Kummer quartic double solids

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Abstract

We study equivariant birational geometry of (rational) quartic double solids ramified over (singular) Kummer surfaces.

Keywords Quartic double solid · Kummer surface · Equivariant birational rigidity · Fano variety

Mathematics Subject Classification 14E07 · 14J45

A Kummer quartic surface is an irreducible normal surface in $\mathbb{P}^3$ of degree 4 that has the maximal possible number of 16 singular points, which are ordinary double singularities. Any such surface is the Kummer variety of the Jacobian surface of a smooth genus 2 curve. Vice versa, the Jacobian surface of a smooth genus 2 curve admits a natural involution such that the quotient surface is a Kummer quartic surface in $\mathbb{P}^3$ (Fig. 1). Let $\mathscr{S}$ be a Kummer surface in $\mathbb{P}^3$, and let $\mathscr{C}$ be the smooth genus 2 curve such that

$$\mathscr{S} \cong \mathrm{J}(\mathscr{C})/\langle\tau\rangle, \tag{1}$$

where τ is the involution of the Jacobian $\mathrm{J}(\mathscr{C})$ that sends a point P to the point $-P$. Recall from [22, 23, 30, 35, 36, 42, 48] that the surface $\mathscr{S}$ can be given by the equation

$$\begin{aligned}
a(x_0^4 + x_1^4 + x_2^4 + x_3^4) &+ 2b(x_0^2x_1^2 + x_2^2x_3^2) \\
&+ 2c(x_0^2x_2^2 + x_1^2x_3^2) + 2d(x_0^2x_3^2 + x_1^2x_2^2) \\
&+ 4ex_0x_1x_2x_3 = 0
\end{aligned} \tag{2}$$
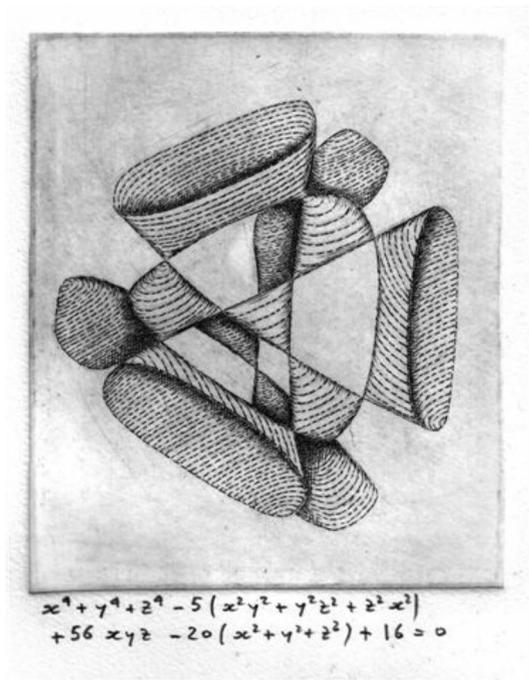
for some $[a : b : c : d : e] \in \mathbb{P}^4$ such that

$$a(a^2 + e^2 - b^2 - c^2 - d^2) + 2bcd = 0. \tag{3}$$

Throughout this paper, all varieties are assumed to be projective and defined over $\mathbb{C}$.

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Fig. 1 A Kummer surface by
Patrice Jeener

Note that the curve $\mathscr{C}$ is hyperelliptic, and equation (3) defines a cubic threefold in $\mathbb{P}^4$, which is projectively equivalent to *the Segre cubic threefold* [22, 48].

Using a formula from the book [11] implemented in Magma [7], we can easily extract an equation of the surface $\mathscr{S}$ from the curve $\mathscr{C}$. However, the resulting equation may differ from (2). For instance, if $\mathscr{C}$ is the unique genus 2 curve such that $\mathrm{Aut}(\mathscr{C}) \cong \boldsymbol{\mu}_2.\mathfrak{S}_4$, then $\mathscr{C}$ is isomorphic to the curve

$$\left\{z^2 = xy(x^4 - y^4)\right\} \subset \mathbb{P}(1, 1, 3)$$

where x, y, z are homogeneous coordinates on $\mathbb{P}(1, 1, 3)$ of weights 1, 1, 2, respectively. In this case, Magma produces the following Kummer quartic surface:

$$\left\{x_0^4 + 2x_0^2 x_2 x_3 - 2x_0^2 x_2^2 + 4x_0 x_2^2 x_2 - 4x_0 x_2 x_3^2 + x_2^2 x_3^2 - 2x_2 x_2^2 x_3 + x_2^4 = 0\right\} \subset \mathbb{P}^3,$$

which is projectively equivalent to the surface given by (2) with parameters $a = b = 1$, $c = d = -1$, $e = -4$ that do not satisfy (3). But this surface is projectively equivalent to

$$\left\{x_0^4 + x_1^4 + x_2^4 + x_3^4 - 4ix_0 x_1 x_2 x_3 = 0\right\} \subset \mathbb{P}^3, \tag{4}$$

which is given by (2) with parameters $a = 1$, $b = c = d = 0$, $e = -i$ that do satisfy (3). Here, we use the following Magma code provided to us by Michela Artebani:

```
R<x>:=PolynomialRing(Rationals());
C:=HyperellipticCurve(x^5-x);
GroupName(GeometricAutomorphismGroup(C));
KummerSurfaceScheme(C);
```

It is not very difficult to recover the hyperelliptic curve $\mathscr{C}$ from the quartic surface $\mathscr{S}$. Indeed, $\mathbb{P}^3$ contains 16 planes $\Pi_1, \ldots, \Pi_{16}$, called *tropes*, such that $\mathscr{S}|_{\Pi_i} = 2\mathcal{C}_i$ for each i, where $\mathcal{C}_i$ is a smooth conic. One can show (see, for example, [31]) that

- each trope Π_i contains exactly six singular points of the surface $\mathscr{S}$,
- each singular point of the surface $\mathscr{S}$ is contained in six planes among $\Pi_1, \ldots, \Pi_{16}$.

Moreover, for every conic $\mathcal{C}_i$, there exists a double cover $\mathscr{C} \to \mathcal{C}_i$ which is ramified over the six points $\mathcal{C}_i \cap \mathrm{Sing}(\mathscr{S})$. This gives us an algorithm how to recover $\mathscr{C}$ from $\mathscr{S}$.

Example 5 Suppose that the surface $\mathscr{S}$ is given by the equation (2) with

$$\begin{cases} a = 2, \\ b = -t^2 - 1, \\ c = -t^2 - 1, \\ d = -t^2 - 1, \\ e = t^3 + 3t, \end{cases}$$

where $t \in \mathbb{C} \setminus \{\pm 1, \pm\sqrt{3}i\}$. Then the surface $\mathscr{S}$ is given by the following equation:

$$\begin{aligned} x_0^4 + x_1^4 + x_2^4 &+ x_3^4 + 2(t^3 + 3t)x_0x_1x_2x_3 \\ &= (t^2 + 1)(x_0^2x_1^2 + x_2^2x_3^2 + x_0^2x_2^2 + x_1^2x_3^2 \\ &\quad + x_0^2x_3^2 + x_1^2x_2^2). \end{aligned} \tag{6}$$

Its singular locus $\mathrm{Sing}(\mathscr{S})$ consists of the following 16 points:

$$[1 : 1 : 1 : t], [-1 : 1 : -1 : t], [-1 : -1 : 1 : t], [1 : -1 : -1 : t],$$
$$[1 : 1 : t : 1], [1 : -1 : t : -1], [-1 : -1 : t : 1], [-1 : 1 : t : -1],$$
$$[t : 1 : 1 : 1], [t : -1 : 1 : -1], [t : 1 : -1 : -1], [t : -1 : -1 : 1],$$
$$[1 : t : 1 : 1], [-1 : t : -1 : 1], [1 : t : -1 : -1], [-1 : t : 1 : -1].$$

Moreover, the tropes $\Pi_1, \ldots, \Pi_{16}$ are listed in the following table:

$\Pi_1 = \{x_0 + x_1 + x_2 + tx_3 = 0\}$	$\Pi_2 = \{x_0 - x_1 + x_2 - tx_3 = 0\}$
$\Pi_3 = \{x_0 + x_1 - x_2 - tx_3 = 0\}$	$\Pi_4 = \{x_0 - x_1 - x_2 + tx_3 = 0\}$
$\Pi_5 = \{x_0 + x_1 + tx_2 + x_3 = 0\}$	$\Pi_6 = \{x_0 - x_1 + tx_2 - x_3 = 0\}$
$\Pi_7 = \{x_0 - tx_2 + x_1 - x_3 = 0\}$	$\Pi_8 = \{x_0 - x_1 - tx_2 + x_3 = 0\}$

$\Pi_9 = \{x_0 + tx_1 + x_2 + x_3 = 0\}$	$\Pi_{10} = \{x_0 - tx_1 - x_2 + x_3 = 0\}$
$\Pi_{11} = \{x_0 - tx_1 + x_2 - x_3 = 0\}$	$\Pi_{12} = \{x_0 + tx_1 - x_2 - x_3 = 0\}$
$\Pi_{13} = \{tx_0 + x_1 + x_2 + x_3 = 0\}$	$\Pi_{14} = \{tx_0 - x_1 - x_2 + x_3 = 0\}$
$\Pi_{15} = \{tx_0 - x_1 + x_2 - x_3 = 0\}$	$\Pi_{16} = \{tx_0 + x_1 - x_2 - x_3 = 0\}$

Then $\mathcal{C}_1$ is the smooth conic

$$\{x_0 + x_1 + x_2 + tx_3 = tx_1x_3 + tx_2x_3 + x_1^2 + x_1x_2 + x_2^2 - x_3^2 = 0\} \subset \mathbb{P}^3.$$

This conic contains the following six singular points of our surface:

$$[1 : -1 : t : -1], [-1 : 1 : t : -1], [t : -1 : 1 : -1],$$
$$[t : 1 : -1 : -1], [1 : t : -1 : -1], [-1 : t : 1 : -1].$$

Projecting from $[t : 1 : -1 : -1]$, we get an isomorphism $\mathcal{C}_1 \cong \mathbb{P}^1$ that maps these points to

$$[t + 1 : -2], [1 : 0], [-1 : 1], [1 - t : 1 + t], [0 : 1], [t - 1 : 2].$$

Therefore, the hyperelliptic curve $\mathscr{C}$ is isomorphic to the curve

$$\{z^2 = xy(x - y)((t - 1)x + 2y)(2x - (t + 1)y)((t + 1)x - (t - 1)y)\} \subset \mathbb{P}(1, 1, 3).$$

In particular, it follows from [9] or Magma computations that

$$\mathrm{Aut}(\mathscr{C}) \cong \begin{cases} \mu_2.\mathfrak{S}_4 \text{ if } t \in \{0, \pm i, 1 \pm 2i, -1 \pm 2i\}, \\ \mu_2.\mathrm{D}_{12} \text{ if } t \in \{0, \pm 3\}, \\ \mu_2 \times \mathfrak{S}_3 \text{ if } t \text{ is general.} \end{cases}$$

For instance, to identify $\mathrm{Aut}(\mathscr{C})$ in the case when $t = i$, one can use the following script:

```
K:=CyclotomicField(4);
R<x>:=PolynomialRing(K);
i:=Roots(x^2+1,K)[1,1];
t:=i;
f:=x*(x-1)*((t-1)*x+2)*(2*x-(t+1))*((t+1)*x-(t-1));
C:=HyperellipticCurve(f);
GroupName(GeometricAutomorphismGroup(C));
```

In this example, we assume that $t \notin \{\pm 1, \pm\sqrt{3}i\}$, because

- if $t = \pm 1$ or $t = \infty$, then the equation (6) defines a union of 4 planes,
- if $t = \pm\sqrt{3}i$, the equation (6) defines a double quadric.

These are semistable degenerations with minimal $\mathrm{PGL}_4(\mathbb{C})$-orbits [53, Theorem 2.4].

Let $\mathrm{Aut}(\mathbb{P}^3, \mathscr{S})$ be the subgroup in $\mathrm{PGL}_4(\mathbb{C})$ consisting of projective transformations that leave $\mathscr{S}$ invariant. Then $\mathrm{Aut}(\mathbb{P}^3, \mathscr{S})$ contains a subgroup $\mathbb{H} \cong \mu_2^4$ generated by

$$A_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, A_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The action of this group on $\mathscr{S}$ is induced by translations of $J(\mathscr{C})$ by two-torsion points [38], so $\mathrm{Sing}(\mathscr{S})$ is an $\mathbb{H}$-orbit. Similarly, we see that $\mathbb{H}$ acts transitively on the set

$$\left\{ \Pi_1, \Pi_2, \Pi_3, \Pi_4, \Pi_5, \Pi_6, \Pi_7, \Pi_8, \Pi_9, \Pi_{10}, \Pi_{11}, \Pi_{12}, \Pi_{13}, \Pi_{14}, \Pi_{15}, \Pi_{16} \right\}. \tag{7}$$

If $\mathscr{S}$ is general, then $\mathrm{Aut}(\mathbb{P}^3, \mathscr{S}) = \mathbb{H}$, and $\mathrm{Aut}(\mathscr{C})$ is generated by the hyperelliptic involution [37, 40]. However, if $\mathscr{S}$ is special, then $\mathrm{Aut}(\mathbb{P}^3, \mathscr{S})$ can be larger than $\mathbb{H}$.

Example 8 Let us use assumptions and notations of Example 5. For $t \in \mathbb{C} \setminus \{\pm 1, \pm\sqrt{3}i\}$, the group $\mathrm{Aut}(\mathbb{P}^3, \mathscr{S})$ contains the subgroup isomorphic to $\boldsymbol{\mu}_2^4 \rtimes \mathfrak{S}_3$ generated by

$$A_1, A_2, A_3, A_4, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In fact, this is the whole group $\mathrm{Aut}(\mathbb{P}^3, \mathscr{S})$ if t is general. On the other hand, if $t = 0$, then it follows from [12, 25] that $\mathrm{Aut}(\mathbb{P}^3, \mathscr{S}) \cong \boldsymbol{\mu}_2^4 \rtimes \mathrm{D}_{12}$, and this group is generated by

$$A_1, A_2, A_3, A_4, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

If $t = \pm i$, then $\mathscr{S}$ is the surface (4), and $\mathrm{Aut}(\mathbb{P}^3, \mathscr{S}) \cong \boldsymbol{\mu}_2^4 \rtimes \mathfrak{S}_4$ is generated by

$$A_1, A_2, A_3, A_4, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Example 9 Suppose that $\mathscr{S}$ is given by the equation (2) with

$$\begin{cases} a = 2\zeta_5^3 + 2\zeta_5^2 + 6\zeta_5 - 1, \\ b = 4\zeta_5^3 + 4\zeta_5^2 - 10\zeta_5 + 9, \\ c = -6\zeta_5^3 - 6\zeta_5^2 + 4\zeta_5 + 3, \\ d = 11, \\ e = -20\zeta_5^3 + 24\zeta_5^2 - 16\zeta_5 + 10. \end{cases}$$

Then $\mathrm{Aut}(\mathscr{C}) \cong \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_5$ and $\mathrm{Aut}(\mathbb{P}^3, \mathscr{S}) \cong \boldsymbol{\mu}_2^4 \rtimes \boldsymbol{\mu}_5$, which is generated by

$$A_1, A_2, A_3, A_4, \begin{pmatrix} -i & 0 & 0 & i \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -i & i & 0 \end{pmatrix}.$$

Looking at Examples 5, 8 and 9, we can see a relation between $\mathrm{Aut}(\mathbb{P}^3, \mathscr{S})$ and $\mathrm{Aut}(\mathscr{C})$. In fact, this relation holds for all Kummer surfaces in $\mathbb{P}^3$ by the following well-known result, about which we learned from Igor Dolgachev.

Lemma 10 *Let $\iota \in \mathrm{Aut}(\mathscr{C})$ be the hyperelliptic involution of the curve $\mathscr{C}$. Then*

$$\mathrm{Aut}\big(\mathbb{P}^3, \mathscr{S}\big) \cong \boldsymbol{\mu}_2^4 \rtimes \big(\mathrm{Aut}(\mathscr{C})/\langle \iota \rangle\big).$$

Proof Let us identify $\mathscr{C}$ with the theta divisor in $\mathrm{J}(\mathscr{C})$ via the Abel–Jacobi map whose base point is one of the fixed points of the involution ι (one of the six Weierstrass points). Then the linear system $|2\mathscr{C}|$ gives a morphism $\mathrm{J}(\mathscr{C}) \to \mathbb{P}^3$ whose image is the surface $\mathscr{S}$. Taking the Stein factorization of the morphism $\mathrm{J}(\mathscr{C}) \to \mathscr{S}$, we get the isomorphism (1).

On the other hand, elements in $\mathrm{Aut}(\mathscr{C})$ give automorphisms in $\mathrm{Aut}(\mathrm{J}(\mathscr{C}))$ that leave the linear system $|2\mathscr{C}|$ invariant. This gives us a homomorphism $\mathrm{Aut}(\mathscr{C}) \to \mathrm{Aut}(\mathbb{P}^3, \mathscr{S})$, whose kernel is the hyperelliptic involution ι, since ι induces the involution $\tau \in \mathrm{Aut}(\mathrm{J}(\mathscr{C}))$.

The image of the group $\mathrm{Aut}(\mathscr{C})$ in $\mathrm{Aut}(\mathbb{P}^3, \mathscr{S})$ normalizes the subgroup $\mathbb{H}$, because elements in $\mathbb{H}$ are induced by the translations of the Jacobian $\mathrm{J}(\mathscr{C})$ by two-torsion points. This gives a monomorphism $\vartheta : \boldsymbol{\mu}_2^4 \rtimes (\mathrm{Aut}(\mathscr{C})/\langle \iota \rangle) \to \mathrm{Aut}(\mathbb{P}^3, \mathscr{S})$.

We claim that ϑ is an epimorphism. Indeed, the action of an element $g \in \mathrm{Aut}(\mathbb{P}^3, \mathscr{S})$ on the surface $\mathscr{S}$ lifts to its its action on the Jacobian $\mathrm{J}(\mathscr{C})$ that leaves $[2\mathscr{C}]$ invariant, so composing g with some $h \in \mathbb{H}$, we obtain an element $g \circ h$ that preserves the class $[\mathscr{C}]$. Thus, since $[\mathscr{C}]$ is a principal polarization, the composition $g \circ h$ preserves $\mathscr{C}$, and it acts faithfully on $\mathscr{C}$, since $\mathscr{C}$ generates $\mathrm{J}(\mathscr{C})$. This gives $g \circ h \in \mathrm{im}(\vartheta)$, so ϑ is surjective. $\qquad\square$

Since $\mathrm{Aut}(\mathscr{C})$ is isomorphic to a group among $\boldsymbol{\mu}_2$, $\boldsymbol{\mu}_2^2$, D_8, D_{12}, $\boldsymbol{\mu}_2.\mathrm{D}_{12}$, $\boldsymbol{\mu}_2.\mathfrak{S}_4$, $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_5$, we conclude that $\mathrm{Aut}(\mathbb{P}^3, \mathscr{S})$ is isomorphic to one of the following groups:

$$\boldsymbol{\mu}_2^4, \boldsymbol{\mu}_2^4 \rtimes \boldsymbol{\mu}_2, \boldsymbol{\mu}_2^4 \rtimes \boldsymbol{\mu}_2^2, \boldsymbol{\mu}_2^4 \rtimes \mathfrak{S}_3, \boldsymbol{\mu}_2^4 \rtimes \mathrm{D}_{12}, \boldsymbol{\mu}_2^4 \rtimes \mathfrak{S}_4, \boldsymbol{\mu}_2^4 \rtimes \boldsymbol{\mu}_5.$$

Note that the group $\mathrm{Aut}(\mathscr{S})$ is always larger that $\mathrm{Aut}(\mathbb{P}^3, \mathscr{S})$ [37, 40].

Remark 11 ([5, 30, 48]) Let $\mathfrak{N}$ be the normalizer of the subgroup $\mathbb{H}$ in the group $\mathrm{PGL}_4(\mathbb{C})$. Then $\mathrm{Aut}(\mathbb{P}^3, \mathscr{C}) \subset \mathfrak{N}$, and there exists an exact sequence $1 \longrightarrow \mathbb{H} \longrightarrow \mathfrak{N} \longrightarrow \mathfrak{S}_6 \longrightarrow 1$, which can be described as follows. Let

$$B_1 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } B_2 = \begin{pmatrix} -i & 0 & 0 & i \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -i & i & 0 \end{pmatrix}.$$

Then $\langle B_1, B_2 \rangle \in \mathfrak{N}$. Since $B_1^2 \in \mathbb{H}$, $B_2^5 = (B_1 B_2)^6 = [B_1, B_2]^3 = \mathrm{Id}_{\mathbb{P}^3}$, $[B_1, B_2 B_1 B_2]^2 \in \mathbb{H}$, the images of B_1 and B_2 in the quotient $\mathfrak{N}/\mathbb{H}$ generate the whole group $\mathfrak{N}/\mathbb{H} \cong \mathfrak{S}_6$. Set

$$S_1 = \{x_0^4 + x_1^4 + x_2^4 + x_3^4 - 6(x_0^2 x_1^2 + x_2^2 x_3^2) - 6(x_0^2 x_2^2 + x_1^2 x_3^2) - 6(x_0^2 x_3^2 + x_1^2 x_2^2) = 0\},$$
$$S_2 = \{x_0^4 + x_1^4 + x_2^4 + x_3^4 - 6(x_0^2 x_1^2 + x_2^2 x_3^2) + 6(x_0^2 x_2^2 + x_1^2 x_3^2) + 6(x_0^2 x_3^2 + x_1^2 x_2^2) = 0\},$$
$$S_3 = \{x_0^4 + x_1^4 + x_2^4 + x_3^4 + 6(x_0^2 x_1^2 + x_2^2 x_3^2) - 6(x_0^2 x_2^2 + x_1^2 x_3^2) + 6(x_0^2 x_3^2 + x_1^2 x_2^2) = 0\},$$
$$S_4 = \{x_0^4 + x_1^4 + x_2^4 + x_3^4 + 6(x_0^2 x_1^2 + x_2^2 x_3^2) + 6(x_0^2 x_2^2 + x_1^2 x_3^2) - 6(x_0^2 x_3^2 + x_1^2 x_2^2) = 0\},$$
$$S_5 = \{x_0^4 + x_1^4 + x_2^4 + x_3^4 - 12 x_0 x_1 x_2 x_3 = 0\},$$
$$S_6 = \{x_0^4 + x_1^4 + x_2^4 + x_3^4 + 12 x_0 x_1 x_2 x_3 = 0\}.$$

Then $S_1, S_2, S_3, S_4, S_5, S_6$ are $\mathbb{H}$-invariant surfaces, and the quotient $\mathfrak{N}/\mathbb{H}$ permutes them. For instance, the transformation B_1 acts on the set $\{S_1, S_2, S_3, S_4, S_5, S_6\}$ as $(1\,2)(3\,4)(5\,6)$, and B_2 acts as the permutation $(1\,2\,6\,3\,5)$. This gives an explicit isomorphism $\mathfrak{N}/\mathbb{H} \cong \mathfrak{S}_6$.

Remark 12 The quotient $\mathrm{Aut}(\mathbb{P}^3, \mathscr{S})/\mathbb{H}$ naturally acts on the threefold (3) fixing the point $[a : b : c : d : e]$ that corresponds to $\mathscr{S}$. Projecting the threefold from this point, we obtain a (rational) double cover of $\mathbb{P}^3$ that is branched along the surface $\mathscr{S}$.

Let $\pi : X \to \mathbb{P}^3$ be the double cover branched along the surface $\mathscr{S}$. Set $H = \pi^*(\mathcal{O}_{\mathbb{P}^3}(1))$. Then $\mathrm{Pic}(X) = \mathbb{Z}[H]$, $H^3 = 2$ and $-K_X \sim 2H$, so X is a del Pezzo threefold of degree 2, which has 16 ordinary double points. We say that X is *a Kummer quartic double solid* [57].

The threefold X is a hypersurface in $\mathbb{P}(1, 1, 1, 1, 2)$ given by

$$\begin{aligned}
w^2 = {} & a(x_0^4 + x_1^4 + x_2^4 + x_3^4) + 2b(x_0^2 x_1^2 + x_2^2 x_3^2) + \\
& + 2c(x_0^2 x_2^2 + x_1^2 x_3^2) + 2d(x_0^2 x_3^2 + x_1^2 x_2^2) + 4e x_0 x_1 x_2 x_3,
\end{aligned} \tag{13}$$

where we consider x_0, x_1, x_2, x_3 as homogeneous coordinates on $\mathbb{P}(1, 1, 1, 1, 2)$ of weight 1, and w is a homogeneous coordinate on $\mathbb{P}(1, 1, 1, 1, 2)$ of weight 2.

It is well-known that the threefold X is rational [15, 50, 52, 57], see also Remark 12. Let us describe one birational map $\mathbb{P}^3 \dashrightarrow X$ following [50, 52]. To do this, fix a twisted cubic curve $C_3 \subset \mathbb{P}^3$. Then C_3 contains six distinct points $P_1, \ldots, P_6$ such that there exists a commutative diagram

$$\begin{array}{ccc}
\widehat{X} & \xrightarrow{\ \varphi\ } & X \\
{\scriptstyle \eta} \downarrow & & \downarrow {\scriptstyle \pi} \\
\mathbb{P}^3 & \dashrightarrow[\ \chi\ ] & \mathbb{P}^3
\end{array} \tag{14}$$

where η is a blow up of the points $P_1, \ldots, P_6$, the morphism φ is a contraction of the proper transform of the curve C_3 and proper transforms of 15 lines in $\mathbb{P}^3$ that pass through two points among $P_1, \ldots, P_6$, and χ is a rational map given by the linear system of quadric surfaces that pass through the points $P_1, \ldots, P_6$.

Corollary 15 ([19, 27]) *One has* $\mathrm{Cl}(X) \cong \mathbb{Z}^7$.

Remark 16 The vertices of the quadric cones in $\mathbb{P}^3$ that pass through $P_1, \ldots, P_6$ span an irreducible singular quartic surface $\mathfrak{S}$ which is known as *the Weddle surface* [35, 57].

This surface is singular at the points $P_1, \dots, P_6$, and χ induces a birational map $\mathfrak{S} \dashrightarrow \mathscr{S}$. On the other hand, the double cover of $\mathbb{P}^3$ branched along $\mathfrak{S}$ is irrational [15, 57].

Let $\sigma \in \mathrm{Aut}(X)$ be the Galois involution of the double cover π. Then σ is contained in the center of the group $\mathrm{Aut}(X)$. Moreover, since π is $\mathrm{Aut}(X)$-equivariant, it induces a homomorphism $\upsilon : \mathrm{Aut}(X) \to \mathrm{Aut}(\mathbb{P}^3, \mathscr{S})$ with $\ker(\upsilon) = \langle \sigma \rangle$, so we have exact sequence

$$1 \longrightarrow \langle \sigma \rangle \longrightarrow \mathrm{Aut}(X) \overset{\upsilon}{\longrightarrow} \mathrm{Aut}\left(\mathbb{P}^3, \mathscr{S}\right) \longrightarrow 1.$$

The main result of this paper is the following theorem (cf. [3, 4, 14]).

Theorem 17 *Let G be any subgroup in $\mathrm{Aut}(X)$ such that $\mathrm{Cl}^G(X) \cong \mathbb{Z}$ and $\mathbb{H} \subseteq \upsilon(G)$. Then the Fano threefold X is G-birationally super-rigid.*

Corollary 18 *Let G be any subgroup in $\mathrm{Aut}(X)$ such that G contains σ and $\mathbb{H} \subseteq \upsilon(G)$. Then X is G-birationally super-rigid.*

The condition $\mathrm{Cl}^G(X) \cong \mathbb{Z}$ in Theorem 17 simply means that X is a G-Mori fibre space, which is required by the definition of G-birational super-rigidity (see [17, Definition 3.1.1]). The condition $\mathbb{H} \subseteq \upsilon(G)$ does not imply that $\mathrm{Cl}^G(X) \cong \mathbb{Z}$, see Examples 28 and 29 below. The following example shows that we cannot remove the condition $\mathbb{H} \subseteq \upsilon(G)$.

Example 19 Observe that $\mathrm{Cl}^{\langle \sigma \rangle}(X) \cong \mathbb{Z}$. Let S_1 and S_2 be two general surfaces in $|H|$, and let $C = S_1 \cap S_2$. Then C is a smooth irreducible $\langle \sigma \rangle$-invariant curve, $\pi(C)$ is a line, and there exists $\langle \sigma \rangle$-commutative diagram
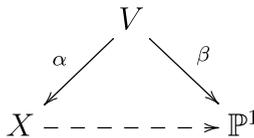
where α is the blow up of the curve C, the dashed arrow $\dashrightarrow$ is given by the pencil generated by the surfaces S_1 and S_2, and β is a fibration into del Pezzo surfaces of degree 2. Therefore, the threefold X is not $\langle \sigma \rangle$-birationally rigid.

Let G be a subgroup in $\mathrm{Aut}(X)$ such that $\upsilon(G)$ contains $\mathbb{H}$. Before proving Theorem 17, let us explain how to check the condition $\mathrm{Cl}^G(X) \cong \mathbb{Z}$. For a homomorphism $\rho : \mathbb{H} \to \mu_2$, consider the action of the group $\mathbb{H}$ on the threefold X given by

$$\begin{aligned}
A_1 &: [x_0 : x_1 : x_2 : x_3 : w] \mapsto [-x_0 : x_1 : -x_2 : x_3 : \rho(A_1)w], \\
A_2 &: [x_0 : x_1 : x_2 : x_3 : w] \mapsto [-x_0 : x_1 : -x_2 : x_3 : \rho(A_2)w], \\
A_3 &: [x_0 : x_1 : x_2 : x_3 : w] \mapsto [x_1 : x_2 : x_3 : x_2 : \rho(A_3)w], \\
A_4 &: [x_0 : x_1 : x_2 : x_3 : w] \mapsto [x_3 : x_2 : x_1 : x_0 : \rho(A_4)w].
\end{aligned}$$

This gives a lift of the subgroup $\mathbb{H}$ to $\mathrm{Aut}(X)$. Let $\mathbb{H}^\rho$ be the resulting subgroup in $\mathrm{Aut}(X)$. Since $\mathbb{H} \subset \upsilon(G)$, we may assume that $\mathbb{H}^\rho \subset G$. If ρ is trivial, we let $\mathbb{H} = \mathbb{H}^\rho$ for simplicity.

For every plane Π_i, one has $\pi^*(\Pi_i) = \Pi_i^+ + \Pi_i^-$, where Π_i^+ and Π_i^- are two irreducible surfaces such that $\Pi_i^+ \neq \Pi_i^-$ and $\sigma(\Pi_i^+) = \Pi_i^-$. Note that we do not have a canonical way to distinguish between the surfaces Π_i^+ and Π_i^-. Namely, if $\pi^*(\Pi_i)$ is given by

$$\begin{cases} h_i(x_0, x_1, x_2, x_3) = 0, \\ w^2 = g_i^2(x_0, x_1, x_2, x_3), \end{cases}$$

where h_i is a linear polynomial such that $\Pi_i = \{h_i = 0\} \subset \mathbb{P}^3$, and g_i is a quadratic polynomial such that the conic $\mathcal{C}_i$ is given by $h_i = g_i = 0$, then

$$\Pi_i^\pm = \left\{ w \pm g_i(x_0, x_1, x_2, x_3) = h_i(x_0, x_1, x_2, x_3) = 0 \right\} \subset \mathbb{P}(1, 1, 1, 1, 2).$$

But the choice of $\pm$ here is not uniquely defined, because we can always swap g_i with $-g_i$.

On the other hand, since $\mathbb{H}$ acts transitively on the set (7), the set

$$\left\{ \Pi_1^+, \Pi_1^-, \Pi_2^+, \Pi_2^-, \Pi_3^+, \Pi_3^- \ldots, \Pi_{14}^+, \Pi_{14}^-, \Pi_{15}^+, \Pi_{15}^-, \Pi_{16}^+, \Pi_{16}^- \right\}$$

splits into two $\mathbb{H}^\rho$-orbits consisting of 16 surfaces such that each of them contains exactly one surface among Π_i^+ and Π_i^- for every i. Hence, we may assume that these $\mathbb{H}^\rho$-orbits are

$$\left\{ \Pi_1^+, \Pi_2^+, \Pi_3^+, \Pi_4^+, \Pi_5^+, \Pi_6^+, \Pi_7^+, \Pi_8^+, \Pi_9^+, \Pi_{10}^+, \Pi_{11}^+, \Pi_{12}^+, \Pi_{13}^+, \Pi_{14}^+, \Pi_{15}^+, \Pi_{16}^+ \right\}$$

and

$$\left\{ \Pi_1^-, \Pi_2^-, \Pi_3^-, \Pi_4^-, \Pi_5^-, \Pi_6^-, \Pi_7^-, \Pi_8^-, \Pi_9^-, \Pi_{10}^-, \Pi_{11}^-, \Pi_{12}^-, \Pi_{13}^-, \Pi_{14}^-, \Pi_{15}^-, \Pi_{16}^- \right\}.$$

Corollary 20 *The surfaces* $\Pi_1^+, \Pi_1^-, \ldots, \Pi_{16}^+, \Pi_{16}^-$ *generate the group* $\mathrm{Cl}(X)$.

Corollary 21 *Either* $\mathrm{Cl}^{\mathbb{H}^\rho}(X) \cong \mathbb{Z}$ *or* $\mathrm{Cl}^{\mathbb{H}^\rho}(X) \cong \mathbb{Z}^2$.

The surfaces $\Pi_1^+, \Pi_1^-, \ldots, \Pi_{16}^+, \Pi_{16}^-$ are called *planes* [52]. They are not Cartier divisors. To describe their strict transforms on the threefold $\widehat{X}$ from the commutative diagram (14), let us introduce the following notations:

(a) let E_i be the η-exceptional surfaces such that $\eta(E_i) = P_i$, where $1 \leqslant i \leqslant 6$;
(b) let $\widehat{\Pi}_{i,j,k}$ be the strict transform on the threefold $\widehat{X}$ of the plane in $\mathbb{P}^3$ that passes through the three distinct points P_i, P_j, P_k, where $1 \leqslant i < j < k \leqslant 6$;
(c) let $\widehat{Q}_i$ be the strict transform of the irreducible quadric cone in $\mathbb{P}^3$ that contains all points $P_1, \ldots, P_6$ and is singular at the point P_i, where $1 \leqslant i \leqslant 6$.

Then strict transforms of the surfaces $\Pi_1^+, \Pi_1^-, \ldots, \Pi_{16}^+, \Pi_{16}^-$ on the threefold $\widehat{X}$ are

$$E_1, E_2, \ldots, E_6, \widehat{\Pi}_{1,2,3}, \widehat{\Pi}_{1,2,4}, \ldots, \widehat{\Pi}_{3,5,6}, \widehat{\Pi}_{4,5,6}, \widehat{Q}_1, \widehat{Q}_2, \ldots \widehat{Q}_6.$$

The involution σ acts birationally on $\widehat{X}$ as a composition of flops of φ-contracted curves, it swaps $E_i \leftrightarrow \widehat{Q}_i$, and it swaps $\widehat{\Pi}_{i,j,k} \leftrightarrow \widehat{\Pi}_{r,s,t}$ such that $\{i, j, k, r, s, t\} = \{1, 2, 3, 4, 5, 6\}$. For

a non-trivial element $g \in \langle \tau, \mathbb{H}^\rho \rangle$, the involution $\varphi^{-1} \circ g \circ \varphi$ is a composition of flops, and either $\eta \circ \varphi^{-1} \circ g \circ \varphi \circ \eta^{-1}$ or $\eta \circ \varphi^{-1} \circ g \circ \sigma \circ \varphi \circ \eta^{-1}$ is a Cremona involution whose fundamental points are four points among $P_1, \ldots, P_6$ that swaps the remaining two points.

Now, let us give a criterion for $\mathrm{Cl}^G(X) \cong \mathbb{Z}$. To do this, we set

$$\Pi^\pm = \sum_{i=1}^{16} \Pi_i^\pm.$$

Then Π^+ and Π^- are $\mathbb{H}^\rho$-invariant divisors, $\sigma(\Pi^+) = \Pi^-$ and $\Pi^+ + \Pi^- \sim 16H$.

Lemma 22 *One has* $\mathrm{Cl}^G(X) \cong \mathbb{Z}$ *is at least one of the following conditions is satisfied*:

(i)   *the group G swaps Π^+ and Π^-;*
(ii)   *the divisor Π^+ is Cartier;*
(iii) *the divisor Π^- is Cartier;*
(iv)   *the surfaces $\Pi_1^+, \ldots, \Pi_{16}^+$ generate the group $\mathrm{Cl}(X)$;*
(v)   *the surfaces $\Pi_1^-, \ldots, \Pi_{16}^-$ generate the group $\mathrm{Cl}(X)$.*

Proof The assertion follows from Corollary 21, since we assume that $\mathbb{H}^\rho \subset G$.   □

This lemma is easy to apply if we fix $\mathscr{S}$ and the group $G \subset \mathrm{Aut}(X)$ such that $\mathbb{H} \subset \upsilon(G)$. For instance, to check whether the surfaces $\Pi_1^+, \ldots, \Pi_{16}^+$ generate the group $\mathrm{Cl}(X)$ or not, we can use the fact that $\mathrm{Cl}(X) \cong \mathbb{Z}^7$ is naturally equipped with an intersection form [50]. Namely, fix a smooth del Pezzo surface $S \in |H|$, and let

$$D_1 \bullet D_2 = D_1\big|_S \cdot D_2\big|_S \in \mathbb{Z}$$

for any two Weil divisors D_1 and D_2 in $\mathrm{Cl}(X)$. Then

$$\Pi_i^\pm \bullet \Pi_j^\pm = \begin{cases} 0 \text{ if } i \neq j \text{ and } \Pi_i^\pm \cap \Pi_j^\pm \text{ does not contain curves,} \\ 1 \text{ if } i \neq j \text{ and } \Pi_i^\pm \cap \Pi_j^\pm \text{ contains a curve,} \\ -1 \text{ if } i = j \text{ and } \Pi_i^\pm = \Pi_j^\pm, \\ 2 \text{ if } i = j \text{ and } \Pi_i^\pm \neq \Pi_j^\pm, \end{cases}$$

where two $\pm$ in $\Pi_i^\pm$ and $\Pi_j^\pm$ are independent.

Remark 23 Let Λ be the sublattice in $\mathrm{Cl}(X)$ consisting of divisors D such that $D \bullet H = 0$. Then Λ is isomorphic to a root lattice of type D_6 by [50, Theorem 1.7], and the natural homomorphism $\mathrm{Aut}(X) \to \mathrm{Aut}(\Lambda)$ is injective [50], where $\mathrm{Aut}(\Lambda) \cong (\boldsymbol{\mu}_2^5 \rtimes \mathfrak{S}_6) \rtimes \boldsymbol{\mu}_2$.

Applying Lemma 22, we get

Corollary 24 *If* $\mathrm{rank}(\Pi_i^+ \bullet \Pi_j^+) = 7$ *or* $\mathrm{rank}(\Pi_i^- \bullet \Pi_j^-) = 7$, *then* $\mathrm{Cl}^{\mathbb{H}^\rho}(X) \cong \mathbb{Z}$.

Let us show how to apply Corollary 24.

Π_1^+	$x_0 + x_1 + x_2 + \frac{2s}{s^2+1}x_3 = 0$ $w = \frac{s^2-1}{(s^2+1)^2}\left((s^2+1)x_1^2 + (s^2+1)x_1x_2 + 2sx_1x_3 + (s^2+1)x_2^2 + 2sx_2x_3 - (s^2+1)x_3^2\right)$
Π_2^+	$x_0 - x_1 + x_2 + \frac{2s}{s^2+1}x_3 = 0$ $w = \frac{s^2-1}{(s^2+1)^2}\left((s^2+1)x_1^2 - (s^2+1)x_1x_2 + 2sx_1x_3 + (s^2+1)x_2^2 - 2sx_2x_3 - (s^2+1)x_3^2\right) = 0$
Π_3^+	$x_0 + x_1 - x_2 - \frac{2s}{s^2+1}x_3 = 0$ $w = \frac{s^2-1}{(s^2+1)^2}\left((s^2+1)x_1^2 - (s^2+1)x_1x_2 - 2sx_1x_3 + (s^2+1)x_2^2 + 2sx_2x_3 - (s^2+1)x_3^2\right) = 0$
Π_4^+	$x_0 - x_1 - x_2 + \frac{2s}{s^2+1}x_3 = 0$ $w = \frac{s^2-1}{(s^2+1)^2}\left((s^2+1)x_1^2 + (s^2+1)x_1x_2 - 2sx_1x_3 + (s^2+1)x_2^2 - 2sx_2x_3 - (s^2+1)x_3^2\right) = 0$
Π_5^+	$x_0 + x_1 + \frac{2s}{s^2+1}x_2 + x_3 = 0$ $w = \frac{s^2-1}{(s^2+1)^2}\left((s^2+1)x_0^2 + 2sx_2x_0 + (s^2+1)x_3x_0 - (s^2+1)x_2^2 + 2sx_2x_3 + (s^2+1)x_3^2\right) = 0$
Π_6^+	$x_0 - x_1 + \frac{2s}{s^2+1}x_2 - x_3 = 0$ $w = \frac{s^2-1}{(s^2+1)^2}\left((s^2+1)x_0^2 + 2sx_2x_0 - (s^2+1)x_3x_0 - (s^2+1)x_2^2 - 2sx_2x_3 + (s^2+1)x_3^2\right) = 0$
Π_7^+	$\frac{2s}{s^2+1}x_2 - x_1 - x_0 + x_3 = 0$ $w = \frac{s^2-1}{(s^2+1)^2}\left((s^2+1)x_0^2 - 2sx_2x_0 - (s^2+1)x_3x_0 - (s^2+1)x_2^2 + 2sx_2x_3 + (s^2+1)x_3^2\right) = 0$
Π_8^+	$x_0 - x_1 - \frac{2s}{s^2+1}x_2 + x_3 = 0$ $w = \frac{s^2-1}{(s^2+1)^2}\left((s^2+1)x_0^2 - 2sx_2x_0 + (s^2+1)x_3x_0 - (s^2+1)x_2^2 - 2sx_2x_3 + (s^2+1)x_3^2\right) = 0$
Π_9^+	$x_0 + \frac{2s}{s^2+1}x_1 + x_2 + x_3 = 0$ $w = \frac{s^2-1}{(s^2+1)^2}\left((s^2+1)x_0^2 + 2sx_1x_0 + (s^2+1)x_3x_0 - (s^2+1)x_1^2 + 2sx_1x_3 + (s^2+1)x_3^2\right) = 0$
Π_{10}^+	$x_0 - \frac{2s}{s^2+1}x_1 - x_2 + x_3 = 0$ $w = \frac{s^2-1}{(s^2+1)^2}\left((s^2+1)x_0^2 - 2sx_1x_0 + (s^2+1)x_3x_0 - (s^2+1)x_1^2 - 2sx_1x_3 + (s^2+1)x_3^2\right) = 0$
Π_{11}^+	$x_0 - \frac{2s}{s^2+1}x_1 + x_2 - x_3 = 0$ $w = \frac{s^2-1}{(s^2+1)^2}\left((s^2+1)x_0^2 - 2sx_1x_0 - (s^2+1)x_3x_0 - (s^2+1)x_1^2 + 2sx_1x_3 + (s^2+1)x_3^2\right) = 0$
Π_{12}^+	$x_0 + \frac{2s}{s^2+1}x_1 - x_2 - x_3 = 0$ $w = \frac{s^2-1}{(s^2+1)^2}\left((s^2+1)x_0^2 + 2sx_1x_0 - (s^2+1)x_3x_0 - (s^2+1)x_1^2 - 2sx_1x_3 + (s^2+1)x_3^2\right) = 0$
Π_{13}^+	$\frac{2s}{s^2+1}x_0 + x_1 + x_2 + x_3 = 0$ $w = \frac{s^2-1}{(s^2+1)^2}\left((s^2+1)x_0^2 - 2sx_1x_0 - 2sx_2x_0 - (s^2+1)x_1^2 - (s^2+1)x_1x_2 - (s^2+1)x_2^2\right) = 0$
Π_{14}^+	$\frac{2s}{s^2+1}x_0 - x_1 - x_2 + x_3 = 0$ $w = \frac{s^2-1}{(s^2+1)^2}\left((s^2+1)x_0^2 + 2sx_1x_0 + 2sx_2x_0 - (s^2+1)x_1^2 - (s^2+1)x_1x_2 - (s^2+1)x_2^2\right) = 0$
Π_{15}^+	$\frac{2s}{s^2+1}x_0 - x_1 + x_2 - x_3 = 0$ $w = \frac{s^2-1}{(s^2+1)^2}\left((s^2+1)x_0^2 + 2sx_1x_0 - 2sx_2x_0 - (s^2+1)x_1^2 + (s^2+1)x_1x_2 - (s^2+1)x_2^2\right) = 0$
Π_{16}^+	$\frac{2s}{s^2+1}x_0 + x_1 - x_2 - x_3 = 0$ $w = \frac{s^2-1}{(s^2+1)^2}\left((s^2+1)x_0^2 - 2sx_1x_0 + 2sx_2x_0 - (s^2+1)x_1^2 + (s^2+1)x_1x_2 - (s^2+1)x_2^2\right) = 0$

Fig. 2 Defining equations of the surfaces $\Pi_1^+, \dots, \Pi_{16}^+$ in Example 25

Example 25 Let us use assumptions and notations of Example 5. Suppose, in addition, that $\rho : \mathbb{H} \to \mu_2$ is the trivial homomorphism. Therefore, we have $\mathbb{H}^\rho = \mathbb{H}$. Set $t = \frac{2s}{s^2+1}$. Observe that $\pi^*(\Pi_1)$ is given in $\mathbb{P}(1,1,1,1,2)$ by the following equations:

$$\begin{cases} x_0 + x_1 + x_2 + \dfrac{2s}{s^2+1}x_3 = 0, \\ w^2 = \dfrac{(s^2-1)^2}{(s^2+1)^4}\left((s^2+1)x_1^2 + (s^2+1)x_1x_2 + 2sx_1x_3 + (s^2+1)x_2^2 + 2sx_2x_3 - (s^2+1)x_3^2\right)^2. \end{cases}$$

Thus, without loss of generality, we may assume that the surface Π_1^+ is given by

$$\begin{cases} x_0 + x_1 + x_2 + \dfrac{2s}{s^2+1}x_3 = 0, \\ w = \dfrac{s^2-1}{(s^2+1)^2}\left((s^2+1)x_1^2 + (s^2+1)x_1x_2 + 2sx_1x_3 + (s^2+1)x_2^2 + 2sx_2x_3 - (s^2+1)x_3^2\right). \end{cases}$$

Then the defining equations of the remaining surfaces $\Pi_2^+, \ldots, \Pi_{16}^+$ are listed in Figure 2. Now, the intersection matrix $(\Pi_i^+ \bullet \Pi_j^+)$ can be computed as follows:

$$\begin{pmatrix}
-1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
1 & -1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & -1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & -1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & -1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 & -1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 1 & -1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & -1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & -1 & 1 & 1 & 0 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & -1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & -1 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & -1 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & -1 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & -1 & 1 \\
0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & -1
\end{pmatrix}$$

The rank of this matrix is 7. Therefore, we conclude that $\mathrm{Cl}^{\mathbb{H}}(X) \cong \mathbb{Z}$ by Corollary 24. Note that we can also prove this using Lemma 22(ii). To do this, it is enough to show that the divisor Π^+ is a Cartier divisor, which can be done locally at any point in $\mathrm{Sing}(X)$. For instance, let $P = [t : 1 : 1 : 1 : 0] \in \mathrm{Sing}(X)$. Among $\Pi_1^+, \ldots, \Pi_{16}^+$, only

$$\Pi_2^+, \Pi_3^+, \Pi_7^+, \Pi_8^+, \Pi_{10}^+, \Pi_{11}^+$$

pass through P. Choosing a generator of the local class group $\mathrm{Cl}_P(X) \cong \mathbb{Z}$, we see that the classes of the surfaces $\Pi_2^+, \Pi_3^+, \Pi_7^+, \Pi_8^+, \Pi_{10}^+, \Pi_{11}^+$ are $1, -1, 1, -1, 1, -1$, respectively. Hence, we see that Π^+ is locally Cartier at P, which implies that Π^+ is globally Cartier, because the group $\mathbb{H}$ acts transitively on the set $\mathrm{Sing}(X)$.

Example 26 Let us use assumptions and notations of Example 5. Then $\mathrm{Aut}(X)$ contains a unique subgroup G such that $G \cong \upsilon(G) \cong \mu_2^4 \rtimes \mu_3$, and $\upsilon(G)$ is generated by

$$A_1, A_2, A_3, A_4, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

One can check that G contains the subgroup $\mathbb{H} = \mathbb{H}^\rho$, where ρ is a trivial homomorphism. Therefore, it follows from Example 25 that $\mathrm{Cl}^G(X) \cong \mathbb{Z}$.

If $\mathrm{Cl}^G(X) \not\cong \mathbb{Z}$, we have the following result.

Lemma 27 *Let G be any subgroup in* $\mathrm{Aut}(X)$ *such that* $\mathrm{Cl}^G(X) \not\cong \mathbb{Z}$ *and* $\mathbb{H} \subseteq \upsilon(G)$. *Then* $\mathrm{Cl}^G(X) \cong \mathbb{Z}^2$, *and exists a G-Sarkisov link*
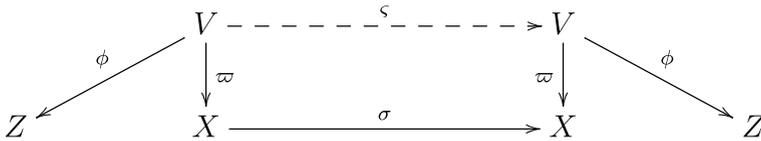
where ϖ is a G-equivariant small resolution, ς is a birational involution that flops sixteen ϖ-contracted curves, ϕ is a $\mathbb{P}^1$-bundle, and Z is a smooth del Pezzo surface of degree 4.

Proof By Corollary 21, one has $\mathrm{Cl}^G(X) \cong \mathbb{Z}^2$, but $\mathrm{Pic}(X) \cong \mathbb{Z}$. Using this and the fact that G acts transitively on the set $\mathrm{Sing}(X)$, we see that there exists a G-equivariant small resolution of singularities $\varpi : V \to X$. Then $-K_V \sim \varpi^*(2H)$.

Now, applying G-equivariant Minimal Model Program to V, we obtain a G-equivariant morphism $\phi : V \to Z$ such that $\phi_*(\mathcal{O}_V) = \mathcal{O}_Z$, and one of the following possibilities hold:

(1) Z is a possibly singular Fano threefold, and ϕ is birational;
(2) Z is a normal surface, and ϕ is a conic bundle;
(3) $Z \cong \mathbb{P}^1$, and ϕ is a del Pezzo fibration.

Note that $\mathrm{rk}\,\mathrm{Cl}^G(V/Z) = 1$. In particular, the divisor $-K_V$ is ϕ-ample.

If ϕ is a del Pezzo fibration, then its general fiber is $\mathbb{P}^1 \times \mathbb{P}^1$ by the adjunction formula, because $-K_V \sim \varpi^*(2H)$. Moreover, in this case all fibers of the morphism ϕ are reduced and irreducible [29], which gives $\mathrm{rk}\,\mathrm{Cl}(V) \leqslant 3$, which is impossible since $\mathrm{Cl}(V) \cong \mathbb{Z}^7$. Therefore, either ϕ is birational, or ϕ is a conic bundle.

Now, applying relative Minimal Model Program to the threefold V over Z, we obtain a commutative diagram
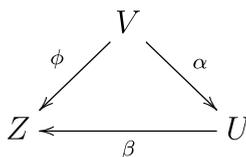
where α is an extremal contraction, and β is a morphism with connected fibers.

Suppose that α is birational. Let E be the α-exceptional surface. Then U is a smooth threefold, and α is a usual blow up of a point in U (see [10, Lemma 2.8] or [21, 46]). Then $E \cong \mathbb{P}^2$ and $E|_E \cong \mathcal{O}_{\mathbb{P}^2}(-1)$. So, the adjunction formula gives $\varpi^*(H)|_E \cong \mathcal{O}_{\mathbb{P}^2}(1)$. Therefore, we conclude that $\pi \circ \varpi(E) \cong \varpi(E) \cong E \cong \mathbb{P}^2$ and $\pi \circ \varpi(E)$ is a plane in $\mathbb{P}^3$, which implies that $\varpi(E) = \Pi_i^+$ or $\varpi(E) = \Pi_i^-$ for some $i \in \{1, \ldots, 16\}$.

On the other hand, for every $g \in G$, we have

$$g(E) \cap E = \varnothing,$$

because all curves in E are contained in one extremal ray of the Mori cone $\overline{NE(V)}$. Moreover, since $\mathbb{H} \subseteq \upsilon(G)$, the G-orbit of the surface E consists of at least 16 surfaces. This implies that rk $\mathrm{Cl}(V) \geqslant 17$, which is impossible, since $\mathrm{Cl}(V) \cong \mathrm{Cl}(X) \cong \mathbb{Z}^7$.

Thus, we conclude that α is not birational, so ϕ is not birational either. Hence, we see that ϕ is a conic bundle. Then α is also a conic bundle, because otherwise α would be birational. Since general fibers of both conic bundles α and β are the same, they generate one extremal ray of the Mori cone $\overline{NE(V)}$. This implies that β must be an isomorphism. Therefore, we may assume that $\alpha = \phi$ and $\beta = \mathrm{Id}_U$.

Now, it follows from [10, Lemma 2.5] that Z is smooth, it is a weak del Pezzo surface, and $V \cong \mathbb{P}(\mathcal{E})$ for some rank two vector bundle $\mathcal{E}$ on the surface Z.

To complete the proof, it is enough to show that Z is a del Pezzo surface of degree 4. First, we observe that $\mathrm{Cl}(Z) \cong \mathbb{Z}^6$, since $\mathrm{Cl}(V) \cong \mathbb{Z}^7$. This gives $K_Z^2 = 4$.

Let C be a general fiber of the $\mathbb{P}^1$-bundle ϕ. Then we have $2H \cdot \varpi(C) = -K_V \cdot C = 2$, which implies that $\pi \circ \phi(C)$ is a line in $\mathbb{P}^3$ that is tangent to the surface $\mathscr{S}$ at two points. Now, for $i \in \{1, \ldots, 16\}$, let $\widetilde{\Pi}_i^\pm$ be the proper transform on V of the surface $\Pi_i^\pm$. Then

$$1 = \varpi^*(H) \cdot C = \left(\widetilde{\Pi}_i^+ + \widetilde{\Pi}_i^- \right) \cdot S = \widetilde{\Pi}_i^+ \cdot C + \widetilde{\Pi}_i^- \cdot C.$$

Thus, without loss of generality, we may assume that $\widetilde{\Pi}_i^+ \cdot C = 1$ and $\widetilde{\Pi}_i^- \cdot C = 0$ for every i. Then $\widetilde{\Pi}_i^-$ is mapped to a curve in Z, and ϕ induces a birational morphism $\widetilde{\Pi}_i^- \to Z$.

Recall that ϖ induces a birational morphism $\widetilde{\Pi}_i^+ \to \Pi_i^+$ that is a blow up of $k \geqslant 0$ points in the intersection $\Pi_i^+ \cap \mathrm{Sing}(X)$, where $k \leqslant 6$, since $\Pi_i^+ \cap \mathrm{Sing}(X)$ consists of six points. Moreover, the morphism ϖ also induces a birational morphism $\widetilde{\Pi}_i^- \to \Pi_i^-$ that blows up the remaining $6 - k$ points in $\Pi_i^+ \cap \mathrm{Sing}(X) = \Pi_i^- \cap \mathrm{Sing}(X)$. We also know that the intersection $\Pi_i^+ \cap \mathrm{Sing}(X)$ is contained in an irreducible conic in $\Pi_i^+ \cong \mathbb{P}^2$. Then

- either $k \leqslant 5$ and $\widetilde{\Pi}_i^+$ is a smooth del Pezzo surface of degree $9 - k$,
- or $k = 6$ and $\widetilde{\Pi}_i^+$ is a smooth weak del Pezzo surface of degree 3.

If $k = 6$, then $\widetilde{\Pi}_i^- \cong \Pi_i^- \cong \mathbb{P}^2$, and the surface $\widetilde{\Pi}_i^-$ cannot be mapped to a curve by ϕ. Thus, we conclude that $k \leqslant 5$, and $\widetilde{\Pi}_i^+$ is a smooth del Pezzo surface of degree $9 - k \geqslant 4$. Since ϕ induces a birational morphism $\widetilde{\Pi}_i^+ \to Z$ and $K_Z^2 = 4$, we get $k = 5$ and $\widetilde{\Pi}_i^+ \cong Z$, which implies that Z is a del Pezzo surface of degree 4, and $\phi(\widetilde{\Pi}_i^-)$ is a (-1)-curve. $\qquad\square$

Before we proceed, let us make few remarks:

(1) In the proof of Lemma 27, we show that general fibers of the $\mathbb{P}^1$-bundle ϕ are mapped by the morphism $\pi \circ \varpi$ to the lines in $\mathbb{P}^3$ tangent to $\mathscr{S}$ at two points. These lines form a classical congruence in $\mathbb{P}^3$ of degree $(2, 2)$, which is related to the quartic surface $\mathscr{S}$, see [8, 23, 36, 43, 44, 56, 2, Example 5.5], [49, Example 2.3]. This implies that the weak Fano threefold V in Lemma 27 is isomorphic to $\mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is a rank two vector

bundle on the quartic del Pezzo surface $Z \subset \mathrm{Gr}(2,4)$, which is obtained by restricting to Z the universal quotient bundle from $\mathrm{Gr}(2,4)$.

(2) It was pointed to us by Alexander Kuznetsov that the diagram in Lemma 27 can be constructed as follows. We know from [55] the existence of the following diagram: where

$$\mathbb{P}(\mathscr{N}) \cong \mathbb{P}(\mathscr{E})$$

$$\pi_{\mathbb{P}^3} \qquad\qquad \pi_{\mathcal{Q}}$$

$$\mathbb{P}^3 \qquad\qquad\qquad\qquad \mathcal{Q}$$
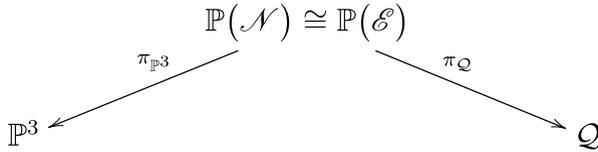
$\mathscr{N}$ is the rank two null-correlation vector bundle on $\mathbb{P}^3$, $\mathcal{Q}$ is a smooth quadric threefold, $\mathscr{E}$ is a rank two vector bundle on the quadric $\mathcal{Q} \subset \mathrm{Gr}(2,4)$ obtained by restricting to $\mathcal{Q}$ the universal quotient bundle from the Grassmannian, and $\pi_{\mathbb{P}^3}$ and $\pi_{\mathcal{Q}}$ are projections. Thus, since Z is cut out on $\mathcal{Q}$ by another quadric, we can identify V with a preimage of the del Pezzo surface Z in the fourfold $\mathbb{P}(\mathscr{E})$. Then $\pi_{\mathbb{P}^3}$ induces a morphism $V \to \mathbb{P}^3$, which is the composition $\pi \circ \varpi$.

(3) We can also describe the $\mathbb{P}^1$-bundle ϕ in Lemma 27 using the diagram (14). Namely, a general fiber of this $\mathbb{P}^1$-bundle is mapped by $\eta \circ \varphi^{-1} \circ \varpi$ to

   - either a line that passes through one of the points $P_1, \ldots, P_6$,
   - or a twisted cubic curve that passes through five points among $P_1, \ldots, P_6$.

(4) If $\widetilde{\mathscr{S}}$ is the preimage of the quartic surface $\mathscr{S}$ on the threefold V in Lemma 27, then $\widetilde{\mathscr{S}}$ is smooth, the morphism $\pi \circ \varpi$ induces a minimal resolution $\widetilde{\mathscr{S}} \to \mathscr{S}$, and the $\mathbb{P}^1$-bundle ϕ induces a G-equivariant double cover $\widetilde{\mathscr{S}} \to Z$, cf. [54].

(5) Using Lemma 27 one can relate the subgroup $G \subset \mathrm{Aut}(X)$ such that $\mathrm{Cl}^G(X) \cong \mathbb{Z}^2$ with a corresponding subgroup in the group $\mathrm{Aut}(Z)$, see [24, 33].

Note that $\mathrm{Cl}^G(X) \cong \mathbb{Z}^2$ is indeed possible. Let us give two (related) examples.

Example 28 Let us use all assumptions and notations of Example 25, and let $G = \mathbb{H}^\rho$, where the homomorphism ρ is defined by $\rho(A_1) = -1$, $\rho(A_2) = 1$, $\rho(A_3) = -1$, $\rho(A_4) = 1$. Then, arguing as in Example 25, we compute $\mathrm{Cl}^G(X) \cong \mathbb{Z}^2$.

Example 29 Let us use all assumptions and notations of Example 9. Then

$$\mathrm{Aut}(X) \cong \boldsymbol{\mu}_2 \times \mathrm{Aut}(\mathbb{P}^3, \mathscr{S}) \cong \boldsymbol{\mu}_2 \times (\boldsymbol{\mu}_2^4 \rtimes \boldsymbol{\mu}_5),$$

and the group $\mathrm{Aut}(X)$ contains a unique subgroup isomorphic to $\mathrm{Aut}(\mathbb{P}^3, \mathscr{S}) \cong \boldsymbol{\mu}_2^4 \rtimes \boldsymbol{\mu}_5$. Suppose that G is this subgroup. It follows from Remark 23 that $\mathrm{Cl}(X) \otimes \mathbb{Q}$ is a faithful seven-dimensional G-representation. Using this, it is easy to see that $\mathrm{Cl}(X) \otimes \mathbb{Q}$ splits as a sum of an irreducible five-dimensional representation and two trivial one-dimensional representations. Hence, we conclude that $\mathrm{Cl}^G(X) \cong \mathbb{Z}^2$.

Before proving Theorem 17, let us prove its two baby cases, which follow from [16, 18].

Proposition 30 *Suppose $G = \mathrm{Aut}(X)$, and $\mathscr{S}$ is the quartic surface from Example 9. Then $\mathrm{Cl}^G(X) \cong \mathbb{Z}$ and X is G-birationally super-rigid.*

Proof Since $\sigma \in G$, we get $\mathrm{Cl}^G(X) \cong \mathbb{Z}$. Let us show that X is G-birationally super-rigid.

Note that the $\upsilon(G)$-equivariant birational geometry of the projective space $\mathbb{P}^3$ has been studied in [18]. In particular, we know from [18, Corollary 4.7] and [18, Theorem 4.16] that

- $\mathbb{P}^3$ does not contain $\upsilon(G)$-orbits of length less that 16,
- $\mathbb{P}^3$ does not contain $\upsilon(G)$-invariant curves of degree less than 8.

Let $\mathcal{M}$ be a G-invariant linear system on X such that $\mathcal{M}$ has no fixed components. Choose a positive integer n such that $\mathcal{M} \subset |nH|$. Then, by [17, Corollary 3.3.3], to prove that the threefold X is G-birationally super-rigid it is enough to show that $(X, \frac{2}{n}\mathcal{M})$ has canonical singularities. Suppose that the singularities of this log pair are not canonical.

Let Z be a center of non-canonical singularities of the pair $(X, \frac{2}{n}\mathcal{M})$ that has the largest dimension. Since the linear system $\mathcal{M}$ does not have fixed components, we conclude that either Z is an irreducible curve, or Z is a point. In both cases, we have

$$\mathrm{mult}_Z(\mathcal{M}) > \frac{n}{2}$$

by [39, Theorem 4.5].

Let M_1 and M_2 be general surfaces in $\mathcal{M}$. If Z is a curve, then

$$M_1 \cdot M_2 = (M_1 \cdot M_2)_Z \mathscr{Z} + \Delta$$

where $\mathscr{Z}$ is the G-irreducible curve in X whose irreducible component is the curve Z, and Δ is an effective one-cycle whose support does not contain $\mathscr{Z}$, which gives

$$
\begin{aligned}
2n^2 = n^2 H^2 = H \cdot M_1 \cdot M_2 &= \left(M_1 \cdot M_2\right)_Z \mathscr{Z} + \Delta \\
&= \left(M_1 \cdot M_2\right)_Z \left(H \cdot \mathscr{Z}\right) + H \cdot \Delta \geqslant \left(M_1 \cdot M_2\right)_Z \left(H \cdot \mathscr{Z}\right) \geqslant \\
&\geqslant \mathrm{mult}_Z^2(\mathcal{M})\left(H \cdot \mathscr{Z}\right) > \frac{n^2}{4}\left(H \cdot \mathscr{Z}\right) \geqslant \frac{n^2}{4}\deg\left(\pi(\mathscr{Z})\right),
\end{aligned}
$$

so $\pi(\mathscr{Z})$ is a $\upsilon(G)$-invariant curve of degree $\leqslant 7$, which contradicts [18, Theorem 4.16].

We see that Z is a point, and $(X, \frac{2}{n}\mathcal{M})$ is canonical away from finitely many points.

We claim that $Z \notin \mathrm{Sing}(X)$. Indeed, suppose Z is a singular point of the threefold X. Let $h : \overline{X} \to X$ be the blow up of the locus $\mathrm{Sing}(X)$, let $E_1, \ldots, E_{16}$ be the h-exceptional surfaces, let $\overline{M}_1$ and $\overline{M}_1$ be the proper transforms on $\overline{X}$ of the surfaces M_1 and M_2, respectively. Write $E = E_1 + \cdots + E_{16}$. Since $\mathrm{Sing}(X)$ is a G-orbit, we have

$$\overline{M}_1 \sim \overline{M}_2 \sim h^*(H) - \epsilon E$$

for some integer $\epsilon \geqslant 0$. Using [20, Theorem 3.10] or [13, Theorem 1.7.20], we get $\epsilon > \frac{n}{2}$. On the other hand, the linear system $|h^*(3H) - E|$ is not empty and does not have base curves away from the locus $E_1 \cup E_2 \cup \cdots \cup E_{16}$, because $\mathrm{Sing}(\mathscr{S})$ is cut out by cubic surfaces in $\mathbb{P}^3$. In particular, the divisor $h^*(3H) - E$ is nef, so

$$0 \leqslant \left(h^*(3H) - E\right) \cdot \overline{M}_1 \cdot \overline{M}_2 = \left(h^*(3H) - E\right) \cdot \left(h^*(3nH) - \epsilon E\right)^2 = 6n^2 - 32\epsilon^2,$$

which is impossible, since $\epsilon > \frac{n}{2}$. So, we see that Z is a smooth point of the threefold X.

Then the pair $(X, \frac{3}{n}\mathcal{M})$ is not log canonical at Z. Let μ be the largest rational number such that the log pair $(X, \mu\mathcal{M})$ is log canonical. Then $\mu < \frac{3}{n}$ and

$$\mathrm{Orb}_G(Z) \subseteq \mathrm{Nklt}\big(X, \mu\mathcal{M}\big).$$

Observe that $\mathrm{Nklt}(X, \mu\mathcal{M})$ is at most one-dimensional, since $\mathcal{M}$ has no fixed components. Moreover, this locus is G-invariant, because $\mathcal{M}$ is G-invariant.

We claim that $\mathrm{Nklt}(X, \mu\mathcal{M})$ does not contain curves. Indeed, suppose this is not true. Then $\mathrm{Nklt}(X, \mu\mathcal{M})$ contains a G-irreducible curve C. We write $M_1 \cdot M_2 = mC + \Omega$, where m is a non-negative integer, and Ω is an effective one-cycle whose support does not contain the curve C. Then it follows from [20, Theorem 3.1] that

$$m \geqslant \frac{4}{\mu^2} > \frac{4n^2}{9}.$$

Therefore, we have

$$2n^2 = n^2 H^3 = H \cdot M_1 \cdot M_2 = m\big(H \cdot C\big) + H \cdot \Omega \geqslant m\big(H \cdot C\big) > \frac{4n^2}{9}\big(H \cdot C\big),$$

which implies that $H \cdot C \leqslant 4$. Then $\pi(C)$ is a $\upsilon(G)$-invariant curve in $\mathbb{P}^3$ of degree $\leqslant 4$, which contradicts [18, Theorem 4.16]. Thus, the locus $\mathrm{Nklt}(X, \mu\mathcal{M})$ contains no curves.

Let $\mathcal{I}$ be the multiplier ideal sheaf of the pair $(X, \mu\mathcal{M})$, and let $\mathcal{L}$ be the corresponding subscheme in X. Then $\mathcal{L}$ is a zero-dimensional (reduced) subscheme such that

$$\mathrm{Orb}_G\big(Z\big) \subseteq \mathrm{Supp}\big(\mathcal{L}\big) = \mathrm{Nklt}\big(X, \mu\mathcal{M}\big).$$

Applying Nadel's vanishing [45, Theorem 9.4.8], we get $h^1(X, \mathcal{I} \otimes \mathcal{O}_X(H)) = 0$. This gives

$$4 = h^0\big(X, \mathcal{O}_X(H)\big) \geqslant h^0\big(\mathcal{O}_\mathcal{L} \otimes \mathcal{O}_X(H)\big) = h^0\big(\mathcal{O}_\mathcal{L}\big) \geqslant |\mathrm{Orb}_G(Z)|.$$

In particular, we conclude that the length of the $\upsilon(G)$-orbit of the point $\pi(Z)$ is at most 4, which is impossible by [18, Corollary 4.7]. □

Proposition 31 *Suppose that $\mathscr{S}$ is the surface from Example 5, and G is the subgroup described in Example 26. Then $\mathrm{Cl}^G(X) \cong \mathbb{Z}$ and X is G-birationally super-rigid.*

Proof Recall from Example 26 that $G \cong \boldsymbol{\mu}_2^4 \rtimes \boldsymbol{\mu}_3$ and $\mathrm{Cl}^G(X) \cong \mathbb{Z}$.

The $\upsilon(G)$-equivariant geometry of the projective space $\mathbb{P}^3$ has been studied in [16]. In particular, we know from [16] that $\mathbb{P}^3$ does not contain $\upsilon(G)$-orbits of length 1, 2 or 3, and the only $\upsilon(G)$-orbits in $\mathbb{P}^3$ of length 4 are

$$\Sigma_4 = \big\{[1:0:0:0], [0:1:0:0], [0:0:1:0], [0:0:0:1]\big\},$$
$$\Sigma_4' = \big\{[1:1:1:-1], [1:1:-1:1], [1:-1:1:1], [-1:1:1:1]\big\},$$
$$\Sigma_4'' = \big\{[1:1:1:1], [1:1:-1:-1], [1:-1:-1:1], [-1:-1:1:1]\big\}$$

We also know from [16] the classification of $\upsilon(G)$-invariant curves in $\mathbb{P}^3$ of degree at most 7. Namely, let $\mathcal{L}_4, \mathcal{L}_4', \mathcal{L}_4'', \mathcal{L}_4''', \mathcal{L}_6, \mathcal{L}_6', \mathcal{L}_6'', \mathcal{L}_6''', \mathcal{L}_6''''$ be $\upsilon(G)$-irreducible curves in $\mathbb{P}^3$ whose irreducible components are the lines

$$\{2x_0 + (1 + \sqrt{3}i)x_2 - (1 - \sqrt{3}i)x_3 = 2x_1 + (1 - \sqrt{3}i)x_2 + (1 + \sqrt{3}i)x_3 = 0\},$$
$$\{2x_0 + (1 - \sqrt{3}i)x_2 - (1 + \sqrt{3}i)x_3 = 2x_1 + (1 + \sqrt{3}i)x_2 + (1 - \sqrt{3}i)x_3 = 0\},$$
$$\{2x_0 - (1 - \sqrt{3}i)x_2 + (1 + \sqrt{3}i)x_3 = 2x_1 + (1 + \sqrt{3}i)x_2 + (1 - \sqrt{3}i)x_3 = 0\},$$
$$\{2x_0 - (1 + \sqrt{3}i)x_2 + (1 - \sqrt{3}i)x_3 = 2x_1 + (1 - \sqrt{3}i)x_2 + (1 + \sqrt{3}i)x_3 = 0\},$$
$$\{x_0 = x_1 = 0\},$$
$$\{x_0 + x_1 = x_2 - x_3 = 0\},$$
$$\{x_0 + x_1 = x_2 + x_3 = 0\},$$
$$\{x_0 + ix_2 = x_1 + ix_3 = 0\},$$
$$\{x_0 + ix_3 = x_1 + ix_2 = 0\},$$

respectively. Then the curves $\mathcal{L}_4, \mathcal{L}_4', \mathcal{L}_4'', \mathcal{L}_4''', \mathcal{L}_6, \mathcal{L}_6', \mathcal{L}_6'', \mathcal{L}_6''', \mathcal{L}_6''''$ are unions of 4, 4, 4, 4, 6, 6, 6, 6, 6 lines, respectively. Moreover, it follows from [16] that
$\mathcal{L}_4, \mathcal{L}_4', \mathcal{L}_4'', \mathcal{L}_4''', \mathcal{L}_6, \mathcal{L}_6', \mathcal{L}_6'', \mathcal{L}_6''', \mathcal{L}_6''''$ are the only $v(G)$-invariant curves in $\mathbb{P}^3$ of degree at most 7.

Now, using the defining equation of the surface $\mathscr{S}$, one can check that any irreducible component of any curve among $\mathcal{L}_4, \mathcal{L}_4', \mathcal{L}_4'', \mathcal{L}_4''', \mathcal{L}_6, \mathcal{L}_6', \mathcal{L}_6'', \mathcal{L}_6''', \mathcal{L}_6''''$ intersects the quartic surface $\mathscr{S}$ transversally by 4 distinct points, so that its preimage in X via the double cover π is a smooth elliptic curve. Thus, if C is a G-invariant curve in X, then $H \cdot C \geqslant 8$.

Suppose that X is not G-birationally super-rigid. It follows from [17, Corollary 3.3.3] that there are a positive integer n and a G-invariant linear subsystem $\mathcal{M} \subset |nH|$ such that $\mathcal{M}$ does not have fixed components, but the log pair $(X, \frac{2}{n}\mathcal{M})$ is not canonical.

Arguing as in the proof of Proposition 30, we see that the log pair $(X, \frac{2}{n}\mathcal{M})$ is canonical away from finitely many points. Let P be a point in X that is a center of non-canonical singularities of the log pair $(X, \frac{2}{n}\mathcal{M})$. Now, arguing as in the proof of Proposition 30 again, we see that P is a smooth point of the threefold X.

Then the log pair $(X, \frac{3}{n}\mathcal{M})$ is not log canonical at P. Let μ be the largest rational number such that $(X, \mu\mathcal{M})$ is log canonical. Then $\mu < \frac{3}{n}$ and

$$\mathrm{Orb}_G(P) \subseteq \mathrm{Nklt}(X, \mu\mathcal{M}).$$

Observe that $\mathrm{Nklt}(X, \mu\mathcal{M})$ is at most one-dimensional, since $\mathcal{M}$ has no fixed components. Moreover, this locus is G-invariant, because $\mathcal{M}$ is G-invariant. Furthermore, arguing as in the proof of Proposition 30, we see that

$$\dim\left(\mathrm{Nklt}(X, \mu\mathcal{M})\right) = 0.$$

Let $\mathcal{I}$ be the multiplier ideal sheaf of the pair $(X, \mu\mathcal{M})$, and let $\mathcal{L}$ be the corresponding subscheme in X. Then $\mathcal{L}$ is a zero-dimensional (reduced) subscheme such that

$$\mathrm{Orb}_G(P) \subseteq \mathrm{Supp}(\mathcal{L}) = \mathrm{Nklt}(X, \mu\mathcal{M}).$$

On the other hand, applying Nadel's vanishing theorem [45, Theorem 9.4.8], we get

$$h^1\left(X, \mathcal{I} \otimes \mathcal{O}_X(H)\right) = 0.$$

This gives

$$4 = h^0\big(X, \mathcal{O}_X(H)\big) \geqslant h^0\big(\mathcal{O}_\mathcal{L} \otimes \mathcal{O}_X(H)\big) = h^0\big(\mathcal{O}_\mathcal{L}\big) \geqslant |\mathrm{Orb}_G(P)|.$$

Thus, we conclude that $|\mathrm{Orb}_G(P)| = 4$ and $\pi(P) \in \Sigma_4 \cup \Sigma_4' \cup \Sigma_4''$.

Let M_1 and M_2 be two general surfaces in $\mathcal{M}$. Using [51] or [20, Corollary 3.4], we get

$$\big(M_1 \cdot M_2\big)_P > n^2. \tag{32}$$

Let $\mathcal{S}$ be a linear subsystem in $|3H|$ that consists of all surfaces that are singular at every point of the G-orbit $\mathrm{Orb}_G(P)$. Then its base locus does not contain curves, which implies that there is a surface $S \in \mathcal{S}$ that does not contain components of the cycle $M_1 \cdot M_2$. Thus, using (32) and $\mathrm{mult}_P(S) \geqslant 2$, we get

$$6n^2 = S \cdot M_1 \cdot M_2 \geqslant \sum_{O \in \mathrm{Orb}_G(P)} 2\big(M_1 \cdot M_2\big)_O = 2|\mathrm{Orb}_G(P)|\big(M_1 \cdot M_2\big)_P$$

$$= 8\big(M_1 \cdot M_2\big)_P > 8n^2,$$

which is absurd. This completes the proof of Proposition 31. $\qquad\square$

In the remaining part of the paper, we prove Theorem 17, and consider one application. Let us recall from [1, 26, 30, 35, 48] basic facts about the $\mathbb{H}$-equivariant geometry of $\mathbb{P}^3$. Set

$$\begin{aligned}
\mathcal{Q}_1 &= \big\{x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0\big\}, \\
\mathcal{Q}_2 &= \big\{x_0^2 + x_1^2 = x_2^2 + x_3^2\big\}, \\
\mathcal{Q}_3 &= \big\{x_0^2 - x_1^2 = x_2^2 - x_3^2\big\}, \\
\mathcal{Q}_4 &= \big\{x_0^2 - x_1^2 = x_3^2 - x_2^2\big\}, \\
\mathcal{Q}_5 &= \big\{x_0 x_2 + x_1 x_3 = 0\big\}, \\
\mathcal{Q}_6 &= \big\{x_0 x_3 + x_1 x_2 = 0\big\}, \\
\mathcal{Q}_7 &= \big\{x_0 x_1 + x_2 x_3 = 0\big\}, \\
\mathcal{Q}_8 &= \big\{x_0 x_2 = x_1 x_3\big\}, \\
\mathcal{Q}_9 &= \big\{x_0 x_3 = x_1 x_2\big\}, \\
\mathcal{Q}_{10} &= \big\{x_0 x_1 = x_2 x_3\big\}.
\end{aligned}$$

Then $\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3, \mathcal{Q}_4, \mathcal{Q}_5, \mathcal{Q}_6, \mathcal{Q}_7, \mathcal{Q}_8, \mathcal{Q}_9, \mathcal{Q}_{10}$ are all $\mathbb{H}$-invariant quadric surfaces in $\mathbb{P}^3$. These quadrics are smooth, and $\mathbb{H} \cong \boldsymbol{\mu}_2^4$ acts *naturally* on each quadric $\mathcal{Q}_i \cong \mathbb{P}^1 \times \mathbb{P}^1$.

For a non-trivial element $g \in \mathbb{H}$, the locus of its fixed points in $\mathbb{P}^3$ consists of two skew lines, which we will denote by L_g and L_g'. For two non-trivial elements $g \neq h$ in $\mathbb{H}$, one has

$$\{L_g, L_g'\} \cap \{L_h, L_h'\} = \varnothing.$$

In total, this gives 30 lines $\ell_1, \dots, \ell_{30}$, whose equations are listed in the following table:

$\ell_1 = \{x_0 = x_1 = 0\}$	$\ell_2 = \{x_2 = x_3 = 0\}$
$\ell_3 = \{x_0 = x_2 = 0\}$	$\ell_4 = \{x_1 = x_3 = 0\}$
$\ell_5 = \{x_0 = x_3 = 0\}$	$\ell_6 = \{x_1 = x_2 = 0\}$

	$\mathcal{Q}_1$	$\mathcal{Q}_2$	$\mathcal{Q}_3$	$\mathcal{Q}_4$	$\mathcal{Q}_5$	$\mathcal{Q}_6$	$\mathcal{Q}_7$	$\mathcal{Q}_8$	$\mathcal{Q}_9$	$\mathcal{Q}_{10}$
ℓ_1	−	−	−	−	+	+	−	+	+	−
ℓ_2	−	−	−	−	+	+	−	+	+	−
ℓ_3	−	−	−	−	−	+	+	−	+	+
ℓ_4	−	−	−	−	−	+	+	−	+	+
ℓ_5	−	−	−	−	+	−	+	+	−	+
ℓ_6	−	−	−	−	+	−	+	+	−	+
ℓ_7	−	−	+	+	−	−	−	+	+	−
ℓ_8	−	−	+	+	−	−	−	+	+	−
ℓ_9	−	+	+	−	−	−	−	−	+	+
ℓ_{10}	−	+	+	−	−	−	−	−	+	+
ℓ_{11}	−	+	−	+	−	−	−	+	−	+
ℓ_{12}	−	+	−	+	−	−	−	+	−	+
ℓ_{13}	−	−	+	+	+	+	−	−	−	−
ℓ_{14}	−	−	+	+	+	+	−	−	−	−
ℓ_{15}	−	+	+	−	−	+	+	−	−	−
ℓ_{16}	−	+	+	−	−	+	+	−	−	−
ℓ_{17}	−	+	−	+	+	−	+	−	−	−
ℓ_{18}	−	+	−	+	+	−	+	−	−	−
ℓ_{19}	+	+	−	−	+	−	−	−	+	−
ℓ_{20}	+	+	−	−	+	−	−	−	+	−
ℓ_{21}	+	−	−	+	−	−	+	−	+	−
ℓ_{22}	+	−	−	+	−	−	+	−	+	−
ℓ_{23}	+	−	+	−	+	−	+	+	−	−
ℓ_{24}	+	−	+	−	−	−	+	+	−	−
ℓ_{25}	+	+	−	−	−	+	−	+	−	−
ℓ_{26}	+	+	−	−	−	+	−	+	−	−
ℓ_{27}	+	−	−	+	−	+	−	−	−	+
ℓ_{28}	+	−	−	+	−	+	−	−	−	+
ℓ_{29}	+	−	+	−	+	−	−	−	−	+
ℓ_{30}	+	−	+	−	+	−	−	−	−	+

Fig. 3 Ten $\mathbb{H}$-invariant quadrics in $\mathbb{P}^3$ and thirty lines in them

$\ell_7 = \{x_0 + x_1 = x_2 + x_3 = 0\}$	$\ell_8 = \{x_0 - x_1 = x_2 - x_3 = 0\}$
$\ell_9 = \{x_0 + x_2 = x_1 + x_3 = 0\}$	$\ell_{10} = \{x_0 - x_2 = x_1 - x_3 = 0\}$
$\ell_{11} = \{x_0 + x_3 = x_1 + x_2 = 0\}$	$\ell_{12} = \{x_0 - x_3 = x_1 - x_2 = 0\}$
$\ell_{13} = \{x_0 + x_1 = x_2 - x_3 = 0\}$	$\ell_{14} = \{x_0 - x_1 = x_2 + x_3 = 0\}$
$\ell_{15} = \{x_0 + x_2 = x_1 - x_3 = 0\}$	$\ell_{16} = \{x_0 - x_2 = x_1 + x_3 = 0\}$

$$\ell_{17} = \{x_0 + x_3 = x_1 - x_2 = 0\} \qquad \ell_{18} = \{x_0 - x_3 = x_1 + x_2 = 0\}$$
$$\ell_{19} = \{x_0 + ix_1 = x_2 + ix_3 = 0\} \qquad \ell_{20} = \{x_0 - ix_1 = x_2 - ix_3 = 0\}$$
$$\ell_{21} = \{x_0 + ix_2 = x_1 + ix_3 = 0\} \qquad \ell_{22} = \{x_0 - ix_2 = x_1 - ix_3 = 0\}$$
$$\ell_{23} = \{x_0 + ix_3 = x_1 + ix_2 = 0\} \qquad \ell_{24} = \{x_0 - ix_3 = x_1 - ix_2 = 0\}$$
$$\ell_{25} = \{x_0 - ix_1 = x_2 + ix_3 = 0\} \qquad \ell_{26} = \{x_0 + ix_1 = x_2 - ix_3 = 0\}$$
$$\ell_{27} = \{x_0 + ix_2 = x_1 - ix_3 = 0\} \qquad \ell_{28} = \{x_0 - ix_2 = x_1 + ix_3 = 0\}$$
$$\ell_{29} = \{x_0 + ix_3 = x_1 - ix_2 = 0\} \qquad \ell_{30} = \{x_0 - ix_3 = x_1 + ix_2 = 0\}$$

Note that $\ell_1, \ldots, \ell_{30}$ are irreducible components of the curves $\mathcal{L}_6, \mathcal{L}'_6, \mathcal{L}''_6, \mathcal{L}'''_6, \mathcal{L}''''_6$ which have been introduced in the proof of Proposition 31. One can check that

- for every $k \in \{1, \ldots, 15\}$, the curve $\ell_{2k-1} + \ell_{2k}$ is $\mathbb{H}$-irreducible,
- each line among $\ell_1, \ldots, \ell_{30}$ is contained in 4 quadrics among $\mathcal{Q}_1, \ldots, \mathcal{Q}_{10}$,
- each quadric among $\mathcal{Q}_1, \ldots, \mathcal{Q}_{10}$ contains 12 lines among $\ell_1, \ldots, \ell_{30}$,
- every two quadrics among $\mathcal{Q}_1, \ldots, \mathcal{Q}_{10}$ intersect by 4 lines among $\ell_1, \ldots, \ell_{30}$.

The incidence relation between $\ell_1, \ldots, \ell_{30}$ and $\mathcal{Q}_1, \ldots, \mathcal{Q}_{10}$ is presented in Figure 3.

Now, let us describe the intersection points of the lines $\ell_1, \ldots, \ell_{30}$. To do this, we set

$$\Sigma_4^1 = \{[1:0:0:0], [0:1:0:0], [0:0:1:0], [0:0:0:1]\},$$
$$\Sigma_4^2 = \{[1:1:1:-1], [1:1:-1:1], [1:-1:1:1], [-1:1:1:1]\},$$
$$\Sigma_4^3 = \{[1:1:1:1], [-1:-1:1:1], [1:-1:-1:1], [-1:1:-1:1]\},$$
$$\Sigma_4^4 = \{[0:0:1:1], [1:1:0:0], [0:0:-1:1], [1:-1:0:0]\},$$
$$\Sigma_4^5 = \{[1:0:1:0], [0:1:0:1], [-1:0:1:0], [0:-1:0:1]\},$$
$$\Sigma_4^6 = \{[0:1:1:0], [1:0:0:1], [0:-1:1:0], [-1:0:0:1]\},$$
$$\Sigma_4^7 = \{[i:0:0:1], [0:i:1:0], [-i:0:0:1], [0:-i:1:0]\},$$
$$\Sigma_4^8 = \{[i:0:1:0], [0:i:0:1], [0:-i:0:1], [-i:0:1:0]\},$$
$$\Sigma_4^9 = \{[i:1:0:0], [0:0:i:1], [-i:1:0:0], [0:0:-i:1]\},$$
$$\Sigma_4^{10} = \{[i:i:1:1], [-i:-i:1:1], [i:-i:-1:1], [-i:i:-1:1]\},$$
$$\Sigma_4^{11} = \{[1:i:i:1], [1:-i:-i:1], [-1:-i:i:1], [-1:i:-i:1]\},$$
$$\Sigma_4^{12} = \{[1:i:-i:1], [-1:i:i:1], [-1:-i:-i:1], [1:-i:i:1]\},$$
$$\Sigma_4^{13} = \{[i:1:i:1], [-i:1:-i:1], [-i:-1:i:1], [i:-1:-i:1]\},$$
$$\Sigma_4^{14} = \{[i:1:-i:1], [i:-1:i:1], [-i:-1:-i:1], [-i:1:i:1]\},$$
$$\Sigma_4^{15} = \{[i:i:-1:1], [-i:-i:-1:1], [i:-i:1:1], [-i:i:1:1]\}.$$

Then the subsets $\Sigma_4^1, \ldots, \Sigma_4^{15}$ are $\mathbb{H}$-orbits of length 4. Moreover, one has

$$\Sigma_4^1 \cup \Sigma_4^2 \cup \cdots \cup \Sigma_4^{15} = \mathrm{Sing}(\ell_1 + \ell_2 + \cdots + \ell_{30}).$$

So, for every ℓ_i and ℓ_j such that $\ell_i \neq \ell_j$ and $\ell_i \cap \ell_j \neq \varnothing$, one has $\ell_i \cap \ell_j \in \Sigma_4^1 \cup \Sigma_4^2 \cup \cdots \cup \Sigma_4^{15}$. Furthermore, one can also check that

- every line among $\ell_1, \ldots, \ell_{30}$ contains 6 points in $\Sigma_4^1 \cup \Sigma_4^2 \cup \cdots \cup \Sigma_4^{15}$,

- every point in $\Sigma_4^1 \cup \Sigma_4^2 \cup \cdots \cup \Sigma_4^{15}$ is contained in 3 lines among $\ell_1, \ldots, \ell_{30}$.

As in Remark 11, let $\mathfrak{N}$ be the normalizer of the subgroup $\mathbb{H}$ in the group $\mathrm{PGL}_4(\mathbb{C})$. Then $\mathrm{Aut}(\mathbb{P}^3, \mathscr{C}) \subset \mathfrak{N}$ and $\mathfrak{N} \cong \mathbb{H}.\mathfrak{S}_6$ by Remark 11. Moreover, one can show that

- the group $\mathfrak{N}$ acts transitively on the set $\{\mathcal{Q}_1, \ldots, \mathcal{Q}_{10}\}$,
- the group $\mathfrak{N}$ acts transitively on the set $\{\ell_1, \ldots, \ell_{30}\}$,
- the group $\mathfrak{N}$ acts transitively on the set $\{\Sigma_4^1, \ldots, \Sigma_4^{15}\}$.

Now, we are ready to describe $\mathbb{H}$-orbits in $\mathbb{P}^3$. They can be described as follows:

(1) $\Sigma_4^1, \ldots, \Sigma_4^{15}$ are $\mathbb{H}$-orbits of length 4;
(2) $\mathbb{H}$-orbit of every point in $(\ell_1 \cup \ell_2 \cup \cdots \cup \ell_{30}) \setminus (\Sigma_4^1 \cup \Sigma_4^2 \cup \cdots \cup \Sigma_4^{15})$ has length 8;
(3) $\mathbb{H}$-orbit of every point in $\mathbb{P}^3 \setminus (\ell_1 \cup \ell_2 \cup \cdots \cup \ell_{30})$ has length 16.

Lemma 33 *The surface $\mathscr{S}$ does not contain $\mathbb{H}$-orbits of length* 4.

Proof The assertion follows from [58, Theorem 3], because the $\mathbb{H}$-action on the minimal resolution of the surface $\mathscr{S}$ is symplectic [32, 47]. Instead, we can check this explicitly. Indeed, it is enough to check that $\mathscr{S}$ does not contain Σ_4^1, since the group $\mathfrak{N}$ transitively permutes the orbits $\Sigma_4^1, \ldots, \Sigma_4^{15}$. If $\Sigma_4^1 \subset \mathscr{S}$, then $\mathscr{S}$ is given by (2) with $a = bcd = 0$, which implies that $\mathscr{S}$ has non-isolated singularities. $\qquad\square$

Corollary 34 *Every line among $\ell_1, \ldots, \ell_{30}$ intersects $\mathscr{S}$ transversally by* 4 *points.*

Proof Fix $k \in \{1, \ldots, 15\}$. If $|\ell_{2k-1} \cap \mathscr{S}| < 4$, then the subset $(\ell_{2k-1} \cup \ell_{2k}) \cap \mathscr{S}$ contains an $\mathbb{H}$-orbit of length 4, which contradicts Lemma 33. Therefore, we have $|\ell_{2k-1} \cap \mathscr{S}| = 4$. Similarly, we see that $|\ell_{2k} \cap \mathscr{S}| = 4$. $\qquad\square$

Now, let us prove one result that plays a crucial role in the proof of Theorem 17.

Lemma 35 *Let C be a possibly reducible $\mathbb{H}$-irreducible curve in $\mathbb{P}^3$ such that $\deg(C) < 8$. Then one of the following two possibilities hold*:

(a) *either $C = \ell_{2k-1} + \ell_{2k}$ for some $k \in \{1, \ldots, 15\}$;*
(b) *or C is a union of 4 disjoint lines and $C \subset \mathcal{Q}_i$ for some $i \in \{1, \ldots, 10\}$.*

Proof Intersecting C with quadric surfaces $\mathcal{Q}_1, \ldots, \mathcal{Q}_{10}$, we conclude that $\deg(C)$ is even. This gives $\deg(C) \in \{2, 4, 6\}$.

Suppose that C is reducible. Since $|\mathbb{H}| = 16$, we have the following possibilities:

(i) C is a union of 2 lines,
(ii) C is a union of 4 lines,
(iii) C is a union of 2 irreducible conics,
(iv) C is a union of 3 irreducible conics.

   (v) C is a union of 2 irreducible plane cubics,
  (vi) C is a union of 2 twisted cubics,

Since $\mathbb{P}^3$ does not have $\mathbb{H}$-orbits of length 2 and 3, cases (iii), (iv) and (v) are impossible. Similarly, case (vi) is also impossible, because $\boldsymbol{\mu}_2^3$ cannot faithfully act on a rational curve. Thus, either C is a union of 2 lines, or C is a union of 4 lines.

Suppose that $C = L_1 + L_2$, where L_1 and L_2 are lines. Then $\mathrm{Stab}_{\mathbb{H}}(L_1) \cong \boldsymbol{\mu}_2^3$, and this group cannot act faithfully on $L_1 \cong \mathbb{P}^1$. Therefore, there exists a non-trivial $g \in \mathrm{Stab}_{\mathbb{H}}(L_1)$ such that g pointwise fixes the line L_1. But this means that L_1 is one of the lines $\ell_1, \ldots, \ell_{30}$, so we have $C = \ell_{2k-1} + \ell_{2k}$ for some $k \in \{1, \ldots, 15\}$ as required.

Suppose $C = L_1 + L_2 + L_3 + L_4$, where L_1, L_2, L_3, L_4 are lines. Then $\mathrm{Stab}_{\mathbb{H}}(L_1) \cong \boldsymbol{\mu}_2^2$. Note that $\mathrm{Stab}_{\mathbb{H}}(L_1)$ must act faithfully on L_1, because L_1 is not one of the lines $\ell_1, \ldots, \ell_{30}$. This implies that L_1 does not have $\mathrm{Stab}_{\mathbb{H}}(L_1)$-fixed points, which implies that $\mathbb{P}^3$ also does not have $\mathrm{Stab}_{\mathbb{H}}(L_1)$-fixed points. All subgroups in $\mathbb{H}$ isomorphic to $\boldsymbol{\mu}^2$ with these property are conjugated by the action of the group $\mathfrak{N}$. Thus, we may assume that

$$\mathrm{Stab}_{\mathbb{H}}(L_1) = \langle A_1 A_2, A_3 \rangle.$$

This subgroup leaves invariant rulings of the quadric surface $\mathcal{Q}_8 \cong \mathbb{P}^1 \times \mathbb{P}^1$. To be precise, for every $[\lambda : \mu] \in \mathbb{P}^1$, the group $\langle A_1 A_2, A_3 \rangle$ leaves invariant the line

$$\left\{ \lambda x_0 + \mu x_3 = \lambda x_1 + \mu x_2 = 0 \right\} \subset \mathcal{Q}_8,$$

and these are all $\langle A_1 A_2, A_3 \rangle$-invariant lines in $\mathbb{P}^3$. So, the lines L_1, L_2, L_3, L_4 are disjoint, and all of them are contained in the quadric $\mathcal{Q}_8$. Thus, we are done in this case.

Therefore, to complete the proof of the lemma, we may assume that C is irreducible. Observe that the curve C is not planar, because $\mathbb{P}^3$ does not contain $\mathbb{H}$-invariant planes. Moreover, the curve C is singular: otherwise its genus is $\leqslant 4$ by the Castelnuovo bound, but $\mathbb{H}$ cannot faithfully act on a smooth curve of genus less than 5 by [16, Lemma 3.2]. Therefore, we conclude that $\deg(C) = 6$, since otherwise the curve C would be planar.

We claim that the curve C does not contain $\mathbb{H}$-orbits of length 4. Suppose that it does. Since $\mathfrak{N}$ transitively permutes the orbits $\Sigma_4^1, \ldots, \Sigma_4^{15}$, we may assume that $\Sigma_4^1 \subset C$. Then

$$\Sigma_4^1 \subset \mathrm{Sing}(C),$$

because the stabilizer in $\mathbb{H}$ of a smooth point in C must be a cyclic group [28, Lemma 2.7]. Let $\iota : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ be the standard Cremona involution, which is given by

$$[x_0 : x_1 : x_2 : x_3] \mapsto [x_1 x_2 x_3 : x_0 x_2 x_3 : x_0 x_1 x_3 : x_0 x_1 x_2].$$

Then ι centralizes $\mathbb{H}$. On the other hand, the curve $\iota(C)$ is a conic, because $\deg(C) = 6$, and C is singular at every point of the $\mathbb{H}$-orbit Σ_4^1. But $\mathbb{P}^3$ contains no $\mathbb{H}$-invariant conics, because it contains no $\mathbb{H}$-invariant planes. Thus, C contains no $\mathbb{H}$-orbits of length 4.

Note that $\mathcal{Q}_1 \cap \mathcal{Q}_2 \cap \cdots \cap \mathcal{Q}_{10} = \varnothing$. So, at least one quadric among $\mathcal{Q}_1, \ldots, \mathcal{Q}_{10}$ does not contain the curve C. Without loss of generality, we may assume that $C \not\subset \mathcal{Q}_1$. Then

$$12 = \mathcal{Q}_1 \cdot C \geqslant |\mathcal{Q}_1 \cap C|,$$

which implies that the intersection $\mathcal{Q}_1 \cap C$ is an $\mathbb{H}$-orbit of length 8, because we already proved that C does not contain $\mathbb{H}$-orbits of length 4. For a point $P \in \mathcal{Q}_1 \cap C$, we have

$$12 = \mathcal{Q}_1 \cdot C = |\mathrm{Orb}_{\mathbb{H}}(P)| \big(\mathcal{Q}_1 \cdot C\big)_P = 8\big(\mathcal{Q}_1 \cdot C\big)_P,$$

which is impossible, since 12 is not divisible by 8. □

Corollary 36 *Let $\mathcal{Q}$ be an $\mathbb{H}$-invariant quadric among $\mathcal{Q}_1, \dots, \mathcal{Q}_{10}$, and let $C = \mathscr{S}|_{\mathcal{Q}}$. Then one of the following possibilities holds*:

- C is a smooth irreducible curve of degree 8 and genus 9;
- C is an $\mathbb{H}$-irreducible curve which is a union of two irreducible smooth quartic elliptic curves that intersect each other transversally by an $\mathbb{H}$-orbit of length 8;
- $C = \mathcal{L}_4 + \mathcal{L}_4'$ for $\mathbb{H}$-irreducible curves $\mathcal{L}_4$ and $\mathcal{L}_4'$ consisting of 4 disjoint lines such that the intersection $\mathcal{L}_4 \cap \mathcal{L}_4'$ is an $\mathbb{H}$-orbit of length 16.

Proof If C is not $\mathbb{H}$-irreducible, then Lemma 35 and Corollary 34 imply the assertion. Thus, we may assume that C is $\mathbb{H}$-irreducible.

Suppose that C is reducible. By Lemma 33, C does not contain $\mathbb{H}$-orbits of length 4. Observe also that $\mathrm{Pic}^{\mathbb{H}}(\mathcal{Q}) \cong \mathbb{Z}^2$. Thus, since C is $\mathbb{H}$-irreducible, we conclude that

(i) either C is a union of 4 irreducible conics,

(ii) or C is an $\mathbb{H}$-irreducible curve which is a union of two irreducible smooth quartic elliptic curves that intersect each other transversally by an $\mathbb{H}$-orbit of length 8.

In the latter case, we are done. In the former case, we have $|\mathrm{Sing}(C)| \leqslant 12$, which implies that $\mathrm{Sing}(C)$ is an $\mathbb{H}$-orbit of length 8, which immediately leads to a contradiction.

Thus, we may assume that C is irreducible. The arithmetic genus of the curve C is 9. If C is smooth, we are done. If C is singular, then the genus of it normalization is $\leqslant 1$, because C does not contain $\mathbb{H}$-orbits of length 4 by Lemma 33. But $\mathbb{H}$ cannot faithfully act on a smooth curve of genus less than 5 by [16, Lemma 3.2]. □

Now, we are ready to prove Theorem 17.

Proof of Theorem 17 Let G be a subgroup in $\mathrm{Aut}(X)$ such that $\mathrm{Cl}^G(X) \cong \mathbb{Z}$ and

$$\mathbb{H} \subseteq \upsilon(G),$$

so G contains a subgroup $\mathbb{H}^\rho$ for some homomorphism $\rho : \mathbb{H} \to \boldsymbol{\mu}_2$. We must prove that the threefold X is G-birationally super-rigid. Suppose it is not G-birationally super-rigid. Then there are a positive integer n and a G-invariant linear subsystem $\mathcal{M} \subset |nH|$ such that the linear system $\mathcal{M}$ does not have fixed components, but $(X, \frac{2}{n}\mathcal{M})$ is not canonical.

Starting from this moment, we are going to forget about the group G. In the following, we will work only with its subgroup $\mathbb{H}^\rho$. Recall that $\upsilon(\mathbb{H}^\rho) = \mathbb{H}$.

Let Z be the center of non-canonical singularities of the log pair $(X, \frac{2}{n}\mathcal{M})$ that has maximal dimension. We claim that Z must be a point. Indeed, suppose that Z is a curve. Let M be sufficiently general surface in the linear system $\mathcal{M}$. Then

$$\mathrm{mult}_Z(M) > \frac{n}{2} \tag{37}$$

by [39, Theorem 4.5]. Let us seek for a contradiction.

Let $\mathscr{Z}$ be an $\mathbb{H}^\rho$-irreducible curve in X whose irreducible components is the curve Z, Then, arguing as in the proof of Proposition 30, we see that

$$H \cdot \mathscr{Z} \leqslant 7.$$

In particular, we conclude that $\pi(\mathscr{Z})$ is a $\mathbb{H}$-invariant curve of degree $\leqslant 7$. By Lemma 35, the curve $\pi(Z)$ is a line, and one of the following two possibilities hold:

(a) either $\pi(\mathscr{Z}) = \ell_{2k-1} + \ell_{2k}$ for some $k \in \{1, \ldots, 15\}$;
(b) or $\pi(\mathscr{Z})$ is a union of 4 disjoint lines and $\pi(\mathscr{Z}) \subset \mathcal{Q}_i$ for some $i \in \{1, \ldots, 10\}$.

Let us deal with these two cases separately.

Suppose we are in case (a). Without loss of generality, we may assume $\pi(\mathscr{Z}) = \ell_1 + \ell_2$. Let C_1 and C_2 be the preimages on the threefold X of the lines ℓ_1 and ℓ_2, respectively. Then it follows from Corollary 34 that C_1 and C_2 are smooth irreducible elliptic curves. In particular, the curves C_1 and C_2 are disjoint and

$$\mathscr{Z} = C_1 + C_2.$$

Let $f : \widetilde{X} \to X$ be the blow up of the curves C_1 and C_2, let E_1 and E_2 be the f-exceptional surfaces such that $f(E_1) = C_1$ and $f(E_2) = C_2$, and let $\widetilde{M}$ be the proper transform on the threefold $\widetilde{X}$ of the surface M. Then $|f^*(2H) - E_1 - E_2|$ is base point free, so

$$
\begin{aligned}
0 \leqslant \left(f^*(2H) - E_1 - E_2\right)^2 \cdot \widetilde{M} \\
= \left(f^*(2H) - E_1 - E_2\right)^2 \cdot \left(f^*(nH) - \mathrm{mult}_Z(M)(E_1 + E_2)\right) = 4n - 8\mathrm{mult}_Z(M),
\end{aligned}
$$

which contradicts (37). This shows that case (a) is impossible.

Suppose we are in case (b). Without loss of generality, we may assume that $\pi(\mathscr{Z}) \subset \mathcal{Q}_1$. Let S be the preimage of the quadric surface $\mathcal{Q}_1$ via the double cover π. Then it follows from Corollary 36 that S is an irreducible normal surface such that

(i) either S is a smooth K3 surface,
(ii) or S is a singular K3 surface that has 8 or 16 ordinary double points.

Note that $\mathscr{Z} \subset S$ by construction. Let $\mathcal{C}$ be the preimage in X of a sufficiently general line in the quadric $\mathcal{Q}_1$ that intersect the line $\pi(Z)$. Then $\mathcal{C}$ is a smooth irreducible elliptic curve, which is contained in the smooth locus of the K3 surface S. Observe that $H \cdot \mathcal{C} = 2$. Moreover, we also have $|\mathcal{C} \cap \mathscr{Z}| \geqslant 4$. Thus, since $\mathcal{C} \not\subset \mathrm{Supp}(M)$, we get

$$2n = nH \cdot \mathcal{C} = M \cdot \mathcal{C} \geqslant \sum_{O \in \mathcal{C} \cap \mathscr{Z}} \mathrm{mult}_O(M) \geqslant \mathrm{mult}_Z(M)|\mathcal{C} \cap \mathscr{Z}| \geqslant 4\mathrm{mult}_Z(M),$$

which contradicts (37). This shows that case (b) is also impossible.

Hence, we see that Z is a point. In particular, the pair $(X, \frac{2}{n}\mathcal{M})$ is canonical away from finitely many points. Now, arguing as in the proof of Proposition 30, we get

$$Z \notin \mathrm{Sing}(X).$$

Let M_1 and M_2 be two general surfaces in $\mathcal{M}$. Using [51] or [20, Corollary 3.4], we get

$$\left(M_1 \cdot M_2\right)_Z > n^2. \tag{38}$$

Let $P = \pi(Z)$. Then, arguing as in the proof of Proposition 31, we get $|\mathrm{Orb}_{\mathbb{H}}(P)| \neq 4$.

We claim that $|\mathrm{Orb}_{\mathbb{H}}(P)| \neq 8$. Indeed, suppose $|\mathrm{Orb}_{\mathbb{H}}(P)| = 8$. Then

$$P \in \ell_1 \cup \ell_2 \cup \cdots \cup \ell_{30}.$$

Without loss of generality, we may assume that $P \in \ell_1$. Let C_1 and C_2 be the preimages on the threefold X of the lines ℓ_1 and ℓ_2, respectively. Recall that C_1 and C_2 are smooth irreducible elliptic curves, and the curve $C_1 + C_2$ is $\mathbb{H}^\rho$-irreducible. Write

$$M_1 \cdot M_2 = m(C_1 + C_2) + \Delta,$$

where m is a non-negative integer, and Δ is an effective one-cycle whose support does not contain the curves C_1 and C_2. Then $m \leqslant \frac{n^2}{2}$, because

$$2n^2 = H \cdot M_1 \cdot M_2 = mH \cdot (C_1 + C_2) + H \cdot \Delta \leqslant mH \cdot (C_1 + C_2) = 4m.$$

On the other hand, since C_1 and C_2 are smooth curves, it follows from (38) that

$$\mathrm{mult}_O\left(\Delta\right) > n^2 - m \tag{39}$$

for every point $O \in \mathrm{Orb}_{\mathbb{H}^\rho}(Z)$. Note also that $Z \in C_1$ and $|\mathrm{Orb}_{\mathbb{H}^\rho}(Z)| \geqslant 8$.

Let $\mathcal{D}$ be the linear subsystem in $|2H|$ that consists of surfaces passing through $C_1 \cup C_2$. Then, as we already implicitly mentioned, the linear system $\mathcal{D}$ does not have base curves except for C_1 and C_2. Therefore, if D is a general surface in $\mathcal{D}$, then D does not contain irreducible components of the one-cycle Δ, so (39) gives

$$4n^2 - 8m = D \cdot \Delta \geqslant \sum_{O \in \mathrm{Orb}_{\mathbb{H}^\rho}(Z)} \mathrm{mult}_O\left(\Delta\right) =$$
$$= |\mathrm{Orb}_{\mathbb{H}^\rho}(Z)| \mathrm{mult}_Z\left(\Delta\right) > |\mathrm{Orb}_{\mathbb{H}^\rho}(Z)|(n^2 - m) \geqslant 8(n^2 - m),$$

which is absurd. This shows that $|\mathrm{Orb}_{\mathbb{H}}(P)| \neq 8$.

In particular, we see that $|\mathrm{Orb}_{\mathbb{H}^\rho}(Z)| = |\mathrm{Orb}_{\mathbb{H}}(P)| = 16$ and $P \notin \ell_1 \cup \ell_2 \cup \cdots \cup \ell_{30}$.

We claim that $P \notin \mathcal{Q}_1 \cup \mathcal{Q}_2 \cup \cdots \cup \mathcal{Q}_{10}$. Indeed, suppose that $P \in \mathcal{Q}_1 \cup \mathcal{Q}_2 \cup \cdots \cup \mathcal{Q}_{10}$. Without loss of generality, we may assume that

$$\pi(Z) = P \in \mathcal{Q}_1.$$

As above, denote by S the preimage of the quadric surface $\mathcal{Q}_1$ via the double cover π. Then S is a K3 surface with at most ordinary double singularities, and it follows from the inversion of adjunction [39, Theorem 5.50] that $(S, \frac{2}{n}\mathcal{M}|_S)$ is not log canonical at Z. Let λ be the largest rational number such that $(S, \lambda\mathcal{M}|_S)$ is log canonical at Z. Then

$$\mathrm{Orb}_{\mathbb{H}^\rho}(Z) \subseteq \mathrm{Nklt}\left(S, \lambda\mathcal{M}|_S\right).$$

Note that the locus $\mathrm{Nklt}(S, \lambda\mathcal{M}|_S)$ is $\mathbb{H}^\rho$-invariant, because $\mathcal{M}$ and S are $\mathbb{H}^\rho$-invariant.

Suppose $\mathrm{Nklt}(S, \lambda\mathcal{M}|_S)$ contains an $\mathbb{H}^\rho$-irreducible curve C that passes through Z. This means that $\lambda\mathcal{M}|_S = C + \Omega$, where Ω is an effective $\mathbb{Q}$-linear system on S. Then

$$H \cdot C \leqslant H \cdot (C + \Omega) = 4n\lambda < 8,$$

hence $\pi(C)$ is a union of 4 disjoint lines in $\mathcal{Q}_1$ by Lemma 35, since $P \notin \ell_1 \cup \ell_2 \cup \cdots \cup \ell_{30}$. Let $\mathcal{C}$ be the preimage in X of a general line in $\mathcal{Q}_1$ that intersect $\pi(C)$. Then

$$4 \leqslant \mathcal{C} \cdot C \leqslant \mathcal{C} \cdot (C + \Omega) = \lambda n (H \cdot \mathcal{C}) = 2\lambda n < 4,$$

which is absurd. So, the locus $\mathrm{Nklt}(S, \lambda\mathcal{M}|_S)$ contains no curves that pass through Z.

Let $\mathcal{I}_S$ be the multiplier ideal sheaf of the pair $(S, \lambda\mathcal{M}|_S)$, let $\mathcal{L}_S$ be the corresponding subscheme in S. Then

$$\mathrm{Supp}(\mathcal{L}_S) = \mathrm{Nklt}(S, \lambda\mathcal{M}|_S).$$

Now, applying Nadel's vanishing theorem [45, Theorem 9.4.8], we get

$$h^1(S, \mathcal{I}_S \otimes \mathcal{O}_S(2H|_S)) = 0.$$

Now, using the Riemann–Roch theorem and Serre's vanishing, we obtain

$$10 = h^0(S, \mathcal{O}_S(2H|_S)) \geqslant h^0(\mathcal{O}_{\mathcal{L}_S} \otimes \mathcal{O}_S(2H|_S)) \geqslant |\mathrm{Orb}_{\mathbb{H}^\rho}(Z)|.$$

because $\mathcal{L}_S$ has at least $|\mathrm{Orb}_{\mathbb{H}^\rho}(Z)|$ disjoint zero-dimensional components, whose supports are points in $\mathrm{Orb}_{\mathbb{H}^\rho}(Z)$, because $\mathrm{Orb}_{\mathbb{H}^\rho}(Z) \subseteq \mathrm{Nklt}(S, \lambda\mathcal{M}|_S)$, and $\mathrm{Nklt}(S, \lambda\mathcal{M}|_S)$ does not contain curves that are not disjoint from $\mathrm{Orb}_{\mathbb{H}^\rho}(Z)$. Hence, we see that $|\mathrm{Orb}_{\mathbb{H}^\rho}(Z)| \leqslant 10$, which is impossible, since $|\mathrm{Orb}_{\mathbb{H}^\rho}(Z)| = 16$. This shows that

$$\pi(Z) = P \notin \mathcal{Q}_1 \cup \mathcal{Q}_2 \cup \cdots \cup \mathcal{Q}_{10}.$$

Let us summarize what we proved so far. Recall that $\mathcal{M}$ is a mobile $\mathbb{H}^\rho$-invariant linear subsystem in $|nH|$, the log pair $(X, \frac{2}{n}\mathcal{M})$ is canonical away from finitely many points, but the singularities of the pair $(X, \frac{2}{n}\mathcal{M})$ are not canonical at the point $Z \in X$ such that

- $Z \notin \mathrm{Sing}(X)$,
- $\pi(Z) \notin \ell_1 \cup \ell_2 \cup \cdots \cup \ell_{30}$,
- $\pi(Z) \notin \mathcal{Q}_1 \cup \mathcal{Q}_2 \cup \cdots \cup \mathcal{Q}_{10}$,
- $|\mathrm{Orb}_{\mathbb{H}^\rho}(Z)| = 16$,
- $|\mathrm{Orb}_{\mathbb{H}}(\pi(Z))| = 16$.

By Lemma 35, $\pi(Z)$ is not contained in any $\mathbb{H}$-invariant curve whose degree is at most 7. Let us use this and Nadel's vanishing [45, Theorem 9.4.8] to derive a contradiction.

As in the proofs of Propositions 30 and 31, we observe that $(X, \frac{3}{n}\mathcal{M})$ is not log canonical at the point Z, because X is smooth at Z. Let μ be the largest rational number such that the log pair $(X, \mu\mathcal{M})$ is log canonical at Z. Then $\mu < \frac{3}{n}$ and

$$\mathrm{Orb}_{\mathbb{H}^\rho}(Z) \subseteq \mathrm{Nklt}(X, \mu\mathcal{M}).$$

Moreover, if the locus $\mathrm{Nklt}(X, \mu\mathcal{M})$ contains an $\mathbb{H}^\rho$-irreducible curve C, then arguing as in the proof of Proposition 30, we see that

$$\deg(\pi(C)) \leqslant H \cdot C \leqslant 4,$$

which implies that the curve C does not pass through Z. Hence, we conclude that every point of the orbit $\mathrm{Orb}_{\mathbb{H}^\rho}(Z)$ is an isolated irreducible component of the locus $\mathrm{Nklt}(X, \mu\mathcal{M})$.

Let $\mathcal{I}$ be the multiplier ideal sheaf of the pair $(X, \mu\mathcal{M})$, and let $\mathcal{L}$ be the corresponding subscheme in X. Then

$$\mathrm{Supp}(\mathcal{L}) = \mathrm{Nklt}(X, \mu\mathcal{M}),$$

so the subscheme $\mathcal{L}$ contains at least $|\mathrm{Orb}_{\mathbb{H}^\rho}(Z)| = 16$ zero-dimensional components whose supports are points in the orbit $\mathrm{Orb}_{\mathbb{H}^\rho}(Z)$. On the other hand, we have

$$h^1(X, \mathcal{I} \otimes \mathcal{O}_X(H)) = 0$$

by Nadel's vanishing theorem [45, Theorem 9.4.8]. This gives

$$4 = h^0(X, \mathcal{O}_X(H)) \geqslant h^0(\mathcal{O}_{\mathcal{L}} \otimes \mathcal{O}_X(H)) \geqslant |\mathrm{Orb}_{\mathbb{H}^\rho}(Z)| = 16,$$

which is absurd. The obtained contradiction completes the proof of Theorem 17. □

Let us conclude this paper with one application of Theorem 17, which was the initial motivation for this paper—we were looking for various embeddings $\mu_2^4 \rtimes \mu_3 \hookrightarrow \mathrm{Bir}(\mathbb{P}^3)$.

Example 40 (cf. Examples 5, 8, 26) Let $G_{48,50}$ be the subgroup in $\mathrm{PGL}_4(\mathbb{C})$ generated by

$$A_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, A_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, A_5 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then one can check that $G_{48,50} \cong \mu_2^4 \rtimes \mu_3$ and the GAP ID of the group $G_{48,50}$ is [48,50]. For every $t \in \mathbb{C} \setminus \{\pm 1, \pm\sqrt{3}i\}$, let S_t be the quartic surface in $\mathbb{P}^3$ given by the equation (6), i.e. the surface S_t is the quartic surface in $\mathbb{P}^3$ given by

$$x_0^4 + x_1^4 + x_2^4 + x_3^4 - (t^2 + 1)(x_0^2 x_1^2 + x_2^2 x_3^2$$
$$+ x_0^2 x_2^2 + x_1^2 x_3^2 + x_0^2 x_3^2 + x_1^2 x_2^2) + 2(t^3 + 3t)x_0 x_1 x_2 x_3 = 0.$$

Then S_t is $G_{48,50}$-invariant, and S_t has 16 ordinary double singularities (see Example 5). Now, let X_t be the hypersurface in $\mathbb{P}(1, 1, 1, 1, 2)$ that is given by

$$w^2 = x_0^4 + x_1^4 + x_2^4 + x_3^4 - (t^2 + 1)(x_0^2 x_1^2 + x_2^2 x_3^2$$
$$+ x_0^2 x_2^2 + x_1^2 x_3^2 + x_0^2 x_3^2 + x_1^2 x_2^2) + 2(t^3 + 3t)x_0 x_1 x_2 x_3,$$

where we consider x_0, x_1, x_2, x_3 as homogeneous coordinates on $\mathbb{P}(1, 1, 1, 1, 2)$ of weight 1, and w is a coordinate of weight 2. Consider the faithful action $G_{48,50} \curvearrowright X_t$ given by

$$A_1 : [x_0 : x_1 : x_2 : x_3 : w] \mapsto [-x_0 : x_1 : -x_2 : x_3 : w],$$
$$A_2 : [x_0 : x_1 : x_2 : x_3 : w] \mapsto [-x_0 : x_1 : -x_2 : x_3 : w],$$
$$A_3 : [x_0 : x_1 : x_2 : x_3 : w] \mapsto [x_1 : x_2 : x_3 : x_2 : w],$$
$$A_4 : [x_0 : x_1 : x_2 : x_3 : w] \mapsto [x_3 : x_2 : x_1 : x_0 : w],$$
$$A_5 : [x_0 : x_1 : x_2 : x_3 : w] \mapsto [x_1 : x_2 : x_0 : x_3 : w].$$

Since the threefold X_t is $G_{48,50}$-invariant, this gives an embedding $G_{48,50} \hookrightarrow \mathrm{Aut}(X_t)$. Then it follows from Theorem 17 that the threefold X_t is $G_{48,50}$-birationally super-rigid. In particular, for any $t_1 \neq t_2$ in $\mathbb{C} \setminus \{\pm 1, \pm\sqrt{3}i\}$, the following conditions are equivalent:

- the threefolds X_{t_1} and X_{t_2} are $G_{48,50}$-birational;
- the surfaces S_{t_1} and S_{t_2} are projectively equivalent.

Recall that X_t is rational. For $t \in \mathbb{C} \setminus \{\pm 1, \pm\sqrt{3}i\}$, fix a birational map $\chi_t : \mathbb{P}^3 \dashrightarrow X_t$, and consider the monomorphism $\eta_L : G_{48,50} \hookrightarrow \mathrm{Bir}(\mathbb{P}^3)$ that is given by $g \mapsto \chi_t^{-1} \circ g \circ \chi_t$. Then, for any $t_1 \neq t_2$ in $\mathbb{C} \setminus \{\pm 1, \pm\sqrt{3}i\}$, we have the following assertion:

$$\eta_{t_1}(G_{48,50}) \text{ and } \eta_{t_1}(G_{48,50}) \text{ are conjugate in } \mathrm{Bir}(\mathbb{P}^3) \iff X_{t_1} \text{ and } X_{t_2} \text{ are } G_{48,50} - birational.$$

Thus, if $t_1 \neq t_2$ are general, then $\eta_{t_1}(G_{48,50})$ and $\eta_{t_1}(G_{48,50})$ are not conjugate in $\mathrm{Bir}(\mathbb{P}^3)$. Similarly, we see that $\eta_t(G_{48,50})$ is not conjugate in $\mathrm{Bir}(\mathbb{P}^3)$ to the group $G_{48,50} \subset \mathrm{PGL}_4(\mathbb{C})$, which also follows from [16]. Can we show this using other obstructions [6, 34, 41]?

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