Main Theorem holds.

Family 2.4. In this case, the threefold X is a blow-up of $\mathbb{P}^3$ along the smooth complete intersection of two cubic surfaces, which implies that $h^{1,2}(X) = 10$. A mirror partner of the threefold X is given by the Minkowski polynomial 3963.1, which is

$$\frac{z^2}{x}+\frac{3z}{x}+\frac{3}{x}+\frac{yz}{x}+\frac{z^2}{y}+\frac{1}{xz}+\frac{2y}{x}+\frac{2z}{y}+\frac{y}{xz}+\frac{1}{y}\\ +4z+\frac{3}{z}+2y+2\frac{xz}{y}+\frac{2y}{z}+\frac{2x}{y}+4x+3\frac{x}{z}+\frac{xy}{z}+\frac{x^2}{y}+\frac{x^2}{z}.$$

The quartic pencil $\mathcal{S}$ is given by

$$(x+z+t)(x+y+z+t)(xy+xz+yz+yt)-7xyzt=\lambda xyzt.$$

This equation is invariant with respect to the permutation $x\leftrightarrow z$.

Suppose that $\lambda\neq\infty$. Let $\mathcal{C}$ be the conic $t=xy+xz+yz=0$. Then

$$\begin{aligned}
H_{\{x\}}\cdot S_\infty&=L_{\{x\},\{y\}}+2L_{\{x\},\{z,t\}}+L_{\{x\},\{y,z,t\}},\\
H_{\{y\}}\cdot S_\infty&=L_{\{x\},\{y\}}+L_{\{y\},\{z\}}+2L_{\{y\},\{x,z,t\}},\\
H_{\{z\}}\cdot S_\infty&=L_{\{y\},\{z\}}+2L_{\{z\},\{x,t\}}+L_{\{z\},\{x,y,t\}},\\
H_{\{t\}}\cdot S_\infty&=L_{\{t\},\{x,z\}}+L_{\{t\},\{x,y,z\}}+\mathcal{C}.
\end{aligned}$$

This shows that

$$S_\infty\cdot S_\lambda=2L_{\{x\},\{y\}}+2L_{\{y\},\{z\}}+2L_{\{x\},\{z,t\}}+2L_{\{z\},\{x,t\}}+L_{\{t\},\{x,z\}}\\ +2L_{\{y\},\{x,z,t\}}+L_{\{x\},\{y,z,t\}}+L_{\{z\},\{x,y,t\}}+L_{\{t\},\{x,y,z\}}+\mathcal{C}.$$

Hence, the base locus of the pencil $\mathcal{S}$ consists of the curves $L_{\{x\},\{y\}}$, $L_{\{y\},\{z\}}$, $L_{\{x\},\{z,t\}}$, $L_{\{z\},\{x,t\}}$, $L_{\{t\},\{x,z\}}$, $L_{\{y\},\{x,z,t\}}$, $L_{\{x\},\{y,z,t\}}$, $L_{\{z\},\{x,y,t\}}$, $L_{\{t\},\{x,y,z\}}$, and $\mathcal{C}$.

Observe that $S_{-7}=H_{\{x,z,t\}}+H_{\{x,y,z,t\}}+\mathsf{Q}$, where Q is an irreducible quadric surface given by $xy+xz+yz+yt=0$. If $\lambda\neq-7,\infty$, then the surface S_λ has isolated singularities, which implies that it is irreducible.

The singular locus of the surface S_{-7} contained in the base locus of the pencil $\mathcal{S}$ consists of the lines $L_{\{x\},\{z,t\}}$, $L_{\{z\},\{x,t\}}$, and $L_{\{y\},\{x,z,t\}}$.

Lemma 2.4.1. *Suppose that $\lambda\neq-7$. Then the singular points of S_λ contained in the base locus of the pencil $\mathcal{S}$ can be described as follows*:

- $P_{\{x\},\{y\},\{z,t\}}$: *type $\mathbb{A}_4$ with quadratic term $(\lambda+7)xy$;*
- $P_{\{x\},\{z\},\{t\}}$: *type $\mathbb{D}_4$ with quadratic term $(x+z+t)^2$;*
- $P_{\{y\},\{t\},\{x,z\}}$: *type $\mathbb{A}_1$ with quadratic term $(\lambda+6)yt-t^2-(x+z)(x+y+z+2t)$;*
- $P_{\{y\},\{z\},\{x,t\}}$: *type $\mathbb{A}_4$ with quadratic term $(\lambda+7)yz$.*

Proof. First let us describe the singularity of the surface S_λ at the point $P_{\{y\},\{z\},\{x,t\}}$. In the chart $t=1$, the surface S_λ is given by

$$(\lambda+7)\overline{z}\,\overline{y}-\overline{x}^2\overline{z}-(\lambda+8)\overline{x}\,\overline{y}\,\overline{z}-2\overline{z}^2\overline{x}-\overline{z}^2\overline{y}-\overline{z}^3+\overline{x}^3\overline{y}+\overline{x}^3\overline{z}\\ +\overline{x}^2\overline{y}^2+4\overline{x}^2\overline{y}\,\overline{z}+2\overline{x}^2\overline{z}^2+2\overline{x}\,\overline{y}^2\overline{z}+4\overline{x}\,\overline{y}\,\overline{z}^2+\overline{z}^3\overline{x}+\overline{y}^2\overline{z}^2+\overline{y}\,\overline{z}^3=0,$$

where $\overline{x}=x+1$, $\overline{y}=y$, and $\overline{z}=z$. Introducing new coordinates $\overline{x}_2=\overline{x}$, $\overline{y}_2=\overline{y}/\overline{x}$, and $\overline{z}_2=\overline{z}/\overline{x}$, we rewrite this equation (after dividing by $\overline{x}_2^2$) as

$$\overline{z}_2\big((\lambda+7)\overline{y}_2-\overline{x}_2\big)+\overline{x}_2^2\overline{y}_2+\overline{x}_2^2\overline{z}_2-(\lambda+8)\overline{x}_2\overline{y}_2\overline{z}_2-2\overline{z}_2^2\overline{x}_2+\overline{x}_2^2\overline{y}_2^2+4\overline{x}_2^2\overline{y}_2\overline{z}_2\\ +2\overline{x}_2^2\overline{z}_2^2-\overline{x}_2\overline{y}_2\overline{z}_2^2-\overline{z}_2^3\overline{x}_2+2\overline{x}_2^2\overline{y}_2^2\overline{z}_2+4\overline{x}_2^2\overline{y}_2\overline{z}_2^2+\overline{x}_2^2\overline{z}_2^3+\overline{x}_2^2\overline{y}_2^2\overline{z}_2^2+\overline{x}_2^2\overline{y}_2\overline{z}_2^3=0.$$

This equation defines (a chart of) the blow-up of the surface S_λ at the point $P_{\{y\},\{z\},\{x,t\}}$. The two exceptional curves of the blow-up are given by the equations $\overline{x}_2=\overline{z}_2=0$ and $\overline{x}_2=\overline{y}_2=0$. They intersect at the point $(0,0,0)$, which is a singular point of the obtained surface. Introducing new coordinates $\widehat{x}_2=(\lambda+7)\overline{y}_2-\overline{x}_2$, $\widehat{y}_2=\overline{y}_2$, and $\widehat{z}_2=\overline{z}_2$, we can rewrite the latter equation as

$$\widehat{x}_2\widehat{z}_2+(\lambda+7)^2\widehat{y}_2^3+\text{Higher order terms}=0$$

with respect to the weights $\mathrm{wt}(\widehat{x}_2)=3$, $\mathrm{wt}(\widehat{z}_2)=2$, and $\mathrm{wt}(\widehat{z}_2)=3$. This shows that the blown-up surface has a singularity of type $\mathbb{A}_2$ at the point $(0,0,0)$, so that $P_{\{y\},\{z\},\{x,t\}}$ is a singular point of S_λ of type $\mathbb{A}_4$.

Since the equation of S_λ is invariant with respect to the permutation $x\leftrightarrow z$, we see that $P_{\{y\},\{x\},\{z,t\}}$ is a singular point of S_λ of type $\mathbb{A}_4$, and the quadratic term of its defining equation is $(\lambda+7)xy$.

To show that $P_{\{y\},\{t\},\{x,z\}}$ is an ordinary double point of the surface S_λ, we simply observe that the quadratic part of the Taylor expansion of the defining equation of S_λ at the point $P_{\{y\},\{t\},\{x,z\}}$ in the chart $z=1$ is

$$(\lambda+6)\acute{t}\acute{y}-t^2-2\acute{t}\acute{x}-\acute{x}^2-\acute{x}\acute{y},$$

where $\acute{x}=x-1$, $\acute{y}=y$, and $\acute{t}=t$. This quadratic form has rank 3, so that $P_{\{y\},\{t\},\{x,z\}}$ is an ordinary double point of the surface S_λ.

Finally, let us show that $P_{\{x\},\{z\},\{t\}}$ is a singular point of S_λ of type $\mathbb{D}_4$. Let us consider the chart $y=1$ and introduce new coordinates $\widetilde{x}=x$, $\widetilde{z}=z$, and $\widetilde{t}=t+x+z$. Then S_λ is given by

$$\widetilde{t}^{\,2}+(\lambda+7)\widetilde{x}^2\widetilde{z}+(\lambda+7)\widetilde{z}^2\widetilde{x}-(\lambda+6)\widetilde{t}\widetilde{x}\widetilde{z}+\widetilde{t}^{\,3}+\widetilde{t}^{\,2}\widetilde{x}\widetilde{z}=0,$$

where $P_{\{x\},\{z\},\{t\}}=(0,0,0)$. Let us blow up S_λ at this point. Introducing new coordinates $\widetilde{x}_6=\widetilde{x}$, $\widetilde{z}_6=\widetilde{z}/\widetilde{x}$, and $\widetilde{t}_6=\widetilde{t}/\widetilde{x}$, we rewrite this equation (after dividing by $\widetilde{x}_6^2$) as

$$\widetilde{t}_6^2+(\lambda+7)\widetilde{x}_6\widetilde{z}_6-(\lambda+6)\widetilde{t}_6\widetilde{x}_6\widetilde{z}_6+(\lambda+7)\widetilde{z}_6^2\widetilde{x}_6+\widetilde{t}_6^3\widetilde{x}_6+\widetilde{t}_6^2\widetilde{x}_6^2\widetilde{z}_6=0.$$

This equation defines (a chart of) the blow-up of the surface S_λ at the point $P_{\{x\},\{z\},\{t\}}$. The exceptional curve of this birational map is given by $\widetilde{x}_6=\widetilde{t}_6=0$. The obtained surface has an ordinary double point at $(0,0,0)$, since its quadratic form $\widetilde{t}_6^2+(\lambda+7)\widetilde{x}_6\widetilde{z}_6$ is of rank 3. Note, however, that this surface is also singular at the point $(\widetilde{x}_6,\widetilde{z}_6,\widetilde{t}_6)=(0,-1,0)$ and is smooth along the curve $\widetilde{x}_6=\widetilde{t}_6=0$ away from these two points. Introducing new coordinates $\breve{x}_6=\widetilde{x}_6$, $\breve{z}_6=\widetilde{z}_6+1$, and $\breve{t}_6=\widetilde{t}_6$, we rewrite the latter equation as

$$\breve{t}_6^2-(\lambda+7)\breve{x}_6\breve{z}_6+(\lambda+6)\breve{t}_6\breve{x}_6-(\lambda+6)\breve{t}_6\breve{x}_6\breve{z}_6+\breve{t}_6^3\breve{x}_6-\breve{t}_6^2\breve{x}_6^2+(\lambda+7)\breve{z}_6^2\breve{x}_6+\breve{t}_6^2\breve{x}_6^2\breve{z}_6=0.$$

Since $\lambda\neq-7$, the quadratic form $\breve{t}_6^2-(\lambda+7)\breve{x}_6\breve{z}_6+(\lambda+6)\breve{t}_6\breve{x}_6$ has rank 3, so that the second singular point is also an ordinary double point of the obtained surface.

Now let us consider another chart of the blow-up of the surface S_λ at the point $P_{\{x\},\{z\},\{t\}}$. To this end, we introduce coordinates $\widetilde{x}_6'=\widetilde{x}/\widetilde{z}$, $\widetilde{z}_6'=\widetilde{z}$ and $\widetilde{t}_6'=\widetilde{t}/\widetilde{z}$. After dividing by $(\widetilde{x}_6')^2$, we obtain the equation

$$(\widetilde{t}_6')^2+(\lambda+7)\widetilde{x}_6'\widetilde{z}_6'-(\lambda+6)\widetilde{t}_6'\widetilde{x}_6'\widetilde{z}_6'+(\lambda+7)(\widetilde{x}_6')^2\widetilde{z}_6'+(\widetilde{t}_6')^3\widetilde{z}_6'+(\widetilde{t}_6')^2\widetilde{x}_6'(\widetilde{z}_6')^2=0.$$

This surface is smooth along the curve $\widetilde{x}_6'=\widetilde{t}_6'=0$ except for two points: the point $(\widetilde{x}_6',\widetilde{z}_6',\widetilde{t}_6')=(0,0,0)$ and the point $(\widetilde{x}_6',\widetilde{z}_6',\widetilde{t}_6')=(-1,0,0)$. Both these points are ordinary double points of the obtained surface. Note also that the point $(\widetilde{x}_6',\widetilde{z}_6',\widetilde{t}_6')=(-1,0,0)$ is the point $(\widetilde{x}_6,\widetilde{z}_6,\widetilde{t}_6)=(0,-1,0)$ in the first chart of the blow-up. This shows that $P_{\{x\},\{z\},\{t\}}$ is a singular point of S_λ of type $\mathbb{D}_4$. $\square$

The surface $S_{\Bbbk}$ is singular at the points $P_{\{x\},\{y\},\{z,t\}}$, $P_{\{x\},\{z\},\{t\}}$, $P_{\{y\},\{t\},\{x,z\}}$, and $P_{\{y\},\{z\},\{x,t\}}$. Their minimal resolutions are described in the proof of Lemma 2.4.1. This gives

Corollary 2.4.2. *One has* $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk})=\operatorname{rk}\operatorname{Pic}(S_{\Bbbk})+13$.

The base locus of the pencil $\mathcal{S}$ consists of the curves $L_{\{x\},\{y\}}$, $L_{\{y\},\{z\}}$, $L_{\{x\},\{z,t\}}$, $L_{\{z\},\{x,t\}}$, $L_{\{t\},\{x,z\}}$, $L_{\{y\},\{x,z,t\}}$, $L_{\{x\},\{y,z,t\}}$, $L_{\{z\},\{x,y,t\}}$, $L_{\{t\},\{x,y,z\}}$, and $\mathcal{C}$. To describe the rank of their intersection matrix on the surface S_λ for $\lambda\neq-7,\infty$, it suffices to compute the rank of the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{y\},\{z\}}$, $L_{\{x\},\{y,z,t\}}$, $L_{\{z\},\{x,y,t\}}$, $L_{\{t\},\{x,y,z\}}$, $L_{\{t\},\{x,z\}}$, and H_λ, because

$$
\begin{aligned}
H_\lambda &\sim L_{\{x\},\{y\}}+2L_{\{x\},\{z,t\}}+L_{\{x\},\{y,z,t\}}\sim L_{\{x\},\{y\}}+L_{\{y\},\{z\}}+2L_{\{y\},\{x,z,t\}}\\
&\sim L_{\{y\},\{z\}}+2L_{\{z\},\{x,t\}}+L_{\{z\},\{x,y,t\}}\sim L_{\{t\},\{x,z\}}+L_{\{t\},\{x,y,z\}}+\mathcal{C}.
\end{aligned}
$$

Moreover, if $\lambda\neq-7$, then

$$
H_{\{x,y,z,t\}}\cdot S_\lambda=L_{\{y\},\{x,z,t\}}+L_{\{x\},\{y,z,t\}}+L_{\{z\},\{x,y,t\}}+L_{\{t\},\{x,y,z\}},
$$

so that $H_\lambda\sim L_{\{y\},\{x,z,t\}}+L_{\{x\},\{y,z,t\}}+L_{\{z\},\{x,y,t\}}+L_{\{t\},\{x,y,z\}}$. Thus, if $\lambda\neq-7$, then the rank of the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{y\},\{z\}}$, $L_{\{x\},\{z,t\}}$, $L_{\{z\},\{x,t\}}$, $L_{\{t\},\{x,z\}}$, $L_{\{y\},\{x,z,t\}}$, $L_{\{x\},\{y,z,t\}}$, $L_{\{z\},\{x,y,t\}}$, $L_{\{t\},\{x,y,z\}}$, and $\mathcal{C}$ on the surface S_λ is the same as the rank of the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{y\},\{z\}}$, $L_{\{x\},\{y,z,t\}}$, $L_{\{z\},\{x,y,t\}}$, $L_{\{t\},\{x,z\}}$, and H_λ. Moreover, we have the following.

Lemma 2.4.3. *Suppose that* $\lambda\neq-7$. *Then the intersection matrix of the curves* $L_{\{x\},\{y\}}$, $L_{\{y\},\{z\}}$, $L_{\{x\},\{y,z,t\}}$, $L_{\{z\},\{x,y,t\}}$, $L_{\{t\},\{x,z\}}$, *and* H_λ *on the surface* S_λ *is given by*

$$
\begin{array}{c}
\\ L_{\{x\},\{y\}}\\ L_{\{y\},\{z\}}\\ L_{\{x\},\{y,z,t\}}\\ L_{\{z\},\{x,y,t\}}\\ L_{\{t\},\{x,z\}}\\ H_\lambda
\end{array}
\begin{array}{c}
\begin{array}{cccccc} L_{\{x\},\{y\}} & L_{\{y\},\{z\}} & L_{\{x\},\{y,z,t\}} & L_{\{z\},\{x,y,t\}} & L_{\{t\},\{x,z\}} & H_\lambda \end{array}\\
\begin{pmatrix}
-4/5 & 1 & 3/5 & 0 & 0 & 1\\
1 & -4/5 & 0 & 3/5 & 0 & 1\\
3/5 & 0 & -6/5 & 1 & 0 & 1\\
0 & 3/5 & 1 & -6/5 & 0 & 1\\
0 & 0 & 0 & 0 & -1/2 & 1\\
1 & 1 & 1 & 1 & 1 & 4
\end{pmatrix}.
\end{array}
$$

Proof. By definition, we have $H_\lambda^2=4$ and

$$
H_\lambda\cdot L_{\{x\},\{y\}}=H_\lambda\cdot L_{\{y\},\{z\}}=H_\lambda\cdot L_{\{x\},\{y,z,t\}}=H_\lambda\cdot L_{\{z\},\{x,y,t\}}=H_\lambda\cdot L_{\{t\},\{x,z\}}=1.
$$

Let us compute $L_{\{x\},\{y\}}^2$. The only singular point of S_λ contained in $L_{\{x\},\{y\}}$ is $P_{\{x\},\{y\},\{z,t\}}$. Moreover, the surface S_λ has a du Val singularity of type $\mathbb{A}_4$ at this point by Lemma 2.4.1. Let us use the notation of Remark A.2.4 (see the Appendix) with $S=S_\lambda$, $O=P_{\{x\},\{y\},\{z,t\}}$, $n=4$, and $C=L_{\{x\},\{y\}}$. Then $\overline{C}$ passes through the point $\overline{G}_1\cap\overline{G}_4$, so that $\widetilde{C}\cap G_2\neq\varnothing$ or $\widetilde{C}\cap G_3\neq\varnothing$. In both cases, we get $L_{\{x\},\{y\}}^2=-4/5$ by Proposition A.1.3.

Similarly, using Remark A.2.4 with $S=S_\lambda$, $O=P_{\{x\},\{y\},\{z,t\}}$, $n=4$, and $C=L_{\{x\},\{y,z,t\}}$, we see that $\overline{C}$ does not pass through the point $\overline{G}_1\cap\overline{G}_4$, so that $L_{\{x\},\{y,z,t\}}^2=-6/5$ by Proposition A.1.3. Keeping in mind the symmetry $x\leftrightarrow z$, we see that $L_{\{y\},\{z\}}^2=L_{\{x\},\{y\}}^2=-4/5$, and $L_{\{z\},\{x,y,t\}}^2=L_{\{x\},\{y,z,t\}}^2=-6/5$. Using Proposition A.1.3 again, we see that $L_{\{t\},\{x,z\}}^2=-1/2$, because $P_{\{x\},\{z\},\{t\}}$ and $P_{\{y\},\{t\},\{x,z\}}$ are the only singular points of S_λ that are contained in the curve $L_{\{t\},\{x,z\}}$.

Observe that $L_{\{x\},\{y\}}\cap L_{\{y\},\{z\}}=P_{\{x\},\{y\},\{z\}}$, which is a smooth point of the surface S_λ. This gives $L_{\{x\},\{y\}}\cdot L_{\{y\},\{z\}}=1$. We also have

$$
L_{\{x\},\{y\}}\cdot L_{\{z\},\{x,y,t\}}=L_{\{x\},\{y\}}\cdot L_{\{t\},\{x,z\}}=0,
$$

because $L_{\{x\},\{y\}}\cap L_{\{z\},\{x,y,t\}}=L_{\{x\},\{y\}}\cap L_{\{t\},\{x,z\}}=\varnothing$.

To compute $L_{\{x\},\{y\}} \cdot L_{\{x\},\{y,z,t\}}$, recall that $L_{\{x\},\{y\}} + 2L_{\{x\},\{z,t\}} + L_{\{x\},\{y,z,t\}} \sim H_\lambda$. Then

$$\begin{aligned} &L_{\{x\},\{y\}} \cdot L_{\{x\},\{y,z,t\}} + 2L_{\{x\},\{z,t\}} \cdot L_{\{x\},\{y,z,t\}} - \frac{6}{5} \\ &\qquad = L_{\{x\},\{y\}} \cdot L_{\{x\},\{y,z,t\}} + 2L_{\{x\},\{z,t\}} \cdot L_{\{x\},\{y,z,t\}} + L^2_{\{x\},\{y,z,t\}} = H_\lambda \cdot L_{\{x\},\{y,z,t\}} = 1. \end{aligned}$$

Using Remark A.2.4 with $S = S_\lambda$, $O = P_{\{x\},\{y\},\{z,t\}}$, $n = 4$, $C = L_{\{x\},\{z,t\}}$, and $Z = L_{\{x\},\{y,z,t\}}$, we see that neither $\overline{C}$ nor $\overline{Z}$ contains the point $\overline{G}_1 \cap \overline{G}_4$, and either $\overline{C} \cap \overline{G}_1 \neq \varnothing \neq \overline{Z} \cap \overline{G}_1$ or $\overline{C} \cap \overline{G}_4 \neq \varnothing \neq \overline{Z} \cap \overline{G}_4$. In both cases, we have $L_{\{x\},\{z,t\}} \cdot L_{\{x\},\{y,z,t\}} = 4/5$ by Proposition A.1.3, which implies $L_{\{x\},\{y\}} \cdot L_{\{x\},\{y,z,t\}} = 3/5$.

Using the symmetry $x \leftrightarrow z$, we see that $L_{\{y\},\{z\}} \cdot L_{\{z\},\{x,y,t\}} = L_{\{x\},\{y\}} \cdot L_{\{x\},\{y,z,t\}} = 3/5$. Since $L_{\{y\},\{z\}} \cap L_{\{x\},\{y,z,t\}} = \varnothing$ and $L_{\{y\},\{z\}} \cap L_{\{t\},\{x,z\}} = \varnothing$, we have $L_{\{y\},\{z\}} \cdot L_{\{x\},\{y,z,t\}} = 0$ and $L_{\{y\},\{z\}} \cdot L_{\{t\},\{x,z\}} = 0$, respectively.

Note that $L_{\{x\},\{y,z,t\}} \cap L_{\{z\},\{x,y,t\}} = P_{\{x\},\{z\},\{y,t\}}$, and $P_{\{x\},\{z\},\{y,t\}}$ is a smooth point of the surface S_λ. This shows that $L_{\{x\},\{y,z,t\}} \cdot L_{\{z\},\{x,y,t\}} = 1$. Since $L_{\{x\},\{y,z,t\}} \cap L_{\{t\},\{x,z\}} = \varnothing$, we have $L_{\{x\},\{y,z,t\}} \cdot L_{\{t\},\{x,z\}} = 0$. Similarly, we have $L_{\{z\},\{x,y,t\}} \cdot L_{\{t\},\{x,z\}} = 0$. $\square$

The rank of the matrix in Lemma 2.4.3 is 5, so that (★) holds by Corollary 2.4.2. Thus, we see that (♢) in the Main Theorem holds in this case.

Using Lemma 2.4.1 and Corollary 1.5.4, we see that $[\mathsf{f}^{-1}(\lambda)] = 1$ for every $\lambda \notin \{\infty, -7\}$. Moreover, we have the following.

Lemma 2.4.4. *One has* $[\mathsf{f}^{-1}(-7)] = 11$.

Proof. Let $C_1 = L_{\{x\},\{y\}}$, $C_2 = L_{\{y\},\{z\}}$, $C_3 = L_{\{x\},\{z,t\}}$, $C_4 = L_{\{z\},\{x,t\}}$, $C_5 = L_{\{t\},\{x,z\}}$, $C_6 = L_{\{y\},\{x,z,t\}}$, $C_7 = L_{\{x\},\{y,z,t\}}$, $C_8 = L_{\{z\},\{x,y,t\}}$, $C_9 = L_{\{t\},\{x,y,z\}}$, and $C_{10} = \mathcal{C}$. Then

$$\mathbf{M}_1^{-7} = \mathbf{M}_2^{-7} = \mathbf{M}_5^{-7} = \mathbf{M}_7^{-7} = \ldots = \mathbf{M}_{10}^{-7} = 1 \qquad \text{and} \qquad \mathbf{M}_3^{-7} = \mathbf{M}_4^{-7} = \mathbf{M}_6^{-7} = 2.$$

On the other hand, we have

$$\mathbf{m}_1^{-7} = \ldots = \mathbf{m}_4^{-7} = \mathbf{m}_6^{-7} = 2 \qquad \text{and} \qquad \mathbf{m}_5^{-7} = \mathbf{m}_7^{-7} = \ldots = \mathbf{m}_{10}^{-7} = 1.$$

Using Lemma 1.8.5 and (1.8.3), we see that

$$\big[\mathsf{f}^{-1}(-7)\big] = 6 + \mathbf{D}^{-7}_{P_{\{x\},\{y\},\{z,t\}}} + \mathbf{D}^{-7}_{P_{\{x\},\{z\},\{t\}}} + \mathbf{D}^{-7}_{P_{\{y\},\{t\},\{x,z\}}} + \mathbf{D}^{-7}_{P_{\{y\},\{z\},\{x,t\}}}.$$

It follows from the proof of Lemma 2.4.1 that the surface S_{-7} has an isolated ordinary double singularity at the point $P_{\{y\},\{t\},\{x,z\}}$. Thus, it follows from Lemma 1.12.1 that its defect is zero, so that $\mathbf{D}^{-7}_{P_{\{y\},\{t\},\{x,z\}}} = 0$. Hence, we conclude that

$$\big[\mathsf{f}^{-1}(-7)\big] = 6 + \mathbf{D}^{-7}_{P_{\{x\},\{y\},\{z,t\}}} + \mathbf{D}^{-7}_{P_{\{y\},\{z\},\{x,t\}}} + \mathbf{D}^{-7}_{P_{\{x\},\{z\},\{t\}}}.$$

The numbers $\mathbf{D}^{-7}_{P_{\{x\},\{y\},\{z,t\}}}$, $\mathbf{D}^{-7}_{P_{\{y\},\{z\},\{x,t\}}}$, and $\mathbf{D}^{-7}_{P_{\{x\},\{z\},\{t\}}}$ can be computed using the algorithm described in Subsection 1.10. To use it, we have to know the structure of the birational morphism α in (1.9.3). Implicitly, it was described in the proof of Lemma 2.4.1. To be more precise, we proved that there exists a commutative diagram

$$\begin{tikzcd}
 & U_3 \arrow[dl, "\alpha_3"'] & U_4 \arrow[l, "\alpha_4"'] & U_5 \arrow[l, "\alpha_5"'] & \\
U_2 \arrow[dr, "\alpha_2"'] & & & & U_6 \arrow[ul, "\alpha_6"'] \\
 & U_1 \arrow[r, "\alpha_1"'] & \mathbb{P}^3 & U \arrow[l, "\alpha"] \arrow[ur, "\gamma"'] &
\end{tikzcd}$$

Here α_1 is the blow-up of the point $P_{\{y\},\{t\},\{x,z\}}$, the morphism α_2 is the blow-up of the preimage of the point $P_{\{y\},\{z\},\{x,t\}}$, the morphism α_3 is the blow-up of a point in $\mathbf{E}_2$, the morphism α_4 is the blow-up of the preimage of the point $P_{\{x\},\{y\},\{z,t\}}$, the morphism α_5 is the blow-up of a point in $\mathbf{E}_4$, the morphism α_6 is the blow-up of the preimage of the point $P_{\{x\},\{z\},\{t\}}$, and γ is the blow-up of three distinct points in $\mathbf{E}_6$ that are described at the very end of the proof of Lemma 2.4.1. In the notation used in the proof of Lemma 2.4.1, these are the points $(\widetilde{x}_6,\widetilde{z}_6,\widetilde{t}_6)=(0,0,0)$, $(\widetilde{x}_6,\widetilde{z}_6,\widetilde{t}_6)=(0,-1,0)$, and $(\widetilde{x}_6',\widetilde{z}_6',\widetilde{t}_6')=(0,0,0)$.

Using Lemma 1.10.7 and (1.10.9), we can find $\mathbf{D}^{-7}_{P_{\{x\},\{y\},\{z,t\}}}$, $\mathbf{D}^{-7}_{P_{\{y\},\{z\},\{x,t\}}}$, and $\mathbf{D}^{-7}_{P_{\{x\},\{z\},\{t\}}}$ by analyzing the base curves of the pencil $\widehat{\mathcal{S}}$. Implicitly, this has already been done in the proof of Lemma 2.4.1, so that we will use the notation introduced in this proof.

Observe that $\widehat{E}_1$ does not contain base curves of the pencil $\widehat{\mathcal{S}}$. To describe the base curves in the surface $\mathbf{E}_2$, note that $\mathcal{S}^2|_{\mathbf{E}_2}$ consists of two lines in $\mathbf{E}_2\cong\mathbb{P}^2$. These curves are given by $\overline{x}_2=\overline{z}_2=0$ and $\overline{x}_2=\overline{y}_2=0$. Denote them by C^2_{11} and C^2_{12}, respectively. Note that $D^2_{-7}=S^2_{-7}+\mathbf{E}_2$, the surface S^2_{-7} contains C^2_{11}, and it does not contain C^2_{12}.

Similarly, the restriction $\mathcal{S}^3|_{\mathbf{E}_3}$ contains one base curve, which is a line in $\mathbf{E}_3\cong\mathbb{P}^2$. Denote this curve by C^3_{13}. Then C^3_{13} is contained in S^3_{-7}, and it is not contained in $\mathbf{E}^3_2$. Moreover, the surface S^3_{-7} is smooth at a general point of the curve C^3_{13}.

The restriction $\mathcal{S}^4|_{\mathbf{E}_4}$ consists of two lines in $\mathbf{E}_4\cong\mathbb{P}^2$, which we denote by C^4_{14} and C^4_{15}. One of them is contained in the surface S^4_{-7}. We may assume that this curve is C^4_{14}. Similarly, the restriction $\mathcal{S}^5|_{\mathbf{E}_5}$ contains one base curve, which is a line in $\mathbf{E}_5\cong\mathbb{P}^2$. Let us denote this curve by C^5_{16}. It is contained in S^5_{-7}, and it is not contained in $\mathbf{E}^5_4$. By construction, we have $D^5_{-7}=S^5_{-7}+\mathbf{E}^5_2+\mathbf{E}^5_4$.

The restriction $\mathcal{S}^6|_{\mathbf{E}_6}$ consists of a single line in $\mathbf{E}_6\cong\mathbb{P}^2$ (taken with multiplicity 2). Denote this line by C^6_{17}. Note that the surface S^6_{-7} is singular along the curve C^6_{17}.

Finally, we observe that $\widehat{E}_7$, $\widehat{E}_8$, and $\widehat{E}_9$ do not contain base curves of the pencil $\widehat{\mathcal{S}}$.

Now we are ready to compute $\mathbf{D}^{-7}_{P_{\{x\},\{y\},\{z,t\}}}$, $\mathbf{D}^{-7}_{P_{\{y\},\{z\},\{x,t\}}}$, and $\mathbf{D}^{-7}_{P_{\{x\},\{z\},\{t\}}}$. First, we observe that $\widehat{D}_{-7}=\widehat{S}_{-7}+\widehat{E}_2+\widehat{E}_4$, so that $\mathbf{A}^{-7}_{P_{\{x\},\{y\},\{z,t\}}}=\mathbf{A}^{-7}_{P_{\{y\},\{z\},\{x,t\}}}=1$ and $\mathbf{A}^{-7}_{P_{\{x\},\{z\},\{t\}}}=0$. Second, we observe that the curves $\widehat{C}_{11},\dots,\widehat{C}_{17}$ are all base curves of the pencil $\widehat{\mathcal{S}}$ that are contained in α-exceptional divisors. The curves $\widehat{C}_{11}$, $\widehat{C}_{12}$, and $\widehat{C}_{13}$ are mapped to the point $P_{\{x\},\{y\},\{z,t\}}$, the curves $\widehat{C}_{14}$, $\widehat{C}_{15}$, and $\widehat{C}_{16}$ are mapped to the point $P_{\{y\},\{z\},\{x,t\}}$, and the curve $\widehat{C}_{17}$ is mapped to the point $P_{\{x\},\{z\},\{t\}}$. Thus, to find $\mathbf{D}^{-7}_{P_{\{x\},\{y\},\{z,t\}}}$, $\mathbf{D}^{-7}_{P_{\{y\},\{z\},\{x,t\}}}$, and $\mathbf{D}^{-7}_{P_{\{x\},\{z\},\{t\}}}$, we have to compute the numbers $\mathbf{C}^{-7}_{11},\dots,\mathbf{C}^{-7}_{17}$ defined in (1.10.5). This can be done using Lemma 1.10.7.

Observe that $\mathbf{M}^{-7}_{12}=\mathbf{M}^{-7}_{13}=\mathbf{M}^{-7}_{15}=\mathbf{M}^{-7}_{16}=1$ and $\mathbf{M}^{-7}_{11}=\mathbf{M}^{-7}_{14}=\mathbf{M}^{-7}_{17}=2$. Let us find the numbers $\mathbf{m}_{11},\dots,\mathbf{m}_{17}$.

Among the base curves of the pencil $\mathcal{S}$, only C_2, C_4, C_6, and C_8 contain the point $P_{\{y\},\{z\},\{x,t\}}$. This shows that

$$
\begin{aligned}
7&=\mathrm{mult}_{P_{\{y\},\{z\},\{x,t\}}}(2C_2+2C_4+2C_6+C_8)=\mathrm{mult}_{P_{\{y\},\{z\},\{x,t\}}}(S_{\lambda_1}\cdot S_{\lambda_2})\\
&=\mathrm{mult}_{P_{\{y\},\{z\},\{x,t\}}}(S_{\lambda_1})\,\mathrm{mult}_{P_{\{y\},\{z\},\{x,t\}}}(S_{\lambda_2})+\mathbf{m}_{11}+\mathbf{m}_{12}=4+\mathbf{m}_{11}+\mathbf{m}_{12}.
\end{aligned}
$$

Moreover, we have $\mathbf{m}_{11}\geq 2$, because $\widehat{D}_{-7}$ is singular along $\widehat{C}_{11}$. This shows that $\mathbf{m}_{11}=2$ and $\mathbf{m}_{12}=1$. Similarly, we see that $\mathbf{m}_{13}=1$. Using the symmetry $x\leftrightarrow z$, we deduce that $\mathbf{m}_{14}=2$ and $\mathbf{m}_{15}=\mathbf{m}_{16}=1$. To find $\mathbf{m}_{17}$, we observe that C_3, C_4, C_5, and C_{10} are the only base curves of the pencil $\mathcal{S}$ that contain the point $P_{\{x\},\{z\},\{t\}}$. This shows that

$$
\mathrm{mult}_{P_{\{x\},\{z\},\{t\}}}(S_{\lambda_1}\cdot S_{\lambda_2})=\mathrm{mult}_{P_{\{x\},\{z\},\{t\}}}(2C_3+2C_4+C_5+C_{10})=6,
$$

which implies $\mathbf{m}_{17}=2$.

Recall from (1.10.9) that

$$\mathbf{D}^{-7}_{P_{\{x\},\{y\},\{z,t\}}} = \mathbf{A}^{-7}_{P_{\{x\},\{y\},\{z,t\}}} + \mathbf{C}^{-7}_{11} + \mathbf{C}^{-7}_{12} + \mathbf{C}^{-7}_{13} = 1 + \mathbf{C}^{-7}_{11} + \mathbf{C}^{-7}_{12} + \mathbf{C}^{-7}_{13},$$

where each of the terms $\mathbf{C}^{-7}_{i}$ is defined in (1.10.5) and can be found using Lemma 1.10.7. This gives $\mathbf{D}^{-7}_{P_{\{x\},\{y\},\{z,t\}}} = \mathbf{C}^{-7}_{11} = 2$. Similarly, we see that $\mathbf{D}^{-7}_{P_{\{y\},\{z\},\{y,t\}}} = \mathbf{C}^{-7}_{14} = 2$. Similarly, we have $\mathbf{D}^{-7}_{P_{\{x\},\{z\},\{t\}}} = \mathbf{C}^{-7}_{17} = 1$. Thus, we see that

$$\big[\mathsf{f}^{-1}(-7)\big] = 6 + \mathbf{D}^{-7}_{P_{\{x\},\{y\},\{z,t\}}} + \mathbf{D}^{-7}_{P_{\{y\},\{z\},\{y,t\}}} + \mathbf{D}^{-7}_{P_{\{x\},\{z\},\{t\}}} = 11,$$

which completes the proof of the lemma. $\square$

Since $h^{1,2}(X) = 10$, we see that (♡) in the Main Theorem also holds in this case.

Family 2.5. In this case, the threefold X is a blow-up of a smooth cubic threefold in $\mathbb{P}^4$ along a smooth plane cubic curve. Note that $h^{1,2}(X) = 6$. A toric Landau–Ginzburg model is given by the Minkowski polynomial 3452, which is

$$\begin{aligned}x + y + z + x^2y^{-1}z^{-1} + 3xz^{-1} + 3yz^{-1} + x^{-1}y^2z^{-1} + 3xy^{-1} + 3x^{-1}y + 3y^{-1}z\\ + 3x^{-1}z + x^{-1}y^{-1}z^2 + xy^{-1}z^{-1} + 2z^{-1} + x^{-1}yz^{-1} + 2y^{-1} + 2x^{-1} + x^{-1}y^{-1}z.\end{aligned}$$

The corresponding quartic pencil $\mathcal{S}$ is given by the equation

$$\begin{aligned}x^2yz + y^2zx + z^2yx + x^3t + 3x^2ty + 3y^2tx + y^3t + 3x^2tz + 3y^2tz + 3z^2tx\\ + 3z^2ty + z^3t + x^2t^2 + 2t^2yx + t^2y^2 + 2t^2zx + 2t^2yz + t^2z^2 = \lambda xyzt.\end{aligned}$$

Observe that this equation is invariant with respect to any permutations of the coordinates x, y, and z. To describe the base locus of the pencil $\mathcal{S}$, we note that

$$\begin{aligned}H_{\{x\}} \cdot S_0 &= L_{\{x\},\{t\}} + 2L_{\{x\},\{y,z\}} + L_{\{x\},\{y,z,t\}},\\ H_{\{y\}} \cdot S_0 &= L_{\{y\},\{t\}} + 2L_{\{y\},\{x,z\}} + L_{\{y\},\{x,z,t\}},\\ H_{\{z\}} \cdot S_0 &= L_{\{z\},\{t\}} + 2L_{\{z\},\{x,y\}} + L_{\{z\},\{x,y,t\}},\\ H_{\{t\}} \cdot S_0 &= L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y,z\}}.\end{aligned}$$

For every $\lambda \notin \{-6, -7, \infty\}$, the surface S_λ has isolated singularities, so that it is irreducible. On the other hand, we have $S_{-6} = H_{\{x,y,z\}} + \mathsf{S}$, where S is a cubic surface given by the equation

$$t^2x + t^2y + t^2z + tx^2 + 2txy + 2txz + ty^2 + 2tyz + tz^2 + xyz = 0.$$

Similarly, we have $S_{-7} = H_{\{x,y,z,t\}} + \mathbf{S}$, where $\mathbf{S}$ is a cubic surface given by $t(x+y+z)^2 + xyz = 0$.

One can show that S is smooth. On the other hand, the surface $\mathbf{S}$ has a unique singular point $P_{\{x\},\{y\},\{z\}}$. The surface $\mathbf{S}$ has a du Val singularity of type $\mathbb{D}_4$ at this point. Observe also that $H_{\{x,y,z\}} \cdot \mathsf{S} = L_{\{x\},\{y,z\}} + L_{\{y\},\{x,z\}} + L_{\{z\},\{x,y\}}$, so that S_{-6} is singular along the lines $L_{\{x\},\{y,z\}}$, $L_{\{y\},\{x,z\}}$, and $L_{\{z\},\{x,y\}}$. Note also that the intersection $H_{\{x,y,z,t\}} \cap \mathbf{S}$ is a smooth cubic curve, which is not contained in the base locus of the pencil $\mathcal{S}$.

Lemma 2.5.1. *Suppose that $\lambda \notin \{-6, -7, \infty\}$. Then the singular points of S_λ contained in the base locus of the pencil $\mathcal{S}$ can be described as follows*:

$P_{\{x\},\{y\},\{z\}}$: *type $\mathbb{D}_4$ with quadratic term $(x+y+z)^2$*;

$P_{\{x\},\{t\},\{y,z\}}$: *type $\mathbb{A}_3$ with quadratic term $x(x+z+y-(\lambda+6)t)$*;

$P_{\{y\},\{t\},\{x,z\}}$: *type $\mathbb{A}_3$ with quadratic term $y(x+z+y-(\lambda+6)t)$*;

$P_{\{z\},\{t\},\{x,y\}}$: *type $\mathbb{A}_3$ with quadratic term $z(x+z+y-(\lambda+6)t)$*.

Proof. First let us describe the singularity of the surface S_λ at the point $P_{\{x\},\{y\},\{z\}}$. In the chart $t=1$, the surface S_λ is given by

$$\widehat{z}^2+\big((\lambda+6)\widehat{x}^2\widehat{y}+(\lambda+6)\widehat{x}\widehat{y}^2-(\lambda+6)\widehat{x}\widehat{y}\widehat{z}+\widehat{z}^3\big)+\big(\widehat{z}^2\widehat{y}\widehat{x}-\widehat{x}^2\widehat{y}\widehat{z}-\widehat{y}^2\widehat{z}\widehat{x}\big)=0,$$

where $\widehat{x}=x$, $\widehat{y}=y$, and $\widehat{z}=x+y+z$. Introducing coordinates $\widehat{x}_4=\widehat{x}$, $\widehat{y}_4=\widehat{y}/\widehat{x}$, and $\widehat{z}_4=\widehat{z}/\widehat{x}$, we can rewrite this equation (after dividing by $\widehat{x}_4^2$) as

$$(\lambda+6)\widehat{x}_4\widehat{y}_4+\widehat{z}_4^2+\big((\lambda+6)\widehat{x}_4\widehat{y}_4^2-(6+\lambda)\widehat{x}_4\widehat{y}_4\widehat{z}_4\big)+\big(\widehat{z}_4^3\widehat{x}_4-\widehat{x}_4^2\widehat{y}_4\widehat{z}_4\big)+\big(\widehat{x}_4^2\widehat{y}_4\widehat{z}_4^2-\widehat{x}_4^2\widehat{y}_4^2\widehat{z}_4\big)=0.$$

This equation defines (a chart of) the blow-up of the surface S_λ at the point $P_{\{x\},\{y\},\{z\}}$. The exceptional curve of the blow-up is given by the equations $\widehat{x}_4=\widehat{z}_4=0$. Observe that the point $(\widehat{x}_4,\widehat{y}_4,\widehat{z}_4)=(0,0,0)$ is an ordinary double point of the obtained surface, because $\lambda\neq-6$. The obtained surface is also singular at the point $(\widehat{x}_4,\widehat{y}_4,\widehat{z}_4)=(0,-1,0)$. This point is also an ordinary double point of this surface. These are all singular points of the obtained surface at this chart of the blow-up. Keeping in mind the symmetry $x\leftrightarrow y$, we see that the exceptional curve of the blow-up of the surface S_λ at the point $P_{\{x\},\{y\},\{z\}}$ contains three ordinary double points of this surface. This shows that $P_{\{x\},\{y\},\{z\}}$ is a singular point of type $\mathbb{D}_4$ of the surface S_λ.

To complete the proof, it suffices to show that $P_{\{y\},\{t\},\{x,z\}}$ is a singular point of S_λ of type $\mathbb{A}_3$, because S_λ is invariant with respect to the permutations of the coordinates x, y, and z. In the chart $z=1$, the surface S_λ is given by

$$\begin{aligned}&\overline{y}\big(\overline{x}+\overline{y}-(6+\lambda)\overline{t}\big)\\ &\qquad=\overline{x}^2\overline{y}+\overline{x}\overline{y}^2-(6+\lambda)\overline{t}\overline{x}\overline{y}+\overline{t}^2\overline{x}^2+2\overline{t}^2\overline{x}\overline{y}+\overline{t}^2\overline{y}^2+\overline{t}\overline{x}^3+3\overline{t}\overline{x}^2\overline{y}+3\overline{t}\overline{x}\overline{y}^2+\overline{t}\overline{y}^3,\end{aligned}$$

where $\overline{x}=x+1$, $\overline{y}=y$, and $\overline{t}=t$. Introducing new coordinates $\breve{x}=\overline{x}+\overline{y}-(6-\lambda)\overline{t}$, $\breve{y}=\overline{y}$, and $\breve{t}=\overline{t}$, we can rewrite this equation as

$$(\lambda+7)(\lambda+6)^2\breve{t}^4=\breve{x}\breve{y}-(\lambda+6)\big(\breve{t}\breve{x}\breve{y}+(3\lambda+20)\breve{t}^3\breve{x}\big)-\breve{x}^2\breve{y}+\breve{x}\breve{y}^2-(3\lambda+19)\breve{t}^2\breve{x}^2-\breve{t}\breve{x}^3,$$

where we grouped together monomials of the same quasihomogeneous degree with respect to the weights $\mathrm{wt}(\breve{x})=2$, $\mathrm{wt}(\breve{y})=2$, and $\mathrm{wt}(\breve{t})=1$. This shows that the surface S_λ has a singularity of type $\mathbb{A}_3$ at the point $P_{\{y\},\{t\},\{x,z\}}$. This completes the proof of the lemma. $\square$

The proof of Lemma 2.5.1 implies that $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk})=\operatorname{rk}\operatorname{Pic}(S_{\Bbbk})+13$.

Lemma 2.5.2. *Suppose that $\lambda\notin\{-6,-7,\infty\}$. Then the intersection matrix of the lines $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{y,z,t\}}$, $L_{\{y\},\{x,z,t\}}$, $L_{\{z\},\{x,y,t\}}$, and $L_{\{t\},\{x,y,z\}}$ on the surface S_λ is given by*

$$\begin{array}{c} \\ L_{\{x\},\{t\}}\\ L_{\{y\},\{t\}}\\ L_{\{z\},\{t\}}\\ L_{\{x\},\{y,z,t\}}\\ L_{\{y\},\{x,z,t\}}\\ L_{\{z\},\{x,y,t\}}\\ L_{\{t\},\{x,y,z\}}\end{array}
\begin{array}{c}
\begin{array}{ccccccc} L_{\{x\},\{t\}} & L_{\{y\},\{t\}} & L_{\{z\},\{t\}} & L_{\{x\},\{y,z,t\}} & L_{\{y\},\{x,z,t\}} & L_{\{z\},\{x,y,t\}} & L_{\{t\},\{x,y,z\}}\end{array}\\
\begin{pmatrix}
-5/4 & 1 & 1 & 3/4 & 0 & 0 & 1/4\\
1 & -5/4 & 1 & 0 & 3/4 & 0 & 1/4\\
1 & 1 & -5/4 & 0 & 0 & 3/4 & 1/4\\
3/4 & 0 & 0 & -5/4 & 1 & 1 & 1/4\\
0 & 3/4 & 0 & 1 & -5/4 & 1 & 1/4\\
0 & 0 & 3/4 & 1 & 1 & -5/4 & 1/4\\
1/4 & 1/4 & 1/4 & 1/4 & 1/4 & 1/4 & 1/4
\end{pmatrix}\end{array}.$$

Proof. Since the equation of the surface S_λ is invariant with respect to the permutations of the coordinates x, y, and z, it suffices to compute $L_{\{x\},\{t\}}^2$, $L_{\{x\},\{y,z,t\}}^2$, $L_{\{t\},\{x,y,z\}}^2$, $L_{\{x\},\{t\}}\cdot L_{\{y\},\{t\}}$, $L_{\{x\},\{t\}}\cdot L_{\{x\},\{y,z,t\}}$, $L_{\{x\},\{t\}}\cdot L_{\{y\},\{x,z,t\}}$, $L_{\{x\},\{t\}}\cdot L_{\{t\},\{x,y,z\}}$, and $L_{\{x\},\{y,z,t\}}\cdot L_{\{t\},\{x,y,z\}}$.

Using Lemma 2.5.1, Proposition A.1.3, and Remark A.2.4, we see that $L_{\{x\},\{t\}}^2=-5/4$, because $P_{\{x\},\{t\},\{y,z\}}$ is the only singular point of S_λ that is contained in the line $L_{\{x\},\{t\}}$. Similarly, we see that $L_{\{x\},\{y,z,t\}}^2 = -5/4$. Similarly, we get $L_{\{t\},\{x,y,z\}}^2 = 1/4$, because the line $L_{\{t\},\{x,y,z\}}$ contains the points $P_{\{x\},\{t\},\{y,z\}}$, $P_{\{y\},\{t\},\{x,z\}}$, and $P_{\{z\},\{t\},\{x,y\}}$. We have $L_{\{x\},\{t\}} \cdot L_{\{y\},\{t\}} = 1$, because $L_{\{x\},\{t\}} \cap L_{\{y\},\{t\}} = P_{\{x\},\{y\},\{t\}}$, which is a smooth point of the surface S_λ by Lemma 2.5.1.

Now applying Remark A.2.4 with $S = S_\lambda$, $O = P_{\{x\},\{t\},\{y,z\}}$, $n = 3$, $C = L_{\{x\},\{t\}}$, and $Z = L_{\{x\},\{y,z,t\}}$, we see that $L_{\{x\},\{t\}} \cdot L_{\{x\},\{y,z,t\}} = 3/4$ by Proposition A.1.2. Similarly, we have

$$L_{\{x\},\{y,z,t\}} \cdot L_{\{t\},\{x,y,z\}} = L_{\{x\},\{y,z,t\}} \cdot L_{\{t\},\{x,y,z\}} = \frac{1}{4}.$$

Finally, we have $L_{\{x\},\{t\}} \cdot L_{\{y\},\{x,z,t\}} = 0$, because $L_{\{x\},\{t\}} \cap L_{\{y\},\{x,z,t\}} = \varnothing$. $\square$

If $\lambda \notin \{-6, -7, \infty\}$, then the intersection matrix of the lines $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{y,z\}}$, $L_{\{y\},\{x,z\}}$, $L_{\{z\},\{x,y\}}$, $L_{\{x\},\{y,z,t\}}$, $L_{\{y\},\{x,z,t\}}$, $L_{\{z\},\{x,y,t\}}$, and $L_{\{t\},\{x,y,z\}}$ on the surface S_λ has the same rank as the intersection matrix of the curves $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{y,z,t\}}$, $L_{\{y\},\{x,z,t\}}$, $L_{\{z\},\{x,y,t\}}$, and $L_{\{t\},\{x,y,z\}}$. This follows from the relations

$$\begin{aligned} H_\lambda &\sim L_{\{x\},\{t\}} + 2L_{\{x\},\{y,z\}} + L_{\{x\},\{y,z,t\}} \sim L_{\{y\},\{t\}} + 2L_{\{y\},\{x,z\}} + L_{\{y\},\{x,z,t\}} \\ &\sim L_{\{z\},\{t\}} + 2L_{\{z\},\{x,y\}} + L_{\{z\},\{x,y,t\}} \sim L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y,z\}}. \end{aligned}$$

The rank of the intersection matrix in Lemma 2.5.2 is 5. Thus, we see that ($\bigstar$) holds. This proves that ($\diamondsuit$) in the Main Theorem holds in this case.

To verify ($\heartsuit$) in the Main Theorem, observe that $[\mathsf{f}^{-1}(\lambda)] = 1$ for every $\lambda \notin \{-6, -7, \infty\}$. This follows from Lemma 2.5.1 and Corollary 1.5.4. Moreover, we have

Lemma 2.5.3. *One has* $[\mathsf{f}^{-1}(-7)] = 2$ *and* $[\mathsf{f}^{-1}(-6)] = 6$.

Proof. Let $C_1 = L_{\{x\},\{t\}}$, $C_2 = L_{\{y\},\{t\}}$, $C_3 = L_{\{z\},\{t\}}$, $C_4 = L_{\{x\},\{y,z\}}$, $C_5 = L_{\{y\},\{x,z\}}$, $C_6 = L_{\{z\},\{x,y\}}$, $C_7 = L_{\{x\},\{y,z,t\}}$, $C_8 = L_{\{y\},\{x,z,t\}}$, $C_9 = L_{\{z\},\{x,y,t\}}$, and $C_{10} = L_{\{t\},\{x,y,z\}}$. Then $\mathbf{m}_1 = \ldots = \mathbf{m}_6 = 2$ and $\mathbf{m}_7 = \ldots = \mathbf{m}_{10} = 1$.

Recall that S_∞ is singular along the curves C_1, C_2, and C_3, and the surface S_{-6} is singular along the curves C_4, C_5, and C_6. Thus, we have

$$\mathbf{M}_4^{-6} = \mathbf{M}_5^{-6} = \mathbf{M}_6^{-6} = 2, \qquad \mathbf{M}_1^{-6} = \mathbf{M}_2^{-6} = \mathbf{M}_3^{-6} = \mathbf{M}_7^{-6} = \ldots = \mathbf{M}_{10}^{-6} = 1,$$

and $\mathbf{M}_1^{-7} = \ldots = \mathbf{M}_{10}^{-7} = 1$.

The birational morphism α in (1.9.3) is described in the proof of Lemma 2.5.1. Namely, it is given by the commutative diagram

$$\begin{tikzcd} U_2 \arrow[d, "\alpha_2"'] & U_3 \arrow[l, "\alpha_3"'] & U_4 \arrow[l, "\alpha_4"'] \\ U_1 \arrow[r, "\alpha_1"'] & \mathbb{P}^3 & U \arrow[l, "\alpha"] \arrow[u, "\gamma"'] \end{tikzcd}$$

Here α_1 is the blow-up of the point $P_{\{x\},\{t\},\{y,z\}}$, the morphism α_2 is the blow-up of the preimage of the point $P_{\{y\},\{t\},\{x,z\}}$, the morphism α_3 is the blow-up of the preimage of the point $P_{\{z\},\{t\},\{x,y\}}$, the morphism α_4 is the blow-up of the preimage of the point $P_{\{x\},\{y\},\{z\}}$, and γ is the blow-up of three distinct points in $\mathbf{E}_4$.

If $\lambda \neq \infty$, then $\widehat{D}_\lambda = \widehat{S}_\lambda$. This follows from the proof of Lemma 2.5.1. It should be pointed out that the surface $\widehat{S}_\lambda$ is singular for every $\lambda \in \mathbb{C}$.

The curves $\widehat{C}_1, \ldots, \widehat{C}_{10}$ are base curves of the pencil $\widehat{\mathcal{S}}$. Let us describe the remaining base curves of $\widehat{\mathcal{S}}$ using the data collected in the proof of Lemma 2.5.1.

For every $\lambda \neq \infty$, the restriction $S_\lambda^2|_{\mathbf{E}_2}$ is given by

$$\overline{y}\big(\overline{x} + \overline{y} - (6 + \lambda)\overline{t}\big) = 0$$

in appropriate homogeneous coordinates $\overline{x}, \overline{y}, \overline{t}$ on $\mathbf{E}_2 \cong \mathbb{P}^2$. This gives us the pencil of conics in $\mathbf{E}_2$ that has a unique base curve, which is given by $\overline{y} = 0$. Thus, the restriction $\mathcal{S}^2|_{\mathbf{E}_2}$ has one base curve. This gives us the base curve of $\widehat{\mathcal{S}}$ that is contained in $\widehat{E}_2$. Let us denote it by $\widehat{C}_{12}$. Similarly, we see that one base curve of $\widehat{\mathcal{S}}$ is contained in the surface $\widehat{E}_1$, and one base curve of $\widehat{\mathcal{S}}$ is contained in the surface $\widehat{E}_3$. Let us denote them by $\widehat{C}_{11}$ and $\widehat{C}_{13}$, respectively.

The restriction $\mathcal{S}^4|_{\mathbf{E}_4}$ consists of one line (taken with multiplicity 2). This gives us one base curve of the pencil $\mathcal{S}^4$ that is contained in the surface $\mathbf{E}_4$. Denote it by C_{14}^4. Observe that the surface S_{-6}^4 is singular at a general point of this curve. Moreover, it follows from the proof of Lemma 2.4.4 that the curves $\widehat{C}_1, \ldots, \widehat{C}_{14}$ are all base curves of the pencil $\widehat{\mathcal{S}}$.

Let us compute $\mathbf{m}_{11}, \ldots, \mathbf{m}_{14}$. Among the base curves of the pencil $\mathcal{S}$, only the curves C_2, C_5, C_8, and C_{10} contain the point $P_{\{y\},\{t\},\{x,z\}}$, This gives

$$\begin{aligned}6 &= \mathrm{mult}_{P_{\{y\},\{t\},\{x,z\}}}(2C_2 + 2C_5 + C_8 + C_{10}) = \mathrm{mult}_{P_{\{y\},\{t\},\{x,z\}}}(S_{\lambda_1} \cdot S_{\lambda_2}) \\ &= \mathrm{mult}_{P_{\{y\},\{t\},\{x,z\}}}(S_{\lambda_1})\, \mathrm{mult}_{P_{\{y\},\{t\},\{x,z\}}}(S_{\lambda_2}) + \mathbf{m}_{11} = 4 + \mathbf{m}_{11},\end{aligned}$$

so that $\mathbf{m}_{11} = 2$. Similarly, we get $\mathbf{m}_{12} = \mathbf{m}_{13} = \mathbf{m}_{14} = 2$.

Observe that $\mathbf{M}_{11}^{-7} = \mathbf{M}_{12}^{-7} = \mathbf{M}_{13}^{-7} = \mathbf{M}_{14}^{-7} = 1$ and $\widehat{D}_{-7} = \widehat{S}_{-7}$ in (1.10.1). Thus, it follows from Corollary 1.10.10 that $[\mathsf{f}^{-1}(-7)] = 2$.

Similarly, we see that $\mathbf{M}_{11}^{-6} = \mathbf{M}_{12}^{-6} = \mathbf{M}_{13}^{-6} = 1$, $\mathbf{M}_{14}^{-6} = 2$, and $\widehat{D}_{-6} = \widehat{S}_{-6}$. Therefore, it follows from (1.10.8) and Lemma 1.10.7 that $[\mathsf{f}^{-1}(-6)] = 6$. $\square$

Since $h^{1,2}(X) = 6$, we see that ($\heartsuit$) in the Main Theorem holds in this case.

Family 2.6. In this case, the threefold X is a divisor of bidegree $(2,2)$ in $\mathbb{P}^2 \times \mathbb{P}^2$, so that $h^{1,2}(X) = 9$. A toric Landau–Ginzburg model is given by

$$\begin{aligned}x + y + \frac{x}{z} + \frac{y}{z} + \frac{xz}{y} + 2z + \frac{yz}{x} + \frac{2x}{y} + \frac{2y}{x} + \frac{x}{yz} + \frac{2}{z} \\ + \frac{y}{xz} + \frac{z^2}{y} + \frac{z^2}{x} + \frac{3z}{y} + \frac{3z}{x} + \frac{3}{y} + \frac{3}{x} + \frac{1}{yz} + \frac{1}{xz},\end{aligned}$$

which is the Minkowski polynomial 3873.2. The pencil $\mathcal{S}$ is given by

$$\begin{aligned}x^2zy + y^2zx + x^2ty + y^2tx + x^2z^2 + 2z^2yx + y^2z^2 + 2x^2tz + 2y^2tz + x^2t^2 \\ + 2t^2yx + t^2y^2 + z^3x + z^3y + 3z^2tx + 3z^2ty + 3t^2zx + 3t^2zy + t^3x + t^3y = \lambda xyzt.\end{aligned}$$

This equation is invariant with respect to the permutations $x \leftrightarrow y$ and $z \leftrightarrow t$.

To describe the base locus of the pencil $\mathcal{S}$, we observe that

$$\begin{aligned}H_{\{x\}} \cdot S_0 &= L_{\{x\},\{y\}} + 2L_{\{x\},\{z,t\}} + L_{\{x\},\{y,z,t\}}, \\ H_{\{y\}} \cdot S_0 &= L_{\{x\},\{y\}} + 2L_{\{y\},\{z,t\}} + L_{\{y\},\{x,z,t\}}, \\ H_{\{z\}} \cdot S_0 &= L_{\{z\},\{t\}} + L_{\{z\},\{x,y\}} + L_{\{z\},\{y,t\}} + L_{\{z\},\{x,t\}}, \\ H_{\{t\}} \cdot S_0 &= L_{\{z\},\{t\}} + L_{\{t\},\{x,y\}} + L_{\{t\},\{y,z\}} + L_{\{t\},\{x,z\}}.\end{aligned}$$

We let $C_1 = L_{\{x\},\{y\}}$, $C_2 = L_{\{z\},\{t\}}$, $C_3 = L_{\{x\},\{z,t\}}$, $C_4 = L_{\{y\},\{z,t\}}$, $C_5 = L_{\{x\},\{y,z,t\}}$, $C_6 = L_{\{y\},\{x,z,t\}}$, $C_7 = L_{\{z\},\{x,y\}}$, $C_8 = L_{\{z\},\{y,t\}}$, $C_9 = L_{\{z\},\{x,t\}}$, $C_{10} = L_{\{t\},\{x,y\}}$, $C_{11} = L_{\{t\},\{y,z\}}$, and $C_{12} = L_{\{t\},\{x,z\}}$. Then $\mathbf{m}_5 = \ldots = \mathbf{m}_{12} = 1$ and $\mathbf{m}_1 = \ldots = \mathbf{m}_4 = 2$. Similarly, we have

$$\mathbf{M}_1^{-4} = \mathbf{M}_2^{-4} = \mathbf{M}_5^{-4} = \ldots = \mathbf{M}_{12}^{-4} = 1 \qquad \text{and} \qquad \mathbf{M}_3^{-4} = \mathbf{M}_4^{-4} = 2,$$

so that S_{-4} is singular along the lines $L_{\{x\},\{z,t\}}$ and $L_{\{y\},\{z,t\}}$.

For every $\lambda \notin \{-4, \infty\}$, the surface S_λ has isolated singularities, which implies, in particular, that S_λ is irreducible. On the other hand, the surface S_{-4} is reducible:

$$S_{-4} = H_{\{x,y\}} + H_{\{z,t\}} + H_{\{y,z,t\}} + H_{\{x,z,t\}}.$$

If $\lambda \notin \{-4, \infty\}$, then the singular points of S_λ contained in the base locus of the pencil $\mathcal{S}$ are the points $P_{\{x\},\{z\},\{t\}}$, $P_{\{y\},\{z\},\{t\}}$, $P_{\{x\},\{y\},\{z,t\}}$, and $P_{\{z\},\{t\},\{x,y\}}$. These are the fixed singular points of the surfaces in $\mathcal{S}$. Let us describe their singularity types and explicitly construct the birational morphism α in (1.9.3). We start with $P_{\{x\},\{z\},\{t\}}$.

In the chart $y = 1$, the surface S_λ is given by

$$(z+t)(x+z+t) + \big(t^3 + 2t^2x + 3t^2z + x^2t + 3z^2t + x^2z + 2xz^2 + z^3 - \lambda txz\big) \\ + \big(xt^3 + x^2t^2 + 3t^2xz + 2tx^2z + 3txz^2 + x^2z^2 + z^3x\big) = 0.$$

For convenience, we rewrite the defining equation of the surface S_λ as

$$\widehat{x}\widehat{t} + \big(4\widehat{t}^{\,2}\widehat{z} + \widehat{t}\widehat{x}^2 - 4\widehat{t}\widehat{x}\widehat{z} - 4\widehat{t}\widehat{z}^2 + 4\widehat{x}\widehat{z}^2 + \lambda\widehat{t}^{\,2}\widehat{z} - \lambda\widehat{t}\widehat{x}\widehat{z} - \lambda\widehat{t}\widehat{z}^2 + \lambda\widehat{x}\widehat{z}^2\big) + \big(\widehat{t}^{\,2}\widehat{x}^2 - \widehat{t}^{\,3}\widehat{x}\big) = 0,$$

where $\widehat{x} = x + z + t$, $\widehat{z} = z$, and $\widehat{t} = z + t$.

Let $\alpha_1\colon U_1 \to \mathbb{P}^3$ be the blow-up of the point $P_{\{x\},\{z\},\{t\}}$. A chart of this blow-up is given by the coordinate change $\widehat{x}_1 = \widehat{x}/\widehat{z}$, $\widehat{z}_1 = \widehat{z}$, $\widehat{t}_1 = \widehat{t}/\widehat{z}$. In this chart, the surface D^1_λ is given by

$$\widehat{t}_1\widehat{x}_1 - (\lambda+4)\widehat{t}_1\widehat{z}_1 + (\lambda+4)\widehat{z}_1\widehat{x}_1 + (\lambda+4)\big(\widehat{t}_1^{\,2}\widehat{z}_1 - \widehat{t}_1\widehat{x}_1\widehat{z}_1\big) + \widehat{t}_1\widehat{x}_1^2\widehat{z}_1 + \big(\widehat{t}_1^{\,2}\widehat{x}_1^2\widehat{z}_1^2 - \widehat{t}_1^{\,3}\widehat{x}_1\widehat{z}_1^2\big) = 0,$$

where $\widehat{z}_1 = 0$ defines the surface $\mathbf{E}_1$. Then $(\widehat{x}_1, \widehat{z}_1, \widehat{t}_1) = (0,0,0)$ is the only singular point of S^1_λ that is contained in $\mathbf{E}_1$. If $\lambda \notin \{-4, \infty\}$, then this point is an ordinary double point of the surface S_λ. Hence, if $\lambda \notin \{-4, \infty\}$, then $P_{\{x\},\{z\},\{t\}}$ is a du Val singular point of S_λ of type $\mathbb{A}_3$.

Notice also that the pencil $\mathcal{S}^1$ has exactly two base curves contained in the surface $\mathbf{E}_1$. Indeed, the restriction $\mathcal{S}^1|_{\mathbf{E}_1}$ consists of the curves $\{\widehat{z}_1 = \widehat{x}_1 = 0\}$ and $\{\widehat{z}_1 = \widehat{t}_1 = 0\}$. Let us denote these curves by C^1_{13} and C^1_{14}, respectively.

Let $\alpha_2\colon U_2 \to U_1$ be the blow-up of the point $(\widehat{x}_1, \widehat{z}_1, \widehat{t}_1) = (0,0,0)$. Then $D^2_\lambda = S^2_\lambda$ for every $\lambda \in \mathbb{C}$. Moreover, the restriction $\mathcal{S}^2|_{\mathbf{E}_2}$ is a pencil of conics in $\mathbf{E}_2 \cong \mathbb{P}^2$ that is given by the equation

$$\widehat{t}_1\widehat{x}_1 - (\lambda+4)\widehat{t}_1\widehat{z}_1 + (\lambda+4)\widehat{z}_1\widehat{x}_1 = 0,$$

where we consider $\widehat{x}_1, \widehat{z}_1, \widehat{t}_1$ as projective coordinates on $\mathbf{E}_2$. This pencil does not have base curves, which implies that $\mathcal{S}^2$ does not have base curves in $\mathbf{E}_2$ either.

Since the defining equation of the surface S_λ is invariant with respect to the permutation $x \leftrightarrow y$, the point $P_{\{y\},\{z\},\{t\}}$ is also a du Val singular point of S_λ of type $\mathbb{A}_3$ provided that $\lambda \notin \{-4, \infty\}$. Let $\alpha_3\colon U_3 \to U_2$ be the blow-up of the preimage of this point. Then $\mathbf{E}_3$ contains two base curves of the pencil $\mathcal{S}^3$. Denote them by C^3_{15} and C^3_{16}. Let $\alpha_4\colon U_4 \to U_3$ be the blow-up of the point $C^3_{15} \cap C^3_{16}$. Then $D^4_\lambda = S^4_\lambda$ for every $\lambda \in \mathbb{C}$. Moreover, the surface $\mathbf{E}_4$ does not contain base curves of the pencil $\mathcal{S}^4$.

Now let us describe the singularity of the surface S_λ at the point $P_{\{x\},\{y\},\{z,t\}}$. In the chart $t = 1$, the surface S_λ is given by

$$(\lambda+4)\overline{x}\,\overline{y} - (\lambda+4)\overline{x}\,\overline{y}\,\overline{z} + \big(\overline{x}^2\overline{z}\,\overline{y} + \overline{x}^2\overline{z}^2 + \overline{y}^2\overline{z}\,\overline{x} + 2\overline{z}^2\overline{y}\,\overline{x} + \overline{z}^3\overline{x} + \overline{y}^2\overline{z}^2 + \overline{y}\,\overline{z}^3\big) = 0,$$

where $\overline{x} = x$, $\overline{y} = y$, and $\overline{z} = z + 1$. Let $\alpha_5\colon U_5 \to U_4$ be the blow-up of the preimage of the point $P_{\{x\},\{y\},\{z,t\}}$. In a neighborhood of $P_{\{x\},\{y\},\{z,t\}}$, one chart of this blow-up is given by the coordinate change $\overline{x}_5 = \overline{x}/\overline{z}$, $\overline{y}_5 = \overline{y}/\overline{z}$, $\overline{z}_5 = \overline{z}$. Then D^5_λ is given by

$$(\lambda+4)\overline{x}_5\overline{y}_5 + \big(\overline{x}_5\overline{z}_5^2 + \overline{z}_5^2\overline{y}_5 - (\lambda+4)\overline{x}_5\overline{y}_5\overline{z}_5\big) + \big(\overline{x}_5^2\overline{z}_5^2 + 2\overline{z}_5^2\overline{y}_5\overline{x}_5 + \overline{y}_5^2\overline{z}_5^2\big) \\ + \big(\overline{x}_5^2\overline{y}_5\overline{z}_5^2 + \overline{x}_5\overline{y}_5^2\overline{z}_5^2\big) = 0,$$

and $\mathbf{E}_5$ is given by $\overline{z}_5 = 0$. Note that $\mathbf{E}_5$ contains one singular point of this surface, namely, the point $(\overline{x}_5, \overline{y}_5, \overline{z}_5) = (0,0,0)$. Note also that $D^5_{-4} = S^5_{-4} + 2\mathbf{E}_5$ and $\mathcal{S}^5|_{\mathbf{E}_5}$ is a union of the curves $\{\overline{z}_5 = \overline{x}_5 = 0\}$ and $\{\overline{z}_5 = \overline{y}_5 = 0\}$. Denote them by C^5_{17} and C^5_{18}, respectively.

Let $\alpha_6\colon U_6 \to U_5$ be the blow-up of the point $C^5_{17} \cap C^5_{18}$. Locally, one chart of this blow-up is given by the coordinate change $\overline{x}_6 = \overline{x}_5/\overline{z}_5$, $\overline{y}_6 = \overline{y}_5/\overline{z}_5$, $\overline{z}_6 = \overline{z}_5$. Moreover, if $\lambda \neq -4$, then S^6_λ in this chart is given by

$$(\lambda+4)\overline{y}_6\overline{x}_6 + \overline{z}_6\overline{x}_6 + \overline{z}_6\overline{y}_6 - (\lambda+4)\overline{x}_6\overline{y}_6\overline{z}_6 + \big(\overline{x}_6^2\overline{z}_6^2 + 2\overline{z}_6^2\overline{y}_6\overline{x}_6 + \overline{y}_6^2\overline{z}_6^2\big) + \big(\overline{x}_6^2\overline{y}_6\overline{z}_6^3 + \overline{x}_6\overline{y}_6^2\overline{z}_6^3\big) = 0.$$

Here, the surface $\mathbf{E}_6$ is given by $\overline{z}_6 = 0$. If $\lambda \neq -4$, then S^6_λ has an ordinary double singularity at the point $(\overline{x}_6, \overline{y}_6, \overline{z}_6) = (0,0,0)$. Therefore, if $\lambda \neq -4$, then $P_{\{x\},\{y\},\{z,t\}}$ is a du Val singular point of S_λ of type $\mathbb{A}_5$.

By construction, we have $D^6_\lambda = S^6_\lambda \sim -K_{U_6}$ for every λ such that $\lambda \neq -4, \infty$. On the other hand, we have $D^6_{-4} = S^6_{-4} + 2\mathbf{E}^6_5$. This follows from the fact that S^5_{-4} contains the point $C^5_{17} \cap C^5_{18}$ and is smooth at it.

Remark 2.6.1. Our computations imply that the proper transform of the line $L_{\{x\},\{y\}}$ on the threefold U_6 passes through the point $(\overline{x}_6, \overline{y}_6, \overline{z}_6) = (0,0,0)$.

The restriction $\mathcal{S}^6|_{\mathbf{E}_6}$ consists of the curves $\{\overline{z}_6 = \overline{x}_6 = 0\}$ and $\{\overline{z}_6 = \overline{y}_6 = 0\}$. Let us denote these curves by C^6_{19} and C^6_{20}, respectively. Let $\alpha_7\colon U_7 \to U_6$ be the blow-up of the point $(\overline{x}_6, \overline{y}_6, \overline{z}_6) = (0,0,0)$. Then $D^7_{-4} = S^7_{-4} + 2\mathbf{E}^7_5 + \mathbf{E}^7_6$. Moreover, the restriction $\mathcal{S}^7|_{\mathbf{E}_7}$ is a pencil of conics in $\mathbf{E}_7 \cong \mathbb{P}^2$ that is given by

$$(\lambda+4)\overline{y}_6\overline{x}_6 + \overline{z}_6\overline{x}_6 + \overline{z}_6\overline{y}_6 = 0,$$

where we consider $\overline{x}_6, \overline{y}_6, \overline{z}_6$ as projective coordinates on $\mathbf{E}_7$. This pencil does not have base curves, so that $\mathcal{S}^7$ does not have base curves contained in the surface $\mathbf{E}_7$ either.

If $\lambda \neq -4$, then $P_{\{z\},\{t\},\{x,y\}}$ is an ordinary double point of the surface S_λ. Indeed, in the chart $y = 1$, the surface S_λ is given by

$$\begin{aligned}\widetilde{x}(\widetilde{z}+\widetilde{t}\,) &- (\lambda+4)\widetilde{z}\widetilde{t} \\ &= \widetilde{x}^2\widetilde{t} - (\lambda+4)\widetilde{t}\widetilde{x}\widetilde{z} + \widetilde{x}^2\widetilde{z} + \widetilde{t}^{\,3}\widetilde{x} + \widetilde{x}^2\widetilde{t}^{\,2} + 3\widetilde{t}^{\,2}\widetilde{x}\widetilde{z} + 2\widetilde{t}\widetilde{x}^2\widetilde{z} + 3\widetilde{t}\widetilde{x}\widetilde{z}^2 + \widetilde{x}^2\widetilde{z}^2 + \widetilde{z}^3\widetilde{x},\end{aligned}$$

where $\widetilde{x} = x - 1$, $\widetilde{z} = z$, and $\widetilde{t} = t$. The quadratic form $\widetilde{x}(\widetilde{z}+\widetilde{t}\,) - (\lambda+4)\widetilde{z}\widetilde{t}$ is not degenerate for $\lambda \neq -4$, so that $P_{\{z\},\{t\},\{x,y\}}$ is an ordinary double point of the surface S_λ.

Corollary 2.6.2. *Suppose that $\lambda \notin \{-4, \infty\}$. Then the singular points of S_λ contained in the base locus of the pencil $\mathcal{S}$ can be described as follows*:

$P_{\{x\},\{z\},\{t\}}$: *type $\mathbb{A}_3$ with quadratic term $(z+t)(x+z+t)$;*

$P_{\{y\},\{z\},\{t\}}$: *type $\mathbb{A}_3$ with quadratic term $(z+t)(y+z+t)$;*

$P_{\{x\},\{y\},\{z,t\}}$: *type $\mathbb{A}_5$ with quadratic term $(\lambda+4)xy$;*

$P_{\{z\},\{t\},\{x,y\}}$: *type $\mathbb{A}_1$ with quadratic term $(\lambda+4)zt - (x+y)(z+t)$.*

Let us finish the description of the birational morphism α. It is given by the following commutative diagram:

$$\begin{tikzcd}
 & U_3 \arrow[dl, "\alpha_3"'] & U_4 \arrow[l, "\alpha_4"'] & U_5 \arrow[l, "\alpha_5"'] & U_6 \arrow[l, "\alpha_6"'] & \\
U_2 \arrow[dr, "\alpha_2"'] & & & & & U_7 \arrow[ul, "\alpha_7"'] \\
 & U_1 \arrow[r, "\alpha_1"'] & \mathbb{P}^3 & & U \arrow[ll, "\alpha"'] \arrow[ur, "\alpha_8"'] &
\end{tikzcd}$$

Here α_8 is the blow-up of the preimage of the point $P_{\{z\},\{t\},\{x,y\}}$.

If $\lambda \neq -4$, then $\widehat{D}_\lambda = \widehat{S}_\lambda \sim -K_U$. On the other hand, we have $\widehat{D}_{-4} = \widehat{S}_{-4} + 2\widehat{E}_5 + \widehat{E}_6$. Moreover, the curves $\widehat{C}_1, \ldots, \widehat{C}_{20}$ are all base curves of the pencil $\widehat{\mathcal{S}}$.

Lemma 2.6.3. *One has* $\mathbf{m}_{13} = \mathbf{m}_{14} = \mathbf{m}_{15} = \mathbf{m}_{16} = \mathbf{m}_{19} = \mathbf{m}_{20} = 1$ *and* $\mathbf{m}_{17} = \mathbf{m}_{18} = 2$.

Proof. To find $\mathbf{m}_{13}$ and $\mathbf{m}_{14}$, we use

$$\begin{aligned}6 &= \mathrm{mult}_{P_{\{x\},\{z\},\{t\}}}(2C_2 + 2C_3 + C_9 + C_{12}) = \mathrm{mult}_{P_{\{x\},\{z\},\{t\}}}(S_0 \cdot S_1)\\ &= \mathrm{mult}_{P_{\{x\},\{z\},\{t\}}}(S_0)\,\mathrm{mult}_{P_{\{x\},\{z\},\{t\}}}(S_1) + \mathbf{m}_{13} + \mathbf{m}_{14} = 4 + \mathbf{m}_{13} + \mathbf{m}_{14},\end{aligned}$$

so that $\mathbf{m}_{13} = \mathbf{m}_{14} = 1$. Similarly, we see that $\mathbf{m}_{15} = \mathbf{m}_{16} = 1$.

Recall that $\widehat{D}_{-4} = \widehat{S}_{-4} + 2\widehat{E}_5 + \widehat{E}_6$, so that $\mathbf{m}_{17} \geq 2$ and $\mathbf{m}_{18} \geq 2$. But

$$\begin{aligned}8 &= \mathrm{mult}_{P_{\{x\},\{y\},\{z,t\}}}(2C_1 + 2C_3 + 2C_4 + C_5 + C_6) = \mathrm{mult}_{P_{\{x\},\{z\},\{t\}}}(S_0 \cdot S_1)\\ &= \mathrm{mult}_{P_{\{x\},\{y\},\{z,t\}}}(S_0)\,\mathrm{mult}_{P_{\{x\},\{y\},\{z,t\}}}(S_1) + \mathbf{m}_{17} + \mathbf{m}_{18} = 4 + \mathbf{m}_{17} + \mathbf{m}_{18},\end{aligned}$$

which implies that $\mathbf{m}_{17} = 2$ and $\mathbf{m}_{18} = 2$.

To find $\mathbf{m}_{19}$ and $\mathbf{m}_{20}$, recall that $\alpha_6\colon U_6 \to U_5$ is the blow-up of the point $C^5_{17} \cap C^5_{18}$. Let $P = C^5_{17} \cap C^5_{18}$. Then

$$6 = \mathrm{mult}_P(2C^5_{17} + 2C^5_{18} + 2C^5_1) = \mathrm{mult}_P(S^5_0 \cdot S^5_1) = 4 + \mathbf{m}_{19} + \mathbf{m}_{20},$$

which gives us $\mathbf{m}_{19} = 1$ and $\mathbf{m}_{20} = 1$. $\square$

For every $\lambda \notin \{-4, \infty\}$, we have $[\mathsf{f}^{-1}(\lambda)] = 1$ by Corollaries 1.5.4 and 2.6.2.

Lemma 2.6.4. *One has* $[\mathsf{f}^{-1}(-4)] = 10$.

Proof. Recall that $[S_{-4}] = 4$ and $[\widehat{D}_{-4}] = 6$. Thus, it follows from (1.10.8) that

$$\left[\mathsf{f}^{-1}(-4)\right] = 6 + \sum_{i=1}^{18} \mathbf{C}_i^{-4}.$$

On the other hand, we have

$$\mathbf{M}_3^{-4} = \mathbf{M}_4^{-4} = \mathbf{M}_{17}^{-4} = \mathbf{M}_{18}^{-4} = 2 \qquad \text{and} \qquad \mathbf{M}_1^{-4} = \mathbf{M}_2^{-4} = \mathbf{M}_5^{-4} = \ldots = \mathbf{M}_{16}^{-4} = 1.$$

But $\mathbf{m}_3 = \mathbf{m}_4 = 2$, and $\mathbf{m}_{17} = \mathbf{m}_{18} = 2$ by Lemma 2.6.3. This shows that

$$\left[\mathsf{f}^{-1}(\lambda)\right] = 6 + \sum_{i=1}^{18} \mathbf{C}_i^{-4} = 6 + \mathbf{C}_3^{-4} + \mathbf{C}_4^{-4} + \mathbf{C}_{17}^{-4} + \mathbf{C}_{18}^{-4} = 10,$$

since $\mathbf{C}_3^{-4} = \mathbf{C}_4^{-4} = \mathbf{C}_{17}^{-4} = \mathbf{C}_{18}^{-4} = 1$ by Lemma 1.10.7. $\square$

Thus, we see that ($\heartsuit$) in the Main Theorem holds in this case.

To prove ($\diamondsuit$) in the Main Theorem, we have to check ($\bigstar$). To this end, note that

$$\mathrm{rk}\,\mathrm{Pic}(\widetilde{S}_{\Bbbk} v) = \mathrm{rk}\,\mathrm{Pic}(S_{\Bbbk}) + 12.$$

This follows from the proof of Corollary 2.6.2. Moreover, if $\lambda \notin \{-4, \infty\}$, then the intersection matrix of the base curves of the pencil $\mathcal{S}$ on the surface S_λ has the same rank as the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{z,t\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{z\},\{y,t\}}$, $L_{\{z\},\{x,t\}}$, $L_{\{t\},\{y,z\}}$, $L_{\{t\},\{x,z\}}$, and H_λ, because

$$\begin{aligned}H_\lambda &\sim L_{\{x\},\{y\}} + 2L_{\{x\},\{z,t\}} + L_{\{x\},\{y,z,t\}} \sim H_{\{y\}} \cdot S_0 = L_{\{x\},\{y\}} + 2L_{\{y\},\{z,t\}} + L_{\{y\},\{x,z,t\}}\\ &\sim L_{\{z\},\{t\}} + L_{\{z\},\{x,y\}} + L_{\{z\},\{y,t\}} + L_{\{z\},\{x,t\}} \sim L_{\{z\},\{t\}} + L_{\{t\},\{x,y\}} + L_{\{t\},\{y,z\}} + L_{\{t\},\{x,z\}}.\end{aligned}$$

This implies ($\bigstar$), because the rank of the intersection matrix in the following lemma is 6.

Lemma 2.6.5. *Suppose that $\lambda \notin \{-4, \infty\}$. Then the intersection form of the curves $L_{\{x\},\{y\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{z,t\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{z\},\{y,t\}}$, $L_{\{z\},\{x,t\}}$, $L_{\{t\},\{y,z\}}$, $L_{\{t\},\{x,z\}}$, and H_λ on the surface S_λ is given by*

$$\begin{array}{c} \\ L_{\{x\},\{y\}} \\ L_{\{z\},\{t\}} \\ L_{\{x\},\{z,t\}} \\ L_{\{y\},\{z,t\}} \\ L_{\{z\},\{y,t\}} \\ L_{\{z\},\{x,t\}} \\ L_{\{t\},\{y,z\}} \\ L_{\{t\},\{x,z\}} \\ H_\lambda \end{array}
\begin{array}{c} \begin{array}{ccccccccc} L_{\{x\},\{y\}} & L_{\{z\},\{t\}} & L_{\{x\},\{z,t\}} & L_{\{y\},\{z,t\}} & L_{\{z\},\{y,t\}} & L_{\{z\},\{x,t\}} & L_{\{t\},\{y,z\}} & L_{\{t\},\{x,z\}} & H_\lambda \end{array} \\
\begin{pmatrix}
-1/2 & 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1/2 & 1/2 & 1/4 & 1/4 & 1/4 & 1/4 & 1 \\
1/2 & 1/2 & -1/6 & 1/6 & 0 & 1/2 & 0 & 1/2 & 1 \\
1/2 & 1/2 & 1/6 & -1/6 & 1/2 & 0 & 1/2 & 0 & 1 \\
0 & 1/4 & 0 & 1/2 & -5/4 & 1 & 3/4 & 0 & 1 \\
0 & 1/4 & 1/2 & 0 & 1 & -5/4 & 0 & 3/4 & 1 \\
0 & 1/4 & 0 & 1/2 & 3/4 & 0 & -5/4 & 1 & 1 \\
0 & 1/4 & 1/2 & 0 & 0 & 3/4 & 1 & -5/4 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 4
\end{pmatrix} \end{array}.$$

Proof. The entries in the last row of the intersection matrix are obvious. Let us compute its diagonal. Using Proposition A.1.3 and Remark 2.6.1, we obtain $L^2_{\{x\},\{y\}} = -1/2$, because $P_{\{x\},\{y\},\{z,t\}}$ is the only singular point of S_λ that is contained in $L_{\{x\},\{y\}}$. Similarly, from Proposition A.1.3 and Remark A.2.4 we have

$$L^2_{\{z\},\{t\}} = -2 + \frac{3}{4} + \frac{3}{4} + \frac{1}{2} = 0,$$

because the line $L_{\{z\},\{t\}}$ contains the points $P_{\{x\},\{z\},\{t\}}$, $P_{\{y\},\{z\},\{t\}}$, and $P_{\{z\},\{t\},\{x,y\}}$.

To compute $L^2_{\{x\},\{z,t\}}$, note that the line $L_{\{x\},\{z,t\}}$ contains the points $P_{\{x\},\{z\},\{t\}}$ and $P_{\{x\},\{y\},\{z,t\}}$. Applying Remark A.2.4 with $S = S_\lambda$, $O = P_{\{x\},\{z\},\{t\}}$, $n = 3$, and $C = L_{\{x\},\{z,t\}}$, we see that $\overline{C}$ contains the point $\overline{G}_1 \cap \overline{G}_2$. Similarly, applying Remark A.2.4 with $S = S_\lambda$, $O = P_{\{x\},\{y\},\{z,t\}}$, $n = 5$, and $C = L_{\{x\},\{z,t\}}$, we see that $\overline{C}$ does not contain the point $\overline{G}_1 \cap \overline{G}_5$. Thus, applying Proposition A.1.3, we get $L^2_{\{x\},\{z,t\}} = -2 + 1 + 5/6 = -1/6$.

To compute $L^2_{\{z\},\{y,t\}}$, notice that $P_{\{y\},\{z\},\{t\}}$ is the only singular point of S_λ that is contained in $L_{\{z\},\{y,t\}}$. Applying Proposition A.1.3, we see that $L^2_{\{z\},\{y,t\}} = -5/4$.

Using the symmetry $x \leftrightarrow y$, we get $L^2_{\{x\},\{z,t\}} = -1/6$ and $L^2_{\{z\},\{x,t\}} = -5/4$. Similarly, using the symmetry $z \leftrightarrow t$, we see that $L^2_{\{t\},\{y,z\}} = L_{\{t\},\{x,z\}} = -5/4$.

Now let us fill in the remaining entries in the first row of the matrix. Clearly, $L_{\{x\},\{y\}} \cdot L_{\{z\},\{t\}} = 0$, $L_{\{x\},\{y\}} \cdot L_{\{z\},\{y,t\}} = 0$, $L_{\{x\},\{y\}} \cdot L_{\{z\},\{x,t\}} = 0$, $L_{\{x\},\{y\}} \cdot L_{\{t\},\{y,z\}} = 0$, and $L_{\{x\},\{y\}} \cdot L_{\{t\},\{x,z\}} = 0$, because $L_{\{x\},\{y\}}$ does not intersect the lines $L_{\{z\},\{t\}}$, $L_{\{z\},\{y,t\}}$, $L_{\{z\},\{x,t\}}$, $L_{\{t\},\{y,z\}}$, and $L_{\{t\},\{x,z\}}$. Using the symmetry $x \leftrightarrow y$, we see that

$$L_{\{x\},\{y\}} \cdot L_{\{x\},\{z,t\}} = L_{\{x\},\{y\}} \cdot L_{\{y\},\{z,t\}}.$$

To find $L_{\{x\},\{y\}} \cdot L_{\{x\},\{z,t\}}$, we observe that $L_{\{x\},\{y\}} \cap L_{\{x\},\{z,t\}} = P_{\{x\},\{y\},\{z,t\}}$. Applying Proposition A.1.2 and Remark 2.6.1, we see that $L_{\{x\},\{y\}} \cdot L_{\{x\},\{z,t\}} = 1/2$.

Let us compute the remaining entries in the second row of the intersection matrix. Since $L_{\{z\},\{t\}} \cap L_{\{x\},\{z,t\}} = P_{\{x\},\{z\},\{t\}}$, we have $L_{\{z\},\{t\}} \cdot L_{\{x\},\{z,t\}} = 1/2$ by Proposition A.1.2 and Remark A.2.4. Using the symmetry $x \leftrightarrow y$, we get $L_{\{z\},\{t\}} \cdot L_{\{y\},\{z,t\}} = 1/2$.

Observe that $L_{\{z\},\{t\}} \cap L_{\{z\},\{x,t\}} = P_{\{x\},\{z\},\{t\}}$. Applying Remark A.2.4 with $S = S_\lambda$, $O = P_{\{x\},\{z\},\{t\}}$, $n = 3$, $C = L_{\{z\},\{t\}}$, and $Z = L_{\{z\},\{x,t\}}$, we see that $\overline{C}$ and $\overline{Z}$ intersect different curves among $\overline{G}_1$ and $\overline{G}_3$. This implies $L_{\{z\},\{t\}} \cdot L_{\{z\},\{x,t\}} = 1/4$ by Proposition A.1.2. Using the symmetry $x \leftrightarrow y$, we get $L_{\{z\},\{t\}} \cdot L_{\{z\},\{y,t\}} = 1/4$. Using the symmetry $z \leftrightarrow t$, we get

$$L_{\{z\},\{t\}} \cdot L_{\{t\},\{y,z\}} = L_{\{z\},\{t\}} \cdot L_{\{t\},\{x,z\}} = \frac{1}{4}.$$

This gives us all entries in the second row of the intersection matrix.

Let us compute the third row. Observe that $L_{\{x\},\{z,t\}} \cap L_{\{y\},\{z,t\}} = P_{\{x\},\{y\},\{z,t\}}$. Applying Remark A.2.4 with $S = S_\lambda$, $O = P_{\{x\},\{y\},\{z,t\}}$, $n = 5$, $C = L_{\{x\},\{z,t\}}$, and $Z = L_{\{y\},\{z,t\}}$, we see that $\overline{C}$ and $\overline{Z}$ intersect different curves among $\overline{G}_1$ and $\overline{G}_5$. Then $L_{\{x\},\{z,t\}} \cdot L_{\{y\},\{z,t\}} = 1/6$ by Proposition A.1.2.

Since $L_{\{x\},\{z,t\}} \cap L_{\{z\},\{y,t\}} = \varnothing$, we have $L_{\{x\},\{z,t\}} \cdot L_{\{z\},\{y,t\}} = 0$. Using the symmetry $z \leftrightarrow t$, we get $L_{\{x\},\{z,t\}} \cap L_{\{t\},\{y,z\}} = 0$. Since $L_{\{x\},\{z,t\}} \cap L_{\{z\},\{x,t\}} = P_{\{x\},\{z\},\{t\}}$, we get

$$L_{\{x\},\{z,t\}} \cdot L_{\{z\},\{x,t\}} = \frac{1}{2}$$

by Proposition A.1.2. Using the symmetry $z \leftrightarrow t$, we get $L_{\{x\},\{z,t\}} \cdot L_{\{t\},\{x,z\}} = 1/2$.

Let us compute the remaining four entries in the fourth row of the intersection matrix. Using the symmetries $x \leftrightarrow y$ and $z \leftrightarrow t$, we get

$$L_{\{y\},\{z,t\}} \cdot L_{\{z\},\{y,t\}} = L_{\{x\},\{z,t\}} \cdot L_{\{z\},\{x,t\}} = L_{\{y\},\{z,t\}} \cdot L_{\{t\},\{y,z\}} = L_{\{y\},\{z,t\}} \cdot L_{\{z\},\{y,t\}} = \frac{1}{2}$$

and

$$L_{\{y\},\{z,t\}} \cdot L_{\{z\},\{x,t\}} = L_{\{x\},\{z,t\}} \cdot L_{\{z\},\{y,t\}} = L_{\{y\},\{z,t\}} \cdot L_{\{t\},\{x,z\}} = L_{\{y\},\{z,t\}} \cdot L_{\{z\},\{x,t\}} = 0.$$

Let us compute the remaining three entries in the fifth row of the intersection matrix. First, we have $L_{\{z\},\{y,t\}} \cdot L_{\{t\},\{x,z\}} = 0$, as $L_{\{z\},\{y,t\}} \cap L_{\{t\},\{x,z\}} = \varnothing$. Second, we have $L_{\{z\},\{y,t\}} \cdot L_{\{z\},\{x,t\}} = 1$, because $L_{\{z\},\{y,t\}} \cap L_{\{z\},\{x,t\}}$ is a smooth point of S_λ. Third, we compute $L_{\{z\},\{y,t\}} \cdot L_{\{t\},\{y,z\}}$. Observe that $L_{\{z\},\{y,t\}} \cap L_{\{t\},\{y,z\}} = P_{\{y\},\{z\},\{t\}}$. Applying Remark A.2.4 with $S = S_\lambda$, $O = P_{\{y\},\{z\},\{t\}}$, $n = 3$, $C = L_{\{z\},\{y,t\}}$, and $Z = L_{\{t\},\{y,z\}}$, we see that $\overline{C}$ and $\overline{Z}$ intersect the same curve among $\overline{G}_1$ and $\overline{G}_3$, and none of them contains the point $\overline{G}_1 \cap \overline{G}_3$. Thus, by Proposition A.1.2 we have $L_{\{z\},\{y,t\}} \cdot L_{\{t\},\{y,z\}} = 3/4$.

Let us compute the remaining three entries of the matrix. Using the symmetry $x \leftrightarrow y$, we get $L_{\{z\},\{x,t\}} \cdot L_{\{t\},\{y,z\}} = L_{\{z\},\{y,t\}} \cdot L_{\{t\},\{x,z\}} = 0$. Similarly, we have

$$L_{\{z\},\{x,t\}} \cdot L_{\{t\},\{x,z\}} = L_{\{z\},\{y,t\}} \cdot L_{\{t\},\{y,z\}} = \frac{3}{4}.$$

Finally, using the symmetry $z \leftrightarrow t$, we get $L_{\{t\},\{y,z\}} \cdot L_{\{t\},\{x,z\}} = L_{\{z\},\{y,t\}} \cdot L_{\{z\},\{x,t\}} = 1$. □

Family 2.7. In this case, the threefold X can be obtained by blowing up a smooth quadric threefold $\mathcal{Q}$ in $\mathbb{P}^4$ along a smooth curve of genus 5. This implies that $h^{1,2}(X) = 5$. A toric Landau–Ginzburg model of this family is given by the Minkowski polynomial 3238. It is

$$x + \frac{x}{y} + z + \frac{z}{y} + \frac{xy}{z} + \frac{2x}{z} + \frac{x}{yz} + 2y + \frac{2}{y} + \frac{yz}{x} + \frac{2z}{x} + \frac{z}{xy} + \frac{2y}{z} + \frac{2}{z} + \frac{2y}{x} + \frac{2}{x} + \frac{y}{xz}.$$

The corresponding pencil of quartic surfaces $\mathcal{S}$ is given by

$$x^2yz + x^2tz + z^2yx + z^2tx + y^2x^2 + 2x^2ty + x^2t^2 + 2y^2zx + 2t^2zx + y^2z^2 \\ + 2z^2ty + t^2z^2 + 2y^2tx + 2t^2yx + 2y^2tz + 2t^2yz + y^2t^2 = \lambda xyzt.$$

This equation is invariant with respect to the permutations $x \leftrightarrow z$ and $y \leftrightarrow t$.

Since the goal is to prove ($\heartsuit$) and ($\diamondsuit$) in the Main Theorem, we may assume that $\lambda \neq \infty$. Let $\mathcal{C}_1$ be the conic $\{x = yz + ty + tz = 0\}$, let $\mathcal{C}_2$ be the conic $\{y = xz + tx + tz = 0\}$, let $\mathcal{C}_3$ be the conic $\{z = xy + tx + ty = 0\}$, and let $\mathcal{C}_4$ be the conic $\{t = xy + xz + yz = 0\}$. Then

$$\begin{aligned} H_{\{x\}} \cdot S_\lambda &= 2\mathcal{C}_1, \\ H_{\{y\}} \cdot S_\lambda &= L_{\{y\},\{t\}} + L_{\{y\},\{x,z\}} + \mathcal{C}_2, \\ H_{\{z\}} \cdot S_\lambda &= 2\mathcal{C}_3, \\ H_{\{t\}} \cdot S_\lambda &= L_{\{y\},\{t\}} + L_{\{t\},\{x,z\}} + \mathcal{C}_4. \end{aligned} \tag{2.7.1}$$

Thus, the base locus of the pencil $\mathcal{S}$ consists of seven smooth rational curves. We let $C_1 = \mathcal{C}_1$, $C_2 = \mathcal{C}_2$, $C_3 = \mathcal{C}_3$, $C_4 = \mathcal{C}_4$, $C_5 = L_{\{y\},\{t\}}$, $C_6 = L_{\{y\},\{x,z\}}$, and $C_7 = L_{\{t\},\{x,z\}}$.

If $\lambda \neq -5$, then S_λ has isolated singularities, so that it is irreducible. On the other hand, one has $S_{-5} = \mathsf{Q} + \mathbf{Q}$, where Q is a quadric surface given by $tx + ty + tz + xy + yz = 0$ and $\mathbf{Q}$ is a quadric surface given by $tx + ty + tz + xy + xz + yz = 0$. Both these quadric surfaces are irreducible. The surface Q is singular at $P_{\{y\},\{t\},\{x,z\}}$, and the surface $\mathbf{Q}$ is smooth. One has $\mathsf{Q} \cap \mathbf{Q} = \mathcal{C}_1 \cup \mathcal{C}_3$, so that S_{-5} is singular along the conics $\mathcal{C}_1$ and $\mathcal{C}_3$.

If $\lambda \neq -5$, then the singular points of S_λ contained in the base locus of the pencil $\mathcal{S}$ are the points $P_{\{y\},\{z\},\{t\}}$, $P_{\{x\},\{z\},\{t\}}$, $P_{\{x\},\{y\},\{t\}}$, $P_{\{x\},\{y\},\{z\}}$, and $P_{\{y\},\{t\},\{x,z\}}$. They are all fixed singular points of the surfaces in $\mathcal{S}$.

Lemma 2.7.2. *Suppose that $\lambda \neq -5$. Then the singular points of S_λ contained in the base locus of the pencil $\mathcal{S}$ can be described as follows*:

$P_{\{y\},\{z\},\{t\}}$: *type $\mathbb{A}_3$ with quadratic term $(y+t)(y+z+t)$*;

$P_{\{x\},\{z\},\{t\}}$: *type $\mathbb{D}_4$ with quadratic term $(x+t+z)^2$*;

$P_{\{x\},\{y\},\{t\}}$: *type $\mathbb{A}_3$ with quadratic term $(y+t)(x+y+t)$*;

$P_{\{x\},\{y\},\{z\}}$: *type $\mathbb{D}_4$ with quadratic term $(x+y+z)^2$*;

$P_{\{y\},\{t\},\{x,z\}}$: *type $\mathbb{A}_1$ with quadratic term $(x+z)(y+t)-(\lambda+4)yt$ for $\lambda \neq -4$, and type $\mathbb{A}_3$ for $\lambda = -4$.*

Let us prove this lemma and explicitly construct the birational morphism α in (1.9.3). To begin with, let us resolve the singularity of the surface S_λ at the point $P_{\{y\},\{z\},\{t\}}$. In the chart $x = 1$, the surface S_λ is given by

$$\widehat{y}\,\widehat{z} + \big((\lambda+4)\widehat{t}^{\,2}\widehat{z} + (\lambda+6)\widehat{t}\,\widehat{y}^2 - (\lambda+4)\widehat{t}\,\widehat{y}\,\widehat{z} - (\lambda+6)\widehat{t}^{\,2}\widehat{y} - \widehat{y}^3 + \widehat{y}\,\widehat{z}^2\big) \\ + \big(\widehat{t}^{\,4} - 2\widehat{t}^{\,3}\widehat{y} + 3\widehat{y}^2\widehat{t}^{\,2} - 2\widehat{t}^{\,2}\widehat{y}\,\widehat{z} + \widehat{y}^4 + 2\widehat{t}\,\widehat{y}^2\widehat{z} - 2\widehat{t}\,\widehat{y}^3 - 2\widehat{y}^3\widehat{z} + \widehat{y}^2\widehat{z}^2\big) = 0,$$

where $\widehat{y} = y + t$, $\widehat{z} = t + z + y$, and $\widehat{t} = t$. Let $\alpha_1 \colon U_1 \to \mathbb{P}^3$ be the blow-up of the point $P_{\{y\},\{z\},\{t\}}$. A chart of the blow-up α_1 is given by the coordinate change $\widehat{y}_1 = \widehat{y}/\widehat{t}$, $\widehat{z}_1 = \widehat{z}/\widehat{t}$, $\widehat{t}_1 = \widehat{t}$. In this chart, the surface S^1_λ is given by the equation

$$\big(\widehat{t}_1^{\,2} - (\lambda+6)\widehat{t}_1\widehat{y}_1 + (\lambda+4)\widehat{t}_1\widehat{z}_1 + \widehat{y}_1\widehat{z}_1\big) - \big(2\widehat{t}_1^{\,2}\widehat{y}_1 - (\lambda+6)\widehat{t}_1\widehat{y}_1^2 + (\lambda+4)\widehat{t}_1\widehat{y}_1\widehat{z}_1\big) \\ + \big(3\widehat{y}_1^2\widehat{t}_1^{\,2} - 2\widehat{t}_1^{\,2}\widehat{y}_1\widehat{z}_1 - \widehat{t}_1\widehat{y}_1^3 + \widehat{t}_1\widehat{y}_1\widehat{z}_1^2\big) + \big(2\widehat{t}_1^{\,2}\widehat{y}_1^2\widehat{z}_1 - 2\widehat{t}_1^{\,2}\widehat{y}_1^3\big) + \big(\widehat{t}_1^{\,2}\widehat{y}_1^4 - 2\widehat{t}_1^{\,2}\widehat{y}_1^3\widehat{z}_1 + \widehat{t}_1^{\,2}\widehat{y}_1^2\widehat{z}_1^2\big) = 0,$$

where $\widehat{t}_1 = 0$ defines the surface $\mathbf{E}_1$. The only singular point of the surface S^1_λ in $\mathbf{E}_1$ is the point $(\widehat{y}_1, \widehat{z}_1, \widehat{t}_1) = (0,0,0)$. If $\lambda \neq -5$, then this point is an ordinary double point of S^1_λ, so that $P_{\{x\},\{z\},\{t\}}$ is a singular point of S_λ of type $\mathbb{A}_3$.

The surface $\mathbf{E}_1$ contains two base curves of the pencil $\mathcal{S}^1$. One of them is $\widehat{t}_1 = \widehat{y}_1 = 0$, and the other is $\widehat{t}_1 = \widehat{z}_1 = 0$. Let us denote these curves by C^1_8 and C^1_9, respectively. Then the proper transform of the line $L_{\{y\},\{t\}}$ on the threefold U_1 does not pass through the point $C^1_8 \cap C^1_9$.

Let $\alpha_2 \colon U_2 \to U_1$ be the blow-up of the point $(\widehat{x}_1, \widehat{z}_1, \widehat{t}_1) = (0,0,0)$. Then $D^2_\lambda \sim S^2_\lambda$ for every $\lambda \in \mathbb{C}$. Moreover, the restriction $\mathcal{S}^2|_{\mathbf{E}_2}$ is a pencil of conics in $\mathbf{E}_2 \cong \mathbb{P}^2$ that does not have base curves. This shows that $\mathbf{E}_2$ contains no base curves of the pencil $\mathcal{S}^2$.

Recall that the defining equation of the surface S_λ is invariant with respect to the permutation $x \leftrightarrow z$. Thus, if $\lambda \neq -5$, then $P_{\{x\},\{y\},\{t\}}$ is a du Val singular point of S_λ of type $\mathbb{A}_3$. Moreover, the surface S_{-5} has a non-isolated ordinary double point at $P_{\{x\},\{y\},\{t\}}$, so that $P_{\{x\},\{y\},\{t\}}$ is a good double point of the surface S_{-5}.

Let $\alpha_3\colon U_3 \to U_2$ be the blow-up of the preimage of the point $P_{\{x\},\{y\},\{t\}}$. Then the pencil $\mathcal{S}^3$ has exactly two base curves contained in the surface $\mathbf{E}_3$. Let us denote these curves by C_{10}^3 and C_{11}^3. Let $\alpha_4\colon U_4 \to U_3$ be the blow-up of the point $C_{10}^3 \cap C_{11}^3$. Then $\mathbf{E}_4$ does not contain base curves of the pencil $\mathcal{S}^4$, and the proper transform of the line $L_{\{y\},\{t\}}$ on the threefold U_3 does not pass through the point $C_{10}^3 \cap C_{11}^3$.

Now let us describe the singularity of the surface S_λ at the point $P_{\{y\},\{t\},\{x,z\}}$. In the chart $z = 1$, the surface S_λ is given by

$$\overline{x}(\overline{t}+\overline{y}) + (\lambda+4)\overline{t}\,\overline{y} - \big(\overline{t}\overline{x}^2 - (\lambda+4)\overline{t}\overline{x}\overline{y} + \overline{x}^2\overline{y}\big) - \big(\overline{x}^2\overline{t}^2 + 2\overline{t}^2\overline{x}\overline{y} + \overline{y}^2\overline{t}^2 + 2\overline{t}\overline{x}^2\overline{y} + 2\overline{t}\overline{x}\overline{y}^2 + \overline{x}^2\overline{y}^2\big) = 0,$$

where $\overline{x} = x+1$, $\overline{y} = y$, and $\overline{t} = t$. Thus, if $\lambda \neq -4$, then $P_{\{y\},\{t\},\{x,z\}}$ is an isolated ordinary double point of S_λ. If $\lambda = -4$, then the latter equation can be rewritten as

$$\breve{x}\breve{t} - \breve{y}^4 + 2\breve{t}\breve{y}^3 - \breve{y}^2\breve{t}^2 + 2\breve{t}\breve{x}\breve{y}^2 - \breve{t}\breve{x}^2 - 2\breve{t}^2\breve{x}\breve{y} - \breve{x}^2\breve{t}^2 = 0,$$

where $\breve{x} = \overline{x}$, $\breve{y} = \overline{y}$, and $\breve{t} = \overline{t} + \overline{y}$. Here, the term $\breve{x}\breve{t} - \breve{y}^4$ has the smallest degree with respect to the weights $\mathrm{wt}(\breve{x}) = 2$, $\mathrm{wt}(\breve{y}) = 1$, and $\mathrm{wt}(\breve{t}) = 2$. This shows that $P_{\{y\},\{t\},\{x,z\}}$ is a singular point of type $\mathbb{A}_3$ of the surface S_{-4}.

Let $\alpha_5\colon U_5 \to U_4$ be the blow-up of the preimage of the point $P_{\{y\},\{t\},\{x,z\}}$. Then the restriction $\mathcal{S}^5|_{\mathbf{E}_5}$ is a pencil of conics that is given by

$$\overline{x}(\overline{t}+\overline{y}) + (\lambda+4)\overline{t}\,\overline{y} = 0,$$

where we consider $\overline{x}, \overline{y}, \overline{t}$ as homogeneous coordinates on $\mathbf{E}_5 \cong \mathbb{P}^2$. This pencil does not have base curves, so that $\mathbf{E}_5$ does not contain base curves of the pencil $\mathcal{S}^5$ either.

Let us show that $P_{\{x\},\{z\},\{t\}}$ is a du Val singular point of S_λ of type $\mathbb{D}_4$. In the chart $y = 1$, the surface S_λ is given by

$$\widetilde{t}^{\,2} + \big(2\widetilde{t}^{\,2}\widetilde{x} + 2\widetilde{t}^{\,2}\widetilde{z} - 2\widetilde{t}\widetilde{x}^2 - (\lambda+8)\widetilde{t}\widetilde{x}\widetilde{z} - 2\widetilde{t}\widetilde{z}^2 + (\lambda+5)\widetilde{x}^2\widetilde{z} + (\lambda+5)\widetilde{x}\widetilde{z}^2\big) + \big(\widetilde{x}^2\widetilde{t}^{\,2} + 2\widetilde{t}^{\,2}\widetilde{x}\widetilde{z} + \widetilde{t}^{\,2}\widetilde{z}^2 - 2\widetilde{x}^3\widetilde{t} - 5\widetilde{t}\widetilde{x}^2\widetilde{z} - 5\widetilde{t}\widetilde{x}\widetilde{z}^2 - 2\widetilde{t}\widetilde{z}^3 + \widetilde{x}^4 + 3\widetilde{z}\widetilde{x}^3 + 4\widetilde{z}^2\widetilde{x}^2 + 3\widetilde{z}^3\widetilde{x} + \widetilde{z}^4\big) = 0,$$

where $\widetilde{x} = x$, $\widetilde{z} = z$, and $\widetilde{t} = x + t + z$. Let $\alpha_6\colon U_6 \to U_5$ be the blow-up of the preimage of the point $P_{\{x\},\{z\},\{t\}}$. A chart of this blow-up is given by the coordinate change $\widetilde{x}_6 = \widetilde{x}/\widetilde{z}$, $\widetilde{z}_6 = \widetilde{z}$, $\widetilde{t}_6 = \widetilde{t}/\widetilde{z}$. Then the equations $\widetilde{z}_6 = \widetilde{t}_6 = 0$ define the exceptional curve of the induced birational morphism $S_\lambda^6 \to S_\lambda^5$. Moreover, if $\lambda \neq -5$, then the quadratic term of the surface S_λ^6 at the point $(\widetilde{x}_6, \widetilde{z}_6, \widetilde{z}) = (0,0,0)$ is

$$\widetilde{t}_6^{\,2} - 2\widetilde{t}_6\widetilde{z}_6 + (\lambda+5)\widetilde{x}_6\widetilde{z}_6 + \widetilde{z}_6^{\,2}.$$

It is not degenerate. Thus, this point is an isolated ordinary double point of the surface S_λ^6. In this case, the chart of the surface S_λ^6 also has an isolated ordinary double singularity at the point $(\widetilde{x}_6, \widetilde{z}_6, \widetilde{t}_6) = (-1,0,0)$, and S_λ^6 is smooth along the curve $\widetilde{z}_6 = \widetilde{t}_6 = 0$ away from these two points.

Now let us consider another chart of the blow-up α_6. To this end, we introduce coordinates $\widetilde{x}_6' = \widetilde{x}$, $\widetilde{z}_6' = \widetilde{z}/\widetilde{x}$, and $\widetilde{t}_6' = \widetilde{t}/\widetilde{x}$. In this chart, the surface S_λ^6 is given by

$$(\widetilde{t}_6')^2 - 2\widetilde{x}_6'\widetilde{t}_6' + (\widetilde{x}_6')^2 + (\lambda+5)\widetilde{x}_6'\widetilde{z}_6' + \text{Higher order terms} = 0,$$

so that S_λ^6 has an isolated ordinary double singularity at the point $(\widetilde{x}_6', \widetilde{z}_6', \widetilde{t}_6') = (0,0,0)$ provided that $\lambda \neq -5$. Therefore, we have proved that if $\lambda \neq -5$, then $P_{\{x\},\{z\},\{t\}}$ is a singular point of S_λ of type $\mathbb{D}_4$.

The surface $\mathbf{E}_6$ contains one base curve of the pencil $\mathcal{S}^6$. This is the curve $\{\widetilde{z}_6 = \widetilde{t}_6 = 0\}$ in the first chart of our blow-up. Denote it by C_{12}^6. Then $\mathbf{M}_{12}^{-5} = 2$, and C_{12}^6 contains three base points of the pencil $\mathcal{S}^6$, which are fixed singular points of this pencil. They are isolated ordinary double points of the surface S^6_λ for $\lambda \neq -5$.

Recall that the defining equation of the surface S_λ is invariant with respect to the permutation $y \leftrightarrow t$. Thus, if $\lambda \neq -5$, then $P_{\{x\},\{y\},\{z\}}$ is a singular point of S_λ of type $\mathbb{D}_4$. Using the symmetry, we see that $(x+y+z)^2$ is the quadratic form of the Taylor expansion of the defining equation of the surface S_λ at the point $P_{\{x\},\{y\},\{z\}}$.

Let $\alpha_7\colon U_7 \to U_6$ be the blow-up of the preimage of the point $P_{\{x\},\{y\},\{z\}}$. Then $\mathbf{E}_7$ contains one base curve of the pencil $\mathcal{S}^7$. Denote it by C^7_{13}. This curve is the exceptional curve of the induced birational morphism $S^7_\lambda \to S^6_\lambda$. If $\lambda \neq -5$, then S^7_λ has three isolated ordinary double points at C^7_{13}. But S^7_{-5} is singular along the curve C^7_{13}.

For a general choice of $\lambda \in \mathbb{C}$, the surface S^7_λ has six singular points. All of them are fixed singular points of the pencil $\mathcal{S}^7$. They are isolated ordinary double points on every surface S^7_λ provided that $\lambda \neq -5$. This proves the assertion of Lemma 2.7.2 and shows the existence of the following commutative diagram:

$$\begin{tikzcd}
 & U_3 \arrow[dl, "\alpha_3"'] & U_4 \arrow[l, "\alpha_4"'] & U_5 \arrow[l, "\alpha_5"'] & U_6 \arrow[l, "\alpha_6"'] & \\
U_2 \arrow[dr, "\alpha_2"'] & & & & & U_7 \arrow[ul, "\alpha_7"'] \\
 & U_1 \arrow[r, "\alpha_1"'] & \mathbb{P}^3 & & U \arrow[ll, "\alpha"] \arrow[ur, "\gamma"'] &
\end{tikzcd}$$

where γ is the blow-up of the six fixed singular points of surfaces in the pencil $\mathcal{S}^7$.

Using Lemma 2.7.2 and Corollary 1.5.4, we see that $[\mathsf{f}^{-1}(\lambda)] = 1$ for every $\lambda \neq -5$. To compute $[\mathsf{f}^{-1}(-5)]$, observe that $\widehat{D}_{-5} = \widehat{S}_{-5}$, so that $[\widehat{D}_{-5}] = [\widehat{S}_{-5}] = 2$. Observe also that the base locus of the pencil $\widehat{\mathcal{S}}$ consists of the curves $\widehat{C}_1, \ldots, \widehat{C}_{13}$. Moreover, we have

$$\mathbf{M}_1^{-5} = \mathbf{M}_2^{-5} = \mathbf{M}_{12}^{-5} = \mathbf{M}_{13}^{-5} = 2 \qquad \text{and} \qquad \mathbf{M}_3^{-5} = \ldots = \mathbf{M}_{11}^{-5} = 1.$$

Arguing as in the proof of Lemma 2.6.3, we see that $\mathbf{m}_8 = \ldots = \mathbf{m}_{11} = 1$ and $\mathbf{m}_{12} = \mathbf{m}_{13} = 2$. Now, using (1.10.8) and Lemma 1.10.7, we conclude that $[\mathsf{f}^{-1}(-5)] = 6$. Thus, we see that ($\heartsuit$) in the Main Theorem holds in this case, because $h^{1,2}(X) = 5$.

To prove ($\diamondsuit$) in the Main Theorem, we have to check ($\bigstar$). To this end, recall that the base locus of the pencil $\mathcal{S}$ consists of the curves $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_4$, $L_{\{y\},\{t\}}$, $L_{\{y\},\{x,z\}}$, and $L_{\{t\},\{x,z\}}$. If $\lambda \neq -5$, then it follows from (2.7.1) that the intersection matrix of these curves on the surface S_λ has the same rank as the intersection matrix of the curves $L_{\{y\},\{t\}}$, $L_{\{y\},\{x,z\}}$, $L_{\{t\},\{x,z\}}$, and H_λ, which is given by

$$\begin{array}{r} \\ L_{\{y\},\{t\}} \\ L_{\{y\},\{x,z\}} \\ L_{\{t\},\{x,z\}} \\ H_\lambda \end{array}
\begin{array}{c} \begin{array}{cccc} L_{\{y\},\{t\}} & L_{\{y\},\{x,z\}} & L_{\{t\},\{x,z\}} & H_\lambda \end{array} \\
\begin{pmatrix} 0 & 1/2 & 1/2 & 1 \\ 1/2 & -1/2 & 1/2 & 1 \\ 1/2 & 1/2 & -1/2 & 1 \\ 1 & 1 & 1 & 4 \end{pmatrix} \end{array}.$$

The rank of this intersection matrix is 3. Moreover, we have $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk}) = \operatorname{rk}\operatorname{Pic}(S_{\Bbbk}) + 15$. Thus, we see that ($\bigstar$) holds, so that ($\diamondsuit$) in the Main Theorem holds in this case.

Family 2.8. One has $h^{1,2}(X) = 9$. In this case, the threefold X is a double cover of the toric Fano threefold obtained by blowing up $\mathbb{P}^3$ at one point. The ramification surface of this double cover is contained in the anticanonical linear system of this toric Fano threefold. A toric Landau–Ginzburg model of the threefold X is given by the Minkowski polynomial 1968. It is

$$\frac{xy}{z} + 2x + \frac{xz}{y} + \frac{2x}{z} + \frac{2x}{y} + \frac{x}{yz} + 2y + 2z + \frac{2}{z} + \frac{2}{y} + \frac{yz}{x} + \frac{2}{yz} + \frac{2}{x} + \frac{1}{xyz}.$$

The pencil of quartic surfaces $\mathcal{S}$ is given by

$$x^2y^2+2x^2zy+x^2z^2+2x^2ty+2x^2tz+x^2t^2+2y^2zx+2z^2yx \\ +2t^2yx+2t^2zx+y^2z^2+2t^3x+2t^2zy+t^4=\lambda xyzt.$$

This equation is invariant with respect to the permutation $y\leftrightarrow z$.

We may assume that $\lambda\neq\infty$. Then

$$H_{\{x\}}\cdot S_\lambda=2\mathcal{C}_1,\qquad H_{\{y\}}\cdot S_\lambda=2\mathcal{C}_2,\qquad H_{\{z\}}\cdot S_\lambda=2\mathcal{C}_3,\qquad H_{\{t\}}\cdot S_\lambda=2\mathcal{C}_4, \tag{2.8.1}$$

where $\mathcal{C}_1$ is a smooth conic given by $x=yz+t^2=0$, the curve $\mathcal{C}_2$ is a smooth conic given by $y=xz+tx+t^2=0$, the curve $\mathcal{C}_3$ is a smooth conic given by $z=xy+tx+t^2=0$, and the curve $\mathcal{C}_4$ is a smooth conic given by $t=xy+xz+yz=0$. Thus, we see that

$$S_\lambda\cdot S_\infty=2\mathcal{C}_1+2\mathcal{C}_2+2\mathcal{C}_3+2\mathcal{C}_4,$$

so that the base locus of the pencil $\mathcal{S}$ consists of four smooth rational curves. To match the notation introduced in Section 1, we let $C_1=\mathcal{C}_1$, $C_2=\mathcal{C}_2$, $C_3=\mathcal{C}_3$, and $C_4=\mathcal{C}_4$.

If $\lambda\neq-2$, then S_λ has isolated singularities, so that it is irreducible. On the other hand, the surface S_{-2} is not reduced. Indeed, one has $S_{-2}=2\mathbf{Q}$, where $\mathbf{Q}$ is an irreducible quadric surface in $\mathbb{P}^3$ given by $t^2+tx+xy+xz+yz=0$. One can check that $\mathbf{Q}$ is smooth.

If $\lambda\neq-2$, then the singular points of S_λ contained in the base locus of the pencil $\mathcal{S}$ are the points $P_{\{x\},\{y\},\{t\}}$, $P_{\{x\},\{z\},\{t\}}$, $P_{\{y\},\{z\},\{t\}}$, and $P_{\{y\},\{z\},\{x,t\}}$. In this case, the surface S_λ has du Val singularities at these points. In fact, we can say more.

Lemma 2.8.2. *If $\lambda\neq-2$, then the singular points of S_λ contained in the base locus of the pencil $\mathcal{S}$ can be described as follows*:

$P_{\{x\},\{y\},\{t\}}$: *type $\mathbb{D}_6$ with quadratic term $(x+y)^2$*;

$P_{\{x\},\{z\},\{t\}}$: *type $\mathbb{D}_6$ with quadratic term $(x+z)^2$*;

$P_{\{y\},\{z\},\{t\}}$: *type $\mathbb{D}_4$ with quadratic term $(y+z+t)^2$*;

$P_{\{y\},\{z\},\{x,t\}}$: *type $\mathbb{A}_1$ with quadratic term $(x+t-y-z)^2+(\lambda+2)yz$.*

Proof. We skip the computations of the quadratic terms, because they are straightforward. If $\lambda\neq-2$, then $P_{\{y\},\{z\},\{x,t\}}$ is an isolated ordinary double point of the surface S_λ. Note that the expressions for quadratic terms are also valid for $\lambda=-2$.

Let us describe the singularity type of the point $P_{\{y\},\{z\},\{t\}}$. In the chart $x=1$, the surface S_λ is given by

$$\widehat{t}^{\,2}+2\widehat{t}^{\,3}-4\widehat{t}^{\,2}\widehat{y}-4\widehat{t}^{\,2}\widehat{z}+2\widehat{y}^{\,2}\widehat{t}+(4-\lambda)\widehat{t}\,\widehat{y}\,\widehat{z}+2\widehat{z}^{\,2}\widehat{t}+(2+\lambda)\widehat{y}^{\,2}\widehat{z} \\ +(2+\lambda)\widehat{z}^{\,2}\widehat{y}+\widehat{t}^{\,4}-4\widehat{y}\widehat{t}^{\,3}-4\widehat{z}\widehat{t}^{\,3}+6\widehat{t}^{\,2}\widehat{y}^{\,2}+14\widehat{t}^{\,2}\widehat{y}\,\widehat{z}+6\widehat{t}^{\,2}\widehat{z}^{\,2}-4\widehat{t}\,\widehat{y}^{\,3} \\ -16\widehat{t}\,\widehat{y}^{\,2}\widehat{z}-16\widehat{t}\,\widehat{y}\,\widehat{z}^{\,2}-4\widehat{t}\,\widehat{z}^{\,3}+\widehat{y}^{\,4}+6\widehat{y}^{\,3}\widehat{z}+11\widehat{y}^{\,2}\widehat{z}^{\,2}+6\widehat{y}\,\widehat{z}^{\,3}+\widehat{z}^{\,4}=0$$

for $\widehat{y}=y$, $\widehat{z}=z$, and $\widehat{t}=y+z+t$. Let $\alpha_1\colon U_1\to\mathbb{P}^3$ be the blow-up of the point $P_{\{y\},\{z\},\{t\}}$. One chart of the blow-up α_1 is given by the coordinate change $\widehat{y}_1=\widehat{y}$, $\widehat{z}_1=\widehat{z}/\widehat{y}$, $\widehat{t}_1=\widehat{t}/\widehat{y}$. In this chart, the surface S^1_λ is given by

$$\widehat{t}_1^{\,2}+2\widehat{t}_1\widehat{y}_1+\widehat{y}_1^{\,2}+(2+\lambda)\widehat{y}_1\widehat{z}_1+\big(6\widehat{y}_1^{\,2}\widehat{z}_1-4\widehat{t}_1^{\,2}\widehat{y}_1-4\widehat{y}_1^{\,2}\widehat{t}_1+(4-\lambda)\widehat{t}_1\widehat{y}_1\widehat{z}_1+(2+\lambda)\widehat{z}_1^{\,2}\widehat{y}_1\big) \\ +\big(2\widehat{y}_1\widehat{t}_1^{\,3}+6\widehat{t}_1^{\,2}\widehat{y}_1^{\,2}-4\widehat{t}_1^{\,2}\widehat{y}_1\widehat{z}_1-16\widehat{t}_1\widehat{y}_1^{\,2}\widehat{z}_1+2\widehat{t}_1\widehat{y}_1\widehat{z}_1^{\,2}+11\widehat{y}_1^{\,2}\widehat{z}_1^{\,2}\big) \\ +\big(14\widehat{t}_1^{\,2}\widehat{y}_1^{\,2}\widehat{z}_1-4\widehat{t}_1^{\,3}\widehat{y}_1^{\,2}-16\widehat{t}_1\widehat{y}_1^{\,2}\widehat{z}_1^{\,2}+6\widehat{y}_1^{\,2}\widehat{z}_1^{\,3}\big) \\ +\big(\widehat{t}_1^{\,4}\widehat{y}_1^{\,2}-4\widehat{t}_1^{\,3}\widehat{y}_1^{\,2}\widehat{z}_1+6\widehat{t}_1^{\,2}\widehat{y}_1^{\,2}\widehat{z}_1^{\,2}-4\widehat{t}_1\widehat{y}_1^{\,2}\widehat{z}_1^{\,3}+\widehat{y}_1^{\,2}\widehat{z}_1^{\,4}\big)=0,$$

and $\mathbf{E}_1$ is given by $\widehat{y}_1=0$. Let $C_5^1=S_\lambda^1\cap\mathbf{E}_1$. Then C_5^1 is the line in $\mathbf{E}_1\cong\mathbb{P}^2$ that is given by $\widehat{y}_1=\widehat{t}_1=0$. Note that $\mathbf{M}_5^{-2}=2$. If $\lambda\neq-2$, then the only singular points of S_λ^1 contained in C_5^1 are the points $(\widehat{y}_1,\widehat{z}_1,\widehat{t}_1)=(0,0,0)$ and $(\widehat{y}_1,\widehat{z}_1,\widehat{t}_1)=(0,0,-1)$. Both of them are isolated ordinary double points of the surface S_λ in this case.

If $\lambda\neq-2$, then S_λ^1 has three isolated ordinary double points in C_5^1. We have already described two of them. The third one can be seen in another chart of the blow-up α_1. Thus, if $\lambda\neq-2$, then $P_{\{y\},\{z\},\{t\}}$ is a singular point of type $\mathbb{D}_4$ of the surface S_λ.

Now let us describe the singularity of the surface S_λ at the point $P_{\{x\},\{z\},\{t\}}$. In the chart $y=1$, the surface S_λ is given by

$$\bar{z}^2+\big(2\bar{t}^2\bar{z}+(2+\lambda)\bar{x}^2\bar{t}-\lambda\bar{t}\bar{x}\bar{z}-2\bar{x}^2\bar{z}+2\bar{x}\bar{z}^2\big)\\ +\big(\bar{t}^4+2\bar{t}^3\bar{x}-\bar{t}^2\bar{x}^2+2\bar{t}^2\bar{x}\bar{z}-2\bar{x}^3\bar{t}+2\bar{t}\bar{x}^2\bar{z}+\bar{x}^4-2\bar{x}^3\bar{z}+\bar{x}^2\bar{z}^2\big)=0,$$

where $\bar{x}=x$, $\bar{z}=x+z$, and $\bar{t}=t$. Let $\alpha_2\colon U_2\to U_1$ be the blow-up of the preimage of the point $P_{\{x\},\{z\},\{t\}}$. A chart of this blow-up is given by the coordinate change $\bar{x}_2=\bar{x}/\bar{t}$, $\bar{z}_2=\bar{z}/\bar{t}$, $\bar{t}_2=\bar{t}$. In this chart, the surface S_λ^2 is given by

$$\big(\bar{t}_2+\bar{z}_2\big)^2+\big(2\bar{t}_2^2\bar{x}_2+(2+\lambda)\bar{x}_2^2\bar{t}_2-\lambda\bar{t}_2\bar{x}_2\bar{z}_2\big)+\big(2\bar{t}_2^2\bar{x}_2\bar{z}_2-\bar{t}_2^2\bar{x}_2^2-2\bar{t}_2\bar{x}_2^2\bar{z}_2+2\bar{t}_2\bar{x}_2\bar{z}_2^2\big)\\ +\big(2\bar{t}_2^2\bar{x}_2^2\bar{z}_2-2\bar{t}_2^2\bar{x}_2^3\big)+\big(\bar{t}_2^2\bar{x}_2^4-2\bar{t}_2^2\bar{x}_2^3\bar{z}_2+\bar{t}_2^2\bar{x}_2^2\bar{z}_2^2\big)=0,$$

and the surface $\mathbf{E}_2$ is given by $\bar{t}_2=0$. Let $C_6^2=S_\lambda^2\cap\mathbf{E}_2$. Then C_6^2 is the line in $\mathbf{E}_2\cong\mathbb{P}^2$ that is given by $\bar{t}_2=\bar{z}_2=0$. Observe that $\mathbf{M}_6^{-2}=2$. On the other hand, if $\lambda\neq-2$, then the point $(\bar{x}_2,\bar{z}_2,\bar{t}_2)=(0,0,0)$ is the only singular point of S_λ^2 that is contained in the curve C_6^2 in this chart. Note that C_6^2 contains another singular point of S_λ^2 that can be seen in another chart of the blow-up α_2. This point is an isolated ordinary double singularity of the surface S_λ^2.

To determine the type of the singular point $(\bar{x}_2,\bar{z}_2,\bar{t}_2)=(0,0,0)$ on the surface S_λ^2 for every $\lambda\neq-2$, we let $\widetilde{x}_2=\bar{x}_2$, $\widetilde{z}_2=\bar{z}_2$, and $\widetilde{t}_2=\bar{t}_2+\bar{z}_2$. Then we can rewrite the defining equation of the surface S_λ as

$$\widetilde{t}_2^2+\big(2\widetilde{t}_2^2\widetilde{x}_2-(2+\lambda)\widetilde{x}_2^2\widetilde{z}_2+(2+\lambda)\widetilde{x}_2^2\widetilde{t}_2+(2+\lambda)\widetilde{x}_2\widetilde{z}_2^2-(\lambda+4)\widetilde{t}_2\widetilde{x}_2\widetilde{z}_2\big)\\ +\big(2\widetilde{t}_2^2\widetilde{x}_2\widetilde{z}_2-\widetilde{t}_2^2\widetilde{x}_2^2-2\widetilde{t}_2\widetilde{x}_2\widetilde{z}_2^2+\widetilde{x}_2^2\widetilde{z}_2^2\big)\\ +\big(2\widetilde{t}_2^2\widetilde{x}_2^2\widetilde{z}_2-2\widetilde{t}_2^2\widetilde{x}_2^3+4\widetilde{t}_2\widetilde{x}_2^3\widetilde{z}_2-4\widetilde{t}_2\widetilde{x}_2^2\widetilde{z}_2^2-2\widetilde{x}_2^3\widetilde{z}_2^2+2\widetilde{x}_2^2\widetilde{z}_2^3\big)\\ +\big(\widetilde{t}_2^2\widetilde{x}_2^4-2\widetilde{t}_2^2\widetilde{x}_2^3\widetilde{z}_2+\widetilde{t}_2^2\widetilde{x}_2^2\widetilde{z}_2^2-2\widetilde{t}_2\widetilde{x}_2^4\widetilde{z}_2+4\widetilde{t}_2\widetilde{x}_2^3\widetilde{z}_2^2-2\widetilde{t}_2\widetilde{x}_2^2\widetilde{z}_2^3+\widetilde{x}_2^4\widetilde{z}_2^2-2\widetilde{x}_2^3\widetilde{z}_2^3+\widetilde{x}_2^2\widetilde{z}_2^4\big)=0.$$

Let $\alpha_3\colon U_3\to U_2$ be the blow-up of the point $(\widetilde{x}_2,\widetilde{z}_2,\widetilde{t}_2)=(0,0,0)$. A chart of this blow-up is given by the coordinate change $\widetilde{x}_3=\widetilde{x}_2$, $\widetilde{z}_3=\widetilde{z}_2/\widetilde{t}_2$, $\widetilde{t}_3=\widetilde{t}_2/\widetilde{x}_2$. In this chart, the surface S_λ^3 is given by

$$(2+\lambda)\widetilde{z}_3\widetilde{x}_3-(2+\lambda)\widetilde{t}_3\widetilde{x}_3-\widetilde{t}_3^2=2\widetilde{t}_3^2\widetilde{x}_3+(2+\lambda)\widetilde{x}_3\widetilde{z}_3^2-(\lambda+4)\widetilde{t}_3\widetilde{x}_3\widetilde{z}_3+\widetilde{x}_3^2\widetilde{z}_3^2-\widetilde{t}_3^2\widetilde{x}_3^2\\ +2\widetilde{t}_3^2\widetilde{x}_3^2\widetilde{z}_3-2\widetilde{t}_3^2\widetilde{x}_3^3+4\widetilde{t}_3\widetilde{x}_3^3\widetilde{z}_3-2\widetilde{t}_3\widetilde{x}_3^2\widetilde{z}_3^2-2\widetilde{x}_3^3\widetilde{z}_3^2+\widetilde{t}_3^2\widetilde{x}_3^4+2\widetilde{t}_3^2\widetilde{x}_3^3\widetilde{z}_3-2\widetilde{t}_3\widetilde{x}_3^4\widetilde{z}_3\\ -4\widetilde{t}_3\widetilde{x}_3^3\widetilde{z}_3^2+\widetilde{x}_3^4\widetilde{z}_3^2+2\widetilde{x}_3^3\widetilde{z}_3^3+4\widetilde{t}_3\widetilde{x}_3^4\widetilde{z}_3^2-2\widetilde{t}_3^2\widetilde{x}_3^4\widetilde{z}_3-2\widetilde{x}_3^4\widetilde{z}_3^3+\widetilde{t}_3^2\widetilde{x}_3^4\widetilde{z}_3^2-2\widetilde{t}_3\widetilde{x}_3^4\widetilde{z}_3^3+\widetilde{x}_3^4\widetilde{z}_3^4,$$

and the surface $\mathbf{E}_3$ is given by $\widetilde{x}_3=0$. Let $C_7^3=S_\lambda^3\cap\mathbf{E}_3$. Then C_7^3 is the line in $\mathbf{E}_3\cong\mathbb{P}^2$ that is given by $\widetilde{x}_3=\widetilde{t}_3=0$ in our chart of the blow-up α_3. Observe that $\mathbf{M}_7^{-2}=2$.

If $\lambda\neq-2$, then the point $(\widetilde{x}_3,\widetilde{z}_3,\widetilde{t}_3)=(0,0,0)$ is an isolated ordinary double point of the surface S_λ^2. This point is contained in the curve C_7^3. Moreover, this curve contains two more singular points of the surface S_λ^2. One of them is the point $(\widetilde{x}_3,\widetilde{z}_3,\widetilde{t}_3)=(0,-1,0)$, and the other lies in another chart of the blow-up α_3. If $\lambda\neq-2$, both these points are isolated ordinary double points of the surface S_λ^3. This means that the surface S_λ^2 has a du Val singularity of type $\mathbb{D}_4$ at the point $(\overline{x}_2, \overline{z}_2, \overline{t}_2) = (0,0,0)$, so that S_λ has a du Val singularity of type $\mathbb{D}_6$ at the point $P_{\{x\},\{z\},\{t\}}$ for every $\lambda \neq -2$.

Keeping in mind that the defining equation of the surface S_λ is symmetric with respect to the permutation $y \leftrightarrow z$, we see that the surface S_λ has a du Val singularity of type $\mathbb{D}_6$ at the point $P_{\{x\},\{y\},\{t\}}$ for every $\lambda \neq -2$. This completes the proof of the lemma. $\square$

The proof of this lemma also gives $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk}) = \operatorname{rk}\operatorname{Pic}(S_{\Bbbk}) + 17$, which implies ($\bigstar$). Indeed, if $\lambda \neq -2$, then $2\mathcal{C}_1 \sim 2\mathcal{C}_2 \sim 2\mathcal{C}_3 \sim 2\mathcal{C}_4 \sim H_\lambda$ on the surface S_λ by (2.8.1), so that the intersection matrix of the conics $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, and $\mathcal{C}_4$ on the surface S_λ has rank 1. Thus, we see that ($\bigstar$) holds, so that ($\diamondsuit$) in the Main Theorem holds in this case.

Lemma 2.8.3. *If $\lambda \neq -2$, then $[\mathsf{f}^{-1}(\lambda)] = 1$. One also has $[\mathsf{f}^{-1}(-2)] = 10$.*

Proof. Using Lemma 2.8.2 and Corollary 1.5.4, we see that $[\mathsf{f}^{-1}(\lambda)] = 1$ for $\lambda \neq -2$. To show that $[\mathsf{f}^{-1}(-2)] = 10$, let us describe the birational morphism α in (1.9.3). Implicitly, this was already done in the proof of Lemma 2.8.2. Because of this, we will use the notation introduced in that proof.

Let $\alpha_4 \colon U_4 \to U_3$ be the blow-up of the preimage of the point $P_{\{x\},\{y\},\{t\}}$. If $\lambda \neq -2$, then the surface S^4_λ has a unique singular point (of type $\mathbb{D}_4$) that is contained in $\mathbf{E}_4$. Let $\alpha_5 \colon U_5 \to U_4$ be the blow-up of this singular point. Then S^5_λ has 12 singular points for general $\lambda \in \mathbb{C}$. One of them is $P_{\{y\},\{z\},\{x,t\}}$, another three are contained in the surface $\mathbf{E}_5$, another one is contained in the surface $\mathbf{E}^5_4$, and the remaining seven were explicitly described in the proof of Lemma 2.8.2. All these 12 points are isolated ordinary double points of the surface S^5_λ provided that $\lambda \neq -2$. Thus, there exists a commutative diagram

$$\begin{tikzcd}
 & U_3 \arrow[dl, "\alpha_3"'] & U_4 \arrow[l, "\alpha_4"'] & U_5 \arrow[l, "\alpha_5"'] & \\
U_2 \arrow[dr, "\alpha_2"'] & & & & U \arrow[ul, "\gamma"'] \arrow[dl, "\alpha"] \\
 & U_1 \arrow[rr, "\alpha_1"'] & & \mathbb{P}^3 &
\end{tikzcd}$$

where $\gamma \colon U \to U_5$ is the blow-up of these 12 points. This gives $\widehat{D}_\lambda = \widehat{S}_\lambda$ for every $\lambda \in \mathbb{C}$.

Let us describe the base curves of the pencil $\widehat{\mathcal{S}}$. Four of them are $\widehat{C}_1$, $\widehat{C}_2$, $\widehat{C}_3$, and $\widehat{C}_4$. The next three are the curves $\widehat{C}_5$, $\widehat{C}_6$, and $\widehat{C}_7$, which are described in the proof of Lemma 2.8.2. The pencil $\widehat{\mathcal{S}}$ contains two more base curves, whose construction is similar to that of the curves $\widehat{C}_6$ and $\widehat{C}_7$. One of them is contained in the surface $\widehat{E}_4$, and the other is contained in the surface $\widehat{E}_5$. Denote the former curve by $\widehat{C}_8$ and the latter by $\widehat{C}_9$. Then $\widehat{C}_1, \dots, \widehat{C}_9$ are all base curves of the pencil $\widehat{\mathcal{S}}$.

Note that $\mathbf{m}_1 = \ldots = \mathbf{m}_9 = 2$ and $\mathbf{M}_1^{-2} = \ldots = \mathbf{M}_9^{-2} = 2$. Thus, using (1.10.8) and Lemma 1.10.7, we conclude that $[\mathsf{f}^{-1}(-2)] = 10$. $\square$

Using Lemma 2.8.3, we see that ($\heartsuit$) in the Main Theorem holds in this case.

Family 2.9. In this case, the threefold X is a blow-up of $\mathbb{P}^3$ along a smooth curve of degree 7 and genus 5. Thus, we have $h^{1,2}(X) = 5$. A toric Landau–Ginzburg model of the threefold X is given by the Minkowski polynomial 3013, which is

$$x + y + z + \frac{x}{z} + \frac{y}{z} + \frac{x}{y} + \frac{y}{x} + 2\frac{z}{y} + 2\frac{z}{x} + \frac{z^2}{xy} + \frac{x}{yz} + \frac{2}{z} + \frac{y}{xz} + \frac{2}{y} + \frac{2}{x} + \frac{z}{xy}.$$

The corresponding pencil $\mathcal{S}$ is given by

$$t^2x^2 + 2t^2xy + 2t^2xz + t^2y^2 + 2t^2yz + t^2z^2 + tx^2y + tx^2z + txy^2 \\ + 2txz^2 + ty^2z + 2tyz^2 + tz^3 + x^2yz + xy^2z + xyz^2 = \lambda xyzt.$$

Observe that this equation is invariant with respect to the permutation $x \leftrightarrow y$.

We may assume that $\lambda \neq \infty$. To describe the base curves of the pencil $\mathcal{S}$, let $\mathcal{C}$ be a smooth conic given by $z = xy + tx + ty = 0$. Then

$$\begin{aligned}
H_{\{x\}} \cdot S_\lambda &= L_{\{x\},\{t\}} + 2L_{\{x\},\{y,z\}} + L_{\{x\},\{z,t\}},\\
H_{\{y\}} \cdot S_\lambda &= L_{\{y\},\{t\}} + 2L_{\{y\},\{x,z\}} + L_{\{y\},\{z,t\}},\\
H_{\{z\}} \cdot S_\lambda &= L_{\{z\},\{t\}} + L_{\{z\},\{x,y\}} + \mathcal{C},\\
H_{\{t\}} \cdot S_\lambda &= L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y,z\}}.
\end{aligned} \tag{2.9.1}$$

So, we let $C_1 = L_{\{x\},\{t\}}$, $C_2 = L_{\{y\},\{t\}}$, $C_3 = L_{\{z\},\{t\}}$, $C_4 = L_{\{x\},\{y,z\}}$, $C_5 = L_{\{y\},\{x,z\}}$, $C_6 = L_{\{x\},\{z,t\}}$, $C_7 = L_{\{y\},\{z,t\}}$, $C_8 = L_{\{z\},\{x,y\}}$, $C_9 = L_{\{t\},\{x,y,z\}}$, and $C_{10} = \mathcal{C}$. These are all base curves of the pencil $\mathcal{S}$.

If $\lambda \neq -3$, then S_λ has isolated singularities, so that it is irreducible. On the other hand, we have $S_{-3} = H_{\{z,t\}} + H_{\{x,y,z\}} + \mathbf{Q}$, where $\mathbf{Q}$ is a smooth quadric surface given by $xy + t(x+y+z) = 0$. Note that S_{-3} is singular along $L_{\{x\},\{y,z\}}$ and $L_{\{y\},\{x,z\}}$, and it is smooth at general points of the remaining base curves of the pencil $\mathcal{S}$.

If $\lambda \neq -3$, then the singular points of S_λ contained in the base locus of the pencil $\mathcal{S}$ are $P_{\{x\},\{y\},\{z\}}$, $P_{\{x\},\{z\},\{t\}}$, $P_{\{y\},\{z\},\{t\}}$, $P_{\{x\},\{t\},\{y,z\}}$, $P_{\{y\},\{t\},\{x,z\}}$, and $P_{\{z\},\{t\},\{x,y\}}$. In this case, all of them are du Val singular points of the surface S_λ by the following.

Lemma 2.9.2. *If $\lambda \neq -3$, then the singular points of S_λ contained in the base locus of the pencil $\mathcal{S}$ can be described as follows*:

$P_{\{x\},\{y\},\{z\}}$: *type $\mathbb{D}_4$ with quadratic term $(x+y+z)^2$*;

$P_{\{x\},\{z\},\{t\}}$: *type $\mathbb{A}_2$ with quadratic term $(x+t)(z+t)$*;

$P_{\{y\},\{z\},\{t\}}$: *type $\mathbb{A}_2$ with quadratic term $(y+t)(z+t)$*;

$P_{\{x\},\{t\},\{y,z\}}$: *type $\mathbb{A}_2$ with quadratic term $x(x+y+z-(\lambda+3)t)$*;

$P_{\{y\},\{t\},\{x,z\}}$: *type $\mathbb{A}_2$ with quadratic term $y(x+y+z-(\lambda+3)t)$*;

$P_{\{z\},\{t\},\{x,y\}}$: *type $\mathbb{A}_1$ with quadratic term $(x+y)(t+z)+z^2-(\lambda+2)tz$.*

Proof. The proof is similar to that of Lemma 2.8.2. Because of this, we will only prove that S_λ has a du Val singularity of type $\mathbb{D}_4$ at the point $P_{\{x\},\{y\},\{z\}}$ for every $\lambda \neq -3$. To this end, we rewrite the defining equation of the surface S_λ in the chart $t=1$ as

$$\overline{z}^2 + (3+\lambda)\overline{x}^2\overline{y} + (3+\lambda)\overline{x}\,\overline{y}^2 - (\lambda+2)\overline{x}\,\overline{y}\,\overline{z} - \overline{x}\overline{z}^2 - \overline{y}\,\overline{z}^2 + \overline{z}^3 - \overline{y}\overline{x}^2\overline{z} - \overline{y}^2\overline{x}\overline{z} + \overline{y}\overline{x}\overline{z}^2 = 0,$$

where $\overline{x} = x$, $\overline{y} = y$, and $\overline{z} = x+y+z$.

Let $\alpha_1 \colon U_1 \to \mathbb{P}^3$ be the blow-up of the point $P_{\{x\},\{y\},\{z\}}$. A chart of this is given by the coordinate change $\overline{x}_1 = \overline{x}$, $\overline{y}_1 = \overline{y}/\overline{x}$, $\overline{z}_1 = \overline{z}/\overline{z}$. In this chart, the surface S^1_λ is given by

$$\begin{aligned}
&(3+\lambda)\overline{x}_1\overline{y}_1 + \overline{z}_1^2\\
&\qquad = \overline{x}_1\overline{z}_1^2 + (3+\lambda)\overline{x}_1\overline{y}_1^2 + (\lambda+2)\overline{x}_1\overline{y}_1\overline{z}_1 - \overline{x}_1\overline{z}_1^3 + \overline{y}_1\overline{x}_1^2\overline{z}_1 + \overline{y}_1\overline{x}_1\overline{z}_1^2 - \overline{y}_1\overline{x}_1^2\overline{z}_1^2 + \overline{x}_1^2\overline{y}_1^2\overline{z}_1,
\end{aligned}$$

and the surface $\mathbf{E}_1$ is given by $\overline{x}_1 = 0$.

Let C^1_{11} be the line in $\mathbf{E}_1 \cong \mathbb{P}^2$ given by $\overline{x}_1 = \overline{z}_1 = 0$. Then $S^1_\lambda \cdot \mathbf{E}_1 = 2C^1_{11}$ and $\mathbf{M}^{-3}_{11} = 2$. If $\lambda \neq -3$, then the curve C^1_{11} contains three singular points of the surface S^1_λ. One of them is the point $(\overline{x}_1, \overline{y}_1, \overline{z}_1) = (0,0,0)$. Another one is the point $(\overline{x}_1, \overline{y}_1, \overline{z}_1) = (0,-1,0)$. The third singular point can be described in another chart of the blow-up α_1. All these points are isolated ordinary double points of the surface S^1_λ in the case when $\lambda \neq -3$. Thus, if $\lambda \neq -3$, then $P_{\{x\},\{y\},\{z\}}$ is a singular point of type $\mathbb{D}_4$ of the surface S_λ. $\square$

The proof of Lemma 2.9.2 implies that $\mathrm{rk}\,\mathrm{Pic}(\widetilde{S}_{\Bbbk}) = \mathrm{rk}\,\mathrm{Pic}(S_{\Bbbk}) + 13$.

If $\lambda \neq -3$, then the intersection form of the curves $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{z,t\}}$, and $L_{\{y\},\{z,t\}}$ on the surface S_λ is given by

$$\begin{array}{c} \\ L_{\{x\},\{t\}} \\ L_{\{y\},\{t\}} \\ L_{\{z\},\{t\}} \\ L_{\{x\},\{z,t\}} \\ L_{\{y\},\{z,t\}} \end{array}
\begin{array}{c} \begin{array}{ccccc} L_{\{x\},\{t\}} & L_{\{y\},\{t\}} & L_{\{z\},\{t\}} & L_{\{x\},\{z,t\}} & L_{\{y\},\{z,t\}} \end{array} \\
\begin{pmatrix} -2/3 & 1 & 1/3 & 1/3 & 0 \\ 1 & -2/3 & 1/3 & 0 & 1/3 \\ 1/3 & 1/3 & -1/6 & 2/3 & 2/3 \\ 1/3 & 0 & 2/3 & -4/3 & 1 \\ 0 & 1/3 & 2/3 & 1 & -4/3 \end{pmatrix} \end{array}.$$

The determinant of this matrix is $34/81$. This easily gives ($\bigstar$). Indeed, the base locus of the pencil $\mathcal{S}$ consists of the lines $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{y,z\}}$, $L_{\{y\},\{x,z\}}$, $L_{\{x\},\{z,t\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{z\},\{x,y\}}$, $L_{\{t\},\{x,y,z\}}$, and the conic $\mathcal{C}$. On the other hand, it follows from (2.9.1) that

$$\begin{aligned} H_\lambda &\sim L_{\{x\},\{t\}} + 2L_{\{x\},\{y,z\}} + L_{\{x\},\{z,t\}} \sim L_{\{y\},\{t\}} + 2L_{\{y\},\{x,z\}} + L_{\{y\},\{z,t\}} \\ &\sim L_{\{z\},\{t\}} + L_{\{z\},\{x,y\}} + \mathcal{C} \sim L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y,z\}} \end{aligned}$$

on the surface S_λ provided that $\lambda \neq -3$. In this case, we also have

$$H_\lambda \sim L_{\{x\},\{y,z\}} + L_{\{y\},\{x,z\}} + L_{\{z\},\{x,y\}} + L_{\{y\},\{x,y,z\}},$$

because $H_{\{x,y,z\}} \cdot S_\lambda = L_{\{x\},\{y,z\}} + L_{\{y\},\{x,z\}} + L_{\{z\},\{x,y\}} + L_{\{y\},\{x,y,z\}}$. Similarly, we have

$$H_\lambda \sim L_{\{x\},\{t,z\}} + L_{\{y\},\{t,z\}} + 2L_{\{z\},\{t\}},$$

because $H_{\{z,t\}} \cdot S_\lambda = L_{\{x\},\{t,z\}} + L_{\{y\},\{t,z\}} + 2L_{\{z\},\{t\}}$. Thus, one can express the classes of the curves $L_{\{x\},\{z,t\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{z\},\{x,y\}}$, $L_{\{t\},\{x,y,z\}}$, and $\mathcal{C}$ in $\mathrm{Pic}(S_\lambda) \otimes \mathbb{Q}$ as linear combinations of the classes of the lines $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{z,t\}}$, and $L_{\{y\},\{z,t\}}$. For instance, we have

$$L_{\{t\},\{x,y,z\}} \sim L_{\{x\},\{z,t\}} + L_{\{y\},\{z,t\}} + L_{\{z\},\{t\}} - L_{\{x\},\{t\}} - L_{\{y\},\{t\}}$$

and

$$L_{\{z\},\{x,y\}} \sim L_{\{z\},\{t\}} + L_{\{x\},\{t\}} + L_{\{y\},\{t\}} - L_{\{x\},\{z,t\}} - L_{\{y\},\{z,t\}}.$$

This shows that the intersection matrix M in Lemma 1.13.1 has rank 5, so that ($\bigstar$) holds in this case. Thus, we see that ($\diamondsuit$) in the Main Theorem holds in this case.

Therefore, to complete the proof of the Main Theorem in this case, we have to prove ($\heartsuit$). Since $h^{1,2}(X) = 5$, the proof is given by the following.

Lemma 2.9.3. *One has* $[\mathsf{f}^{-1}(\lambda)] = 1$ *for every* $\lambda \neq -3$. *One also has* $[\mathsf{f}^{-1}(-3)] = 6$.

Proof. If $\lambda \neq -3$, then we have $[\mathsf{f}^{-1}(\lambda)] = 1$ by Lemma 2.9.2 and Corollary 1.5.4. To show that $[\mathsf{f}^{-1}(-3)] = 6$, observe that

$$\mathbf{M}_4^{-3} = \mathbf{M}_5^{-3} = 2 \qquad \text{and} \qquad \mathbf{M}_1^{-3} = \mathbf{M}_2^{-3} = \mathbf{M}_3^{-3} = \mathbf{M}_6^{-3} = \mathbf{M}_7^{-3} = \mathbf{M}_8^{-3} = \mathbf{M}_9^{-3} = 1.$$

We also have $\mathbf{m}_1 = \ldots = \mathbf{m}_5 = 2$ and $\mathbf{m}_6 = \ldots = \mathbf{m}_9 = 1$.

Observe that $[S_{-3}] = 3$ and the set Σ consists of the points $P_{\{x\},\{y\},\{z\}}$, $P_{\{x\},\{z\},\{t\}}$, $P_{\{y\},\{z\},\{t\}}$, $P_{\{x\},\{t\},\{y,z\}}$, $P_{\{y\},\{t\},\{x,z\}}$, and $P_{\{z\},\{t\},\{x,y\}}$. Thus, it follows from (1.8.3) and Lemma 1.8.5 that

$$\left[\mathsf{f}^{-1}(-3)\right] = 5 + \sum_{P \in \Sigma} \mathbf{D}_P^{-3}.$$

Moreover, it follows from the proof of Lemma 2.9.2 that $P_{\{x\},\{z\},\{t\}}$, $P_{\{y\},\{z\},\{t\}}$, $P_{\{x\},\{t\},\{y,z\}}$, $P_{\{y\},\{t\},\{x,z\}}$, and $P_{\{z\},\{t\},\{x,y\}}$ are good double points of the surface S_{-3}. Thus, their defects vanish by Lemma 1.12.1. Therefore, we conclude that

$$\big[\mathsf{f}^{-1}(-3)\big] = 5 + \mathbf{D}^{-3}_{P_{\{x\},\{y\},\{z\}}}.$$

Let us show that $\mathbf{D}^{-3}_{P_{\{x\},\{y\},\{z\}}} = 1$. To this end, we use the notation introduced in the proof of Lemma 2.9.2. Then there exists a commutative diagram

$$\begin{tikzcd} & U \arrow[dl, "\gamma"'] \arrow[dr, "\alpha"] & \\ U_1 \arrow[rr, "\alpha_1"'] & & \mathbb{P}^3 \end{tikzcd}$$

for some birational morphism γ. On the other hand, the curve $\widehat{C}_{11}$ is the only base of the pencil $\widehat{\mathcal{S}}$ that is mapped to $P_{\{x\},\{y\},\{z\}}$ by the birational morphism α. This follows from the proof of Lemma 2.9.2. Using Corollary 1.10.4 and (1.10.9), we see that $\mathbf{D}^{-3}_{P_{\{x\},\{y\},\{z\}}} = \mathbf{C}^{-3}_{11}$. By Lemma 1.10.7, we have $\mathbf{C}^{-3}_{11} = 1$, so that $\mathbf{D}^{-3}_{P_{\{x\},\{y\},\{z\}}} = 1$ and $[\mathsf{f}^{-1}(-3)] = 6$. $\square$

Family 2.10. In this case, the threefold X is a blow-up of a complete intersection of two quadrics in $\mathbb{P}^5$ along a smooth elliptic curve of degree 4. This implies that $h^{1,2}(X) = 3$. A toric Landau–Ginzburg model of the threefold X is given by the Minkowski polynomial 3018, which is

$$x + y + \frac{x}{z} + 2z + \frac{yz}{x} + \frac{x}{y} + \frac{y}{x} + \frac{x}{yz} + \frac{2}{z} + \frac{z^2}{x} + \frac{z}{y} + \frac{3z}{x} + \frac{2}{y} + \frac{3}{x} + \frac{1}{yz} + \frac{1}{xz}.$$

The quartic pencil $\mathcal{S}$ is given by

$$\begin{aligned} x^2zy + y^2zx + x^2ty + 2z^2xy + y^2z^2 + x^2tz + y^2tz + x^2t^2 + 2t^2xy \\ + z^3y + z^2tx + 3z^2ty + 2t^2zx + 3t^2zy + t^3x + t^3y = \lambda xyzt. \end{aligned}$$

We may assume that $\lambda \neq \infty$. Let $\mathcal{C}_1$ be a smooth conic given by $x = yz + z^2 + 2zt + t^2 = 0$, and let $\mathcal{C}_2$ be a smooth conic given by $z = xy + xt + yt = 0$. Then

$$\begin{aligned} H_{\{x\}} \cdot S_\lambda &= L_{\{x\},\{y\}} + L_{\{x\},\{z,t\}} + \mathcal{C}_1, \\ H_{\{y\}} \cdot S_\lambda &= L_{\{x\},\{y\}} + L_{\{y\},\{t\}} + L_{\{y\},\{z,t\}} + L_{\{y\},\{x,z,t\}}, \\ H_{\{z\}} \cdot S_\lambda &= L_{\{z\},\{t\}} + L_{\{z\},\{x,t\}} + \mathcal{C}_2, \\ H_{\{t\}} \cdot S_\lambda &= L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,z\}} + L_{\{t\},\{x,y,z\}}. \end{aligned} \tag{2.10.1}$$

Let $C_1 = \mathcal{C}_1$, $C_2 = \mathcal{C}_2$, $C_3 = L_{\{x\},\{y\}}$, $C_4 = L_{\{y\},\{t\}}$, $C_5 = L_{\{z\},\{t\}}$, $C_6 = L_{\{x\},\{z,t\}}$, $C_7 = L_{\{y\},\{z,t\}}$, $C_8 = L_{\{z\},\{x,t\}}$, $C_9 = L_{\{t\},\{x,z\}}$, $C_{10} = L_{\{y\},\{x,z,t\}}$, and $C_{11} = L_{\{t\},\{x,y,z\}}$. These are all base curves of the pencil $\mathcal{S}$.

If $\lambda \neq -4, -5$, then S_λ has isolated singularities, so that it is irreducible. On the other hand, both surfaces S_{-4} and S_{-5} are reducible. Indeed, one has $S_{-4} = H_{\{x,z,t\}} + \mathsf{S}$, where S is a cubic surface given by

$$t^2x + t^2y + txy + txz + 2tyz + xyz + y^2z + yz^2 = 0.$$

Similarly, we have $S_{-5} = \mathsf{Q} + \mathbf{Q}$, where $\mathbf{Q}$ and Q are quadric surfaces given by the equations $t^2 + tx + 2tz + xz + yz + z^2 = 0$ and $tx + ty + xy + yz = 0$, respectively.

Both quadric surfaces $\mathbf{Q}$ and Q are smooth. On the other hand, the surface S has two singular points: $P_{\{x\},\{y\},\{z,t\}}$ and $P_{\{y\},\{z\},\{t\}}$. One can show that S has an ordinary double singularity at $P_{\{x\},\{y\},\{z,t\}}$, and it has a singularity of type $\mathbb{A}_2$ at $P_{\{y\},\{z\},\{t\}}$.

If $\lambda \neq -4, -5$, then the singular points of S_λ contained in the base locus of the pencil $\mathcal{S}$ are the points $P_{\{x\},\{z\},\{t\}}$, $P_{\{y\},\{z\},\{t\}}$, $P_{\{x\},\{y\},\{z,t\}}$, and $P_{\{y\},\{t\},\{x,z\}}$. In this case, all of them are du Val singular points of the surface S_λ by the following lemma.

Lemma 2.10.2. *If $\lambda \neq -4, -5$, then the singular points of S_λ contained in the base locus of the pencil $\mathcal{S}$ can be described as follows*:

$P_{\{x\},\{z\},\{t\}}$: *type $\mathbb{A}_4$ with quadratic term $z(x+z+t)$*;

$P_{\{y\},\{z\},\{t\}}$: *type $\mathbb{A}_2$ with quadratic term $(y+t)(z+t)$*;

$P_{\{x\},\{y\},\{z,t\}}$: *type $\mathbb{A}_4$ with quadratic term $(\lambda+4)xy$*;

$P_{\{y\},\{t\},\{x,z\}}$: *type $\mathbb{A}_2$ with quadratic term $t(x+z+t-(\lambda+4)y)$.*

Proof. We will only describe the singularity of the surface S_λ at the point $P_{\{x\},\{y\},\{z,t\}}$. To this end, we rewrite the defining equation of the surface S_λ in the chart $t=1$ as

$$(\lambda+4)\overline{x}\,\overline{y} + \big(\overline{x}^2\overline{z} - \overline{x}\,\overline{y}^2 - (\lambda+4)\overline{x}\,\overline{y}\,\overline{z} + \overline{z}^2\overline{x} - \overline{y}^2\overline{z}\big) + \big(\overline{x}^2\overline{z}\,\overline{y} + \overline{y}^2\overline{z}\,\overline{x} + 2\overline{z}^2\overline{x}\,\overline{y} + \overline{y}^2\overline{z}^2 + \overline{z}^3\overline{y}\big) = 0,$$

where $\overline{x} = x$, $\overline{y} = y$, and $\overline{z} = z+t$.

Let $\alpha_1\colon U_1 \to \mathbb{P}^3$ be the blow-up of the point $P_{\{x\},\{y\},\{z,t\}}$. One chart of this blow-up is given by the coordinate change $\overline{x}_1 = \overline{x}/\overline{z}$, $\overline{y}_1 = \overline{y}/\overline{z}$, $\overline{z}_1 = \overline{z}$. In this chart, the surface $\mathbf{E}_1$ is given by $\overline{z}_1 = 0$. If $\lambda \neq -4$, then the surface S^1_λ is given by

$$\begin{aligned}&\overline{x}_1\big(\overline{z}_1 + (\lambda+4)\overline{y}_1\big)\\ &\qquad = (\lambda+4)\overline{x}_1\overline{y}_1\overline{z}_1 - \overline{x}_1^2\overline{z}_1 + \overline{y}_1^2\overline{z}_1 - \overline{z}_1^2\overline{y}_1 - 2\overline{x}_1\overline{y}_1\overline{z}_1^2 - \overline{y}_1^2\overline{z}_1^2 + \overline{x}_1\overline{y}_1^2\overline{z}_1 - \overline{x}_1^2\overline{y}_1\overline{z}_1^2 - \overline{x}_1\overline{y}_1^2\overline{z}_1^2.\end{aligned}$$

If $\lambda = -4$, then this equation defines $D^1_{-4} = S^1_{-4} + \mathbf{E}_1$.

Let C^1_{12} and C^1_{13} be the lines in $\mathbf{E}_1 \cong \mathbb{P}^2$ that are given by $\overline{z}_1 = \overline{x}_1 = 0$ and $\overline{z}_1 = \overline{y}_1 = 0$, respectively. Then S^1_{-4} does not contain them.

Let $\alpha_2\colon U_2 \to U_1$ be the blow-up of the point $C^1_{12} \cap C^1_{13}$. If $\lambda \neq -4, -5$, then the surface S^2_λ is smooth along $\mathbf{E}_2$. Hence, in this case, the surface S_λ has a singular point of type $\mathbb{A}_4$ at the point $P_{\{x\},\{y\},\{z,t\}}$. $\square$

The base locus of the pencil $\mathcal{S}$ consists of the curves $L_{\{x\},\{y\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{z,t\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{z\},\{x,t\}}$, $L_{\{t\},\{x,z\}}$, $L_{\{y\},\{x,z,t\}}$, $L_{\{t\},\{x,y,z\}}$, $\mathcal{C}_1$, and $\mathcal{C}_2$. If $\lambda \in \{-4,-5\}$, then

$$\begin{aligned} H_\lambda &\sim L_{\{x\},\{y\}} + L_{\{x\},\{z,t\}} + \mathcal{C}_1 \sim L_{\{x\},\{y\}} + L_{\{y\},\{t\}} + L_{\{y\},\{z,t\}} + L_{\{y\},\{x,z,t\}}\\ &\sim L_{\{z\},\{t\}} + L_{\{z\},\{x,t\}} + \mathcal{C}_2 \sim L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,z\}} + L_{\{t\},\{x,y,z\}}\end{aligned}$$

on the surface S_λ. This follows from (2.10.1). Moreover, in this case, we also have

$$H_\lambda \sim L_{\{x\},\{z,t\}} + L_{\{z\},\{x,t\}} + L_{\{t\},\{x,z\}} + L_{\{y\},\{x,z,t\}},$$

because $H_{\{x,z,t\}} \cdot S_\lambda = L_{\{x\},\{z,t\}} + L_{\{z\},\{x,t\}} + L_{\{t\},\{x,z\}} + L_{\{y\},\{x,z,t\}}$. This shows that the intersection matrix M in Lemma 1.13.1 has the same rank as the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{z,t\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{z\},\{x,t\}}$, and $L_{\{t\},\{x,z\}}$ on the surface S_λ. If $\lambda \neq -4, -5$, then the latter matrix is given by

$$\begin{array}{c} \\ L_{\{x\},\{y\}} \\ L_{\{y\},\{t\}} \\ L_{\{z\},\{t\}} \\ L_{\{x\},\{z,t\}} \\ L_{\{y\},\{z,t\}} \\ L_{\{z\},\{x,t\}} \\ L_{\{t\},\{x,z\}} \end{array}
\begin{array}{c} \begin{array}{ccccccc} L_{\{x\},\{y\}} & L_{\{y\},\{t\}} & L_{\{z\},\{t\}} & L_{\{x\},\{z,t\}} & L_{\{y\},\{z,t\}} & L_{\{z\},\{x,t\}} & L_{\{t\},\{x,z\}} \end{array} \\
\begin{pmatrix} -4/5 & 1 & 0 & 3/5 & 2/5 & 0 & 0 \\ 1 & -2/3 & 1/3 & 0 & 1/3 & 0 & 2/3 \\ 0 & 1/3 & -8/15 & 1/5 & 2/3 & 3/5 & 1/5 \\ 3/5 & 0 & 1/5 & -2/5 & 1/5 & 2/5 & 4/5 \\ 2/5 & 1/3 & 2/3 & 1/5 & -8/15 & 0 & 0 \\ 0 & 0 & 3/5 & 2/5 & 0 & -4/5 & 2/5 \\ 0 & 2/3 & 1/5 & 4/5 & 0 & 2/5 & -8/15 \end{pmatrix} \end{array}.$$

The rank of this matrix is 6. We see that ($\bigstar$) holds, because $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk}) = \operatorname{rk}\operatorname{Pic}(S_{\Bbbk}) + 12$. Thus, we conclude that ($\diamondsuit$) in the Main Theorem also holds in this case.

Lemma 2.10.3. *One has* $[\mathsf{f}^{-1}(\lambda)] = 1$ *for* $\lambda \notin \{-4,-5\}$, $[\mathsf{f}^{-1}(-4)] = 3$, *and* $[\mathsf{f}^{-1}(-5)] = 2$.

Proof. If $\lambda \notin \{-4,-5\}$, then $[\mathsf{f}^{-1}(\lambda)] = 1$ by Lemma 2.10.2 and Corollary 1.5.4. Moreover, it follows from Corollary 1.12.2 that $[\mathsf{f}^{-1}(-5)] = 2$, because $P_{\{x\},\{z\},\{t\}}$, $P_{\{y\},\{z\},\{t\}}$, $P_{\{x\},\{y\},\{z,t\}}$, and $P_{\{y\},\{t\},\{x,z\}}$ are good double points of the surface S_{-5}.

To complete the proof, we have to show that $[\mathsf{f}^{-1}(-4)] = 3$. Using (1.8.3), we see that

$$\left[\mathsf{f}^{-1}(-4)\right] = 2 + \mathbf{D}^{-4}_{P_{\{x\},\{y\},\{z,t\}}}.$$

Here, we also used Lemmas 1.8.5 and 1.12.1.

To compute the defect $\mathbf{D}^{-4}_{P_{\{x\},\{y\},\{z,t\}}}$, let us use the proof of Lemma 2.10.2 and the notation adopted in this proof. First, we have $D^2_{-4} = S^2_{-4} + \mathbf{E}^2_1$, so that $\mathbf{A}^{-4}_{P_{\{x\},\{y\},\{z,t\}}} = 1$, where $\mathbf{A}^{-4}_{P_{\{x\},\{y\},\{z,t\}}}$ is the number defined in (1.10.3).

The curves C^2_{12} and C^2_{13} are the base curves of the pencil $\mathcal{S}^2$. Along with these curves, this pencil has one more base curve contained in $\mathbf{E}_2 \cup \mathbf{E}^2_1$. However, the divisor D^2_{-4} is smooth at the general points of these three curves. Now, using Lemma 1.10.7 and (1.10.9), we conclude that $\mathbf{D}^{-4}_{P_{\{x\},\{y\},\{z,t\}}} = \mathbf{A}^{-4}_{P_{\{x\},\{y\},\{z,t\}}} = 1$, so that $[\mathsf{f}^{-1}(-4)] = 3$. $\square$

Note that Lemma 2.10.3 implies ($\heartsuit$) in the Main Theorem, because $h^{1,2}(X) = 3$.

Family 2.11. In this case, the threefold X is a blow-up of a smooth cubic threefold along a line, so that $h^{1,2}(X) = 5$. A toric Landau–Ginzburg model of the threefold X is given by the Minkowski polynomial 1700, which is

$$y + \frac{x}{z} + \frac{y}{z} + z + \frac{yz}{x} + \frac{2y}{x} + \frac{2z^2}{x} + \frac{x}{yz} + \frac{2}{z} + \frac{y}{xz} + \frac{2z}{y} + \frac{2z}{x} + \frac{z^3}{xy}.$$

The pencil of quartic surfaces $\mathcal{S}$ is given by the equation

$$y^2zx + x^2ty + y^2tx + z^2xy + y^2z^2 + 2y^2tz + 2z^3y \\ + x^2t^2 + 2t^2xy + t^2y^2 + 2z^2tx + 2z^2ty + z^4 = \lambda xyzt.$$

In the remaining part of this subsection, we will assume that $\lambda \neq \infty$.

Let $\mathcal{C}_1$ be the smooth conic given by $x = ty + yz + z^2 = 0$, let $\mathcal{C}_2$ be the smooth conic given by $y = tx + z^2 = 0$, let $\mathcal{C}_3$ be the smooth conic given by $z = tx + ty + xy = 0$, and let $\mathcal{C}_4$ be the smooth conic given by $t = tx + ty + xy = 0$. Then

$$\begin{aligned} H_{\{x\}} \cdot S_\lambda &= 2\mathcal{C}_1, & H_{\{z\}} \cdot S_\lambda &= L_{\{z\},\{t\}} + L_{\{z\},\{x,y\}} + \mathcal{C}_3, \\ H_{\{y\}} \cdot S_\lambda &= 2\mathcal{C}_2, & H_{\{t\}} \cdot S_\lambda &= L_{\{z\},\{t\}} + L_{\{t\},\{y,z\}} + \mathcal{C}_4, \end{aligned} \tag{2.11.1}$$

so that $L_{\{z\},\{t\}}$, $L_{\{z\},\{x,y\}}$, $L_{\{t\},\{y,z\}}$, $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, and $\mathcal{C}_4$ are all base curves of the pencil $\mathcal{S}$. To match the notation used in Subsection 1.8, we let $C_1 = \mathcal{C}_1$, $C_2 = \mathcal{C}_2$, $C_3 = \mathcal{C}_3$, $C_4 = \mathcal{C}_4$, $C_5 = L_{\{z\},\{t\}}$, $C_6 = L_{\{z\},\{x,y\}}$, and $C_7 = L_{\{t\},\{y,z\}}$.

If $\lambda \neq -2$, then the surface S_λ has isolated singularities, so that S_λ is irreducible. On the other hand, we have $S_{-2} = \mathbf{Q} + \mathsf{Q}$, where $\mathbf{Q}$ and Q are irreducible quadric surfaces given by the equations $xy + yz + z^2 + xt + yt = 0$ and $xt + ty + yz + z^2 = 0$, respectively. Both these quadric surfaces are smooth. Note that $\mathbf{Q} \cap \mathsf{Q} = \mathcal{C}_1 \cup \mathcal{C}_2$.

If $\lambda\neq -2$, then the singular points of S_λ contained in the base locus of the pencil $\mathcal{S}$ are the points $P_{\{x\},\{z\},\{t\}}$, $P_{\{x\},\{y\},\{z\}}$, $P_{\{y\},\{z\},\{t\}}$, and $P_{\{x\},\{t\},\{y,z\}}$, which are du Val singular points of the surface S_λ. In fact, we can say more.

Lemma 2.11.2. *If $\lambda\neq -2$, then the singular points of S_λ contained in the base locus of the pencil $\mathcal{S}$ can be described as follows*:

$P_{\{x\},\{y\},\{z\}}$: *type $\mathbb{D}_6$ with quadratic term $(x+y)^2$*;

$P_{\{x\},\{z\},\{t\}}$: *type $\mathbb{A}_3$ with quadratic term $(z+t)(x+z+t)$*;

$P_{\{y\},\{z\},\{t\}}$: *type $\mathbb{A}_5$ with quadratic term $t(y+t)$*;

$P_{\{x\},\{t\},\{y,z\}}$: *type $\mathbb{A}_1$ with quadratic term $(\lambda+2)xt+(y+z-t)(y+z-x-t)$.*

Proof. We will only describe the singularity of the surface S_λ at the point $P_{\{x\},\{y\},\{z\}}$. To this end, we rewrite the defining equation of the surface S_λ in the chart $t=1$ as

$$\overline{x}^2+\big(\overline{x}^2\overline{y}-\lambda\overline{x}\,\overline{y}\,\overline{z}+(\lambda+2)\overline{y}^2\overline{z}-\overline{x}\,\overline{y}^2+2\overline{x}\,\overline{z}^2\big)+\big(\overline{y}^2\overline{z}\,\overline{x}+\overline{z}^2\overline{x}\,\overline{y}-\overline{y}^3\overline{z}+2\overline{z}^3\overline{y}+\overline{z}^4\big)=0,$$

where $\overline{x}=x+y$, $\overline{y}=y$, and $\overline{z}=z$. Let $\alpha_1\colon U_1\to\mathbb{P}^3$ be the blow-up of the point $P_{\{x\},\{y\},\{z,t\}}$. One chart of this blow-up is given by the coordinate change $\overline{x}_1=\overline{x}/\overline{z}$, $\overline{y}_1=\overline{y}/\overline{z}$, $\overline{z}_1=\overline{z}$. In this chart, the surface $\mathbf{E}_1$ is given by $\overline{z}_1=0$. Then S^1_λ is given by

$$(\overline{x}_1+\overline{z}_1)^2+\big(2\overline{y}_1\overline{z}_1^2-\lambda\overline{x}_1\overline{y}_1\overline{z}_1+(\lambda+2)\overline{y}_1^2\overline{z}_1\big)\\+\big(\overline{x}_1^2\overline{y}_1\overline{z}_1-\overline{y}_1^2\overline{z}_1\overline{x}_1+\overline{z}_1^2\overline{x}_1\overline{y}_1\big)+\big(\overline{x}_1\overline{y}_1^2\overline{z}_1^2-\overline{y}_1^3\overline{z}_1^2\big)=0.$$

Denote by C^1_8 the line in $\mathbf{E}_1\cong\mathbb{P}^2$ that is given by $\overline{z}_1=\overline{x}_1=0$. Then S^1_{-2} is singular along this line. If $\lambda\neq-2$, then S^1_λ has two singular points in $\mathbf{E}_1$. One of them is the point $(\overline{x}_1,\overline{y}_1,\overline{z}_1)=(0,0,0)$. The second singular point lies in another chart of the blow-up α_1. If $\lambda\neq-2$, then this point is an isolated ordinary double point of the surface S^1_λ.

Let $\widehat{x}_1=\overline{x}_1+\overline{z}_1$, $\widehat{y}_1=\overline{y}_1$, and $\widehat{z}_1=\overline{z}_1$. Then we can rewrite the (local) defining equation of the surface S^1_λ as

$$\widehat{x}_1^2+\big((\lambda+2)\widehat{y}_1\widehat{z}_1^2-\lambda\widehat{x}_1\widehat{y}_1\widehat{z}_1+(\lambda+2)\widehat{y}_1^2\widehat{z}_1\big)\\+\big(\widehat{x}_1^2\widehat{y}_1\widehat{z}_1-\widehat{y}_1^2\widehat{z}_1\widehat{x}_1-\widehat{z}_1^2\widehat{x}_1\widehat{y}_1+\widehat{y}_1^2\widehat{z}_1^2\big)+\big(\widehat{x}_1\widehat{y}_1^2\widehat{z}_1^2-\widehat{y}_1^3\widehat{z}_1^2-\widehat{y}_1^2\widehat{z}_1^3\big)=0.$$

Let $\alpha_2\colon U_2\to U_1$ be the blow-up of the point $(\widehat{x}_1,\widehat{y}_1,\widehat{z}_1)=(0,0,0)$. One chart of this blow-up is given by the coordinate change $\widehat{x}_2=\widehat{x}_1/\widehat{z}_1$, $\widehat{y}_2=\widehat{y}_1/\widehat{z}_1$, $\widehat{z}_2=\widehat{z}_1$. In this chart, the surface S^2_λ is given by

$$\widehat{x}_2^2+(\lambda+2)\widehat{y}_2\widehat{z}_2+\big((\lambda+2)\widehat{y}_2^2\widehat{z}_2-\lambda\widehat{x}_2\widehat{y}_2\widehat{z}_2\big)+\big(\widehat{y}_2^2\widehat{z}_2^2-\widehat{z}_2^2\widehat{x}_2\widehat{y}_2\big)\\+\big(\widehat{x}_2^2\widehat{y}_2\widehat{z}_2^2-\widehat{x}_2\widehat{y}_2^2\widehat{z}_2^2-\widehat{y}_2^2\widehat{z}_2^3\big)+\big(\widehat{x}_2\widehat{y}_2^2\widehat{z}_2^3-\widehat{y}_2^3\widehat{z}_2^3\big)=0,$$

and the surface $\mathbf{E}_2$ is given by $\widehat{z}_2=0$. Thus, if $\lambda\neq-2$, then S^2_λ has an isolated ordinary double singularity at the point $(\widehat{x}_2,\widehat{y}_2,\widehat{z}_2)=(0,0,0)$.

Denote by C^2_9 the line in $\mathbf{E}_2\cong\mathbb{P}^2$ that is given by $\widehat{z}_2=\widehat{x}_2=0$. Then S^2_{-2} is singular along this line. If $\lambda\neq-2$, then $\mathbf{E}_2$ contains three singular points of the surface S^2_λ. One of them is the point $(\widehat{x}_2,\widehat{y}_2,\widehat{z}_2)=(0,0,0)$. The second one is $(\widehat{x}_2,\widehat{y}_2,\widehat{z}_2)=(0,-1,0)$. The third point is contained in another chart of the blow-up α_2. All of them are isolated ordinary double singularities of the surface S^2_λ. Hence, if $\lambda\neq-2$, then S_λ has a singular point of type $\mathbb{D}_6$ at the point $P_{\{x\},\{y\},\{z\}}$. $\square$

The proof of Lemma 2.11.2 implies

$$\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk})=\operatorname{rk}\operatorname{Pic}(S_{\Bbbk})+15. \tag{2.11.3}$$

By Lemma 1.13.1, to verify ($\diamondsuit$) in the Main Theorem, we have to compute the rank of the intersection matrix of the curves $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_4$, $L_{\{z\},\{t\}}$, $L_{\{z\},\{t\}}$, and $L_{\{t\},\{y,z\}}$ on a general surface in the pencil $\mathcal{S}$. On the other hand, if $\lambda \neq -2$, then it follows from (2.11.1) that

$$H_\lambda \sim 2\mathcal{C}_1 \sim 2\mathcal{C}_2 \sim L_{\{z\},\{t\}} + L_{\{z\},\{x,y\}} + \mathcal{C}_3 \sim L_{\{z\},\{t\}} + L_{\{t\},\{y,z\}} + \mathcal{C}_4$$

on the surface S_λ. We have $\mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3 + \mathcal{C}_4 \sim 2H_\lambda$, because $\mathbf{Q} \cdot S_\lambda = \mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3 + \mathcal{C}_4$. Similarly, we have $\mathsf{Q} \cdot S_\lambda = 2L_{\{z\},\{t\}} + L_{\{z\},\{x,y\}} + L_{\{t\},\{y,z\}} + \mathcal{C}_1 + \mathcal{C}_2$, so that

$$2L_{\{z\},\{t\}} + L_{\{z\},\{x,y\}} + L_{\{t\},\{y,z\}} + \mathcal{C}_1 + \mathcal{C}_2 \sim 2H_\lambda,$$

which implies $2L_{\{z\},\{t\}} + L_{\{z\},\{x,y\}} + L_{\{t\},\{y,z\}} \sim_{\mathbb{Q}} H_\lambda$. Thus, if $\lambda \neq -2$, then the rank of the intersection matrix of the curves $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_4$, $L_{\{z\},\{t\}}$, $L_{\{z\},\{t\}}$, and $L_{\{t\},\{y,z\}}$ on the surface S_λ is the same as the rank of the intersection matrix of the curves $L_{\{z\},\{t\}}$, $L_{\{z\},\{x,y\}}$ and H_λ, which is very easy to compute.

Lemma 2.11.4. *Suppose that $\lambda \neq -2$. Then the intersection form of the curves $L_{\{z\},\{t\}}$, $L_{\{z\},\{x,y\}}$, and H_λ on the surface S_λ is given by*

$$\begin{array}{r c} & \begin{array}{ccc} L_{\{z\},\{t\}} & L_{\{z\},\{x,y\}} & H_\lambda \end{array} \\ \begin{array}{r} L_{\{z\},\{t\}} \\ L_{\{z\},\{x,y\}} \\ H_\lambda \end{array} & \begin{pmatrix} -5/12 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 4 \end{pmatrix} \end{array}.$$

Proof. Since $L_{\{z\},\{t\}} \cap L_{\{z\},\{x,y\}} = P_{\{z\},\{t\},\{x,y\}}$ and S_λ is smooth at this point, we conclude that $L_{\{z\},\{t\}} \cdot L_{\{z\},\{x,y\}} = 1$. So, to complete the proof, we have to find $L^2_{\{z\},\{t\}}$ and $L^2_{\{z\},\{x,y\}}$.

Observe that $P_{\{x\},\{z\},\{t\}}$ and $P_{\{y\},\{z\},\{t\}}$ are the only singular points of S_λ that are contained in the line $L_{\{z\},\{t\}}$. Applying Remark A.2.4 with $S = S_\lambda$, $O = P_{\{x\},\{z\},\{t\}}$, $n = 3$, and $C = L_{\{z\},\{t\}}$, we see that $\overline{C}$ does not contain the point $\overline{G}_1 \cap \overline{G}_3$. Similarly, applying Remark A.2.4 with $S = S_\lambda$, $O = P_{\{y\},\{z\},\{t\}}$, $n = 5$, and $C = L_{\{z\},\{t\}}$, we see that $\overline{C}$ does not contain the point $\overline{G}_1 \cap \overline{G}_5$. Thus, it follows from Proposition A.1.3 that

$$L^2_{\{z\},\{t\}} = -2 + \frac{3}{4} + \frac{5}{6} = -\frac{5}{12}.$$

Note that $P_{\{x\},\{y\},\{z\}}$ is the only singular point of S_λ that is contained in the line $L^2_{\{z\},\{x,y\}}$. To find $L^2_{\{z\},\{x,y\}}$, let us use the notation of Lemma A.3.2 with $S = S_\lambda$, $O = P_{\{x\},\{y\},\{z\}}$, $n = 6$, and $C = L_{\{z\},\{x,y\}}$. Let us also use the notation of the proof of Lemma 2.11.2. It follows from this proof that the proper transform of the line $L_{\{z\},\{x,y\}}$ on the surface S^1_λ does not contain the point $(\overline{x}_1, \overline{y}_1, \overline{z}_1) = (0, 0, 0)$. Thus, in the notation of Lemma A.3.2, we have $\widetilde{C} \cdot G_6 = 1$, which implies that $L^2_{\{z\},\{x,y\}} = -1$ by Lemma A.3.2. $\square$

The determinant of the intersection matrix in Lemma 2.11.4 is 13/12. Using (2.11.3), we get ($\bigstar$), so that ($\diamondsuit$) in the Main Theorem holds in this case. Moreover, since $h^{1,2}(X) = 5$, the assertion ($\heartsuit$) in the Main Theorem is given by

Lemma 2.11.5. *If $\lambda \neq -2$, then $[\mathsf{f}^{-1}(\lambda)] = 1$. One also has $[\mathsf{f}^{-1}(-2)] = 6$.*

Proof. If $\lambda \neq -2$, then $[\mathsf{f}^{-1}(\lambda)] = 1$ by Lemma 2.11.2 and Corollary 1.5.4. Hence, to complete the proof, we need to show that $[\mathsf{f}^{-1}(-2)] = 6$. Observe that $\mathbf{m}_3 = \mathbf{m}_4 = \mathbf{m}_6 = \mathbf{m}_7 = 1$ and $\mathbf{m}_1 = \mathbf{m}_2 = \mathbf{m}_5 = 2$. Note that $\mathbf{M}_3^{-2} = \ldots = \mathbf{M}_7^{-2} = 1$ and $\mathbf{M}_1^{-2} = \mathbf{M}_2^{-2} = 2$. Thus, applying Lemmas 1.8.5 and 1.12.1 and using (1.8.3), we see that

$$\left[\mathsf{f}^{-1}(-2)\right] = 4 + \mathbf{D}^{-2}_{P_{\{x\},\{y\},\{z\}}},$$

where $\mathbf{D}^{-2}_{P_{\{x\},\{y\},\{z\}}}$ is the defect of the singular point $P_{\{x\},\{y\},\{z\}}$.

To calculate $\mathbf{D}^{-2}_{P_{\{x\},\{y\},\{z\}}}$, we will use the local computations made in the proof of Lemma 2.11.2. They give $\mathbf{M}_8^{-2}=\mathbf{M}_9^{-2}=2$ and $D^2_{-2}=S^2_{-2}$. Now, using Lemma 1.10.7 and (1.10.9), we conclude that $\mathbf{D}^{-2}_{P_{\{x\},\{y\},\{z\}}}\geq 2$. In fact, the proof of Lemma 2.11.2 implies that $\mathbf{D}^{-2}_{P_{\{x\},\{y\},\{z\}}}=2$, so that $[\mathsf{f}^{-1}(-2)]=6$. $\square$

Family 2.12. In this case, the threefold X is a blow-up of $\mathbb{P}^3$ along a smooth curve of genus 3 and degree 6, so that $h^{1,2}(X)=3$. Here, we choose its toric Landau–Ginzburg model to be given by the Minkowski polynomial 1193, which is

$$x+\frac{xy}{z}+z+y+\frac{2x}{z}+\frac{2y}{z}+\frac{x}{yz}+\frac{2}{y}+\frac{2}{z}+\frac{z}{xy}+\frac{2}{x}+\frac{y}{xz}.$$

The quartic pencil $\mathcal{S}$ is given by

$$x^2zy+x^2y^2+z^2xy+y^2zx+2x^2ty+2y^2tx+x^2t^2\\ +2t^2zx+2t^2xy+t^2z^2+2t^2zy+t^2y^2=\lambda xyzt.$$

This equation is symmetric with respect to the permutation $x\leftrightarrow y$.

Let $\mathcal{C}$ be a conic given by $z=xy+xt+yt=0$. If $\lambda\neq\infty$, then

$$\begin{aligned} H_{\{x\}}\cdot S_\lambda&=2L_{\{x\},\{t\}}+2L_{\{x\},\{y,z\}},\\ H_{\{y\}}\cdot S_\lambda&=2L_{\{y\},\{t\}}+2L_{\{y\},\{x,z\}},\\ H_{\{z\}}\cdot S_\lambda&=2\mathcal{C},\\ H_{\{t\}}\cdot S_\lambda&=L_{\{x\},\{t\}}+L_{\{y\},\{t\}}+L_{\{t\},\{x,z\}}+L_{\{t\},\{y,z\}}.\end{aligned}\tag{2.12.1}$$

So, we may assume that $C_1=L_{\{x\},\{t\}}$, $C_2=L_{\{y\},\{t\}}$, $C_3=L_{\{x\},\{y,z\}}$, $C_4=L_{\{y\},\{x,z\}}$, $C_5=L_{\{t\},\{x,z\}}$, $C_6=L_{\{t\},\{y,z\}}$, and $C_7=\mathcal{C}$. These are all base curves of the pencil $\mathcal{S}$.

If $\lambda\neq\infty,-2$, then the surface S_λ has isolated singularities, so that S_λ is irreducible. On the other hand, the surface S_{-2} is singular along $L_{\{x\},\{y,z\}}$ and $L_{\{y\},\{x,z\}}$.

Lemma 2.12.2. *The surface S_{-2} is irreducible.*

Proof. Let Π be a plane in $\mathbb{P}^3$ that is given by $z=t$. Then the intersection $S_{-2}\cap\Pi$ is a plane quartic curve that is singular at the points $\Pi\cap L_{\{x\},\{y,z\}}$ and $\Pi\cap L_{\{y\},\{x,z\}}$. Moreover, this curve is smooth away from these points. Furthermore, both of these points are isolated ordinary double points of the curve $S_{-2}\cap\Pi$. This implies that this curve is irreducible, so that the surface S_{-2} is also irreducible. $\square$

The fixed singular points of the surfaces in the pencil $\mathcal{S}$ are $P_{\{x\},\{y\},\{z\}}$, $P_{\{x\},\{y\},\{t\}}$, $P_{\{x\},\{z\},\{t\}}$, $P_{\{y\},\{z\},\{t\}}$, $P_{\{x\},\{t\},\{y,z\}}$, and $P_{\{y\},\{t\},\{x,z\}}$. If $\lambda\neq\infty,-2$, then the singularities of the surface S_λ at these points can be described as follows:

$P_{\{x\},\{y\},\{z\}}$: type $\mathbb{D}_4$ with quadratic term $(x+y+z)^2$;
$P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_1$ with quadratic term $xy+t^2$;
$P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_1$ with quadratic term $x^2+xz+2xt+t^2$;
$P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_1$ with quadratic term $y^2+yz+2yt+t^2$;
$P_{\{x\},\{t\},\{y,z\}}$: type $\mathbb{A}_3$ with quadratic term $x(y+z-(\lambda+2)t)$;
$P_{\{y\},\{t\},\{x,z\}}$: type $\mathbb{A}_3$ with quadratic term $y(x+z-(\lambda+2)t)$.

The surfaces in $\mathcal{S}$ also have floating singular points. They are contained in the conic $\mathcal{C}$. To describe them nicely, we introduce a new parameter $\mu \in \mathbb{C} \cup \{\infty\}$ such that

$$\lambda = \frac{2\mu^2 - 2\mu - 1}{\mu(1-\mu)}.$$

Then S_λ is singular at the points $[1-\mu : \mu : 0 : \mu(\mu-1)]$ and $[\mu : 1-\mu : 0 : \mu(\mu-1)]$. Denote these two points by P_μ and $P_{1-\mu}$, respectively. Then $P_\mu \neq P_{1-\mu} \Leftrightarrow \mu \notin \{\infty, 1/2\}$. If $\mu = 1/2$, then $P_\mu = P_{1-\mu} = [-2 : -2 : 0 : 1]$. If $\mu = \infty$, then $P_\mu = P_{1-\mu} = P_{\{x\},\{y\},\{z\}}$.

Lemma 2.12.3. *Suppose that $\lambda \neq \infty, -2$. If $\mu \neq 1/2$, then S_λ has isolated ordinary double singularities at the points P_μ and $P_{1-\mu}$. If $\mu = 1/2$, then $\lambda = -6$ and S_{-6} has a du Val singularity of type $\mathbb{A}_3$ at the point $P_{1/2}$.*

Proof. Due to the symmetry $x \leftrightarrow y$, it suffices to describe the singularity of the surface S_λ at the point P_μ. Moreover, we may assume that $\mu \neq 0, 1$, since $P_0 = P_{\{y\},\{z\},\{t\}}$ and $P_1 = P_{\{x\},\{z\},\{t\}}$. Then $P_\mu = [1/\mu : 1/(\mu-1) : 0 : 1]$. In the chart $t = 1$, the surface S_λ is given by

$$\begin{aligned}(\mu-1)^4\overline{y}^2 + \mu(\mu-1)(\mu^2-\mu-1)\overline{z}^2 + \mu(\mu-1)(2\mu^2-1)\overline{z}\,\overline{x} + 2\mu^2(\mu-1)^2\overline{x}\,\overline{y}\\ + \mu(\mu-1)(2\mu^2-4\mu+1)\overline{y}\,\overline{z} + \mu^4\overline{x}^2 + \text{Higher order terms} = 0,\end{aligned}$$

where $\overline{x} = x - 1/\mu$, $\overline{y} = y - 1/(\mu-1)$, and $\overline{z} = z$. If $\mu \neq 1/2$, this quadratic form is nondegenerate, so that S_λ has an isolated ordinary double singularity at P_μ.

To complete the proof, we may assume that $\mu = 1/2$. Then $P_{1/2} = [-2 : -2 : 0 : 1]$. Note that $\lambda = -6$ in this case. In the chart $t = 1$, the surface S_{-6} is given by

$$\begin{aligned}\widehat{x}^2 + 2\widehat{z}\,\widehat{x} + 5\widehat{z}^2 + \big(2\widehat{x}\widehat{y}^2 - 2\widehat{x}^2\widehat{y} - 2\widehat{x}^2\widehat{z} + 2\widehat{x}\widehat{y}\,\widehat{z} - 2\widehat{x}\widehat{z}^2 - 2\widehat{y}^2\widehat{z}\big)\\ + \big(\widehat{x}^2\widehat{y}^2 + \widehat{z}\widehat{y}\widehat{x}^2 - 2\widehat{y}^3\widehat{x} - \widehat{z}\widehat{y}^2\widehat{x} + \widehat{z}^2\widehat{y}\widehat{x} + \widehat{y}^4 - \widehat{y}^2\widehat{z}^2\big) = 0,\end{aligned}$$

where $\widehat{x} = x + y + 4$, $\widehat{y} = y + 2$, and $\widehat{z} = z$. Introducing new coordinates $\widehat{x}_1 = \widehat{x}/\widehat{y}$, $\widehat{y}_1 = \widehat{y}$, and $\widehat{z}_1 = \widehat{z}/\widehat{y}$, we can rewrite this equation (after dividing by $\widehat{y}_1^2$) as

$$\begin{aligned}\widehat{x}_1^2 + 2\widehat{x}_1\widehat{y}_1 + 2\widehat{z}_1\widehat{x}_1 + \widehat{y}_1^2 - 2\widehat{y}_1\widehat{z}_1 + 5\widehat{z}_1^2 + \big(2\widehat{x}_1\widehat{y}_1\widehat{z}_1 - 2\widehat{x}_1^2\widehat{y}_1 - 2\widehat{x}_1\widehat{y}_1^2\big)\\ + \big(\widehat{x}_1^2\widehat{y}_1^2 - 2\widehat{z}_1\widehat{y}_1\widehat{x}_1^2 - \widehat{z}_1\widehat{y}_1^2\widehat{x}_1 - 2\widehat{z}_1^2\widehat{y}_1\widehat{x}_1 - \widehat{y}_1^2\widehat{z}_1^2\big) + \widehat{z}_1\widehat{y}_1^2\widehat{x}_1^2 + \widehat{x}_1\widehat{y}_1^2\widehat{z}_1^2.\end{aligned}$$

This equation defines a chart of a blow-up of the surface S_{-6} at the point $P_{1/2}$. Its quadratic part is not degenerate, which shows that $P_{1/2}$ is a du Val singular point of type $\mathbb{A}_3$ of the surface S_{-6}. This completes the proof of the lemma. $\square$

If $\lambda \neq \infty, -2$, then the singular points of S_λ contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

- $P_{\{x\},\{y\},\{z\}}$, $P_{\{x\},\{y\},\{t\}}$, $P_{\{x\},\{z\},\{t\}}$, $P_{\{y\},\{z\},\{t\}}$, $P_{\{x\},\{t\},\{y,z\}}$, and $P_{\{y\},\{t\},\{x,z\}}$;
- P_μ and $P_{1-\mu}$, where $\mu \in \mathbb{C} \cup \{\infty\}$ is such that $\lambda = (2\mu^2 - 2\mu - 1)/(\mu(1-\mu))$;
- $P_{\{t\},\{x,z\},\{y,z\}}$, which is an isolated ordinary double point of the surface S_{-4}.

If $\lambda \neq -4$, then S_λ is smooth at the point $P_{\{t\},\{x,z\},\{y,z\}}$.

Note that fixed singular points of the quartic surfaces in the pencil $\mathcal{S}$ can be considered as singular points of the surface $S_{\Bbbk}$. In our case, all exceptional curves of the minimal resolution of the surface $S_{\Bbbk}$ at these singular points are geometrically irreducible. Similarly, we can consider the union $P_\mu \cup P_{1-\mu}$ as a (geometrically reducible) singular point of the surface $S_{\Bbbk}$. This gives $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk}) = \operatorname{rk}\operatorname{Pic}(S_{\Bbbk}) + 14$. Thus, to prove ($\bigstar$), we have to show that the intersection matrix of the curves $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{x\},\{y,z\}}$, $L_{\{y\},\{x,z\}}$, $L_{\{t\},\{x,z\}}$, $L_{\{t\},\{y,z\}}$, and $\mathcal{C}$ on a general surface in $\mathcal{S}$ is of rank 4. If $\lambda \neq \infty, -2$, then it follows from (2.12.1) that

$$H_\lambda \sim 2L_{\{x\},\{t\}} + 2L_{\{x\},\{y,z\}} \sim 2L_{\{y\},\{t\}} + 2L_{\{y\},\{x,z\}}$$
$$\sim 2\mathcal{C} \sim L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{t\},\{x,z\}} + L_{\{t\},\{y,z\}}$$

on the surface S_λ. Therefore, in this case, the intersection matrix of the curves $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{x\},\{y,z\}}$, $L_{\{y\},\{x,z\}}$, $L_{\{t\},\{x,z\}}$, $L_{\{t\},\{y,z\}}$, and $\mathcal{C}$ on the surface S_λ has the same rank as the intersection matrix of the four lines $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{t\},\{x,z\}}$, and $L_{\{t\},\{y,z\}}$. If $\lambda \notin \{\infty, -2, -4, -6\}$, then the latter matrix is given by

$$\begin{array}{c} \\ L_{\{x\},\{t\}} \\ L_{\{y\},\{t\}} \\ L_{\{t\},\{x,z\}} \\ L_{\{t\},\{y,z\}} \end{array}
\begin{array}{c} \begin{array}{cccc} L_{\{x\},\{t\}} & L_{\{y\},\{t\}} & L_{\{t\},\{x,z\}} & L_{\{t\},\{y,z\}} \end{array} \\ \begin{pmatrix} -1/4 & 1/2 & 1/2 & 1/4 \\ 1/2 & -1/4 & 1/4 & 1/2 \\ 1/2 & 1/4 & -3/4 & 1 \\ 1/4 & 1/2 & 1 & -3/4 \end{pmatrix} \end{array}.$$

Its determinant is $-5/8$, so that ($\diamondsuit$) in the Main Theorem holds by Lemma 1.13.1.

Lemma 2.12.4. *If* $\lambda \neq -2$, *then* $[\mathsf{f}^{-1}(\lambda)] = 1$. *One also has* $[\mathsf{f}^{-1}(-2)] = 4$.

Proof. If $\lambda \neq -2$, then S_λ has du Val singularities in the base locus of the pencil $\mathcal{S}$, so that $[\mathsf{f}^{-1}(\lambda)] = 1$ by Corollary 1.5.4. Hence, to complete the proof, we have to show that $[\mathsf{f}^{-1}(-2)] = 4$. To this end, we observe that $\mathbf{m}_1 = \mathbf{m}_2 = 3$, $\mathbf{m}_3 = \mathbf{m}_4 = \mathbf{m}_7 = 2$, and $\mathbf{m}_5 = \mathbf{m}_6 = 1$. Similarly, we have $\mathbf{M}_1^{-2} = \mathbf{M}_2^{-2} = \mathbf{M}_5^{-2} = \mathbf{M}_6^{-2} = \mathbf{M}_7^{-2} = 1$ and $\mathbf{M}_3^{-2} = \mathbf{M}_4^{-2} = 2$. Thus, using (1.8.3) and Lemma 1.8.5, we see that

$$\big[\mathsf{f}^{-1}(-2)\big] = 3 + \sum_{P \in \Sigma} \mathbf{D}_P^{-2},$$

where the set Σ consists of the points $P_{\{x\},\{y\},\{z\}}$, $P_{\{x\},\{y\},\{t\}}$, $P_{\{x\},\{z\},\{t\}}$, $P_{\{y\},\{z\},\{t\}}$, $P_{\{x\},\{t\},\{y,z\}}$, and $P_{\{y\},\{t\},\{x,z\}}$. Using Lemma 1.12.1, we see that

$$\mathbf{D}^{-2}_{P_{\{x\},\{y\},\{t\}}} = \mathbf{D}^{-2}_{P_{\{x\},\{z\},\{t\}}} = \mathbf{D}^{-2}_{P_{\{y\},\{z\},\{t\}}} = \mathbf{D}^{-2}_{P_{\{x\},\{t\},\{y,z\}}} = \mathbf{D}^{-2}_{P_{\{y\},\{t\},\{x,z\}}} = 0.$$

Thus, we conclude that $[\mathsf{f}^{-1}(-2)] = 3 + \mathbf{D}^{-2}_{P_{\{x\},\{y\},\{z\}}}$. Let us show that $\mathbf{D}^{-2}_{P_{\{x\},\{y\},\{z\}}} = 1$.

Let $\alpha_1\colon U_1 \to \mathbb{P}^3$ be a blow-up of the point $P_{\{x\},\{y\},\{z\}}$. Then $D^1_\lambda = S^1_\lambda$ for every $\lambda \neq \infty$. Moreover, the surface $\mathbf{E}_1$ contains a unique base curve of the pencil $\mathcal{S}^1$. Denote it by C^1_8. Then $\mathbf{m}_8 = 2$ and $\mathbf{M}_8^{-2} = 2$. Thus, using (1.10.9), we see that $\mathbf{D}^{-2}_{P_{\{x\},\{y\},\{z\}}} \geq 1$.

To show that $\mathbf{D}^{-2}_{P_{\{x\},\{y\},\{z\}}} = 1$, observe that there exists a commutative diagram

$$\begin{tikzcd} & U \arrow[dl, "\gamma"'] \arrow[dr, "\alpha"] & \\ U_1 \arrow[rr, "\alpha_1"'] & & \mathbb{P}^3 \end{tikzcd}$$

for some birational morphism γ. Then $\widehat{C}_8$ is the unique base curve of $\widehat{\mathcal{S}}$ that is mapped to $P_{\{x\},\{y\},\{z\}}$ by the morphism α. Thus, using Lemma 1.10.7 and (1.10.9), we conclude that $\mathbf{D}^{-2}_{P_{\{x\},\{y\},\{z\}}} = 1$. $\square$

Recall that $h^{1,2}(X) = 3$. Then ($\heartsuit$) in the Main Theorem follows from Lemma 2.12.4.

Family 2.13. In this case, the threefold X is a blow-up of a smooth quadric threefold in $\mathbb{P}^4$ along a smooth curve of genus 2 and degree 6, which gives $h^{1,2}(X) = 2$. Its toric Landau–Ginzburg model is given by the Minkowski polynomial 1392. Replacing x by x/y, we rewrite it as

$$x + y + \frac{xz}{y} + \frac{x}{yz} + \frac{z}{y} + \frac{yz}{x} + \frac{2}{z} + \frac{2}{y} + \frac{2y}{x} + \frac{1}{yz} + \frac{y}{xz}.$$

The quartic pencil $\mathcal{S}$ is given by

$$xt^3+x^2t^2+2xyt^2+2xzt^2+y^2t^2+x^2tz+xz^2t+2y^2tz+x^2yz+xy^2z+xyz^2+y^2z^2=\lambda xyzt.$$

To prove the Main Theorem in this case, we may assume that $\lambda\neq\infty$. Let $\mathcal{C}$ be a smooth conic given by $z=x^2+2xy+y^2+xt=0$. Then

$$\begin{aligned}
H_{\{x\}}\cdot S_\lambda&=2L_{\{x\},\{y\}}+2L_{\{x\},\{z,t\}},\\
H_{\{y\}}\cdot S_\lambda&=L_{\{x\},\{y\}}+L_{\{y\},\{t\}}+L_{\{y\},\{z,t\}}+L_{\{y\},\{x,z,t\}},\\
H_{\{z\}}\cdot S_\lambda&=2L_{\{z\},\{t\}}+\mathcal{C},\\
H_{\{t\}}\cdot S_\lambda&=L_{\{y\},\{t\}}+L_{\{z\},\{t\}}+L_{\{t\},\{x,z\}}+L_{\{t\},\{x,y\}}.
\end{aligned}\tag{2.13.1}$$

So, we may assume that $C_1=L_{\{x\},\{y\}}$, $C_2=L_{\{y\},\{t\}}$, $C_3=L_{\{z\},\{t\}}$, $C_4=L_{\{x\},\{z,t\}}$, $C_5=L_{\{y\},\{z,t\}}$, $C_6=L_{\{t\},\{x,y\}}$, $C_7=L_{\{t\},\{x,z\}}$, $C_8=L_{\{y\},\{x,z,t\}}$, and $C_9=\mathcal{C}$. These are all base curves of the pencil $\mathcal{S}$.

For every $\lambda\in\mathbb{C}$, the surface S_λ has isolated singularities, so that S_λ is irreducible.

If $\lambda\neq-3$, then the singularities of S_λ that are contained in the base locus of the pencil $\mathcal{S}$ are all du Val and can be described as follows:

$P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_1$ with quadratic term $xy+y^2+xt$;

$P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_1$ with quadratic term $xz+z^2+2tz+t^2$;

$P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_1$ with quadratic term $yz+zt+t^2$;

$P_{\{x\},\{y\},\{z,t\}}$: type $\mathbb{A}_5$ with quadratic term $(\lambda+3)xy$;

$P_{\{y\},\{t\},\{x,z\}}$: type $\mathbb{A}_1$ with quadratic term $(x+z+t)(y+t)-(\lambda+1)yt$ for $\lambda\neq-1$, and type $\mathbb{A}_2$ for $\lambda=-1$;

$P_{\{z\},\{t\},\{x,y\}}$: type $\mathbb{A}_2$ with quadratic term $z(x+y-(\lambda+3)t)$;

$[0:\lambda+3:-1:1]$: type $\mathbb{A}_1$;

$P_{\{t\},\{x,y\},\{x,z\}}$: smooth point for $\lambda\neq-2$, and type $\mathbb{A}_2$ for $\lambda=-2$.

Therefore, the points $P_{\{x\},\{z\},\{t\}}$, $P_{\{x\},\{y\},\{t\}}$, $P_{\{y\},\{z\},\{t\}}$, $P_{\{x\},\{y\},\{z,t\}}$, $P_{\{y\},\{t\},\{x,z\}}$, and $P_{\{z\},\{t\},\{x,y\}}$ are the fixed singular points of the surfaces in the pencil $\mathcal{S}$.

Lemma 2.13.2. *If* $\lambda\neq-3$, *then* $[\mathsf{f}^{-1}(\lambda)]=1$. *One also has* $[\mathsf{f}^{-1}(-3)]=3$.

Proof. If $\lambda\neq-3$, then S_λ has du Val singularities in the base locus of the pencil $\mathcal{S}$, so that $[\mathsf{f}^{-1}(\lambda)]=1$ by Corollary 1.5.4. Hence, we need to show that $[\mathsf{f}^{-1}(-3)]=3$.

Recall that S_{-3} has isolated singularities. Moreover, the points $P_{\{x\},\{z\},\{t\}}$, $P_{\{x\},\{y\},\{t\}}$, $P_{\{y\},\{z\},\{t\}}$, $P_{\{y\},\{t\},\{x,z\}}$, and $P_{\{z\},\{t\},\{x,y\}}$ are good double points of this surface (see Subsection 1.12). Thus, it follows from (1.8.3) and Lemmas 1.8.5 and 1.12.1 that

$$[\mathsf{f}^{-1}(-3)]=1+\mathbf{D}^{-3}_{P_{\{x\},\{y\},\{z,t\}}},$$

where $\mathbf{D}^{-3}_{\mathbf{D}_{P_{\{x\},\{y\},\{z,t\}}}}$ is the defect of the singular point $P_{\{x\},\{y\},\{z,t\}}$.

Let $\alpha_1\colon U_1\to\mathbb{P}^3$ be a blow-up of the point $P_{\{x\},\{y\},\{z,t\}}$. Then $D^1_{-3}=S^1_{-3}+\mathbf{E}_1$. In the chart $t=1$, the surface S_λ is given by

$$(\lambda+3)\overline{x}\,\overline{y}+\big(\overline{x}^2\overline{z}-\overline{x}^2\overline{y}-\overline{x}\,\overline{y}^2-(\lambda+2)\overline{x}\,\overline{y}\,\overline{z}+\overline{x}\,\overline{z}^2\big)+\big(\overline{x}^2\overline{y}\,\overline{z}+\overline{x}\,\overline{y}^2\overline{z}+\overline{x}\,\overline{y}\,\overline{z}^2+\overline{y}^2\overline{z}^2\big)=0,$$

where $\overline{x}=x$, $\overline{y}=y$, and $\overline{z}=z+1$. Then a chart of the blow-up α_1 is given by the coordinate change $\overline{x}_1=\overline{x}/\overline{z}$, $\overline{y}_1=\overline{y}/\overline{z}$, $\overline{z}_1=\overline{x}$. Then D^1_λ is given by

$$\overline{x}_1\big(\overline{z}_1+(\lambda+3)\overline{y}_1\big)+\big(\overline{x}_1^2\overline{z}_1-(\lambda+2)\overline{x}_1\overline{y}_1\overline{z}_1\big)\\ +\big(\overline{x}_1\overline{y}_1\overline{z}_1^2-\overline{x}_1^2\overline{y}_1\overline{z}_1-\overline{x}_1\overline{y}_1^2\overline{z}_1+\overline{y}_1^2\overline{z}_1^2\big)+\big(\overline{x}_1^2\overline{y}_1\overline{z}_1^2+\overline{x}_1\overline{y}_1^2\overline{z}_1^2\big)=0,$$

and the surface $\mathbf{E}_1$ is given by $\overline{z}_1=0$.

The surface $\mathbf{E}_1$ contains two base curves of the pencil $\mathcal{S}^1$. They are given by $\overline{z}_1=\overline{x}_1=0$ and $\overline{z}_1=\overline{y}_1=0$. Denote them by C^1_9 and C^1_{10}, respectively. Then $\mathbf{M}_9^{-3}=2$, $\mathbf{M}_{10}^{-3}=1$, and $\mathbf{m}_9=2$. Thus, using (1.10.9) and Lemma 1.10.7, we see that $\mathbf{D}^{-3}_{P_{\{x\},\{y\},\{z,t\}}}\geq 2$.

To show that $\mathbf{D}^{-3}_{P_{\{x\},\{y\},\{z,t\}}}=2$, we have to blow-up U_2 at the point $(\overline{x}_1,\overline{x}_2,\overline{z}_1)=(0,0,0)$. Namely, let $\alpha_2\colon U_2\to U_1$ be this blow-up. Then $D^2_{-3}=S^2_{-3}+\mathbf{E}^2_1$, and $\mathbf{E}_2$ contains a unique base curve of the pencil $\mathcal{S}^2$. Denote it by C^2_{11}. Then $\mathbf{M}_{11}^{-3}=1$. Now, using (1.10.9) and Lemma 1.10.7 again, we obtain $\mathbf{D}^{-3}_{P_{\{x\},\{y\},\{z,t\}}}=2$. This gives $[\mathsf{f}^{-1}(-3)]=3$. $\square$

Note that Lemma 2.13.2 implies (♡) in the Main Theorem, since $h^{1,2}(X)=2$.

To verify (♡) in the Main Theorem, recall that the base curves of the pencil $\mathcal{S}$ are $L_{\{x\},\{y\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{z,t\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{t\},\{x,y\}}$, $L_{\{t\},\{x,z\}}$, $L_{\{y\},\{x,z,t\}}$, and $\mathcal{C}$. On a general quartic surface in this pencil, the intersection matrix of these curves has the same rank as the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{t\},\{x,y\}}$, and H_λ. This follows from (2.13.1). On the other hand, if $\lambda\notin\{-1,-2,-3\}$, then the intersection form of the curves $L_{\{x\},\{y\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{t\},\{x,y\}}$, and H_λ on the surface S_λ is given by

$$\begin{array}{c} \\ L_{\{x\},\{y\}}\\ L_{\{y\},\{t\}}\\ L_{\{z\},\{t\}}\\ L_{\{y\},\{z,t\}}\\ L_{\{t\},\{x,y\}}\\ H_\lambda\end{array}\begin{array}{c}\begin{array}{cccccc} L_{\{x\},\{y\}} & L_{\{y\},\{t\}} & L_{\{z\},\{t\}} & L_{\{y\},\{z,t\}} & L_{\{t\},\{x,y\}} & H_\lambda\end{array}\\ \begin{pmatrix} -1/6 & 1/2 & 0 & 1/3 & 1/2 & 1\\ 1/2 & -1/2 & 1/2 & 1/2 & 1/2 & 1\\ 0 & 1/2 & -1/3 & 1/2 & 1/3 & 1\\ 1/3 & 1/2 & 1/2 & -2/3 & 0 & 1\\ 1/2 & 1/2 & 1/3 & 0 & -5/6 & 1\\ 1 & 1 & 1 & 1 & 1 & 4\end{pmatrix}\end{array}.$$

Since the determinant of this matrix is $-5/12$, we see that its rank is 6. On the other hand, we have $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk})=\operatorname{rk}\operatorname{Pic}(S_{\Bbbk})+12$. Hence, we see that (★) holds, so that (♢) in the Main Theorem also holds by Lemma 1.13.1.

Family 2.14. Let V_5 be a smooth threefold such that $-K_{V_5}\sim 2H$ and $H^3=5$, where H is an ample Cartier divisor. Then V_5 is determined by these properties uniquely up to isomorphism. A general surface in $|H|$ is a smooth del Pezzo surface of degree 5. This linear system is base point free and gives an embedding $V_5\hookrightarrow\mathbb{P}^6$.

In our case, the threefold X is a blow-up of the threefold V_5 along an elliptic curve that is a complete intersection of two general surfaces in the linear system $|H|$. Its toric Landau–Ginzburg model is given by the Minkowski polynomial 1658, which is

$$x+\frac{xy}{z}+z+\frac{2y}{z}+\frac{z^2}{xy}+\frac{z}{x}+\frac{2}{z}+\frac{3z}{xy}+\frac{3}{x}+\frac{y}{xz}+\frac{3}{xy}+\frac{2}{xz}+\frac{1}{xyz}.$$

The quartic pencil $\mathcal{S}$ is given by

$$x^2zy+y^2x^2+z^2yx+2y^2tx+z^3t+z^2ty+2t^2yx+3t^2z^2\\ +3t^2zy+t^2y^2+3t^3z+2t^3y+t^4=\lambda xyzt.$$

Suppose that $\lambda \neq \infty$. Let $\mathcal{C}_1$ be a conic given by $x = t^2 + ty + 2tz + z^2 = 0$, let $\mathcal{C}_2$ be a conic given by $z = xy + ty + t^2 = 0$, and let $\mathcal{C}_3$ be a conic given by $t = xy + xz + z^2 = 0$. Then

$$\begin{aligned} H_{\{x\}} \cdot S_\lambda &= L_{\{x\},\{t\}} + L_{\{x\},\{y,z,t\}} + \mathcal{C}_1, \\ H_{\{y\}} \cdot S_\lambda &= L_{\{y\},\{t\}} + 3L_{\{y\},\{z,t\}}, \\ H_{\{z\}} \cdot S_\lambda &= 2\mathcal{C}_2, \\ H_{\{t\}} \cdot S_\lambda &= L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + \mathcal{C}_3. \end{aligned} \tag{2.14.1}$$

Let $C_1 = \mathcal{C}_1$, $C_2 = \mathcal{C}_2$, $C_3 = \mathcal{C}_3$, $C_4 = L_{\{x\},\{t\}}$, $C_5 = L_{\{y\},\{t\}}$, $C_6 = L_{\{x\},\{y,z,t\}}$, and $C_7 = L_{\{y\},\{z,t\}}$. These are all base curves of the pencil $\mathcal{S}$.

For every $\lambda \in \mathbb{C}$, the surface S_λ has isolated singularities, so that S_λ is irreducible.

If $\lambda \neq -4$, then the singularities of S_λ that are contained in the base locus of the pencil $\mathcal{S}$ are all du Val and can be described as follows:

$P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term $y(z + y)$;

$P_{\{x\},\{z\},\{t\}}$: type $\mathbb{D}_5$ with quadratic term $(x + t)^2$;

$P_{\{x\},\{z\},\{y,t\}}$: type $\mathbb{A}_1$ with quadratic term
$$((\lambda + 3)x - t)z + (x - y - 2z)(x - y - z - 2t) + t^2 \text{ for } \lambda \neq -3, \text{ and}$$
type $\mathbb{A}_3$ for $\lambda = -3$;

$P_{\{x\},\{y\},\{z,t\}}$: type $\mathbb{A}_2$ with quadratic term $y((\lambda + 3)x + y + z + t)$ for $\lambda \neq -3$, and type $\mathbb{A}_5$ for $\lambda = -3$;

$[\lambda + 3 : 0 : -1 : 1]$: type $\mathbb{A}_2$ for $\lambda \neq -3$;

$[(\lambda + 4)(\lambda + 3) : -1 : 0 : \lambda + 4]$: type $\mathbb{A}_1$ for $\lambda \neq -3$.

Therefore, if $\lambda \neq -4$, then S_λ has du Val singularities in the base locus of the pencil $\mathcal{S}$, so that the fiber $\mathsf{f}^{-1}(\lambda)$ is irreducible by Corollary 1.5.4. On the other hand, we have

Lemma 2.14.2. *One has* $[\mathsf{f}^{-1}(-4)] = 2$.

Proof. The points $P_{\{y\},\{z\},\{t\}}$, $P_{\{x\},\{z\},\{y,t\}}$, and $P_{\{x\},\{y\},\{z,t\}}$ are good double points of the surface S_{-4}. Thus, it follows from (1.8.3) and Lemmas 1.8.5 and 1.12.1 that

$$\left[\mathsf{f}^{-1}(-4)\right] = 1 + \mathbf{D}^{-4}_{P_{\{x\},\{z\},\{t\}}}.$$

Here, the number $\mathbf{D}^{-4}_{P_{\{x\},\{z\},\{t\}}}$ is the defect of the point $P_{\{x\},\{z\},\{t\}}$. To compute it, we have to (partially) resolve the singularity of the surface S_{-4} at the point $P_{\{x\},\{z\},\{t\}}$.

Let $\alpha_1 \colon U_1 \to \mathbb{P}^3$ be a blow-up of the point $P_{\{x\},\{z\},\{t\}}$. In the chart $y = 1$, the surface S_λ is given by

$$\bar{x}^2 + \left(2\bar{t}^2\bar{x} + (\lambda + 4)\bar{t}^2\bar{z} - (\lambda + 2)\bar{t}\bar{x}\bar{z} + \bar{x}^2\bar{z} + \bar{x}\bar{z}^2\right) + \left(\bar{t}^4 + 3\bar{z}\bar{t}^3 + 3\bar{t}^2\bar{z}^2 + \bar{t}\bar{z}^3\right) = 0,$$

where $\bar{x} = x + t$, $\bar{z} = z$, and $\bar{t} = t$. Then a chart of the blow-up α_1 is given by the coordinate change $\bar{x}_1 = \bar{x}/\bar{z}$, $\bar{z}_1 = \bar{x}$, $\bar{t}_1 = \bar{t}/\bar{z}$. Let $\widehat{x}_1 = \bar{x}_1$, $\widehat{z}_1 = \bar{x}_1 + \bar{z}_1$, and $\widehat{t}_1 = \bar{t}_1$. In these coordinates, the surface S^1_λ is given by

$$\begin{aligned} \widehat{x}_1\widehat{z}_1 &+ \left(\widehat{t}_1\widehat{z}_1^2 - \widehat{x}_1^3 + \widehat{x}_1^2\widehat{z}_1 - (\lambda + 4)\widehat{t}_1^2\widehat{x}_1 + (\lambda + 4)\widehat{t}_1^2\widehat{z}_1 + (3 + \lambda)\widehat{x}_1^2\widehat{t}_1 - (\lambda + 4)\widehat{t}_1\widehat{x}_1\widehat{z}_1\right) \\ &+ \widehat{t}_1^2\widehat{x}_1^2 - 4\widehat{t}_1^2\widehat{x}_1\widehat{z}_1 + 3\widehat{t}_1^2\widehat{z}_1^2 + 3\widehat{x}_1^2\widehat{t}_1^3 - 6\widehat{x}_1\widehat{z}_1\widehat{t}_1^3 + 3\widehat{t}_1^3\widehat{z}_1^2 + \widehat{t}_1^4\widehat{x}_1^2 - 2\widehat{t}_1^4\widehat{x}_1\widehat{z}_1 + \widehat{t}_1^4\widehat{z}_1^2 = 0. \end{aligned}$$

If $\lambda \neq -4$, the surface S^1_λ has an isolated singularity at $(\widehat{x}_1, \widehat{z}_1, \widehat{t}_1) = (0, 0, 0)$. In this case, the surface $\mathbf{E}_1$ contains another singular point of S^1_λ, which lies in another chart of the blow-up α_1. If

$\lambda \neq -4$, then this point is an isolated ordinary double point of the surface S^1_λ. On the other hand, the surface S^1_{-4} is singular along the curve $\overline{z}_1 = \overline{x}_1 = 0$. This explains why the singularity of the surface S_{-4} at the point $P_{\{x\},\{z\},\{t\}}$ is not du Val.

Let $\alpha_2\colon U_2 \to U_1$ be a blow-up of the point $(\widehat{x}_1, \widehat{z}_1, \widehat{t}_1) = (0,0,0)$. A chart of this blow-up is given by the coordinate change $\widehat{x}_2 = \widehat{x}_1/\widehat{t}_1$, $\widehat{y}_2 = \widehat{y}_1/\widehat{t}_1$, $\widehat{t}_2 = \widehat{t}_1$. In these coordinates, the surface S^2_λ is given by

$$\begin{aligned}\widehat{x}_2\widehat{z}_2 - (\lambda+4)\widehat{x}_2\widehat{t}_2 + (\lambda+4)\widehat{z}_2\widehat{t}_2 + \big(\widehat{t}_2\widehat{z}_2^2 + (3+\lambda)\widehat{x}_2^2\widehat{t}_2 - (\lambda+4)\widehat{t}_2\widehat{x}_2\widehat{z}_2\big) + \widehat{t}_2^2\widehat{x}_2^2 - 4\widehat{t}_2^2\widehat{x}_2\widehat{z}_2 \\ + 3\widehat{t}_2^2\widehat{z}_2^2 - \widehat{t}_2\widehat{x}_2^3 + \widehat{t}_2\widehat{x}_2^2\widehat{z}_2 + 3\widehat{x}_2^2\widehat{t}_2^3 - 6\widehat{x}_2\widehat{z}_2\widehat{t}_2^3 + 3\widehat{t}_2^3\widehat{z}_2^2 + \widehat{t}_2^4\widehat{x}_2^2 - 2\widehat{t}_2^4\widehat{x}_2\widehat{z}_2 + \widehat{t}_2^4\widehat{z}_2^2 = 0,\end{aligned}$$

and the surface $\mathbf{E}_2$ is given by $\widehat{t}_2 = 0$. Note that $D^2_\lambda = S^2_\lambda \sim -K_{U_2}$ for every $\lambda \in \mathbb{C}$.

If $\lambda \neq -4$, then the quadric form $\widehat{x}_2\widehat{z}_2 - (\lambda+4)\widehat{x}_2\widehat{t}_2 + (\lambda+4)\widehat{z}_2\widehat{t}_2$ is not degenerate, so that S^2_λ has an isolated ordinary double singularity at $(\widehat{x}_2, \widehat{z}_2, \widehat{t}_2) = (0,0,0)$. Thus, in this case, the surface S^1_λ has a du Val singularity of type $\mathbb{A}_3$ at the point $(\widehat{x}_1, \widehat{z}_1, \widehat{t}_1) = (0,0,0)$. Therefore, if $\lambda \neq -4$, then S_λ has a du Val singularity of type $\mathbb{D}_5$ at the point $P_{\{x\},\{z\},\{t\}}$.

Now we are ready to compute $\mathbf{D}^{-4}_{P_{\{x\},\{z\},\{t\}}}$ using the algorithm described in Subsection 1.10. Observe that $\mathbf{E}_1$ contains one base curve of the pencil $\mathcal{S}^1$. It is given by $\overline{z}_1 = \overline{x}_1 = 0$. Denote this curve by C^1_8. Then $\mathbf{m}_8 = 2$ and $\mathbf{M}^{-4}_8 = 2$. Similarly, the surface $\mathbf{E}_2$ contains two base curves of the pencil $\mathcal{S}^2$. They are given by $\widehat{t}_2 = \widehat{x}_2 = 0$ and $\widehat{t}_2 = \widehat{z}_2 = 0$. Denote them by C^2_9 and C^2_{10}, respectively. Then $\mathbf{M}^{-4}_9 = \mathbf{M}^{-4}_{10} = 1$. Now, using (1.10.9) and Lemma 1.10.7, we deduce that $\mathbf{D}^{-4}_{P_{\{x\},\{z\},\{t\}}} = 1$, so that $[\mathsf{f}^{-1}(-4)] = 2$. $\square$

Note that Lemma 2.14.2 implies (♡) in the Main Theorem, since $h^{1,2}(X) = 2$.

Lemma 2.14.3. *Suppose that $\lambda \neq -4, -3$. Then the intersection form of the curves $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{x\},\{y,z,t\}}$, and H_λ on the surface S_λ is given by*

$$\begin{array}{r c}
 & \begin{array}{cccc} L_{\{x\},\{t\}} & L_{\{y\},\{t\}} & L_{\{x\},\{y,z,t\}} & H_\lambda \end{array} \\
\begin{array}{r} L_{\{x\},\{t\}} \\ L_{\{y\},\{t\}} \\ L_{\{x\},\{y,z,t\}} \\ H_\lambda \end{array} & \begin{pmatrix} -3/4 & 1 & 1 & 1 \\ 1 & 5/4 & 0 & 1 \\ 1 & 0 & -5/6 & 1 \\ 1 & 1 & 1 & 4 \end{pmatrix}.
\end{array}$$

Proof. To compute $L^2_{\{x\},\{t\}}$, let us use the notation of Lemma A.3.2 with $S = S_\lambda$, $n = 5$, $O = P_{\{x\},\{z\},\{t\}}$, and $C = L_{\{x\},\{t\}}$. Then $\overline{C}$ contains the point $\alpha(G_1) = \alpha(G_2) = \alpha(G_3)$, and either $\widetilde{C} \cdot G_2 = 1$ or $\widetilde{C} \cdot G_3 = 1$. This follows from the proof of Lemma 2.14.2. Thus, we have $L^2_{\{x\},\{t\}} = -3/4$ by Lemma A.3.2.

To find $L^2_{\{y\},\{t\}}$, we observe that $P_{\{y\},\{z\},\{t\}}$ is the only singular point of S_λ that is contained in $L_{\{y\},\{t\}}$. Thus, it follows from Proposition A.1.2 that $L^2_{\{y\},\{t\}} = -5/4$.

To find $L^2_{\{x\},\{y,z,t\}}$, we note that $P_{\{x\},\{z\},\{y,t\}}$ and $P_{\{x\},\{y\},\{z,t\}}$ are the only singular points of S_λ that are contained in the line $L_{\{x\},\{y,z,t\}}$. Thus, it follows from Proposition A.1.2 that $L^2_{\{y\},\{t\}} = -5/6$.

Since $L_{\{x\},\{t\}} \cap L_{\{y\},\{t\}} = P_{\{x\},\{y\},\{t\}}$ and $L_{\{x\},\{t\}} \cap L_{\{x\},\{y,z,t\}} = P_{\{x\},\{t\},\{y,z\}}$, we obtain

$$L_{\{x\},\{t\}} \cdot L_{\{y\},\{t\}} = L_{\{x\},\{t\}} \cdot L_{\{x\},\{y,z,t\}} = 1,$$

because S_λ is smooth at the points $P_{\{x\},\{y\},\{t\}}$ and $P_{\{x\},\{t\},\{y,z\}}$.

Finally, we have $L_{\{y\},\{t\}} \cdot L_{\{x\},\{y,z,t\}} = 0$, since $L_{\{y\},\{t\}} \cap L_{\{x\},\{y,z,t\}} = \varnothing$. $\square$

Recall that the base curves of the pencil $\mathcal{S}$ are the curves $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{x\},\{y,z,t\}}$, $L_{\{y\},\{z,t\}}$, $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$. If follows from (2.14.1) that the intersection matrix of these curves on S_λ has the same rank as the intersection matrix of the curves $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{x\},\{y,z,t\}}$, and H_λ. On the other hand, the determinant of the intersection matrix in Lemma 2.14.3 is 25/16. Thus, if $\lambda\neq-4,-3$, then the intersection matrix of the curves $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{x\},\{y,z,t\}}$, $L_{\{y\},\{z,t\}}$, $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ on the surface S_λ has rank 4. We also have $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk})=\operatorname{rk}\operatorname{Pic}(S_{\Bbbk})+14$. Hence, we see that (★) holds, so that (♢) in the Main Theorem also holds by Lemma 1.13.1.

Family 2.15. In this case, the Fano threefold X is a blow-up of $\mathbb{P}^3$ along a smooth curve of degree 6 and genus 4. Thus, we have $h^{1,2}(X)=4$. A toric Landau–Ginzburg model of this family is given by the Minkowski polynomial 910, which is

$$x+y+z+\frac{x}{z}+\frac{y}{z}+\frac{x}{yz}+\frac{2}{z}+\frac{y}{xz}+\frac{2}{y}+\frac{2}{x}+\frac{z}{xy}.$$

The pencil $\mathcal{S}$ is given by

$$x^2zy+y^2zx+z^2yx+x^2ty+y^2tx+x^2t^2+2t^2yx+t^2y^2+2t^2zx+2t^2zy+t^2z^2=\lambda xyzt.$$

Observe that this equation is symmetric with respect to the permutation $x\leftrightarrow y$.

We may assume that $\lambda\neq\infty$. Let $\mathcal{C}$ be the conic $\{z=xy+xt+yt=0\}$. Then

$$\begin{aligned}
&H_{\{x\}}\cdot S_\lambda=2L_{\{x\},\{t\}}+2L_{\{x\},\{y,z\}},\\
&H_{\{y\}}\cdot S_\lambda=2L_{\{y\},\{t\}}+2L_{\{y\},\{x,z\}},\\
&H_{\{z\}}\cdot S_\lambda=L_{\{z\},\{t\}}+L_{\{z\},\{x,y\}}+\mathcal{C},\\
&H_{\{t\}}\cdot S_\lambda=L_{\{x\},\{t\}}+L_{\{y\},\{t\}}+L_{\{z\},\{t\}}+L_{\{t\},\{x,y,z\}}.
\end{aligned}\tag{2.15.1}$$

So, we may assume that $C_1=L_{\{x\},\{t\}}$, $C_2=L_{\{y\},\{t\}}$, $C_3=L_{\{z\},\{t\}}$, $C_4=L_{\{x\},\{y,z\}}$, $C_5=L_{\{y\},\{x,z\}}$, $C_6=L_{\{z\},\{x,y\}}$, $C_7=L_{\{t\},\{x,y,z\}}$, and $C_8=\mathcal{C}$. These are all base curves of the pencil $\mathcal{S}$.

If $\lambda\neq-1$, then the surface S_λ has isolated singularities, so that S_λ is irreducible. On the other hand, we have $S_{-1}=H_{\{x,y,z\}}+\mathbf{S}$, where $\mathbf{S}$ is an irreducible cubic surface given by the equation $xyz+xyt+xt^2+yt^2+zt^2=0$. The surface $\mathbf{S}$ is singular at $P_{\{y\},\{z\},\{t\}}$, $P_{\{x\},\{z\},\{t\}}$, and $P_{\{x\},\{y\},\{t\}}$. These are isolated ordinary double points of this surface. Note also that

$$H_{\{x,y,z\}}\cap\mathbf{S}=L_{\{x\},\{y,z\}}+L_{\{y\},\{x,z\}}+\ell,$$

where ℓ is the line $\{y+x-t=z+t=0\}$.

If $\lambda\neq-1$, then the singularities of S_λ that are contained in the base locus of the pencil $\mathcal{S}$ are all du Val and can be described as follows:

$P_{\{x\},\{y\},\{z\}}$: type $\mathbb{D}_4$ with quadratic term $(x+y+z)^2$;

$P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_1$ with quadratic term $xy+t^2$;

$P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_1$ with quadratic term $xz+xt+t^2$;

$P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_1$ with quadratic term $yz+yt+t^2$;

$P_{\{x\},\{t\},\{y,z\}}$: type $\mathbb{A}_3$ with quadratic term $x(x+y+z-(\lambda+1)t)$;

$P_{\{y\},\{t\},\{x,z\}}$: type $\mathbb{A}_3$ with quadratic term $y(x+y+z-(\lambda+1)t)$;

$P_{\{z\},\{t\},\{x,y\}}$: type $\mathbb{A}_1$ with quadratic term $(x+y)(z+t)+z^2-\lambda zt$.

If $\lambda\neq-1$, then $[\mathsf{f}^{-1}(\lambda)]=1$ by Corollary 1.5.4, so that (♡) in the Main Theorem follows from

Lemma 2.15.2. *One has* $[\mathsf{f}^{-1}(-1)]=5$.

Proof. It follows from (1.8.3) and Lemmas 1.8.5 and 1.12.1 that

$$\big[\mathsf{f}^{-1}(-1)\big]=4+\mathbf{D}^{-1}_{P_{\{x\},\{y\},\{z\}}}.$$

Observe that $\mathbf{S}$ is smooth at $P_{\{x\},\{y\},\{z\}}$, and $H_{\{x,y,z\}}$ is tangent to $\mathbf{S}$ at this point. Thus, the proper transforms of these surfaces on the blow-up of $\mathbb{P}^3$ at the point $P_{\{x\},\{y\},\{z\}}$ both pass through the base curve of the proper transform of the pencil $\mathcal{S}$ that is contained in the exceptional divisor. Using (1.10.9), we conclude that $\mathbf{D}^{-1}_{P_{\{x\},\{y\},\{z\}}} \geq 1$. Arguing as in the proof of Lemma 2.5.3, we see that $\mathbf{D}^{-1}_{P_{\{x\},\{y\},\{z\}}} = 1$, so that $[\mathsf{f}^{-1}(-1)] = 5$. $\square$

If $\lambda \neq -1$, then the intersection form of the curves $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{z\},\{x,y\}}$, and H_λ on the surface S_λ is given by

$$
\begin{array}{c} \\ L_{\{x\},\{t\}} \\ L_{\{y\},\{t\}} \\ L_{\{z\},\{t\}} \\ L_{\{z\},\{x,y\}} \\ H_\lambda \end{array}
\begin{array}{c}
\begin{array}{ccccc} L_{\{x\},\{t\}} & L_{\{y\},\{t\}} & L_{\{z\},\{t\}} & L_{\{z\},\{x,y\}} & H_\lambda \end{array} \\
\begin{pmatrix}
-1/4 & 1/2 & 1/2 & 0 & 1 \\
1/2 & -1/4 & 1/2 & 0 & 1 \\
1/2 & 1/2 & -1/2 & 1/2 & 1 \\
0 & 0 & 1/2 & -1/2 & 1 \\
1 & 1 & 1 & 1 & 4
\end{pmatrix}
\end{array}.
$$

The rank of this matrix is 4. Thus, if $\lambda \neq -1$, then it follows from (2.15.1) that the intersection matrix of the base curves of the pencil $\mathcal{S}$ on the surface S_λ also has rank 4. On the other hand, we have $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk}) = \operatorname{rk}\operatorname{Pic}(S_{\Bbbk}) + 14$. Hence, we see that ($\bigstar$) holds, so that ($\diamondsuit$) in the Main Theorem also holds by Lemma 1.13.1.

Family 2.16. In this case, the Fano threefold X is a blow-up of a smooth complete intersection of two quadrics in a conic. We have $h^{1,2}(X) = 2$. A toric Landau–Ginzburg model of this family is given by the Minkowski polynomial 1939, which is

$$
x + z + \frac{y}{z} + \frac{y}{x} + \frac{x}{z} + \frac{z}{x} + \frac{x}{y} + \frac{1}{z} + \frac{z}{y} + \frac{1}{x} + \frac{x}{yz} + \frac{2}{y} + \frac{z}{xy}.
$$

The quartic pencil $\mathcal{S}$ is given by

$$
x^2yz + xy^2t + xyz^2 + y^2zt + x^2yt + yz^2t + x^2zt \\
+ xyt^2 + xz^2t + yzt^2 + x^2t^2 + 2xzt^2 + z^2t^2 = \lambda xyzt.
$$

As usual, we suppose that $\lambda \neq \infty$. If $\lambda \neq -2$, then the surface S_λ has isolated singularities, so that it is irreducible. One also has $S_{-2} = H_{\{x,z\}} + H_{\{y,t\}} + \mathbf{Q}$, where $\mathbf{Q}$ is an irreducible quadric surface given by $xz + xt + yt + zt = 0$.

Let $\mathcal{C}$ be the conic in $\mathbb{P}^3$ that is given by $y = xz + xt + zt = 0$. Then

$$
\begin{aligned}
H_{\{x\}} \cdot S_\lambda &= L_{\{x\},\{z\}} + L_{\{x\},\{t\}} + L_{\{x\},\{y,z\}} + L_{\{x\},\{y,t\}}, \\
H_{\{y\}} \cdot S_\lambda &= L_{\{y\},\{t\}} + L_{\{y\},\{x,z\}} + \mathcal{C}, \\
H_{\{z\}} \cdot S_\lambda &= L_{\{x\},\{z\}} + L_{\{z\},\{t\}} + L_{\{z\},\{x,y\}} + L_{\{z\},\{y,t\}}, \\
H_{\{t\}} \cdot S_\lambda &= L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,z\}},
\end{aligned}
\tag{2.16.1}
$$

so that the base locus of the pencil $\mathcal{S}$ consists of the curves $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{y,z\}}$, $L_{\{x\},\{y,t\}}$, $L_{\{y\},\{x,z\}}$, $L_{\{z\},\{x,y\}}$, $L_{\{z\},\{y,t\}}$, $L_{\{t\},\{x,z\}}$, and $\mathcal{C}$.

If $\lambda \neq -2$, then the singular points of S_λ contained in the base locus of the pencil $\mathcal{S}$ are all du Val and can be described as follows:

$P_{\{x\},\{y\},\{z\}}$: type $\mathbb{A}_3$ with quadratic term $(x+z)(x+y+z)$ for $\lambda \neq -3$, and type $\mathbb{A}_5$ for $\lambda = -3$;

$P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_2$ with quadratic term $(x+t)(y+t)$;

$P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term $t(x+z)$;

$P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_2$ with quadratic term $(y+t)(z+t)$;

$P_{\{x\},\{z\},\{y,t\}}$: type $\mathbb{A}_1$ with quadratic term $(x+z)(y+t)-(\lambda+2)xz$;

$P_{\{y\},\{t\},\{x,z\}}$: type $\mathbb{A}_1$ with quadratic term $(x+z)(y+t)-(\lambda+2)yt$.

Moreover, the singularities of the surface S_{-2} at these points are non-isolated ordinary double points. Thus, if $\lambda \neq -2$, then the fiber $\mathsf{f}^{-1}(\lambda)$ is irreducible by Corollary 1.5.4. Similarly, it follows from (1.8.3) and Lemma 1.12.1 that $[\mathsf{f}^{-1}(-2)] = 3$. This confirms (♡) in the Main Theorem, since $h^{1,2}(X) = 2$.

If $\lambda \neq -2$, then the intersection matrix of the base curves of the pencil $\mathcal{S}$ on the surface S_λ has the same rank as the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{y\},\{t\}}$, $L_{\{x\},\{y,z\}}$, $L_{\{x\},\{y,t\}}$, $L_{\{y\},\{x,z\}}$, and H_λ, because

$$L_{\{x\},\{z\}} + L_{\{x\},\{t\}} + L_{\{x\},\{y,z\}} + L_{\{x\},\{y,t\}} \sim L_{\{y\},\{t\}} + L_{\{y\},\{x,z\}} + \mathcal{C}$$
$$\sim L_{\{x\},\{z\}} + L_{\{z\},\{t\}} + L_{\{z\},\{x,y\}} + L_{\{z\},\{y,t\}} \sim L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,z\}} \sim H_\lambda$$

on the surface S_λ by (2.16.1), and

$$2L_{\{y\},\{t\}} + L_{\{z\},\{y,t\}} + L_{\{x\},\{y,t\}} \sim 2L_{\{x\},\{z\}} + L_{\{y\},\{x,z\}} + L_{\{t\},\{x,z\}} \sim H_\lambda,$$

since

$$H_{\{y,t\}} \cdot S_\lambda = 2L_{\{y\},\{t\}} + L_{\{z\},\{y,t\}} + L_{\{x\},\{y,t\}}$$

and

$$H_{\{x,z\}} \cdot S_\lambda = 2L_{\{x\},\{z\}} + L_{\{y\},\{x,z\}} + L_{\{t\},\{x,z\}}.$$

If $\lambda \neq -2, -3$, then the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{y\},\{t\}}$, $L_{\{x\},\{y,z\}}$, $L_{\{x\},\{y,t\}}$, $L_{\{y\},\{x,z\}}$, and H_λ on the surface S_λ is given by

$$\begin{array}{r} \\ L_{\{x\},\{y\}} \\ L_{\{y\},\{t\}} \\ L_{\{x\},\{y,z\}} \\ L_{\{x\},\{y,t\}} \\ L_{\{y\},\{x,z\}} \\ H_\lambda \end{array}
\begin{array}{c} \begin{array}{cccccc} L_{\{x\},\{y\}} & L_{\{y\},\{t\}} & L_{\{x\},\{y,z\}} & L_{\{x\},\{y,t\}} & L_{\{y\},\{x,z\}} & H_\lambda \end{array} \\
\begin{pmatrix} 0 & 0 & 1/4 & 1/4 & 1/2 & 1 \\ 0 & -1/6 & 0 & 2/3 & 1/2 & 1 \\ 1/4 & 0 & -5/4 & 1 & 1/2 & 1 \\ 1/2 & 2/3 & 1 & -5/6 & 0 & 1 \\ 1/2 & 1/2 & 1/2 & 0 & -1/2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 4 \end{pmatrix} \end{array}.$$

This matrix has rank 6. Hence, we see that (★) holds, because $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk}) = \operatorname{rk}\operatorname{Pic}(S_{\Bbbk}) + 12$. Then (♢) in the Main Theorem holds by Lemma 1.13.1.

Family 2.17. The threefold X is a blow-up of a smooth quadric threefold along a smooth elliptic curve of degree 5, so that $h^{1,2}(X) = 1$. A toric Landau–Ginzburg model of this family is given by the Minkowski polynomial 1926, which is

$$x + y + z + \frac{x}{y} + \frac{y}{x} + \frac{z}{y} + \frac{z}{x} + \frac{1}{z} + \frac{2}{y} + \frac{1}{x} + \frac{z}{xy} + \frac{1}{xz} + \frac{1}{xy}.$$

The quartic pencil $\mathcal{S}$ is given by

$$x^2yz + xy^2z + xyz^2 + x^2zt + y^2zt + xz^2t + yz^2t + xyt^2 + 2xzt^2 + yzt^2 + z^2t^2 + yt^3 + zt^3 = \lambda xyzt.$$

As usual, we suppose that $\lambda\neq\infty$. Let $\mathcal{C}$ be the conic $\{x=yz+tz+t^2=0\}$. Then

$$
\begin{aligned}
H_{\{x\}}\cdot S_\lambda &= L_{\{x\},\{t\}}+L_{\{x\},\{y,z\}}+\mathcal{C},\\
H_{\{y\}}\cdot S_\lambda &= L_{\{y\},\{z\}}+L_{\{y\},\{t\}}+L_{\{y\},\{x,t\}}+L_{\{y\},\{x,z,t\}},\\
H_{\{z\}}\cdot S_\lambda &= L_{\{y\},\{z\}}+2L_{\{z\},\{t\}}+L_{\{z\},\{x,t\}},\\
H_{\{t\}}\cdot S_\lambda &= L_{\{x\},\{t\}}+L_{\{y\},\{t\}}+L_{\{z\},\{t\}}+L_{\{t\},\{x,y,z\}}.
\end{aligned}
\tag{2.17.1}
$$

So, we may assume that $C_1=L_{\{x\},\{t\}}$, $C_2=L_{\{y\},\{z\}}$, $C_3=L_{\{y\},\{t\}}$, $C_4=L_{\{z\},\{t\}}$, $C_5=L_{\{x\},\{y,z\}}$, $C_6=L_{\{y\},\{x,t\}}$, $C_7=L_{\{z\},\{x,t\}}$, $C_8=L_{\{y\},\{x,z,t\}}$, $C_9=L_{\{t\},\{x,y,z\}}$, and $C_{10}=\mathcal{C}$. These are all base curves of the pencil $\mathcal{S}$. Note that $\mathbf{m}_1=\mathbf{m}_2=\mathbf{m}_3=2$, $\mathbf{m}_4=3$, and $\mathbf{m}_5=\ldots=\mathbf{m}_{10}=1$.

If $\lambda\neq-2$, then the surface $S_\lambda\in\mathcal{S}$ has isolated singularities, so that it is irreducible. On the other hand, one also has $S_{-2}=H_{\{x,t\}}+\mathbf{S}$, where $\mathbf{S}$ is an irreducible cubic surface given by the equation $xyz+xzt+y^2z+yz^2+yzt+yt^2+z^2t+zt^2=0$.

If $\lambda\neq-2$, then the singular points of S_λ contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

- $P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_2$ with quadratic term $(x+t)(y+t)$;
- $P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term $z(x+t)$;
- $P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_2$ with quadratic term $z(y+t)$;
- $P_{\{x\},\{t\},\{y,z\}}$: type $\mathbb{A}_1$ with quadratic term $(x+y+z)(x+t)-(\lambda+2)xt$;
- $P_{\{y\},\{t\},\{x,z\}}$: type $\mathbb{A}_1$ with quadratic term $(y+t)(x+y+z+t)-(\lambda+3)yt$ for $\lambda\neq-3$, and type $\mathbb{A}_2$ for $\lambda=-3$;
- $P_{\{y\},\{z\},\{x,t\}}$: type $\mathbb{A}_2$ with quadratic term $y(x+t+(\lambda+2)z)$;
- $P_{\{z\},\{t\},\{x,y\}}$: type $\mathbb{A}_1$ with quadratic term $z(x+y+z-(\lambda+2)t)+t^2$;
- $[0:-2:2:-1\pm\sqrt{5}]$: smooth point for $\lambda\neq(-1\mp\sqrt{5})/2$, and type $\mathbb{A}_1$ for $\lambda=(-1\mp\sqrt{5})/2$.

Thus, if $\lambda\neq-2$, then $[\mathsf{f}^{-1}(\lambda)]=1$ by Corollary 1.5.4. To find $[\mathsf{f}^{-1}(-2)]$, observe that the set Σ consists of the points $P_{\{y\},\{z\},\{t\}}$, $P_{\{x\},\{z\},\{t\}}$, $P_{\{x\},\{y\},\{t\}}$, $P_{\{x\},\{t\},\{y,z\}}$, $P_{\{y\},\{z\},\{x,t\}}$, $P_{\{y\},\{t\},\{x,z\}}$, and $P_{\{z\},\{t\},\{x,y\}}$. Thus, it follows from (1.8.3) and Lemma 1.8.5 that

$$\left[\mathsf{f}^{-1}(-2)\right]=2+\sum_{P\in\Sigma}\mathbf{D}_P^{-2}.$$

Moreover, the quadratic terms of the surface S_λ at the singular points $P_{\{y\},\{z\},\{t\}}$, $P_{\{x\},\{z\},\{t\}}$, $P_{\{x\},\{y\},\{t\}}$, $P_{\{x\},\{t\},\{y,z\}}$, $P_{\{y\},\{z\},\{x,t\}}$, $P_{\{y\},\{t\},\{x,z\}}$, and $P_{\{z\},\{t\},\{x,y\}}$ given above are also valid for $\lambda=-2$. This shows that all these points are good double points of the surface S_{-2}, so that their defects vanish by Lemma 1.12.1. Hence, we have $[\mathsf{f}^{-1}(-2)]=2$. This confirms (♡) in the Main Theorem, since $h^{1,2}(X)=1$.

If $\lambda\neq-2$, then the intersection matrix of the base curves of the pencil $\mathcal{S}$ on the surface S_λ has the same rank as the intersection matrix of the curves $L_{\{x\},\{y,z\}}$, $L_{\{y\},\{x,t\}}$, $L_{\{z\},\{x,t\}}$, $L_{\{y\},\{x,z,t\}}$, $L_{\{y\},\{z\}}$, and H_λ, because

$$
\begin{aligned}
L_{\{x\},\{t\}}+L_{\{x\},\{y,z\}}+\mathcal{C}&\sim L_{\{y\},\{z\}}+L_{\{y\},\{t\}}+L_{\{y\},\{x,t\}}+L_{\{y\},\{x,z,t\}}\\
&\sim L_{\{y\},\{z\}}+2L_{\{z\},\{t\}}+L_{\{z\},\{x,t\}}\sim L_{\{x\},\{t\}}+L_{\{y\},\{t\}}+L_{\{z\},\{t\}}+L_{\{t\},\{x,y,z\}}\sim H_\lambda
\end{aligned}
$$

on the surface S_λ by (2.17.1), and

$$2L_{\{x\},\{t\}}+L_{\{y\},\{x,t\}}+L_{\{z\},\{x,t\}}\sim H_\lambda,$$

since $H_{\{x,t\}} \cdot S_\lambda = 2L_{\{x\},\{t\}} + L_{\{y\},\{x,t\}} + L_{\{z\},\{x,t\}}$. Moreover, if $\lambda \notin \{-2,-3,(-1\pm\sqrt{5})/2\}$, then the intersection matrix of the curves $L_{\{x\},\{y,z\}}$, $L_{\{y\},\{x,t\}}$, $L_{\{z\},\{x,t\}}$, $L_{\{y\},\{x,z,t\}}$, $L_{\{y\},\{z\}}$, and H_λ on the surface S_λ is given by

$$
\begin{array}{c}
\\
L_{\{x\},\{y,z\}}\\
L_{\{y\},\{x,t\}}\\
L_{\{z\},\{x,t\}}\\
L_{\{y\},\{x,z,t\}}\\
L_{\{y\},\{z\}}\\
H_\lambda
\end{array}
\begin{array}{c}
\begin{array}{cccccc}
L_{\{x\},\{y,z\}} & L_{\{y\},\{x,t\}} & L_{\{z\},\{x,t\}} & L_{\{y\},\{x,z,t\}} & L_{\{y\},\{z\}} & H_\lambda
\end{array}\\
\begin{pmatrix}
-3/2 & 0 & 0 & 0 & 1 & 1\\
0 & -2/3 & 1/3 & 2/3 & 2/3 & 1\\
0 & 1/3 & -1/3 & 1/3 & 1/3 & 1\\
0 & 2/3 & 1/3 & -5/6 & 2/3 & 1\\
1 & 2/3 & 1/3 & 2/3 & -2/3 & 1\\
1 & 1 & 1 & 1 & 1 & 4
\end{pmatrix}
\end{array}.
$$

The determinant of this matrix is $-7/18$. But $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk}) = \operatorname{rk}\operatorname{Pic}(S_{\Bbbk}) + 12$. Thus, we see that ($\bigstar$) holds, so that ($\diamondsuit$) in the Main Theorem also holds by Lemma 1.13.1.

Family 2.18. In this case, the threefold X is a double cover of $\mathbb{P}^1 \times \mathbb{P}^2$ ramified in a divisor of bidegree $(2,2)$. We have $h^{1,2}(X) = 2$. A toric Landau–Ginzburg model of this family is given by the Minkowski polynomial 1922, which is

$$x + y + z + \frac{y}{x} + \frac{z}{x} + \frac{x}{yz} + \frac{1}{z} + \frac{1}{y} + \frac{1}{x} + \frac{2}{yz} + \frac{1}{xz} + \frac{1}{xy} + \frac{1}{xyz}.$$

The quartic pencil $\mathcal{S}$ is given by

$$x^2yz + xy^2z + xyz^2 + y^2zt + yz^2t + x^2t^2 + xyt^2 + xzt^2 + yzt^2 + 2xt^3 + yt^3 + zt^3 + t^4 = \lambda xyzt.$$

Suppose that $\lambda \neq \infty$. Let $\mathcal{C}$ be the conic given by $x = yz + t^2 = 0$. Then

$$
\begin{aligned}
H_{\{x\}} \cdot S_\lambda &= L_{\{x\},\{t\}} + L_{\{x\},\{y,z,t\}} + \mathcal{C},\\
H_{\{y\}} \cdot S_\lambda &= 2L_{\{y\},\{t\}} + L_{\{y\},\{x,t\}} + L_{\{y\},\{x,z,t\}},\\
H_{\{z\}} \cdot S_\lambda &= 2L_{\{z\},\{t\}} + L_{\{z\},\{x,t\}} + L_{\{z\},\{x,y,t\}},\\
H_{\{t\}} \cdot S_\lambda &= L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y,z\}}.
\end{aligned}
\tag{2.18.1}
$$

So, we may assume that $C_1 = L_{\{x\},\{t\}}$, $C_2 = L_{\{y\},\{t\}}$, $C_3 = L_{\{z\},\{t\}}$, $C_4 = L_{\{y\},\{x,t\}}$, $C_5 = L_{\{z\},\{x,t\}}$, $C_6 = L_{\{x\},\{y,z,t\}}$, $C_7 = L_{\{y\},\{x,z,t\}}$, $C_8 = L_{\{z\},\{x,y,t\}}$, $C_9 = L_{\{t\},\{x,y,z\}}$, and $C_{10} = \mathcal{C}$. These are all base curves of the pencil $\mathcal{S}$.

Note that $S_{-2} = H_{\{x,t\}} + H_{\{x,y,z,t\}} + \mathbf{Q}$, where $\mathbf{Q}$ is an irreducible quadric cone in $\mathbb{P}^3$ that is given by $yz + t^2 = 0$. On the other hand, if $\lambda \neq -2$, then S_λ has isolated singularities. Furthermore, if $\lambda \neq -2$, then the singular points of S_λ contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

$P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_1$ with quadratic term $yz + t^2$;

$P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term $z(x+t)$;

$P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term $y(x+t)$;

$P_{\{x\},\{t\},\{y,z\}}$: type $\mathbb{A}_1$ with quadratic term $(x+t)(x+y+z+t) - (\lambda+2)xt$;

$P_{\{y\},\{z\},\{x,t\}}$: type $\mathbb{A}_1$ with quadratic term $(x+t)(x+y+z+t) + (\lambda+2)yz$;

$P_{\{y\},\{t\},\{x,z\}}$: type $\mathbb{A}_2$ with quadratic term $y(x+y+z-t-\lambda t)$;

$P_{\{z\},\{t\},\{x,y\}}$: type $\mathbb{A}_2$ with quadratic term $z(x+y+z-t-\lambda t)$.

Thus, the set Σ consists of the points $P_{\{y\},\{z\},\{t\}}$, $P_{\{x\},\{z\},\{t\}}$, $P_{\{x\},\{y\},\{t\}}$, $P_{\{x\},\{t\},\{y,z\}}$, $P_{\{y\},\{z\},\{x,t\}}$, $P_{\{y\},\{t\},\{x,z\}}$, and $P_{\{z\},\{t\},\{x,y\}}$.

If $\lambda \neq -2$, then the fiber $\mathsf{f}^{-1}(\lambda)$ is irreducible by Corollary 1.5.4. Similarly, it follows from (1.8.3) and Lemmas 1.8.5 and 1.12.1 that $[\mathsf{f}^{-1}(-2)] = [S_{-2}] = 3$, because $\mathbf{M}_1^{-2} = \ldots = \mathbf{M}_{10}^{-2} = 1$, and every point of the set Σ is a good double point of the surface S_{-2}. This confirms ($\heartsuit$) in the Main Theorem, since $h^{1,2}(X) = 2$.

To verify ($\diamondsuit$) in the Main Theorem, observe that

$$H_{\{x,t\}} \cdot S_\lambda = 2L_{\{x\},\{t\}} + L_{\{y\},\{x,t\}} + L_{\{z\},\{x,t\}}$$

for $\lambda \neq -2$. Thus, if $\lambda \neq -2$, then $2L_{\{x\},\{t\}} + L_{\{y\},\{x,t\}} + L_{\{z\},\{x,t\}} \sim H_\lambda$ on the surface S_λ. Similarly, if $\lambda \neq -2$, then

$$2L_{\{x\},\{y,z,t\}} + L_{\{y\},\{x,z,t\}} + L_{\{z\},\{x,y,t\}} + L_{\{t\},\{x,y,z\}} \sim H_\lambda,$$

since $H_{\{x,y,z,t\}} \cdot S_\lambda = 2L_{\{x\},\{y,z,t\}} + L_{\{y\},\{x,z,t\}} + L_{\{z\},\{x,y,t\}} + L_{\{t\},\{x,y,z\}}$. If $\lambda \neq -2$, then

$$\begin{aligned} H_\lambda &\sim L_{\{x\},\{t\}} + L_{\{x\},\{y,z,t\}} + \mathcal{C} \sim 2L_{\{y\},\{t\}} + L_{\{y\},\{x,t\}} + L_{\{y\},\{x,z,t\}} \\ &\sim 2L_{\{z\},\{t\}} + L_{\{z\},\{x,t\}} + L_{\{z\},\{x,y,t\}} \sim L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y,z\}} \end{aligned}$$

on the surface S_λ. This follows from (2.18.1). So, if $\lambda \neq -2$, then the rank of the intersection matrix of the curves $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{y\},\{x,t\}}$, $L_{\{z\},\{x,t\}}$, $L_{\{x\},\{y,z,t\}}$, $L_{\{y\},\{x,z,t\}}$, $L_{\{z\},\{x,y,t\}}$, $L_{\{t\},\{x,y,z\}}$, and $\mathcal{C}$ on the surface S_λ is 5, since the intersection matrix of the curves $L_{\{x\},\{t\}}$, $L_{\{x\},\{y,z,t\}}$, $L_{\{y\},\{x,z,t\}}$, $L_{\{y\},\{x,t\}}$, and H_λ on the surface S_λ is given by

$$\begin{array}{c} \\ L_{\{x\},\{t\}} \\ L_{\{x\},\{y,z,t\}} \\ L_{\{y\},\{x,z,t\}} \\ L_{\{y\},\{x,t\}} \\ H_\lambda \end{array}
\begin{array}{c} \begin{array}{ccccc} L_{\{x\},\{t\}} & L_{\{x\},\{y,z,t\}} & L_{\{y\},\{x,z,t\}} & L_{\{y\},\{x,t\}} & H_\lambda \end{array} \\ \begin{pmatrix} 0 & 1/2 & 0 & 1/2 & 1 \\ 1/2 & -3/2 & 1 & 0 & 1 \\ 0 & 1 & -5/6 & 1/2 & 1 \\ 1/2 & 0 & 1/2 & -1/2 & 1 \\ 1 & 1 & 1 & 1 & 4 \end{pmatrix} \end{array}.$$

On the other hand, we have $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk}) = \operatorname{rk}\operatorname{Pic}(S_{\Bbbk}) + 13$, so that ($\bigstar$) holds in this case. Then ($\diamondsuit$) in the Main Theorem holds by Lemma 1.13.1.

Family 2.19. In this case, the threefold X can be obtained by blowing up a smooth complete intersection of two quadrics in $\mathbb{P}^5$ along a line, so that $h^{1,2}(X) = 2$. A toric Landau–Ginzburg model of this family is given by

$$x + y + z + \frac{x}{y} + \frac{z}{y} + \frac{yz}{x} + \frac{x}{z} + \frac{x}{yz} + \frac{y}{z} + \frac{1}{y} + \frac{y}{x},$$

which is the Minkowski polynomial 1108. Then the pencil $\mathcal{S}$ is given by

$$x^2yz + xyz^2 + x^2zt + xy^2z + xz^2t + y^2z^2 + x^2yt + x^2t^2 + xy^2t + xzt^2 + y^2zt = \lambda xyzt.$$

For simplicity, we assume that $\lambda \neq \infty$. If $\lambda \neq -1$, then the surface S_λ has isolated singularities, so that it is irreducible. On the other hand, we have $S_{-1} = H_{\{x,z\}} + H_{\{x,t\}} + \mathbf{Q}$, where $\mathbf{Q}$ is a smooth quadric in $\mathbb{P}^3$ that is given by $xy + xt + y^2 = 0$.

Let $\mathcal{C}$ be the conic in $\mathbb{P}^3$ that is given by $z = xy + xt + y^2 = 0$. Then

$$\begin{aligned} H_{\{x\}} \cdot S_\lambda &= 2L_{\{x\},\{y\}} + L_{\{x\},\{z\}} + L_{\{x\},\{z,t\}}, \\ H_{\{y\}} \cdot S_\lambda &= L_{\{x\},\{y\}} + L_{\{y\},\{t\}} + L_{\{y\},\{x,z\}} + L_{\{y\},\{z,t\}}, \\ H_{\{z\}} \cdot S_\lambda &= L_{\{x\},\{z\}} + L_{\{z\},\{t\}} + \mathcal{C}, \\ H_{\{t\}} \cdot S_\lambda &= L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y\}} + L_{\{t\},\{x,z\}}. \end{aligned}$$

This shows that the base locus of the pencil $\mathcal{S}$ consists of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{z,t\}}$, $L_{\{y\},\{x,z\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{t\},\{x,y\}}$, $L_{\{t\},\{x,z\}}$, and $\mathcal{C}$. This also gives

$$2L_{\{x\},\{y\}} + L_{\{x\},\{z\}} + L_{\{x\},\{z,t\}} \sim L_{\{x\},\{y\}} + L_{\{y\},\{t\}} + L_{\{y\},\{x,z\}} + L_{\{y\},\{z,t\}} \\ \sim L_{\{x\},\{z\}} + L_{\{z\},\{t\}} + \mathcal{C} \sim L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y\}} + L_{\{t\},\{x,z\}} \sim H_\lambda$$

on the surface S_λ with $\lambda \neq -1$.

If $\lambda \neq -1$, then the singularities of S_λ contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

$P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_2$ with quadratic term $(y+t)(z+t)$;

$P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_2$ with quadratic term $(x+z)(z+t)$;

$P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_1$ with quadratic term $xy + xt + y^2$;

$P_{\{x\},\{y\},\{z\}}$: type $\mathbb{A}_4$ with quadratic term $x(x+z)$;

$P_{\{x\},\{y\},\{z,t\}}$: type $\mathbb{A}_2$ with quadratic term $x(y+\lambda y - z - t)$;

$P_{\{y\},\{t\},\{x,z\}}$: type $\mathbb{A}_1$ with quadratic term $(x+z)(y+t) - (\lambda+1)yt$;

$P_{\{z\},\{t\},\{x,y\}}$: type $\mathbb{A}_1$ with quadratic term $(z+t)(x+y-t) - (\lambda+1)zt$.

These quadratic terms also remain valid for $\lambda = -1$. Thus, using (1.8.3) and applying Lemmas 1.8.5 and 1.12.1, we see that $[\mathsf{f}^{-1}(-1)] = [S_{-1}] = 3$, because S_{-1} is smooth at general points of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{z,t\}}$, $L_{\{y\},\{x,z\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{t\},\{x,y\}}$, $L_{\{t\},\{x,z\}}$, and $\mathcal{C}$. This confirms (♡) in the Main Theorem, since $h^{1,2}(X) = 2$.

To verify (♢) in the Main Theorem, observe that $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk}) = \operatorname{rk}\operatorname{Pic}(S_{\Bbbk}) + 13$. Moreover, if $\lambda \neq -1$, then the rank of the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{z,t\}}$, $L_{\{y\},\{x,z\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{t\},\{x,y\}}$, $L_{\{t\},\{x,z\}}$, and $\mathcal{C}$ on the surface S_λ is the same as the rank of the intersection matrix of the curves $L_{\{x\},\{z,t\}}$, $L_{\{y\},\{t\}}$, $L_{\{y\},\{x,z\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{z\},\{t\}}$, $L_{\{t\},\{x,y\}}$, and H_λ. Thus, using Lemma 1.13.1, we see that (♢) in the Main Theorem holds in this case, because the matrix in the following lemma has rank 5.

Lemma 2.19.1. *Suppose that $\lambda \neq -1$. Then the intersection matrix of the curves $L_{\{x\},\{z,t\}}$, $L_{\{y\},\{t\}}$, $L_{\{y\},\{x,z\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{z\},\{t\}}$, $L_{\{t\},\{x,y\}}$, and H_λ on the surface S_λ is given by*

$$\begin{array}{r} \\ L_{\{x\},\{z,t\}} \\ L_{\{y\},\{t\}} \\ L_{\{y\},\{x,z\}} \\ L_{\{y\},\{z,t\}} \\ L_{\{z\},\{t\}} \\ L_{\{t\},\{x,y\}} \\ H_\lambda \end{array}
\begin{array}{c} \begin{array}{ccccccc} L_{\{x\},\{z,t\}} & L_{\{y\},\{t\}} & L_{\{y\},\{x,z\}} & L_{\{y\},\{z,t\}} & L_{\{z\},\{t\}} & L_{\{t\},\{x,y\}} & H_\lambda \end{array} \\
\begin{pmatrix} -2/3 & 0 & 0 & 1/3 & 2/3 & 0 & 1 \\ 0 & -1/3 & 1/2 & 1/3 & 1/3 & 1/2 & 1 \\ 0 & 1/2 & -7/10 & 1 & 0 & 0 & 1 \\ 1/3 & 1/3 & 1 & -2/3 & 2/3 & 0 & 1 \\ 2/3 & 1/3 & 0 & 2/3 & -1/6 & 1/2 & 1 \\ 0 & 1/2 & 0 & 0 & 1/2 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 4 \end{pmatrix} \end{array}.$$

Proof. The last column and the last row in this matrix are obvious. To find its diagonal entries, we use Proposition A.1.3. For instance, the line $L_{\{x\},\{z,t\}}$ contains two singular points of the surface S_λ. These are the points $P_{\{x\},\{z\},\{t\}}$ and $P_{\{x\},\{y\},\{z,t\}}$. Both of them are singular points of type $\mathbb{A}_2$. Thus, by Proposition A.1.3, we have

$$L^2_{\{x\},\{z\}} = -2 + \frac{2}{3} + \frac{2}{3} = -\frac{2}{3}.$$

Similarly, we obtain the remaining diagonal entries.

To find the remaining entries of the intersection matrix, observe that the line $L_{\{x\},\{z,t\}}$ does not intersect the lines $L_{\{y\},\{t\}}$, $L_{\{y\},\{x,z\}}$, and $L_{\{t\},\{x,y\}}$, so that

$$L_{\{x\},\{z,t\}} \cdot L_{\{y\},\{t\}} = L_{\{x\},\{z,t\}} \cdot L_{\{y\},\{x,z\}} = L_{\{x\},\{z,t\}} \cdot L_{\{t\},\{x,y\}} = 0.$$

Now observe that $L_{\{x\},\{z,t\}} \cap L_{\{y\},\{z,t\}} = P_{\{x\},\{y\},\{z,t\}}$, which is a singular point of S_λ of type $\mathbb{A}_2$. Moreover, the strict transforms of the lines $L_{\{x\},\{z,t\}}$ and $L_{\{y\},\{z,t\}}$ on the minimal resolution of singularities of the surface S_λ at the point $P_{\{x\},\{y\},\{z,t\}}$ intersect different exceptional curves. This implies that $L_{\{x\},\{z,t\}} \cdot L_{\{y\},\{z,t\}} = 1/3$ by Proposition A.1.3. Similarly, we get $L_{\{x\},\{z,t\}} \cdot L_{\{z\},\{t\}} = 2/3$, $L_{\{y\},\{t\}} \cdot L_{\{y\},\{x,z\}} = 1/2$, $L_{\{y\},\{t\}} \cdot L_{\{y\},\{z,t\}} = 1/3$, $L_{\{y\},\{t\}} \cdot L_{\{z\},\{t\}} = 1/3$, $L_{\{y\},\{t\}} \cdot L_{\{t\},\{x,y\}} = 1/2$, and $L_{\{y\},\{z,t\}} \cdot L_{\{z\},\{t\}} = 2/3$.

Observe that the line $L_{\{y\},\{x,z\}}$ does not intersect the lines $L_{\{z\},\{t\}}$ and $L_{\{t\},\{x,y\}}$, and the line $L_{\{y\},\{z,t\}}$ does not intersect the line $L_{\{t\},\{x,y\}}$, so that

$$L_{\{y\},\{x,z\}} \cdot L_{\{z\},\{t\}} = L_{\{y\},\{x,z\}} \cdot L_{\{t\},\{x,y\}} = L_{\{y\},\{z,t\}} \cdot L_{\{t\},\{x,y\}} = 0.$$

Moreover, the intersection $L_{\{y\},\{x,z\}} \cap L_{\{y\},\{z,t\}}$ consists of a smooth point of the surface S_λ. Thus, we have $L_{\{y\},\{x,z\}} \cdot L_{\{y\},\{z,t\}} = 1$.

Finally, observe that $L_{\{z\},\{t\}} \cap L_{\{t\},\{x,y\}} = P_{\{z,t\},\{x,y\}}$, which is a singular point of S_λ of type $\mathbb{A}_1$. Thus, we have $L_{\{z\},\{t\}} \cdot L_{\{t\},\{x,y\}} = 1/2$ by Proposition A.1.3. ☐

Family 2.20. In this case, the threefold X is a blow-up of the threefold V_5 along a twisted cubic (see family 2.14). Thus, we have $h^{1,2}(X) = 0$. A toric Landau–Ginzburg model of this family is given by the Minkowski polynomial 1109, which is

$$\frac{y}{z} + x + y + \frac{1}{z} + \frac{y}{xz} + \frac{y}{x} + \frac{1}{xz} + \frac{xz}{y} + z + \frac{1}{y} + \frac{1}{x}.$$

The pencil $\mathcal{S}$ is given by the equation

$$y^2tx + x^2zy + y^2zx + t^2xy + t^2y^2 + y^2zt + t^3y + x^2z^2 + z^2xy + t^2zx + t^2zy = \lambda xyzt.$$

Suppose that $\lambda \neq \infty$. Then the surface S_λ has isolated singularities. In particular, we see that S_λ is irreducible.

Let $\mathcal{C}$ be a conic in $\mathbb{P}^3$ that is given by $y = xz + t^2 = 0$. Then

$$\begin{aligned}
H_{\{x\}} \cdot S_\lambda &= L_{\{x\},\{y\}} + L_{\{x\},\{t\}} + L_{\{x\},\{y,t\}} + L_{\{x\},\{z,t\}},\\
H_{\{y\}} \cdot S_\lambda &= L_{\{x\},\{y\}} + L_{\{y\},\{z\}} + \mathcal{C},\\
H_{\{z\}} \cdot S_\lambda &= L_{\{y\},\{z\}} + L_{\{z\},\{t\}} + L_{\{z\},\{x,t\}} + L_{\{z\},\{y,t\}},\\
H_{\{t\}} \cdot S_\lambda &= L_{\{x\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y\}} + L_{\{t\},\{y,z\}}.
\end{aligned} \tag{2.20.1}$$

This shows that $L_{\{x\},\{y\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{y,t\}}$, $L_{\{x\},\{z,t\}}$, $L_{\{z\},\{x,t\}}$, $L_{\{z\},\{y,t\}}$, $L_{\{t\},\{x,y\}}$, $L_{\{t\},\{y,z\}}$, and $\mathcal{C}$ are all base curves of the pencil $\mathcal{S}$.

If $\lambda \neq -2, -3$, then the singular points of S_λ contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

$P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_4$ with quadratic term $z(y+z)$;

$P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_2$ with quadratic term $(x+t)(z+t)$;

$P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_4$ with quadratic term $x(x+y)$.

The surface S_{-3} has the same singularities at $P_{\{y\},\{z\},\{t\}}$, $P_{\{x\},\{z\},\{t\}}$, and $P_{\{x\},\{y\},\{t\}}$. In addition to them, it is also singular at the points $[0:1:1:-1]$ and $[1:1:0:-1]$, which are isolated ordinary double points of the surface S_{-3}. Similarly, the singular points of the surface S_{-2} are $P_{\{y\},\{z\},\{t\}}$,

$P_{\{x\},\{z\},\{t\}}$, $P_{\{x\},\{y\},\{t\}}$, and $[1:-1:1:0]$. They are singular points of the surface S_{-2} of types $\mathbb{A}_6$, $\mathbb{A}_2$, $\mathbb{A}_6$, and $\mathbb{A}_1$, respectively.

We see that every surface S_λ has du Val singularities at every base point of the pencil $\mathcal{S}$. Thus, by Corollary 1.5.4, every fiber $\mathsf{f}^{-1}(\lambda)$ is irreducible. This confirms ($\heartsuit$) in the Main Theorem, since $h^{1,2}(X)=0$. To verify ($\diamondsuit$) in the Main Theorem, we need the following.

Lemma 2.20.2. *Suppose that $\lambda\neq -2,-3$. Then the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{y,t\}}$, $L_{\{x\},\{z,t\}}$, $L_{\{y\},\{z\}}$, $L_{\{z\},\{x,t\}}$, $L_{\{z\},\{y,t\}}$, $L_{\{t\},\{y,z\}}$, and H_λ on the surface S_λ is given by*

$$
\begin{array}{c|cccccccc}
 & L_{\{x\},\{y\}} & L_{\{x\},\{y,t\}} & L_{\{x\},\{z,t\}} & L_{\{y\},\{z\}} & L_{\{z\},\{x,t\}} & L_{\{z\},\{y,t\}} & L_{\{t\},\{y,z\}} & H_\lambda \\
\hline
L_{\{x\},\{y\}} & -4/5 & 2/5 & 1 & 1 & 0 & 0 & 0 & 1 \\
L_{\{x\},\{y,t\}} & 2/5 & -6/5 & 1 & 0 & 0 & 1 & 0 & 1 \\
L_{\{x\},\{z,t\}} & 1 & 1 & -4/3 & 0 & 1/3 & 0 & 0 & 1 \\
L_{\{y\},\{z\}} & 1 & 0 & 0 & -4/5 & 1 & 2/5 & 3/5 & 1 \\
L_{\{z\},\{x,t\}} & 0 & 0 & 1/3 & 1 & -4/3 & 1 & 0 & 1 \\
L_{\{z\},\{y,t\}} & 0 & 1 & 0 & 2/5 & 1 & -6/5 & 1/5 & 1 \\
L_{\{t\},\{y,z\}} & 0 & 0 & 0 & 3/5 & 0 & 1/5 & -6/5 & 1 \\
H_\lambda & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 4
\end{array}.
$$

Proof. All diagonal entries here can be found using Proposition A.1.3. For instance, the only singular point of the surface S_λ that is contained in $L_{\{x\},\{y\}}$ is the point $P_{\{x\},\{y\},\{t\}}$, which is a singular point of type $\mathbb{A}_4$ of S_λ. Applying Remark A.2.4 with $S=S_\lambda$, $n=4$, $O=P_{\{x\},\{y\},\{t\}}$, and $C=L_{\{x\},\{y\}}$, we see that $\overline{C}$ contains the point $\overline{G}_1\cap\overline{G}_4$, because the quadratic term of the surface S_λ at the point $P_{\{x\},\{y\},\{t\}}$ is $x(x+y)$. This shows that $\widetilde{C}$ intersects either G_2 or G_3. Then $L_{\{x\},\{y\}}^2=-4/5$ by Proposition A.1.3.

Applying Proposition A.1.2, we can find the remaining entries of the intersection matrix. For instance, observe that

$$L_{\{x\},\{y\}}\cap L_{\{x\},\{t\}}=L_{\{x\},\{y\}}\cap L_{\{x\},\{y,t\}}=P_{\{x\},\{y\},\{t\}}.$$

Thus, it follows from Proposition A.1.2 that $L_{\{x\},\{y\}}\cdot L_{\{x\},\{t\}}$ and $L_{\{x\},\{y\}}\cdot L_{\{x\},\{y,t\}}$ are among $2/5$ and $3/5$. But $L_{\{x\},\{y\}}+L_{\{x\},\{t\}}+L_{\{x\},\{y,t\}}+L_{\{x\},\{z,t\}}\sim H_\lambda$ by (2.20.1), so that

$$
\begin{aligned}
1&=\big(L_{\{x\},\{y\}}+L_{\{x\},\{t\}}+L_{\{x\},\{y,t\}}+L_{\{x\},\{z,t\}}\big)\cdot L_{\{x\},\{y\}}\\
&=L_{\{x\},\{y\}}\cdot L_{\{x\},\{t\}}+L_{\{x\},\{y\}}\cdot L_{\{x\},\{y,t\}}-\frac{1}{5}.
\end{aligned}
$$

Hence, we deduce that $L_{\{x\},\{y\}}\cdot L_{\{x\},\{t\}}=2/5$ and $L_{\{x\},\{y\}}\cdot L_{\{x\},\{y,t\}}=2/5$. Similarly, we can find all remaining entries of the intersection matrix. $\square$

The matrix in Lemma 2.20.2 has rank 8. But $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk})=\operatorname{rk}\operatorname{Pic}(S_{\Bbbk})+10$, so that ($\bigstar$) holds. By Lemma 1.13.1, this shows that ($\diamondsuit$) in the Main Theorem also holds.

Family 2.21. In this case, the threefold X can be obtained from a smooth quadric threefold in $\mathbb{P}^4$ by blowing up a smooth rational curve of degree 4. Then $h^{1,2}(X)=0$. A toric Landau–Ginzburg model of this family is given by

$$\frac{x}{z}+x+\frac{y}{z}+\frac{x}{y}+\frac{1}{z}+y+z+\frac{y}{x}+\frac{z}{y}+\frac{1}{x},$$

which is the Minkowski polynomial 730. Then the pencil $\mathcal{S}$ is given by

$$x^2ty+x^2zy+y^2tx+x^2zt+t^2yx+y^2zx+z^2yx+y^2zt+z^2tx+t^2zy=\lambda xyzt.$$

As usual, we assume that $\lambda \neq \infty$. Then

$$H_{\{x\}} \cdot S_\lambda = L_{\{x\},\{y\}} + L_{\{x\},\{z\}} + L_{\{x\},\{t\}} + L_{\{x\},\{y,t\}},$$
$$H_{\{y\}} \cdot S_\lambda = L_{\{x\},\{y\}} + L_{\{y\},\{z\}} + L_{\{y\},\{t\}} + L_{\{y\},\{x,z\}},$$
$$H_{\{z\}} \cdot S_\lambda = L_{\{x\},\{z\}} + L_{\{y\},\{z\}} + L_{\{z\},\{t\}} + L_{\{z\},\{x,y,t\}},$$
$$H_{\{t\}} \cdot S_\lambda = L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y,z\}}.$$

This shows that $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{y,t\}}$, $L_{\{y\},\{x,z\}}$, $L_{\{z\},\{x,y,t\}}$, and $L_{\{t\},\{x,y,z\}}$ are all base curves of the pencil $\mathcal{S}$.

For every $\lambda \in \mathbb{C}$, the surface S_λ is irreducible, it has isolated singularities, and its singular points contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

$P_{\{x\},\{y\},\{z\}}$: type $\mathbb{A}_3$ with quadratic term $y(x+z)$;

$P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term $x(y+t)$ for $\lambda \neq -1$, and type $\mathbb{A}_5$ for $\lambda = -1$;

$P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_1$;

$P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_1$;

$P_{\{x\},\{z\},\{y,t\}}$: type $\mathbb{A}_1$ for $\lambda \neq -2$, and type $\mathbb{A}_2$ for $\lambda = -2$;

$P_{\{y\},\{t\},\{x,z\}}$: type $\mathbb{A}_1$ for $\lambda \neq -2$, and type $\mathbb{A}_2$ for $\lambda = -2$;

$P_{\{z\},\{t\},\{x,y\}}$: type $\mathbb{A}_1$ for $\lambda \neq -4$, and type $\mathbb{A}_2$ for $\lambda = -4$.

Then $[\mathsf{f}^{-1}(\lambda)] = 1$ for every $\lambda \in \mathbb{C}$ by Corollary 1.5.4. This confirms ($\heartsuit$) in the Main Theorem.

The rank of the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{y,t\}}$, $L_{\{y\},\{x,z\}}$, $L_{\{z\},\{x,y,t\}}$, and $L_{\{t\},\{x,y,z\}}$ on the surface S_λ is the same as the rank of the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{z\},\{t\}}$, $L_{\{z\},\{x,y,t\}}$, $L_{\{t\},\{x,y,z\}}$, and H_λ. If $\lambda \notin \{-1,-2,-4\}$, then the latter matrix is given by

$$\begin{array}{r c}
 & \begin{array}{ccccccc} L_{\{x\},\{y\}} & L_{\{x\},\{t\}} & L_{\{y\},\{z\}} & L_{\{z\},\{t\}} & L_{\{z\},\{x,y,t\}} & L_{\{t\},\{x,y,z\}} & H_\lambda \end{array} \\
\begin{array}{r} L_{\{x\},\{y\}} \\ L_{\{x\},\{t\}} \\ L_{\{y\},\{z\}} \\ L_{\{z\},\{t\}} \\ L_{\{z\},\{x,y,t\}} \\ L_{\{t\},\{x,y,z\}} \\ H_\lambda \end{array} &
\begin{pmatrix}
-1/2 & 3/4 & 3/4 & 0 & 0 & 0 & 1 \\
3/4 & -3/4 & 0 & 1/2 & 0 & 1 & 1 \\
3/4 & 0 & -3/4 & 1/2 & 1 & 0 & 1 \\
0 & 1/2 & 1/2 & -1 & 1/2 & 1/2 & 1 \\
0 & 0 & 1 & 1/2 & -1 & 1/2 & 1 \\
0 & 1 & 0 & 1/2 & 1/2 & -1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 4
\end{pmatrix}.
\end{array}$$

Thus, its determinant is $-45/16 \neq 0$. Moreover, we have $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk}) = \operatorname{rk}\operatorname{Pic}(S_{\Bbbk}) + 11$. Hence, we see that ($\bigstar$) holds, so that ($\diamondsuit$) in the Main Theorem also holds by Lemma 1.13.1.

Family 2.22. The threefold X is a blow-up of the threefold V_5 along a conic (see family 2.14), so that $h^{1,2}(X) = 0$. A toric Landau–Ginzburg model of this family is given by the Minkowski polynomial 413, which is

$$\frac{y}{x} + \frac{1}{x} + y + z + \frac{1}{xz} + \frac{1}{z} + \frac{1}{y} + x + \frac{xz}{y}.$$

The quartic pencil $\mathcal{S}$ is given by

$$y^2zt + t^2zy + y^2zx + z^2yx + t^3y + t^2yx + t^2zx + x^2zy + x^2z^2 = \lambda xyzt.$$

Suppose that $\lambda \neq \infty$. Let $\mathcal{C}_1$ be the conic in $\mathbb{P}^3$ that is given by $x = yz + zt + t^2 = 0$, and let $\mathcal{C}_2$ be the conic in $\mathbb{P}^3$ that is given by $y = xz + t^2 = 0$.

Then

$$
\begin{aligned}
H_{\{x\}}\cdot S_\lambda &= L_{\{x\},\{y\}} + L_{\{x\},\{t\}} + \mathcal{C}_1,\\
H_{\{y\}}\cdot S_\lambda &= L_{\{x\},\{y\}} + L_{\{y\},\{z\}} + \mathcal{C}_2,\\
H_{\{z\}}\cdot S_\lambda &= L_{\{y\},\{z\}} + 2L_{\{z\},\{t\}} + L_{\{z\},\{x,t\}},\\
H_{\{t\}}\cdot S_\lambda &= L_{\{x\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y\}} + L_{\{t\},\{y,z\}}.
\end{aligned}
\tag{2.22.1}
$$

Thus, the base locus of the pencil $\mathcal{S}$ consists of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{z\},\{t\}}$, $L_{\{z\},\{x,t\}}$, $L_{\{t\},\{x,y\}}$, $L_{\{t\},\{y,z\}}$, $\mathcal{C}_1$, and $\mathcal{C}_2$.

For every $\lambda\in\mathbb{C}$, the surface S_λ is irreducible and has isolated singularities. Moreover, the singular points of S_λ contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

- $P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_4$ with quadratic term $x(x+y)$ for $\lambda\neq-2$, and type $\mathbb{A}_5$ for $\lambda=-2$;
- $P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term $z(x+t)$ for $\lambda\neq-2$, and type $\mathbb{A}_5$ for $\lambda=-2$;
- $P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_4$ with quadratic term $z(y+z)$ for $\lambda\neq-1$, and type $\mathbb{A}_5$ for $\lambda=-1$;
- $P_{\{z\},\{t\},\{x,y\}}$: type $\mathbb{A}_1$;
- $[1:-1:1:0]$: smooth point for $\lambda\neq-1$, and type $\mathbb{A}_1$ for $\lambda=-1$.

Then $[\mathsf{f}^{-1}(\lambda)]=1$ for every $\lambda\in\mathbb{C}$ by Corollary 1.5.4. This confirms (♡) in the Main Theorem.

If $\lambda\neq-1,-2$, then the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{z\},\{t\}}$, $L_{\{z\},\{x,t\}}$, $L_{\{t\},\{x,y\}}$, $L_{\{t\},\{y,z\}}$, and H_λ on the surface S_λ is given by

$$
\begin{array}{c}
\\ L_{\{x\},\{y\}}\\ L_{\{z\},\{t\}}\\ L_{\{z\},\{x,t\}}\\ L_{\{t\},\{x,y\}}\\ L_{\{t\},\{y,z\}}\\ H_\lambda
\end{array}
\begin{array}{c}
\begin{array}{cccccc} L_{\{x\},\{y\}} & L_{\{z\},\{t\}} & L_{\{z\},\{x,t\}} & L_{\{t\},\{x,y\}} & L_{\{t\},\{y,z\}} & H_\lambda \end{array}\\
\begin{pmatrix}
-4/5 & 0 & 0 & 3/5 & 0 & 1\\
0 & 1/20 & 1/2 & 1/2 & 1/5 & 1\\
0 & 1/2 & -1 & 0 & 0 & 1\\
3/5 & 1/2 & 0 & -7/10 & 1 & 1\\
0 & 1/5 & 0 & 1 & -6/5 & 1\\
1 & 1 & 1 & 1 & 1 & 4
\end{pmatrix}
\end{array}.
$$

This matrix has rank 6. Hence, using (2.22.1), we see that the rank of the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{z\},\{t\}}$, $L_{\{z\},\{x,t\}}$, $L_{\{t\},\{x,y\}}$, $L_{\{t\},\{y,z\}}$, $\mathcal{C}_1$, and $\mathcal{C}_2$ is also 6. But $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk})=\operatorname{rk}\operatorname{Pic}(S_{\Bbbk})+12$, so we conclude that (★) holds. Then (♢) in the Main Theorem holds by Lemma 1.13.1.

Family 2.23. The threefold X is a blow-up of a smooth quadric threefold in $\mathbb{P}^4$ along a smooth elliptic curve of degree 4. Then $h^{1,2}(X)=1$. A toric Landau–Ginzburg model of this family is given by the Minkowski polynomial 410, which is

$$
x+y+z+\frac{z}{x}+\frac{z}{y}+\frac{x}{z}+\frac{y}{z}+\frac{1}{x}+\frac{1}{y}.
$$

In this case, the pencil $\mathcal{S}$ is given by the equation

$$
xyz^2+x^2yz+xy^2z+xz^2t+yz^2t+x^2yt+xy^2t+xzt^2+yzt^2=\lambda xyzt.
$$

As usual, we assume that $\lambda\neq\infty$. Then

$$
\begin{aligned}
H_{\{x\}}\cdot S_\lambda &= L_{\{x\},\{y\}} + L_{\{x\},\{z\}} + L_{\{x\},\{t\}} + L_{\{x\},\{z,t\}},\\
H_{\{y\}}\cdot S_\lambda &= L_{\{x\},\{y\}} + L_{\{y\},\{z\}} + L_{\{y\},\{t\}} + L_{\{y\},\{z,t\}},\\
H_{\{z\}}\cdot S_\lambda &= L_{\{x\},\{z\}} + L_{\{y\},\{z\}} + L_{\{z\},\{t\}} + L_{\{z\},\{x,y\}},\\
H_{\{t\}}\cdot S_\lambda &= L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y,z\}}.
\end{aligned}
\tag{2.23.1}
$$

This shows that the base locus of the pencil $\mathcal{S}$ is a union of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{z,t\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{z\},\{x,y\}}$, and $L_{\{t\},\{x,y,z\}}$.

Observe that $S_{-1} = H_{\{z,t\}} + \mathbf{S}$, where $\mathbf{S}$ is an irreducible cubic surface in $\mathbb{P}^3$ that is given by $xzt + yzt + x^2y + xy^2 + xyz = 0$. On the other hand, if $\lambda \neq -1$, then S_λ has isolated singularities, so that it is irreducible. Moreover, in this case, the singular points of S_λ contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

$P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term $y(z+t)$;

$P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term $x(z+t)$;

$P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_1$ with quadratic term $xy + xt + yt$;

$P_{\{x\},\{y\},\{z\}}$: type $\mathbb{A}_3$ with quadratic term $z(x+y)$ for $\lambda \neq 0$, and type $\mathbb{A}_5$ for $\lambda = 0$;

$P_{\{z\},\{t\},\{x,y\}}$: type $\mathbb{A}_1$ with quadratic term $(x+y+z)(z+t) - (\lambda+1)zt$;

$P_{\{x\},\{y\},\{z,t\}}$: type $\mathbb{A}_1$ with quadratic term $(x+y)(z+t) - (\lambda+1)xy$.

Thus, it follows from Corollary 1.5.4 that the fiber $\mathsf{f}^{-1}(\lambda)$ is irreducible for every $\lambda \neq -1$. Moreover, the surface S_{-1} has good double points at $P_{\{y\},\{z\},\{t\}}$, $P_{\{x\},\{z\},\{t\}}$, $P_{\{x\},\{y\},\{t\}}$, $P_{\{x\},\{y\},\{z\}}$, $P_{\{z\},\{t\},\{x,y\}}$, and $P_{\{x\},\{y\},\{z,t\}}$. Furthermore, it is smooth at general points of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{z,t\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{z\},\{x,y\}}$, and $L_{\{t\},\{x,y,z\}}$. This gives $[\mathsf{f}^{-1}(-1)] = [S_{-1}] = 2$ by (1.8.3) and Lemmas 1.8.5 and 1.12.1 and confirms ($\heartsuit$) in the Main Theorem, since $h^{1,2}(X) = 1$.

To verify ($\diamondsuit$) in the Main Theorem, we may assume that $\lambda \neq 0, -1$. Then the intersection matrix of the curves $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{x\},\{z,t\}}$, $L_{\{y\},\{t\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{z\},\{x,y\}}$, and H_λ on the surface S_λ is given by

$$
\begin{array}{r c}
 & \begin{array}{ccccccc} L_{\{x\},\{z\}} & L_{\{x\},\{t\}} & L_{\{x\},\{z,t\}} & L_{\{y\},\{t\}} & L_{\{y\},\{z,t\}} & L_{\{z\},\{x,y\}} & H_\lambda \end{array} \\
\begin{array}{r} L_{\{x\},\{z\}} \\ L_{\{x\},\{t\}} \\ L_{\{x\},\{z,t\}} \\ L_{\{y\},\{t\}} \\ L_{\{y\},\{z,t\}} \\ L_{\{z\},\{x,y\}} \\ H_\lambda \end{array} &
\begin{pmatrix}
-1/2 & 3/4 & 1/2 & 0 & 0 & 1/2 & 1 \\
3/4 & -3/4 & 1/2 & 1/2 & 0 & 0 & 1 \\
1/2 & 1/2 & -1/2 & 0 & 1/2 & 0 & 1 \\
0 & 1/2 & 0 & -3/4 & 1/2 & 0 & 1 \\
0 & 0 & 1/2 & 1/2 & -1/2 & 0 & 1 \\
1/2 & 0 & 0 & 0 & 0 & -1/2 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 4
\end{pmatrix}.
\end{array}
$$

The rank of this matrix is 6. Thus, using (2.23.1), we see that the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{z,t\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{z\},\{x,y\}}$, and $L_{\{t\},\{x,y,z\}}$ is also of rank 6. But $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk}) = \operatorname{rk}\operatorname{Pic}(S_{\Bbbk}) + 12$. This shows that ($\bigstar$) holds, so that ($\diamondsuit$) in the Main Theorem also holds by Lemma 1.13.1.

Family 2.24. The threefold X is a smooth divisor in $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(1,2)$, which implies that $h^{1,2}(X) = 0$. A toric Landau–Ginzburg model of this family is given by the Minkowski polynomial 411, which is

$$\frac{xy}{z} + x + y + z + \frac{x}{z} + \frac{y}{x} + \frac{z}{y} + \frac{1}{y} + \frac{1}{x}.$$

Then the pencil $\mathcal{S}$ is given by the equation

$$x^2y^2 + x^2yz + y^2xz + z^2xy + x^2yt + y^2tz + z^2xt + t^2xz + t^2yz = \lambda xyzt.$$

Moreover, the base locus of this pencil consists of the lines $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{x\},\{y,t\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{z\},\{y,t\}}$, $L_{\{t\},\{y,z\}}$, and $L_{\{t\},\{x,z\}}$, because

$$\begin{aligned} H_{\{x\}}\cdot S_\lambda &= L_{\{x\},\{y\}} + L_{\{x\},\{z\}} + L_{\{x\},\{t\}} + L_{\{x\},\{y,t\}},\\ H_{\{y\}}\cdot S_\lambda &= L_{\{x\},\{y\}} + L_{\{y\},\{z\}} + L_{\{y\},\{t\}} + L_{\{y\},\{z,t\}},\\ H_{\{z\}}\cdot S_\lambda &= 2L_{\{x\},\{z\}} + L_{\{y\},\{z\}} + L_{\{z\},\{y,t\}},\\ H_{\{t\}}\cdot S_\lambda &= L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{t\},\{y,z\}} + L_{\{t\},\{x,z\}}. \end{aligned} \tag{2.24.1}$$

Here, as usual, we assume that $\lambda \neq \infty$.

For every $\lambda \in \mathbb{C}$, the surface S_λ has isolated singularities, so that S_λ is irreducible. Its singular points contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

$P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_1$;

$P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term $x(y+t)$ for $\lambda \neq -2$, and type $\mathbb{A}_4$ for $\lambda = -2$;

$P_{\{x\},\{y\},\{z\}}$: type $\mathbb{A}_2$ with quadratic term $z(x+y)$;

$P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term $y(y+z+t)$ for $\lambda \neq -2$, and type $\mathbb{A}_4$ for $\lambda = -2$;

$P_{\{x\},\{z\},\{y,t\}}$: type $\mathbb{A}_2$ with quadratic term $z((\lambda+2)x - y - t)$ for $\lambda \neq -2$, and type $\mathbb{A}_3$ for $\lambda = -2$;

$[1:1:-1:0]$: smooth point for $\lambda \neq -1$, and type $\mathbb{A}_1$ for $\lambda = -1$.

Then $[\mathsf{f}^{-1}(\lambda)] = 1$ for every $\lambda \in \mathbb{C}$ by Corollary 1.5.4. This confirms (♡) in the Main Theorem.

To verify (♢) in the Main Theorem, we may assume that $\lambda \notin \{-1,-2\}$. Then the intersection matrix of the curves $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{x\},\{y,t\}}$, $L_{\{y\},\{t\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{t\},\{x,z\}}$, and H_λ on the surface S_λ is given by

$$\begin{array}{c} \\ L_{\{x\},\{z\}} \\ L_{\{x\},\{t\}} \\ L_{\{x\},\{y,t\}} \\ L_{\{y\},\{t\}} \\ L_{\{y\},\{z,t\}} \\ L_{\{t\},\{x,z\}} \\ H_\lambda \end{array}
\begin{array}{c} \begin{array}{ccccccc} L_{\{x\},\{z\}} & L_{\{x\},\{t\}} & L_{\{x\},\{y,t\}} & L_{\{y\},\{t\}} & L_{\{y\},\{z,t\}} & L_{\{t\},\{x,z\}} & H_\lambda \end{array} \\
\begin{pmatrix} -1/6 & 1/2 & 1/3 & 0 & 0 & 1/2 & 1\\ 1/2 & -3/4 & 1/2 & 1/4 & 0 & 1/2 & 1\\ 1/3 & 1/2 & -1/3 & 1/2 & 0 & 0 & 1\\ 0 & 1/4 & 1/2 & -1/2 & 1/2 & 1 & 1\\ 0 & 0 & 0 & 1/2 & -1 & 0 & 1\\ 1/2 & 1/2 & 0 & 1 & 0 & -3/2 & 1\\ 1 & 1 & 1 & 1 & 1 & 1 & 4 \end{pmatrix} \end{array}.$$

The rank of this intersection matrix is 7. Thus, using (2.24.1), we see that the rank of the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{x\},\{y,t\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{z\},\{y,t\}}$, $L_{\{t\},\{y,z\}}$, and $L_{\{t\},\{x,z\}}$ on the surface S_λ is also 7. On the other hand, we have $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk}) = \operatorname{rk}\operatorname{Pic}(S_{\Bbbk}) + 11$. Hence, we see that (★) holds. By Lemma 1.13.1, we see that (♢) in the Main Theorem also holds.

Family 2.25. In this case, the threefold X is a blow-up of $\mathbb{P}^3$ along a smooth elliptic curve that is an intersection of two quadrics. This shows that $h^{1,2}(X) = 1$. A toric Landau–Ginzburg model of this family is given by the Minkowski polynomial 198, which is

$$x + y + z + \frac{yz}{x} + \frac{x}{z} + \frac{1}{y} + \frac{1}{x} + \frac{1}{yz}.$$

Thus, the pencil of quartic surfaces $\mathcal{S}$ is given by the equation

$$x^2yz + y^2xz + xyz^2 + y^2z^2 + x^2yt + zxt^2 + yzt^2 + xt^3 = \lambda xyzt.$$

As usual, we assume that $\lambda \neq \infty$.

Let $\mathcal{C}_1$ be the conic in $\mathbb{P}^3$ that is given by $x = yz + t^2 = 0$, and let $\mathcal{C}_2$ be the conic in $\mathbb{P}^3$ that is given by $z = xy + t^2 = 0$. Then

$$\begin{aligned} H_{\{x\}} \cdot S_\lambda &= L_{\{x\},\{y\}} + L_{\{x\},\{z\}} + \mathcal{C}_1, \\ H_{\{y\}} \cdot S_\lambda &= L_{\{x\},\{y\}} + 2L_{\{y\},\{t\}} + L_{\{y\},\{z,t\}}, \\ H_{\{z\}} \cdot S_\lambda &= L_{\{x\},\{z\}} + L_{\{z\},\{t\}} + \mathcal{C}_2, \\ H_{\{t\}} \cdot S_\lambda &= L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y\}} + L_{\{t\},\{x,z\}}. \end{aligned} \tag{2.25.1}$$

This shows that the base locus of the pencil $\mathcal{S}$ consists of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{t\},\{x,y\}}$, $L_{\{t\},\{x,z\}}$, $\mathcal{C}_1$, and $\mathcal{C}_2$.

To describe the singularities of the surfaces in the pencil $\mathcal{S}$, observe that

$$S_{-1} = \mathsf{Q} + \mathbf{Q},$$

where Q is an irreducible quadric surface given by $yz + xt + xz = 0$ and $\mathbf{Q}$ is an irreducible quadric surface given by $yz + xy + t^2 = 0$. Thus, the singularities of the surface S_{-1} are not isolated. On the other hand, if $\lambda \neq -1$, then the surface S_λ has isolated singularities, so that it is irreducible. Moreover, in this case, the singular points of S_λ contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

$P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term $y(z + t)$;

$P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_5$ with quadratic term $z(x + z)$;

$P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_4$ with quadratic term $y(x + y)$;

$P_{\{y\},\{t\},\{x,z\}}$: type $\mathbb{A}_1$ with quadratic term $xy + yz - (\lambda + 1)yt + t^2$.

By Corollary 1.5.4, we have $[\mathsf{f}^{-1}(\lambda)] = 1$ for every $\lambda \neq -1$. Moreover, the points $P_{\{y\},\{z\},\{t\}}$, $P_{\{x\},\{z\},\{t\}}$, $P_{\{x\},\{y\},\{t\}}$, and $P_{\{y\},\{t\},\{x,z\}}$ are good double points of the surface S_{-1}. Furthermore, the surface S_{-1} is smooth at general points of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{t\},\{x,y\}}$, $L_{\{t\},\{x,z\}}$, $\mathcal{C}_1$, and $\mathcal{C}_2$. Thus, it follows from (1.8.3) and Lemmas 1.8.5 and 1.12.1 that $[\mathsf{f}^{-1}(-1)] = [S_{-1}] = 2$. This confirms ($\heartsuit$) in the Main Theorem, since $h^{1,2}(X) = 1$.

To verify ($\diamondsuit$) in the Main Theorem, we may assume that $\lambda \neq -1$. Then, using (2.25.1), we see that the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{t\},\{x,y\}}$, $L_{\{t\},\{x,z\}}$, $\mathcal{C}_1$, and $\mathcal{C}_2$ on the surface S_λ has the same rank as the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{t\},\{x,y\}}$, $L_{\{t\},\{x,z\}}$, and H_λ, which is given by

$$\begin{array}{c} \\ L_{\{x\},\{y\}} \\ L_{\{x\},\{z\}} \\ L_{\{y\},\{z,t\}} \\ L_{\{t\},\{x,y\}} \\ L_{\{t\},\{x,z\}} \\ H_\lambda \end{array}
\begin{array}{c} \begin{array}{cccccc} L_{\{x\},\{y\}} & L_{\{x\},\{z\}} & L_{\{y\},\{z,t\}} & L_{\{t\},\{x,y\}} & L_{\{t\},\{x,z\}} & H_\lambda \end{array} \\ \begin{pmatrix} -4/5 & 1 & 1 & 3/5 & 0 & 1 \\ 1 & -2/3 & 0 & 0 & 1/3 & 1 \\ 1 & 0 & -1 & 0 & 0 & 1 \\ 3/5 & 0 & 0 & -6/5 & 1 & 1 \\ 0 & 1/3 & 0 & 1 & -2/3 & 1 \\ 1 & 1 & 1 & 1 & 1 & 4 \end{pmatrix} \end{array}.$$

The rank of this matrix is 5. On the other hand, using the description of the singular points of the surface S_λ, we conclude that $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk}) = \operatorname{rk}\operatorname{Pic}(S_{\Bbbk}) + 13$. Hence, we see that ($\bigstar$) holds. By Lemma 1.13.1, we see that ($\diamondsuit$) in the Main Theorem also holds.

Family 2.26. In this case, the threefold X is a blow-up of the threefold V_5 along a line (see family 2.14). Then $h^{1,2}(X) = 0$. A toric Landau–Ginzburg model of this family is given by

$$\frac{y}{x} + \frac{1}{x} + y + z + \frac{1}{z} + \frac{1}{y} + x + \frac{x}{yz},$$

which is the Minkowski polynomial 201. The quartic pencil $\mathcal{S}$ is given by

$$y^2zt+t^2yz+y^2xz+z^2xy+t^2xy+t^2xz+x^2yz+x^2t^2=\lambda xyzt.$$

Suppose that $\lambda\neq\infty$. Then

$$\begin{aligned}H_{\{x\}}\cdot S_\lambda&=L_{\{x\},\{y\}}+L_{\{x\},\{z\}}+L_{\{x\},\{t\}}+L_{\{x\},\{y,t\}},\\ H_{\{y\}}\cdot S_\lambda&=L_{\{x\},\{y\}}+2L_{\{y\},\{t\}}+L_{\{y\},\{x,z\}},\\ H_{\{z\}}\cdot S_\lambda&=L_{\{x\},\{z\}}+2L_{\{z\},\{t\}}+L_{\{z\},\{x,y\}},\\ H_{\{t\}}\cdot S_\lambda&=L_{\{x\},\{t\}}+L_{\{y\},\{t\}}+L_{\{z\},\{t\}}+L_{\{t\},\{x,y,z\}}.\end{aligned}\tag{2.26.1}$$

Thus, the base locus of the pencil $\mathcal{S}$ consists of the lines $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{x\},\{y,t\}}$, $L_{\{y\},\{t\}}$, $L_{\{y\},\{x,z\}}$, $L_{\{z\},\{t\}}$, $L_{\{z\},\{x,y\}}$, and $L_{\{t\},\{x,y,z\}}$.

For every $\lambda\in\mathbb{C}$, the surface S_λ is irreducible, it has isolated singularities, and its singular points contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

$P_{\{x\},\{y\},\{z\}}$: type $\mathbb{A}_2$ with quadratic term $(x+y)(x+z)$ for $\lambda\neq-1$, and type $\mathbb{A}_3$ for $\lambda=-1$;

$P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term xy;

$P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_2$ with quadratic term $z(x+t)$;

$P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_1$;

$P_{\{y\},\{t\},\{x,z\}}$: type $\mathbb{A}_2$ with quadratic term $y(x+y+z-\lambda t)$ for $\lambda\neq0$, and type $\mathbb{A}_3$ for $\lambda=0$;

$P_{\{z\},\{t\},\{x,y\}}$: type $\mathbb{A}_2$ with quadratic term $z(x+y+z-t-\lambda t)$.

Then $[\mathsf{f}^{-1}(\lambda)]=1$ for every $\lambda\in\mathbb{C}$ by Corollary 1.5.4. This confirms ($\heartsuit$) in the Main Theorem.

By Lemma 1.13.1, to verify ($\diamondsuit$) in the Main Theorem, we have to prove ($\bigstar$). Observe that the intersection matrix of the lines $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{x\},\{y,t\}}$, $L_{\{y\},\{t\}}$, $L_{\{y\},\{x,z\}}$, $L_{\{z\},\{t\}}$, $L_{\{z\},\{x,y\}}$, and $L_{\{t\},\{x,y,z\}}$ on the surface S_λ has the same rank as the intersection matrix of the curves $L_{\{x\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{y,t\}}$, $L_{\{y\},\{x,z\}}$, $L_{\{z\},\{x,y\}}$, $L_{\{t\},\{x,y,z\}}$, and H_λ, since

$$\begin{aligned}L_{\{x\},\{y\}}+L_{\{x\},\{z\}}+L_{\{x\},\{t\}}+L_{\{x\},\{y,t\}}&\sim L_{\{x\},\{y\}}+2L_{\{y\},\{t\}}+L_{\{y\},\{x,z\}}\\ \sim L_{\{x\},\{z\}}+2L_{\{z\},\{t\}}+L_{\{z\},\{x,y\}}&\sim L_{\{x\},\{t\}}+L_{\{y\},\{t\}}+L_{\{z\},\{t\}}+L_{\{t\},\{x,y,z\}}\sim H_\lambda,\end{aligned}$$

which follows from (2.26.1). On the other hand, if $\lambda\neq0,-1$, then the intersection matrix of the curves $L_{\{x\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{y,t\}}$, $L_{\{y\},\{x,z\}}$, $L_{\{z\},\{x,y\}}$, $L_{\{t\},\{x,y,z\}}$, and H_λ on the surface S_λ is given by

$$\begin{array}{c|c} & \begin{array}{ccccccc} L_{\{x\},\{t\}} & L_{\{z\},\{t\}} & L_{\{x\},\{y,t\}} & L_{\{y\},\{x,z\}} & L_{\{z\},\{x,y\}} & L_{\{t\},\{x,y,z\}} & H_\lambda\end{array}\\ \hline \begin{array}{c} L_{\{x\},\{t\}}\\ L_{\{z\},\{t\}}\\ L_{\{x\},\{y,t\}}\\ L_{\{y\},\{x,z\}}\\ L_{\{z\},\{x,y\}}\\ L_{\{t\},\{x,y,z\}}\\ H_\lambda\end{array} & \begin{pmatrix} -7/12 & 1/3 & 3/4 & 0 & 0 & 1 & 1\\ 1/3 & -1/6 & 0 & 0 & 2/3 & 1/3 & 1\\ 3/4 & 0 & -5/4 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & -2/3 & 1/3 & 1/3 & 1\\ 0 & 2/3 & 0 & 1/3 & -2/3 & 1/3 & 1\\ 1 & 1/3 & 0 & 1/3 & 1/3 & -2/3 & 1\\ 1 & 1 & 1 & 1 & 1 & 1 & 4\end{pmatrix}\end{array}.$$

The rank of this matrix is 6. But $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk})=\operatorname{rk}\operatorname{Pic}(S_{\Bbbk})+12$. We conclude that ($\bigstar$) holds, so that ($\diamondsuit$) in the Main Theorem holds by Lemma 1.13.1.

Family 2.27. In this case, the threefold X is a blow-up of $\mathbb{P}^3$ in a twisted cubic, so that $h^{1,2}(X)=0$. A toric Landau–Ginzburg model of this family is given by the Minkowski polynomial 70, which is

$$x+y+z+\frac{x}{z}+\frac{1}{x}+\frac{1}{yz}+\frac{1}{xy}.$$

The quartic pencil $\mathcal{S}$ is given by the equation

$$x^2zy+y^2zx+z^2xy+x^2ty+t^2zy+t^3x+t^3z=\lambda xyzt.$$

Suppose that $\lambda\neq\infty$. Let $\mathcal{C}$ be the conic in $\mathbb{P}^3$ that is given by $z=xy+t^2=0$. Then

$$\begin{aligned} H_{\{x\}}\cdot S_\lambda &= L_{\{x\},\{z\}}+2L_{\{x\},\{t\}}+L_{\{x\},\{y,t\}},\\ H_{\{y\}}\cdot S_\lambda &= 3L_{\{y\},\{t\}}+L_{\{y\},\{x,z\}},\\ H_{\{z\}}\cdot S_\lambda &= L_{\{x\},\{z\}}+L_{\{z\},\{t\}}+\mathcal{C},\\ H_{\{t\}}\cdot S_\lambda &= L_{\{x\},\{t\}}+L_{\{y\},\{t\}}+L_{\{z\},\{t\}}+L_{\{t\},\{x,y,z\}}. \end{aligned} \tag{2.27.1}$$

Thus, the base locus of the pencil $\mathcal{S}$ consists of the curves $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{y,t\}}$, $L_{\{y\},\{x,z\}}$, $L_{\{t\},\{x,y,z\}}$, and $\mathcal{C}$.

For every $\lambda\in\mathbb{C}$, the surface S_λ is irreducible, it has isolated singularities, and its singular points contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

- $P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_2$ with quadratic term xy;
- $P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_5$ with quadratic term xz for $\lambda\neq-1$, and type $\mathbb{A}_6$ for $\lambda=-1$;
- $P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_2$ with quadratic term $y(z+t)$;
- $P_{\{x\},\{t\},\{y,z\}}$: type $\mathbb{A}_1$;
- $P_{\{y\},\{t\},\{x,z\}}$: type $\mathbb{A}_3$ with quadratic term $y(x+y+z-t-\lambda t)$ for $\lambda\neq-1$, and type $\mathbb{A}_4$ for $\lambda=-1$.

By Corollary 1.5.4, each fiber $\mathsf{f}^{-1}(\lambda)$ is irreducible. This confirms ($\heartsuit$) in the Main Theorem.

Lemma 2.27.2. *Suppose that $\lambda\neq-1$. Then the intersection matrix of the curves $L_{\{x\},\{t\}}$, $L_{\{x\},\{y,t\}}$, $L_{\{y\},\{x,z\}}$, $L_{\{z\},\{t\}}$, and H_λ on the surface S_λ is given by*

$$\begin{array}{c} \\ L_{\{x\},\{t\}}\\ L_{\{x\},\{y,t\}}\\ L_{\{y\},\{x,z\}}\\ L_{\{z\},\{t\}}\\ H_\lambda \end{array}\begin{array}{c} \begin{array}{ccccc} L_{\{x\},\{t\}} & L_{\{x\},\{y,t\}} & L_{\{y\},\{x,z\}} & L_{\{z\},\{t\}} & H_\lambda \end{array}\\ \begin{pmatrix} 0 & 2/3 & 0 & 1/6 & 1\\ 2/3 & -4/3 & 0 & 0 & 1\\ 0 & 0 & -5/4 & 0 & 1\\ 1/6 & 0 & 0 & -1/2 & 1\\ 1 & 1 & 1 & 1 & 4 \end{pmatrix} \end{array}.$$

Proof. The entries of the last row and the last column in the intersection matrix are obvious. To find its diagonal entries, we use Proposition A.1.3. For instance, to compute $L^2_{\{x\},\{t\}}$, observe that the only singular points of S_λ contained in the line $L_{\{x\},\{t\}}$ are the points $P_{\{x\},\{z\},\{t\}}$, $P_{\{x\},\{y\},\{t\}}$, and $P_{\{x\},\{t\},\{y,z\}}$. Using Remark A.2.4 with $S=S_\lambda$, $n=5$, $O=P_{\{x\},\{z\},\{t\}}$, and $C=L_{\{x\},\{t\}}$, we see that $\overline{C}$ does not contain the point $\overline{G}_1\cap\overline{G}_5$. Thus, it follows from Proposition A.1.3 that $L^2_{\{x\},\{t\}}=0$. Similarly, we see that $L^2_{\{x\},\{y,t\}}=-4/3$, $L^2_{\{y\},\{x,z\}}=-5/4$, and $L^2_{\{z\},\{t\}}=-1/2$.

Note that $L_{\{x\},\{y,t\}}\cap L_{\{y\},\{x,z\}}=L_{\{x\},\{y,t\}}\cap L_{\{z\},\{t\}}=L_{\{y\},\{x,z\}}\cap L_{\{z\},\{t\}}=\varnothing$, so that

$$L_{\{x\},\{y,t\}}\cdot L_{\{y\},\{x,z\}}=L_{\{x\},\{y,t\}}\cdot L_{\{z\},\{t\}}=L_{\{y\},\{x,z\}}\cdot L_{\{z\},\{t\}}=0.$$

Similarly, we see that $L_{\{x\},\{t\}}\cdot L_{\{y\},\{x,z\}}=0$.

To find the remaining entries of the intersection matrix, we use Proposition A.1.2. To begin with, let us compute $L_{\{x\},\{t\}}\cdot L_{\{z\},\{t\}}$. Observe that $L_{\{x\},\{t\}}\cap L_{\{z\},\{t\}}=P_{\{x\},\{z\},\{t\}}$. Using Remark A.2.4 with $S=S_\lambda$, $n=5$, $O=P_{\{x\},\{z\},\{t\}}$, $C=L_{\{x\},\{t\}}$, and $Z=L_{\{z\},\{t\}}$, we see that none of the curves $\overline{C}$ and $\overline{Z}$ contains the point $\overline{G}_1\cap\overline{G}_5$. Moreover, since the quadratic term of the surface S_λ at the singular point $P_{\{x\},\{z\},\{t\}}$ is xz, we see that either $\overline{C}\cdot\overline{G}_1=\overline{Z}\cdot\overline{G}_5=1$ or $\overline{C}\cdot\overline{G}_5=\overline{Z}\cdot\overline{G}_1=1$. Thus, using Proposition A.1.2, we conclude that $L_{\{x\},\{t\}}\cdot L_{\{z\},\{t\}}=1/6$.

Finally, let us compute $L_{\{x\},\{t\}}\cdot L_{\{x\},\{y,t\}}$. Observe that $L_{\{x\},\{t\}}\cap L_{\{x\},\{y,t\}}=P_{\{x\},\{y\},\{t\}}$ and $P_{\{x\},\{y\},\{t\}}$ is a singular point of S_λ of type $\mathbb{A}_2$. Let us use the notation of Subsection A.2 in the Appendix with $S=S_\lambda$, $n=2$, $O=P_{\{x\},\{y\},\{t\}}$, $C=L_{\{x\},\{t\}}$, and $Z=L_{\{x\},\{y,t\}}$. Then π is the blow-up of the point O, and either both curves $\widetilde{C}$ and $\widetilde{Z}$ intersect G_1, or they intersect the curve G_2. Thus, we have $L_{\{x\},\{t\}}\cdot L_{\{x\},\{y,t\}}=2/3$ by Proposition A.1.2. $\square$

Using (2.27.1), we see that the intersection matrix of the curves $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{y,t\}}$, $L_{\{y\},\{x,z\}}$, $L_{\{t\},\{x,y,z\}}$, and $\mathcal{C}$ on the surface S_λ has the same rank as the intersection matrix of the curves $L_{\{x\},\{t\}}$, $L_{\{x\},\{y,t\}}$, $L_{\{y\},\{x,z\}}$, $L_{\{z\},\{t\}}$, and H_λ. But the matrix in Lemma 2.27.2 has rank 5. Thus, since $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk})=\operatorname{rk}\operatorname{Pic}(S_{\Bbbk})+12$, we conclude that ($\bigstar$) holds. Hence, it follows from Lemma 1.13.1 that ($\diamondsuit$) in the Main Theorem holds.

Family 2.28. In this case, the threefold X is a blow-up of $\mathbb{P}^3$ in a smooth plane cubic curve, which implies that $h^{1,2}(X)=1$. A toric Landau–Ginzburg model of this family is given by the Minkowski polynomial 68, which is

$$x+\frac{x}{z}+\frac{x}{yz}+\frac{y}{z}+z+\frac{1}{y}+\frac{y}{x}.$$

The quartic pencil $\mathcal{S}$ is given by

$$x^2yz+x^2yt+x^2t^2+xy^2t+xyz^2+xzt^2+y^2zt=\lambda xyzt.$$

Suppose that $\lambda\neq\infty$. Let $\mathcal{C}$ be the conic given by $z=xy+xt+y^2=0$. Then

$$\begin{aligned}
&H_{\{x\}}\cdot S_\lambda=2L_{\{x\},\{y\}}+L_{\{x\},\{z\}}+L_{\{x\},\{t\}},\\
&H_{\{y\}}\cdot S_\lambda=L_{\{x\},\{y\}}+2L_{\{y\},\{t\}}+L_{\{y\},\{x,z\}},\\
&H_{\{z\}}\cdot S_\lambda=L_{\{x\},\{z\}}+L_{\{z\},\{t\}}+\mathcal{C},\\
&H_{\{t\}}\cdot S_\lambda=L_{\{x\},\{t\}}+L_{\{y\},\{t\}}+L_{\{z\},\{t\}}+L_{\{t\},\{x,z\}}.
\end{aligned}\tag{2.28.1}$$

Thus, the base locus of the pencil $\mathcal{S}$ consists of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{y\},\{x,z\}}$, $L_{\{t\},\{x,z\}}$, and $\mathcal{C}$.

If $\lambda\neq-1$, then S_λ is irreducible, it has isolated singularities, and its singular points contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

- $P_{\{x\},\{y\},\{z\}}$: type $\mathbb{A}_4$ with quadratic term $x(x+z)$;
- $P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_4$ with quadratic term xy;
- $P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term $t(x+z)$;
- $P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_1$ with quadratic term $yz+yt+t^2$;
- $P_{\{y\},\{t\},\{x,z\}}$: type $\mathbb{A}_2$ with quadratic term $y(x+z-t-\lambda t)$.

Thus, if $\lambda\neq-1$, then the fiber $\mathsf{f}^{-1}(\lambda)$ is irreducible by Corollary 1.5.4. On the other hand, we have $S_{-1}=H_{\{x,z\}}+\mathbf{S}$, where $\mathbf{S}$ is an irreducible cubic surface in $\mathbb{P}^3$ given by $xyz+xyt+xt^2+y^2t=0$. Nevertheless, the points $P_{\{y\},\{z\},\{t\}}$, $P_{\{x\},\{z\},\{t\}}$, $P_{\{x\},\{y\},\{t\}}$, $P_{\{x\},\{y\},\{z\}}$, and $P_{\{y\},\{t\},\{x,z\}}$ are good double points of the surface S_{-1}. Moreover, the surface S_{-1} is smooth at general points of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{y\},\{x,z\}}$, $L_{\{t\},\{x,z\}}$, and $\mathcal{C}$. Thus, using (1.8.3) and Lemmas 1.8.5 and 1.12.1, we conclude that $[\mathsf{f}^{-1}(-1)] = [S_{-1}] = 2$. This confirms ($\heartsuit$) in the Main Theorem.

Now let us verify ($\diamondsuit$) in the Main Theorem. By Lemma 1.13.1, it suffices to show that equality ($\bigstar$) holds. If $\lambda \neq -1$, then it follows from (2.28.1) that the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{y\},\{x,z\}}$, $L_{\{t\},\{x,z\}}$, and $\mathcal{C}$ on the surface S_λ has the same rank as the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, and H_λ. If $\lambda \neq -1$, then the latter matrix is given by

$$\begin{array}{c} \\ L_{\{x\},\{y\}} \\ L_{\{x\},\{t\}} \\ L_{\{y\},\{t\}} \\ L_{\{z\},\{t\}} \\ H_\lambda \end{array}
\begin{array}{c}
\begin{array}{ccccc} L_{\{x\},\{y\}} & L_{\{x\},\{t\}} & L_{\{y\},\{t\}} & L_{\{z\},\{t\}} & H_\lambda \end{array} \\
\begin{pmatrix}
0 & 3/5 & 2/5 & 0 & 1 \\
3/5 & -9/20 & 1/5 & 3/4 & 1 \\
2/5 & 1/5 & -1/30 & 1/2 & 1 \\
0 & 3/4 & 1/2 & -3/4 & 1 \\
1 & 1 & 1 & 1 & 4
\end{pmatrix}
\end{array}.$$

Its rank is 4. On the other hand, it follows from the description of the singular points of the surface S_λ that $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk}) = \operatorname{rk}\operatorname{Pic}(S_{\Bbbk}) + 14$. Thus, we can conclude that ($\bigstar$) holds, so that ($\diamondsuit$) in the Main Theorem also holds.

Family 2.29. In this case, the threefold X is a blow-up of a smooth quadric threefold in $\mathbb{P}^4$ along a conic. This implies that $h^{1,2}(X) = 0$. A toric Landau–Ginzburg model of this family is given by the Minkowski polynomial 71, which is

$$x + y + z + \frac{x}{z} + \frac{y}{z} + \frac{1}{y} + \frac{1}{x}.$$

The quartic pencil $\mathcal{S}$ is given by

$$x^2zy + y^2zx + z^2yx + x^2ty + y^2tx + t^2zx + t^2zy = \lambda xyzt.$$

Its base locus consists of the lines $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{z\},\{x,y\}}$, and $L_{\{t\},\{x,y,z\}}$, because

$$\begin{aligned}
H_{\{x\}} \cdot S_\lambda &= L_{\{x\},\{y\}} + L_{\{x\},\{z\}} + 2L_{\{x\},\{t\}},\\
H_{\{y\}} \cdot S_\lambda &= L_{\{x\},\{y\}} + L_{\{y\},\{z\}} + 2L_{\{y\},\{t\}},\\
H_{\{z\}} \cdot S_\lambda &= L_{\{x\},\{z\}} + L_{\{y\},\{z\}} + L_{\{z\},\{t\}} + L_{\{z\},\{x,y\}},\\
H_{\{t\}} \cdot S_\lambda &= L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y,z\}}.
\end{aligned}$$

Here, as usual, we assume that $\lambda \neq \infty$.

For every $\lambda \in \mathbb{C}$, the surface S_λ has isolated singularities, so that it is irreducible. Its singular points contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

- $P_{\{x\},\{y\},\{z\}}$: type $\mathbb{A}_3$ with quadratic term $z(x+y)$ for $\lambda \neq 0$, and type $\mathbb{A}_5$ for $\lambda = 0$;
- $P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term xy;
- $P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_2$ with quadratic term $x(z+t)$;
- $P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_2$ with quadratic term $y(z+t)$;
- $P_{\{x\},\{t\},\{y,z\}}$: type $\mathbb{A}_1$;
- $P_{\{y\},\{t\},\{x,z\}}$: type $\mathbb{A}_1$;
- $P_{\{z\},\{t\},\{x,y\}}$: type $\mathbb{A}_1$ for $\lambda \neq -1$, and type $\mathbb{A}_3$ for $\lambda = -1$.

By Corollary 1.5.4, every fiber $\mathsf{f}^{-1}(\lambda)$ is irreducible. This confirms ($\heartsuit$) in the Main Theorem.

If $\lambda \neq 0, -1$, then the intersection matrix of the lines $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{z\},\{x,y\}}$, and $L_{\{t\},\{x,y,z\}}$ on the surface S_λ has the same rank as the following intersection matrix:

$$\begin{array}{c} \\ L_{\{x\},\{y\}} \\ L_{\{x\},\{z\}} \\ L_{\{y\},\{z\}} \\ L_{\{z\},\{t\}} \\ H_\lambda \end{array} \begin{array}{c} \begin{array}{ccccc} L_{\{x\},\{y\}} & L_{\{x\},\{z\}} & L_{\{y\},\{z\}} & L_{\{z\},\{t\}} & H_\lambda \end{array} \\ \begin{pmatrix} -1/4 & 1/4 & 1/4 & 0 & 1 \\ 1/4 & -7/12 & 3/4 & 1/3 & 1 \\ 1/4 & 3/4 & -7/12 & 1/3 & 1 \\ 0 & 1/3 & 1/3 & -1/6 & 1 \\ 1 & 1 & 1 & 1 & 4 \end{pmatrix} \end{array}.$$

This matrix has rank 5. On the other hand, we have $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk}) = \operatorname{rk}\operatorname{Pic}(S_{\Bbbk}) + 13$. Thus, we conclude that ($\bigstar$) holds, so that ($\diamondsuit$) in the Main Theorem also holds.

Family 2.30. In this case, the threefold X is a blow-up of $\mathbb{P}^3$ along a conic, so that $h^{1,2}(X) = 0$. A toric Landau–Ginzburg model of this family is given by the Minkowski polynomial 22, which is

$$\frac{x}{yz} + x + \frac{1}{y} + \frac{1}{z} + \frac{z}{x} + \frac{y}{x}.$$

The pencil $\mathcal{S}$ is given by the equation

$$x^2t^2 + x^2yz + t^2zx + t^2yx + z^2yt + y^2zt = \lambda xyzt.$$

Suppose for simplicity that $\lambda \neq \infty$. Then

$$\begin{aligned} H_{\{x\}} \cdot S_\lambda &= L_{\{x\},\{y\}} + L_{\{x\},\{z\}} + L_{\{x\},\{t\}} + L_{\{x\},\{y,z\}}, \\ H_{\{y\}} \cdot S_\lambda &= L_{\{x\},\{y\}} + 2L_{\{y\},\{t\}} + L_{\{y\},\{x,z\}}, \\ H_{\{z\}} \cdot S_\lambda &= L_{\{x\},\{z\}} + 2L_{\{z\},\{t\}} + L_{\{z\},\{x,y\}}, \\ H_{\{t\}} \cdot S_\lambda &= 2L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}}. \end{aligned} \tag{2.30.1}$$

Thus, the base locus of the pencil $\mathcal{S}$ is a union of the lines $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{y,z\}}$, $L_{\{y\},\{x,z\}}$, and $L_{\{z\},\{x,y\}}$.

For every $\lambda \in \mathbb{C}$, the surface S_λ has isolated singularities, so that it is irreducible. Its singular points contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

- $P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_1$;
- $P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_4$ with quadratic term zt;
- $P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_4$ with quadratic term yt;
- $P_{\{x\},\{y\},\{z\}}$: type $\mathbb{A}_3$ with quadratic term $x(x+y+z)$ for $\lambda \neq -1$, and type $\mathbb{A}_5$ for $\lambda = -1$;
- $P_{\{x\},\{t\},\{y,z\}}$: type $\mathbb{A}_1$.

Then $[\mathsf{f}^{-1}(\lambda)] = 1$ for every $\lambda \in \mathbb{C}$ by Corollary 1.5.4. This confirms ($\heartsuit$) in the Main Theorem.

Let us verify ($\diamondsuit$) in the Main Theorem. If $\lambda \neq -1$, then the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{y\},\{x,z\}}$, $L_{\{z\},\{x,y\}}$, and H_λ on the surface S_λ is given by

$$\begin{array}{c} \\ L_{\{x\},\{y\}} \\ L_{\{x\},\{z\}} \\ L_{\{y\},\{x,z\}} \\ L_{\{z\},\{x,y\}} \\ H_\lambda \end{array} \begin{array}{c} \begin{array}{ccccc} L_{\{x\},\{y\}} & L_{\{x\},\{z\}} & L_{\{y\},\{x,z\}} & L_{\{z\},\{x,y\}} & H_\lambda \end{array} \\ \begin{pmatrix} -9/20 & 3/4 & 1/4 & 1/4 & 1 \\ 3/4 & -9/20 & 1/4 & 1/4 & 1 \\ 1/4 & 1/4 & -5/4 & 3/4 & 1 \\ 1/4 & 1/4 & 3/4 & -5/4 & 1 \\ 1 & 1 & 1 & 1 & 4 \end{pmatrix} \end{array}.$$

Observe that this matrix has rank 5. Thus, if $\lambda \neq 1$, then the intersection matrix of the lines $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{y,z\}}$, $L_{\{y\},\{x,z\}}$, and $L_{\{z\},\{x,y\}}$ on the surface S_λ also has rank 5, because

$$L_{\{x\},\{y\}} + L_{\{x\},\{z\}} + L_{\{x\},\{t\}} + L_{\{x\},\{y,z\}} \sim L_{\{x\},\{y\}} + 2L_{\{y\},\{t\}} + L_{\{y\},\{x,z\}} \\ \sim L_{\{x\},\{z\}} + 2L_{\{z\},\{t\}} + L_{\{z\},\{x,y\}} \sim 2L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} \sim H_\lambda$$

on the surface S_λ by (2.30.1). On the other hand, we have $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk}) = \operatorname{rk}\operatorname{Pic}(S_{\Bbbk}) + 13$. Hence, we see that ($\bigstar$) holds, so that ($\diamondsuit$) in the Main Theorem also holds by Lemma 1.13.1.

Family 2.31. In this case, the threefold X is a blow-up of the smooth quadric threefold in $\mathbb{P}^4$ along a line. This shows that $h^{1,2}(X) = 0$. A toric Landau–Ginzburg model of this family is given by the Minkowski polynomial 20, which is

$$x + y + z + \frac{x}{y} + \frac{1}{x} + \frac{1}{yz}.$$

The pencil $\mathcal{S}$ is given by the equation

$$x^2yz + y^2xz + z^2yx + x^2tz + t^2yz + t^3x = \lambda xyzt.$$

We suppose that $\lambda \neq \infty$. Let $\mathcal{C}$ be the conic $\{y = xz + t^2 = 0\}$. Then

$$\begin{aligned} H_{\{x\}} \cdot S_\lambda &= L_{\{x\},\{y\}} + L_{\{x\},\{z\}} + 2L_{\{x\},\{t\}}, \\ H_{\{y\}} \cdot S_\lambda &= L_{\{x\},\{y\}} + L_{\{y\},\{t\}} + \mathcal{C}, \\ H_{\{z\}} \cdot S_\lambda &= L_{\{x\},\{z\}} + 3L_{\{z\},\{t\}}, \\ H_{\{t\}} \cdot S_\lambda &= L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y,z\}}. \end{aligned} \tag{2.31.1}$$

Therefore, the base locus of the pencil $\mathcal{S}$ consists of the lines $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{t\},\{x,y,z\}}$, and the conic $\mathcal{C}$.

For every $\lambda \in \mathbb{C}$, the surface S_λ has isolated singularities, so that it is irreducible. Moreover, the singular points of S_λ contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

$P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_5$ with quadratic term xy for $\lambda \neq 0$, and type $\mathbb{A}_6$ for $\lambda = 0$;

$P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_4$ with quadratic term xz;

$P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_2$ with quadratic term $z(y+t)$;

$P_{\{z\},\{t\},\{x,y\}}$: type $\mathbb{A}_2$ with quadratic term $z(x+y+z-t-\lambda t)$;

$P_{\{x\},\{t\},\{y,z\}}$: type $\mathbb{A}_1$.

In particular, every fiber $\mathsf{f}^{-1}(\lambda)$ is irreducible by Corollary 1.5.4. This confirms ($\heartsuit$) in the Main Theorem, since $h^{1,2}(X) = 0$.

Now let us verify ($\diamondsuit$) in the Main Theorem. If $\lambda \neq 0$, then the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, and H_λ on the surface S_λ is given by

$$\begin{array}{c} \\ L_{\{x\},\{y\}} \\ L_{\{y\},\{t\}} \\ L_{\{z\},\{t\}} \\ H_\lambda \end{array} \begin{array}{c} \begin{array}{cccc} L_{\{x\},\{y\}} & L_{\{y\},\{t\}} & L_{\{z\},\{t\}} & H_\lambda \end{array} \\ \begin{pmatrix} -2/3 & 2/3 & 0 & 1 \\ 2/3 & -1/2 & 1/3 & 1 \\ 0 & 1/3 & 2/15 & 1 \\ 1 & 1 & 1 & 4 \end{pmatrix} \end{array}.$$

This matrix has rank 4. On the other hand, if $\lambda \neq 0$, then the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{t\},\{x,y,z\}}$, and $\mathcal{C}$ on the surface S_λ has the same rank as the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, and H_λ, because

$$L_{\{x\},\{y\}} + L_{\{x\},\{z\}} + 2L_{\{x\},\{t\}} \sim L_{\{x\},\{y\}} + L_{\{y\},\{t\}} + \mathcal{C}$$
$$\sim L_{\{x\},\{z\}} + 3L_{\{z\},\{t\}} \sim L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y,z\}} \sim H_\lambda$$

on the surface S_λ by (2.31.1). Moreover, it follows from the description of singularities of the surface S_λ that $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk}) = \operatorname{rk}\operatorname{Pic}(S_{\Bbbk}) + 14$. Therefore, we conclude that ($\bigstar$) holds, so that ($\diamondsuit$) in the Main Theorem also holds by Lemma 1.13.1.

Family 2.32. In this case, the threefold X is a divisor of bidegree $(1,1)$ on $\mathbb{P}^2 \times \mathbb{P}^2$, so that $h^{1,2}(X) = 0$. A toric Landau–Ginzburg model of this family is given by the Minkowski polynomial 21, which is

$$x + y + z + \frac{1}{y} + \frac{1}{x} + \frac{1}{xyz}.$$

The quartic pencil $\mathcal{S}$ is given by the equation

$$x^2yz + y^2xz + z^2yx + t^2xz + t^2yz + t^4 = \lambda xyzt.$$

As usual, we suppose that $\lambda \neq \infty$.

Let $\mathcal{C}_1$ be the conic in $\mathbb{P}^3$ that is given by $x = yz + t^2 = 0$, and let $\mathcal{C}_2$ be the conic in $\mathbb{P}^3$ that is given by $y = xz + t^2 = 0$. Then

$$\begin{aligned} H_{\{x\}} \cdot S_\lambda &= 2L_{\{x\},\{t\}} + \mathcal{C}_1, \\ H_{\{y\}} \cdot S_\lambda &= 2L_{\{y\},\{t\}} + \mathcal{C}_2, \\ H_{\{z\}} \cdot S_\lambda &= 4L_{\{z\},\{t\}}, \\ H_{\{t\}} \cdot S_\lambda &= L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y,z\}}. \end{aligned} \tag{2.32.1}$$

This shows that the base locus of the pencil $\mathcal{S}$ consists of the curves $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{t\},\{x,y,z\}}$, $\mathcal{C}_1$, and $\mathcal{C}_2$.

For every $\lambda \in \mathbb{C}$, the surface S_λ has isolated singularities, so that it is irreducible. Its singularities contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

$P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_4$ with quadratic term xy for $\lambda \neq 0$, and type $\mathbb{A}_5$ for $\lambda = 0$;

$P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term xz;

$P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term yz;

$P_{\{x\},\{t\},\{y,z\}}$: type $\mathbb{A}_1$;

$P_{\{y\},\{t\},\{x,z\}}$: type $\mathbb{A}_1$;

$P_{\{z\},\{t\},\{x,y\}}$: type $\mathbb{A}_3$ with quadratic term $z(x + y + z - \lambda t)$.

In particular, every fiber $\mathsf{f}^{-1}(\lambda)$ is irreducible by Corollary 1.5.4. This confirms ($\heartsuit$) in the Main Theorem, since $h^{1,2}(X) = 0$.

To verify ($\diamondsuit$) in the Main Theorem, we need the following result.

Lemma 2.32.2. *Suppose that $\lambda \neq 0$. Then the intersection matrix of the curves $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, and H_λ on the surface S_λ is given by*

$$\begin{array}{c} \\ L_{\{x\},\{t\}} \\ L_{\{y\},\{t\}} \\ H_\lambda \end{array} \begin{array}{c} \begin{array}{ccc} L_{\{x\},\{t\}} & L_{\{y\},\{t\}} & H_\lambda \end{array} \\ \begin{pmatrix} 1/20 & 1/5 & 1 \\ 1/5 & 1/20 & 1 \\ 1 & 1 & 4 \end{pmatrix} \end{array}.$$

Proof. To find $L_{\{x\},\{t\}}^2$, observe that the singular points of S_λ contained in the line $L_{\{x\},\{t\}}$ are the points $P_{\{x\},\{z\},\{t\}}$, $P_{\{x\},\{y\},\{t\}}$, and $P_{\{x\},\{t\},\{y,z\}}$. These points are singular points of S_λ of types $\mathbb{A}_3$, $\mathbb{A}_4$, and $\mathbb{A}_1$, respectively. Applying Proposition A.1.3, we see that

$$L_{\{x\},\{t\}}^2 = -2 + \frac{3}{4} + \frac{4}{5} + \frac{1}{2} = \frac{1}{20}.$$

Similarly, we find $L_{\{y\},\{t\}}^2 = 1/20$. Finally, observe that

$$L_{\{x\},\{t\}} \cap L_{\{y\},\{t\}} = P_{\{x\},\{y\},\{t\}}.$$

Using Remark A.2.4 with $S = S_\lambda$, $n = 4$, $O = P_{\{x\},\{y\},\{t\}}$, $C = L_{\{x\},\{t\}}$, and $Z = L_{\{y\},\{t\}}$, we see that none of the curves $\overline{C}$ and $\overline{Z}$ contains the point $\overline{G}_1 \cap \overline{G}_4$. Moreover, since the quadratic term of the surface S_λ at the singular point $P_{\{x\},\{y\},\{t\}}$ is xy, we see that either $\overline{C}\cdot\overline{G}_1 = \overline{Z}\cdot\overline{G}_4 = 1$ or $\overline{C}\cdot\overline{G}_4 = \overline{Z}\cdot\overline{G}_1 = 1$. Thus, using Proposition A.1.2, we conclude that $L_{\{x\},\{t\}}\cdot L_{\{y\},\{t\}} = 1/5$. ☐

If $\lambda \neq 0$, then the intersection matrix of the curves $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{t\},\{x,y,z\}}$, $\mathcal{C}_1$, and $\mathcal{C}_2$ on the surface S_λ has the same rank as the intersection matrix of the curves $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, and H_λ, because

$$2L_{\{x\},\{t\}} + \mathcal{C}_1 \sim 2L_{\{y\},\{t\}} + \mathcal{C}_2 \sim 4L_{\{z\},\{t\}} \sim L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y,z\}} \sim H_\lambda$$

on the surface S_λ by (2.32.1). On the other hand, the matrix in Lemma 2.32.2 has rank 3. Moreover, we have $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk}) = \operatorname{rk}\operatorname{Pic}(S_{\Bbbk}) + 15$. Hence, we see that (★) holds, so that (♢) in the Main Theorem also holds by Lemma 1.13.1.

Family 2.33. The threefold X is a blow-up of $\mathbb{P}^3$ in a line, so that $h^{1,2}(X) = 0$. A toric Landau–Ginzburg model of this family is given by the Minkowski polynomial 6, which is

$$x + y + z + \frac{x}{z} + \frac{1}{xy}.$$

The quartic pencil $\mathcal{S}$ is given by the equation

$$x^2yz + y^2xz + z^2yx + x^2ty + t^3z = \lambda xyzt.$$

Suppose that $\lambda \neq \infty$. Then

$$\begin{aligned}
H_{\{x\}}\cdot S_\lambda &= L_{\{x\},\{z\}} + 3L_{\{x\},\{t\}},\\
H_{\{y\}}\cdot S_\lambda &= L_{\{y\},\{z\}} + 3L_{\{y\},\{t\}},\\
H_{\{z\}}\cdot S_\lambda &= 2L_{\{x\},\{z\}} + L_{\{y\},\{z\}} + L_{\{z\},\{t\}},\\
H_{\{t\}}\cdot S_\lambda L_{\{x\},\{t\}} &+ L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y,z\}}.
\end{aligned}$$

Thus, the base locus of the pencil $\mathcal{S}$ consists of the lines $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, and $L_{\{t\},\{x,y,z\}}$.

Observe that the surface S_λ has isolated singularities for every $\lambda\in\mathbb{C}$, so that it is irreducible. Moreover, the singular points of S_λ contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

$P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_2$ with quadratic term xy;

$P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_6$ with quadratic term xz;

$P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term $y(z+t)$;

$P_{\{x\},\{t\},\{y,z\}}$: type $\mathbb{A}_2$ with quadratic term $x(x+y+z-\lambda t)$;

$P_{\{y\},\{t\},\{x,z\}}$: type $\mathbb{A}_2$ with quadratic term $y(x+y+z-t-\lambda t)$.

Then $[\mathsf{f}^{-1}(\lambda)] = 1$ for every $\lambda\in\mathbb{C}$ by Corollary 1.5.4. This confirms (♡) in the Main Theorem.

Lemma 2.33.1. *The intersection matrix of the curves $L_{\{x\},\{z\}}$, $L_{\{y\},\{z\}}$, and H_λ on the surface S_λ is given by*

$$\begin{array}{r c} & \begin{array}{ccc} L_{\{x\},\{z\}} & L_{\{y\},\{z\}} & H_\lambda \end{array} \\ \begin{array}{r} L_{\{x\},\{z\}} \\ L_{\{y\},\{z\}} \\ H_\lambda \end{array} & \begin{pmatrix} -2/7 & 1 & 1 \\ 1 & -5/4 & 1 \\ 1 & 1 & 4 \end{pmatrix} \end{array}.$$

Proof. The only singular point of S_λ contained in $L_{\{x\},\{z\}}$ is the point $P_{\{x\},\{z\},\{t\}}$. Let us use the notation of Subsection A.2 in the Appendix with $S = S_\lambda$, $n = 6$, $O = P_{\{x\},\{z\},\{t\}}$, and $C = L_{\{x\},\{z\}}$. Then it follows from explicit computations that $\widetilde{C}$ intersects one of the curves G_3 or G_4. Then $L^2_{\{x\},\{z\}} = -2/7$ by Proposition A.1.3.

The only singular point of S_λ contained in $L_{\{y\},\{z\}}$ is the point $P_{\{y\},\{z\},\{t\}}$. Using Remark A.2.4 with $S = S_\lambda$, $n = 3$, $O = P_{\{y\},\{z\},\{t\}}$, and $C = L_{\{y\},\{z\}}$, we see that the curve $\overline{C}$ does not contain the point $\overline{G}_1 \cap \overline{G}_3$. Then $L^2_{\{y\},\{z\}} = -5/4$ by Proposition A.1.3.

Finally, observe that $L_{\{x\},\{z\}} \cap L_{\{y\},\{z\}} = P_{\{x\},\{y\},\{z\}}$ and S_λ is smooth at $P_{\{x\},\{y\},\{z\}}$. Thus, we conclude that $L_{\{x\},\{z\}} \cdot L_{\{y\},\{z\}} = 1$. $\square$

The intersection matrix of the lines $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, and $L_{\{t\},\{x,y,z\}}$ on the surface S_λ has the same rank as the intersection matrix in Lemma 2.32.2, because

$$\begin{aligned} H_\lambda &\sim L_{\{x\},\{z\}} + 3L_{\{x\},\{t\}} \sim L_{\{y\},\{z\}} + 3L_{\{y\},\{t\}} \\ &\sim 2L_{\{x\},\{z\}} + L_{\{y\},\{z\}} + L_{\{z\},\{t\}} \sim L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y,z\}}. \end{aligned}$$

On the other hand, the matrix in Lemma 2.33.1 has rank 3. Moreover, it follows from the description of singularities of the surface S_λ that $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk}) = \operatorname{rk}\operatorname{Pic}(S_{\Bbbk}) + 15$. Hence, we see that ($\bigstar$) holds, so that ($\diamondsuit$) in the Main Theorem also holds by Lemma 1.13.1.

Family 2.34. One has $X \cong \mathbb{P}^1 \times \mathbb{P}^2$. We discussed this case in Example 1.13.2, where we described the pencil $\mathcal{S}$ and its base locus. In this example, we also verified ($\diamondsuit$) in the Main Theorem, so that now we will only check ($\heartsuit$) in the Main Theorem.

If $\lambda \neq \infty$, then S_λ is irreducible, it has isolated singularities, and its singular points contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

$P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_4$ with quadratic term xy;

$P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_4$ with quadratic term xz;

$P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_2$ with quadratic term yz;

$P_{\{y\},\{t\},\{x,z\}}$: type $\mathbb{A}_2$ with quadratic term $y(x+y+z-\lambda t)$;

$P_{\{z\},\{t\},\{x,y\}}$: type $\mathbb{A}_2$ with quadratic term $z(x+y+z-\lambda t)$;

$P_{\{x\},\{t\},\{y,z\}}$: type $\mathbb{A}_1$.

Then $[\mathsf{f}^{-1}(\lambda)] = 1$ for every $\lambda \in \mathbb{C}$ by Corollary 1.5.4. This confirms ($\heartsuit$) in the Main Theorem.

Family 2.35. We have $X \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1))$. A toric Landau–Ginzburg model of this family is given by the Minkowski polynomial 5, which is

$$x + y + z + \frac{x}{yz} + \frac{1}{x}.$$

The quartic pencil $\mathcal{S}$ is given by

$$x^2yz + y^2zx + z^2yx + x^2t^2 + t^2yz = \lambda xyzt.$$

Suppose that $\lambda \neq \infty$. Then

$$\begin{aligned}
H_{\{x\}} \cdot S_\lambda &= L_{\{x\},\{y\}} + L_{\{x\},\{z\}} + 2L_{\{x\},\{t\}},\\
H_{\{y\}} \cdot S_\lambda &= 2L_{\{x\},\{y\}} + 2L_{\{y\},\{t\}},\\
H_{\{z\}} \cdot S_\lambda &= 2L_{\{x\},\{z\}} + 2L_{\{z\},\{t\}},\\
H_{\{t\}} \cdot S_\lambda &= L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y,z\}}.
\end{aligned}$$

Thus, the base locus of the pencil $\mathcal{S}$ consists of the lines $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, and $L_{\{t\},\{x,y,z\}}$.

For every $\lambda \in \mathbb{C}$, the surface $S_\lambda \in \mathcal{S}$ has isolated singularities. In particular, it is irreducible. Moreover, its singular points contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

$$\begin{aligned}
P_{\{x\},\{y\},\{z\}}&\colon\ \text{type } \mathbb{A}_1;\\
P_{\{x\},\{y\},\{t\}}&\colon\ \text{type } \mathbb{A}_5 \text{ with quadratic term } xy;\\
P_{\{x\},\{z\},\{t\}}&\colon\ \text{type } \mathbb{A}_5 \text{ with quadratic term } xz;\\
P_{\{y\},\{z\},\{t\}}&\colon\ \text{type } \mathbb{A}_1;\\
P_{\{x\},\{t\},\{y,z\}}&\colon\ \text{type } \mathbb{A}_1;\\
P_{\{y\},\{t\},\{x,z\}}&\colon\ \text{type } \mathbb{A}_1;\\
P_{\{z\},\{t\},\{x,y\}}&\colon\ \text{type } \mathbb{A}_1.
\end{aligned}$$

In particular, every fiber $\mathsf{f}^{-1}(\lambda)$ is irreducible by Corollary 1.5.4. This confirms ($\heartsuit$) in the Main Theorem, since $h^{1,2}(X) = 0$.

To verify ($\diamondsuit$) in the Main Theorem, observe that the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, and H_λ on the surface S_λ is given by

$$\begin{array}{c}
\\ L_{\{x\},\{y\}} \\ L_{\{x\},\{z\}} \\ L_{\{x\},\{t\}} \\ L_{\{y\},\{t\}} \\ L_{\{z\},\{t\}} \\ H_\lambda
\end{array}
\begin{array}{c}
\begin{array}{cccccc} L_{\{x\},\{y\}} & L_{\{x\},\{z\}} & L_{\{x\},\{t\}} & L_{\{y\},\{t\}} & L_{\{z\},\{t\}} & H_\lambda \end{array}\\
\begin{pmatrix}
-1/6 & 1/2 & 1/3 & 2/3 & 0 & 1\\
1/2 & -1/6 & 1/3 & 0 & 2/3 & 1\\
1/3 & 1/3 & 1/6 & 1/6 & 1/6 & 1\\
2/3 & 0 & 1/6 & -1/6 & 1/2 & 1\\
0 & 2/3 & 1/6 & 1/2 & -1/6 & 1\\
1 & 1 & 1 & 1 & 1 & 4
\end{pmatrix}
\end{array}.$$

This matrix has rank 3. On the other hand, the intersection matrix of the lines $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, and $L_{\{t\},\{x,y,z\}}$ on the surface S_λ has the same rank as the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, and H_λ, because

$$\begin{aligned}
H_\lambda &\sim L_{\{x\},\{y\}} + L_{\{x\},\{z\}} + 2L_{\{x\},\{t\}} \sim 2L_{\{x\},\{y\}} + 2L_{\{y\},\{t\}}\\
&\sim 2L_{\{x\},\{z\}} + 2L_{\{z\},\{t\}} \sim L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y,z\}}.
\end{aligned}$$

Moreover, we have $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk}) = \operatorname{rk}\operatorname{Pic}(S_{\Bbbk}) + 15$. Therefore, we conclude that ($\bigstar$) holds, so that ($\diamondsuit$) in the Main Theorem also holds by Lemma 1.13.1.

Family 2.36. In this case, we have $X \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2))$, so that $h^{1,2}(X) = 0$. A toric Landau–Ginzburg model of this family is given by the Minkowski polynomial 7, which is

$$x + y + z + \frac{x^2}{yz} + \frac{1}{x}.$$

The pencil $\mathcal{S}$ is given by the equation

$$x^2yz + y^2zx + z^2yx + x^3t + t^2yz = \lambda xyzt.$$

Suppose that $\lambda \neq \infty$. Then

$$\begin{aligned} H_{\{x\}} \cdot S_\lambda &= L_{\{x\},\{y\}} + L_{\{x\},\{z\}} + 2L_{\{x\},\{t\}}, \\ H_{\{y\}} \cdot S_\lambda &= 3L_{\{x\},\{y\}} + L_{\{y\},\{t\}}, \\ H_{\{z\}} \cdot S_\lambda &= 3L_{\{x\},\{z\}} + L_{\{z\},\{t\}}, \\ H_{\{t\}} \cdot S_\lambda &= L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y,z\}}. \end{aligned} \tag{2.36.1}$$

Thus, the base locus of the pencil $\mathcal{S}$ consists of the lines $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, and $L_{\{t\},\{x,y,z\}}$.

For every $\lambda \in \mathbb{C}$, the surface S_λ has isolated singularities, so that it is irreducible. Moreover, its singular points contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

$P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_6$ with quadratic term xy;

$P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_6$ with quadratic term xz;

$P_{\{x\},\{t\},\{y,z\}}$: type $\mathbb{A}_1$;

$P_{\{x\},\{y\},\{z\}}$: type $\mathbb{A}_2$ with quadratic term yz.

In particular, every fiber $\mathsf{f}^{-1}(\lambda)$ is irreducible by Corollary 1.5.4. This confirms ($\heartsuit$) in the Main Theorem, since $h^{1,2}(X) = 0$.

On the surface S_λ, the intersection matrix of the lines $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, and $L_{\{t\},\{x,y,z\}}$ has the same rank as the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, and H_λ, because

$$\begin{aligned} H_\lambda &\sim L_{\{x\},\{y\}} + L_{\{x\},\{z\}} + 2L_{\{x\},\{t\}} \sim 3L_{\{x\},\{y\}} + L_{\{y\},\{t\}} \\ &\sim 3L_{\{x\},\{z\}} + L_{\{z\},\{t\}} \sim L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y,z\}}. \end{aligned}$$

These rational equivalences follow from (2.36.1).

Lemma 2.36.2. *The intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, and H_λ on the surface S_λ is given by*

$$\begin{array}{c c} & \begin{array}{ccc} L_{\{x\},\{y\}} & L_{\{x\},\{z\}} & H_\lambda \end{array} \\ \begin{array}{c} L_{\{x\},\{y\}} \\ L_{\{x\},\{z\}} \\ H_\lambda \end{array} & \begin{pmatrix} 2/21 & 1/3 & 1 \\ 1/3 & 2/21 & 1 \\ 1 & 1 & 4 \end{pmatrix} \end{array}.$$

Proof. By definition, we have $H_\lambda^2 = 4$ and $H_\lambda \cdot L_{\{x\},\{y\}} = H_\lambda \cdot L_{\{x\},\{z\}} = 1$. Note that

$$L_{\{x\},\{y\}} \cap L_{\{x\},\{z\}} = P_{\{x\},\{y\},\{z\}}.$$

Recall that $P_{\{x\},\{y\},\{z\}}$ is a singular point of S_λ of type $\mathbb{A}_2$. Then one gets $L_{\{x\},\{y\}} \cdot L_{\{x\},\{z\}} = 1/3$ by Proposition A.1.2.

We claim that $L_{\{y\},\{t\}}^2 = -8/7$. Indeed, the point $P_{\{x\},\{y\},\{t\}}$ is the only singular point of S_λ that is contained in $L_{\{y\},\{t\}}$. Using Remark A.2.4 with $S = S_\lambda$, $n = 6$, $O = P_{\{x\},\{y\},\{t\}}$, and $C = L_{\{y\},\{t\}}$, we see that $\overline{C}$ does not contain the point $\overline{G}_1 \cap \overline{G}_6$, because the quadratic term of the surface S_λ at the point $P_{\{x\},\{y\},\{t\}}$ is xy. Thus, we have $L_{\{y\},\{t\}}^2 = -8/7$ by Proposition A.1.3.

Since $L_{\{y\},\{t\}}^2=-8/7$, we get $L_{\{x\},\{y\}}\cdot L_{\{y\},\{t\}}=5/7$, because

$$1=H_\lambda\cdot L_{\{y\},\{t\}}=\big(3L_{\{x\},\{y\}}+L_{\{y\},\{t\}}\big)\cdot L_{\{y\},\{t\}}=3L_{\{x\},\{y\}}\cdot L_{\{y\},\{t\}}-\frac{8}{7}.$$

Since $L_{\{x\},\{y\}}\cdot L_{\{y\},\{t\}}=5/7$, we get $L_{\{x\},\{y\}}^2=2/21$, because

$$1=H_\lambda\cdot L_{\{y\},\{t\}}=\big(3L_{\{x\},\{y\}}+L_{\{y\},\{t\}}\big)\cdot L_{\{x\},\{y\}}=3L_{\{x\},\{y\}}^2+\frac{5}{7}.$$

Similarly, we see that $L_{\{z\},\{t\}}^2=2/21$. $\square$

The matrix in Lemma 2.36.2 has rank 3. Moreover, we have $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk})=\operatorname{rk}\operatorname{Pic}(S_{\Bbbk})+15$. Hence, we see that ($\bigstar$) holds, so that ($\diamondsuit$) in the Main Theorem also holds by Lemma 1.13.1.

3. FANO THREEFOLDS OF PICARD RANK 3

Family 3.1. In this case, the threefold X is a double cover of $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$ branched over a smooth divisor of tridegree $(2,2,2)$, which implies that $h^{1,2}(X)=8$. The toric Landau–Ginzburg model is given by the Minkowski polynomial 3873.4, which is the Laurent polynomial

$$x+y+\frac{x}{z}+\frac{y}{z}+\frac{xz}{y}+3z+\frac{yz}{x}+\frac{2x}{y}+\frac{2y}{x}+\frac{x}{yz}\\ +\frac{3}{z}+\frac{y}{xz}+\frac{z^2}{y}+\frac{z^2}{x}+\frac{3z}{y}+\frac{3z}{x}+\frac{3}{y}+\frac{3}{x}+\frac{1}{yz}+\frac{1}{xz}.$$

The quartic pencil $\mathcal{S}$ is given by

$$x^2yz+y^2zx+x^2ty+y^2tx+x^2z^2+3z^2yx+y^2z^2+2x^2tz+2y^2tz+x^2t^2+3t^2yx\\ +t^2y^2+z^3x+z^3y+3z^2tx+3z^2ty+3t^2zx+3t^2yz+t^3x+t^3y=\lambda xyzt.$$

This equation is symmetric with respect to the permutations of variables $x\leftrightarrow y$ and $z\leftrightarrow t$.

To prove the Main Theorem in this case, we may assume that $\lambda\neq\infty$. Then

$$\begin{aligned}H_{\{x\}}\cdot S_\lambda&=L_{\{x\},\{y\}}+2L_{\{x\},\{z,t\}}+L_{\{x\},\{y,z,t\}},\\ H_{\{y\}}\cdot S_\lambda&=L_{\{x\},\{y\}}+2L_{\{y\},\{z,t\}}+L_{\{y\},\{x,z,t\}},\\ H_{\{z\}}\cdot S_\lambda&=L_{\{z\},\{t\}}+L_{\{z\},\{x,y,t\}}+\mathcal{C}_1,\\ H_{\{t\}}\cdot S_\lambda&=L_{\{z\},\{t\}}+L_{\{t\},\{x,y,z\}}+\mathcal{C}_2,\end{aligned}\tag{3.1.1}$$

where $\mathcal{C}_1$ is a smooth conic given by $z=xy+xt+yt=0$ and $\mathcal{C}_2$ is a smooth conic given by $t=xy+xz+yz=0$. Hence, since $\lambda\neq\infty$, we have

$$S_\lambda\cdot S_\infty=2L_{\{x\},\{y\}}+2L_{\{z\},\{t\}}+2L_{\{x\},\{z,t\}}+2L_{\{y\},\{z,t\}}\\ +L_{\{x\},\{y,z,t\}}+L_{\{y\},\{x,z,t\}}+L_{\{z\},\{x,y,t\}}+L_{\{t\},\{x,y,z\}}+\mathcal{C}_1+\mathcal{C}_2.$$

Let $C_1=\mathcal{C}_1$, $C_2=\mathcal{C}_2$, $C_3=L_{\{x\},\{y\}}$, $C_4=L_{\{z\},\{t\}}$, $C_5=L_{\{x\},\{z,t\}}$, $C_6=L_{\{y\},\{z,t\}}$, $C_7=L_{\{x\},\{y,z,t\}}$, $C_8=L_{\{y\},\{x,z,t\}}$, $C_9=L_{\{z\},\{x,y,t\}}$, and $C_{10}=L_{\{t\},\{x,y,z\}}$. These are all base curves of the pencil $\mathcal{S}$.

For every $\lambda\neq-6$, the surface S_λ has isolated singularities, so that S_λ is irreducible. On the other hand, we have $S_{-6}=H_{\{z,t\}}+H_{\{x,y,z,t\}}+\mathbf{Q}$, where $\mathbf{Q}$ is an irreducible quadric given by $xy+xz+yz+xt+yt=0$. This quadric is singular at the point $P_{\{y\},\{z,t\}}$, which is also contained in the planes $H_{\{z,t\}}$ and $H_{\{x,y,z,t\}}$.

If $\lambda \neq -6$, then the singularities of S_λ that are contained in the base locus of the pencil $\mathcal{S}$ are all du Val and can be described as follows:

$P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term $(z+t)(y+z+t)$;

$P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term $(z+t)(x+z+t)$;

$P_{\{x\},\{y\},\{z,t\}}$: type $\mathbb{A}_5$ with quadratic term $(\lambda+6)xy$;

$P_{\{z\},\{t\},\{x,y\}}$: type $\mathbb{A}_1$ with quadratic term $(\lambda+6)zt-(z+t)(x+y+z+t)$.

In the notation of Subsection 1.8, the points $P_{\{y\},\{z\},\{t\}}$, $P_{\{x\},\{z\},\{t\}}$, $P_{\{x\},\{y\},\{z,t\}}$, and $P_{\{z\},\{t\},\{x,y\}}$ are the fixed singular points of the quartic surfaces in the pencil $\mathcal{S}$.

By Corollary 1.5.4, the fiber $\mathsf{f}^{-1}(\lambda)$ is irreducible for every $\lambda \neq -6$. Therefore, the assertion ($\heartsuit$) in the Main Theorem follows from

Lemma 3.1.2. *One has* $[\mathsf{f}^{-1}(-6)] = 9$.

Proof. Recall that $[S_{-6}] = 3$. Moreover, we have

$$\mathbf{M}_5^{-6} = \mathbf{M}_6^{-6} = 2 \qquad \text{and} \qquad \mathbf{M}_1^{-6} = \ldots = \mathbf{M}_4^{-6} = \mathbf{M}_7^{-6} = \ldots = \mathbf{M}_{10}^{-6} = 1.$$

But $\mathbf{m}_1 = \mathbf{m}_2 = \mathbf{m}_7 = \ldots = \mathbf{m}_{10} = 1$ and $\mathbf{m}_3 = \ldots = \mathbf{m}_6 = 2$, and the points $P_{\{y\},\{z\},\{t\}}$, $P_{\{x\},\{z\},\{t\}}$, and $P_{\{z\},\{t\},\{x,y\}}$ are non-isolated ordinary double points of the surface S_{-6}. Thus, it follows from (1.8.3) and Lemmas 1.8.5 and 1.12.1 that

$$\left[\mathsf{f}^{-1}(-6)\right] = 5 + \mathbf{D}^{-6}_{P_{\{x\},\{y\},\{z,t\}}}.$$

Let $\alpha_1\colon U_1 \to \mathbb{P}^3$ be a blow-up of the point $P_{\{x\},\{y\},\{z,t\}}$. Then $D^1_{-6} = S^1_{-6} + 2\mathbf{E}_1$. The surface $\mathbf{E}_1$ contains two base curves of the pencil $\mathcal{S}^1$. Denote them by C^1_{11} and C^1_{12}. Then $\mathbf{m}_{11} = \mathbf{m}_{12} = 2$ and $\mathbf{M}_{11}^{-6} = \mathbf{M}_{12}^{-6} = 2$.

Let $\alpha_2\colon U_2 \to U_1$ be the blow-up of the point $C^1_{11} \cap C^1_{12}$. Then $D^2_{-6} = S^2_{-6} + 2\mathbf{E}^2_1 + \mathbf{E}_2$. The surface $\mathbf{E}_2$ contains two base curves of the pencil $\mathcal{S}^2$. Denote them by C^2_{13} and C^2_{14}. Then $\mathbf{M}_{13}^{-6} = \mathbf{M}_{14}^{-6} = 2$.

Note that there exists a commutative diagram

$$\begin{tikzcd} & U_2 \arrow[ld, "\alpha_2"'] & \\ U_1 \arrow[rd, "\alpha_1"'] & & U \arrow[lu, "\gamma"'] \arrow[ld, "\alpha"] \\ & \mathbb{P}^3 & \end{tikzcd}$$

for some birational morphism γ. Moreover, the only base curves of the pencil $\widehat{\mathcal{S}}$ that are mapped to the singular point $P_{\{x\},\{y\},\{z,t\}}$ are the curves $\widehat{C}_{11}, \ldots, \widehat{C}_{14}$. Furthermore, our computations also give $\mathbf{A}^{-6}_{P_{\{x\},\{y\},\{z,t\}}} = 2$. Thus, it follows from (1.10.9) and Lemma 1.10.7 that $\mathbf{D}^{-6}_{P_{\{x\},\{y\},\{z,t\}}} = 4$, so that $[\mathsf{f}^{-1}(-6)] = 9$. $\square$

If $\lambda \neq -6$, then the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{z,t\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{x\},\{y,z,t\}}$, $L_{\{y\},\{x,z,t\}}$, $L_{\{z\},\{x,y,t\}}$, $L_{\{t\},\{x,y,z\}}$, $\mathcal{C}_1$, and $\mathcal{C}_2$ on the surface S_λ has the same rank as the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{z,t\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{z\},\{x,y,t\}}$, $L_{\{t\},\{x,y,z\}}$, and H_λ. This follows from (3.1.1). On the other hand, if $\lambda \neq -6$, then

$$L_{\{x\},\{z,t\}} + L_{\{y\},\{z,t\}} + 2L_{\{z\},\{t\}} \sim H_\lambda$$

on the surface S_λ, because $H_{\{z,t\}} \cdot S_\lambda = L_{\{x\},\{z,t\}} + L_{\{y\},\{z,t\}} + 2L_{\{z\},\{t\}}$. Similarly, if $\lambda \neq -6$, then

$$L_{\{x\},\{y,z,t\}} + L_{\{y\},\{x,z,t\}} + L_{\{z\},\{x,y,t\}} + L_{\{t\},\{x,y,z\}} \sim H_\lambda.$$

Using this, we can easily compute the intersection form of the curves $L_{\{x\},\{y\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{z,t\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{z\},\{x,y,t\}}$, $L_{\{t\},\{x,y,z\}}$, and H_λ on the surface S_λ. If $\lambda \neq -6$, it is given by the following matrix:

$$
\begin{array}{r}
\\
L_{\{x\},\{y\}}\\
L_{\{z\},\{t\}}\\
L_{\{x\},\{z,t\}}\\
L_{\{y\},\{z,t\}}\\
L_{\{z\},\{x,y,t\}}\\
L_{\{t\},\{x,y,z\}}\\
H_\lambda
\end{array}
\begin{array}{c}
\begin{array}{ccccccc}
L_{\{x\},\{y\}} & L_{\{z\},\{t\}} & L_{\{x\},\{z,t\}} & L_{\{y\},\{z,t\}} & L_{\{z\},\{x,y,t\}} & L_{\{t\},\{x,y,z\}} & H_\lambda
\end{array}\\
\begin{pmatrix}
-1/2 & 0 & 1/2 & 1/2 & 0 & 0 & 1\\
0 & 0 & 1/2 & 1/2 & 1/2 & 1/2 & 1\\
1/2 & 1/2 & -1/6 & 1/6 & 0 & 0 & 1\\
1/2 & 1/2 & 1/6 & -1/6 & 0 & 0 & 1\\
0 & 1/2 & 0 & 0 & -3/2 & 1/2 & 1\\
0 & 1/2 & 0 & 0 & 1/2 & -3/2 & 1\\
1 & 1 & 1 & 1 & 1 & 1 & 4
\end{pmatrix}
\end{array}.
$$

The rank of this intersection matrix is 5. Moreover, we have $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk}) = \operatorname{rk}\operatorname{Pic}(S_{\Bbbk}) + 12$. Hence, we see that ($\bigstar$) holds, so that ($\diamondsuit$) in the Main Theorem also holds by Lemma 1.13.1.

Family 3.2. We already discussed this case in Example 1.8.6. Because of this, let us use the notation of this example. Note that $h^{1,2}(X) = 3$ and the defining equation of the surface S_λ is symmetric with respect to the permutations $x \leftrightarrow y$ and $z \leftrightarrow t$.

To prove the Main Theorem in this case, we may assume that $\lambda \neq \infty$. Then

$$
\begin{aligned}
H_{\{x\}} \cdot S_\lambda &= L_{\{x\},\{t\}} + L_{\{x\},\{y,z\}} + \mathcal{C}_1,\\
H_{\{y\}} \cdot S_\lambda &= L_{\{y\},\{t\}} + L_{\{y\},\{x,z\}} + \mathcal{C}_2,\\
H_{\{z\}} \cdot S_\lambda &= L_{\{z\},\{t\}} + 3L_{\{z\},\{x,y\}},\\
H_{\{t\}} \cdot S_\lambda &= L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y,z\}}.
\end{aligned}
\tag{3.2.1}
$$

For every $\lambda \neq -6$, the surface S_λ is irreducible, it has isolated singularities, and its singularities contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

$P_{\{x\},\{y\},\{z\}}$: type $\mathbb{A}_5$ with quadratic term $z(x+y+z)$;
$P_{\{x\},\{t\},\{y,z\}}$: type $\mathbb{A}_2$ with quadratic term $x(x+y+z+(\lambda+6)t)$;
$P_{\{y\},\{t\},\{x,z\}}$: type $\mathbb{A}_2$ with quadratic term $y(x+y+z+(\lambda+6)t)$;
$P_{\{z\},\{t\},\{x,y\}}$: type $\mathbb{A}_3$ with quadratic term $z(x+y+z+(\lambda+6)t)$.

By Corollary 1.5.4, one has $[\mathsf{f}^{-1}(\lambda)] = 1$ for every $\lambda \neq -6$.

Recall that $S_{-6} = H_{\{x,y,z\}} + \mathbf{S}$, where $\mathbf{S}$ is a cubic surface whose singular locus consists of the points $P_{\{z\},\{x,y\},\{x,t\}}$ and $P_{\{z\},\{x,y\},\{y,t\}}$. Observe also that $H_{\{x,y,z\}} \cap \mathbf{S}$ consists of the line $L_{\{z\},\{x,y\}}$ and an irreducible conic $x+y+z = xy+t^2$. Then S_{-6} has good double points at $P_{\{x\},\{y\},\{z\}}$, $P_{\{x\},\{t\},\{y,z\}}$, $P_{\{y\},\{t\},\{x,z\}}$, and $P_{\{z\},\{t\},\{x,y\}}$. Hence, using (1.8.3) and Lemmas 1.8.5 and 1.12.1, we get $[\mathsf{f}^{-1}(-6)] = 4$. This confirms ($\heartsuit$) in the Main Theorem.

Let us verify ($\diamondsuit$) in the Main Theorem. We may assume that $\lambda \neq -6$. Then

$$
L_{\{x\},\{y,z\}} + L_{\{y\},\{x,z\}} + L_{\{z\},\{x,y\}} \sim L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}}
$$

on the surface S_λ. This follows from (3.2.1) and the fact that

$$
H_{\{x,y,z\}} \cdot S_\lambda = L_{\{x\},\{y,z\}} + L_{\{y\},\{x,z\}} + L_{\{z\},\{x,y\}} + L_{\{t\},\{x,y,z\}}.
$$

Using this, we can compute the intersection form of the curves $L_{\{x\},\{t\}}$, $L_{\{x\},\{y,z\}}$, $L_{\{y\},\{t\}}$, $L_{\{y\},\{x,z\}}$, $L_{\{z\},\{t\}}$, and H_λ on the surface S_λ. Namely, it is given by the following matrix:

$$\begin{array}{c} \\ L_{\{x\},\{t\}} \\ L_{\{x\},\{y,z\}} \\ L_{\{y\},\{t\}} \\ L_{\{y\},\{x,z\}} \\ L_{\{z\},\{t\}} \\ H_\lambda \end{array} \begin{array}{c} \begin{array}{cccccc} L_{\{x\},\{t\}} & L_{\{x\},\{y,z\}} & L_{\{y\},\{t\}} & L_{\{y\},\{x,z\}} & L_{\{z\},\{t\}} & H_\lambda \end{array} \\ \begin{pmatrix} -4/3 & 2/3 & 1 & 0 & 1 & 1 \\ 2/3 & -1/2 & 0 & 5/6 & 0 & 1 \\ 1 & 0 & -4/3 & 2/3 & 1 & 1 \\ 0 & 5/6 & 2/3 & -1/2 & 0 & 1 \\ 1 & 0 & 1 & 0 & -5/4 & 1 \\ 1 & 1 & 1 & 1 & 1 & 4 \end{pmatrix} \end{array}.$$

The rank of this matrix is 5. Thus, if $\lambda \neq -6$, then it follows from (3.2.1) that the intersection matrix of the curves $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{y,z\}}$, $L_{\{y\},\{x,z\}}$, $L_{\{z\},\{x,y\}}$, $L_{\{t\},\{x,y,z\}}$, $\mathcal{C}_1$, and $\mathcal{C}_2$ on the surface S_λ also has rank 5. But $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk}) = \operatorname{rk}\operatorname{Pic}(S_{\Bbbk}) + 12$. Hence, we see that ($\bigstar$) holds, so that ($\diamondsuit$) in the Main Theorem also holds by Lemma 1.13.1.

Family 3.3. The threefold X is a divisor of tridegree $(1,1,2)$ on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, which implies that $h^{1,2}(X) = 3$. Its toric Landau–Ginzburg model is given by the Minkowski polynomial 1804. Using the coordinate change $x \mapsto y/x$, $z \mapsto z/x$, we can rewrite it as

$$\frac{yz}{x} + \frac{y}{x} + y + \frac{z}{x} + z + \frac{2}{x} + 2x + \frac{1}{z} + \frac{2x}{z} + \frac{x^2}{z} + \frac{1}{xy} + \frac{2}{y} + \frac{x}{y}.$$

The quartic pencil $\mathcal{S}$ is given by the equation

$$t^3z + t^2xy + 2t^2xz + 2t^2yz + 2tx^2y + tx^2z + ty^2z + tyz^2 \\ + x^3y + 2x^2yz + xy^2z + xyz^2 + y^2z^2 = \lambda xyzt.$$

This equation is symmetric with respect to the involution $[x:y:z:t] \leftrightarrow [t:z:y:x]$.

Suppose that $\lambda \neq \infty$. Let $\mathcal{C}_1$ be a smooth conic given by $x = yz + yt + t^2 = 0$, and let $\mathcal{C}_2$ be a smooth conic given by $t = x^2 + xz + yz = 0$. Then

$$\begin{aligned} H_{\{x\}} \cdot S_\lambda &= L_{\{x\},\{z\}} + L_{\{x\},\{y,t\}} + \mathcal{C}_1, \\ H_{\{y\}} \cdot S_\lambda &= L_{\{y\},\{t\}} + L_{\{y\},\{z\}} + 2L_{\{y\},\{x,t\}}, \\ H_{\{z\}} \cdot S_\lambda &= L_{\{x\},\{z\}} + L_{\{y\},\{z\}} + 2L_{\{z\},\{x,t\}}, \\ H_{\{t\}} \cdot S_\lambda &= L_{\{y\},\{t\}} + L_{\{t\},\{x,z\}} + \mathcal{C}_2. \end{aligned} \tag{3.3.1}$$

Hence, we conclude that $L_{\{x\},\{z\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{x\},\{y,t\}}$, $L_{\{y\},\{x,t\}}$, $L_{\{z\},\{x,t\}}$, $L_{\{t\},\{x,z\}}$, $\mathcal{C}_1$, and $\mathcal{C}_2$ are all base curves of the pencil $\mathcal{S}$.

The surface S_{-4} is irreducible. However, its singularities are not isolated: it is singular along the lines $L_{\{z\},\{x,t\}}$ and $L_{\{y\},\{x,t\}}$, and smooth away from them.

If $\lambda \neq -4$, then S_λ has isolated singularities, so that S_λ is irreducible. In this case, the singularities of S_λ that are contained in the base locus of the pencil $\mathcal{S}$ are all du Val and can be described as follows:

$P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_4$ with quadratic term $y(y+z+t)$;

$P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_4$ with quadratic term $z(x+z+t)$;

$P_{\{y\},\{z\},\{x,t\}}$: type $\mathbb{A}_3$ with quadratic term $(\lambda+4)yz$.

Thus, it follows from Corollary 1.5.4 that $[\mathsf{f}^{-1}(\lambda)] = 1$ for every $\lambda \neq -4$.

Lemma 3.3.2. *One has* $[\mathsf{f}^{-1}(-4)] = 4$.

Proof. Let $C_1 = \mathcal{C}_1$, $C_2 = \mathcal{C}_2$, $C_3 = L_{\{x\},\{z\}}$, $C_4 = L_{\{y\},\{z\}}$, $C_5 = L_{\{y\},\{t\}}$, $C_6 = L_{\{x\},\{y,t\}}$, $C_7 = L_{\{y\},\{x,t\}}$, $C_8 = L_{\{z\},\{x,t\}}$, and $C_9 = L_{\{t\},\{x,z\}}$. Then $\mathbf{m}_1 = \mathbf{m}_2 = \mathbf{m}_6 = \mathbf{m}_9 = 1$ and $\mathbf{m}_3 = \mathbf{m}_4 = \mathbf{m}_5 = \mathbf{m}_7 = \mathbf{m}_8 = 2$. Similarly, we have

$$\mathbf{M}_7^{-4} = \mathbf{M}_8^{-4} = 2 \qquad \text{and} \qquad \mathbf{M}_1^{-4} = \ldots = \mathbf{M}_6^{-4} = \mathbf{M}_9^{-4} = 1.$$

Thus, it follows from (1.8.3) and Lemma 1.8.5 that

$$\left[\mathsf{f}^{-1}(-4)\right] = 3 + \mathbf{D}^{-4}_{P_{\{x\},\{z\},\{t\}}} + \mathbf{D}^{-4}_{P_{\{x\},\{y\},\{t\}}} + \mathbf{D}^{-4}_{P_{\{y\},\{z\},\{x,t\}}}.$$

The surface S_{-4} has (non-isolated) ordinary double singularities at the points $P_{\{x\},\{z\},\{t\}}$ and $P_{\{x\},\{y\},\{t\}}$. Thus, it follows from Lemma 1.12.1 that $\mathbf{D}^{-4}_{P_{\{x\},\{z\},\{t\}}} = \mathbf{D}^{-4}_{P_{\{x\},\{y\},\{t\}}} = 0$.

Let $\alpha_1\colon U_1 \to \mathbb{P}^3$ be a blow-up of the point $P_{\{y\},\{z\},\{x,t\}}$. Then $D^1_{-4} = S^1_{-4} + \mathbf{E}_1$. The surface $\mathbf{E}_1$ contains two base curves of the pencil $\mathcal{S}^1$. Denote them by C^1_{10} and C^1_{11}. Then $\mathbf{M}_{10}^{-4} = \mathbf{M}_{11}^{-4} = 1$, $\mathbf{A}^{-4}_{P_{\{y\},\{z\},\{x,t\}}} = 1$, and the only base curves of the pencil $\widehat{\mathcal{S}}$ that are mapped to the singular point $P_{\{y\},\{z\},\{x,t\}}$ are the curves $\widehat{C}_{10}$ and $\widehat{C}_{11}$. Then

$$\mathbf{D}^{-4}_{P_{\{y\},\{z\},\{x,t\}}} = 1$$

by (1.10.9) and Lemma 1.10.7. We conclude that $[\mathsf{f}^{-1}(-4)] = 4$. $\square$

Recall that $h^{1,2}(X) = 3$. Since the fiber $\mathsf{f}^{-1}(\lambda)$ is irreducible for every $\lambda \neq -4$, we see that (♡) in the Main Theorem follows from Lemma 3.3.2.

Now let us prove (♢) in the Main Theorem. We may assume that $\lambda \neq -4$. Then the intersection form of the curves $L_{\{x\},\{z\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{x\},\{t,z\}}$, $L_{\{t\},\{x,z\}}$, and H_λ on the surface S_λ is given by

$$\begin{array}{c} \\ L_{\{x\},\{z\}} \\ L_{\{y\},\{z\}} \\ L_{\{y\},\{t\}} \\ L_{\{x\},\{t,z\}} \\ L_{\{t\},\{x,z\}} \\ H_\lambda \end{array}
\begin{array}{c} \begin{array}{cccccc} L_{\{x\},\{z\}} & L_{\{y\},\{z\}} & L_{\{y\},\{t\}} & L_{\{x\},\{t,z\}} & L_{\{t\},\{x,z\}} & H_\lambda \end{array} \\
\begin{pmatrix} -6/5 & 1 & 0 & 1 & 1/5 & 1 \\ 1 & -1 & 1 & 0 & 0 & 1 \\ 0 & 1 & -6/5 & 1/5 & 1 & 1 \\ 1 & 0 & 1/5 & -6/5 & 0 & 1 \\ 1/5 & 0 & 1 & 0 & -6/5 & 1 \\ 1 & 1 & 1 & 1 & 1 & 4 \end{pmatrix} \end{array}.$$

The determinant of this matrix is $-112/25$. Thus, it follows from (3.3.1) that the intersection matrix of the curves $L_{\{x\},\{z\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{x\},\{y,t\}}$, $L_{\{y\},\{x,t\}}$, $L_{\{z\},\{x,t\}}$, $L_{\{t\},\{x,z\}}$, $\mathcal{C}_1$, and $\mathcal{C}_2$ on the surface S_λ also has rank 6. But $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk}) = \operatorname{rk}\operatorname{Pic}(S_{\Bbbk}) + 11$. Summarizing, we see that (★) holds, so that (♢) in the Main Theorem also holds by Lemma 1.13.1.

Family 3.4. The threefold X is a blow-up of a double cover of $\mathbb{P}^1 \times \mathbb{P}^2$ in a divisor of bidegree $(2,2)$ along a smooth fiber of the projection to $\mathbb{P}^2$. One has $h^{1,2}(X) = 2$. The toric Landau–Ginzburg model is given by the Minkowski polynomial 1724, which is

$$y + \frac{yz}{x} + \frac{2y}{x} + x + \frac{y}{xz} + 2z + \frac{z}{x} + \frac{2}{z} + \frac{xz}{y} + \frac{2}{x} + \frac{2x}{y} + \frac{1}{xz} + \frac{x}{yz}.$$

The pencil $\mathcal{S}$ is given by

$$y^2xz + y^2z^2 + 2y^2tz + x^2yz + t^2y^2 + 2xyz^2 + z^2yt + 2t^2xy \\ + x^2z^2 + 2t^2yz + 2x^2tz + t^3y + x^2t^2 = \lambda xyzt.$$

As usual, we will assume that $\lambda \neq \infty$.

Let $\mathcal{C}_1$ be a conic given by $z = x^2 + 2xy + y^2 + yt = 0$, and let $\mathcal{C}_2$ be a conic given by $t = xy + xz + yz = 0$. Then

$$\begin{aligned}
H_{\{x\}} \cdot S_\lambda &= L_{\{x\},\{y\}} + 2L_{\{x\},\{z,t\}} + L_{\{x\},\{y,t\}},\\
H_{\{y\}} \cdot S_\lambda &= 2L_{\{x\},\{y\}} + 2L_{\{y\},\{z,t\}},\\
H_{\{z\}} \cdot S_\lambda &= 2L_{\{z\},\{t\}} + \mathcal{C}_1,\\
H_{\{t\}} \cdot S_\lambda &= L_{\{z\},\{t\}} + L_{\{t\},\{x,y\}} + \mathcal{C}_2.
\end{aligned} \tag{3.4.1}$$

This shows that $L_{\{x\},\{y\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{y,t\}}$, $L_{\{x\},\{z,t\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{t\},\{x,y\}}$, $\mathcal{C}_1$, and $\mathcal{C}_2$ are all base curves of the pencil $\mathcal{S}$.

For every $\lambda \in \mathbb{C}$, the surface S_λ has isolated singularities, so that S_λ is irreducible. Moreover, if $\lambda \neq -4$, then the singularities of S_λ that are contained in the base locus of the pencil $\mathcal{S}$ are all du Val and can be described as follows:

$P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_1$ with quadratic term $x^2 + 2xy + y^2 + yt$;

$P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_1$ with quadratic term $xz + z^2 + 2zt + t^2$;

$P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_1$ with quadratic term $z^2 + yz + 2zt + t^2$;

$P_{\{x\},\{y\},\{z,t\}}$: type $\mathbb{A}_5$ with quadratic term $(\lambda+4)xy$;

$P_{\{z\},\{t\},\{x,y\}}$: type $\mathbb{A}_2$ with quadratic term $z(x + y - (\lambda+4)t)$;

$[\lambda+4 : 0 : -1 : 1]$: type $\mathbb{A}_1$;

$[0 : \lambda+4 : -1 : 1]$: type $\mathbb{A}_1$ for $\lambda \neq -5$, and type $\mathbb{A}_3$ for $\lambda = -5$.

Therefore, it follows from Corollary 1.5.4 that $[\mathsf{f}^{-1}(\lambda)] = 1$ for every $\lambda \neq -4$. Thus, the assertion (♡) in the Main Theorem follows from

Lemma 3.4.2. *One has* $[\mathsf{f}^{-1}(-4)] = 3$.

Proof. The surface S_{-4} has isolated ordinary double singularities at the points $P_{\{x\},\{y\},\{t\}}$, $P_{\{x\},\{z\},\{t\}}$, and $P_{\{y\},\{z\},\{t\}}$, and it has a du Val singularity of type $\mathbb{A}_2$ at the point $P_{\{z\},\{t\},\{x,y\}}$. Thus, using (1.8.3), we see that

$$\left[\mathsf{f}^{-1}(-4)\right] = 1 + \mathbf{D}^{-4}_{P_{\{x\},\{y\},\{z,t\}}}$$

by Lemmas 1.8.5 and 1.12.1. To compute $\mathbf{D}^{-4}_{P_{\{x\},\{y\},\{z,t\}}}$, we have to (partially) describe the birational morphism α in (1.9.3).

In the chart $t = 1$, the surface S_{-4} is given by

$$\overline{y}\,\overline{z}^2 - \overline{x}^2\overline{y} - \overline{x}\,\overline{y}^2 + \left(\overline{x}^2\overline{y}\,\overline{z} + \overline{x}^2\overline{z}^2 + \overline{x}\,\overline{y}^2\overline{z} + 2\overline{x}\,\overline{y}\,\overline{z}^2 + \overline{y}^2\overline{z}^2\right) = 0,$$

where $\overline{x} = x$, $\overline{y} = y$, and $\overline{z} = z + 1$. In particular, the singularity of the surface S_{-4} at the point $P_{\{x\},\{y\},\{z,t\}}$ is not du Val. Since $P_{\{x\},\{y\},\{z,t\}}$ is a singular point of S_{-4} of multiplicity 3, we can use (1.8.3) to conclude that $\mathbf{D}^{-4}_{P_{\{x\},\{y\},\{z,t\}}} > 0$.

Let $\alpha_1 \colon U_1 \to \mathbb{P}^3$ be the blow-up of the point $P_{\{x\},\{y\},\{z,t\}}$. A local chart of this blow-up is given by the coordinate change $\overline{x}_1 = \overline{x}/\overline{z}$, $\overline{y}_1 = \overline{y}/\overline{z}$, $\overline{z}_1 = \overline{z}$. In this chart, the surface $\mathbf{E}_1$ is given by $\overline{z}_1 = 0$, and D^1_λ is given by

$$\begin{aligned}
(\lambda+4)\overline{x}_1\overline{y}_1 + \overline{y}_1\overline{z}_1 - (\lambda+4)\overline{x}_1\overline{y}_1\overline{z}_1&\\
+ \left(-\overline{x}_1^2\overline{y}_1\overline{z}_1 + \overline{x}_1^2\overline{z}_1^2 - \overline{x}_1\overline{y}_1^2\overline{z}_1 + 2\overline{x}_1\overline{y}_1\overline{z}_1^2 + \overline{y}_1^2\overline{z}_1^2\right) &+ \left(\overline{x}_1^2\overline{y}_1\overline{z}_1^2 + \overline{y}_1^2\overline{x}_1\overline{z}_1^2\right) = 0.
\end{aligned}$$

Thus, we see that $D^1_{-4} = S^1_{-4} + \mathbf{E}_1$.

The surface $\mathbf{E}_1$ contains two base curves of the pencil $\mathcal{S}^1$. One of them is given by $\bar{z}_1=\bar{x}_1=0$, and the other, by $\bar{z}_1=\bar{y}_1=0$. Denote the former curve by C_9^1 and the latter by C_{10}^1. Then $\mathbf{M}_9^{-4}=1$ and $\mathbf{M}_{10}^{-4}=2$. Hence, using (1.10.9) and Lemma 1.10.7, we conclude that $\mathbf{D}^{-4}_{P_{\{x\},\{y\},\{z,t\}}}\geq 2$.

Let $\alpha_2\colon U_2\to U_1$ be the blow-up of the point $C_9^1\cap C_{10}^1$. Then $D^2_{-4}=S^2_{-4}+\mathbf{E}_1^2$. Moreover, the surface $\mathbf{E}_2$ contains two base curves of the pencil $\mathcal{S}^2$. Denote them by C_{11}^2 and C_{12}^2. Then $\mathbf{M}_{11}^{-4}=\mathbf{M}_{12}^{-4}=1$. Moreover, one can show that the only base curves of the pencil $\widehat{\mathcal{S}}$ that are mapped to $P_{\{x\},\{y\},\{z,t\}}$ are the curves $\widehat{C}_9$, $\widehat{C}_{10}$, $\widehat{C}_{11}$, and $\widehat{C}_{12}$. Finally, local computations imply that $\mathbf{A}^{-4}_{P_{\{x\},\{y\},\{z,t\}}}=1$. Thus, using (1.10.9) and Lemma 1.10.7 we get $\mathbf{D}^{-4}_{P_{\{x\},\{y\},\{z,t\}}}=2$, so that $[\mathsf{f}^{-1}(-4)]=3$. $\square$

To prove ($\diamondsuit$) in the Main Theorem, we need the following result.

Lemma 3.4.3. *Suppose that $\lambda\neq -4,-5$. Then the intersection form of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{y,t\}}$, $L_{\{z\},\{t\}}$, $L_{\{t\},\{x,y\}}$, and H_λ on the surface S_λ is given by*

$$\begin{array}{c}
\\
\begin{array}{r}L_{\{x\},\{y\}}\\ L_{\{x\},\{y,t\}}\\ L_{\{z\},\{t\}}\\ L_{\{t\},\{x,y\}}\\ H_\lambda\end{array}
\end{array}
\begin{array}{c}
\begin{array}{ccccc}L_{\{x\},\{y\}} & L_{\{x\},\{y,t\}} & L_{\{z\},\{t\}} & L_{\{t\},\{x,y\}} & H_\lambda\end{array}\\
\begin{pmatrix}
-1/6 & 1/2 & 0 & 1/2 & 1\\
1/2 & -3/2 & 0 & 1/2 & 1\\
0 & 0 & -5/6 & 1/3 & 1\\
1/2 & 1/2 & 1/3 & -5/6 & 1\\
1 & 1 & 1 & 1 & 4
\end{pmatrix}
\end{array}.$$

Proof. First, let us compute non-diagonal entries. Since $L_{\{x\},\{y\}}\cap L_{\{x\},\{y,t\}}=P_{\{x\},\{y\},\{t\}}$ and $P_{\{x\},\{y\},\{t\}}$ is an ordinary double point of the surface S_λ, we get $L_{\{x\},\{y\}}\cdot L_{\{x\},\{y,t\}}=1/2$ by Proposition A.1.2. Similarly, we have $L_{\{x\},\{y\}}\cdot L_{\{t\},\{x,y\}}=L_{\{x\},\{y,t\}}\cdot L_{\{t\},\{x,y\}}=1/2$.

Since $L_{\{x\},\{y\}}\cap L_{\{z\},\{t\}}=L_{\{x\},\{y,t\}}\cap L_{\{z\},\{t\}}=\varnothing$, we have

$$L_{\{x\},\{y\}}\cdot L_{\{z\},\{t\}}=L_{\{x\},\{y,t\}}\cdot L_{\{z\},\{t\}}=0.$$

To compute $L_{\{z\},\{t\}}\cdot L_{\{t\},\{x,y\}}$, observe that $L_{\{z\},\{t\}}\cap L_{\{t\},\{x,y\}}=P_{\{z\},\{t\},\{x,y\}}$. Moreover, the surface S_λ has a du Val singularity of type $\mathbb{A}_2$ at the point $P_{\{z\},\{t\},\{x,y\}}$. Furthermore, the quadratic term of its defining equation at this point is $z(x+y-(\lambda+4)t)$. Thus, using Remark A.2.4 with $S=S_\lambda$, $O=P_{\{z\},\{t\},\{x,y\}}$, $n=3$, $C=L_{\{z\},\{t\}}$, and $Z=L_{\{t\},\{x,y\}}$, we see that $\widetilde{C}$ and $\widetilde{Z}$ intersect different curves among G_1 and G_2. Then $L_{\{z\},\{t\}}\cdot L_{\{t\},\{x,y\}}=1/3$ by Proposition A.1.2.

Now let us compute the diagonal entries. Since $P_{\{x\},\{y\},\{t\}}$ and $P_{\{z\},\{t\},\{x,y\}}$ are the only singular points of S_λ that are contained in the line $L_{\{t\},\{x,y\}}$, we see that

$$L^2_{\{z\},\{t\}}=-2+\frac{1}{2}+\frac{2}{3}=-\frac{5}{6}$$

by Proposition A.1.3. Similarly, we have $L^2_{\{t\},\{x,y\}}=-5/6$. We also have $L^2_{\{x\},\{y,t\}}=-3/2$, because $P_{\{x\},\{y\},\{t\}}$ is the only singular point of S_λ that is contained in $L_{\{x\},\{y,t\}}$.

To compute $L^2_{\{x\},\{y\}}$, let us use the notation of the proof of Lemma 3.4.2. Note that the proper transform of the line $L_{\{x\},\{y\}}$ on the surface S^1_λ passes through the point $C_9^1\cap C_{10}^1$. On the other hand, its proper transform on the surface S^2_λ does not pass through the intersection of C_{11}^2 and C_{12}^2. Applying Remark A.2.4 with $S=S_\lambda$, $O=P_{\{x\},\{y\},\{z,t\}}$, $n=5$, and $C=L_{\{x\},\{y\}}$, we see that $\widetilde{C}$ intersects either the curve G_2 or the curve G_4. Thus, it follows from Proposition A.1.3 that $L^2_{\{x\},\{y\}}=-1/6$, because $P_{\{x\},\{y\},\{z,t\}}$ and $P_{\{x\},\{y\},\{t\}}$ are the only singular points of S_λ contained in the line $L_{\{x\},\{y\}}$. $\square$

The determinant of the matrix in Lemma 3.4.3 is $-16/9$. Thus, if $\lambda\neq-4,-5$, then it follows from (3.4.1) that the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{y,t\}}$, $L_{\{x\},\{z,t\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{t\},\{x,y\}}$, $\mathcal{C}_1$, and $\mathcal{C}_2$ also has rank 5. On the other hand, one can easily see that

$\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk}) = \operatorname{rk}\operatorname{Pic}(S_{\Bbbk}) + 12$. Hence, we conclude that ($\bigstar$) holds, so that ($\diamondsuit$) in the Main Theorem also holds by Lemma 1.13.1.

Family 3.5. The threefold X can be obtained by blowing up $\mathbb{P}^1 \times \mathbb{P}^2$ along a smooth rational curve of bidegree $(5, 2)$. Then $h^{1,2}(X) = 0$. A toric Landau–Ginzburg model of this family is given by the Minkowski polynomial 1819, which is

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{2y}{x} + \frac{2x}{y} + \frac{y}{z} + \frac{x}{z} + \frac{yz}{x} + z + \frac{y^2}{x} + 3y + 3x + \frac{x^2}{y}.$$

The quartic pencil $\mathcal{S}$ is given by

$$t^2xy + t^2xz + t^2yz + tx^2y + 2tx^2z + txy^2 + 2ty^2z + x^3z \\ + 3x^2yz + 3xy^2z + xyz^2 + y^3z + y^2z^2 = \lambda xyzt.$$

Suppose that $\lambda \neq \infty$. Then S_λ has isolated singularities, so that it is irreducible.

Let $\mathcal{C}_1$ be the conic in $\mathbb{P}^3$ that is given by $x = (y+t)^2 + yz = 0$, and let $\mathcal{C}_2$ be the conic given by $t = (x+y)^2 + yz = 0$. Then

$$\begin{aligned} H_{\{x\}} \cdot S_\lambda &= L_{\{x\},\{y\}} + L_{\{x\},\{z\}} + \mathcal{C}_1, \\ H_{\{y\}} \cdot S_\lambda &= L_{\{x\},\{y\}} + L_{\{y\},\{z\}} + 2L_{\{y\},\{x,t\}}, \\ H_{\{z\}} \cdot S_\lambda &= L_{\{x\},\{z\}} + L_{\{y\},\{z\}} + L_{\{z\},\{t\}} + L_{\{z\},\{x,y,t\}}, \\ H_{\{t\}} \cdot S_\lambda &= L_{\{z\},\{t\}} + L_{\{t\},\{x,y\}} + \mathcal{C}_2. \end{aligned}$$

Therefore, the base locus of the pencil $\mathcal{S}$ consists of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{y\},\{z\}}$, $L_{\{z\},\{t\}}$, $L_{\{y\},\{x,t\}}$, $L_{\{t\},\{x,y\}}$, $L_{\{z\},\{x,y,t\}}$, $\mathcal{C}_1$, and $\mathcal{C}_2$.

For every $\lambda \in \mathbb{C}$, the singular points of S_λ contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

$$\begin{aligned} P_{\{x\},\{y\},\{z\}}&\colon \text{ type } \mathbb{A}_1; \\ P_{\{x\},\{y\},\{t\}}&\colon \text{ type } \mathbb{A}_4 \text{ for } \lambda \neq -4, \text{ and type } \mathbb{A}_5 \text{ for } \lambda = -4; \\ P_{\{x\},\{z\},\{y,t\}}&\colon \text{ type } \mathbb{A}_1; \\ P_{\{y\},\{z\},\{x,t\}}&\colon \text{ type } \mathbb{A}_2 \text{ for } \lambda \neq -4, \text{ and type } \mathbb{A}_4 \text{ for } \lambda = -4; \\ P_{\{z\},\{t\},\{x,y\}}&\colon \text{ type } \mathbb{A}_2 \text{ for } \lambda \neq -4, \text{ and type } \mathbb{A}_3 \text{ for } \lambda = -4; \\ [1:0:\lambda+4:-1]&\colon \text{ type } \mathbb{A}_1 \text{ for } \lambda \neq -4. \end{aligned}$$

In particular, it follows from Corollary 1.5.4 that $\mathsf{f}^{-1}(\lambda)$ is irreducible for every $\lambda \neq \infty$. Thus, since $h^{1,2}(X) = 0$, we see that ($\heartsuit$) in the Main Theorem holds in this case.

Lemma 3.5.1. *Suppose that $\lambda \neq -4$. Then the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{y\},\{z\}}$, $L_{\{z\},\{t\}}$, $L_{\{y\},\{x,t\}}$, $L_{\{t\},\{x,y\}}$, and H_λ on the surface S_λ is given by*

$$\begin{array}{r} \\ L_{\{x\},\{y\}} \\ L_{\{x\},\{z\}} \\ L_{\{y\},\{z\}} \\ L_{\{z\},\{t\}} \\ L_{\{y\},\{x,t\}} \\ L_{\{t\},\{x,y\}} \\ H_\lambda \end{array} \begin{array}{c} \begin{array}{ccccccc} L_{\{x\},\{y\}} & L_{\{x\},\{z\}} & L_{\{y\},\{z\}} & L_{\{z\},\{t\}} & L_{\{y\},\{x,t\}} & L_{\{t\},\{x,y\}} & H_\lambda \end{array} \\ \begin{pmatrix} -3/10 & 1/2 & 1/2 & 0 & 2/5 & 3/5 & 1 \\ 1/2 & -1 & 1/2 & 1 & 0 & 0 & 1 \\ 1/2 & 1/2 & -5/6 & 1 & 2/3 & 0 & 1 \\ 0 & 1 & 1 & -4/3 & 0 & 2/3 & 1 \\ 2/5 & 0 & 2/3 & 0 & -1/30 & 1/5 & 1 \\ 3/5 & 0 & 0 & 2/3 & 1/5 & -8/15 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 4 \end{pmatrix} \end{array}.$$

Proof. Observe that $L_{\{x\},\{y\}} \cap L_{\{z\},\{t\}} = \varnothing$, so that $L_{\{x\},\{y\}} \cdot L_{\{z\},\{t\}} = 0$. Similarly, we see that $L_{\{x\},\{z\}} \cdot L_{\{z\},\{x,y,t\}} = 0$, $L_{\{x\},\{z\}} \cdot L_{\{t\},\{x,y\}} = 0$, and $L_{\{y\},\{z\}} \cdot L_{\{t\},\{x,y\}} = 0$. Since $L_{\{x\},\{z\}} \cap L_{\{z\},\{t\}} = P_{\{x\},\{z\},\{t\}}$ and S_λ is smooth at $P_{\{x\},\{z\},\{t\}}$, we have $L_{\{x\},\{z\}} \cdot L_{\{z\},\{t\}} = 1$. Similarly, we have $L_{\{y\},\{z\}} \cdot L_{\{z\},\{t\}} = 1$.

The points $P_{\{x\},\{y\},\{z\}}$ and $P_{\{x\},\{z\},\{y,t\}}$ are the only singular points of S_λ that are contained in $L_{\{x\},\{z\}}$. Thus, we have $L^2_{\{x\},\{z\}} = -1$ by Proposition A.1.3. Similarly, we see that $L^2_{\{y\},\{z\}} = -5/6$, because $P_{\{x\},\{y\},\{z\}}$ and $P_{\{y\},\{z\},\{x,t\}}$ are the only singular points of the surface S_λ that are contained in $L_{\{y\},\{z\}}$. Similarly, we have $L^2_{\{z\},\{t\}} = -4/3$, because $P_{\{z\},\{t\},\{x,y\}}$ is the only singular point of S_λ contained in $L_{\{z\},\{t\}}$.

Since $L_{\{x\},\{y\}} \cap L_{\{x\},\{z\}} = P_{\{x\},\{y\},\{z\}}$, we have $L_{\{x\},\{y\}} \cdot L_{\{x\},\{z\}} = 1/2$ by Proposition A.1.2. Similarly, we have $L_{\{x\},\{y\}} \cdot L_{\{y\},\{z\}} = 1/2$.

Let us show that $L_{\{y\},\{z\}} \cdot L_{\{y\},\{x,t\}} = 2/3$. To this end, let us use the notation of the Appendix with $S = S_\lambda$, $O = P_{\{y\},\{z\},\{x,t\}}$, $n = 2$, $C = L_{\{y\},\{x,t\}}$, and $Z = L_{\{y\},\{z\}}$. We may assume that $\widetilde{C} \cap E_1 \neq \varnothing$. If $\widetilde{Z} \cap E_1 \neq \varnothing$, then $L_{\{y\},\{z\}} \cdot L_{\{y\},\{x,t\}} = 2/3$ by Proposition A.1.2. Otherwise, we have $L_{\{y\},\{z\}} \cdot L_{\{y\},\{x,t\}} = 1/3$. In the chart $t = 1$, the surface S_λ is given by

$$\overline{y}\big(\overline{x} + \overline{y} - (\lambda+4)\overline{z}\big) + \text{Higher order terms} = 0,$$

where $\overline{x} = x + 1$, $\overline{y} = y$, and $\overline{z} = z$. Here $O = (0,0,0)$. In these coordinates, the line $L_{\{y\},\{x,t\}}$ is given by $\overline{y} = \overline{x} = 0$, and the line $L_{\{y\},\{z\}}$ is given by $\overline{y} = \overline{z} = 0$. This shows that $\widetilde{Z} \cap E_1 \neq \varnothing$, so that $L_{\{y\},\{z\}} \cdot L_{\{y\},\{x,t\}} = 2/3$.

Let us compute $L^2_{\{y\},\{x,t\}}$, $L^2_{\{x\},\{y\}}$, and $L_{\{x\},\{y\}} \cdot L_{\{y\},\{x,t\}}$. Using Remark A.2.4 with $S = S_\lambda$, $O = P_{\{x\},\{y\},\{t\}}$, $n = 4$, $C = L_{\{y\},\{x,t\}}$, and $Z = L_{\{x\},\{y\}}$, we see that $\overline{C}$ does not pass through the point $\overline{G}_1 \cap \overline{G}_4$, and $\overline{Z}$ passes through the point $\overline{G}_1 \cap \overline{G}_4$. Now, using Proposition A.1.3, we obtain

$$L^2_{\{y\},\{x,t\}} = -2 + \frac{1}{2} + \frac{2}{3} + \frac{4}{5} = -\frac{1}{30},$$

because $P_{\{x\},\{y\},\{t\}}$, $P_{\{y\},\{z\},\{x,t\}}$, and $[1:0:\lambda+4:-1]$ are the only singular points of S_λ that are contained in the line $L_{\{y\},\{x,t\}}$. Similarly, we get

$$L^2_{\{x\},\{y\}} = -2 + \frac{1}{2} + \frac{6}{5} = -\frac{3}{10},$$

because $P_{\{x\},\{y\},\{z\}}$ and $P_{\{x\},\{y\},\{t\}}$ are the only singular points of S_λ that are contained in the line $L_{\{x\},\{y\}}$. Moreover, using Proposition A.1.2, we see that either $L_{\{x\},\{y\}} \cdot L_{\{y\},\{x,t\}} = 2/5$ or $L_{\{x\},\{y\}} \cdot L_{\{y\},\{x,t\}} = 3/5$. In fact, we have $L_{\{x\},\{y\}} \cdot L_{\{y\},\{x,t\}} = 2/5$, because

$$\begin{aligned} 1 = H_\lambda \cdot L_{\{y\},\{x,t\}} &= \big(L_{\{x\},\{y\}} + L_{\{y\},\{z\}} + 2L_{\{y\},\{x,t\}}\big) \cdot L_{\{y\},\{x,t\}} \\ &= L_{\{x\},\{y\}} \cdot L_{\{y\},\{x,t\}} + L_{\{y\},\{z\}} \cdot L_{\{y\},\{x,t\}} + 2L^2_{\{y\},\{x,t\}} = L_{\{x\},\{y\}} \cdot L_{\{y\},\{x,t\}} + \frac{3}{5}, \end{aligned}$$

since $H_\lambda \sim L_{\{x\},\{y\}} + L_{\{y\},\{z\}} + 2L_{\{y\},\{x,t\}}$ on the surface S_λ.

To complete the proof of the lemma, we must find $L_{\{t\},\{x,y\}} \cdot L_{\{x\},\{y\}}$, $L_{\{t\},\{x,y\}} \cdot L_{\{z\},\{t\}}$, $L_{\{t\},\{x,y\}} \cdot L_{\{y\},\{x,t\}}$, and $L^2_{\{t\},\{x,y\}}$. Observe that $P_{\{x\},\{y\},\{t\}}$ and $P_{\{z\},\{t\},\{x,y\}}$ are the only singular points of S_λ that are contained in the line $L_{\{t\},\{x,y\}}$. Thus, since $L_{\{t\},\{x,y\}} \cap L_{\{z\},\{t\}} = P_{\{z\},\{t\},\{x,y\}}$, we get $L_{\{t\},\{x,y\}} \cdot L_{\{z\},\{t\}} = 2/3$ by Proposition A.1.2.

To find the remaining entries of the intersection matrix, let us use Remark A.2.4 with $S = S_\lambda$, $O = P_{\{x\},\{y\},\{t\}}$, $n = 4$, $C = L_{\{y\},\{x,t\}}$, and $Z = L_{\{t\},\{x,y\}}$. As we have already checked above, the curve $\overline{C}$ does not pass through the point $\overline{G}_1 \cap \overline{G}_4$. Similarly, the curve $\overline{Z}$ does not pass through this point, so we may assume that $\widetilde{C} \cap G_1 \neq \varnothing$ and $\widetilde{Z} \cap G_4 \neq \varnothing$. Hence, by Proposition A.1.2 we have

$L_{\{t\},\{x,y\}} \cdot L_{\{y\},\{x,t\}} = 1/5$. Similarly, it follows from Proposition A.1.3 that $L^2_{\{t\},\{x,y\}} = -8/15$. This gives $L_{\{x\},\{y\}} \cdot L_{\{t\},\{x,y\}} = 3/5$, because

$$\begin{aligned}1 = H_\lambda \cdot L_{\{t\},\{x,y\}} &= \big(L_{\{x\},\{y\}} + L_{\{y\},\{z\}} + 2L_{\{y\},\{x,t\}}\big) \cdot L_{\{t\},\{x,y\}}\\ &= L_{\{x\},\{y\}} \cdot L_{\{t\},\{x,y\}} + L_{\{y\},\{z\}} \cdot L_{\{t\},\{x,y\}} + 2L^2_{\{t\},\{x,y\}} = L_{\{x\},\{y\}} \cdot L_{\{t\},\{x,y\}} + \frac{2}{5},\end{aligned}$$

since $L_{\{x\},\{y\}} + L_{\{y\},\{z\}} + 2L_{\{y\},\{x,t\}} \sim H_\lambda$. $\square$

The matrix in Lemma 3.5.1 has rank 6. Moreover, we have $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk}) = \operatorname{rk}\operatorname{Pic}(S_{\Bbbk}) + 11$. Hence, we see that ($\bigstar$) holds, so that ($\diamondsuit$) in the Main Theorem also holds by Lemma 1.13.1.

Family 3.6. In this case, the Main Theorem is proved in Example 1.14.1.

Family 3.7. In this case, the threefold X can be obtained by blowing up a hypersurface of bidegree $(1,1)$ in $\mathbb{P}^2 \times \mathbb{P}^2$ along a smooth elliptic curve, so that $h^{1,2}(X) = 1$. A toric Landau–Ginzburg model of the threefold X is given by

$$x + y + z + \frac{y}{z} + \frac{y}{x} + \frac{z}{y} + \frac{z}{x} + \frac{1}{z} + \frac{y}{xz} + \frac{1}{y} + \frac{2}{x} + \frac{z}{xy} + \frac{1}{xz} + \frac{1}{xy},$$

which is the Minkowski polynomial 2354.2. The pencil $\mathcal{S}$ is given by

$$\begin{aligned}&x^2yz + xy^2z + xyz^2 + xy^2t + y^2zt + xz^2t + yz^2t + xyt^2 + y^2t^2\\ &\qquad\qquad + xt^2z + 2yzt^2 + z^2t^2 + yt^3 + zt^3 = \lambda xyzt.\end{aligned}$$

As usual, we suppose that $\lambda \neq \infty$.

For every $\lambda \neq -3$, the surface S_λ has isolated singularities, so that S_λ is irreducible. On the other hand, one has $S_{-3} = H_{\{x,t\}} + \mathsf{S}$, where S is an irreducible cubic surface given by

$$xyz + yt^2 + zt^2 + y^2t + z^2t + 2yzt + y^2z + yz^2 = 0.$$

To describe the base locus of the pencil $\mathcal{S}$, we observe that

$$\begin{aligned}H_{\{x\}} \cdot S_\lambda &= L_{\{x\},\{t\}} + L_{\{x\},\{y,z\}} + L_{\{x\},\{y,t\}} + L_{\{x\},\{z,t\}},\\ H_{\{y\}} \cdot S_\lambda &= L_{\{y\},\{z\}} + L_{\{y\},\{t\}} + L_{\{y\},\{x,t\}} + L_{\{y\},\{z,t\}},\\ H_{\{z\}} \cdot S_\lambda &= L_{\{y\},\{z\}} + L_{\{z\},\{t\}} + L_{\{z\},\{x,t\}} + L_{\{z\},\{y,t\}},\\ H_{\{t\}} \cdot S_\lambda &= L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y,z\}}.\end{aligned} \tag{3.7.1}$$

Thus, the lines $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{y,z\}}$, $L_{\{x\},\{y,t\}}$, $L_{\{x\},\{z,t\}}$, $L_{\{y\},\{x,t\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{z\},\{x,t\}}$, $L_{\{z\},\{y,t\}}$, and $L_{\{t\},\{x,y,z\}}$ are all base curves of the pencil $\mathcal{S}$.

If $\lambda \neq -2, -3$, then the singular points of S_λ contained in the base locus of the pencil $\mathcal{S}$ are all du Val and can be described as follows:

- $P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term yz;
- $P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_2$ with quadratic term $(x+t)(z+t)$;
- $P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_2$ with quadratic term $(x+t)(y+t)$;
- $P_{\{x\},\{t\},\{y,z\}}$: type $\mathbb{A}_1$ with quadratic term $(x+y+z)(x+t) - (\lambda+3)xt$;
- $P_{\{y\},\{z\},\{x,t\}}$: type $\mathbb{A}_1$ with quadratic term $(x+t)(y+z) + (\lambda+3)yz$.

Thus, it follows from Corollary 1.5.4 that $[\mathsf{f}^{-1}(\lambda)] = 1$ for every $\lambda \neq -3, -2$.

The surface S_{-2} has the same singularities at the points $P_{\{y\},\{z\},\{t\}}$, $P_{\{x\},\{z\},\{t\}}$, $P_{\{x\},\{y\},\{t\}}$, $P_{\{x\},\{t\},\{y,z\}}$, and $P_{\{y\},\{z\},\{x,t\}}$. In addition to them, it also has isolated ordinary double singularities at the points $[0:-1:1:1]$, $[0:1:-1:1]$, and $[0:1:1:-1]$. Thus, using Corollary 1.5.4, we conclude that $[\mathsf{f}^{-1}(-2)]=1$.

The surface S_{-3} has good double points at $P_{\{y\},\{z\},\{t\}}$, $P_{\{x\},\{z\},\{t\}}$, $P_{\{x\},\{y\},\{t\}}$, $P_{\{x\},\{t\},\{y,z\}}$, and $P_{\{y\},\{z\},\{x,t\}}$, and it is smooth at general points of the lines $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{y,z\}}$, $L_{\{x\},\{y,t\}}$, $L_{\{x\},\{z,t\}}$, $L_{\{y\},\{x,t\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{z\},\{x,t\}}$, $L_{\{z\},\{y,t\}}$, and $L_{\{t\},\{x,y,z\}}$. Thus, it follows from (1.8.3) and Lemmas 1.8.5 and 1.12.1 that $[\mathsf{f}^{-1}(-3)]=[S_{-3}]=2$. Hence, we see that ($\heartsuit$) in the Main Theorem holds in this case, because $h^{1,2}(X)=1$.

To prove ($\diamondsuit$) in the Main Theorem, we may assume that $\lambda\neq-2,-3$. Then

$$H_{\{x,t\}}\cdot S_\lambda=2L_{\{x\},\{t\}}+L_{\{y\},\{x,t\}}+L_{\{z\},\{x,t\}},$$

so that $2L_{\{x\},\{t\}}+L_{\{y\},\{x,t\}}+L_{\{z\},\{x,t\}}\sim H_\lambda$ on the surface S_λ. It follows from (3.7.1) that the intersection matrix of the lines $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{y,z\}}$, $L_{\{x\},\{y,t\}}$, $L_{\{x\},\{z,t\}}$, $L_{\{y\},\{x,t\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{z\},\{x,t\}}$, $L_{\{z\},\{y,t\}}$, and $L_{\{t\},\{x,y,z\}}$ on the surface S_λ has the same rank as the intersection matrix of the curves $L_{\{x\},\{y,z\}}$, $L_{\{x\},\{y,t\}}$, $L_{\{x\},\{z,t\}}$, $L_{\{y\},\{x,t\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{z\},\{y,t\}}$, $L_{\{t\},\{x,y,z\}}$, and H_λ. The latter matrix is given by

$$\begin{array}{c c}
 & \begin{array}{cccccccc} L_{\{x\},\{y,z\}} & L_{\{x\},\{y,t\}} & L_{\{x\},\{z,t\}} & L_{\{y\},\{x,t\}} & L_{\{y\},\{z,t\}} & L_{\{z\},\{y,t\}} & L_{\{t\},\{x,y,z\}} & H_\lambda \end{array} \\
\begin{array}{c} L_{\{x\},\{y,z\}} \\ L_{\{x\},\{y,t\}} \\ L_{\{x\},\{z,t\}} \\ L_{\{y\},\{x,t\}} \\ L_{\{y\},\{z,t\}} \\ L_{\{z\},\{y,t\}} \\ L_{\{t\},\{x,y,z\}} \\ H_\lambda \end{array} &
\begin{pmatrix}
-3/2 & 1 & 1 & 0 & 0 & 0 & 1/2 & 1\\
1 & -4/3 & 1 & 1/3 & 10 & 1 & 0 & 1\\
1 & 1 & -4/3 & 0 & 1 & 0 & 0 & 1\\
0 & 1/3 & 0 & -5/6 & 1 & 0 & 0 & 1\\
0 & 0 & 1 & 1 & -5/4 & 1/4 & 0 & 1\\
0 & 1 & 0 & 0 & 1/4 & -5/4 & 0 & 1\\
1/2 & 0 & 0 & 0 & 0 & 0 & -3/2 & 1\\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 4
\end{pmatrix}
\end{array}.$$

This matrix has rank 8, and $\mathrm{rk}\,\mathrm{Pic}(\widetilde{S}_{\Bbbk})=\mathrm{rk}\,\mathrm{Pic}(S_{\Bbbk})+9$. Hence, we see that ($\bigstar$) holds, so that ($\diamondsuit$) in the Main Theorem also holds by Lemma 1.13.1.

Family 3.8. For a description of the threefold X, see [8]. In this case, we have $h^{1,2}(X)=0$, and a toric Landau–Ginzburg model of the threefold X is given by

$$x+y+z+\frac{xz}{y}+\frac{x}{y}+\frac{y}{x}+\frac{z}{y}+\frac{1}{z}+\frac{2}{y}+\frac{2}{x}+\frac{1}{xz}+\frac{1}{xy},$$

which is the Minkowski polynomial 1504. The pencil $\mathcal{S}$ is given by

$$x^2yz+xy^2z+x^2z^2+xyz^2+x^2zt+y^2zt+xz^2t+xyt^2+2xzt^2+2yzt^2+yt^3+zt^3=\lambda xyzt.$$

Suppose that $\lambda\neq\infty$. Then S_λ has isolated singularities, so that it is irreducible.

Let $\mathcal{C}_1$ be a plane cubic curve given by $x=y^2z+2yzt+yt^2+zt^2=0$. Then $\mathcal{C}_1$ is singular at $P_{\{x\},\{y\},\{t\}}$. Let $\mathcal{C}_2$ be a conic given by $y=xz+xt+t^2=0$. Then

$$\begin{aligned}
H_{\{x\}}\cdot S_\lambda&=L_{\{x\},\{t\}}+\mathcal{C}_1,\\
H_{\{y\}}\cdot S_\lambda&=L_{\{y\},\{z\}}+L_{\{y\},\{x,t\}}+\mathcal{C}_2,\\
H_{\{z\}}\cdot S_\lambda&=L_{\{y\},\{z\}}+2L_{\{z\},\{t\}}+L_{\{z\},\{x,t\}},\\
H_{\{t\}}\cdot S_\lambda&=L_{\{x\},\{t\}}+L_{\{z\},\{t\}}+L_{\{t\},\{x,y\}}+L_{\{t\},\{y,z\}}.
\end{aligned}\tag{3.8.1}$$

Thus, the base locus of the pencil $\mathcal{S}$ consists of the curves $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{z\},\{t\}}$, $L_{\{y\},\{x,t\}}$, $L_{\{z\},\{x,t\}}$, $L_{\{t\},\{x,y\}}$, $L_{\{t\},\{y,z\}}$, $\mathcal{C}_1$, and $\mathcal{C}_2$.

For every $\lambda \in \mathbb{C}$, the singular points of S_λ contained in the base locus of the pencil $\mathcal{S}$ are du Val and can be described as follows:

$P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term $x(x+y+t)$;

$P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term $z(x+t)$ for $\lambda \neq -3$, and type $\mathbb{A}_4$ for $\lambda = -3$;

$P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_2$ with quadratic term $z(y+z+t)$;

$P_{\{y\},\{z\},\{x,t\}}$: type $\mathbb{A}_2$ with quadratic term $y(x+3z+\lambda z+t)$ for $\lambda \neq -3,-4$, and type $\mathbb{A}_3$ for $\lambda \in \{-3,-4\}$;

$P_{\{z\},\{t\},\{x,y\}}$: type $\mathbb{A}_1$;

$P_{\{t\},\{x,y\},\{y,z\}}$: smooth point for $\lambda \neq -3$, and type $\mathbb{A}_1$ for $\lambda = 3$.

Thus, it follows from Corollary 1.5.4 that the fiber $\mathsf{f}^{-1}(\lambda)$ is irreducible for every $\lambda \neq \infty$. Hence, we see that ($\heartsuit$) in the Main Theorem holds in this case.

To prove ($\diamondsuit$) in the Main Theorem, we may assume that $\lambda \neq -3,-4$. Then it follows from (3.8.1) that the intersection matrix of the curves $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{x,t\}}$, $L_{\{y\},\{z\}}$, $L_{\{z\},\{t\}}$, $L_{\{z\},\{x,t\}}$, $L_{\{x\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{t\},\{x,y\}}$, $L_{\{t\},\{y,z\}}$, $\mathcal{C}_1$, and $\mathcal{C}_2$ on the surface S_λ has the same rank as the intersection matrix of the curves $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{x,t\}}$, $L_{\{t\},\{x,y\}}$, $L_{\{t\},\{y,z\}}$, and H_λ. The latter matrix is given by

$$
\begin{array}{c c}
 & \begin{array}{cccccc} L_{\{x\},\{t\}} & L_{\{y\},\{z\}} & L_{\{y\},\{x,t\}} & L_{\{t\},\{x,y\}} & L_{\{t\},\{y,z\}} & H_\lambda \end{array} \\
\begin{array}{c} L_{\{x\},\{t\}} \\ L_{\{y\},\{z\}} \\ L_{\{y\},\{x,t\}} \\ L_{\{t\},\{x,y\}} \\ L_{\{t\},\{y,z\}} \\ H_\lambda \end{array} &
\begin{pmatrix}
-1/2 & 0 & 1/4 & 1/4 & 0 & 1 \\
0 & -2/3 & 2/3 & 0 & 1/3 & 1 \\
1/4 & 2/3 & -7/12 & 3/4 & 0 & 1 \\
1/4 & 0 & 3/4 & -3/4 & 1 & 1 \\
0 & 1/3 & 0 & 1 & -4/3 & 1 \\
1 & 1 & 1 & 1 & 1 & 4
\end{pmatrix}
\end{array}.
$$

This matrix has rank 6, and $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk}) = \operatorname{rk}\operatorname{Pic}(S_{\Bbbk}) + 11$. Hence, we see that ($\bigstar$) holds, so that ($\diamondsuit$) in the Main Theorem also holds by Lemma 1.13.1.

Family 3.9. In this case, the threefold X is a blow-up of a cone over a Veronese surface in $\mathbb{P}^5$ in a disjoint union of the vertex and a smooth curve of genus 3. Thus, we have $h^{1,2}(X) = 3$. A toric Landau–Ginzburg model of this family is given by

$$x + y + z + \frac{x^2}{yz} + \frac{y}{x} + \frac{z}{x} + \frac{2x}{yz} + \frac{1}{x} + \frac{1}{yz},$$

which is the Minkowski polynomial 373. The pencil $\mathcal{S}$ is given by

$$x^2yz + xy^2z + xyz^2 + x^3t + y^2zt + yz^2t + 2x^2t^2 + yzt^2 + xt^3 = \lambda xyzt.$$

As usual, we assume that $\lambda \neq \infty$.

If $\lambda \neq -2$, then the surface S_λ has isolated singularities, so that it is irreducible. But $S_{-2} = H_{\{x,t\}} + \mathbf{S}$, where $\mathbf{S}$ is an irreducible cubic surface given by $xt^2 + x^2t + yzt + y^2z + yz^2 + xyz = 0$. The surface $\mathbf{S}$ has isolated singularities, and $H_{\{x,t\}} \cdot \mathbf{S} = L_{\{y\},\{x,t\}} + L_{\{z\},\{x,t\}} + L_{\{x,t\},\{y,z\}}$.

To describe the base locus of the pencil $\mathcal{S}$, we observe that

$$H_{\{x\}} \cdot S_\lambda = L_{\{x\},\{y\}} + L_{\{x\},\{z\}} + L_{\{x\},\{t\}} + L_{\{x\},\{y,z,t\}},$$

$$H_{\{y\}} \cdot S_\lambda = L_{\{x\},\{y\}} + L_{\{y\},\{t\}} + 2L_{\{y\},\{x,t\}},$$

$$H_{\{z\}} \cdot S_\lambda = L_{\{x\},\{z\}} + L_{\{z\},\{t\}} + 2L_{\{z\},\{x,t\}},$$

$$H_{\{t\}} \cdot S_\lambda = L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y,z\}}.$$

Therefore, the lines $L_{\{x\},\{z\}}$, $L_{\{x\},\{y\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{z\},\{x,t\}}$, $L_{\{y\},\{x,t\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, and $L_{\{t\},\{x,y,z\}}$ are all base curves of the pencil $\mathcal{S}$.

If $\lambda \neq -2$, then the singular points of S_λ contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

$P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_5$ with quadratic term $z(x+t)$;

$P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_5$ with quadratic term $z(y+t)$;

$P_{\{x\},\{t\},\{y,z\}}$: type $\mathbb{A}_1$ with quadratic term $x^2+xy+xz+yt+zt+t^2+\lambda xt$;

$P_{\{y\},\{z\},\{x,t\}}$: type $\mathbb{A}_1$ with quadratic term $(\lambda+2)yz-(x+t)^2$.

Thus, it follows from Corollary 1.5.4 that the fiber $\mathsf{f}^{-1}(\lambda)$ is irreducible for every $\lambda \neq -2$.

Note that the surface S_{-2} consists of two irreducible components, and it is singular along the lines $L_{\{y\},\{x,t\}}$ and $L_{\{z\},\{x,t\}}$. Thus, it follows from (1.8.3) and Lemma 1.8.5 that

$$\big[\mathsf{f}^{-1}(-2)\big] = 4 + \mathbf{D}^{-2}_{P_{\{x\},\{z\},\{t\}}} + \mathbf{D}^{-2}_{P_{\{x\},\{y\},\{t\}}} + \mathbf{D}^{-2}_{P_{\{x\},\{t\},\{y,z\}}} + \mathbf{D}^{-2}_{P_{\{y\},\{z\},\{x,t\}}}.$$

Moreover, the surface S_{-2} has good double points at $P_{\{x\},\{z\},\{t\}}$, $P_{\{x\},\{y\},\{t\}}$, $P_{\{x\},\{t\},\{y,z\}}$, and $P_{\{y\},\{z\},\{x,t\}}$. By Lemma 1.12.1, this implies

$$\mathbf{D}^{-2}_{P_{\{x\},\{z\},\{t\}}} = \mathbf{D}^{-2}_{P_{\{x\},\{y\},\{t\}}} = \mathbf{D}^{-2}_{P_{\{x\},\{t\},\{y,z\}}} = \mathbf{D}^{-2}_{P_{\{y\},\{z\},\{x,t\}}} = 0,$$

so that $[\mathsf{f}^{-1}(-2)] = 4$. Hence, we see that ($\heartsuit$) in the Main Theorem holds in this case.

If $\lambda \neq -2$, then the intersection matrix of the lines $L_{\{x\},\{z\}}$, $L_{\{x\},\{y\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{z\},\{x,t\}}$, $L_{\{y\},\{x,t\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, and $L_{\{t\},\{x,y,z\}}$ on the surface S_λ has the same rank as the intersection matrix of the curves $L_{\{x\},\{z\}}$, $L_{\{x\},\{y,z,t\}}$, $L_{\{y\},\{x,t\}}$, $L_{\{z\},\{x,t\}}$, $L_{\{t\},\{x,y,z\}}$, and H_λ. The latter matrix is given by

$$\begin{array}{c} \\ L_{\{x\},\{z\}} \\ L_{\{x\},\{y,z,t\}} \\ L_{\{y\},\{x,t\}} \\ L_{\{z\},\{x,t\}} \\ L_{\{t\},\{x,y,z\}} \\ H_\lambda \end{array}
\begin{array}{c} \begin{array}{cccccc} L_{\{x\},\{z\}} & L_{\{x\},\{y,z,t\}} & L_{\{y\},\{x,t\}} & L_{\{z\},\{x,t\}} & L_{\{t\},\{x,y,z\}} & H_\lambda \end{array} \\
\begin{pmatrix} -7/6 & 1 & 0 & 2/3 & 0 & 1 \\ 1 & -3/2 & 0 & 0 & 1/2 & 1 \\ 0 & 0 & -1/6 & 1/2 & 0 & 1 \\ 2/3 & 0 & 1/2 & -1/6 & 0 & 1 \\ 0 & 1/2 & 0 & 0 & -3/2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 4 \end{pmatrix} \end{array}.$$

Its determinant vanishes. The geometric reason for this is the following: if $\lambda \neq -2$, then

$$H_{\{x,t\}} \cdot S_\lambda = 2L_{\{x\},\{t\}} + L_{\{y\},\{x,t\}} + L_{\{y\},\{x,t\}},$$

which implies that $2L_{\{x\},\{t\}} + L_{\{y\},\{x,t\}} + L_{\{y\},\{x,t\}} \sim H_\lambda$ on the surface S_λ. In fact, one can check that the rank of this matrix is 5. Moreover, we have $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk}) = \operatorname{rk}\operatorname{Pic}(S_{\Bbbk}) + 12$. Hence, we see that ($\bigstar$) holds, so that ($\diamondsuit$) in the Main Theorem also holds by Lemma 1.13.1.

Family 3.10. In this case, the threefold X is a blow-up of a smooth quadric hypersurface in $\mathbb{P}^4$ along a disjoint union of two irreducible conics. Thus, we have $h^{1,2}(X) = 0$. A toric Landau–Ginzburg model is given by the Laurent polynomial

$$\frac{z}{y} + x + \frac{1}{y} + z + \frac{z}{xy} + \frac{x}{z} + \frac{z}{x} + \frac{xy}{z} + \frac{1}{z} + y + \frac{1}{x},$$

which is the Minkowski polynomial 1112. The pencil $\mathcal{S}$ is given by

$$z^2tx + x^2yz + t^2zx + z^2yx + t^2z^2 + x^2yt + z^2yt + x^2y^2 + t^2yx + y^2zx + t^2yz = \lambda xyzt.$$

If $\lambda \neq \infty$, then S_λ has isolated singularities, so that, in particular, it is irreducible.

To describe the base locus of the pencil $\mathcal{S}$, we observe that

$$\begin{aligned} H_{\{x\}} \cdot S_\lambda &= L_{\{x\},\{z\}} + L_{\{x\},\{t\}} + \mathcal{C}_1, \\ H_{\{y\}} \cdot S_\lambda &= L_{\{y\},\{z\}} + L_{\{y\},\{t\}} + \mathcal{C}_2, \\ H_{\{z\}} \cdot S_\lambda &= L_{\{x\},\{z\}} + L_{\{y\},\{z\}} + \mathcal{C}_3, \\ H_{\{t\}} \cdot S_\lambda &= L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{t\},\{x,z\}} + L_{\{t\},\{y,z\}}, \end{aligned} \tag{3.10.1}$$

where $\mathcal{C}_1$ is the conic $\{x = ty + tz + yz = 0\}$, the curve $\mathcal{C}_2$ is the conic $\{y = tx + tz + xz = 0\}$, and $\mathcal{C}_3$ is the conic $\{z = t^2 + tx + xy = 0\}$. Thus, the curves $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{t\},\{x,z\}}$, $L_{\{t\},\{y,z\}}$, $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ are all base curves of the pencil $\mathcal{S}$.

Lemma 3.10.2. *Suppose that $\lambda \neq \infty$. Then the singular points of S_λ contained in the base locus of the pencil $\mathcal{S}$ can be described as follows*:

$P_{\{y\},\{z\},\{t\}}$: *type $\mathbb{A}_3$ for $\lambda \neq -4$, and type $\mathbb{A}_4$ for $\lambda = -4$;*

$P_{\{x\},\{z\},\{t\}}$: *type $\mathbb{A}_4$ for $\lambda \neq -2$, and type $\mathbb{A}_6$ for $\lambda = -2$;*

$P_{\{x\},\{y\},\{t\}}$: *type $\mathbb{A}_2$ for $\lambda \neq -4$, and type $\mathbb{A}_3$ for $\lambda = -4$;*

$P_{\{x\},\{y\},\{z\}}$: *type $\mathbb{A}_2$ for $\lambda \neq -3$, and type $\mathbb{A}_4$ for $\lambda = -3$;*

$P_{\{t\},\{x,z\},\{y,z\}}$: *smooth point for $\lambda \neq -3$, and type $\mathbb{A}_2$ for $\lambda = -3$.*

Proof. Taking partial derivatives shows that $P_{\{y\},\{z\},\{t\}}$, $P_{\{x\},\{z\},\{t\}}$, $P_{\{x\},\{y\},\{t\}}$, and $P_{\{x\},\{y\},\{z\}}$ are the singular points of the surface S_λ. Moreover, if $\lambda \neq -3$, then these points are the only singular points of S_λ that are contained in the base locus of the pencil $\mathcal{S}$. If $\lambda = -3$, then $P_{\{t\},\{x,z\},\{y,z\}}$ is also a singular point of S_λ. In this case, the surface S_λ does not have other singular points which are contained in the base locus of the pencil $\mathcal{S}$.

In the chart $x = 1$, the surface S_λ is given by the equation

$$y(y + z + t) + y^2z + yz^2 - \lambda tyz + t^2y + t^2z + tz^2 + t^2yz + t^2z^2 + tyz^2 = 0.$$

Introducing coordinates $\overline{y} = y$, $\overline{z} = z$, and $\overline{t} = t + y + z$, we can rewrite this equation as

$$\begin{aligned} &\overline{t}\,\overline{y} + \overline{t}^2\overline{y} + \overline{t}^2\overline{z} - 2\overline{t}\,\overline{y}^2 - (\lambda+4)\overline{z}\,\overline{t}\,\overline{y} - \overline{z}^2\overline{t} + \overline{y}^3 + (\lambda+4)\overline{z}\,\overline{y}^2 + (\lambda+3)\overline{z}^2\overline{y} \\ &\qquad + \overline{t}^2\overline{y}\,\overline{z} + \overline{t}^2\overline{z}^2 - 2\overline{t}\,\overline{y}^2\overline{z} - 3\overline{t}\,\overline{y}\,\overline{z}^2 - 2\overline{t}\,\overline{z}^3 + \overline{y}^3\overline{z} + 2\overline{y}^2\overline{z}^2 + 2\overline{y}\,\overline{z}^3 + \overline{z}^4 = 0. \end{aligned}$$

Here, we have $P_{\{y\},\{z\},\{t\}} = (0, 0, 0)$. Let us blow up this point.

Let $\widehat{z} = z$, $\widehat{y} = y/z$, and $\widehat{t} = t/z$. We can rewrite the latter equation (after dividing by $\widehat{z}^2$) as

$$\begin{aligned} &\widehat{t}\,\widehat{y} - \widehat{t}\,\widehat{z} + (\lambda+3)\widehat{y}\,\widehat{z} + \widehat{z}^2 + \big(\widehat{t}^{\,2}\widehat{z} - (\lambda+4)\widehat{z}\,\widehat{t}\,\widehat{y} - 2\widehat{z}^2\widehat{t} + (\lambda+4)\widehat{z}\,\widehat{y}^2 + 2\widehat{z}^2\widehat{y}\big) \\ &\qquad + \big(\widehat{t}^{\,2}\widehat{y}\,\widehat{z} + \widehat{t}^{\,2}\widehat{z}^2 - 2\widehat{t}\,\widehat{y}^2\widehat{z} - 3\widehat{t}\,\widehat{y}\,\widehat{z}^2 + \widehat{y}^3\widehat{z} + 2\widehat{y}^2\widehat{z}^2\big) + \big(\widehat{t}^{\,2}\widehat{y}\,\widehat{z}^2 - 2\widehat{t}\,\widehat{y}^2\widehat{z}^2 + \widehat{y}^3\widehat{z}^2\big) = 0. \end{aligned}$$

This equation defines (a chart of) the blow-up of the surface S_λ at $P_{\{y\},\{z\},\{t\}}$. The two exceptional curves of the blow-up are given by the equations $\widehat{z} = \widehat{t} = 0$ and $\widehat{z} = \widehat{y} = 0$. They intersect at the point $(0, 0, 0)$, which is a singular point of the obtained surface.

If $\lambda \neq -4$, then $\widehat{t}\widehat{y} - \widehat{t}\widehat{z} + (\lambda+3)\widehat{y}\widehat{z} + \widehat{z}^2$ is nondegenerate, so that $P_{\{y\},\{z\},\{t\}}$ is a singular point of S_λ of type $\mathbb{A}_3$. If $\lambda = -4$, then this form splits as $(\widehat{y} - \widehat{z})(\widehat{t} - \widehat{z})$. In this case, introducing new coordinates $\widetilde{y} = \widehat{t} - \widehat{z}$, $\widetilde{z} = \widehat{y} - \widehat{z}$, and $\widetilde{t} = \widehat{t}$, we rewrite the latter equation (with $\lambda = -4$) as

$$\widetilde{y}\widetilde{z} + \widetilde{t}^3 + \text{Higher order terms} = 0,$$

where we order the monomials with respect to the weights $\mathrm{wt}(\widetilde{y}) = 3$, $\mathrm{wt}(\widetilde{z}) = 3$, and $\mathrm{wt}(\widetilde{t}) = 2$. We see that this point is a singular point of type $\mathbb{A}_2$. Therefore, if $\lambda = -4$, then $P_{\{y\},\{z\},\{t\}}$ is a singular point of S_λ of type $\mathbb{A}_4$.

We leave the proofs of the remaining assertions of the lemma to the reader. $\square$

Thus, it follows from Corollary 1.5.4 that the fiber $\mathsf{f}^{-1}(\lambda)$ is irreducible for every $\lambda \neq \infty$. This implies ($\heartsuit$) in the Main Theorem. To prove ($\diamondsuit$) in the Main Theorem, we need the following.

Lemma 3.10.3. *Suppose that $\lambda \notin \{-2, -3, -4, \infty\}$. Then the intersection matrix of the curves $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{t\},\{x,z\}}$, and H_λ on the surface S_λ is given by*

$$\begin{array}{c} \\ L_{\{x\},\{z\}} \\ L_{\{x\},\{t\}} \\ L_{\{y\},\{z\}} \\ L_{\{y\},\{t\}} \\ L_{\{t\},\{x,z\}} \\ H_\lambda \end{array}
\begin{array}{c} \begin{array}{cccccc} L_{\{x\},\{z\}} & L_{\{x\},\{t\}} & L_{\{y\},\{z\}} & L_{\{y\},\{t\}} & L_{\{t\},\{x,z\}} & H_\lambda \end{array} \\
\begin{pmatrix} -2/15 & 2/5 & 1/3 & 0 & 3/5 & 1 \\ 2/5 & -8/5 & 0 & 1/3 & 1/5 & 1 \\ 1/3 & 0 & -7/2 & 3/4 & 0 & 1 \\ 0 & 1/3 & 3/4 & -7/12 & 1 & 1 \\ 3/5 & 1/5 & 0 & 1 & -6/5 & 1 \\ 1 & 1 & 1 & 1 & 1 & 4 \end{pmatrix} \end{array}.$$

Proof. Let us show how to compute the diagonal entries of the intersection matrix. To begin with, let us compute $L^2_{\{x\},\{z\}}$. Observe that $P_{\{x\},\{z\},\{t\}}$ and $P_{\{x\},\{y\},\{z\}}$ are the only singular points of S_λ that are contained in $L_{\{x\},\{z\}}$. Thus, by Proposition A.1.3, one has $L^2_{\{x\},\{z\}} = -2 + 2/3 + k/5$, where either $k = 4$ or $k = 6$. In fact, we have $k = 6$ here. Indeed, let us use the notation of Remark A.2.4 with $S = S_\lambda$, $O = P_{\{x\},\{z\},\{t\}}$, $n = 4$, and $C = L_{\{x\},\{z\}}$. In the chart $y = 1$, the surface S_λ is given by

$$x(x+z) + \text{Higher order terms} = 0,$$

and $L_{\{x\},\{z\}}$ is given by $x = z = 0$. This shows that $\overline{C}$ contains the point $\overline{G}_1 \cap \overline{G}_4$. Thus, either $\widetilde{C} \cap G_2 \neq \varnothing$ or $\widetilde{C} \cap G_3 \neq \varnothing$. In both cases, we have $k = 6$ by Proposition A.1.3. Thus, we have $L^2_{\{x\},\{z\}} = -2/15$.

Similarly, it follows from Proposition A.1.3 that $L^2_{\{x\},\{t\}} = -8/15$, as $P_{\{x\},\{z\},\{t\}}$ and $P_{\{x\},\{y\},\{t\}}$ are the only singular points of S_λ contained in $L_{\{x\},\{t\}}$. Similarly, we see that $L^2_{\{t\},\{x,z\}} = -6/5$, because $P_{\{x\},\{z\},\{t\}}$ is the only singular point of S_λ that is contained in $L_{\{t\},\{x,z\}}$. Using Proposition A.1.3 again, we get $L^2_{\{y\},\{t\}} = L^2_{\{y\},\{z\}} = -7/12$.

Now let us compute the remaining entries of the first row in the intersection matrix. Since $L_{\{x\},\{z\}} \cap L_{\{y\},\{t\}} = \varnothing$, we have $L_{\{x\},\{z\}} \cdot L_{\{y\},\{t\}} = 0$. To compute $L_{\{x\},\{z\}} \cdot L_{\{y\},\{z\}}$, observe that $L_{\{x\},\{z\}} \cap L_{\{y\},\{z\}} = P_{\{x\},\{y\},\{t\}}$. In the chart $t = 1$, the surface S_λ is given by

$$(x+z)(z+y) + \text{Higher order terms} = 0.$$

Thus, using Proposition A.1.2 and Remark A.2.4 with $S = S_\lambda$, $O = P_{\{y\},\{z\},\{t\}}$, $n = 2$, $C = L_{\{x\},\{z\}}$, and $Z = L_{\{y\},\{z\}}$, we see that $L_{\{x\},\{z\}} \cdot L_{\{y\},\{z\}} = 1/3$.

To find $L_{\{x\},\{z\}} \cdot L_{\{x\},\{t\}}$ and $L_{\{x\},\{z\}} \cdot L_{\{t\},\{y,z\}}$, we notice that

$$L_{\{x\},\{z\}} \cap L_{\{x\},\{t\}} = L_{\{x\},\{z\}} \cap L_{\{t\},\{y,z\}} = P_{\{x\},\{z\},\{t\}}.$$

Let us use the notation of Remark A.2.4 with $O=P_{\{x\},\{z\},\{t\}}$, $n=4$, $C=L_{\{x\},\{t\}}$, and $Z=L_{\{t\},\{y,z\}}$. Keeping in mind the equation of the surface S_λ in the chart $y=1$, we see that neither $\overline{C}$ nor $\overline{Z}$ contains the point $\overline{G}_1\cap\overline{G}_4$. By Proposition A.1.2, this implies, in particular, that $L_{\{x\},\{t\}}\cdot L_{\{t\},\{y,z\}}=1/5$. On the other hand, we have already checked above that the proper transform of the line $L_{\{x\},\{z\}}$ on the surface $\overline{S}$ does contain the point $\overline{G}_1\cap\overline{G}_4$. This implies that $L_{\{x\},\{z\}}\cdot L_{\{x\},\{t\}}$ and $L_{\{x\},\{z\}}\cdot L_{\{t\},\{y,z\}}$ are among $2/5$ and $3/5$. Moreover, one has

$$\begin{aligned}1=H_{\{t\}}\cdot L_{\{x\},\{z\}}&=\big(L_{\{x\},\{t\}}+L_{\{y\},\{t\}}+L_{\{t\},\{x,z\}}+L_{\{t\},\{y,z\}}\big)\cdot L_{\{x\},\{z\}}\\&=L_{\{x\},\{t\}}\cdot L_{\{x\},\{z\}}+L_{\{y\},\{t\}}\cdot L_{\{x\},\{z\}}+L_{\{t\},\{x,z\}}\cdot L_{\{x\},\{z\}}+L_{\{t\},\{y,z\}}\cdot L_{\{x\},\{z\}}\\&=L_{\{x\},\{t\}}\cdot L_{\{x\},\{z\}}+L_{\{t\},\{y,z\}}\cdot L_{\{x\},\{z\}},\end{aligned}$$

because $L_{\{y\},\{t\}}\cdot L_{\{x\},\{z\}}=0$ and $L_{\{t\},\{x,z\}}\cdot L_{\{x\},\{z\}}=0$. Similarly, we have

$$\begin{aligned}H_{\{x\}}\cdot L_{\{x\},\{t\}}&=\big(L_{\{x\},\{z\}}+L_{\{x\},\{t\}}+C_1\big)\cdot L_{\{x\},\{t\}}\\&=L_{\{x\},\{z\}}\cdot L_{\{x\},\{t\}}+L_{\{x\},\{t\}}^2+C_1\cdot L_{\{x\},\{t\}}\\&=L_{\{x\},\{z\}}\cdot L_{\{x\},\{t\}}-\frac{8}{5}+C_1\cdot L_{\{x\},\{t\}}.\end{aligned}$$

Moreover, we have $C_1\cap L_{\{x\},\{t\}}=P_{\{x\},\{z\},\{t\}}\cup P_{\{x\},\{y\},\{t\}}$. Thus, applying Proposition A.1.2 and Remark A.2.4, we see that

$$C_1\cdot L_{\{x\},\{t\}}=\frac{1}{3}+\frac{4}{5}=\frac{17}{15},$$

so that $L_{\{x\},\{z\}}\cdot L_{\{x\},\{t\}}=2/5$. Thus, we have $L_{\{x\},\{z\}}\cdot L_{\{t\},\{y,z\}}=3/5$.

To compute the remaining entries of the second row in the intersection matrix, we have to find $L_{\{x\},\{t\}}\cdot L_{\{y\},\{z\}}$ and $L_{\{x\},\{t\}}\cdot L_{\{y\},\{t\}}$. But $L_{\{x\},\{t\}}\cap L_{\{y\},\{z\}}=\varnothing$, so that $L_{\{x\},\{t\}}\cdot L_{\{y\},\{z\}}=0$. Moreover, we have $L_{\{x\},\{t\}}\cap L_{\{y\},\{t\}}=P_{\{x\},\{y\},\{t\}}$, so that $L_{\{x\},\{t\}}\cdot L_{\{y\},\{t\}}=1/3$ by Proposition A.1.2.

To complete the proof of the lemma, we have to find $L_{\{y\},\{z\}}\cdot L_{\{y\},\{t\}}$, $L_{\{y\},\{z\}}\cdot L_{\{t\},\{x,z\}}$, and $L_{\{y\},\{t\}}\cdot L_{\{t\},\{x,z\}}$. Since $L_{\{y\},\{z\}}\cap L_{\{t\},\{x,z\}}=\varnothing$, we have $L_{\{y\},\{z\}}\cdot L_{\{t\},\{x,z\}}=0$. Similarly, we have $L_{\{y\},\{t\}}\cdot L_{\{t\},\{x,z\}}=1$, since $L_{\{y\},\{t\}}\cap L_{\{t\},\{x,z\}}=P_{\{y\},\{z\},\{x,t\}}$ and the surface S_λ is smooth at the point $[1:0:-1:0]$. Finally, observe that $L_{\{y\},\{z\}}\cdot L_{\{y\},\{t\}}=3/4$ by Proposition A.1.2, since $L_{\{y\},\{z\}}\cap L_{\{y\},\{t\}}=P_{\{y\},\{z\},\{t\}}$. $\square$

If $\lambda\notin\{-2,-3,-4,\infty\}$, then it follows from (3.10.1) that the intersection matrix of the curves $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{t\},\{x,z\}}$, $L_{\{t\},\{y,z\}}$, $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ on the surface S_λ has the same rank as the intersection matrix of the curves $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{t\},\{x,z\}}$, and H_λ. On the other hand, the determinant of the matrix in Lemma 3.10.3 is $-2/9$, and $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk})=\operatorname{rk}\operatorname{Pic}(S_{\Bbbk})+11$. Hence, we see that ($\bigstar$) holds, so that ($\diamondsuit$) in the Main Theorem also holds by Lemma 1.13.1.

Family 3.11. The threefold X can be obtained from $\mathbb{P}^3$ by blowing up a disjoint union of a point and a smooth elliptic curve. We discussed this case in Example 1.12.3, where we described the pencil $\mathcal{S}$ and its base locus. Let us use the notation introduced in this example. As usual, we assume that $\lambda\neq\infty$. Observe that

$$\begin{aligned}H_{\{x\}}\cdot S_\lambda&=L_{\{x\},\{t\}}+L_{\{x\},\{z,t\}}+\mathcal{C},\\H_{\{y\}}\cdot S_\lambda&=L_{\{y\},\{z\}}+L_{\{y\},\{t\}}+L_{\{y\},\{x,t\}}+L_{\{y\},\{z,t\}},\\H_{\{z\}}\cdot S_\lambda&=2L_{\{y\},\{z\}}+L_{\{z\},\{t\}}+L_{\{z\},\{x,t\}},\\H_{\{t\}}\cdot S_\lambda&=L_{\{x\},\{t\}}+L_{\{y\},\{t\}}+L_{\{z\},\{t\}}+L_{\{t\},\{x,y,z\}}.\end{aligned}\tag{3.11.1}$$

If $\lambda \neq -2$, then S_λ is irreducible, it has isolated singularities, and its singular points contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

$$
\begin{aligned}
P_{\{y\},\{z\},\{t\}}&\colon && \text{type } \mathbb{A}_4 \text{ with quadratic term } yz;\\
P_{\{x\},\{z\},\{t\}}&\colon && \text{type } \mathbb{A}_2 \text{ with quadratic term } (x+t)(z+t);\\
P_{\{x\},\{y\},\{t\}}&\colon && \text{type } \mathbb{A}_2 \text{ with quadratic term } (x+t)(y+t);\\
P_{\{x\},\{t\},\{y,z\}}&\colon && \text{type } \mathbb{A}_1 \text{ with quadratic term } (x+t)(x+y+z-t)-(\lambda+2)xt;\\
P_{\{y\},\{z\},\{x,t\}}&\colon && \text{type } \mathbb{A}_2 \text{ with quadratic term } z(x+t+(\lambda+2)y);\\
[0:1\pm\sqrt{5}:-2:2]&\colon && \text{smooth point for } \lambda\neq-1, \text{ and type } \mathbb{A}_1 \text{ for } \lambda=-1.
\end{aligned}
$$

Then $[\mathsf{f}^{-1}(\lambda)]=1$ for every $\lambda\neq-2$ by Corollary 1.5.4.

Recall that $S_{-2}=H_{\{x,t\}}+\mathbf{S}$, where $\mathbf{S}$ is an irreducible cubic surface that has good double points at $P_{\{y\},\{z\},\{t\}}$, $P_{\{x\},\{z\},\{t\}}$, $P_{\{x\},\{y\},\{t\}}$, $P_{\{x\},\{t\},\{y,z\}}$, and $P_{\{y\},\{z\},\{x,t\}}$. Moreover, the surface S_{-2} is smooth at general points of the base curves $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{z,t\}}$, $L_{\{y\},\{x,t\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{z\},\{x,t\}}$, $L_{\{t\},\{x,y,z\}}$, and $\mathcal{C}$. Thus, it follows from (1.8.3) and Lemmas 1.8.5 and 1.12.1 that $[\mathsf{f}^{-1}(-2)]=[S_{-3}]=2$. Therefore, we conclude that ($\heartsuit$) in the Main Theorem holds in this case.

To verify ($\diamondsuit$) in the Main Theorem, we may assume that $\lambda\neq-2$ and $\lambda\neq(-1\pm\sqrt{5})/2$. Then, using (3.11.1), we see that the intersection matrix of the curves $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{z,t\}}$, $L_{\{y\},\{x,t\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{z\},\{x,t\}}$, $L_{\{t\},\{x,y,z\}}$, and $\mathcal{C}$ on the surface S_λ has the same rank as the intersection matrix of the curves $L_{\{x\},\{t\}}$, $L_{\{x\},\{z,t\}}$, $L_{\{y\},\{x,t\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{z\},\{x,t\}}$, $L_{\{t\},\{x,y,z\}}$, and H_λ. The latter matrix is given by

$$
\begin{array}{c}
\\ L_{\{x\},\{t\}} \\ L_{\{x\},\{z,t\}} \\ L_{\{y\},\{x,t\}} \\ L_{\{y\},\{z,t\}} \\ L_{\{z\},\{x,t\}} \\ L_{\{t\},\{x,y,z\}} \\ H_\lambda
\end{array}
\begin{array}{c}
\begin{array}{ccccccc}
L_{\{x\},\{t\}} & L_{\{x\},\{z,t\}} & L_{\{y\},\{x,t\}} & L_{\{y\},\{z,t\}} & L_{\{z\},\{x,t\}} & L_{\{t\},\{x,y,z\}} & H_\lambda
\end{array}\\
\begin{pmatrix}
-1/6 & 1/3 & 2/3 & 0 & 2/3 & 1/2 & 1\\
1/3 & -4/3 & 0 & 1 & 1/3 & 0 & 1\\
2/3 & 0 & -2/3 & 1 & 1/3 & 0 & 1\\
0 & 1 & 1 & -6/5 & 0 & 0 & 1\\
2/3 & 1/3 & 1/3 & 0 & -2/3 & 0 & 1\\
1/2 & 0 & 0 & 0 & 0 & -3/2 & 1\\
1 & 1 & 1 & 1 & 1 & 1 & 4
\end{pmatrix}
\end{array}.
$$

Its rank is 6. Note also that $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk})=\operatorname{rk}\operatorname{Pic}(S_{\Bbbk})+11$. Hence, we see that ($\bigstar$) holds, so that ($\diamondsuit$) in the Main Theorem also holds by Lemma 1.13.1.

Family 3.12. In this case, the threefold X can be obtained from $\mathbb{P}^3$ by blowing up a disjoint union of a line and a twisted cubic curve. Its toric Landau–Ginzburg model is given by

$$
\frac{z}{x}+\frac{1}{x}+y+z+\frac{y}{z}+\frac{z}{y}+\frac{1}{z}+\frac{xy}{z}+\frac{1}{y}+x,
$$

which is the Minkowski polynomial 737. The pencil $\mathcal{S}$ is given by

$$
z^2yt+t^2yz+y^2zx+z^2yx+y^2tx+z^2tx+t^2yx+x^2y^2+t^2zx+x^2yz=\lambda xyzt.
$$

As usual, we suppose that $\lambda\neq\infty$.

Let $\mathcal{C}$ be the conic $z=xy+yt+t^2=0$. Then

$$
\begin{aligned}
H_{\{x\}}\cdot S_\lambda &= L_{\{x\},\{y\}}+L_{\{x\},\{z\}}+L_{\{x\},\{t\}}+L_{\{x\},\{z,t\}},\\
H_{\{y\}}\cdot S_\lambda &= L_{\{x\},\{y\}}+L_{\{y\},\{z\}}+L_{\{y\},\{t\}}+L_{\{y\},\{z,t\}},\\
H_{\{z\}}\cdot S_\lambda &= L_{\{x\},\{z\}}+L_{\{y\},\{z\}}+\mathcal{C},\\
H_{\{t\}}\cdot S_\lambda &= L_{\{x\},\{t\}}+L_{\{y\},\{t\}}+L_{\{t\},\{x,z\}}+L_{\{t\},\{y,z\}}.
\end{aligned}
\tag{3.12.1}
$$

Thus, the base locus of the pencil $\mathcal{S}$ consists of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{x\},\{z,t\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{t\},\{x,z\}}$, $L_{\{t\},\{y,z\}}$, and $\mathcal{C}$.

For every $\lambda \in \mathbb{C}$, the surface S_λ has isolated singularities, so that it is irreducible. Moreover, the singular points of S_λ contained in the base locus of the pencil $\mathcal{S}$ are du Val and can be described as follows:

$P_{\{x\},\{y\},\{z\}}$: type $\mathbb{A}_1$;

$P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_1$;

$P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term $x(x+z+t)$ for $\lambda \neq -3$, and type $\mathbb{A}_4$ for $\lambda = -3$;

$P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_4$ with quadratic term $y(y+z)$ for $\lambda \neq -2$, and type $\mathbb{A}_5$ for $\lambda = -2$;

$P_{\{x\},\{y\},\{z,t\}}$: type $\mathbb{A}_1$ for $\lambda \neq -2$, and type $\mathbb{A}_2$ for $\lambda = -2$;

$P_{\{t\},\{x,z\},\{y,z\}}$: smooth point for $\lambda \neq -3$, and type $\mathbb{A}_1$ for $\lambda = -3$.

Thus, it follows from Corollary 1.5.4 that the fiber $\mathsf{f}^{-1}(\lambda)$ is irreducible for every $\lambda \in \mathbb{C}$. Since $h^{1,2}(X) = 0$, we see that ($\heartsuit$) in the Main Theorem holds in this case.

Now let us verify ($\diamondsuit$) in the Main Theorem. We may assume that $\lambda \neq -2, -3$. Then the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{t\},\{x,z\}}$, and H_λ on the surface S_λ is given by

$$\begin{array}{c} \\ L_{\{x\},\{y\}} \\ L_{\{x\},\{z\}} \\ L_{\{x\},\{t\}} \\ L_{\{y\},\{t\}} \\ L_{\{y\},\{z,t\}} \\ L_{\{t\},\{x,z\}} \\ H_\lambda \end{array}
\begin{array}{c} \begin{array}{ccccccc} L_{\{x\},\{y\}} & L_{\{x\},\{z\}} & L_{\{x\},\{t\}} & L_{\{y\},\{t\}} & L_{\{y\},\{z,t\}} & L_{\{t\},\{x,z\}} & H_\lambda \end{array} \\ \begin{pmatrix} -1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 0 & 1 \\ 1/2 & -3/4 & 3/4 & 0 & 0 & 1/4 & 1 \\ 1/2 & 3/4 & -3/4 & 1/2 & 0 & 1/4 & 1 \\ 1/2 & 0 & 1/2 & -7/10 & 3/5 & 1 & 1 \\ 1/2 & 0 & 0 & 3/5 & -7/10 & 0 & 1 \\ 0 & 1/4 & 1/4 & 1 & 0 & -5/4 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 4 \end{pmatrix} \end{array}.$$

The rank of this matrix is 7. On the other hand, it follows from (3.12.1) that

$$\begin{aligned} L_{\{x\},\{y\}} + L_{\{x\},\{z\}} + L_{\{x\},\{t\}} + L_{\{x\},\{z,t\}} &\sim L_{\{x\},\{y\}} + L_{\{y\},\{z\}} + L_{\{y\},\{t\}} + L_{\{y\},\{z,t\}} \\ &\sim L_{\{x\},\{z\}} + L_{\{y\},\{z\}} + \mathcal{C} \sim L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{t\},\{x,z\}} + L_{\{t\},\{y,z\}} \sim H_\lambda. \end{aligned}$$

This implies that the rank of the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{x\},\{z,t\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{t\},\{x,z\}}$, $L_{\{t\},\{y,z\}}$, and $\mathcal{C}$ is also 7. As we have seen above, $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk}) = \operatorname{rk}\operatorname{Pic}(S_{\Bbbk}) + 10$. Hence, we see that ($\bigstar$) holds, so that ($\diamondsuit$) in the Main Theorem holds by Lemma 1.13.1.

Family 3.13. The threefold X is a blow-up of a smooth hypersurface in $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(1,1)$ in a smooth rational curve of bidegree $(2,2)$. Thus, we have $h^{1,2}(X) = 0$. A toric Landau–Ginzburg model of this family is given by the Minkowski polynomial 420, which is

$$\frac{x}{y} + x + \frac{1}{y} + z + \frac{z}{x} + \frac{1}{z} + y + \frac{1}{x} + \frac{y}{z}.$$

The quartic pencil $\mathcal{S}$ is given by

$$x^2zt + x^2yz + t^2zx + z^2yx + z^2yt + t^2yx + y^2zx + t^2yz + y^2tx = \lambda xyzt.$$

As usual, we assume that $\lambda \neq \infty$. Then

$$
\begin{aligned}
H_{\{x\}} \cdot S_\lambda &= L_{\{x\},\{y\}} + L_{\{x\},\{z\}} + L_{\{x\},\{t\}} + L_{\{x\},\{z,t\}},\\
H_{\{y\}} \cdot S_\lambda &= L_{\{x\},\{y\}} + L_{\{y\},\{z\}} + L_{\{y\},\{t\}} + L_{\{y\},\{x,t\}},\\
H_{\{z\}} \cdot S_\lambda &= L_{\{x\},\{z\}} + L_{\{y\},\{z\}} + L_{\{z\},\{t\}} + L_{\{z\},\{y,t\}},\\
H_{\{t\}} \cdot S_\lambda &= L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y,z\}}.
\end{aligned}
$$

Thus, the lines $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{z,t\}}$, $L_{\{y\},\{x,t\}}$, $L_{\{z\},\{y,t\}}$, and $L_{\{t\},\{x,y,z\}}$ are all base curves of the pencil $\mathcal{S}$.

Each surface S_λ is irreducible, it has isolated singularities, and its singular points contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

- $P_{\{x\},\{y\},\{z\}}$: type $\mathbb{A}_1$;
- $P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term $y(x+t)$ for $\lambda \neq -2$, and type $\mathbb{A}_4$ for $\lambda = -2$;
- $P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term $x(z+t)$ for $\lambda \neq -2$, and type $\mathbb{A}_4$ for $\lambda = -2$;
- $P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term $z(y+t)$ for $\lambda \neq -2$, and type $\mathbb{A}_4$ for $\lambda = -2$.

Then $[\mathsf{f}^{-1}(\lambda)] = 1$ for every $\lambda \in \mathbb{C}$ by Corollary 1.5.4. This confirms ($\heartsuit$) in the Main Theorem.

Now we suppose that $\lambda \neq -2$. Then the rank of the intersection matrix of the lines $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{z,t\}}$, $L_{\{y\},\{x,t\}}$, $L_{\{z\},\{y,t\}}$, and $L_{\{t\},\{x,y,z\}}$ on the surface S_λ is the same as the rank of the following matrix:

$$
\begin{array}{c}
\\ L_{\{x\},\{z\}} \\ L_{\{x\},\{t\}} \\ L_{\{x\},\{z,t\}} \\ L_{\{y\},\{x,t\}} \\ L_{\{z\},\{y,t\}} \\ L_{\{t\},\{x,y,z\}} \\ H_\lambda
\end{array}
\begin{array}{c}
\begin{array}{ccccccc} L_{\{x\},\{z\}} & L_{\{x\},\{t\}} & L_{\{x\},\{z,t\}} & L_{\{y\},\{x,t\}} & L_{\{z\},\{y,t\}} & L_{\{t\},\{x,y,z\}} & H_\lambda \end{array}\\
\begin{pmatrix}
-3/4 & 3/4 & 1 & 0 & 1 & 0 & 1\\
3/4 & -1/2 & 1 & 1 & 0 & 1 & 1\\
1 & 1 & -1 & 0 & 0 & 0 & 1\\
0 & 1 & 0 & -1 & 0 & 0 & 1\\
1 & 0 & 0 & 0 & -1 & 0 & 1\\
0 & 1 & 0 & 0 & 0 & -2 & 1\\
1 & 1 & 1 & 1 & 1 & 1 & 4
\end{pmatrix}
\end{array}.
$$

Its rank is 7. On the other hand, we have $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk}) = \operatorname{rk}\operatorname{Pic}(S_{\Bbbk}) + 10$. Hence, we see that ($\bigstar$) holds. By Lemma 1.13.1, we see that ($\diamondsuit$) in the Main Theorem also holds.

Family 3.14. The threefold X is $\mathbb{P}^3$ blown up in a union of a smooth plane cubic and a point that does not lie on the plane containing the cubic, so that $h^{1,2}(X) = 1$. A toric Landau–Ginzburg model of this family is given by

$$
x + y + z + \frac{x^2}{yz} + \frac{y}{x} + \frac{z}{x} + \frac{x}{yz} + \frac{1}{x},
$$

which is the Minkowski polynomial 202. The quartic pencil $\mathcal{S}$ is given by

$$
x^2yz + xy^2z + xyz^2 + x^3t + y^2zt + yz^2t + x^2t^2 + yzt^2 = \lambda xyzt.
$$

Suppose that $\lambda \neq \infty$. Then

$$
\begin{aligned}
H_{\{x\}} \cdot S_\lambda &= L_{\{x\},\{y\}} + L_{\{x\},\{z\}} + L_{\{x\},\{t\}} + L_{\{x\},\{y,z,t\}},\\
H_{\{y\}} \cdot S_\lambda &= 2L_{\{x\},\{y\}} + L_{\{y\},\{t\}} + L_{\{y\},\{x,t\}},\\
H_{\{z\}} \cdot S_\lambda &= 2L_{\{x\},\{z\}} + L_{\{z\},\{t\}} + L_{\{z\},\{x,t\}},\\
H_{\{t\}} \cdot S_\lambda &= L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y,z\}}.
\end{aligned}
\tag{3.14.1}
$$

Thus, the base locus of the pencil $\mathcal{S}$ consists of the lines $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{y,z,t\}}$, $L_{\{y\},\{x,t\}}$, $L_{\{z\},\{x,t\}}$, and $L_{\{t\},\{x,y,z\}}$.

Let $\mathbf{S}$ be the cubic surface in $\mathbb{P}^3$ that is given by

$$xyz + x^2t + y^2z + yz^2 + yzt = 0.$$

Then $\mathbf{S}$ is irreducible and $S_{-2} = H_{\{x,t\}} + \mathbf{S}$. On the other hand, if $\lambda \neq -2$, then the surface S_λ has isolated singularities, so that it is irreducible. In this case, its singular points contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

$P_{\{x\},\{y\},\{z\}}$: type $\mathbb{A}_1$ with quadratic term $x^2 + yz$;

$P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_4$ with quadratic term $y(x+t)$;

$P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_4$ with quadratic term $z(x+t)$;

$P_{\{x\},\{y\},\{z,t\}}$: type $\mathbb{A}_1$ with quadratic term $x^2 - y^2 - yz - yt + (\lambda+1)xy$;

$P_{\{x\},\{z\},\{y,t\}}$: type $\mathbb{A}_1$ with quadratic term $x^2 - z^2 - yz - zt + (\lambda+1)xz$;

$P_{\{x\},\{t\},\{y,z\}}$: type $\mathbb{A}_1$ with quadratic term $(x+t)(x+y+z+t) - (\lambda+2)xt$.

By Corollary 1.5.4, we have $[\mathsf{f}^{-1}(\lambda)] = 1$ for every $\lambda \neq -2$. Moreover, the points $P_{\{x\},\{z\},\{t\}}$, $P_{\{x\},\{y\},\{t\}}$, $P_{\{x\},\{y\},\{z\}}$, $P_{\{x\},\{y\},\{z,t\}}$, $P_{\{x\},\{z\},\{y,t\}}$, and $P_{\{x\},\{t\},\{y,z\}}$ are good double points of the surface S_{-2}. Furthermore, one can check that the surface S_{-2} is smooth at general points of the lines $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{y,z,t\}}$, $L_{\{y\},\{x,t\}}$, $L_{\{z\},\{x,t\}}$, and $L_{\{t\},\{x,y,z\}}$. Therefore, using (1.8.3) and applying Lemmas 1.8.5 and 1.12.1, we conclude that $[\mathsf{f}^{-1}(-2)] = [S_{-2}] = 2$. This confirms ($\heartsuit$) in the Main Theorem.

To verify ($\diamondsuit$) in the Main Theorem, we suppose that $\lambda \neq -2$. Then (3.14.1) gives

$$\begin{aligned} H_\lambda &\sim L_{\{x\},\{y\}} + L_{\{x\},\{z\}} + L_{\{x\},\{t\}} + L_{\{x\},\{y,z,t\}} \sim 2L_{\{x\},\{y\}} + L_{\{y\},\{t\}} + L_{\{y\},\{x,t\}} \\ &\sim 2L_{\{x\},\{z\}} + L_{\{z\},\{t\}} + L_{\{z\},\{x,t\}} \sim L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y,z\}}. \end{aligned}$$

Therefore, the intersection matrix of the lines $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{y,z,t\}}$, $L_{\{y\},\{x,t\}}$, $L_{\{z\},\{x,t\}}$, and $L_{\{t\},\{x,y,z\}}$ on the surface S_λ has the same rank as the intersection matrix of the curves $L_{\{y\},\{x,t\}}$, $L_{\{x\},\{y,z,t\}}$, $L_{\{z\},\{x,t\}}$, $L_{\{t\},\{x,y,z\}}$, $L_{\{x\},\{y\}}$, and H_λ. The latter matrix is given by

$$\begin{array}{c} \\ L_{\{y\},\{x,t\}} \\ L_{\{x\},\{y,z,t\}} \\ L_{\{z\},\{x,t\}} \\ L_{\{t\},\{x,y,z\}} \\ L_{\{x\},\{y\}} \\ H_\lambda \end{array} \begin{array}{c} \begin{array}{cccccc} L_{\{y\},\{x,t\}} & L_{\{x\},\{y,z,t\}} & L_{\{z\},\{x,t\}} & L_{\{t\},\{x,y,z\}} & L_{\{x\},\{y\}} & H_\lambda \end{array} \\ \begin{pmatrix} -4/5 & 0 & 1 & 0 & 3/5 & 1 \\ 0 & -1/2 & 0 & 1/2 & 1/2 & 1 \\ 1 & 0 & -4/5 & 0 & 0 & 1 \\ 0 & 1/2 & 0 & -3/2 & 0 & 1 \\ 3/5 & 1/2 & 0 & 0 & -1/5 & 1 \\ 1 & 1 & 1 & 1 & 1 & 4 \end{pmatrix} \end{array}.$$

The rank of this matrix is 5. We can see that the determinant of this matrix is 0 without computing it. Indeed, we have $H_\lambda \sim 2L_{\{x\},\{t\}} + L_{\{y\},\{x,t\}} + L_{\{z\},\{x,t\}}$ on the surface S_λ, because $H_{\{x,t\}} \cdot S_\lambda = 2L_{\{x\},\{t\}} + L_{\{y\},\{x,t\}} + L_{\{z\},\{x,t\}}$.

Observe that $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk}) = \operatorname{rk}\operatorname{Pic}(S_{\Bbbk}) + 12$. Therefore, we conclude that ($\bigstar$) holds. Using Lemma 1.13.1, we see that ($\diamondsuit$) in the Main Theorem also holds.

Family 3.15. In this case, the threefold X is a blow-up of a quadric in a disjoint union of a line and a conic, so that $h^{1,2}(X)=0$. A toric Landau–Ginzburg model of this family is given by the Minkowski polynomial 419, which is

$$x+y+z+\frac{x}{z}+\frac{z}{y}+\frac{1}{z}+\frac{1}{y}+\frac{1}{x}+\frac{z}{xy}.$$

The quartic pencil $\mathcal{S}$ is given by

$$x^2zy+y^2zx+z^2yx+x^2ty+z^2tx+t^2yx+t^2zx+t^2zy+t^2z^2=\lambda xyzt.$$

Suppose that $\lambda\neq\infty$. Let $\mathcal{C}$ be a conic given by $y=xz+xt+zt=0$. Then

$$\begin{aligned}
H_{\{x\}}\cdot S_\lambda &= L_{\{x\},\{z\}}+2L_{\{x\},\{t\}}+L_{\{x\},\{y,z\}},\\
H_{\{y\}}\cdot S_\lambda &= L_{\{y\},\{z\}}+L_{\{y\},\{t\}}+\mathcal{C},\\
H_{\{z\}}\cdot S_\lambda &= L_{\{x\},\{z\}}+L_{\{y\},\{z\}}+L_{\{z\},\{t\}}+L_{\{z\},\{x,t\}},\\
H_{\{t\}}\cdot S_\lambda &= L_{\{x\},\{t\}}+L_{\{y\},\{t\}}+L_{\{z\},\{t\}}+L_{\{t\},\{x,y,z\}}.
\end{aligned}\tag{3.15.1}$$

Thus, the base locus of the pencil $\mathcal{S}$ consists of the curves $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{x\},\{y,z\}}$, $L_{\{z\},\{t\}}$, $L_{\{z\},\{x,t\}}$, $L_{\{t\},\{x,y,z\}}$, and $\mathcal{C}$.

For every $\lambda\in\mathbb{C}$, the surface S_λ is irreducible, it has isolated singularities, and its singular points contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

- $P_{\{x\},\{y\},\{z\}}$: type $\mathbb{A}_2$ with quadratic term $(x+z)(y+z)$ for $\lambda\neq-2$, and type $\mathbb{A}_3$ for $\lambda=-2$;
- $P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_1$;
- $P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term xz;
- $P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term $y(z+t)$ for $\lambda\neq-3$, and type $\mathbb{A}_4$ for $\lambda=-3$;
- $P_{\{x\},\{t\},\{y,z\}}$: type $\mathbb{A}_2$ with quadratic term $x(x+y+z-t-\lambda t)$.

So, by Corollary 1.5.4, each fiber $\mathsf{f}^{-1}(\lambda)$ is irreducible. This confirms ($\heartsuit$) in the Main Theorem.

If $\lambda\neq-2,-3$, then the intersection matrix of the curves $L_{\{x\},\{z\}}$, $L_{\{x\},\{y,z\}}$, $L_{\{y\},\{z\}}$, $L_{\{z\},\{x,t\}}$, $L_{\{t\},\{x,y,z\}}$, and H_λ on the surface S_λ is given by

$$\begin{array}{c|cccccc}
 & L_{\{x\},\{z\}} & L_{\{x\},\{y,z\}} & L_{\{y\},\{z\}} & L_{\{z\},\{x,t\}} & L_{\{t\},\{x,y,z\}} & H_\lambda\\
\hline
L_{\{x\},\{z\}} & -1/3 & 1/3 & 1/3 & 1/2 & 0 & 1\\
L_{\{x\},\{y,z\}} & 1/3 & -2/3 & 2/3 & 0 & 1/3 & 1\\
L_{\{y\},\{z\}} & 1/3 & 2/3 & -7/12 & 0 & 0 & 1\\
L_{\{z\},\{x,t\}} & 1/2 & 0 & 0 & -5/4 & 0 & 1\\
L_{\{t\},\{x,y,z\}} & 0 & 1/3 & 0 & 0 & -4/3 & 1\\
H_\lambda & 1 & 1 & 1 & 1 & 1 & 4
\end{array}.$$

This matrix has rank 6. On the other hand, using (3.15.1), we see that

$$\begin{aligned}
H_\lambda &\sim L_{\{x\},\{z\}}+2L_{\{x\},\{t\}}+L_{\{x\},\{y,z\}}\sim L_{\{y\},\{z\}}+L_{\{y\},\{t\}}+\mathcal{C}\\
&\sim L_{\{x\},\{z\}}+L_{\{y\},\{z\}}+L_{\{z\},\{t\}}+L_{\{z\},\{x,t\}}\sim L_{\{x\},\{t\}}+L_{\{y\},\{t\}}+L_{\{z\},\{t\}}+L_{\{t\},\{x,y,z\}}
\end{aligned}$$

on the surface S_λ. Thus, the rank of the intersection matrix of the curves $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{x\},\{y,z\}}$, $L_{\{z\},\{t\}}$, $L_{\{z\},\{x,t\}}$, $L_{\{t\},\{x,y,z\}}$, and $\mathcal{C}$ is also 6. On the other hand, we have $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk})=\operatorname{rk}\operatorname{Pic}(S_{\Bbbk})+11$. Thus, we see that ($\bigstar$) holds, so that ($\diamondsuit$) in the Main Theorem holds by Lemma 1.13.1.

Family 3.16. In this case, the threefold X can be obtained from $\mathbb{P}^3$ blown up at a point by blowing up a proper transform of a twisted cubic curve passing through the point. Thus, we see that $h^{1,2}(X)=0$. A toric Landau–Ginzburg model of this family is given by the Minkowski polynomial 212, which is

$$x+y+z+\frac{y}{z}+\frac{x}{y}+\frac{y}{xz}+\frac{1}{y}+\frac{1}{x}.$$

The pencil $\mathcal{S}$ is given by the equation

$$x^2zy+y^2zx+z^2yx+y^2tx+x^2tz+t^2y^2+t^2zx+t^2zy=\lambda xyzt.$$

Suppose that $\lambda\neq\infty$. Then

$$\begin{aligned}
H_{\{x\}}\cdot S_\lambda&=L_{\{x\},\{y\}}+2L_{\{x\},\{t\}}+L_{\{x\},\{y,z\}},\\
H_{\{y\}}\cdot S_\lambda&=L_{\{x\},\{y\}}+L_{\{y\},\{z\}}+L_{\{y\},\{t\}}+L_{\{y\},\{x,t\}},\\
H_{\{z\}}\cdot S_\lambda&=2L_{\{y\},\{z\}}+L_{\{z\},\{t\}}+L_{\{z\},\{x,t\}},\\
H_{\{t\}}\cdot S_\lambda&=L_{\{x\},\{t\}}+L_{\{y\},\{t\}}+L_{\{z\},\{t\}}+L_{\{t\},\{x,y,z\}}.
\end{aligned}\tag{3.16.1}$$

Thus, the base locus of the pencil $\mathcal{S}$ consists of the lines $L_{\{x\},\{y\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{y,z\}}$, $L_{\{y\},\{x,t\}}$, $L_{\{z\},\{x,t\}}$, and $L_{\{t\},\{x,y,z\}}$.

For every $\lambda\in\mathbb{C}$, the surface $S_\lambda\in\mathcal{S}$ has isolated singularities, so that it is irreducible.

The singular points of S_λ contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

$P_{\{x\},\{y\},\{z\}}$: type $\mathbb{A}_1$;

$P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_1$;

$P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term xy;

$P_{\{x\},\{t\},\{y,z\}}$: type $\mathbb{A}_2$ with quadratic term $x(x+y+z-t-\lambda t)$ for $\lambda\neq-1$, and type $\mathbb{A}_3$ for $\lambda=-1$;

$P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_2$ with quadratic term $z(y+t)$;

$P_{\{y\},\{z\},\{x,t\}}$: type $\mathbb{A}_2$ with quadratic term $z(x+t-2y-\lambda y)$ for $\lambda\neq-2$, and type $\mathbb{A}_3$ for $\lambda=-2$.

Therefore, every fiber $\mathsf{f}^{-1}(\lambda)$ is irreducible by Corollary 1.5.4. This confirms ($\heartsuit$) in the Main Theorem, because $h^{1,2}(X)=0$.

Now let us verify ($\diamondsuit$) in the Main Theorem. We may assume that $\lambda\neq-1,-2$. Using (3.16.1), we see that the intersection matrix of the lines $L_{\{x\},\{y\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{y,z\}}$, $L_{\{y\},\{x,t\}}$, $L_{\{z\},\{x,t\}}$, and $L_{\{t\},\{x,y,z\}}$ on the surface S_λ has the same rank as the intersection matrix of the curves $L_{\{x\},\{t\}}$, $L_{\{x\},\{y,z\}}$, $L_{\{y\},\{z\}}$, $L_{\{z\},\{t\}}$, $L_{\{t\},\{x,y,z\}}$, and H_λ. But the latter matrix is given by

$$\begin{array}{c@{\,}c}
 & \begin{array}{cccccc} L_{\{x\},\{t\}} & L_{\{x\},\{y,z\}} & L_{\{y\},\{z\}} & L_{\{z\},\{t\}} & L_{\{t\},\{x,y,z\}} & H_\lambda \end{array}\\
\begin{array}{c} L_{\{x\},\{t\}}\\ L_{\{x\},\{y,z\}}\\ L_{\{y\},\{z\}}\\ L_{\{z\},\{t\}}\\ L_{\{t\},\{x,y,z\}}\\ H_\lambda \end{array} &
\begin{pmatrix}
-1/12 & 2/3 & 0 & 1/2 & 1/3 & 1\\
2/3 & -5/6 & 1/2 & 0 & 1/3 & 1\\
0 & 1/2 & -1/6 & 2/3 & 0 & 1\\
1/2 & 0 & 2/3 & -5/6 & 1 & 1\\
1/3 & 1/3 & 0 & 1 & -4/3 & 1\\
1 & 1 & 1 & 1 & 1 & 4
\end{pmatrix}
\end{array}.$$

Its rank is 6. On the other hand, the description of the singular points of the surface S_λ easily gives $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk})=\operatorname{rk}\operatorname{Pic}(S_{\Bbbk})+11$. Thus, we can conclude that ($\bigstar$) holds in this case, so that ($\diamondsuit$) in the Main Theorem also holds by Lemma 1.13.1.

Family 3.17. The threefold X is a divisor of tridegree $(1,1,1)$ in $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^2$, so that $h^{1,2}(X)=0$. A toric Landau–Ginzburg model of this family is given by the Minkowski polynomial 208, which is

$$\frac{z}{y}+x+\frac{1}{y}+z+y+\frac{1}{x}+\frac{1}{xz}+\frac{y}{xz}.$$

The pencil of quartic surfaces $\mathcal{S}$ is given by the equation

$$z^2tx+x^2zy+t^2zx+z^2yx+y^2zx+t^2zy+t^3y+t^2y^2=\lambda xyzt.$$

To describe the base locus of the pencil $\mathcal{S}$, we observe that

$$\begin{aligned}
H_{\{x\}}\cdot S_\lambda&=L_{\{x\},\{y\}}+2L_{\{x\},\{t\}}+L_{\{x\},\{y,z,t\}},\\
H_{\{y\}}\cdot S_\lambda&=L_{\{x\},\{y\}}+L_{\{y\},\{z\}}+L_{\{y\},\{t\}}+L_{\{y\},\{z,t\}},\\
H_{\{z\}}\cdot S_\lambda&=L_{\{y\},\{z\}}+2L_{\{z\},\{t\}}+L_{\{z\},\{y,t\}},\\
H_{\{t\}}\cdot S_\lambda&=L_{\{x\},\{t\}}+L_{\{y\},\{t\}}+L_{\{z\},\{t\}}+L_{\{t\},\{x,y,z\}}.
\end{aligned}$$

Thus, the base locus of the pencil $\mathcal{S}$ consists of the lines $L_{\{x\},\{y\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{y,z,t\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{z\},\{y,t\}}$, and $L_{\{t\},\{x,y,z\}}$.

If $\lambda\neq\infty$, then S_λ has isolated singularities, so that it is irreducible. In this case, the singular points of S_λ contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

$P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_2$ with quadratic term $x(y+t)$;

$P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_1$;

$P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_4$ with quadratic term yz;

$P_{\{x\},\{y\},\{z,t\}}$: type $\mathbb{A}_1$ for $\lambda\neq-2$, and type $\mathbb{A}_2$ for $\lambda=-2$;

$P_{\{x\},\{t\},\{y,z\}}$: type $\mathbb{A}_2$ with quadratic term $x(x+y+z-t-\lambda t)$;

$P_{\{z\},\{t\},\{x,y\}}$: type $\mathbb{A}_1$.

Thus, by Corollary 1.5.4, the fiber $\mathsf{f}^{-1}(\lambda)$ is irreducible for every $\lambda\neq\infty$. This confirms ($\heartsuit$) in the Main Theorem, since $h^{1,2}(X)=0$.

Let us check ($\diamondsuit$) in the Main Theorem. To this end, we may assume that $\lambda\neq\infty,-2$. Then the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{t\},\{x,y,z\}}$, and H_λ on the surface S_λ is given by

$$\begin{array}{c}
\\ L_{\{x\},\{y\}}\\ L_{\{x\},\{t\}}\\ L_{\{y\},\{t\}}\\ L_{\{y\},\{z,t\}}\\ L_{\{t\},\{x,y,z\}}\\ H_\lambda
\end{array}
\begin{array}{c}
\begin{array}{cccccc} L_{\{x\},\{y\}} & L_{\{x\},\{t\}} & L_{\{y\},\{t\}} & L_{\{y\},\{z,t\}} & L_{\{t\},\{x,y,z\}} & H_\lambda \end{array}\\
\begin{pmatrix}
-5/6 & 2/3 & 1/3 & 1/2 & 0 & 1\\
2/3 & -1/6 & 1/3 & 0 & 1/3 & 1\\
1/3 & 1/3 & -8/15 & 4/5 & 1 & 1\\
1/2 & 0 & 4/5 & -7/10 & 0 & 1\\
0 & 1/3 & 1 & 0 & -5/6 & 1\\
1 & 1 & 1 & 1 & 1 & 4
\end{pmatrix}
\end{array}.$$

This matrix has rank 6. Thus, the intersection matrix of the lines $L_{\{x\},\{y\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{y,z,t\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{z\},\{y,t\}}$, and $L_{\{t\},\{x,y,z\}}$ on the surface S_λ also has rank 6, because

$$\begin{aligned}
H_\lambda&\sim L_{\{x\},\{y\}}+2L_{\{x\},\{t\}}+L_{\{x\},\{y,z,t\}}\sim L_{\{x\},\{y\}}+L_{\{y\},\{z\}}+L_{\{y\},\{t\}}+L_{\{y\},\{z,t\}}\\
&\sim L_{\{y\},\{z\}}+2L_{\{z\},\{t\}}+L_{\{z\},\{y,t\}}\sim L_{\{x\},\{t\}}+L_{\{y\},\{t\}}+L_{\{z\},\{t\}}+L_{\{t\},\{x,y,z\}}.
\end{aligned}$$

On the other hand, one has $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk})=\operatorname{rk}\operatorname{Pic}(S_{\Bbbk})+11$. We conclude that ($\bigstar$) holds. By Lemma 1.13.1, this implies that ($\diamondsuit$) in the Main Theorem also holds.

Family 3.18. The threefold X can be obtained by blowing up $\mathbb{P}^3$ in a disjoint union of a line and a conic. This shows that $h^{1,2}(X)=0$. A toric Landau–Ginzburg model of this family is given by the Minkowski polynomial 211, which is

$$\frac{x}{y}+x+\frac{1}{y}+z+\frac{x}{z}+y+\frac{1}{x}+\frac{y}{z}.$$

Thus, the pencil $\mathcal{S}$ is given by the equation

$$x^2tz+x^2zy+t^2zx+z^2yx+x^2ty+y^2zx+t^2zy+y^2tx=\lambda xyzt.$$

As usual, we suppose that $\lambda\neq\infty$. Then

$$\begin{aligned}
H_{\{x\}}\cdot S_\lambda&=L_{\{x\},\{y\}}+L_{\{x\},\{z\}}+2L_{\{x\},\{t\}},\\
H_{\{y\}}\cdot S_\lambda&=L_{\{x\},\{y\}}+L_{\{y\},\{z\}}+L_{\{y\},\{t\}}+L_{\{y\},\{x,t\}},\\
H_{\{z\}}\cdot S_\lambda&=L_{\{x\},\{z\}}+L_{\{y\},\{z\}}+L_{\{z\},\{t\}}+L_{\{z\},\{x,y\}},\\
H_{\{t\}}\cdot S_\lambda&=L_{\{x\},\{t\}}+L_{\{y\},\{t\}}+L_{\{z\},\{t\}}+L_{\{t\},\{x,y,z\}}.
\end{aligned}\tag{3.18.1}$$

Thus, the base locus of the pencil $\mathcal{S}$ consists of the lines $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{y\},\{x,t\}}$, $L_{\{z\},\{x,y\}}$, and $L_{\{t\},\{x,y,z\}}$.

For every $\lambda\in\mathbb{C}$, the surface S_λ is irreducible, it has isolated singularities, and its singular points contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

- $P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_1$;
- $P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_2$ with quadratic term $x(z+t)$;
- $P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term xy;
- $P_{\{x\},\{y\},\{z\}}$: type $\mathbb{A}_3$ with quadratic term $x(x+y)$ for $\lambda\neq-1$, and type $\mathbb{A}_5$ for $\lambda=-1$;
- $P_{\{x\},\{t\},\{y,z\}}$: type $\mathbb{A}_1$;
- $P_{\{z\},\{t\},\{x,y\}}$: type $\mathbb{A}_1$ for $\lambda\neq-2$, and type $\mathbb{A}_2$ for $\lambda=-2$.

Then $[\mathsf{f}^{-1}(\lambda)]=1$ for every $\lambda\in\mathbb{C}$ by Corollary 1.5.4. This confirms ($\heartsuit$) in the Main Theorem.

To verify ($\diamondsuit$) in the Main Theorem, we may assume that $\lambda\neq-1,-2$. In this case, the intersection matrix of the curves $L_{\{y\},\{t\}}$, $L_{\{y\},\{x,t\}}$, $L_{\{z\},\{t\}}$, $L_{\{z\},\{x,y\}}$, $L_{\{t\},\{x,y,z\}}$, and H_λ on the surface S_λ is given by

$$\begin{array}{c} \\ L_{\{y\},\{t\}}\\ L_{\{y\},\{x,t\}}\\ L_{\{z\},\{t\}}\\ L_{\{z\},\{x,y\}}\\ L_{\{t\},\{x,y,z\}}\\ H_\lambda\end{array}
\begin{array}{c}
\begin{array}{cccccc} L_{\{y\},\{t\}} & L_{\{y\},\{x,t\}} & L_{\{z\},\{t\}} & L_{\{z\},\{x,y\}} & L_{\{t\},\{x,y,z\}} & H_\lambda\end{array}\\
\begin{pmatrix}
-3/4 & 3/4 & 1/2 & 0 & 1 & 1\\
3/4 & -5/4 & 0 & 0 & 0 & 1\\
1/2 & 0 & -1/3 & 1/2 & 1/2 & 1\\
0 & 0 & 1/2 & -3/4 & 1/2 & 1\\
1 & 0 & 1/2 & 1/2 & -1 & 1\\
1 & 1 & 1 & 1 & 1 & 4
\end{pmatrix}
\end{array}.$$

This matrix has rank 6. On the other hand, it follows from (3.18.1) that

$$\begin{aligned}
H_\lambda&\sim L_{\{x\},\{y\}}+L_{\{x\},\{z\}}+2L_{\{x\},\{t\}}\sim L_{\{x\},\{y\}}+L_{\{y\},\{z\}}+L_{\{y\},\{t\}}+L_{\{y\},\{x,t\}}\\
&\sim L_{\{x\},\{z\}}+L_{\{y\},\{z\}}+L_{\{z\},\{t\}}+L_{\{z\},\{x,y\}}\sim L_{\{x\},\{t\}}+L_{\{y\},\{t\}}+L_{\{z\},\{t\}}+L_{\{t\},\{x,y,z\}}.
\end{aligned}$$

Thus, the intersection matrix of the lines $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{y\},\{x,t\}}$, $L_{\{z\},\{x,y\}}$, and $L_{\{t\},\{x,y,z\}}$ on the surface S_λ has the same rank as the intersection matrix of the curves $L_{\{y\},\{t\}}$, $L_{\{y\},\{x,t\}}$, $L_{\{z\},\{t\}}$, $L_{\{z\},\{x,y\}}$, $L_{\{t\},\{x,y,z\}}$, and H_λ. Moreover, we have $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk})=\operatorname{rk}\operatorname{Pic}(S_{\Bbbk})+11$. Thus, we see that ($\bigstar$) holds, so that ($\diamondsuit$) in the Main Theorem holds by Lemma 1.13.1.

Family 3.19. The threefold X can be obtained by blowing up a smooth quadric hypersurface in $\mathbb{P}^3$ at two points, so that $h^{1,2}(X)=0$. A toric Landau–Ginzburg model of this family is given by the Minkowski polynomial 74, which is

$$\frac{z}{x}+\frac{1}{x}+y+z+x+\frac{1}{yz}+\frac{x}{yz}.$$

The quartic pencil $\mathcal{S}$ is given by the following equations:

$$z^2ty+t^2yz+y^2xz+z^2xy+x^2yz+t^3x+x^2t^2=\lambda xyzt.$$

Suppose that $\lambda\neq\infty$. Then

$$\begin{aligned}
H_{\{x\}}\cdot S_\lambda &= L_{\{x\},\{y\}}+L_{\{x\},\{z\}}+L_{\{x\},\{t\}}+L_{\{x\},\{z,t\}},\\
H_{\{y\}}\cdot S_\lambda &= L_{\{x\},\{y\}}+2L_{\{y\},\{t\}}+L_{\{y\},\{x,t\}},\\
H_{\{z\}}\cdot S_\lambda &= L_{\{x\},\{z\}}+2L_{\{z\},\{t\}}+L_{\{z\},\{x,t\}},\\
H_{\{t\}}\cdot S_\lambda &= L_{\{x\},\{t\}}+L_{\{y\},\{t\}}+L_{\{z\},\{t\}}+L_{\{t\},\{x,y,z\}}.
\end{aligned}\tag{3.19.1}$$

Thus, the base locus of the pencil $\mathcal{S}$ consists of the lines $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{z,t\}}$, $L_{\{y\},\{x,t\}}$, $L_{\{z\},\{x,t\}}$, and $L_{\{t\},\{x,y,z\}}$.

For every $\lambda\in\mathbb{C}$, the quartic surface S_λ has isolated singularities, so that it is irreducible. Moreover, the singular points of S_λ contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

$P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_4$ with quadratic term $y(x+t)$;

$P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_4$ with quadratic term xz;

$P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_1$;

$P_{\{y\},\{t\},\{x,z\}}$: type $\mathbb{A}_1$;

$P_{\{z\},\{t\},\{x,y\}}$: type $\mathbb{A}_1$.

Then each fiber $\mathsf{f}^{-1}(\lambda)$ is irreducible by Corollary 1.5.4. This confirms ($\heartsuit$) in the Main Theorem.

Let us verify ($\diamondsuit$) in the Main Theorem. It follows from (3.19.1) that the intersection matrix of the lines $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{z,t\}}$, $L_{\{y\},\{x,t\}}$, $L_{\{z\},\{x,t\}}$, and $L_{\{t\},\{x,y,z\}}$ on the surface S_λ has the same rank as the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{z,t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{t\},\{x,y,z\}}$, and H_λ. The latter matrix is given by

$$\begin{array}{c@{}c}
 & \begin{array}{cccccc} L_{\{x\},\{y\}} & L_{\{x\},\{z,t\}} & L_{\{y\},\{t\}} & L_{\{z\},\{t\}} & L_{\{t\},\{x,y,z\}} & H_\lambda \end{array}\\
\begin{array}{r} L_{\{x\},\{y\}}\\ L_{\{x\},\{z,t\}}\\ L_{\{y\},\{t\}}\\ L_{\{z\},\{t\}}\\ L_{\{t\},\{x,y,z\}}\\ H_\lambda \end{array} &
\begin{pmatrix}
-6/5 & 1 & 4/5 & 0 & 0 & 1\\
1 & -6/5 & 0 & 1/5 & 0 & 1\\
4/5 & 0 & -1/5 & 1/2 & 1/2 & 1\\
0 & 1/5 & 1/2 & -1/5 & 1/2 & 1\\
0 & 1 & 1/2 & 1/2 & -1 & 1\\
1 & 1 & 1 & 1 & 1 & 4
\end{pmatrix}
\end{array}.$$

The rank of this matrix is 6. Moreover, we have $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk})=\operatorname{rk}\operatorname{Pic}(S_{\Bbbk})+11$. Thus, we conclude that ($\bigstar$) holds. Then ($\diamondsuit$) in the Main Theorem holds by Lemma 1.13.1.

Family 3.20. In this case, the threefold X is a blow-up of the smooth quadric threefold along a disjoint union of two lines, so that $h^{1,2}(X)=0$. A toric Landau–Ginzburg model of this family is given by the Minkowski polynomial 79, which is

$$\frac{y}{x}+\frac{1}{x}+y+z+\frac{1}{y}+x+\frac{x}{yz}.$$

The quartic pencil $\mathcal{S}$ is given by the following equation:

$$y^2tz+t^2yz+y^2xz+z^2xy+t^2xz+x^2yz+x^2t^2=\lambda xyzt.$$

As usual, we suppose that $\lambda\neq\infty$. Then

$$\begin{aligned}
H_{\{x\}}\cdot S_\lambda&=L_{\{x\},\{y\}}+L_{\{x\},\{z\}}+L_{\{x\},\{t\}}+L_{\{x\},\{y,t\}},\\
H_{\{y\}}\cdot S_\lambda&=L_{\{x\},\{y\}}+2L_{\{y\},\{t\}}+L_{\{y\},\{x,z\}},\\
H_{\{z\}}\cdot S_\lambda&=2L_{\{x\},\{z\}}+2L_{\{z\},\{t\}},\\
H_{\{t\}}\cdot S_\lambda&=L_{\{x\},\{t\}}+L_{\{y\},\{t\}}+L_{\{z\},\{t\}}+L_{\{t\},\{x,y,z\}}.
\end{aligned}\tag{3.20.1}$$

Thus, the base locus of the pencil $\mathcal{S}$ consists of the lines $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{y,t\}}$, $L_{\{y\},\{x,z\}}$, and $L_{\{t\},\{x,y,z\}}$.

For every $\lambda\in\mathbb{C}$, the surface S_λ has isolated singularities, so that it is irreducible. The singular points of S_λ contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

$P_{\{x\},\{y\},\{z\}}$: type $\mathbb{A}_1$;

$P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term xy;

$P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term $z(x+t)$;

$P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_1$;

$P_{\{x\},\{y\},\{z,t\}}$: type $\mathbb{A}_1$ for $\lambda\neq-1$, and type $\mathbb{A}_2$ for $\lambda=-1$;

$P_{\{y\},\{t\},\{x,z\}}$: type $\mathbb{A}_2$ with quadratic term $y(x+y+z-\lambda t)$ for $\lambda\neq0$, and type $\mathbb{A}_3$ for $\lambda=0$;

$P_{\{z\},\{t\},\{x,y\}}$: type $\mathbb{A}_1$.

By Corollary 1.5.4, each fiber $\mathsf{f}^{-1}(\lambda)$ is irreducible. This confirms ($\heartsuit$) in the Main Theorem.

If $\lambda\neq0,-1$, then the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{y,t\}}$, $L_{\{y\},\{x,z\}}$, $L_{\{t\},\{x,y,z\}}$, and H_λ on the surface S_λ is given by

$$\begin{array}{c}
\\ L_{\{x\},\{y\}}\\ L_{\{x\},\{y,t\}}\\ L_{\{y\},\{x,z\}}\\ L_{\{t\},\{x,y,z\}}\\ H_\lambda
\end{array}
\begin{array}{c}
\begin{array}{ccccc} L_{\{x\},\{y\}} & L_{\{x\},\{y,t\}} & L_{\{y\},\{x,z\}} & L_{\{t\},\{x,y,z\}} & H_\lambda \end{array}\\
\begin{pmatrix}
0 & 1/2 & 1/2 & 0 & 1\\
1/2 & -5/4 & 0 & 0 & 1\\
1/2 & 0 & -5/6 & 1/3 & 1\\
0 & 0 & 1/3 & -5/6 & 1\\
1 & 1 & 1 & 1 & 4
\end{pmatrix}
\end{array}.$$

The rank of this matrix is 5. On the other hand, it follows from (3.20.1) that

$$\begin{aligned}
L_{\{x\},\{y\}}+L_{\{x\},\{z\}}+L_{\{x\},\{t\}}+L_{\{x\},\{y,t\}}&\sim L_{\{x\},\{y\}}+2L_{\{y\},\{t\}}+L_{\{y\},\{x,z\}}\\
&\sim 2L_{\{x\},\{z\}}+2L_{\{z\},\{t\}}\sim L_{\{x\},\{t\}}+L_{\{y\},\{t\}}+L_{\{z\},\{t\}}+L_{\{t\},\{x,y,z\}}\sim H_\lambda
\end{aligned}$$

on the surface S_λ. Thus, if $\lambda\neq0,-1$, then the rank of the intersection matrix of the lines $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{y,t\}}$, $L_{\{y\},\{x,z\}}$, and $L_{\{t\},\{x,y,z\}}$ on the surface S_λ is also 5. Moreover, we have $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk})=\operatorname{rk}\operatorname{Pic}(S_{\Bbbk})+12$. Thus, we see that ($\bigstar$) holds, so that ($\diamondsuit$) in the Main Theorem holds by Lemma 1.13.1.

Family 3.21. In this case, the threefold X is a blow-up of $\mathbb{P}^1 \times \mathbb{P}^2$ in a curve of bidegree $(2,1)$, so that $h^{1,2}(X) = 0$. A toric Landau–Ginzburg model of this family is given by the Minkowski polynomial 213, which is

$$\frac{z}{y} + x + \frac{1}{y} + z + \frac{z}{xy} + \frac{1}{z} + y + \frac{1}{x}.$$

The quartic pencil $\mathcal{S}$ is given by the equation

$$z^2xt + x^2yz + t^2xz + z^2xy + t^2z^2 + t^2xy + y^2xz + t^2yz = \lambda xyzt.$$

Suppose that $\lambda \neq \infty$. Let $\mathcal{C}$ be the conic in $\mathbb{P}^3$ that is given by $y = xz + xt + zt = 0$. Then

$$\begin{aligned} H_{\{x\}} \cdot S_\lambda &= L_{\{x\},\{z\}} + 2L_{\{x\},\{t\}} + L_{\{x\},\{y,z\}}, \\ H_{\{y\}} \cdot S_\lambda &= L_{\{y\},\{z\}} + L_{\{y\},\{t\}} + \mathcal{C}, \\ H_{\{z\}} \cdot S_\lambda &= L_{\{x\},\{z\}} + L_{\{y\},\{z\}} + 2L_{\{z\},\{t\}}, \\ H_{\{t\}} \cdot S_\lambda &= L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y,z\}}. \end{aligned} \tag{3.21.1}$$

Thus, the base locus of the pencil $\mathcal{S}$ consists of the curves $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{y,z\}}$, $L_{\{t\},\{x,y,z\}}$, and $\mathcal{C}$.

For every $\lambda \in \mathbb{C}$, the surface S_λ has isolated singularities, so that it is irreducible. Moreover, the singular points of S_λ contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

- $P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_1$;
- $P_{\{x\},\{y\},\{z\}}$: type $\mathbb{A}_2$ with quadratic term $(x+z)(y+z)$ for $\lambda \neq -1$, and type $\mathbb{A}_3$ for $\lambda = -1$;
- $P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term xz;
- $P_{\{x\},\{t\},\{y,z\}}$: type $\mathbb{A}_2$ with quadratic term $x(x+y+z-t-\lambda t)$ for $\lambda \neq -1$, and type $\mathbb{A}_3$ for $\lambda = -1$;
- $P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term yz;
- $P_{\{z\},\{t\},\{x,y\}}$: type $\mathbb{A}_1$.

By Corollary 1.5.4, each fiber $\mathsf{f}^{-1}(\lambda)$ is irreducible. This confirms ($\heartsuit$) in the Main Theorem.

Now let us show that ($\diamondsuit$) in the Main Theorem also holds in this case. To this end, we may assume that $\lambda \neq -1$. Then the intersection matrix of the curves $L_{\{x\},\{z\}}$, $L_{\{x\},\{y,z\}}$, $L_{\{y\},\{t\}}$, $L_{\{t\},\{x,y,z\}}$, and H_λ on the surface S_λ is given by

$$\begin{array}{r c} & \begin{array}{ccccc} L_{\{x\},\{z\}} & L_{\{x\},\{y,z\}} & L_{\{y\},\{t\}} & L_{\{t\},\{x,y,z\}} & H_\lambda \end{array} \\ \begin{array}{r} L_{\{x\},\{z\}} \\ L_{\{x\},\{y,z\}} \\ L_{\{y\},\{t\}} \\ L_{\{t\},\{x,y,z\}} \\ H_\lambda \end{array} & \begin{pmatrix} -1/3 & 1/3 & 0 & 0 & 1 \\ 1/3 & -2/3 & 0 & 1/3 & 1 \\ 0 & 0 & -3/4 & 1 & 1 \\ 0 & 1/3 & 1 & -5/6 & 1 \\ 1 & 1 & 1 & 1 & 4 \end{pmatrix} \end{array}.$$

The rank of this intersection matrix is 5. On the other hand, it follows from (3.21.1) that

$$\begin{aligned} H_\lambda &\sim L_{\{x\},\{z\}} + 2L_{\{x\},\{t\}} + L_{\{x\},\{y,z\}} \sim L_{\{y\},\{z\}} + L_{\{y\},\{t\}} + \mathcal{C} \\ &\sim L_{\{x\},\{z\}} + L_{\{y\},\{z\}} + 2L_{\{z\},\{t\}} \sim L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y,z\}} \end{aligned}$$

on the surface S_λ. Thus, the rank of the intersection matrix of the curves $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{y,z\}}$, $L_{\{t\},\{x,y,z\}}$, and $\mathcal{C}$ on the surface S_λ is also 5. Moreover, we have $\mathrm{rk}\,\mathrm{Pic}(\widetilde{S}_{\Bbbk}) = \mathrm{rk}\,\mathrm{Pic}(S_{\Bbbk}) + 12$. Thus, we see that ($\bigstar$) holds, so that ($\diamondsuit$) in the Main Theorem holds by Lemma 1.13.1.

Family 3.22. In this case, the threefold X is a blow-up of $\mathbb{P}^1 \times \mathbb{P}^2$ in a conic contained in a fiber of the projection $\mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^1$. Thus, we have $h^{1,2}(X) = 0$. A toric Landau–Ginzburg model of this family is given by the Minkowski polynomial 75, which is

$$\frac{z}{x} + \frac{1}{x} + y + z + \frac{1}{xyz} + x + \frac{1}{yz}.$$

The quartic pencil $\mathcal{S}$ is given by

$$z^2ty + t^2yz + y^2xz + z^2xy + t^4 + x^2yz + t^3x = \lambda xyzt.$$

Let $\mathcal{C}$ be a cubic curve in $\mathbb{P}^3$ that is given by $x = yz^2 + yzt + t^3 = 0$. Then $\mathcal{C}$ is singular at the point $P_{\{x\},\{z\},\{t\}}$. Moreover, if $\lambda \neq \infty$, then

$$\begin{aligned}
&H_{\{x\}} \cdot S_\lambda = L_{\{x\},\{t\}} + \mathcal{C},\\
&H_{\{y\}} \cdot S_\lambda = 3L_{\{y\},\{t\}} + L_{\{y\},\{x,t\}},\\
&H_{\{z\}} \cdot S_\lambda = 3L_{\{z\},\{t\}} + L_{\{z\},\{x,t\}},\\
&H_{\{t\}} \cdot S_\lambda = L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y,z\}}.
\end{aligned} \tag{3.22.1}$$

Therefore, the base locus of the pencil $\mathcal{S}$ consists of the curves $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{y\},\{x,t\}}$, $L_{\{z\},\{x,t\}}$, $L_{\{t\},\{x,y,z\}}$, and $\mathcal{C}$.

For every $\lambda \neq \infty$, the surface S_λ has isolated singularities, which implies that S_λ is irreducible. In this case, the singular points of S_λ contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

$P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_4$ with quadratic term $y(x+t)$ for $\lambda \neq -2$, and type $\mathbb{A}_6$ for $\lambda = -2$;

$P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term xz;

$P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_2$ with quadratic term yz;

$P_{\{y\},\{t\},\{x,z\}}$: type $\mathbb{A}_2$ with quadratic term $y(x+y+z-t-\lambda t)$;

$P_{\{z\},\{t\},\{x,y\}}$: type $\mathbb{A}_2$ with quadratic term $z(x+y+z-\lambda t)$.

Thus, it follows from Corollary 1.5.4 that $[\mathsf{f}^{-1}(\lambda)] = 1$ for every $\lambda \in \mathbb{C}$. This confirms ($\heartsuit$) in the Main Theorem, since $h^{1,2}(X) = 0$.

Let us verify ($\diamondsuit$) in the Main Theorem. If $\lambda \neq \infty$, then

$$\begin{aligned}
H_\lambda &\sim L_{\{x\},\{t\}} + \mathcal{C} \sim 3L_{\{y\},\{t\}} + L_{\{y\},\{x,t\}}\\
&\sim 3L_{\{z\},\{t\}} + L_{\{z\},\{x,t\}} \sim L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y,z\}}
\end{aligned}$$

on the surface S_λ. This follows from (3.22.1). Thus, if $\lambda \neq \infty$, then the intersection matrix of $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{y\},\{x,t\}}$, $L_{\{z\},\{x,t\}}$, $L_{\{t\},\{x,y,z\}}$, and $\mathcal{C}$ on the surface S_λ has the same rank as the intersection matrix of the curves $L_{\{y\},\{x,t\}}$, $L_{\{z\},\{x,t\}}$, $L_{\{t\},\{x,y,z\}}$, and H_λ. If $\lambda \neq \infty, -2$, then the latter matrix is

$$\begin{array}{c c}
 & \begin{array}{cccc} L_{\{y\},\{x,t\}} & L_{\{z\},\{x,t\}} & L_{\{t\},\{x,y,z\}} & H_\lambda \end{array}\\
\begin{array}{c} L_{\{y\},\{x,t\}} \\ L_{\{z\},\{x,t\}} \\ L_{\{t\},\{x,y,z\}} \\ H_\lambda \end{array} &
\begin{pmatrix} -4/5 & 1 & 0 & 1\\ 1 & -5/4 & 0 & 1\\ 0 & 0 & -2/3 & 1\\ 1 & 1 & 1 & 4 \end{pmatrix}
\end{array}.$$

The rank of this matrix is 4. On the other hand, we have $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk}) = \operatorname{rk}\operatorname{Pic}(S_{\Bbbk}) + 13$. Thus, we see that ($\bigstar$) holds, so that ($\diamondsuit$) in the Main Theorem holds by Lemma 1.13.1.

Family 3.23. In this case, the threefold X is a blow-up of $\mathbb{P}^3$ blown up at a point at the proper transform of a conic passing through this point. A toric Landau–Ginzburg model of this family is given by the Minkowski polynomial 76, which is

$$\frac{z}{x}+\frac{1}{x}+y+z+\frac{1}{xy}+x+\frac{1}{yz}.$$

The pencil $\mathcal{S}$ is given by the following equation:

$$z^2ty+t^2yz+y^2xz+z^2xy+t^3z+x^2yz+t^3x=\lambda xyzt.$$

Suppose that $\lambda\neq\infty$. Let $\mathcal{C}$ be the conic in $\mathbb{P}^3$ given by $x=yz+yt+t^2=0$. Then

$$\begin{aligned}
&H_{\{x\}}\cdot S_\lambda=L_{\{x\},\{z\}}+L_{\{x\},\{t\}}+\mathcal{C},\\
&H_{\{y\}}\cdot S_\lambda=3L_{\{y\},\{t\}}+L_{\{y\},\{x,z\}},\\
&H_{\{z\}}\cdot S_\lambda=L_{\{x\},\{z\}}+3L_{\{z\},\{t\}},\\
&H_{\{t\}}\cdot S_\lambda=L_{\{x\},\{t\}}+L_{\{y\},\{t\}}+L_{\{z\},\{t\}}+L_{\{t\},\{x,y,z\}}.
\end{aligned}\tag{3.23.1}$$

Thus, the base locus of the pencil $\mathcal{S}$ consists of the curves $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{y\},\{x,z\}}$, $L_{\{t\},\{x,y,z\}}$, and $\mathcal{C}$.

Observe that S_λ has isolated singularities. In particular, it is irreducible. Moreover, its singular points contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

$P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_2$ with quadratic term $y(x+t)$;
$P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_4$ with quadratic term xz;
$P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_2$ with quadratic term yz;
$P_{\{y\},\{t\},\{x,z\}}$: type $\mathbb{A}_3$ with quadratic term $y(x+y+z-t-\lambda t)$ for $\lambda\neq-1$, and type $\mathbb{A}_4$ for $\lambda=-1$;
$P_{\{z\},\{t\},\{x,y\}}$: type $\mathbb{A}_2$ with quadratic term $z(x+y+z-\lambda t)$.

Thus, by Corollary 1.5.4, every fiber $\mathsf{f}^{-1}(\lambda)$ is irreducible. This confirms (♡) in the Main Theorem, since $h^{1,2}(X)=0$.

To check (♢) in the Main Theorem, we may assume that $\lambda\neq-1$. Then the intersection matrix of the curves $L_{\{x\},\{t\}}$, $L_{\{y\},\{x,z\}}$, $L_{\{t\},\{x,y,z\}}$, and H_λ on the surface S_λ is given by

$$\begin{array}{r c}
 & \begin{array}{cccc} L_{\{x\},\{t\}} & L_{\{y\},\{x,z\}} & L_{\{t\},\{x,y,z\}} & H_\lambda \end{array}\\
\begin{array}{r} L_{\{x\},\{t\}}\\ L_{\{y\},\{x,z\}}\\ L_{\{t\},\{x,y,z\}}\\ H_\lambda \end{array} & \begin{pmatrix} -8/15 & 0 & 1 & 1\\ 0 & -5/4 & 1/4 & 1\\ 1 & 1/4 & -1/2 & 1\\ 1 & 1 & 1 & 4 \end{pmatrix}
\end{array}.$$

This matrix has rank 4. On the other hand, it follows from (3.23.1) that

$$\begin{aligned}
H_\lambda&\sim L_{\{x\},\{z\}}+L_{\{x\},\{t\}}+\mathcal{C}\sim 3L_{\{y\},\{t\}}+L_{\{y\},\{x,z\}}\\
&\sim L_{\{x\},\{z\}}+3L_{\{z\},\{t\}}\sim L_{\{x\},\{t\}}+L_{\{y\},\{t\}}+L_{\{z\},\{t\}}+L_{\{t\},\{x,y,z\}}
\end{aligned}$$

on the surface S_λ. Hence, the rank of the intersection matrix of the curves $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{y\},\{x,z\}}$, $L_{\{t\},\{x,y,z\}}$, and $\mathcal{C}$ on the surface S_λ is also 4. Using the description of the singular points of the surface S_λ, we see that $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk})=\operatorname{rk}\operatorname{Pic}(S_{\Bbbk})+13$. Thus, we see that (★) holds, so that (♢) in the Main Theorem holds by Lemma 1.13.1.

Family 3.24. In this case, the threefold X is a complete intersection of divisors of tridegrees $(1,1,0)$ and $(0,1,1)$ on $\mathbb{P}^1\times\mathbb{P}^2\times\mathbb{P}^2$. Its toric Landau–Ginzburg model is given by the Minkowski polynomial 77, which is

$$x+y+z+\frac{y}{x}+\frac{1}{y}+\frac{1}{x}+\frac{1}{xyz}.$$

Thus, the quartic pencil $\mathcal{S}$ is given by

$$x^2yz+y^2xz+z^2xy+y^2tz+t^2xz+t^2yz+t^4=\lambda xyzt.$$

Let $\mathcal{C}_1$ be the cubic curve in $\mathbb{P}^3$ that is given by $x=y^2z+yzt+t^3=0$. Then $\mathcal{C}_1$ is singular at the point $P_{\{x\},\{y\},\{t\}}$, but its proper transform on U is a smooth rational curve. Let $\mathcal{C}_2$ be the conic in $\mathbb{P}^3$ that is given by $y=xz+t^2=0$. If $\lambda\neq\infty$, then

$$\begin{aligned}
H_{\{x\}}\cdot S_\lambda&=L_{\{x\},\{t\}}+\mathcal{C}_1,\\
H_{\{y\}}\cdot S_\lambda&=2L_{\{y\},\{t\}}+\mathcal{C}_2,\\
H_{\{z\}}\cdot S_\lambda&=4L_{\{z\},\{t\}},\\
H_{\{t\}}\cdot S_\lambda&=L_{\{x\},\{t\}}+L_{\{y\},\{t\}}+L_{\{z\},\{t\}}+L_{\{t\},\{x,y,z\}}.
\end{aligned}\tag{3.24.1}$$

Thus, the base locus of the pencil $\mathcal{S}$ is a union of the curves $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{t\},\{x,y,z\}}$, $\mathcal{C}_1$, and $\mathcal{C}_2$.

If $\lambda\neq\infty$, then the quartic surface S_λ has isolated singularities, so that it is irreducible. In this case, its singular points contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

$P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term xy;

$P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term $z(x+t)$;

$P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term yz;

$P_{\{x\},\{t\},\{y,z\}}$: type $\mathbb{A}_1$ for $\lambda\neq-5/2$, and type $\mathbb{A}_2$ for $\lambda=-5/2$;

$P_{\{y\},\{t\},\{x,z\}}$: type $\mathbb{A}_1$;

$P_{\{z\},\{t\},\{x,y\}}$: type $\mathbb{A}_3$ with quadratic term $z(x+y+z-t-\lambda t)$.

Thus, by Corollary 1.5.4, every fiber $\mathsf{f}^{-1}(\lambda)$ is irreducible. This confirms (♡) in the Main Theorem, since $h^{1,2}(X)=0$ in this case.

If $\lambda\neq\infty$, then it follows from (3.24.1) that

$$L_{\{x\},\{t\}}+\mathcal{C}_1\sim 2L_{\{y\},\{t\}}+\mathcal{C}_2\sim 4L_{\{z\},\{t\}}\sim L_{\{x\},\{t\}}+L_{\{y\},\{t\}}+L_{\{z\},\{t\}}+L_{\{t\},\{x,y,z\}}\sim H_\lambda.$$

In this case, the intersection matrix of the curves $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{t\},\{x,y,z\}}$, $\mathcal{C}_1$, and $\mathcal{C}_2$ on the surface S_λ has the same rank as the intersection matrix of the curves $L_{\{x\},\{t\}}$, $L_{\{t\},\{x,y,z\}}$, and H_λ. On the other hand, if $\lambda\neq\infty,-5/2$, then the latter matrix is given by

$$\begin{array}{c|ccc}
 & L_{\{x\},\{t\}} & L_{\{t\},\{x,y,z\}} & H_\lambda\\
\hline
L_{\{x\},\{t\}} & 0 & 1/2 & 1\\
L_{\{t\},\{x,y,z\}} & 1/2 & -1/4 & 1\\
H_\lambda & 1 & 1 & 4
\end{array}.$$

The rank of this matrix is 3. Thus, if $\lambda\neq\infty,-5/2$, then the rank of the intersection matrix of the curves $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{t\},\{x,y,z\}}$, $\mathcal{C}_1$, and $\mathcal{C}_2$ on the surface S_λ is also 3. This implies (★), because $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk})=\operatorname{rk}\operatorname{Pic}(S_{\Bbbk})+14$. Then (♢) in the Main Theorem holds by Lemma 1.13.1.

Family 3.25. In this case, the threefold X is a blow-up of $\mathbb{P}^3$ in a disjoint union of two lines. Thus, we have $h^{1,2}(X)=0$. A toric Landau–Ginzburg model of this family is given by the Minkowski polynomial 24, which is

$$x+y+z+\frac{x}{z}+\frac{1}{x}+\frac{1}{xy}.$$

Hence, the quartic pencil $\mathcal{S}$ is given by the following equation:

$$x^2yz+y^2xz+z^2xy+x^2ty+t^2yz+t^3z=\lambda xyzt.$$

Suppose that $\lambda\neq\infty$. Then

$$\begin{aligned}
H_{\{x\}}\cdot S_\lambda&=L_{\{x\},\{z\}}+2L_{\{x\},\{t\}}+L_{\{x\},\{y,t\}},\\
H_{\{y\}}\cdot S_\lambda&=L_{\{y\},\{z\}}+3L_{\{y\},\{t\}},\\
H_{\{z\}}\cdot S_\lambda&=2L_{\{x\},\{z\}}+L_{\{y\},\{z\}}+L_{\{z\},\{t\}},\\
H_{\{t\}}\cdot S_\lambda&=L_{\{x\},\{t\}}+L_{\{y\},\{t\}}+L_{\{z\},\{t\}}+L_{\{t\},\{x,y,z\}}.
\end{aligned}\tag{3.25.1}$$

Thus, the base locus of the pencil $\mathcal{S}$ consists of the lines $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{y,t\}}$, and $L_{\{t\},\{x,y,z\}}$.

Observe that S_λ has isolated singularities, so that it is irreducible. Moreover, its singular points contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

$P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_2$ with quadratic term xy;

$P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_4$ with quadratic term xz;

$P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term $y(z+t)$;

$P_{\{x\},\{z\},\{y,t\}}$: type $\mathbb{A}_1$;

$P_{\{x\},\{t\},\{y,z\}}$: type $\mathbb{A}_1$;

$P_{\{y\},\{t\},\{x,z\}}$: type $\mathbb{A}_2$ with quadratic term $y(x+y+z+(\lambda+1)t)$.

Therefore, by Corollary 1.5.4, every fiber $\mathsf{f}^{-1}(\lambda)$ is irreducible. This confirms ($\heartsuit$) in the Main Theorem, since $h^{1,2}(X)=0$.

Let us verify ($\diamondsuit$) in the Main Theorem. It follows from (3.25.1) that

$$\begin{aligned}
H_\lambda&\sim L_{\{x\},\{z\}}+2L_{\{x\},\{t\}}+L_{\{x\},\{y,t\}}\sim L_{\{y\},\{z\}}+3L_{\{y\},\{t\}}\\
&\sim 2L_{\{x\},\{z\}}+L_{\{y\},\{z\}}+L_{\{z\},\{t\}}\sim L_{\{x\},\{t\}}+L_{\{y\},\{t\}}+L_{\{z\},\{t\}}+L_{\{t\},\{x,y,z\}}
\end{aligned}$$

on the surface S_λ. Thus, the intersection matrix of the lines $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{y,t\}}$, and $L_{\{t\},\{x,y,z\}}$ has the same rank as the intersection matrix of the curves $L_{\{x\},\{y,t\}}$, $L_{\{y\},\{z\}}$, $L_{\{t\},\{x,y,z\}}$, and H_λ. The latter matrix is given by

$$\begin{array}{c}
\\ L_{\{x\},\{y,t\}}\\ L_{\{y\},\{t\}}\\ L_{\{t\},\{x,y,z\}}\\ H_\lambda
\end{array}
\begin{array}{c}
\begin{array}{cccc} L_{\{x\},\{y,t\}} & L_{\{z\},\{t\}} & L_{\{t\},\{x,y,z\}} & H_\lambda \end{array}\\
\begin{pmatrix}
-5/6 & 0 & 0 & 1\\
0 & -4/3 & 1/2 & 1\\
0 & 1/2 & -1/2 & 1\\
1 & 1 & 1 & 4
\end{pmatrix}
\end{array}.$$

This matrix has rank 4. This gives ($\bigstar$), since $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk})=\operatorname{rk}\operatorname{Pic}(S_{\Bbbk})+13$. Thus, we see that ($\diamondsuit$) in the Main Theorem holds by Lemma 1.13.1.

Family 3.26. The threefold X can be obtained from $\mathbb{P}^3$ by blowing up a disjoint union of a point and a line, so that $h^{1,2}(X) = 0$. A toric Landau–Ginzburg model of this family is given by the Minkowski polynomial 25, which is

$$\frac{y}{x} + \frac{1}{x} + y + z + x + \frac{1}{yz}.$$

Then the pencil $\mathcal{S}$ is given by the following equation:

$$y^2tz + t^2yz + y^2xz + z^2xy + x^2yz + t^3x = \lambda xyzt.$$

Suppose that $\lambda \neq \infty$. Then

$$\begin{aligned}
H_{\{x\}} \cdot S_\lambda &= L_{\{x\},\{y\}} + L_{\{x\},\{z\}} + L_{\{x\},\{t\}} + L_{\{x\},\{y,t\}},\\
H_{\{y\}} \cdot S_\lambda &= L_{\{x\},\{y\}} + 3L_{\{y\},\{t\}},\\
H_{\{z\}} \cdot S_\lambda &= L_{\{x\},\{z\}} + 3L_{\{z\},\{t\}},\\
H_{\{t\}} \cdot S_\lambda &= L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y,z\}}.
\end{aligned} \tag{3.26.1}$$

Thus, the base locus of the pencil $\mathcal{S}$ consists of the lines $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{y,t\}}$, and $L_{\{t\},\{x,y,z\}}$.

The surface S_λ has isolated singularities, so that it is irreducible. Its singular points contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

$P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_4$ with quadratic term xy;
$P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term $z(x+t)$;
$P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_2$ with quadratic term yz;
$P_{\{y\},\{t\},\{x,z\}}$: type $\mathbb{A}_2$ with quadratic term $y(x+y-\lambda t)$;
$P_{\{z\},\{t\},\{x,y\}}$: type $\mathbb{A}_2$ with quadratic term $z(x+y+z-t-\lambda t)$.

By Corollary 1.5.4, every fiber $\mathsf{f}^{-1}(\lambda)$ is irreducible. This confirms ($\heartsuit$) in the Main Theorem.

To verify ($\diamondsuit$) in the Main Theorem, observe that the intersection form of the curves $L_{\{x\},\{y,t\}}$, $L_{\{z\},\{t\}}$, $L_{\{t\},\{x,y,z\}}$, and H_λ on the surface S_λ is given by the following matrix:

$$\begin{array}{r c}
 & \begin{array}{cccc} L_{\{x\},\{y,t\}} & L_{\{z\},\{t\}} & L_{\{t\},\{x,y,z\}} & H_\lambda \end{array} \\
\begin{array}{r} L_{\{x\},\{y,t\}} \\ L_{\{z\},\{t\}} \\ L_{\{t\},\{x,y,z\}} \\ H_\lambda \end{array} &
\begin{pmatrix} -5/4 & 0 & 0 & 1 \\ 0 & 1/12 & 1/3 & 1 \\ 0 & 1/3 & -2/3 & 1 \\ 1 & 1 & 1 & 4 \end{pmatrix}
\end{array}.$$

The rank of this matrix is 4. On the other hand, it follows from (3.26.1) that

$$\begin{aligned}
L_{\{x\},\{y\}} + L_{\{x\},\{z\}} + L_{\{x\},\{t\}} + L_{\{x\},\{y,t\}} &\sim L_{\{x\},\{y\}} + 3L_{\{y\},\{t\}}\\
&\sim L_{\{x\},\{z\}} + 3L_{\{z\},\{t\}} \sim L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y,z\}} \sim H_\lambda.
\end{aligned}$$

Thus, the rank of the intersection matrix of the lines $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{y,t\}}$, and $L_{\{t\},\{x,y,z\}}$ is also 4. Moreover, the description of the singular points of the surface S_λ easily implies $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk}) = \operatorname{rk}\operatorname{Pic}(S_{\Bbbk}) + 13$. Thus, we see that ($\bigstar$) holds, so that ($\diamondsuit$) in the Main Theorem holds by Lemma 1.13.1.

Family 3.27. We already discussed this case in Example 1.7.1, where we also described the pencil $\mathcal{S}$. Suppose that $\lambda \neq \infty$. Then

$$\begin{aligned} H_{\{x\}} \cdot S_\lambda &= L_{\{x\},\{y\}} + L_{\{x\},\{z\}} + 2L_{\{x\},\{t\}}, \\ H_{\{y\}} \cdot S_\lambda &= L_{\{x\},\{y\}} + L_{\{y\},\{z\}} + 2L_{\{y\},\{t\}}, \\ H_{\{z\}} \cdot S_\lambda &= L_{\{x\},\{z\}} + L_{\{y\},\{z\}} + 2L_{\{z\},\{t\}}, \\ H_{\{t\}} \cdot S_\lambda &= L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y,z\}}. \end{aligned} \tag{3.27.1}$$

Thus, the base locus of the pencil $\mathcal{S}$ consists of the lines $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, and $L_{\{t\},\{x,y,z\}}$.

The surface S_λ has isolated singularities, so that it is irreducible. Moreover, its singular points contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

$P_{\{x\},\{y\},\{z\}}$: type $\mathbb{A}_1$;
$P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term xy;
$P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term xz;
$P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term yz;
$P_{\{x\},\{t\},\{y,z\}}$: type $\mathbb{A}_1$;
$P_{\{y\},\{t\},\{x,z\}}$: type $\mathbb{A}_1$;
$P_{\{z\},\{t\},\{x,y\}}$: type $\mathbb{A}_1$.

By Corollary 1.5.4, we have $[\mathsf{f}^{-1}(\lambda)] = 1$. This confirms ($\heartsuit$) in the Main Theorem.

To prove ($\diamondsuit$) in the Main Theorem, observe that the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{y\},\{z\}}$, and H_λ on the surface S_λ is given by

$$\begin{array}{c} \\ L_{\{x\},\{y\}} \\ L_{\{x\},\{z\}} \\ L_{\{y\},\{z\}} \\ H_\lambda \end{array} \begin{array}{c} \begin{array}{cccc} L_{\{x\},\{y\}} & L_{\{x\},\{z\}} & L_{\{y\},\{z\}} & H_\lambda \end{array} \\ \begin{pmatrix} 1 & 1/2 & 1/2 & 1 \\ 1/2 & 1 & 1/2 & 1 \\ 1/2 & 1/2 & 1 & 1 \\ 1 & 1 & 1 & 4 \end{pmatrix} \end{array}.$$

The determinant of this matrix is $5/4$. On the other hand, it follows from (3.27.1) that

$$\begin{aligned} H_\lambda &\sim L_{\{x\},\{y\}} + L_{\{x\},\{z\}} + 2L_{\{x\},\{t\}} \sim L_{\{x\},\{y\}} + L_{\{y\},\{z\}} + 2L_{\{y\},\{t\}} \\ &\sim L_{\{x\},\{z\}} + L_{\{y\},\{z\}} + 2L_{\{z\},\{t\}} \sim L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y,z\}}. \end{aligned}$$

Thus, the rank of the intersection matrix of the lines $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, and $L_{\{t\},\{x,y,z\}}$ is 4. As we have seen above, the description of the singular points of the surface S_λ implies $\mathrm{rk}\,\mathrm{Pic}(\widetilde{S}_{\Bbbk}) = \mathrm{rk}\,\mathrm{Pic}(S_{\Bbbk}) + 13$, so that ($\bigstar$) holds. This gives ($\diamondsuit$) in the Main Theorem by Lemma 1.13.1.

Family 3.28. The threefold X is $\mathbb{P}^1 \times \mathbb{F}_1$, where $\mathbb{F}_1$ is a blow-up of $\mathbb{P}^2$ at a point. Thus, we have $h^{1,2}(X) = 0$. A toric Landau–Ginzburg model of this family is given by the Minkowski polynomial 29, which is

$$x + y + z + \frac{x}{z} + \frac{1}{x} + \frac{1}{y}.$$

Then the pencil $\mathcal{S}$ is given by

$$x^2yz + y^2xz + z^2xy + x^2ty + t^2xz + t^2yz = \lambda xyzt.$$

As usual, we suppose that $\lambda \neq \infty$. Then

$$
\begin{aligned}
H_{\{x\}} \cdot S_\lambda &= L_{\{x\},\{y\}} + L_{\{x\},\{z\}} + 2L_{\{x\},\{t\}},\\
H_{\{y\}} \cdot S_\lambda &= L_{\{x\},\{y\}} + L_{\{y\},\{z\}} + 2L_{\{y\},\{t\}},\\
H_{\{z\}} \cdot S_\lambda &= 2L_{\{x\},\{z\}} + L_{\{y\},\{z\}} + L_{\{z\},\{t\}},\\
H_{\{t\}} \cdot S_\lambda &= L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y,z\}}.
\end{aligned}
\tag{3.28.1}
$$

Thus, the base locus of the pencil $\mathcal{S}$ consists of the lines $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, and $L_{\{t\},\{x,y,z\}}$.

Each surface S_λ has isolated singularities. In particular, it is irreducible. Its singular points contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

$P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_2$ with quadratic term $y(z+t)$;

$P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_4$ with quadratic term xz;

$P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term xy;

$P_{\{x\},\{y\},\{z\}}$: type $\mathbb{A}_2$ with quadratic term $x(x+y)$;

$P_{\{x\},\{t\},\{y,z\}}$: type $\mathbb{A}_1$;

$P_{\{y\},\{t\},\{x,z\}}$: type $\mathbb{A}_1$.

Thus, each fiber $\mathsf{f}^{-1}(\lambda)$ is irreducible by Corollary 1.5.4. This confirms ($\heartsuit$) in the Main Theorem.

To verify ($\diamondsuit$) in the Main Theorem, observe that the intersection matrix of the curves $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{t\},\{x,y,z\}}$, and H_λ on the surface S_λ is given by

$$
\begin{array}{c}
\\ L_{\{y\},\{z\}} \\ L_{\{y\},\{t\}} \\ L_{\{t\},\{x,y,z\}} \\ H_\lambda
\end{array}
\begin{array}{c}
\begin{array}{cccc} L_{\{y\},\{z\}} & L_{\{y\},\{t\}} & L_{\{t\},\{x,y,z\}} & H_\lambda \end{array}\\
\begin{pmatrix}
-2/3 & 2/3 & 0 & 1\\
2/3 & -1/12 & 1/2 & 1\\
0 & 1/2 & -1 & 1\\
1 & 1 & 1 & 4
\end{pmatrix}.
\end{array}
$$

This matrix has rank 4. Using (3.28.1), we see that

$$
\begin{aligned}
H_\lambda &\sim L_{\{x\},\{y\}} + L_{\{x\},\{z\}} + 2L_{\{x\},\{t\}} \sim L_{\{x\},\{y\}} + L_{\{y\},\{z\}} + 2L_{\{y\},\{t\}}\\
&\sim 2L_{\{x\},\{z\}} + L_{\{y\},\{z\}} + L_{\{z\},\{t\}} \sim L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y,z\}}
\end{aligned}
$$

on the surface S_λ. Thus, the rank of the intersection matrix of the lines $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, and $L_{\{t\},\{x,y,z\}}$ on the surface S_λ is also 4. On the other hand, we have $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk}) = \operatorname{rk}\operatorname{Pic}(S_{\Bbbk}) + 13$. Thus, we see that ($\bigstar$) holds, so that ($\diamondsuit$) in the Main Theorem holds by Lemma 1.13.1.

Family 3.29. In this case, the threefold X is $\mathbb{P}^3$ blown up in a point and a line lying on the exceptional divisor of the blow-up of a point. We have $h^{1,2}(X) = 0$. A toric Landau–Ginzburg model of this family is given by the Minkowski polynomial 26, which is

$$\frac{y}{x} + \frac{1}{x} + y + z + \frac{1}{xyz} + x.$$

Hence, the pencil $\mathcal{S}$ is given by the equation

$$y^2tz + t^2yz + y^2xz + z^2xy + t^4 + x^2yz = \lambda xyzt.$$

Let $\mathcal{C}$ be the cubic curve in $\mathbb{P}^3$ that is given by $x = y^2z + yzt + t^3 = 0$. Then $\mathcal{C}$ is singular at the point $P_{\{x\},\{y\},\{t\}}$. If $\lambda \neq \infty$, then

$$\begin{aligned} H_{\{x\}} \cdot S_\lambda &= L_{\{x\},\{t\}} + \mathcal{C}, \\ H_{\{y\}} \cdot S_\lambda &= 4L_{\{y\},\{t\}}, \\ H_{\{z\}} \cdot S_\lambda &= 4L_{\{z\},\{t\}}, \\ H_{\{t\}} \cdot S_\lambda &= L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y,z\}}. \end{aligned} \tag{3.29.1}$$

Thus, the base locus of the pencil $\mathcal{S}$ consists of the curves $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{t\},\{x,y,z\}}$, and $\mathcal{C}$.

If $\lambda \neq \infty$, then S_λ has isolated singularities, so that it is irreducible. In this case, the singular points of S_λ contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

$P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term xy;

$P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term $z(x+t)$;

$P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term yz;

$P_{\{y\},\{t\},\{x,z\}}$: type $\mathbb{A}_3$ with quadratic term $y(x+y+z-\lambda t)$;

$P_{\{z\},\{t\},\{x,y\}}$: type $\mathbb{A}_3$ with quadratic term $z(x+y+z-t-\lambda t)$.

By Corollary 1.5.4, we have $[\mathsf{f}^{-1}(\lambda)]$ for every $\lambda \in \mathbb{C}$. This confirms ($\heartsuit$) in the Main Theorem.

To verify ($\diamondsuit$) in the Main Theorem, observe that

$$L_{\{x\},\{t\}} + \mathcal{C} \sim 4L_{\{y\},\{t\}} \sim 4L_{\{z\},\{t\}} \sim L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y,z\}}$$

on the surface S_λ with $\lambda \neq \infty$. This follows from (3.29.1). Thus, the intersection matrix of the curves $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{t\},\{x,y,z\}}$, and $\mathcal{C}$ on the surface S_λ has the same rank as the intersection matrix of the lines $L_{\{x\},\{t\}}$ and $L_{\{y\},\{t\}}$. The rank of the latter matrix is 2, because we have $L_{\{x\},\{t\}}^2 = L_{\{x\},\{t\}} \cdot L_{\{y\},\{t\}} = 1/4$ and $L_{\{x\},\{t\}}^2 = 1/2$. Moreover, we have $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk}) = \operatorname{rk}\operatorname{Pic}(S_{\Bbbk}) + 15$. This shows that ($\bigstar$) holds. Then ($\diamondsuit$) in the Main Theorem also holds by Lemma 1.13.1.

Family 3.30. The threefold X can be obtained from $\mathbb{P}^3$ blown up at a point by blowing up the proper transform of a line passing through this point. This shows that $h^{1,2}(X) = 0$. A toric Landau–Ginzburg model of this family is given by the Minkowski polynomial 28, which is

$$x + y + z + \frac{y}{z} + \frac{x}{y} + \frac{1}{x}.$$

In this case, the quartic pencil $\mathcal{S}$ is given by the equation

$$x^2yz + y^2xz + z^2xy + y^2xt + x^2tz + t^2yz = \lambda xyzt.$$

Suppose that $\lambda \neq \infty$. Then

$$\begin{aligned} H_{\{x\}} \cdot S_\lambda &= L_{\{x\},\{y\}} + L_{\{x\},\{z\}} + 2L_{\{x\},\{t\}}, \\ H_{\{y\}} \cdot S_\lambda &= 2L_{\{x\},\{y\}} + L_{\{y\},\{z\}} + L_{\{y\},\{t\}}, \\ H_{\{z\}} \cdot S_\lambda &= L_{\{x\},\{z\}} + 2L_{\{y\},\{z\}} + L_{\{z\},\{t\}}, \\ H_{\{t\}} \cdot S_\lambda &= L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y,z\}}. \end{aligned} \tag{3.30.1}$$

Thus, the base locus of the pencil $\mathcal{S}$ consists of the lines $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, and $L_{\{t\},\{x,y,z\}}$.

Each surface S_λ is irreducible and has isolated singularities. Moreover, its singular points contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

$P_{\{x\},\{y\},\{z\}}$: type $\mathbb{A}_4$ with quadratic term yz;

$P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_4$ with quadratic term xy;

$P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_2$ with quadratic term $x(z+t)$;

$P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_2$ with quadratic term $z(y+t)$;

$P_{\{x\},\{t\},\{y,z\}}$: type $\mathbb{A}_1$.

By Corollary 1.5.4, each fiber $\mathsf{f}^{-1}(\lambda)$ is irreducible. This confirms ($\heartsuit$) in the Main Theorem.

To verify ($\diamondsuit$) in the Main Theorem, observe that $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk}) = \operatorname{rk}\operatorname{Pic}(S_{\Bbbk}) + 13$. On the other hand, it follows from (3.30.1) that

$$\begin{aligned}H_\lambda &\sim L_{\{x\},\{y\}} + L_{\{x\},\{z\}} + 2L_{\{x\},\{t\}} \sim 2L_{\{x\},\{y\}} + L_{\{y\},\{z\}} + L_{\{y\},\{t\}}\\ &\sim L_{\{x\},\{z\}} + 2L_{\{y\},\{z\}} + L_{\{z\},\{t\}} \sim L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y,z\}}\end{aligned}$$

on the surface S_λ. Thus, the intersection matrix of the lines $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, and $L_{\{t\},\{x,y,z\}}$ has the same rank as the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{t\},\{x,y,z\}}$, and H_λ. The latter matrix is given by

$$\begin{array}{r c}
 & \begin{array}{cccc} L_{\{x\},\{y\}} & L_{\{x\},\{z\}} & L_{\{t\},\{x,y,z\}} & H_\lambda \end{array}\\
\begin{array}{r} L_{\{x\},\{y\}}\\ L_{\{x\},\{z\}}\\ L_{\{t\},\{x,y,z\}}\\ H_\lambda \end{array} & \begin{pmatrix} 0 & 1/4 & 0 & 1\\ 1/4 & -7/12 & 0 & 1\\ 0 & 0 & -3/2 & 1\\ 1 & 1 & 1 & 4 \end{pmatrix}.
\end{array}$$

Its rank is 4, so that ($\bigstar$) holds. Then ($\diamondsuit$) in the Main Theorem holds by Lemma 1.13.1.

Family 3.31. The threefold X can be obtained by blowing up an irreducible quadric cone in $\mathbb{P}^4$ at its vertex. This implies that $h^{1,2}(X)=0$. A toric Landau–Ginzburg model of this family is given by the Minkowski polynomial 27, which is

$$x+y+z+\frac{x}{z}+\frac{x}{y}+\frac{1}{x}.$$

Then the pencil $\mathcal{S}$ is given by the following equation:

$$t^2yz+tx^2y+tx^2z+x^2yz+xy^2z+xyz^2=\lambda xyzt.$$

Suppose that $\lambda\neq\infty$. Then

$$\begin{aligned}
H_{\{x\}}\cdot S_\lambda &= L_{\{x\},\{y\}} + L_{\{x\},\{z\}} + 2L_{\{x\},\{t\}},\\
H_{\{y\}}\cdot S_\lambda &= 2L_{\{x\},\{y\}} + L_{\{y\},\{z\}} + L_{\{y\},\{t\}},\\
H_{\{z\}}\cdot S_\lambda &= 2L_{\{x\},\{z\}} + L_{\{y\},\{z\}} + L_{\{z\},\{t\}},\\
H_{\{t\}}\cdot S_\lambda &= L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y,z\}}.
\end{aligned}\tag{3.31.1}$$

Thus, the base locus of the pencil $\mathcal{S}$ consists of the lines $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, and $L_{\{t\},\{x,y,z\}}$.

Each surface S_λ is irreducible, it has isolated singularities, and its singular points contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

$P_{\{x\},\{y\},\{z\}}$: type $\mathbb{A}_3$ with quadratic term yz;

$P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_4$ with quadratic term xy;

$P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_4$ with quadratic term xz;

$P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_1$;

$P_{\{x\},\{t\},\{y,z\}}$: type $\mathbb{A}_1$.

Thus, by Corollary 1.5.4, every fiber $\mathsf{f}^{-1}(\lambda)$ is irreducible. This confirms ($\heartsuit$) in the Main Theorem, since $h^{1,2}(X)=0$.

To verify ($\diamondsuit$) in the Main Theorem, observe that $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk})=\operatorname{rk}\operatorname{Pic}(S_{\Bbbk})+13$. Moreover, it follows from (3.31.1) that

$$H_\lambda \sim L_{\{x\},\{y\}}+L_{\{x\},\{z\}}+2L_{\{x\},\{t\}} \sim 2L_{\{x\},\{y\}}+L_{\{y\},\{z\}}+L_{\{y\},\{t\}}$$
$$\sim 2L_{\{x\},\{z\}}+L_{\{y\},\{z\}}+L_{\{z\},\{t\}} \sim L_{\{x\},\{t\}}+L_{\{y\},\{t\}}+L_{\{z\},\{t\}}+L_{\{t\},\{x,y,z\}}$$

on the surface S_λ. Thus, the intersection matrix of the lines $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, and $L_{\{t\},\{x,y,z\}}$ has the same rank as the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{y\},\{z\}}$, and H_λ. The latter matrix is given by

$$\begin{array}{c} \\ L_{\{x\},\{y\}} \\ L_{\{x\},\{z\}} \\ L_{\{y\},\{z\}} \\ H_\lambda \end{array}\begin{array}{c} \begin{array}{cccc} L_{\{x\},\{y\}} & L_{\{x\},\{z\}} & L_{\{y\},\{z\}} & H_\lambda \end{array} \\ \begin{pmatrix} -1/20 & 1/4 & 1/2 & 1 \\ 1/4 & -1/20 & 1/2 & 1 \\ 1/2 & 1/2 & -1/2 & 1 \\ 1 & 1 & 1 & 4 \end{pmatrix} \end{array}.$$

The determinant of this matrix is $-3/25$. Thus, we see that ($\bigstar$) holds. Then ($\diamondsuit$) in the Main Theorem holds by Lemma 1.13.1.

4. FANO THREEFOLDS OF PICARD RANK 4

Family 4.1. The threefold X is a divisor of degree $(1,1,1,1)$ on $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$. In this case, we have $h^{1,2}(X)=1$. A toric Landau–Ginzburg model of this family is given by the Minkowski polynomial 2354.1, which is

$$x+y+z+\frac{y}{z}+\frac{y}{x}+\frac{z}{y}+\frac{z}{x}+\frac{1}{z}+\frac{y}{xz}+\frac{1}{y}+\frac{3}{x}+\frac{z}{xy}+\frac{1}{xz}+\frac{1}{xy}.$$

The quartic pencil $\mathcal{S}$ is given by the following equation:

$$x^2yz+xy^2z+xyz^2+xy^2t+y^2zt+xz^2t+yz^2t+xyt^2+y^2t^2+xt^2z$$
$$+3yzt^2+z^2t^2+yt^3+zt^3=\lambda xyzt.$$

As usual, we assume that $\lambda\neq\infty$ (just for simplicity).

Let $\mathcal{C}$ be the conic in $\mathbb{P}^3$ given by $x=yz+yt+zt=0$. Then

$$\begin{aligned} H_{\{x\}}\cdot S_\lambda &= L_{\{x\},\{t\}}+L_{\{x\},\{y,z,t\}}+\mathcal{C},\\ H_{\{y\}}\cdot S_\lambda &= L_{\{y\},\{z\}}+L_{\{y\},\{t\}}+L_{\{y\},\{x,t\}}+L_{\{y\},\{z,t\}},\\ H_{\{z\}}\cdot S_\lambda &= L_{\{y\},\{z\}}+L_{\{z\},\{t\}}+L_{\{z\},\{x,t\}}+L_{\{z\},\{y,t\}},\\ H_{\{t\}}\cdot S_\lambda &= L_{\{x\},\{t\}}+L_{\{y\},\{t\}}+L_{\{z\},\{t\}}+L_{\{t\},\{x,y,z\}}. \end{aligned}\tag{4.1.1}$$

Thus, the base locus of the pencil $\mathcal{S}$ consists of the curves $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{y\},\{x,t\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{z\},\{x,t\}}$, $L_{\{z\},\{y,t\}}$, $L_{\{x\},\{y,z,t\}}$, $L_{\{t\},\{x,y,z\}}$, and $\mathcal{C}$.

Observe that $S_{-4} = H_{\{x,t\}} + \mathbf{S}$, where $\mathbf{S}$ is a cubic surface in $\mathbb{P}^3$ that is given by

$$yt^2 + zt^2 + z^2t + y^2t + 3yzt + y^2z + yz^2 + xyz = 0.$$

On the other hand, if $\lambda \neq -4$, then S_λ is irreducible and has isolated singularities. Moreover, if $\lambda \neq -3, -4$, then the singular points of S_λ contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

$P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_2$ with quadratic term $(x+t)(y+t)$;

$P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_2$ with quadratic term $(x+t)(z+t)$;

$P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term yz;

$P_{\{x\},\{t\},\{y,z\}}$: type $\mathbb{A}_1$ with quadratic term $(x+t)(x+y+z+t) - (\lambda+4)xt$;

$P_{\{y\},\{z\},\{x,t\}}$: type $\mathbb{A}_1$ with quadratic term $(x+t)(y+z) + (\lambda+4)yz$.

Furthermore, the surface S_{-3} has the same types of singularities at the points $P_{\{x\},\{z\},\{t\}}$, $P_{\{x\},\{y\},\{t\}}$, $P_{\{y\},\{z\},\{t\}}$, $P_{\{x\},\{t\},\{y,z\}}$, and $P_{\{y\},\{z\},\{x,t\}}$. In addition to this, the surface S_{-3} is also singular at the points $[0:\xi_3:1:\xi_3^2]$ and $[0:\xi_3^2:1:\xi_3]$, where ξ_3 is a primitive cube root of unity. Both these points are singular points of the surface S_{-3} of type $\mathbb{A}_1$.

For $\lambda \neq -4$, the surface S_λ has du Val singularities at the base points of the pencil $\mathcal{S}$. Therefore, by Corollary 1.5.4, the fiber $\mathsf{f}^{-1}(\lambda)$ is irreducible for every $\lambda \neq -4$. Moreover, the points $P_{\{x\},\{z\},\{t\}}$, $P_{\{x\},\{y\},\{t\}}$, $P_{\{y\},\{z\},\{t\}}$, $P_{\{x\},\{t\},\{y,z\}}$, and $P_{\{y\},\{z\},\{x,t\}}$ are good double points of the surface S_{-4}. Furthermore, the surface S_{-4} is smooth at general points of the curves $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{y\},\{x,t\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{z\},\{x,t\}}$, $L_{\{z\},\{y,t\}}$, $L_{\{x\},\{y,z,t\}}$, $L_{\{t\},\{x,y,z\}}$, and $\mathcal{C}$. Thus, we see that

$$\big[\mathsf{f}^{-1}(-4)\big] = \big[S_{-4}\big] = 2$$

by (1.8.3) and Lemmas 1.8.5 and 1.12.1. This confirms ($\heartsuit$) in the Main Theorem.

To verify ($\diamondsuit$) in the Main Theorem, we may assume that $\lambda \neq -4$. Then the intersection matrix of the curves $L_{\{x\},\{t\}}$, $L_{\{x\},\{y,z,t\}}$, $L_{\{y\},\{x,t\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{z\},\{y,t\}}$, $L_{\{t\},\{x,y,z\}}$, and H_λ on the surface S_λ is given by

$$\begin{array}{c} \\ L_{\{x\},\{t\}} \\ L_{\{x\},\{y,z,t\}} \\ L_{\{y\},\{x,t\}} \\ L_{\{y\},\{z,t\}} \\ L_{\{z\},\{y,t\}} \\ L_{\{t\},\{x,y,z\}} \\ H_\lambda \end{array}
\begin{array}{c} \begin{array}{ccccccc} L_{\{x\},\{t\}} & L_{\{x\},\{y,z,t\}} & L_{\{y\},\{x,t\}} & L_{\{y\},\{z,t\}} & L_{\{z\},\{y,t\}} & L_{\{t\},\{x,y,z\}} & H_\lambda \end{array} \\ \begin{pmatrix} -1/6 & 1/2 & 2/3 & 0 & 0 & 1/2 & 1 \\ 1/2 & -3/2 & 0 & 1 & 1 & 1/2 & 1 \\ 2/3 & 0 & -5/6 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & -5/4 & 1/4 & 0 & 1 \\ 1 & 1 & 0 & 1/4 & -5/4 & 0 & 1 \\ 1/2 & 1/2 & 0 & 0 & 0 & -3/2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 4 \end{pmatrix} \end{array}.$$

This matrix has rank 7. On the other hand, it follows from (4.1.1) that

$$\begin{aligned} H_\lambda &\sim L_{\{x\},\{t\}} + L_{\{x\},\{y,z,t\}} + \mathcal{C} \sim L_{\{y\},\{z\}} + L_{\{y\},\{t\}} + L_{\{y\},\{x,t\}} + L_{\{y\},\{z,t\}} \\ &\sim L_{\{y\},\{z\}} + L_{\{z\},\{t\}} + L_{\{z\},\{x,t\}} + L_{\{z\},\{y,t\}} \sim L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y,z\}}. \end{aligned}$$

Moreover, we also have $2L_{\{x\},\{t\}} + L_{\{y\},\{x,t\}} + L_{\{z\},\{x,t\}} \sim H_\lambda$, because

$$H_{\{x,t\}} \cdot S_\infty = 2L_{\{x\},\{t\}} + L_{\{y\},\{x,t\}} + L_{\{z\},\{x,t\}}.$$

Therefore, the intersection matrix of the curves $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{y\},\{x,t\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{z\},\{x,t\}}$, $L_{\{z\},\{y,t\}}$, $L_{\{x\},\{y,z,t\}}$, $L_{\{t\},\{x,y,z\}}$, and $\mathcal{C}$ on the surface S_λ has the same rank as the intersection matrix of the curves $L_{\{x\},\{t\}}$, $L_{\{x\},\{y,z,t\}}$, $L_{\{y\},\{x,t\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{z\},\{y,t\}}$, $L_{\{t\},\{x,y,z\}}$, and H_λ. But $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_\Bbbk) = \operatorname{rk}\operatorname{Pic}(S_\Bbbk) + 9$. Therefore, we conclude that ($\bigstar$) holds. Then ($\diamondsuit$) in the Main Theorem holds by Lemma 1.13.1.

Family 4.2. In this case, the threefold X is a blow-up of the irreducible quadric cone in $\mathbb{P}^4$ at its vertex and a smooth elliptic curve that does not pass through the vertex. This shows that $h^{1,2}(X) = 1$. A toric Landau–Ginzburg model of this family is given by the Minkowski polynomial 663, which is

$$x + y + \frac{z}{y} + \frac{z}{x} + \frac{x}{y} + \frac{y}{x} + \frac{2}{x} + \frac{2}{y} + \frac{1}{yz} + \frac{1}{xz}.$$

The quartic pencil $\mathcal{S}$ is given by the equation

$$x^2yz + xy^2z + xz^2t + yz^2t + x^2zt + y^2zt + 2xzt^2 + 2yzt^2 + xt^3 + yt^3 = \lambda xyzt.$$

For simplicity, we assume that $\lambda \neq \infty$.

If $\lambda \neq -2$, then S_λ is irreducible and has isolated singularities. On the other hand, we have $S_{-2} = H_{\{x,y\}} + \mathbf{S}$, where $\mathbf{S}$ is an irreducible cubic surface given by the equation $xyz + xzt + t^3 + z^2t + 2zt^2 + yzt = 0$.

Let $\mathcal{C}_1$ be the conic in $\mathbb{P}^3$ that is given by $x = yz + (z+t)^2 = 0$, and let $\mathcal{C}_2$ be the conic in $\mathbb{P}^3$ that is given by $y = xz + (z+t)^2 = 0$. Then

$$\begin{aligned} H_{\{x\}} \cdot S_\lambda &= L_{\{x\},\{y\}} + L_{\{x\},\{t\}} + \mathcal{C}_1, \\ H_{\{y\}} \cdot S_\lambda &= L_{\{x\},\{y\}} + L_{\{y\},\{t\}} + \mathcal{C}_2, \\ H_{\{z\}} \cdot S_\lambda &= 3L_{\{z\},\{t\}} + L_{\{z\},\{x,y\}}, \\ H_{\{t\}} \cdot S_\lambda &= L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y\}}. \end{aligned} \tag{4.2.1}$$

Thus, the base locus of the pencil $\mathcal{S}$ consists of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{z\},\{x,y\}}$, $L_{\{t\},\{x,y\}}$, $\mathcal{C}_1$, and $\mathcal{C}_2$.

If $\lambda \neq -2$, then the singular points of S_λ contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

$P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term $t(x+y)$;

$P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_2$ with quadratic term $z(x+t)$;

$P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_2$ with quadratic term $z(y+t)$;

$P_{\{x\},\{y\},\{z,t\}}$: type $\mathbb{A}_3$ with quadratic term $x^2 + y^2 + \lambda xy$;

$P_{\{z\},\{t\},\{x,y\}}$: type $\mathbb{A}_3$ with quadratic term $z(x + y - 2t - \lambda t)$.

Therefore, by Corollary 1.5.4, the fiber $\mathsf{f}^{-1}(\lambda)$ is irreducible for every $\lambda \neq -2$. Moreover, the points $P_{\{y\},\{z\},\{t\}}$, $P_{\{x\},\{z\},\{t\}}$, $P_{\{x\},\{y\},\{t\}}$, $P_{\{x\},\{y\},\{z,t\}}$, and $P_{\{z\},\{t\},\{x,y\}}$ are good double points of the surface S_{-2}. Furthermore, the surface S_{-2} is smooth at general points of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{z\},\{x,y\}}$, $L_{\{t\},\{x,y\}}$, $\mathcal{C}_1$, and $\mathcal{C}_2$. Thus, we see that $[\mathsf{f}^{-1}(-2)] = [S_{-2}] = 2$ by (1.8.3) and Lemmas 1.8.5 and 1.12.1. This confirms ($\heartsuit$) in the Main Theorem.

To verify ($\diamondsuit$) in the Main Theorem, we may assume that $\lambda \neq -2$. By (4.2.1), we have

$$\begin{aligned} H_\lambda &\sim L_{\{x\},\{y\}} + L_{\{x\},\{t\}} + \mathcal{C}_1 \sim L_{\{x\},\{y\}} + L_{\{y\},\{t\}} + \mathcal{C}_2 \\ &\sim 3L_{\{z\},\{t\}} + L_{\{z\},\{x,y\}} \sim L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y\}} \end{aligned}$$

on the surface S_λ. Since $H_{\{x,y\}} \cdot S_\lambda = 2L_{\{x\},\{y\}} + L_{\{z\},\{x,y\}} + L_{\{t\},\{x,y\}}$, we also have

$$2L_{\{x\},\{y\}} + L_{\{z\},\{x,y\}} + L_{\{t\},\{x,y\}} \sim H_\lambda.$$

Thus, the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{z\},\{x,y\}}$, $L_{\{t\},\{x,y\}}$, $\mathcal{C}_1$, and $\mathcal{C}_2$ on the surface S_λ has the same rank as the intersection matrix

$$\begin{array}{c} \\ L_{\{z\},\{x,y\}} \\ L_{\{z\},\{x,t\}} \\ L_{\{t\},\{x,y\}} \\ H_\lambda \end{array} \begin{array}{c} \begin{array}{cccc} L_{\{z\},\{x,y\}} & L_{\{z\},\{x,t\}} & L_{\{t\},\{x,y\}} & H_\lambda \end{array} \\ \begin{pmatrix} -5/4 & 1 & 1/4 & 1 \\ 1 & -7/12 & 1/2 & 1 \\ 1/4 & 1/2 & -1 & 1 \\ 1 & 1 & 1 & 4 \end{pmatrix} \end{array}.$$

Its rank is 4. On the other hand, we have $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk}) = \operatorname{rk}\operatorname{Pic}(S_{\Bbbk}) + 12$, because the quadratic term of the defining equation of the surface S_λ at $P_{\{x\},\{y\},\{z,t\}}$ is $x^2 + y^2 + \lambda xy$, which is irreducible over $\Bbbk$. Then ($\diamondsuit$) in the Main Theorem holds by Lemma 1.13.1.

Family 4.3. In this case, the threefold X is a blow-up of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ along a smooth rational curve of tridegree $(1, 1, 2)$. Thus, we have $h^{1,2}(X) = 0$. A toric Landau–Ginzburg model of this family is given by the Minkowski polynomial 740, which is

$$x + y + z + \frac{y}{z} + \frac{y}{x} + \frac{z}{y} + \frac{z}{x} + \frac{1}{z} + \frac{1}{y} + \frac{1}{x}.$$

The quartic pencil $\mathcal{S}$ is given by the following equation:

$$x^2yz + y^2zx + z^2yx + y^2tx + y^2tz + z^2tx + z^2ty + t^2yx + t^2zx + t^2yz = \lambda xyzt.$$

As usual, we suppose that $\lambda \neq \infty$.

The base locus of the pencil $\mathcal{S}$ consists of the lines $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{y,z,t\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{z\},\{y,t\}}$, and $L_{\{t\},\{x,y,z\}}$, because

$$\begin{aligned} H_{\{x\}} \cdot S_\lambda &= L_{\{x\},\{y\}} + L_{\{x\},\{z\}} + L_{\{x\},\{t\}} + L_{\{x\},\{y,z,t\}}, \\ H_{\{y\}} \cdot S_\lambda &= L_{\{x\},\{y\}} + L_{\{y\},\{z\}} + L_{\{y\},\{t\}} + L_{\{y\},\{z,t\}}, \\ H_{\{z\}} \cdot S_\lambda &= L_{\{x\},\{z\}} + L_{\{y\},\{z\}} + L_{\{z\},\{t\}} + L_{\{z\},\{y,t\}}, \\ H_{\{t\}} \cdot S_\lambda &= L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y,z\}}. \end{aligned} \tag{4.3.1}$$

For every $\lambda \in \mathbb{C}$, the surface S_λ is irreducible, it has isolated singularities, and its singular points contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

$P_{\{x\},\{y\},\{z\}}$: type $\mathbb{A}_1$;

$P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_1$;

$P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_1$;

$P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term yz;

$P_{\{x\},\{y\},\{z,t\}}$: type $\mathbb{A}_1$ for $\lambda \neq -3$, and type $\mathbb{A}_2$ for $\lambda = -3$;

$P_{\{x\},\{z\},\{y,t\}}$: type $\mathbb{A}_1$ for $\lambda \neq -3$, and type $\mathbb{A}_2$ for $\lambda = -3$;

$P_{\{x\},\{t\},\{y,z\}}$: type $\mathbb{A}_1$ for $\lambda \neq -3$, and type $\mathbb{A}_2$ for $\lambda = -3$.

Then $[\mathsf{f}^{-1}(\lambda)] = 1$ for every $\lambda \in \mathbb{C}$ by Corollary 1.5.4. This confirms ($\heartsuit$) in the Main Theorem.

To verify ($\diamondsuit$) in the Main Theorem, observe that $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk}) = \operatorname{rk}\operatorname{Pic}(S_{\Bbbk}) + 9$. On the other hand, it follows from (4.3.1) that the intersection matrix of the lines $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{y,z,t\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{z\},\{y,t\}}$, and $L_{\{t\},\{x,y,z\}}$ on the surface S_λ has the same rank as the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{t\},\{x,y,z\}}$, and H_λ. If $\lambda\neq -3$, the latter matrix is given by

$$\begin{array}{c} \\ L_{\{x\},\{y\}} \\ L_{\{x\},\{z\}} \\ L_{\{x\},\{t\}} \\ L_{\{y\},\{z\}} \\ L_{\{y\},\{t\}} \\ L_{\{t\},\{x,y,z\}} \\ H_\lambda \end{array}
\begin{array}{c}
\begin{array}{ccccccc} L_{\{x\},\{y\}} & L_{\{x\},\{z\}} & L_{\{x\},\{t\}} & L_{\{y\},\{z\}} & L_{\{y\},\{t\}} & L_{\{t\},\{x,y,z\}} & H_\lambda \end{array}\\
\begin{pmatrix}
-1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 0 & 1\\
1/2 & -1/2 & 1/2 & 1/2 & 0 & 0 & 1\\
1/2 & 1/2 & -1/2 & 0 & 1/2 & 1/2 & 1\\
1/2 & 1/2 & 0 & -1 & 1/2 & 0 & 1\\
1/2 & 0 & 1/2 & 1/2 & -1/4 & 1 & 1\\
0 & 0 & 1/2 & 0 & 1 & -3/2 & 1\\
1 & 1 & 1 & 1 & 1 & 1 & 4
\end{pmatrix}
\end{array}.$$

Its rank is 7, so that ($\bigstar$) holds. Then ($\diamondsuit$) in the Main Theorem holds by Lemma 1.13.1.

Family 4.4. In this case, the threefold X is a blow-up of a quadric in two non-collinear points and the proper transform of a conic passing through these two points. We have $h^{1,2}(X)=0$. A toric Landau–Ginzburg model of this family is given by the Minkowski polynomial 426, which is

$$x+y+z+\frac{x}{z}+\frac{y}{z}+\frac{x}{y}+\frac{y}{x}+\frac{1}{y}+\frac{1}{x}.$$

The quartic pencil $\mathcal{S}$ is given by

$$x^2yz+y^2zx+z^2yx+x^2ty+y^2tx+x^2tz+y^2tz+t^2zx+t^2yz=\lambda xyzt.$$

Suppose that $\lambda\neq\infty$. Then

$$\begin{aligned}
H_{\{x\}}\cdot S_\lambda&=L_{\{x\},\{y\}}+L_{\{x\},\{z\}}+L_{\{x\},\{t\}}+L_{\{x\},\{y,t\}},\\
H_{\{y\}}\cdot S_\lambda&=L_{\{x\},\{y\}}+L_{\{y\},\{z\}}+L_{\{y\},\{t\}}+L_{\{y\},\{x,t\}},\\
H_{\{z\}}\cdot S_\lambda&=L_{\{x\},\{z\}}+L_{\{y\},\{z\}}+L_{\{z\},\{t\}}+L_{\{z\},\{x,y\}},\\
H_{\{t\}}\cdot S_\lambda&=L_{\{x\},\{t\}}+L_{\{y\},\{t\}}+L_{\{z\},\{t\}}+L_{\{t\},\{x,y,z\}}.
\end{aligned}\tag{4.4.1}$$

Each surface S_λ is irreducible, it has isolated singularities, and its singular points contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

$P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term xy;

$P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_1$;

$P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_1$;

$P_{\{x\},\{y\},\{z\}}$: type $\mathbb{A}_3$ with quadratic term yz for $\lambda\neq-2$, and type $\mathbb{A}_5$ for $\lambda=-2$;

$P_{\{z\},\{t\},\{x,y\}}$: type $\mathbb{A}_1$ for $\lambda\neq-3$, and type $\mathbb{A}_3$ for $\lambda=-3$.

Then each fiber $\mathsf{f}^{-1}(\lambda)$ is irreducible by Corollary 1.5.4. This confirms ($\heartsuit$) in the Main Theorem.

Let us prove ($\diamondsuit$) in the Main Theorem. We may assume that $\lambda\neq-2,-3$. Then the intersection matrix of the curves $L_{\{x\},\{y,t\}}$, $L_{\{y\},\{x,t\}}$, $L_{\{y\},\{z\}}$, $L_{\{z\},\{t\}}$, $L_{\{z\},\{x,y\}}$, $L_{\{t\},\{x,y,z\}}$, and H_λ on the surface S_λ is given by

$$\begin{array}{c} \\ L_{\{x\},\{y,t\}} \\ L_{\{y\},\{x,t\}} \\ L_{\{y\},\{z\}} \\ L_{\{z\},\{t\}} \\ L_{\{z\},\{x,y\}} \\ L_{\{t\},\{x,y,z\}} \\ H_\lambda \end{array}
\begin{array}{c}
\begin{array}{ccccccc} L_{\{x\},\{y,t\}} & L_{\{y\},\{x,t\}} & L_{\{y\},\{z\}} & L_{\{z\},\{t\}} & L_{\{z\},\{x,y\}} & L_{\{t\},\{x,y,z\}} & H_\lambda \end{array}\\
\begin{pmatrix}
-5/4 & 1/4 & 0 & 0 & 0 & 0 & 1\\
1/4 & -5/4 & 1 & 0 & 0 & 0 & 1\\
0 & 1 & -1/2 & 1/2 & 1/2 & 0 & 1\\
0 & 0 & 1/2 & -1/2 & 1/2 & 1/2 & 1\\
0 & 0 & 1/2 & 1/2 & -3/4 & 1/2 & 1\\
0 & 0 & 0 & 1/2 & 1/2 & -3/2 & 1\\
1 & 1 & 1 & 1 & 1 & 1 & 4
\end{pmatrix}
\end{array}.$$

This matrix has rank 7. Thus, it follows from (4.4.1) that the rank of the intersection matrix of the lines $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{z\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{x\},\{y,t\}}$, $L_{\{y\},\{x,t\}}$, $L_{\{z\},\{x,y\}}$, and $L_{\{t\},\{x,y,z\}}$ is also 7. But $\mathrm{rk}\,\mathrm{Pic}(\widetilde{S}_{\Bbbk}) = \mathrm{rk}\,\mathrm{Pic}(S_{\Bbbk}) + 9$, so that ($\bigstar$) holds. Then ($\diamondsuit$) in the Main Theorem holds by Lemma 1.13.1.

Family 4.5. In this case, the threefold X is a blow-up of $\mathbb{P}^1 \times \mathbb{P}^2$ in a disjoint union of curves of bidegrees $(2,1)$ and $(1,0)$. We have $h^{1,2}(X) = 0$. A toric Landau–Ginzburg model of this family is given by the Minkowski polynomial 425, which is

$$x + y + z + \frac{y}{z} + \frac{y}{x} + \frac{z}{y} + \frac{1}{z} + \frac{1}{y} + \frac{1}{x}.$$

Then the pencil $\mathcal{S}$ is given by the equation

$$x^2yz + y^2zx + z^2yx + y^2tx + y^2tz + z^2tx + t^2yx + t^2zx + t^2yz = \lambda xyzt.$$

Suppose that $\lambda \neq \infty$. Then

$$\begin{aligned} H_{\{x\}} \cdot S_\lambda &= L_{\{x\},\{y\}} + L_{\{x\},\{z\}} + L_{\{x\},\{t\}} + L_{\{x\},\{y,t\}},\\ H_{\{y\}} \cdot S_\lambda &= L_{\{x\},\{y\}} + L_{\{y\},\{z\}} + L_{\{y\},\{t\}} + L_{\{y\},\{z,t\}},\\ H_{\{z\}} \cdot S_\lambda &= L_{\{x\},\{z\}} + L_{\{y\},\{z\}} + L_{\{z\},\{t\}} + L_{\{z\},\{y,t\}},\\ H_{\{t\}} \cdot S_\lambda &= L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y,z\}}. \end{aligned} \tag{4.5.1}$$

Observe that S_λ is irreducible, it has isolated singularities, and its singular points contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

$P_{\{x\},\{y\},\{z\}}$: type $\mathbb{A}_1$;
$P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term $x(y+t)$ for $\lambda \neq -2$, and type $\mathbb{A}_4$ for $\lambda = -2$;
$P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_1$;
$P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term yz;
$P_{\{x\},\{z\},\{y,t\}}$: type $\mathbb{A}_1$ for $\lambda \neq -2$, and type $\mathbb{A}_2$ for $\lambda = -2$.

Then each fiber $\mathsf{f}^{-1}(\lambda)$ is irreducible by Corollary 1.5.4. This confirms ($\heartsuit$) in the Main Theorem.

It follows from (4.5.1) that the intersection matrix of the base curves of the pencil $\mathcal{S}$ on the surface S_λ has the same rank as the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{t\},\{x,y,z\}}$, and H_λ. If $\lambda \neq -2$, the latter matrix is given by

$$\begin{array}{c} \\ L_{\{x\},\{y\}} \\ L_{\{x\},\{z\}} \\ L_{\{x\},\{t\}} \\ L_{\{y\},\{z\}} \\ L_{\{y\},\{z,t\}} \\ L_{\{t\},\{x,y,z\}} \\ H_\lambda \end{array}
\begin{array}{c} \begin{array}{ccccccc} L_{\{x\},\{y\}} & L_{\{x\},\{z\}} & L_{\{x\},\{t\}} & L_{\{y\},\{z\}} & L_{\{y\},\{z,t\}} & L_{\{t\},\{x,y,z\}} & H_\lambda \end{array} \\
\begin{pmatrix} -3/4 & 1/2 & 3/4 & 1/2 & 1 & 0 & 1\\ 1/2 & -1/2 & 1/2 & 1/2 & 0 & 0 & 1\\ 3/4 & 1/2 & -3/4 & 0 & 0 & 1 & 1\\ 1/2 & 1/2 & 0 & -1/2 & 1/2 & 0 & 1\\ 1 & 0 & 0 & 1/2 & -5/4 & 0 & 1\\ 0 & 0 & 1 & 0 & 0 & -2 & 1\\ 1 & 1 & 1 & 1 & 1 & 1 & 4 \end{pmatrix} \end{array}.$$

The determinant of this matrix is $39/128$. However, we also have $\mathrm{rk}\,\mathrm{Pic}(\widetilde{S}_{\Bbbk}) = \mathrm{rk}\,\mathrm{Pic}(S_{\Bbbk}) + 9$. Therefore, we see that ($\diamondsuit$) in the Main Theorem holds by Lemma 1.13.1.

Family 4.6. In this case, the threefold X is a blow-up of $\mathbb{P}^3$ in a disjoint union of three lines. Thus, we have $h^{1,2}(X)=0$. A toric Landau–Ginzburg model of this family is given by the Minkowski polynomial 423, which is

$$x+y+z+\frac{z}{y}+\frac{1}{z}+\frac{1}{y}+\frac{1}{x}+\frac{1}{xz}+\frac{1}{xy}.$$

The quartic pencil $\mathcal{S}$ is given by the following equation:

$$x^2yz+y^2zx+z^2yx+z^2tx+t^2yx+t^2zx+t^2yz+t^3y+t^3z=\lambda xyzt.$$

As usual, we assume that $\lambda\neq\infty$.

Let $\mathcal{C}_1$ be the conic in $\mathbb{P}^3$ that is given by $x=yz+yt+zt=0$, and let $\mathcal{C}_2$ be the conic in $\mathbb{P}^3$ that is given by $y=xz+xt+t^2=0$. Then

$$\begin{aligned} H_{\{x\}}\cdot S_\lambda &= 2L_{\{x\},\{t\}}+\mathcal{C}_1,\\ H_{\{y\}}\cdot S_\lambda &= L_{\{y\},\{z\}}+L_{\{y\},\{t\}}+\mathcal{C}_2,\\ H_{\{z\}}\cdot S_\lambda &= L_{\{y\},\{z\}}+2L_{\{z\},\{t\}}+L_{\{z\},\{x,t\}},\\ H_{\{t\}}\cdot S_\lambda &= L_{\{x\},\{t\}}+L_{\{y\},\{t\}}+L_{\{z\},\{t\}}+L_{\{t\},\{x,y,z\}}. \end{aligned} \tag{4.6.1}$$

For every λ, the surface S_λ is irreducible, it has isolated singularities, and its singular points contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

$P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term $x(y+t)$ for $\lambda\neq-3$, and type $\mathbb{A}_5$ for $\lambda=-3$;
$P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_2$ with quadratic term xz;
$P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term yz;
$P_{\{x\},\{t\},\{y,z\}}$: type $\mathbb{A}_1$;
$P_{\{y\},\{z\},\{x,t\}}$: type $\mathbb{A}_1$ for $\lambda\neq-3$, and type $\mathbb{A}_3$ for $\lambda=-3$;
$P_{\{z\},\{t\},\{x,y\}}$: type $\mathbb{A}_1$.

By Corollary 1.5.4, each fiber $\mathsf{f}^{-1}(\lambda)$ is irreducible. This confirms ($\heartsuit$) in the Main Theorem.

To verify ($\diamondsuit$) in the Main Theorem, observe that $\mathrm{rk}\,\mathrm{Pic}(\widetilde{S}_{\Bbbk})=\mathrm{rk}\,\mathrm{Pic}(S_{\Bbbk})+11$. On the other hand, it follows from (4.6.1) that the intersection matrix of the curves $2L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{z\},\{x,t\}}$, $L_{\{t\},\{x,y,z\}}$, $\mathcal{C}_1$, and $\mathcal{C}_2$ has the same rank as the intersection matrix of the curves $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, and H_λ. If $\lambda\neq-3$, then the latter matrix is given by

$$\begin{array}{c} \\ L_{\{x\},\{t\}}\\ L_{\{y\},\{z\}}\\ L_{\{y\},\{t\}}\\ L_{\{z\},\{t\}}\\ H_\lambda \end{array}\begin{array}{c} \begin{array}{ccccc} L_{\{x\},\{t\}} & L_{\{y\},\{z\}} & L_{\{y\},\{t\}} & L_{\{z\},\{t\}} & H_\lambda \end{array}\\ \begin{pmatrix} -1/12 & 0 & 1/4 & 1/3 & 1\\ 0 & -1/2 & 1/2 & 1/2 & 1\\ 1/4 & 1/2 & -1/2 & 1/4 & 1\\ 1/3 & 1/2 & 1/4 & -1/12 & 1\\ 1 & 1 & 1 & 1 & 4 \end{pmatrix} \end{array}.$$

Its rank is 5, so that ($\bigstar$) holds. Then ($\diamondsuit$) in the Main Theorem holds by Lemma 1.13.1.

Family 4.7. In this case, the threefold X can be obtained by blowing up a smooth hypersurface in $\mathbb{P}^2\times\mathbb{P}^2$ of bidegree $(1,1)$ in a disjoint union of two smooth rational curves. This shows that $h^{1,2}(X)=0$. A toric Landau–Ginzburg model of this family is given by the Minkowski polynomial 215, which is

$$x+y+z+\frac{z}{y}+\frac{1}{z}+\frac{1}{y}+\frac{1}{x}+\frac{1}{xz}.$$

The quartic pencil $\mathcal{S}$ is given by

$$x^2yz+y^2zx+z^2yx+z^2tx+t^2yx+t^2zx+t^2yz+t^3y=\lambda xyzt.$$

Suppose that $\lambda\neq\infty$. Then

$$\begin{aligned}
H_{\{x\}}\cdot S_\lambda &= L_{\{x\},\{y\}}+2L_{\{x\},\{t\}}+L_{\{x\},\{z,t\}},\\
H_{\{y\}}\cdot S_\lambda &= L_{\{x\},\{y\}}+L_{\{y\},\{z\}}+L_{\{y\},\{t\}}+L_{\{y\},\{z,t\}},\\
H_{\{z\}}\cdot S_\lambda &= L_{\{y\},\{z\}}+2L_{\{z\},\{t\}}+L_{\{z\},\{x,t\}},\\
H_{\{t\}}\cdot S_\lambda &= L_{\{x\},\{t\}}+L_{\{y\},\{t\}}+L_{\{z\},\{t\}}+L_{\{t\},\{x,y,z\}}.
\end{aligned}\tag{4.7.1}$$

Thus, the base locus of the pencil $\mathcal{S}$ consists of the lines $L_{\{x\},\{y\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{z,t\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{z\},\{x,t\}}$, and $L_{\{t\},\{x,y,z\}}$.

For every $\lambda\in\mathbb{C}$, the surface S_λ has isolated singularities. Thus, we conclude that every surface S_λ is irreducible. Moreover, the singular points of S_λ contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

$P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_2$ with quadratic term $x(y+t)$;
$P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_2$ with quadratic term xz;
$P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term yz;
$P_{\{x\},\{y\},\{z,t\}}$: type $\mathbb{A}_1$ for $\lambda\neq-2$, and type $\mathbb{A}_2$ for $\lambda=-2$;
$P_{\{x\},\{t\},\{y,z\}}$: type $\mathbb{A}_1$;
$P_{\{z\},\{t\},\{x,y\}}$: type $\mathbb{A}_1$.

By Corollary 1.5.4, each fiber $\mathsf{f}^{-1}(\lambda)$ is irreducible. This confirms ($\heartsuit$) in the Main Theorem.

To verify ($\diamondsuit$) in the Main Theorem, observe first that $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk})=\operatorname{rk}\operatorname{Pic}(S_{\Bbbk})+10$. On the other hand, it follows from (4.7.1) that the intersection matrix of the lines $L_{\{x\},\{y\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{z,t\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{z\},\{x,t\}}$, and $L_{\{t\},\{x,y,z\}}$ on the surface S_λ has the same rank as the intersection matrix of the curves $L_{\{x\},\{t\}}$, $L_{\{x\},\{z,t\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{z\},\{x,t\}}$, $L_{\{t\},\{x,y,z\}}$, and H_λ. If $\lambda\neq-2$, then the latter matrix is given by

$$\begin{array}{c} \\ L_{\{x\},\{t\}}\\ L_{\{x\},\{z,t\}}\\ L_{\{y\},\{z,t\}}\\ L_{\{z\},\{x,t\}}\\ L_{\{t\},\{x,y,z\}}\\ H_\lambda \end{array}
\begin{array}{c}
\begin{array}{cccccc} L_{\{x\},\{t\}} & L_{\{x\},\{z,t\}} & L_{\{y\},\{z,t\}} & L_{\{z\},\{x,t\}} & L_{\{t\},\{x,y,z\}} & H_\lambda \end{array}\\
\begin{pmatrix}
-1/6 & 2/3 & 0 & 1/3 & 1/2 & 1\\
2/3 & -5/6 & 1/2 & 1/3 & 1/2 & 1\\
0 & 1/2 & -5/4 & 0 & 0 & 1\\
1/3 & 1/3 & 0 & -4/3 & 0 & 1\\
1/2 & 1/2 & 0 & 0 & -1 & 1\\
1 & 1 & 1 & 1 & 1 & 4
\end{pmatrix}
\end{array}.$$

Its rank is 6, so that ($\bigstar$) holds. Then ($\diamondsuit$) in the Main Theorem holds by Lemma 1.13.1.

Family 4.8. The threefold X can be obtained by blowing up $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ along a smooth rational curve of tridegree $(1,1,0)$. Then $h^{1,2}(X) = 0$. A toric Landau–Ginzburg model of this family is given by the Minkowski polynomial 216, which is

$$x + y + z + \frac{z}{y} + \frac{z}{x} + \frac{1}{z} + \frac{1}{y} + \frac{1}{x}.$$

The quartic pencil $\mathcal{S}$ is given by

$$x^2yz + y^2zx + z^2yx + z^2tx + z^2ty + t^2yx + t^2zx + t^2yz = \lambda xyzt.$$

Suppose that $\lambda \neq \infty$. Then

$$\begin{aligned} H_{\{x\}} \cdot S_\lambda &= L_{\{x\},\{y\}} + L_{\{x\},\{z\}} + L_{\{x\},\{t\}} + L_{\{x\},\{z,t\}}, \\ H_{\{y\}} \cdot S_\lambda &= L_{\{x\},\{y\}} + L_{\{y\},\{z\}} + L_{\{y\},\{t\}} + L_{\{y\},\{z,t\}}, \\ H_{\{z\}} \cdot S_\lambda &= L_{\{x\},\{z\}} + L_{\{y\},\{z\}} + 2L_{\{z\},\{t\}}, \\ H_{\{t\}} \cdot S_\lambda &= L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y,z\}}. \end{aligned} \tag{4.8.1}$$

Thus, the base locus of the pencil $\mathcal{S}$ consists of the lines $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{z,t\}}$, $L_{\{y\},\{z,t\}}$, and $L_{\{t\},\{x,y,z\}}$.

For every $\lambda \in \mathbb{C}$, the surface S_λ is irreducible, it has isolated singularities and its singular points contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

- $P_{\{x\},\{y\},\{z\}}$: type $\mathbb{A}_1$;
- $P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_1$;
- $P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term xz;
- $P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term yz;
- $P_{\{x\},\{y\},\{z,t\}}$: type $\mathbb{A}_1$ for $\lambda \neq -2$, and type $\mathbb{A}_5$ for $\lambda = -2$;
- $P_{\{z\},\{t\},\{x,y\}}$: type $\mathbb{A}_1$.

By Corollary 1.5.4, each fiber $\mathsf{f}^{-1}(\lambda)$ is irreducible. This confirms ($\heartsuit$) in the Main Theorem.

To verify ($\diamondsuit$) in the Main Theorem, observe that $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk}) = \operatorname{rk}\operatorname{Pic}(S_{\Bbbk}) + 10$. On the other hand, it follows from (4.8.1) that the intersection matrix of the lines $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{z,t\}}$, $L_{\{y\},\{z,t\}}$, and $L_{\{t\},\{x,y,z\}}$ on the surface S_λ has the same rank as the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{z,t\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{y\},\{t\}}$, and H_λ. If $\lambda \neq -2$, then the latter matrix is given by

$$\begin{array}{c} \\ L_{\{x\},\{y\}} \\ L_{\{x\},\{z\}} \\ L_{\{x\},\{z,t\}} \\ L_{\{y\},\{z,t\}} \\ L_{\{y\},\{t\}} \\ H_\lambda \end{array} \begin{array}{c} \begin{array}{cccccc} L_{\{x\},\{y\}} & L_{\{x\},\{z\}} & L_{\{x\},\{z,t\}} & L_{\{y\},\{z,t\}} & L_{\{y\},\{t\}} & H_\lambda \end{array} \\ \begin{pmatrix} -1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1 \\ 1/2 & -1/2 & 1/2 & 0 & 0 & 1 \\ 1/2 & 1/2 & -3/4 & 1/2 & 0 & 1 \\ 1/2 & 0 & 1/2 & -3/4 & 3/4 & 1 \\ 1/2 & 0 & 0 & 3/4 & -3/4 & 1 \\ 1 & 1 & 1 & 1 & 1 & 4 \end{pmatrix} \end{array}.$$

Its rank is 6, so that ($\bigstar$) holds. Then ($\diamondsuit$) in the Main Theorem holds by Lemma 1.13.1.

Family 4.9. In this case, the threefold X is a blow-up of $\mathbb{P}^3$ in a disjoint union of two lines and an exceptional curve of the blow-up. We have $h^{1,2}(X)=0$. A toric Landau–Ginzburg model of this family is given by the Minkowski polynomial 81, which is

$$\frac{y}{x}+\frac{1}{x}+y+z+\frac{1}{y}+x+\frac{1}{yz}.$$

The quartic pencil $\mathcal{S}$ is given by

$$y^2tz+t^2yz+y^2zx+z^2yx+t^2zx+x^2yz+t^3x=\lambda xyzt.$$

As usual, we suppose that $\lambda\neq\infty$. Then

$$\begin{aligned}
H_{\{x\}}\cdot S_\lambda &= L_{\{x\},\{y\}}+L_{\{x\},\{z\}}+L_{\{x\},\{t\}}+L_{\{x\},\{y,t\}},\\
H_{\{y\}}\cdot S_\lambda &= L_{\{x\},\{y\}}+2L_{\{y\},\{t\}}+2L_{\{y\},\{z,t\}},\\
H_{\{z\}}\cdot S_\lambda &= L_{\{x\},\{z\}}+3L_{\{z\},\{t\}},\\
H_{\{t\}}\cdot S_\lambda &= L_{\{x\},\{t\}}+L_{\{y\},\{t\}}+L_{\{z\},\{t\}}+L_{\{t\},\{x,y,z\}}.
\end{aligned}\tag{4.9.1}$$

Thus, we see that the base locus of the pencil $\mathcal{S}$ consists of the eight lines $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{y,t\}}$, $L_{\{y\},\{z,t\}}$, and $L_{\{t\},\{x,y,z\}}$.

For every $\lambda\in\mathbb{C}$, the surface S_λ is irreducible and has isolated singularities. Moreover, the singular points of S_λ contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

- $P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term xy;
- $P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term $z(x+t)$;
- $P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_2$ with quadratic term yz;
- $P_{\{y\},\{t\},\{x,z\}}$: type $\mathbb{A}_1$;
- $P_{\{z\},\{t\},\{x,y\}}$: type $\mathbb{A}_2$ with quadratic term $z(x+y+z-t-\lambda t)$.

Therefore, it follows from Corollary 1.5.4 that the fiber $\mathsf{f}^{-1}(\lambda)$ is irreducible for every $\lambda\in\mathbb{C}$. This confirms ($\heartsuit$) in the Main Theorem in this case, since $h^{1,2}(X)=0$.

To verify ($\diamondsuit$) in the Main Theorem, observe that $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_\Bbbk)=\operatorname{rk}\operatorname{Pic}(S_\Bbbk)+11$. This immediately follows from the description of singular points of the surface S_λ given above. Note also that

$$\begin{aligned}
L_{\{x\},\{y\}}+L_{\{x\},\{z\}}+L_{\{x\},\{t\}}+L_{\{x\},\{y,t\}} &\sim L_{\{x\},\{y\}}+2L_{\{y\},\{t\}}+2L_{\{y\},\{z,t\}}\\
&\sim L_{\{x\},\{z\}}+3L_{\{z\},\{t\}}\sim L_{\{x\},\{t\}}+L_{\{y\},\{t\}}+L_{\{z\},\{t\}}+L_{\{t\},\{x,y,z\}}\sim H_\lambda
\end{aligned}$$

on the surface S_λ. This follows from (4.9.1). Using this, we see that the intersection matrix of the lines $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{y,t\}}$, $L_{\{y\},\{z,t\}}$, and $L_{\{t\},\{x,y,z\}}$ on the surface S_λ has the same rank as the intersection matrix of the curves $L_{\{x\},\{y,t\}}$, $L_{\{y\},\{t\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{t\},\{x,y,z\}}$, and H_λ. The later matrix is not hard to compute:

$$\begin{array}{r@{}c}
 & \begin{array}{ccccc} L_{\{x\},\{y,t\}} & L_{\{y\},\{t\}} & L_{\{y\},\{z,t\}} & L_{\{t\},\{x,y,z\}} & H_\lambda \end{array}\\
\begin{array}{r} L_{\{x\},\{y,t\}}\\ L_{\{y\},\{t\}}\\ L_{\{y\},\{z,t\}}\\ L_{\{t\},\{x,y,z\}}\\ H_\lambda \end{array} &
\begin{pmatrix}
-5/4 & 1/3 & 0 & 0 & 1\\
1/3 & -1/12 & 2/3 & 1/2 & 1\\
0 & 2/3 & -4/3 & 0 & 1\\
0 & 1/2 & 0 & -5/6 & 1\\
1 & 1 & 1 & 1 & 4
\end{pmatrix}
\end{array}.$$

The rank of this matrix is 5. Thus, we conclude that ($\bigstar$) holds in this case, so that ($\diamondsuit$) in the Main Theorem also holds by Lemma 1.13.1.

Family 4.10. In this case, we have $X \cong \mathbb{P}^1 \times \mathbf{S}_7$, where $\mathbf{S}_7$ is a smooth del Pezzo surface of degree 7. This shows that $h^{1,2}(X) = 0$. A toric Landau–Ginzburg model of this family is given by the Minkowski polynomial 84, which is

$$\frac{y}{x} + \frac{1}{x} + y + z + \frac{1}{z} + \frac{1}{y} + x.$$

Thus, the quartic pencil $\mathcal{S}$ is given by the following equation:

$$y^2tz + t^2zy + y^2xz + z^2xy + t^2xy + t^2xz + x^2zy = \lambda xyzt.$$

Suppose that $\lambda \neq \infty$. Then

$$\begin{aligned} H_{\{x\}} \cdot S_\lambda &= L_{\{x\},\{y\}} + L_{\{x\},\{z\}} + L_{\{x\},\{t\}} + L_{\{x\},\{y,t\}}, \\ H_{\{y\}} \cdot S_\lambda &= L_{\{x\},\{y\}} + L_{\{y\},\{z\}} + 2L_{\{y\},\{t\}}, \\ H_{\{z\}} \cdot S_\lambda &= L_{\{x\},\{z\}} + L_{\{y\},\{z\}} + 2L_{\{z\},\{t\}}, \\ H_{\{t\}} \cdot S_\lambda &= L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y,z\}}. \end{aligned} \tag{4.10.1}$$

Thus, the base locus of the pencil $\mathcal{S}$ consists of the lines $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{y,t\}}$, and $L_{\{t\},\{x,y,z\}}$.

For every $\lambda \in \mathbb{C}$, the surface S_λ is irreducible and has isolated singularities. Moreover, the singular points of S_λ contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

$P_{\{x\},\{y\},\{z\}}$: type $\mathbb{A}_1$;

$P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term xy;

$P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_2$ with quadratic term $z(x+t)$;

$P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term yz;

$P_{\{y\},\{t\},\{x,z\}}$: type $\mathbb{A}_1$;

$P_{\{z\},\{t\},\{x,y\}}$: type $\mathbb{A}_1$.

Therefore, using Corollary 1.5.4, we see that the fiber $\mathsf{f}^{-1}(\lambda)$ is irreducible for every $\lambda \in \mathbb{C}$. This confirms ($\heartsuit$) in the Main Theorem in this case, because $h^{1,2}(X) = 0$.

Let us prove ($\diamondsuit$) in the Main Theorem. Observe that the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{y,t\}}$, $L_{\{t\},\{x,y,z\}}$, and H_λ on the surface S_λ is given by

$$\begin{array}{c} \\ L_{\{x\},\{y\}} \\ L_{\{x\},\{z\}} \\ L_{\{x\},\{y,t\}} \\ L_{\{t\},\{x,y,z\}} \\ H_\lambda \end{array} \begin{array}{c} \begin{array}{ccccc} L_{\{x\},\{y\}} & L_{\{x\},\{z\}} & L_{\{x\},\{y,t\}} & L_{\{t\},\{x,y,z\}} & H_\lambda \end{array} \\ \begin{pmatrix} -1/2 & 1/2 & 1/2 & 0 & 1 \\ 1/2 & -5/6 & 1 & 0 & 1 \\ 1/2 & 1 & -5/4 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \\ 1 & 1 & 1 & 1 & 4 \end{pmatrix} \end{array}.$$

The rank of this matrix is 5. On the other hand, it follows from (3.30.1) that

$$\begin{aligned} H_\lambda &\sim L_{\{x\},\{y\}} + L_{\{x\},\{z\}} + L_{\{x\},\{t\}} + L_{\{x\},\{y,t\}} \sim L_{\{x\},\{y\}} + L_{\{y\},\{z\}} + 2L_{\{y\},\{t\}} \\ &\sim L_{\{x\},\{z\}} + L_{\{y\},\{z\}} + 2L_{\{z\},\{t\}} \sim L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y,z\}} \end{aligned}$$

on the surface S_λ. Therefore, the intersection matrix of the lines $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{y,t\}}$, and $L_{\{t\},\{x,y,z\}}$ has the same rank as the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{y,t\}}$, $L_{\{t\},\{x,y,z\}}$, and H_λ. On the other hand, we also have $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk}) = \operatorname{rk}\operatorname{Pic}(S_{\Bbbk}) + 11$. Thus, we see that ($\bigstar$) holds. Then we use Lemma 1.13.1 to conclude that ($\diamondsuit$) in the Main Theorem also holds in this case.

Family 4.11. For a description of the threefold X, see [8]. In this case, we have $h^{1,2}(X)=0$. A toric Landau–Ginzburg model of this family is given by the Minkowski polynomial 82, which is

$$\frac{y}{x}+\frac{1}{x}+y+z+\frac{1}{xz}+\frac{1}{y}+x.$$

Then the quartic pencil $\mathcal{S}$ is given by the following equation:

$$y^2tz+t^2zy+y^2xz+z^2xy+t^3y+t^2xz+x^2zy=\lambda xyzt.$$

As usual, we assume that $\lambda\neq\infty$.

Let $\mathcal{C}$ be the conic in $\mathbb{P}^3$ that is given by $x=yz+zt+t^2=0$. Then

$$\begin{aligned}H_{\{x\}}\cdot S_\lambda &= L_{\{x\},\{y\}}+L_{\{x\},\{t\}}+\mathcal{C},\\ H_{\{y\}}\cdot S_\lambda &= L_{\{x\},\{y\}}+L_{\{y\},\{z\}}+2L_{\{y\},\{t\}},\\ H_{\{z\}}\cdot S_\lambda &= L_{\{y\},\{z\}}+3L_{\{z\},\{t\}},\\ H_{\{t\}}\cdot S_\lambda &= L_{\{x\},\{t\}}+L_{\{y\},\{t\}}+L_{\{z\},\{t\}}+L_{\{t\},\{x,y,z\}}.\end{aligned}\tag{4.11.1}$$

For every λ, the surface S_λ is irreducible, it has isolated singularities, and its singular points contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

$P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term xy;
$P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_2$ with quadratic term $z(x+t)$;
$P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_4$ with quadratic term yz;
$P_{\{y\},\{t\},\{x,z\}}$: type $\mathbb{A}_1$;
$P_{\{z\},\{t\},\{x,y\}}$: type $\mathbb{A}_2$ with quadratic term $z(x+y+z-t-\lambda t)$.

By Corollary 1.5.4, the fiber $\mathsf{f}^{-1}(\lambda)$ is irreducible for every $\lambda\in\mathbb{C}$. This confirms ($\heartsuit$) in the Main Theorem, since $h^{1,2}(X)=0$.

The description of the singular points of the surface S_λ gives $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk})=\operatorname{rk}\operatorname{Pic}(S_{\Bbbk})+12$. On the other hand, it follows from (4.11.1) that

$$\begin{aligned}H_\lambda &\sim L_{\{x\},\{y\}}+L_{\{x\},\{t\}}+\mathcal{C}\sim L_{\{x\},\{y\}}+L_{\{y\},\{z\}}+2L_{\{y\},\{t\}}\\ &\sim L_{\{y\},\{z\}}+3L_{\{z\},\{t\}}\sim L_{\{x\},\{t\}}+L_{\{y\},\{t\}}+L_{\{z\},\{t\}}+L_{\{t\},\{x,y,z\}}\end{aligned}$$

on the surface S_λ. Therefore, the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{t\},\{x,y,z\}}$, and $\mathcal{C}$ has the same rank as the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{t\}}$, $L_{\{t\},\{x,y,z\}}$, and H_λ. The latter matrix is given by

$$\begin{array}{r c}
 & \begin{array}{cccc} L_{\{x\},\{y\}} & L_{\{x\},\{t\}} & L_{\{t\},\{x,y,z\}} & H_\lambda \end{array}\\
\begin{array}{r} L_{\{x\},\{y\}}\\ L_{\{x\},\{t\}}\\ L_{\{t\},\{x,y,z\}}\\ H_\lambda \end{array} & \begin{pmatrix} -1 & 1/2 & 0 & 1\\ 1/2 & -7/12 & 1 & 1\\ 0 & 1 & -5/6 & 1\\ 1 & 1 & 1 & 4 \end{pmatrix}
\end{array}.$$

The rank of this matrix is 4. Thus, we see that ($\bigstar$) holds in this case. Then ($\diamondsuit$) in the Main Theorem also holds by Lemma 1.13.1.

Family 4.12. In this case, the threefold X is a blow-up of $\mathbb{P}^3$ in a line and two exceptional curves of the blow-up. We have $h^{1,2}(X) = 0$. A toric Landau–Ginzburg model is given by the Minkowski polynomial 83, which is

$$\frac{y}{x} + \frac{y}{xz} + \frac{1}{x} + y + z + \frac{1}{y} + x.$$

Then the quartic pencil $\mathcal{S}$ is given by the following equation:

$$y^2tz + t^2y^2 + t^2zy + y^2xz + z^2xy + t^2xz + x^2zy = \lambda xyzt.$$

Here, for simplicity, we suppose that $\lambda \neq \infty$.

Let $\mathcal{C}$ be the conic in $\mathbb{P}^3$ that is given by $x = yz + yt + zt = 0$. Then

$$\begin{aligned} H_{\{x\}} \cdot S_\lambda &= L_{\{x\},\{y\}} + L_{\{x\},\{t\}} + \mathcal{C}, \\ H_{\{y\}} \cdot S_\lambda &= L_{\{x\},\{y\}} + L_{\{y\},\{z\}} + 2L_{\{y\},\{t\}}, \\ H_{\{z\}} \cdot S_\lambda &= 2L_{\{y\},\{z\}} + 2L_{\{z\},\{t\}}, \\ H_{\{t\}} \cdot S_\lambda &= L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y,z\}}. \end{aligned} \tag{4.12.1}$$

This shows that the base locus of the pencil $\mathcal{S}$ consists of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{t\},\{x,y,z\}}$, and $\mathcal{C}$.

Every surface S_λ is irreducible, it has isolated singularities, and its singular points contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

$P_{\{x\},\{y\},\{z\}}$: type $\mathbb{A}_1$;
$P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term xy;
$P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_1$;
$P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_5$ with quadratic term yz;
$P_{\{y\},\{t\},\{x,z\}}$: type $\mathbb{A}_1$;
$P_{\{z\},\{t\},\{x,y\}}$: type $\mathbb{A}_1$.

By Corollary 1.5.4, every fiber $\mathsf{f}^{-1}(\lambda)$ is irreducible. This confirms (♡) in the Main Theorem.

One has $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk}) = \operatorname{rk}\operatorname{Pic}(S_{\Bbbk}) + 12$. On the other hand, it follows from (4.12.1) that

$$\begin{aligned} H_\lambda &\sim L_{\{x\},\{y\}} + L_{\{x\},\{t\}} + \mathcal{C} \sim L_{\{x\},\{y\}} + L_{\{y\},\{z\}} + 2L_{\{y\},\{t\}} \\ &\sim 2L_{\{y\},\{z\}} + 2L_{\{z\},\{t\}} \sim L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y,z\}} \end{aligned}$$

on the surface S_λ. Thus, the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{t\},\{x,y,z\}}$, and $\mathcal{C}$ on the surface S_λ has the same rank as the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{t\}}$, $L_{\{t\},\{x,y,z\}}$, and H_λ. The latter matrix is given by

$$\begin{array}{c} \\ L_{\{x\},\{y\}} \\ L_{\{x\},\{t\}} \\ L_{\{t\},\{x,y,z\}} \\ H_\lambda \end{array} \begin{array}{c} \begin{array}{cccc} L_{\{x\},\{y\}} & L_{\{x\},\{t\}} & L_{\{t\},\{x,y,z\}} & H_\lambda \end{array} \\ \begin{pmatrix} -1/2 & 1/2 & 0 & 1 \\ 1/2 & -3/4 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 1 & 1 & 1 & 4 \end{pmatrix} \end{array}.$$

Its rank is 4, so that (★) holds. Then (◇) in the Main Theorem holds by Lemma 1.13.1.

Family 4.13. Recall that this family was missed in [8] and later found in [12]. In this case, the threefold X is a blow-up of $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$ along a smooth rational curve of tridegree $(1,1,3)$. Thus, we have $h^{1,2}(X)=0$. A toric Landau–Ginzburg model of this family is given by the Minkowski polynomial 1080, which is

$$x+y+z+\frac{x}{y}+\frac{y}{x}+\frac{x}{yz}+\frac{1}{z}+\frac{2}{y}+\frac{2}{x}+\frac{1}{xy}+\frac{1}{yz}.$$

The quartic pencil $\mathcal{S}$ is given by

$$x^2yz+xy^2z+xyz^2+x^2zt+y^2zt+x^2t^2+xyt^2+2xzt^2+2yzt^2+xt^3+zt^3=\lambda xyzt.$$

As usual, we suppose that $\lambda\neq\infty$.

Let $\mathcal{C}$ be the conic in $\mathbb{P}^3$ that is given by $y=xz+xt+tz=0$. Then

$$\begin{aligned}
H_{\{x\}}\cdot S_\lambda&=L_{\{x\},\{z\}}+L_{\{x\},\{t\}}+2L_{\{x\},\{y,t\}},\\
H_{\{y\}}\cdot S_\lambda&=L_{\{y\},\{t\}}+L_{\{y\},\{x,t\}}+\mathcal{C},\\
H_{\{z\}}\cdot S_\lambda&=L_{\{x\},\{z\}}+2L_{\{z\},\{t\}}+L_{\{z\},\{x,y,t\}},\\
H_{\{t\}}\cdot S_\lambda&=L_{\{x\},\{t\}}+L_{\{y\},\{t\}}+L_{\{z\},\{t\}}+L_{\{t\},\{x,y,z\}}.
\end{aligned}\tag{4.13.1}$$

Therefore, we conclude that the base locus of the pencil $\mathcal{S}$ consists of the curves $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{y,t\}}$, $L_{\{y\},\{x,t\}}$, $L_{\{z\},\{x,y,t\}}$, $L_{\{t\},\{x,y,z\}}$, and $\mathcal{C}$.

For every $\lambda\in\mathbb{C}$, the surface S_λ is irreducible, it has isolated singularities, and its singular points contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

$P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_2$ with quadratic term xy;
$P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_2$ with quadratic term $z(x+t)$;
$P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_1$;
$P_{\{x\},\{z\},\{y,t\}}$: type $\mathbb{A}_2$ with quadratic term $x(x+y+t+3z+\lambda z)$ for $\lambda\neq-3$, and type $\mathbb{A}_4$ for $\lambda=-3$;
$P_{\{z\},\{t\},\{x,y\}}$: type $\mathbb{A}_2$ with quadratic term $z(x+y+z-2t-\lambda t)$ for $\lambda\neq-3$, and type $\mathbb{A}_3$ for $\lambda=-3$;
$[0:1:-3-\lambda:-1]$: type $\mathbb{A}_1$ for $\lambda\neq-3$, and type $\mathbb{A}_4$ for $\lambda=-3$.

Thus, using Corollary 1.5.4, we conclude that the fiber $\mathsf{f}^{-1}(\lambda)$ is irreducible for every $\lambda\in\mathbb{C}$. This confirms (♡) in the Main Theorem, since $h^{1,2}(X)=0$.

Using the above description of the singular points of the surface S_λ, we see that $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk})=\operatorname{rk}\operatorname{Pic}(S_{\Bbbk})+10$. On the other hand, it follows from (4.13.1) that

$$\begin{aligned}
H_\lambda&\sim L_{\{x\},\{z\}}+L_{\{x\},\{t\}}+2L_{\{x\},\{y,t\}}\sim L_{\{y\},\{t\}}+L_{\{y\},\{x,t\}}+\mathcal{C}\\
&\sim L_{\{x\},\{z\}}+2L_{\{z\},\{t\}}+L_{\{z\},\{x,y,t\}}\sim L_{\{x\},\{t\}}+L_{\{y\},\{t\}}+L_{\{z\},\{t\}}+L_{\{t\},\{x,y,z\}}
\end{aligned}$$

on the surface S_λ. Hence, the intersection matrix of the curves $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{y,t\}}$, $L_{\{y\},\{x,t\}}$, $L_{\{z\},\{x,y,t\}}$, $L_{\{t\},\{x,y,z\}}$, and $\mathcal{C}$ on the surface S_λ has the same rank as the intersection matrix of the curves $L_{\{x\},\{y,t\}}$, $L_{\{y\},\{x,t\}}$, $L_{\{z\},\{t\}}$, $L_{\{z\},\{x,y,t\}}$, $L_{\{t\},\{x,y,z\}}$, and H_λ. Using

Propositions A.1.2 and A.1.3, we see that the latter matrix is

$$\begin{array}{r}\\ L_{\{x\},\{y,t\}}\\ L_{\{y\},\{x,t\}}\\ L_{\{z\},\{t\}}\\ L_{\{z\},\{x,y,t\}}\\ L_{\{t\},\{x,y,z\}}\\ H_\lambda\end{array}\begin{array}{c}\begin{array}{cccccc} L_{\{x\},\{y,t\}} & L_{\{y\},\{x,t\}} & L_{\{z\},\{t\}} & L_{\{z\},\{x,y,t\}} & L_{\{t\},\{x,y,z\}} & H_\lambda\end{array}\\ \begin{pmatrix} -5/6 & 1/3 & 1 & 1/3 & 0 & 1\\ 1/3 & -4/3 & 0 & 1 & 0 & 1\\ 1 & 0 & -1/6 & 1/3 & 1/3 & 1\\ 1/3 & 1 & 1/3 & -2/3 & 1/3 & 1\\ 0 & 0 & 1/3 & 1/3 & -4/3 & 1\\ 1 & 1 & 1 & 1 & 1 & 4\end{pmatrix}\end{array}.$$

Observe that the rank of this matrix is 6. Thus, we see that (★) holds. Then it follows from Lemma 1.13.1 that (♢) in the Main Theorem also holds in this case.

5. FANO THREEFOLDS OF PICARD RANK 5

Family 5.1. In this case, the threefold X is a quadric blown up in a conic and three exceptional curves of the blow-up. We have $h^{1,2}(X) = 0$. A toric Landau–Ginzburg model of this family is given by the Minkowski polynomial 1082, which is

$$x + y + \frac{1}{z} + \frac{x}{y} + \frac{y}{x} + \frac{2}{y} + \frac{2}{x} + \frac{z}{y} + \frac{z}{x} + \frac{1}{xy} + \frac{z}{xy}.$$

The quartic pencil $\mathcal{S}$ is given by

$$x^2zy + y^2xz + xyt^2 + x^2zt + y^2zt + 2zxt^2 + 2zyt^2 + z^2xt + z^2yt + zt^3 + z^2t^2 = \lambda xyzt.$$

Suppose that $\lambda \neq \infty$. Then

$$\begin{aligned} H_{\{x\}} \cdot S_\lambda &= L_{\{x\},\{z\}} + L_{\{x\},\{t\}} + L_{\{x\},\{y,t\}} + L_{\{x\},\{y,z,t\}},\\ H_{\{y\}} \cdot S_\lambda &= L_{\{y\},\{z\}} + L_{\{y\},\{t\}} + L_{\{y\},\{x,t\}} + L_{\{y\},\{x,z,t\}},\\ H_{\{z\}} \cdot S_\lambda &= L_{\{x\},\{z\}} + L_{\{y\},\{z\}} + 2L_{\{z\},\{t\}},\\ H_{\{t\}} \cdot S_\lambda &= L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y\}}. \end{aligned} \tag{5.1.1}$$

Thus, the base locus of the pencil $\mathcal{S}$ consists of the lines $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{y,t\}}$, $L_{\{y\},\{x,t\}}$, $L_{\{x\},\{y,z,t\}}$, $L_{\{y\},\{x,z,t\}}$, and $L_{\{t\},\{x,y\}}$.

For every $\lambda \in \mathbb{C}$, the surface S_λ has isolated singularities, so that it is irreducible. The singular points of S_λ contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

- $P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_2$ with quadratic term $z(y+t)$;
- $P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_2$ with quadratic term $z(x+t)$;
- $P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term $t(x+y+t)$ for $\lambda \neq -3$, and type $\mathbb{A}_5$ for $\lambda = -3$;
- $P_{\{z\},\{t\},\{x,y\}}$: type $\mathbb{A}_1$.

Thus, it follows from Corollary 1.5.4 that the fiber $\mathsf{f}^{-1}(\lambda)$ is irreducible for every $\lambda \in \mathbb{C}$. This confirms (♡) in the Main Theorem, because $h^{1,2}(X) = 0$.

To verify (♢) in the Main Theorem, observe that $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk}) = \operatorname{rk}\operatorname{Pic}(S_{\Bbbk}) + 8$. This follows from the description of the singular points of the surface S_λ for $\lambda \neq -3$. On the other hand, it follows from (5.1.1) that

$$\begin{aligned} L_{\{x\},\{z\}} + L_{\{x\},\{t\}} + L_{\{x\},\{y,t\}} + L_{\{x\},\{y,z,t\}} &\sim L_{\{y\},\{z\}} + L_{\{y\},\{t\}} + L_{\{y\},\{x,t\}} + L_{\{y\},\{x,z,t\}}\\ &\sim L_{\{x\},\{z\}} + L_{\{y\},\{z\}} + 2L_{\{z\},\{t\}} \sim L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y\}} \sim H_\lambda \end{aligned}$$

on the surface S_λ. Thus, the intersection matrix of the lines $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{y,t\}}$, $L_{\{y\},\{x,t\}}$, $L_{\{x\},\{y,z,t\}}$, $L_{\{y\},\{x,z,t\}}$, and $L_{\{t\},\{x,y\}}$ on the surface S_λ has the same rank as the intersection matrix of the curves $L_{\{x\},\{y,t\}}$, $L_{\{x\},\{y,z,t\}}$, $L_{\{y\},\{x,t\}}$, $L_{\{y\},\{x,z,t\}}$, $L_{\{z\},\{t\}}$, $L_{\{y\},\{t\}}$, and H_λ. If $\lambda \neq -3$, then the latter matrix is given by

$$\begin{array}{c} \\ L_{\{x\},\{y,t\}} \\ L_{\{x\},\{y,z,t\}} \\ L_{\{y\},\{x,t\}} \\ L_{\{y\},\{x,z,t\}} \\ L_{\{z\},\{t\}} \\ L_{\{y\},\{t\}} \\ H_\lambda \end{array}
\begin{array}{c}
\begin{array}{ccccccc} L_{\{x\},\{y,t\}} & L_{\{x\},\{y,z,t\}} & L_{\{y\},\{x,t\}} & L_{\{y\},\{x,z,t\}} & L_{\{z\},\{t\}} & L_{\{y\},\{t\}} & H_\lambda \end{array} \\
\begin{pmatrix}
-5/4 & 1 & 3/4 & 0 & 0 & 1/4 & 1 \\
1 & -2 & 0 & 1 & 0 & 0 & 1 \\
3/4 & 0 & -5/4 & 1 & 0 & 1/4 & 1 \\
0 & 1 & 1 & -2 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & -1/6 & 1/3 & 1 \\
1/4 & 0 & 1/4 & 1 & 1/3 & -7/12 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 4
\end{pmatrix}
\end{array}.$$

Its determinant is $7/9$. This shows that ($\bigstar$) holds. Thus, we can use Lemma 1.13.1 to conclude that ($\diamondsuit$) in the Main Theorem also holds in this case.

Family 5.2. In this case, the threefold X is a blow-up of $\mathbb{P}^3$ in a disjoint union of two lines and a disjoint union of two exceptional curves of the blow-up that lie on the same irreducible component of the blow-up. We have $h^{1,2}(X) = 0$. A toric Landau–Ginzburg model of this family is given by the Minkowski polynomial 219, which is

$$x + y + z + \frac{x}{z} + \frac{x}{y} + \frac{y}{x} + \frac{1}{y} + \frac{1}{x}.$$

Thus, the quartic pencil $\mathcal{S}$ is given by the equation

$$x^2zy + y^2xz + z^2xy + x^2ty + x^2tz + y^2tz + t^2xz + t^2zy = \lambda xyzt.$$

Suppose that $\lambda \neq \infty$. Then

$$\begin{aligned}
H_{\{x\}} \cdot S_\lambda &= L_{\{x\},\{y\}} + L_{\{x\},\{z\}} + L_{\{x\},\{t\}} + L_{\{x\},\{y,t\}}, \\
H_{\{y\}} \cdot S_\lambda &= L_{\{x\},\{y\}} + L_{\{y\},\{z\}} + L_{\{y\},\{t\}} + L_{\{y\},\{x,t\}}, \\
H_{\{z\}} \cdot S_\lambda &= 2L_{\{x\},\{z\}} + L_{\{y\},\{z\}} + L_{\{z\},\{t\}}, \\
H_{\{t\}} \cdot S_\lambda &= L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y,z\}}.
\end{aligned} \tag{5.2.1}$$

For every λ, the surface S_λ is irreducible, it has isolated singularities, and its singular points contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

$P_{\{x\},\{y\},\{z\}}$: type $\mathbb{A}_2$ with quadratic term $z(x+y)$;

$P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term xy;

$P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_2$ with quadratic term $z(x+t)$;

$P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_1$;

$P_{\{x\},\{t\},\{y,z\}}$: type $\mathbb{A}_1$.

Therefore, using Corollary 1.5.4, we see that the fiber $\mathsf{f}^{-1}(\lambda)$ is irreducible for every $\lambda \in \mathbb{C}$. This confirms ($\heartsuit$) in the Main Theorem, since $h^{1,2}(X) = 0$.

Using (5.2.1), we see that the intersection matrix of the lines $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{y,t\}}$, $L_{\{y\},\{x,t\}}$, and $L_{\{t\},\{x,y,z\}}$ on the surface S_λ has the same rank as the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{x\},\{t\}}$, and H_λ. The latter matrix is given by

$$\begin{array}{c} \\ L_{\{x\},\{y\}} \\ L_{\{x\},\{z\}} \\ L_{\{y\},\{z\}} \\ L_{\{y\},\{t\}} \\ L_{\{x\},\{t\}} \\ H_\lambda \end{array} \begin{array}{c} \begin{array}{cccccc} L_{\{x\},\{y\}} & L_{\{x\},\{z\}} & L_{\{y\},\{z\}} & L_{\{y\},\{t\}} & L_{\{x\},\{t\}} & H_\lambda \end{array} \\ \begin{pmatrix} -1/3 & 2/3 & 2/3 & 1/2 & 1/2 & 1 \\ 2/3 & -1/6 & 2/3 & 0 & 2/3 & 1 \\ 2/3 & 2/3 & -3/2 & 1/2 & 0 & 1 \\ 1/2 & 0 & 1/2 & -3/4 & 1/4 & 1 \\ 1/2 & 2/3 & 0 & 1/4 & -1/12 & 1 \\ 1 & 1 & 1 & 1 & 1 & 4 \end{pmatrix} \end{array}.$$

Note that this matrix has rank 6. Moreover, using the description of the singular points of the surface S_λ, we see that $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk}) = \operatorname{rk}\operatorname{Pic}(S_{\Bbbk}) + 9$. This shows that ($\bigstar$) holds in this case. Then ($\diamondsuit$) in the Main Theorem holds by Lemma 1.13.1.

Family 5.3. In this case, we have $X \cong \mathbb{P}^1 \times \mathbf{S}_6$, where $\mathbf{S}_6$ is a smooth del Pezzo surface of degree 6. This implies that $h^{1,2}(X) = 0$. A toric Landau–Ginzburg model of this family is given by the Minkowski polynomial 218, which is

$$x + y + z + \frac{y}{z} + \frac{z}{y} + \frac{1}{z} + \frac{1}{y} + \frac{1}{x}.$$

Therefore, the corresponding pencil $\mathcal{S}$ is given by the following equation:

$$x^2zy + y^2xz + z^2xy + y^2xt + z^2xt + t^2xy + t^2xz + t^2zy = \lambda xyzt.$$

Suppose that $\lambda \neq \infty$. Then

$$\begin{aligned} H_{\{x\}} \cdot S_\lambda &= L_{\{x\},\{y\}} + L_{\{x\},\{z\}} + 2L_{\{x\},\{t\}}, \\ H_{\{y\}} \cdot S_\lambda &= L_{\{x\},\{y\}} + L_{\{y\},\{z\}} + L_{\{y\},\{t\}} + L_{\{y\},\{z,t\}}, \\ H_{\{z\}} \cdot S_\lambda &= L_{\{x\},\{z\}} + L_{\{y\},\{z\}} + L_{\{z\},\{t\}} + L_{\{z\},\{y,t\}}, \\ H_{\{t\}} \cdot S_\lambda &= L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y,z\}}. \end{aligned} \tag{5.3.1}$$

Thus, the base locus of the pencil $\mathcal{S}$ consists of the lines $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{z\},\{y,t\}}$, and $L_{\{t\},\{x,y,z\}}$.

For every λ, the surface S_λ is irreducible, it has isolated singularities, and its singular points contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

$P_{\{x\},\{y\},\{z\}}$: type $\mathbb{A}_1$;
$P_{\{x\},\{z\},\{t\}}$: type $\mathbb{A}_2$ with quadratic term $x(z+t)$;
$P_{\{x\},\{y\},\{t\}}$: type $\mathbb{A}_2$ with quadratic term $x(y+t)$;
$P_{\{y\},\{z\},\{t\}}$: type $\mathbb{A}_3$ with quadratic term yz;
$P_{\{x\},\{t\},\{y,z\}}$: type $\mathbb{A}_1$.

Thus, by Corollary 1.5.4, the fiber $\mathsf{f}^{-1}(\lambda)$ is irreducible for every $\lambda \in \mathbb{C}$. This confirms ($\heartsuit$) in the Main Theorem, since $h^{1,2}(X) = 0$.

Now let us verify ($\diamondsuit$) in the Main Theorem. On the surface S_λ, we have

$$\begin{aligned} H_\lambda &\sim L_{\{x\},\{y\}} + L_{\{x\},\{z\}} + 2L_{\{x\},\{t\}} \sim L_{\{x\},\{y\}} + L_{\{y\},\{z\}} + L_{\{y\},\{t\}} + L_{\{y\},\{z,t\}} \\ &\sim L_{\{x\},\{z\}} + L_{\{y\},\{z\}} + L_{\{z\},\{t\}} + L_{\{z\},\{y,t\}} \sim L_{\{x\},\{t\}} + L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + L_{\{t\},\{x,y,z\}}. \end{aligned}$$

This follows from (5.3.1). Thus, the intersection matrix of the lines $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, $L_{\{y\},\{z,t\}}$, $L_{\{z\},\{y,t\}}$, and $L_{\{t\},\{x,y,z\}}$ has the same rank as the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{z\},\{t\}}$, and H_λ. The latter matrix is given by

$$\begin{array}{c} \\ L_{\{x\},\{y\}} \\ L_{\{x\},\{z\}} \\ L_{\{y\},\{z\}} \\ L_{\{y\},\{t\}} \\ L_{\{z\},\{t\}} \\ H_\lambda \end{array}
\begin{array}{c}
\begin{array}{cccccc} L_{\{x\},\{y\}} & L_{\{x\},\{z\}} & L_{\{y\},\{z\}} & L_{\{y\},\{t\}} & L_{\{z\},\{t\}} & H_\lambda \end{array} \\
\begin{pmatrix}
-2/3 & 1/2 & 1/2 & 1/3 & 0 & 1 \\
1/2 & -2/3 & 1/2 & 0 & 1/3 & 1 \\
1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1 \\
1/3 & 0 & 1/2 & -7/12 & 1/4 & 1 \\
0 & 1/3 & 1/2 & 1/4 & -7/12 & 1 \\
1 & 1 & 1 & 1 & 1 & 4
\end{pmatrix}
\end{array}.$$

Its rank is 6. On the other hand, it follows from the description of the singular points of the surface S_λ that $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk}) = \operatorname{rk}\operatorname{Pic}(S_{\Bbbk}) + 9$, so that ($\bigstar$) holds in this case. Thus, by Lemma 1.13.1, we see that ($\diamondsuit$) in the Main Theorem also holds in this case.

6. FANO THREEFOLDS OF PICARD RANK 6

Family 6.1. We have $X \cong \mathbb{P}^1 \times \mathbf{S}_5$, where $\mathbf{S}_5$ is the smooth del Pezzo surface of degree 5. In particular, we have $h^{1,2}(X) = 0$. A toric Landau–Ginzburg model of this family is given by the Minkowski polynomial 283, which is

$$x + \frac{1}{y} + z + \frac{1}{xy} + \frac{1}{z} + 2y + \frac{3}{x} + \frac{3y}{x} + \frac{y^2}{x}.$$

Thus, the corresponding pencil $\mathcal{S}$ is given by the equation

$$x^2yz + xzt^2 + xyz^2 + zt^3 + xyt^2 + 2xy^2z + 3yzt^2 + 3y^2zt + y^3z = \lambda xyzt.$$

For simplicity, we suppose that $\lambda \neq \infty$.

Let $\mathcal{C}$ be the conic in $\mathbb{P}^3$ that is given by $t = (x+y)^2 + xz = 0$. Then

$$\begin{aligned}
H_{\{x\}} \cdot S_\lambda &= L_{\{x\},\{z\}} + 3L_{\{x\},\{y,t\}}, \\
H_{\{y\}} \cdot S_\lambda &= L_{\{y\},\{z\}} + 2L_{\{y\},\{t\}} + L_{\{y\},\{x,t\}}, \\
H_{\{z\}} \cdot S_\lambda &= L_{\{x\},\{z\}} + L_{\{y\},\{z\}} + 2L_{\{z\},\{t\}}, \\
H_{\{t\}} \cdot S_\lambda &= L_{\{y\},\{t\}} + L_{\{z\},\{t\}} + \mathcal{C}.
\end{aligned} \tag{6.1.1}$$

Thus, the base locus of the pencil $\mathcal{S}$ consists of the curves $L_{\{x\},\{z\}}$, $L_{\{x\},\{y,t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{y\},\{x,t\}}$, $L_{\{z\},\{t\}}$, and $\mathcal{C}$.

For every $\lambda \in \mathbb{C}$, the surface S_λ is irreducible and has isolated singularities. Moreover, if $\lambda \neq -1, -5$, then the singular points of S_λ contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

$$\begin{aligned}
P_{\{x\},\{y\},\{t\}}&\colon && \text{type } \mathbb{A}_2 \text{ with quadratic term } xy; \\
P_{\{y\},\{z\},\{t\}}&\colon && \text{type } \mathbb{A}_3 \text{ with quadratic term } yz; \\
P_{\{y\},\{z\},\{x,t\}}&\colon && \text{type } \mathbb{A}_1; \\
P_{\{z\},\{t\},\{x,y\}}&\colon && \text{type } \mathbb{A}_3 \text{ with quadratic term } z^2 + t^2 + (\lambda+3)tz; \\
[0 : -2 : \lambda + 3 \pm \sqrt{\lambda^2 + 6\lambda + 5} : 2]&\colon && \text{type } \mathbb{A}_2.
\end{aligned}$$

Thus, in the notation of Subsection 1.8, the set Σ consists of the points $P_{\{y\},\{z\},\{t\}}$, $P_{\{x\},\{y\},\{t\}}$, $P_{\{y\},\{z\},\{x,t\}}$, and $P_{\{z\},\{t\},\{x,y\}}$.

The description of the singular points of the surface S_λ also gives

$$\mathrm{rk}\,\mathrm{Pic}(\widetilde{S}_{\Bbbk}) = \mathrm{rk}\,\mathrm{Pic}(S_{\Bbbk}) + 10. \tag{6.1.2}$$

Observe that the singular point $P_{\{z\},\{t\},\{x,y\}}$ contributes ② to this formula. Similarly, the singular points $[0\colon -2\colon \lambda+3\pm\sqrt{\lambda^2+6\lambda+5}\colon 2]$ also contribute ② to (6.1.2).

To verify ($\heartsuit$) in the Main Theorem, observe that the surface S_λ has du Val singularities at the base points of the pencil $\mathcal{S}$ provided that $\lambda\neq -1,-5$. Thus, by Corollary 1.5.4, the fiber $\mathsf{f}^{-1}(\lambda)$ is irreducible for every $\lambda\in\mathbb{C}$ such that $\lambda\neq -1,-5$. Moreover, we have

Lemma 6.1.3. *One has* $[\mathsf{f}^{-1}(-1)]=[\mathsf{f}^{-1}(-5)]=1$.

Proof. It is enough to prove that $[\mathsf{f}^{-1}(-1)]=1$, since the proof in the other case is identical. Observe that the points $P_{\{y\},\{z\},\{t\}}$, $P_{\{x\},\{y\},\{t\}}$, and $P_{\{y\},\{z\},\{x,t\}}$ are good double points of the surface S_{-1}. Thus, it follows from (1.8.3) and Lemmas 1.8.5 and 1.12.1 that

$$\bigl[\mathsf{f}^{-1}(-1)\bigr]=\bigl[S_{-1}\bigr]+\mathbf{D}^{-1}_{P_{\{z\},\{t\},\{x,y\}}}=1+\mathbf{D}^{-1}_{P_{\{z\},\{t\},\{x,y\}}}.$$

In the neighborhood of the point $P_{\{z\},\{t\},\{x,y\}}$, the morphism α in (1.9.3) is just a blow-up of this point. Moreover, its exceptional surface that is mapped to $P_{\{z\},\{t\},\{x,y\}}$ does not contain base curves of the pencil $\widehat{\mathcal{S}}$, because the quadratic term of the surface S_λ at this point is $z^2+t^2+(\lambda+3)tz$. Furthermore, the point $P_{\{z\},\{t\},\{x,y\}}$ is a double point of the surface S_{-1}. In fact, the surface S_{-1} has a singularity of type $\mathbb{D}_4$ at $P_{\{z\},\{t\},\{x,y\}}$. We see that $A^{-1}_{P_{\{z\},\{t\},\{x,y\}}}=0$. Then $\mathbf{D}^{-1}_{P_{\{z\},\{t\},\{x,y\}}}=0$ by (1.10.9), so that $[\mathsf{f}^{-1}(-1)]=1$. ☐

Thus, we conclude that $\mathsf{f}^{-1}(\lambda)$ is irreducible for every $\lambda\in\mathbb{C}$. This confirms ($\heartsuit$) in the Main Theorem, since $h^{1,2}(X)=0$.

To verify ($\diamondsuit$) in the Main Theorem, observe that

$$\begin{aligned} H_\lambda &\sim L_{\{x\},\{z\}}+3L_{\{x\},\{y,t\}}\sim L_{\{y\},\{z\}}+2L_{\{y\},\{t\}}+L_{\{y\},\{x,t\}}\\ &\sim L_{\{x\},\{z\}}+L_{\{y\},\{z\}}+2L_{\{z\},\{t\}}\sim L_{\{y\},\{t\}}+L_{\{z\},\{t\}}+\mathcal{C}\end{aligned}$$

on the surface S_λ. This follows from (6.1.1). Thus, the intersection matrix of the curves $L_{\{x\},\{z\}}$, $L_{\{x\},\{y,t\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, $L_{\{y\},\{x,t\}}$, $L_{\{z\},\{t\}}$, and $\mathcal{C}$ on the surface S_λ has the same rank as the intersection matrix of the curves $L_{\{x\},\{z\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{t\}}$, and H_λ. If $\lambda\neq -1,-5$, then the latter matrix is given by

$$\begin{array}{r c} & \begin{array}{cccc} L_{\{x\},\{z\}} & L_{\{y\},\{z\}} & L_{\{y\},\{t\}} & H_\lambda \end{array}\\ \begin{array}{r} L_{\{x\},\{z\}}\\ L_{\{y\},\{z\}}\\ L_{\{y\},\{t\}}\\ H_\lambda \end{array} & \begin{pmatrix} -2 & 1 & 0 & 1\\ 1 & -1/2 & 1/2 & 1\\ 0 & 1/2 & -7/12 & 1\\ 1 & 1 & 1 & 4 \end{pmatrix} \end{array}.$$

The rank of this matrix is 4. Thus, using (6.1.2), we see that ($\bigstar$) holds. Then ($\diamondsuit$) in the Main Theorem holds by Lemma 1.13.1.

7. FANO THREEFOLDS OF PICARD RANK 7

Family 7.1. In this case, we have $X\cong\mathbb{P}^1\times\mathbf{S}_4$, where $\mathbf{S}_4$ is a smooth del Pezzo surface of degree 4. This implies that $h^{1,2}(X)=0$. A toric Landau–Ginzburg model of this family is given by the Minkowski polynomial 505, which is

$$\frac{1}{x}+\frac{1}{y}+z+\frac{2y}{x}+\frac{2x}{y}+\frac{1}{z}+\frac{y^2}{x}+3y+3x+\frac{x^2}{y}.$$

Hence, the corresponding pencil $\mathcal{S}$ is given by the following equation:

$$t^2zy + t^2xz + z^2xy + 2y^2tz + 2x^2tz + t^2xy + y^3z + 3y^2xz + 3x^2zy + x^3z = \lambda xyzt.$$

For simplicity, we suppose that $\lambda \neq \infty$.

Let $\mathcal{C}$ be the cubic curve in $\mathbb{P}^3$ that is given by $t = xyz + (x+y)^3 = 0$. This curve is singular at the point $P_{\{x\},\{y\},\{t\}}$. Moreover, we have

$$\begin{aligned} H_{\{x\}} \cdot S_\lambda &= L_{\{x\},\{y\}} + L_{\{x\},\{z\}} + 2L_{\{x\},\{y,t\}}, \\ H_{\{y\}} \cdot S_\lambda &= L_{\{x\},\{y\}} + L_{\{y\},\{z\}} + 2L_{\{y\},\{x,t\}}, \\ H_{\{z\}} \cdot S_\lambda &= L_{\{x\},\{z\}} + L_{\{y\},\{z\}} + 2L_{\{z\},\{t\}}, \\ H_{\{t\}} \cdot S_\lambda &= L_{\{z\},\{t\}} + \mathcal{C}. \end{aligned} \tag{7.1.1}$$

Thus, the base locus of the pencil $\mathcal{S}$ consists of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{y\},\{z\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{y,t\}}$, $L_{\{y\},\{x,t\}}$, and $\mathcal{C}$.

If $\lambda \neq -2, -6$, then S_λ is irreducible and has isolated singularities. In this case, the singular points of S_λ contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

$$\begin{aligned} P_{\{x\},\{y\},\{z\}}&\colon\ \text{type } \mathbb{A}_1; \\ P_{\{x\},\{y\},\{t\}}&\colon\ \text{type } \mathbb{A}_3 \text{ with quadratic term } xy; \\ P_{\{z\},\{t\},\{x,y\}}&\colon\ \text{type } \mathbb{A}_5 \text{ with quadratic term } z^2 + t^2 - (\lambda+4)zt; \\ [0 : -2 : \lambda + 4 \pm \sqrt{\lambda^2 + 8\lambda + 12} : 2]&\colon\ \text{type } \mathbb{A}_1; \\ [-2 : 0 : \lambda + 4 \pm \sqrt{\lambda^2 + 8\lambda + 12} : 2]&\colon\ \text{type } \mathbb{A}_1. \end{aligned}$$

Thus, the set Σ consists of the points $P_{\{y\},\{z\},\{t\}}$, $P_{\{x\},\{y\},\{t\}}$, $P_{\{y\},\{z\},\{x,t\}}$, and $P_{\{z\},\{t\},\{x,y\}}$.

The description of the singular points of the surface S_λ also gives

$$\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk}) = \operatorname{rk}\operatorname{Pic}(S_{\Bbbk}) + 9. \tag{7.1.2}$$

Note that the singular point $P_{\{z\},\{t\},\{x,y\}}$ contributes ③ to this formula. Similarly, the singular points $[0 : -2 : \lambda + 4 \pm \sqrt{\lambda^2 + 8\lambda + 12} : 2]$ contribute ① to this formula. Similarly, the singular points $[-2 : 0 : \lambda + 4 \pm \sqrt{\lambda^2 + 8\lambda + 12} : 2]$ also contribute ① to (7.1.2).

To verify (♡) in the Main Theorem, observe that the surface S_λ has du Val singularities at the base points of the pencil $\mathcal{S}$ provided that $\lambda \neq -2, -6$. Thus, by Corollary 1.5.4, the fiber $\mathsf{f}^{-1}(\lambda)$ is irreducible for every $\lambda \in \mathbb{C}$ such that $\lambda \neq -2, -6$. On the other hand, we have

Lemma 7.1.3. *One has* $[\mathsf{f}^{-1}(-2)] = [\mathsf{f}^{-1}(-6)] = 1$.

Proof. Observe that both surfaces S_{-2} and S_{-6} have non-isolated singularities. Namely, the surface S_{-2} is singular along the line $x + y + z = x + y + t = 0$, and S_{-6} is singular along the line $x + y - z = x + y + t = 0$. However, both these surfaces are irreducible. This can be checked by analyzing their hyperplane sections.

It is enough to prove that $[\mathsf{f}^{-1}(-2)] = 1$, since the proof in the other case is identical. Observe that the points $P_{\{y\},\{z\},\{t\}}$, $P_{\{x\},\{y\},\{t\}}$, and $P_{\{y\},\{z\},\{x,t\}}$ are good double points of the surface S_{-2}. Thus, it follows from (1.8.3) and Lemmas 1.8.5 and 1.12.1 that

$$\big[\mathsf{f}^{-1}(-2)\big] = 1 + \mathbf{D}^{-2}_{P_{\{z\},\{t\},\{x,y\}}}.$$

Arguing as in the proof of Lemma 6.1.3, we get $\mathbf{D}^{-2}_{P_{\{z\},\{t\},\{x,y\}}} = 0$, so that $[\mathsf{f}^{-1}(-2)] = 1$. $\square$

We see that $\mathsf{f}^{-1}(\lambda)$ is irreducible for every $\lambda \in \mathbb{C}$. This confirms (♡) in the Main Theorem.

Let us verify ($\diamondsuit$) in the Main Theorem. It follows from (7.1.1) that the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{y\},\{z\}}$, $L_{\{z\},\{t\}}$, $L_{\{x\},\{y,t\}}$, $L_{\{y\},\{x,t\}}$, and $\mathcal{C}$ on the surface S_λ has the same rank as the intersection matrix of the curves $L_{\{x\},\{z\}}$, $L_{\{y\},\{z\}}$, $L_{\{y\},\{x,t\}}$, and H_λ. If $\lambda\neq -2,-6$, then the latter matrix is given by

$$\begin{array}{r c} & \begin{array}{cccc} L_{\{x\},\{z\}} & L_{\{y\},\{z\}} & L_{\{y\},\{x,t\}} & H_\lambda \end{array} \\ \begin{array}{r} L_{\{x\},\{z\}} \\ L_{\{y\},\{z\}} \\ L_{\{y\},\{x,t\}} \\ H_\lambda \end{array} & \begin{pmatrix} -3/2 & 1/2 & 0 & 1 \\ 1/2 & -2 & 1 & 1 \\ 0 & 1 & -5/4 & 1 \\ 1 & 1 & 1 & 4 \end{pmatrix} \end{array}.$$

Its rank is 4, so that ($\diamondsuit$) in the Main Theorem holds by (7.1.2) and Lemma 1.13.1.

8. FANO THREEFOLDS OF PICARD RANK 8

Family 8.1. We discussed this case in Example 1.10.11, where we also described the pencil $\mathcal{S}$. Let us use the notation of this example and assume that $\lambda\neq\infty$. Then

$$\begin{aligned} H_{\{x\}}\cdot S_\lambda &= L_{\{x\},\{y\}}+L_{\{x\},\{z\}}+2L_{\{x\},\{t\}},\\ H_{\{y\}}\cdot S_\lambda &= L_{\{x\},\{y\}}+3L_{\{y\},\{t,z\}},\\ H_{\{z\}}\cdot S_\lambda &= L_{\{x\},\{z\}}+3L_{\{z\},\{t,y\}},\\ H_{\{t\}}\cdot S_\lambda &= L_{\{x\},\{t\}}+\mathcal{C}. \end{aligned} \tag{8.1.1}$$

If $\lambda\neq -4,-8$, then the surface S_λ is irreducible and has isolated singularities. In fact, in this case, we can say more.

Lemma 8.1.2. *Suppose that $\lambda\neq -4,-8$. Then the singular points of S_λ contained in the base locus of the pencil $\mathcal{S}$ can be described as follows*:

$$\begin{aligned} P_{\{y\},\{z\},\{t\}}:&\quad \textit{type } \mathbb{A}_2;\\ P_{\{x\},\{t\},\{y,z\}}:&\quad \textit{type } \mathbb{A}_5;\\ [\lambda+3\pm\sqrt{\lambda^2+12\lambda+32}:0:-2:2]:&\quad \textit{type } \mathbb{A}_2;\\ [\lambda+3\pm\sqrt{\lambda^2+12\lambda+32}:-2:0:2]:&\quad \textit{type } \mathbb{A}_2. \end{aligned}$$

Proof. Taking partial derivatives, we see that the singular points of S_λ contained in the base locus of the pencil $\mathcal{S}$ are the points described in the statement of the lemma. To describe their types, we start with $P_{\{y\},\{z\},\{t\}}$. In the chart $x=1$, the surface S_λ is given by

$$yz+t^3+\text{Higher order terms}=0,$$

where we order the monomials with respect to the weights $\mathrm{wt}(y)=3$, $\mathrm{wt}(z)=3$, and $\mathrm{wt}(t)=2$. This implies that $P_{\{y\},\{z\},\{t\}}$ is a singular point of type $\mathbb{A}_2$.

To describe the type of the singular point $P_{\{x\},\{t\},\{y,z\}}$, we consider the chart $y=1$ and change coordinates as follows: $\overline{x}=x$, $\overline{z}=z+1$, $\overline{t}=t$. Then S_λ is given by

$$-\overline{x}^2+(\lambda+6)\overline{x}\overline{t}-\overline{t}^2+\text{Higher order terms}=0.$$

Note that

$$-\overline{x}^2+(\lambda+6)\overline{x}\overline{t}-\overline{t}^2=-\big(\overline{x}-\big(\lambda+3+\sqrt{\lambda^2+12\lambda+32}\big)\overline{t}\big)\big(\overline{x}-\big(\lambda+3-\sqrt{\lambda^2+12\lambda+32}\big)\overline{t}\big),$$

and this quadratic form has rank 2, because $\lambda\neq -4,-8$. Introducing new coordinates $\widehat{z}=\overline{z}$, $\widehat{y}=\overline{y}/\overline{z}$, and $\widehat{t}=\overline{t}/\overline{z}$, we obtain the equation of the blow-up of the surface S_λ at $P_{\{x\},\{t\},\{y,z\}}$. It is

$$\widehat{x}^2-(\lambda+6)\widehat{t}\widehat{x}+\widehat{t}^{\,2}=\widehat{x}^2\widehat{z}-(\lambda+6)\widehat{t}\widehat{x}\widehat{z}+\widehat{z}^2\widehat{x}+\widehat{t}^{\,2}\widehat{z}+3\widehat{t}\widehat{x}\widehat{z}^2+\widehat{t}^{\,3}\widehat{x}\widehat{z}^2+3\widehat{t}^{\,2}\widehat{x}\widehat{z}^2.$$

The two exceptional curves of the blow-up are given by $\widehat{z}=\widehat{t}=0$ and $\widehat{z}=\widehat{y}=0$. They intersect at the point $(0,0,0)$, which is a singular point of the obtained surface. To blow up the latter point, we introduce new coordinates $\widetilde{z}=\widehat{z}$, $\widetilde{y}=\widehat{y}/\widehat{z}$, and $\widetilde{t}=\widehat{t}/\widehat{z}$. After dividing by $\widehat{z}^2$, we rewrite the latter equation as

$$\widetilde{x}^2-\widetilde{z}\widetilde{x}-(\lambda+6)\widetilde{t}\widetilde{x}=\widetilde{x}^2\widetilde{z}-\widetilde{t}^{\,2}+\widetilde{t}^{\,2}\widetilde{z}-(\lambda+6)\widetilde{t}\widetilde{x}\widetilde{z}+3\widetilde{t}\widetilde{x}\widetilde{z}^2+3\widetilde{t}^{\,2}\widetilde{x}\widetilde{z}^3+\widetilde{t}^{\,3}\widetilde{x}\widetilde{z}^4.$$

The quadratic form of this equation has rank 3, so that this surface has an ordinary double point at $(0,0,0)$. This implies that $P_{\{x\},\{t\},\{y,z\}}$ is a singular point of type $\mathbb{A}_5$.

Now we describe the type of the floating singular points. We will only consider the singular point $[\lambda+3+\sqrt{\lambda^2+12\lambda+32}:0:-2:2]$, because the calculations in the other cases are similar. Let us introduce an auxiliary parameter $\mu\in\mathbb{C}$ such that $\lambda=-(4\mu^2-4\mu-1)/(\mu(\mu-1))$. We assume that $\mu\neq 0,1$. Then

$$[\lambda+3+\sqrt{\lambda^2+12\lambda+32}:0:-2:2]=[\mu-1:0:-\mu:\mu].$$

Taking the chart $t=1$ and introducing new coordinates $\overline{x}=x+(\mu-1)/\mu$, $\overline{y}=y$, and $\overline{z}=z-1$, we see that S_λ is given by

$$(2\mu-1)\overline{x}\,\overline{y}+(\mu-1)^2\overline{z}^3+\text{Higher order terms}=0.$$

Here, as above, we order the monomials with respect to the weights $\mathrm{wt}(\overline{x})=3$, $\mathrm{wt}(\overline{y})=3$, and $\mathrm{wt}(\overline{z})=2$. This implies that $[\mu-1:0:-\mu:\mu]$ is a singular point of type $\mathbb{A}_2$. ☐

Note that the singular locus of the surface S_{-4} consists of the point $P_{\{x\},\{y\},\{z\}}$ and the line $\{x-t=y+z+t=0\}$. Similarly, the singular locus of the surface S_{-8} consists of the point $P_{\{x\},\{y\},\{z\}}$ and the line $\{x+t=y+z+t=0\}$. Moreover, we have

Lemma 8.1.3. *Both surfaces S_{-8} and S_{-4} are irreducible.*

Proof. It suffices to prove that S_{-4} is irreducible, because the other case can be handled in a similar way. Let Π be the plane $\{t=z\}$. Denote by C_4 the intersection $S_{-4}\cap\Pi$. Then C_4 is the quartic curve in $\Pi\cong\mathbb{P}^2$ that it is given by

$$x^2yz+xy^3+6xy^2z+10xyz^2+8xz^3+yz^3=0.$$

This curve has exactly two singular points: $[1:0:0:0]$ and $[1:-2:1:1]$. Moreover, the point $[1:0:0:0]$ is an ordinary double point of the curve C_4, and the point $[1:-2:1:1]$ is an ordinary cusp of the curve C_4. This implies that the curve C_4 is irreducible, so that the surface S_{-4} is also irreducible. ☐

In Example 1.10.11, we proved that $[\mathsf{f}^{-1}(\lambda)]=1$ for every $\lambda\in\mathbb{C}$. Thus, we conclude that (♡) in the Main Theorem holds in this case.

Let us verify (♢) in the Main Theorem. It follows from (8.1.1) that the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, $L_{\{x\},\{t\}}$, $L_{\{y\},\{t,z\}}$, $L_{\{z\},\{t,y\}}$, and $\mathcal{C}$ on the surface S_λ has the same rank as the intersection matrix of the curves $L_{\{x\},\{y\}}$, $L_{\{x\},\{z\}}$, and H_λ. If $\lambda\neq-4,-8$, then the latter matrix is given by

$$\begin{array}{c}\\ L_{\{x\},\{y\}}\\ L_{\{x\},\{z\}}\\ H_\lambda\end{array}\begin{array}{c}\begin{array}{ccc}L_{\{x\},\{y\}} & L_{\{x\},\{z\}} & H_\lambda\end{array}\\ \begin{pmatrix}-2 & 1 & 1\\ 1 & -2 & 1\\ 1 & 1 & 4\end{pmatrix}\end{array}.$$

Its determinant is $18\neq 0$. On the other hand, we have $\mathrm{rk}\,\mathrm{Pic}(\widetilde{S}_{\Bbbk})=\mathrm{rk}\,\mathrm{Pic}(S_{\Bbbk})+9$. Thus, we see that (★) holds. Then (♢) in the Main Theorem holds by Lemma 1.13.1.

9. FANO THREEFOLDS OF PICARD RANK 9

Family 9.1. In this case, we have $X \cong \mathbb{P}^1 \times \mathbf{S}_2$, where $\mathbf{S}_2$ is a smooth del Pezzo surface of degree 2. In particular, we have $h^{1,2}(X) = 0$. This case is somewhat similar to the cases of families 2.2 and 2.3. As in these two cases, this family does not have toric Landau–Ginzburg models with reflexive Newton polytope. Let p be the Laurent polynomial

$$\frac{(a+b+1)^4}{ab} + c + \frac{1}{c}.$$

Then p gives the commutative diagram (✠) by [16, Proposition 16].

Let $\gamma\colon \mathbb{C}^3 \dashrightarrow \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$ be a birational transformation given by the change of coordinates

$$a = xz, \qquad b = x - xz - 1, \qquad c = \frac{z}{y}.$$

As for family 2.2, we can use γ to expand (✠) to the commutative diagram (2.2.1). The only difference is that now the pencil $\mathcal{S}$ is given by the equation

$$x^3y = (\lambda yz - y^2 - z^2)(xt - xz - t^2), \tag{9.1.1}$$

where $\lambda \in \mathbb{C} \cup \{\infty\}$. As for family 2.2, we will follow the scheme described in Section 1. The only difference is that now S_λ is the quartic surface given by equation (9.1.1).

Let $\mathbf{Q}$ be the quadric in $\mathbb{P}^3$ given by $xt - xz - t^2 = 0$. Then

$$S_\infty = H_{\{y\}} + H_{\{z\}} + \mathbf{Q}.$$

On the other hand, if $\lambda \neq \infty$, then S_λ is irreducible and has isolated singularities.

Let $\mathcal{C}_1$ be the conic in $\mathbb{P}^3$ given by $y = xt - xz - t^2 = 0$, and let $\mathcal{C}_2$ be the cubic curve in $\mathbb{P}^3$ given by $z = x^3 + yt(x+t) = 0$. If $\lambda \neq \infty$, then

$$H_{\{y\}} \cdot S_\lambda = 2L_{\{y\},\{z\}} + \mathcal{C}_1, \qquad H_{\{z\}} \cdot S_\lambda = L_{\{y\},\{z\}} + \mathcal{C}_2, \qquad \mathbf{Q} \cdot S_\lambda = 6L_{\{x\},\{t\}} + \mathcal{C}_1. \tag{9.1.2}$$

Thus, the base locus of the pencil $\mathcal{S}$ consists of the curves $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $\mathcal{C}_1$, and $\mathcal{C}_2$.

If $\lambda \neq \infty$, then the singular points of S_λ contained in the base locus of the pencil $\mathcal{S}$ can be described as follows:

$$\begin{aligned} P_{\{x\},\{z\},\{t\}}&\colon\ \text{type } \mathbb{A}_1;\\ P_{\{x\},\{y\},\{z\}}&\colon\ \text{type } \mathbb{A}_5 \text{ for } \lambda \neq \pm 2, \text{ and non-du Val for } \lambda = \pm 2;\\ [0 : \lambda \pm \sqrt{\lambda^2 - 4} : 2 : 0]&\colon\ \text{type } \mathbb{A}_5 \text{ for } \lambda \neq \pm 2, \text{ and non-du Val for } \lambda = \pm 2. \end{aligned}$$

Thus, it follows from (1.10.8) and Lemma 1.12.1 that the fiber $\mathsf{f}^{-1}(\lambda)$ is irreducible for every $\lambda \in \mathbb{C}$. This confirms (♡) in the Main Theorem.

To verify (◇) in the Main Theorem, observe that $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk}) = \operatorname{rk}\operatorname{Pic}(S_{\Bbbk}) + 9$. Indeed, the minimal resolution $\widetilde{S}_{\Bbbk} \to S_{\Bbbk}$ of the point $P_{\{x\},\{y\},\{z\}}$ is given by three consecutive blow-ups that have three irreducible (over $\Bbbk$) exceptional curves. Two of them are geometrically reducible, and one is geometrically irreducible. Similarly, the minimal resolution $\widetilde{S}_{\Bbbk} \to S_{\Bbbk}$ of the point $[0 : \lambda \pm \sqrt{\lambda^2 - 4} : 2 : 0]$ has five exceptional curves, and the minimal resolution of the point $P_{\{x\},\{z\},\{t\}}$ has one exceptional curve.

If $\lambda \neq \infty$, then it follows from (9.1.2) that

$$H_\lambda \sim 2L_{\{y\},\{z\}} + \mathcal{C}_1 \sim L_{\{y\},\{z\}} + \mathcal{C}_2 \sim_{\mathbb{Q}} 3L_{\{x\},\{t\}} + \frac{1}{2}\mathcal{C}_1$$

on the surface S_λ. Thus, if $\lambda \neq \infty$, then the intersection matrix of the curves $L_{\{x\},\{t\}}$ and H_λ on the surface S_λ has the same rank as the intersection matrix of the curves $L_{\{x\},\{t\}}$, $L_{\{y\},\{z\}}$, $\mathcal{C}_1$, $\mathcal{C}_2$, and H_λ. On the other hand, if $\lambda \neq \infty, \pm 2$, the former matrix is given by

$$\begin{array}{c c} & \begin{array}{cc} L_{\{x\},\{t\}} & H_\lambda \end{array} \\ \begin{array}{c} L_{\{x\},\{t\}} \\ H_\lambda \end{array} & \begin{pmatrix} -3/2 & 1 \\ 1 & 4 \end{pmatrix} \end{array}.$$

The rank of this matrix is 2. Thus, we see that ($\bigstar$) holds in this case. By Lemma 1.13.1, this confirms ($\diamondsuit$) in the Main Theorem.

10. FANO THREEFOLDS OF PICARD RANK 10

Family 10.1. In this case, we have $X \cong \mathbb{P}^1 \times \mathbf{S}_1$, where $\mathbf{S}_1$ is a smooth del Pezzo surface of degree 1. In particular, we have $h^{1,2}(X) = 0$. This case is very similar to the case of family 2.1. As in that case, this family does not have toric Landau–Ginzburg models with reflexive Newton polytope. However, there are Laurent polynomials with non-reflexive Newton polytopes that give the commutative diagram (✠). One of them is the Laurent polynomial

$$\frac{(x+y+1)^6}{xy^2} + z + \frac{1}{z},$$

which we also denote by p.

Let $\gamma\colon \mathbb{C}^3 \dashrightarrow \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$ be a birational transformation given by the change of coordinates

$$x = \frac{1}{b} - \frac{1}{b^2c} - 1, \qquad y = \frac{1}{b^2c}, \qquad z = y.$$

Arguing as in Subsection 1.9, we can expand (✠) to the commutative diagram (2.1.1). The only difference is that now the pencil $\mathcal{S}$ is given by the equation

$$xyc^3 = (\lambda xy - x^2 - y^2)(abc - b^2c - a^3), \tag{10.1.1}$$

where $\lambda \in \mathbb{C} \cup \{\infty\}$. Here $([x:y],[a:b:c])$ is a point in $\mathbb{P}^1 \times \mathbb{P}^2$.

As for family 2.1, we will follow the scheme described in Section 1, and we will use the assumptions and notation introduced in that section. The only difference is that $\mathbb{P}^3$ is now replaced by $\mathbb{P}^1 \times \mathbb{P}^2$ and that S_λ is now the surface in $\mathbb{P}^1 \times \mathbb{P}^2$ given by (10.1.1). As for family 2.1, we will extend our handy notation in Subsection 1.6 to bilinear sections of $\mathbb{P}^1 \times \mathbb{P}^2$.

Let S be the surface in $\mathbb{P}^1 \times \mathbb{P}^2$ given by $abc - b^2c - a^3 = 0$. Then S is irreducible. Moreover, we have

$$S_\infty = H_{\{x\}} + H_{\{y\}} + \mathsf{S}.$$

On the other hand, if $\lambda \neq \infty$, then S_λ is irreducible and has isolated singularities.

Let $\mathcal{C}_1$ be the curve in $\mathbb{P}^1 \times \mathbb{P}^2$ given by $x = abc - b^2c - a^3 = 0$, and let $\mathcal{C}_2$ be the curve in $\mathbb{P}^1 \times \mathbb{P}^2$ given by $y = abc - b^2c - a^3 = 0$. Then

$$H_{\{x\}} \cdot S_\lambda = \mathcal{C}_1, \qquad H_{\{y\}} \cdot S_\lambda = \mathcal{C}_2, \qquad \mathsf{S} \cdot S_\lambda = \mathcal{C}_1 + \mathcal{C}_2 + 9L_{\{a\},\{c\}}. \tag{10.1.2}$$

Thus, the base locus of the pencil $\mathcal{S}$ consists of the curves $\mathcal{C}_1$, $\mathcal{C}_2$, and $L_{\{a\},\{c\}}$.

If $\lambda \neq \infty$, then the only singular points of S_λ contained in the base locus of the pencil $\mathcal{S}$ are the points

$$\big([\lambda \pm \sqrt{\lambda^2-4} : 2], [0:1:0]\big). \tag{10.1.3}$$

If $\lambda \neq \pm 2$, then the surface S_λ has a singularity of type $\mathbb{A}_8$ at each of the points (10.1.3). If $\lambda = \pm 2$, then (10.1.3) gives the points $([\pm 1:1],[0:1:0])$. One can check that the surface $S_{\pm 2}$ has triple singularities at these points.

Remark 10.1.4. There exists a commutative diagram

$$\begin{tikzcd}
& V_2 \arrow[dl, "\beta_2"'] & V_3 \arrow[l, "\beta_3"'] & V_4 \arrow[l, "\beta_4"'] & \\
V_1 \arrow[d, "\beta_1"'] & & & & V \arrow[ul, "\gamma"'] \arrow[dllll, "\pi"'] \arrow[d, "\mathbf{g}"] \\
\mathbb{P}^1 \times \mathbb{P}^2 \arrow[rrrr, dashed, "\phi"] & & & & \mathbb{P}^1
\end{tikzcd}$$

where ϕ is a rational map given by the pencil $\mathcal{S}$, the morphism β_1 is the blow-up of the curve $\mathcal{C}_1$, the morphism β_2 is the blow-up of the proper transform of the curve $\mathcal{C}_2$, the morphism β_3 is the blow-up of a curve that dominates the curve $\mathcal{C}_1$, the morphism β_4 is the blow-up of a curve that dominates the curve $\mathcal{C}_2$, and γ is a birational morphism that is a composition of nine blow-ups of smooth curves that dominate the curve $L_{\{a\},\{c\}}$. Note that the curve $\mathcal{C}_1$ has a node at the point $P_{\{x\},\{a\},\{b\}}$. Similarly, the curve $\mathcal{C}_1$ has a node at the point $P_{\{y\},\{a\},\{b\}}$. Thus, both threefolds V_1 and V_2 are singular. Moreover, the morphism β_3 blows up a nodal curve that is contained in the smooth locus of the threefold V_2. Similarly, the morphism β_4 blows up a nodal curve contained in the smooth locus of the threefold V_3. Thus, the threefold V has four isolated ordinary double points. However, all of them are contained in the fiber $\mathbf{g}^{-1}(\infty)$, which consists of the proper transforms on V of the following surfaces: $H_{\{x\}}$, $H_{\{y\}}$, S, the exceptional surface of the morphism β_1, and the exceptional surface of the morphism β_2. Thus, the singularities of the threefold V are not important for the proof of the Main Theorem in this case. Note that

$$-K_V \sim \mathbf{g}^{-1}(\infty).$$

If we want to keep this condition and make V smooth, we must compose π with a small resolution of singular points of the threefold V. However, the resulting smooth threefold would not be projective (cf. the proof of [16, Proposition 29]). Indeed, by construction, the threefold V is $\mathbb{Q}$-factorial, so it does not admit projective small resolutions.

Note that the surfaces in the pencil $\mathcal{S}$ do not have fixed singular points, so that $\Sigma = \varnothing$. Thus, using (1.8.3), we get $[\mathsf{f}^{-1}(\lambda)] = 1$ for every $\lambda \in \mathbb{C}$. This confirms ($\heartsuit$) in the Main Theorem, since $h^{1,2}(X) = 0$.

To verify ($\diamondsuit$) in the Main Theorem, observe that $\operatorname{rk}\operatorname{Pic}(\widetilde{S}_{\Bbbk}) = \operatorname{rk}\operatorname{Pic}(S_{\Bbbk}) + 8$. On the other hand, if $\lambda \neq \infty, \pm 2$, the rank of the intersection matrix of the curves $\mathcal{C}_1$, $\mathcal{C}_2$, and $L_{\{a\},\{c\}}$ on the surface S_λ is 2. This follows from (10.1.2). Thus, we see that ($\bigstar$) holds in this case. By Lemma 1.13.1, this confirms ($\diamondsuit$) in the Main Theorem.

Appendix. CURVES ON SINGULAR SURFACES

Let S be a normal surface, and let C and Z be distinct irreducible curves in S. For every point $P \in S$, one can define the intersection multiplicity $(C \cdot Z)_P \in \mathbb{Q}_{\geq 0}$ as in [18]. As in the case when S is smooth, one has

$$C \cdot Z = \sum_{P \in C \cap Z} (C \cdot Z)_P.$$

In this appendix, we present two simple results (probably well-known to many experts) that can be used to compute the (local) intersection multiplicity $(C \cdot Z)_P$ and the (global) self-intersection C^2 in simple cases. These results are Propositions A.1.2 and A.1.3 below.

A.1. Intersection multiplicity. Fix a point $O \in C \cap Z$. Let $\pi\colon \widetilde{S} \to S$ be the minimal resolution of singularity of the point O, and let $G_1, \dots, G_n$ be the exceptional curves of the birational morphism π. Denote by $\widetilde{C}$ and $\widetilde{Z}$ the proper transforms of the curves C and Z on the surface $\widetilde{S}$, respectively. Following [18], one can define $\pi^*(C)$ as

$$\pi^*(C) = \widetilde{C} + \sum_{i=1}^{n} \mathbf{a}_i G_i$$

for some positive rational numbers $\mathbf{a}_1, \dots, \mathbf{a}_n$ such that

$$\left(\widetilde{C} + \sum_{i=1}^{n} \mathbf{a}_i G_i\right) \cdot G_i = 0.$$

Similarly, we have

$$\pi^*(Z) = \widetilde{Z} + \sum_{i=1}^{n} \mathbf{b}_i G_i$$

for some positive rational numbers $\mathbf{b}_1, \dots, \mathbf{b}_n$. We define

$$C \cdot Z = \left(\widetilde{C} + \sum_{i=1}^{n} \mathbf{a}_i G_i\right) \cdot \left(\widetilde{Z} + \sum_{i=1}^{n} \mathbf{b}_i G_i\right) = \pi^*(C) \cdot \pi^*(Z) = \pi^*(C) \cdot \widetilde{Z} = \widetilde{C} \cdot \pi^*(Z).$$

Let $\mathbf{G} = G_1 \cup \ldots \cup G_n$. Then one can define $(C \cdot Z)_O$ as

$$(C \cdot Z)_O = C \cdot Z - \widetilde{C} \cdot \widetilde{Z} + \sum_{P \in \widetilde{C} \cap \widetilde{Z} \cap \mathbf{G}} (\widetilde{C} \cdot \widetilde{Z})_P. \tag{A.1.1}$$

The main goal of this appendix is to prove the following two simple results.

Proposition A.1.2. *Suppose that O is a du Val singular point of the surface S, both curves C and Z are smooth at O, and C intersects Z transversally at the point O. Then the following assertions hold*:

(i) *The point O is a singular point of S of type $\mathbb{A}_n$ or $\mathbb{D}_n$.*

(ii) *If O is a singular point of type $\mathbb{A}_n$ and the proper transforms of the curves C and Z on the surface $\widetilde{S}$ intersect the k-th and r-th exceptional curves in the chain of exceptional curves of the minimal resolution of O, then*

$$(C \cdot Z)_O = \begin{cases} \dfrac{r(n+1-k)}{n+1} & \text{for } \ r \le k, \\[2mm] \dfrac{k(n+1-r)}{n+1} & \text{for } \ r > k. \end{cases}$$

(iii) *If O is of type $\mathbb{D}_n$, then $(C \cdot Z)_O = 1/2$.*

Proposition A.1.3. *Suppose that O is a du Val singular point of the surface S and the curve C is smooth at the point O. Then the following holds*:

(i) *The point O is a singular point of S of type $\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{E}_6$, or $\mathbb{E}_7$.*

(ii) *If O is a singular point of type $\mathbb{A}_n$ and $\widetilde{C}$ intersects the k-th exceptional curve in the chain of exceptional curves of the minimal resolution of O, then*

$$C^2 = \widetilde{C}^2 + \frac{k(n+1-k)}{n+1}.$$

(iii) *If O is a singular point of type $\mathbb{D}_n$, then either $C^2 = \widetilde{C}^2 + 1$ or $C^2 = \widetilde{C}^2 + n/4$.*

(iv) *If O is a singular point of type $\mathbb{E}_6$, then* $C^2 = \widetilde{C}^2 + 4/3$.

(v) *If O is a singular point of type $\mathbb{E}_7$, then* $C^2 = \widetilde{C}^2 + 3/2$.

Propositions A.1.2 and A.1.3 follow from Corollaries A.2.2 and A.2.3 and Lemmas A.3.1, A.3.2, and A.4.1–A.4.3, which we will prove below.

A.2. Singular points of type $\mathbb{A}$. In this subsection, we suppose that the surface S has a du Val singularity of type $\mathbb{A}_n$ at the point O, where $n \geq 1$. Then we may assume that

$$G_i \cdot G_j = \begin{cases} -2 & \text{if } \ i = j, \\ 0 & \text{if } \ |i-j| > 1, \\ 1 & \text{if } \ |i-j| = 1. \end{cases}$$

If the curve C is smooth at O, then $\widetilde{C}$ is smooth along $\mathbf{G}$, it intersects exactly one curve among $G_1, \dots, G_n$, and this intersection is transversal and consists of one point. The same holds for $\widetilde{Z}$ in the case when Z is smooth at O. This is well known (see [2]).

Lemma A.2.1. *Suppose that C is smooth at O and $\widetilde{C} \cap G_k \neq \varnothing$. Then*

$$\mathbf{a}_i = \begin{cases} \dfrac{i(n+1-k)}{n+1} & \textit{for } \ i \leq k, \\[2ex] \dfrac{k(n+1-i)}{n+1} & \textit{for } \ i > k. \end{cases}$$

In particular, one has $\mathbf{a}_k = k(n+1-k)/(n+1)$.

Proof. We may assume that $n \geq 2$, since the assertion for $n = 1$ is obvious. Replacing k by $n+1-l$, we may assume that $k \leq (n+1)/2$. Then

$$0 = \widetilde{C} \cdot G_n = -2\mathbf{a}_n + \mathbf{a}_{n-1}.$$

If $k = 1$, then $1 = \widetilde{C} \cdot G_1 = -2\mathbf{a}_1 + \mathbf{a}_2$ and

$$0 = \widetilde{C} \cdot G_i = -2\mathbf{a}_i + \mathbf{a}_{i-1} + \mathbf{a}_{i+1}$$

in the case when $n > i > 1$. This gives $\mathbf{a}_i = (n+1-i)/(n+1)$ in this case.

Thus we may assume that $k \geq 2$, so that $n \geq 3$. Then $0 = \widetilde{C} \cdot G_1 = -2\mathbf{a}_1 + \mathbf{a}_2$ and

$$1 = \widetilde{C} \cdot G_k = -2\mathbf{a}_k + \mathbf{a}_{k-1} + \mathbf{a}_{k+1}.$$

For every $i \neq k$ such that $i \neq 1$ and $i \neq n-1$, we also have

$$0 = \widetilde{C} \cdot G_i = -2\mathbf{a}_i + \mathbf{a}_{i-1} + \mathbf{a}_{i+1}.$$

Solving this system of equations, we obtain the required assertion. $\square$

Corollary A.2.2. *Suppose that both C and Z are smooth at O. Suppose that C intersects the curve Z transversally at O. Suppose also that $\widetilde{C} \cap G_k \neq \varnothing$ and $\widetilde{Z} \cap G_r \neq \varnothing$. Then*

$$(C \cdot Z)_O = \begin{cases} \dfrac{r(n+1-k)}{n+1} & \textit{for } \ r \leq k, \\[2ex] \dfrac{k(n+1-r)}{n+1} & \textit{for } \ r > k. \end{cases}$$

Proof. Since C intersects Z transversally at O, we have $\widetilde{C} \cap \widetilde{Z} \cap \mathbf{G} = \varnothing$. But

$$\widetilde{C} \cdot \widetilde{Z} = \left(\pi^*(C) - \sum_{i=1}^{n} \mathbf{a}_i G_i \right) \cdot \widetilde{Z} = C \cdot Z - \mathbf{a}_k.$$

Thus, the required assertion follows from (A.1.1) and Lemma A.2.1. $\square$

Corollary A.2.3. *Suppose that C is smooth at O and $\widetilde{C}\cap G_k\neq\varnothing$. Then*

$$C^2=\widetilde{C}^2+\frac{k(n+1-k)}{n+1}.$$

Proof. One has

$$\widetilde{C}^2=\left(\pi^*(C)-\sum_{i=1}^{n}\mathbf{a}_iG_i\right)^2=C^2-\mathbf{a}_1=C^2-\frac{n}{n+1}$$

by Lemma A.2.1. □

Remark A.2.4. Suppose that $n\geq 3$. Then there exists a commutative diagram

$$\begin{tikzcd} & \overline{S} \arrow[dr, "\beta"] & \\ \widetilde{S} \arrow[ur, "\alpha"] \arrow[rr, "\pi"'] & & S \end{tikzcd}$$

such that β is the blow-up of the point O and α is a birational morphism that contracts the curves $G_2,\ldots,G_{n-1}$ to the singular point of type $\mathbb{A}_{n-2}$. Denote by $\overline{G}_1$, $\overline{G}_n$, $\overline{C}$, and $\overline{Z}$ the proper transforms of the curves G_1, G_n, $\widetilde{C}$, and $\widetilde{Z}$ on the surface $\overline{S}$, respectively. If C and Z are smooth at O and the curve C intersects Z transversally at O, then the curves $\overline{C}$ and $\overline{Z}$ are smooth and at most one of them passes through the intersection point $\overline{G}_1\cap\overline{G}_n$.

A.3. Singular points of type $\mathbb{D}$. Now we suppose that the surface S has a du Val singularity of type $\mathbb{D}_n$ at the point O, where $n\geq 4$. We start with the following.

Lemma A.3.1. *Suppose that $n=4$, both C and Z are smooth at O, and C intersects the curve Z transversally at O. Then $(C\cdot Z)_O=1/2$ and $C^2=\widetilde{C}^2+1$.*

Proof. We may assume that the intersection form of the curves G_1, G_2, G_3, G_4 is given by

$$\begin{array}{c c} & \begin{array}{cccc} G_1 & G_2 & G_3 & G_4 \end{array} \\ \begin{array}{c} G_1 \\ G_2 \\ G_3 \\ G_4 \end{array} & \begin{pmatrix} -2 & 1 & 1 & 1 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & -2 \end{pmatrix} \end{array}.$$

Then $2G_1+G_2+G_3+G_4$ is the *fundamental cycle* of the singular point O (see [2]). This implies

$$\widetilde{C}\cdot\big(2G_1+G_2+G_3+G_4\big)=\mathrm{mult}_O(C)=1.$$

Thus, we see that $\widetilde{C}\cap G_1=\varnothing$. Hence, we may assume that $\widetilde{C}\cdot G_2=1$, which implies that $\widetilde{C}\cdot G_1=\widetilde{C}\cdot G_3=\widetilde{C}\cdot G_4=0$, which gives

$$\begin{cases} 1+\mathbf{a}_1-2\mathbf{a}_2=0, \\ \mathbf{a}_2+\mathbf{a}_3+\mathbf{a}_4-2\mathbf{a}_1=0, \\ \mathbf{a}_1-2\mathbf{a}_3=0, \\ \mathbf{a}_1-2\mathbf{a}_4=0. \end{cases}$$

Solving this system of equations, we see that $\mathbf{a}_1=1$, $\mathbf{a}_2=1$, $\mathbf{a}_3=1/2$, and $\mathbf{a}_4=1/2$. This implies $C^2=\widetilde{C}^2+1$. Note also that there exists a commutative diagram

$$\begin{tikzcd} & \overline{S} \arrow[dr, "\beta"] & \\ \widetilde{S} \arrow[ur, "\alpha"] \arrow[rr, "\pi"'] & & S \end{tikzcd}$$

such that β is the blow-up of the point O and α is a birational morphism that contracts the curves G_2, G_3, and G_4 to three ordinary double points of the surface $\overline{S}$. Denote by $\overline{C}$ and $\overline{Z}$ the proper transforms of the curves $\widetilde{C}$ and $\widetilde{Z}$ on the surface $\overline{S}$, respectively. If C and Z are smooth at O and the curve C intersects Z transversally at O, then $(C \cdot Z)_O = 1/2$, because $\overline{C} \cap \overline{Z} \cap \alpha(G_1) = \varnothing$, the curves $\overline{C}$ and $\overline{Z}$ are smooth along $\alpha(G_1)$, and each of them contains a singular point of $\overline{S}$ contained in $\alpha(G_1)$. $\square$

Now we suppose that $n \geq 5$. In this case, we may assume that the intersection form of the exceptional curves $G_1, \ldots, G_n$ is given by the following matrix:

$$
\begin{array}{c} \\ G_1 \\ G_2 \\ G_3 \\ G_4 \\ G_5 \\ \vdots \\ G_{n-1} \\ G_n \end{array}
\begin{array}{c}
\begin{array}{cccccccc} G_1 & G_2 & G_3 & G_4 & G_5 & \ldots & G_{n-1} & G_n \end{array} \\
\begin{pmatrix}
-2 & 1 & 1 & 1 & 0 & \ldots & 0 & 0 \\
1 & -2 & 0 & 0 & 0 & \ldots & 0 & 0 \\
1 & 0 & -2 & 0 & 0 & \ldots & 0 & 0 \\
1 & 0 & 0 & -2 & 1 & \ldots & 0 & 0 \\
0 & 0 & 0 & 1 & -2 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \ldots & -2 & 1 \\
0 & 0 & 0 & 0 & 0 & \ldots & 1 & -2
\end{pmatrix}
\end{array}.
$$

Lemma A.3.2. *Suppose that C and Z are smooth at O and C intersects Z transversally at the point O. Then*

$$(C \cdot Z)_O = \frac{1}{2}.$$

If $C \cap G_n \neq \varnothing$, then $C^2 = \widetilde{C}^2 + 1$. Otherwise, one has $C^2 = \widetilde{C}^2 + n/4$.

Proof. Recall from [2] that $2G_1 + G_2 + G_3 + 2G_4 + \ldots + 2G_{n-1} + G_n$ is the fundamental cycle of the singular point O. Then

$$\widetilde{C} \cdot \big(2G_1 + G_2 + G_3 + 2G_4 + \ldots + 2G_{n-1} + G_n\big) = \mathrm{mult}_O(C) = 1.$$

This shows that $\widetilde{C} \cdot G_1 = \widetilde{C} \cdot G_4 = \ldots = \widetilde{C} \cdot G_{n-1} = 0$ and $\widetilde{C} \cdot G_2 + \widetilde{C} \cdot G_3 + \widetilde{C} \cdot G_n = 1$. Hence, the curve $\widetilde{C}$ intersects exactly one of the curves G_2, G_3, or G_n, and it intersects this curve transversally at a single point. Similarly, the same holds for the curve Z.

Let $\beta\colon \overline{S} \to S$ be the blow-up of the point O. Then we have the following commutative diagram:

$$
\begin{tikzcd}
 & \overline{S} \arrow[dr, "\beta"] & \\
\widetilde{S} \arrow[ur, "\alpha"] \arrow[rr, "\pi"'] & & S
\end{tikzcd}
$$

where α is a birational morphism that contracts the curves $G_1, \ldots, G_{n-2}$ and G_n. Thus, we see that $\alpha(G_{n-1})$ is the exceptional curve of the blow-up β. Note that $\alpha(G_n)$ is an isolated ordinary double point of the surface $\overline{S}$. Similarly, we see that the surface $\overline{S}$ has a du Val singular point of type $\mathbb{D}_{n-2}$ at the point $\alpha(G_1) = \ldots = \alpha(G_{n-2})$. Here, we assume that $\mathbb{D}_3 = \mathbb{A}_3$.

Denote by $\overline{C}$ and $\overline{Z}$ the proper transforms on $\overline{S}$ of the curves $\widetilde{C}$ and $\widetilde{Z}$, respectively. Since neither $\widetilde{C}$ nor $\widetilde{Z}$ intersects the curve G_{n-1}, each of the curves $\overline{C}$ and $\overline{Z}$ must pass through some singular point of $\overline{S}$ contained in $\beta(G_{n-1})$. Furthermore, we have $\overline{C} \cap \overline{Z} = \varnothing$, since the curve C intersects the curve Z transversally at O. Thus, without loss of generality, one can assume that

$\widetilde{C} \cdot G_n = 1$ and $\widetilde{Z} \cdot G_2 = 1$. This gives the following system of equations:

$$\begin{cases} 2\mathbf{a}_1 - \mathbf{a}_2 - \mathbf{a}_3 - \mathbf{a}_4 = \widetilde{C} \cdot G_1 = 0, \\ 2\mathbf{a}_2 - \mathbf{a}_1 = \widetilde{C} \cdot G_2 = 0, \\ 2\mathbf{a}_3 - \mathbf{a}_1 = \widetilde{C} \cdot G_3 = 0, \\ 2\mathbf{a}_4 - \mathbf{a}_1 - \mathbf{a}_5 = \widetilde{C} \cdot G_4 = 0, \\ 2\mathbf{a}_5 - \mathbf{a}_4 - \mathbf{a}_6 = \widetilde{C} \cdot G_5 = 0, \\ \dots \\ 2\mathbf{a}_{n-1} - \mathbf{a}_{n-2} - \mathbf{a}_n = \widetilde{C} \cdot G_{n-1} = 0, \\ 2\mathbf{a}_n - \mathbf{a}_{n-1} = \widetilde{C} \cdot G_n = 1. \end{cases}$$

Solving it, we obtain $\mathbf{a}_1 = 1$, $\mathbf{a}_2 = \mathbf{a}_3 = 1/2$, and $\mathbf{a}_4 = \ldots = \mathbf{a}_n = 1$. In particular, we have

$$\widetilde{C} \cdot \widetilde{Z} = \left(\pi^*(C) - \sum_{i=1}^{n} \mathbf{a}_i G_i\right) \cdot \widetilde{Z} = C \cdot Z - \mathbf{a}_2 = C \cdot Z - \frac{1}{2}.$$

Hence, we see that $(C \cdot Z)_O = 1/2$. Similarly, we get $C^2 = \widetilde{C}^2 + 1$.

Similarly, we have the following system of equations:

$$\begin{cases} 2\mathbf{b}_1 - \mathbf{b}_2 - \mathbf{b}_3 - \mathbf{b}_4 = \widetilde{Z} \cdot G_1 = 0, \\ 2\mathbf{b}_2 - \mathbf{b}_1 = \widetilde{Z} \cdot G_2 = 1, \\ 2\mathbf{b}_3 - \mathbf{b}_1 = \widetilde{Z} \cdot G_3 = 0, \\ 2\mathbf{b}_4 - \mathbf{b}_1 - \mathbf{b}_5 = \widetilde{Z} \cdot G_4 = 0, \\ 2\mathbf{b}_5 - \mathbf{b}_4 - \mathbf{b}_6 = \widetilde{Z} \cdot G_5 = 0, \\ \dots \\ 2\mathbf{b}_{n-1} - \mathbf{b}_{n-2} - \mathbf{b}_n = \widetilde{Z} \cdot G_{n-1} = 0, \\ 2\mathbf{b}_n - \mathbf{b}_{n-1} = \widetilde{Z} \cdot G_n = 0. \end{cases}$$

Solving it, we see that

$$\mathbf{b}_1 = \frac{n-2}{4}, \quad \mathbf{b}_2 = \frac{n}{4}, \quad \mathbf{b}_3 = \frac{n-2}{4}, \quad \mathbf{b}_4 = \frac{n-3}{2}, \quad \mathbf{b}_5 = \frac{n-4}{2}, \quad \ldots, \quad \mathbf{b}_n = \frac{1}{2}.$$

As above, this gives $Z^2 = \widetilde{Z}^2 + n/4$. This completes the proof of the lemma. □

A.4. Singular points of type $\mathbb{E}$. Now we consider the case when S has a du Val singularity of type $\mathbb{E}_6$, $\mathbb{E}_7$, or $\mathbb{E}_8$ at the point O. We start with the following fact.

Lemma A.4.1. *Suppose that S has a du Val singularity of type $\mathbb{E}_6$ at the point O and both curves C and Z are smooth at O. Then C is tangent to Z at the point O and*

$$C^2 = \widetilde{C}^2 + \frac{4}{3}.$$

Proof. We have $n = 6$. We may assume that the intersection form of the curves $G_1, \ldots, G_6$ is given by the following matrix:

$$\begin{array}{c|cccccc} & G_1 & G_2 & G_3 & G_4 & G_5 & G_6 \\ \hline G_1 & -2 & 1 & 1 & 1 & 0 & 0 \\ G_2 & 1 & -2 & 0 & 0 & 0 & 0 \\ G_3 & 1 & 0 & -2 & 0 & 1 & 0 \\ G_4 & 1 & 0 & 0 & -2 & 0 & 1 \\ G_5 & 0 & 0 & 1 & 0 & -2 & 0 \\ G_6 & 0 & 0 & 0 & 1 & 0 & -2 \end{array}.$$

Thus, the curve G_1 is a *fork* curve.

Let $\beta\colon \overline{S} \to S$ be the blow-up of the point O. Then there exists a commutative diagram

$$\begin{tikzcd} & \overline{S} \arrow[dr, "\beta"] & \\ \widetilde{S} \arrow[ur, "\alpha"] \arrow[rr, "\pi"'] & & S \end{tikzcd}$$

where α is a contraction of the curves G_1 and $G_3, \ldots, G_6$. We see that $\alpha(G_2)$ is the exceptional curve of the blow-up β. This curve contains one singular point of the surface $\overline{S}$. Denote it by P. Then P is the image of the curves G_1 and $G_3, \ldots, G_6$. Note that $\overline{S}$ has a du Val singular point of type $\mathbb{A}_5$ at the point P.

Let $\overline{C}$ and $\overline{Z}$ be the proper transforms on $\overline{S}$ of the curves C and Z, respectively. Then both $\overline{C}$ and $\overline{Z}$ are smooth along $\alpha(G_2)$. We claim that $\overline{C} \cap \overline{Z} = P$. Indeed, the fundamental cycle of the singular point O is $G_5 + G_6 + 2G_2 + 2G_3 + 2G_4 + 3G_1$. Thus, the curve $\widetilde{C}$ does not intersect the curves $G_1, \ldots, G_4$. Similarly, we see that the curve $\widetilde{Z}$ does not intersect the curves $G_1, \ldots, G_4$. Hence, without loss of generality, we may assume that $\widetilde{C} \cap G_5 \neq \varnothing$. Then $\widetilde{C} \cap G_6 = \varnothing$ and $\widetilde{C} \cdot G_5 = 1$. Similarly, we see that either $\widetilde{Z} \cap G_5 \neq \varnothing$ or $\widetilde{Z} \cap G_6 \neq \varnothing$. In both cases, we have $\overline{C} \cap \overline{Z} = P$, so that the curve C is tangent to Z at the point O.

Since $\widetilde{C} \cdot G_5 = 1$ and $\widetilde{C} \cdot G_1 = \widetilde{C} \cdot G_2 = \widetilde{C} \cdot G_3 = \widetilde{C} \cdot G_4 = \widetilde{C} \cdot G_6$, we get the following system of equations:

$$\begin{cases} 2\mathbf{a}_1 - \mathbf{a}_2 - \mathbf{a}_3 - \mathbf{a}_4 = \widetilde{C} \cdot G_1 = 0, \\ \mathbf{a}_2 - \mathbf{a}_1 = \widetilde{C} \cdot G_2 = 0, \\ 2\mathbf{a}_3 - \mathbf{a}_1 - \mathbf{a}_5 = \widetilde{C} \cdot G_3 = 0, \\ 2\mathbf{a}_4 - \mathbf{a}_1 - \mathbf{a}_6 = \widetilde{C} \cdot G_4 = 0, \\ 2\mathbf{a}_5 - \mathbf{a}_3 = \widetilde{C} \cdot G_5 = 1, \\ 2\mathbf{a}_6 - \mathbf{a}_4 = \widetilde{C} \cdot G_6 = 0. \end{cases}$$

Solving it, we see that $\mathbf{a}_1 = 2$, $\mathbf{a}_2 = 1$, $\mathbf{a}_3 = 5/3$, $\mathbf{a}_4 = \mathbf{a}_5 = 4/3$, and $\mathbf{a}_6 = 2/3$. Thus

$$\widetilde{C}^2 = \left(\pi^*(C) - 2G_1 - G_2 - \frac{5}{3}G_3 - \frac{4}{3}G_4 - \frac{4}{3}G_5 - \frac{2}{3}G_6\right) \cdot \widetilde{C} = C^2 - \frac{4}{3},$$

which gives $C^2 = \widetilde{C}^2 + 4/3$. $\square$

Lemma A.4.2. *Suppose that S has a du Val singularity of type $\mathbb{E}_7$ at the point O and both curves C and Z are smooth at O. Then C is tangent to Z at the point O and*

$$C^2 = \widetilde{C}^2 + \frac{3}{2}.$$

Proof. We may assume that the intersection form of the curves $G_1,\dots,G_7$ is given by the following matrix:

$$\begin{array}{c} \\ G_1 \\ G_2 \\ G_3 \\ G_4 \\ G_5 \\ G_6 \\ G_7 \end{array} \begin{array}{c} \begin{array}{ccccccc} G_1 & G_2 & G_3 & G_4 & G_5 & G_6 & G_7 \end{array} \\ \begin{pmatrix} -2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix} \end{array}.$$

Thus, the curve G_1 is a fork curve.

The fundamental cycle of the singular point O is $2G_5+2G_6+2G_2+3G_3+3G_4+4G_1+G_7$. This shows that $\widetilde{C}\cdot G_7=1$ and $\widetilde{C}\cdot G_1=\widetilde{C}\cdot G_2=\widetilde{C}\cdot G_3=\widetilde{C}\cdot G_4=\widetilde{C}\cdot G_5=\widetilde{C}\cdot G_6=0$, which gives the following system of equations:

$$\begin{cases} 2\mathbf{a}_1-\mathbf{a}_2-\mathbf{a}_3-\mathbf{a}_4=\widetilde{C}\cdot G_1=0, \\ \mathbf{a}_2-\mathbf{a}_1=\widetilde{C}\cdot G_2=0, \\ 2\mathbf{a}_3-\mathbf{a}_1-\mathbf{a}_5=\widetilde{C}\cdot G_3=0, \\ 2\mathbf{a}_4-\mathbf{a}_1-\mathbf{a}_6=\widetilde{C}\cdot G_4=0, \\ 2\mathbf{a}_5-\mathbf{a}_3=\widetilde{C}\cdot G_5=0, \\ 2\mathbf{a}_6-\mathbf{a}_4-\mathbf{a}_7=\widetilde{C}\cdot G_6=0, \\ 2\mathbf{a}_7-\mathbf{a}_6=\widetilde{C}\cdot G_7=1. \end{cases}$$

Then $\mathbf{a}_1=3$, $\mathbf{a}_2=3/2$, $\mathbf{a}_3=2$, $\mathbf{a}_4=5/2$, $\mathbf{a}_5=1$, $\mathbf{a}_6=2$, and $\mathbf{a}_7=3/2$. This gives $C^2=\widetilde{C}^2+3/2$. Arguing as in the proof of Lemma A.4.1, we see that C is tangent to Z at the point O. $\square$

Finally, we conclude this appendix by proving the following.

Lemma A.4.3. *If S has a du Val singularity of type $\mathbb{E}_8$ at O, then C is singular at O.*

Proof. This follows from the fact that the coefficients of all exceptional curves of the minimal resolution of O in the fundamental cycle are greater than 1. $\square$

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

REFERENCES

1. M. Akhtar, T. Coates, S. Galkin, and A. M. Kasprzyk, “Minkowski polynomials and mutations,” SIGMA, Symmetry Integrability Geom. Methods Appl. **8**, 094 (2012).
2. M. Artin, “On isolated rational singularities of surfaces,” Am. J. Math. **88** (1), 129–136 (1966).

3. D. Auroux, L. Katzarkov, and D. Orlov, "Mirror symmetry for Del Pezzo surfaces: Vanishing cycles and coherent sheaves," Invent. Math. **166** (3), 537–582 (2006).
4. T. Coates, A. Corti, S. Galkin, V. Golyshev, and A. Kasprzyk, "Fano varieties and extremal Laurent polynomials: A collaborative research blog," http://coates.ma.ic.ac.uk/fanosearch/.
5. T. Coates, A. Corti, S. Galkin, and A. Kasprzyk, "Quantum periods for 3-dimensional Fano manifolds," Geom. Topol. **20** (1), 103–256 (2016).
6. A. Harder, "Hodge numbers of Landau–Ginzburg models," Adv. Math. **378**, 107436 (2021); arXiv: 1708.01174 [math.AG].
7. N. O. Ilten, J. Lewis, and V. Przyjalkowski, "Toric degenerations of Fano threefolds giving weak Landau–Ginzburg models," J. Algebra **374**, 104–121 (2013).
8. V. A. Iskovskikh and Yu. G. Prokhorov, *Fano Varieties* (Springer, Berlin, 1999), Encycl. Math. Sci. **47**.
9. L. Katzarkov, M. Kontsevich, and T. Pantev, "Bogomolov–Tian–Todorov theorems for Landau–Ginzburg models," J. Diff. Geom. **105** (1), 55–117 (2017).
10. J. Kollár, "Singularities of pairs," in *Algebraic Geometry: Proc. Summer Res. Inst., Santa Cruz, CA, 1995* (Am. Math. Soc., Providence, RI, 1997), Proc. Symp. Pure Math. **62** (Part 1), pp. 221–287.
11. V. Lunts and V. Przyjalkowski, "Landau–Ginzburg Hodge numbers for mirrors of del Pezzo surfaces," Adv. Math. **329**, 189–216 (2018).
12. S. Mori and S. Mukai, "Extremal rays and Fano 3-folds," in *The Fano Conference, Turin, 2002* (Univ. Torino, Turin, 2004), pp. 37–50.
13. C. Peters and J. Stienstra, "A pencil of K3-surfaces related to Apéry's recurrence for $\zeta(3)$ and Fermi surfaces for potential zero," in *Arithmetic of Complex Manifolds: Proc. Conf., Erlangen, FRG, 1988* (Springer, Berlin, 1989), Lect. Notes Math. **1399**, pp. 110–127.
14. V. Przyjalkowski, "On Landau–Ginzburg models for Fano varieties," Commun. Number Theory Phys. **1** (4), 713–728 (2007).
15. V. V. Przyjalkowski, "Weak Landau–Ginzburg models for smooth Fano threefolds," Izv. Math. **77** (4), 772–794 (2013) [transl. from Izv. Ross. Akad. Nauk, Ser. Mat. **77** (4), 135–160 (2013)].
16. V. V. Przyjalkowski, "Calabi–Yau compactifications of toric Landau–Ginzburg models for smooth Fano threefolds," Sb. Math. **208** (7), 992–1013 (2017) [transl. from Mat. Sb. **208** (7), 84–108 (2017)].
17. V. V. Przyjalkowski, "Toric Landau–Ginzburg models," Russ. Math. Surv. **73** (6), 1033–1118 (2018) [transl. from Usp. Mat. Nauk **73** (6), 95–190 (2018)].
18. F. Sakai, "Weil divisors on normal surfaces," Duke Math. J. **51** (4), 877–887 (1984).