Basis log canonical thresholds, local intersection estimates, and asymptotically log del Pezzo surfaces

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Abstract

The purpose of this article is to develop techniques for estimating basis log canonical thresholds on logarithmic surfaces. To that end, we develop new local intersection estimates that imply log canonicity. Our main motivation and application is to show the existence of Kähler–Einstein edge metrics on all but finitely many families of asymptotically log del Pezzo surfaces, partially confirming a conjecture of two of us. In an appendix we show that the basis log canonical threshold of Fujita–Odaka coincides with the greatest lower Ricci bound invariant of Tian.

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1 Introduction

1.1 Estimating basis log canonical thresholds

Global and local log canonical thresholds naturally play a crucial rôle in algebraic geometry. For instance, Shokurov's conjecture [59] on the ascending chain condition

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for local log canonical thresholds (proved in [41]) implies the inductive step in termination of higher-dimensional log flips [2]. Likewise, Birkar's boundedness results [3] for global log canonical threshold imply the Borisov–Alexeev–Borisov conjecture in all dimensions: the set of Fano varieties of dimension d with ϵ-log canonical singularities forms a bounded family for given $d \in \mathbb{N}$ and $\epsilon > 0$. This conjecture implies that the birational automorphism group of any rationally connected variety is Jordan [53], so that, in particular, all Cremona groups are Jordan (in dimension 2 this was proved by Serre in [58]). Global log canonical thresholds are used to prove irrationality of Fano varieties [11,54] the absence of non-trivial fiber-wise birational maps between Mori fiber spaces [10,13,17], the uniqueness of a Kollár component of a Kawamata log terminal singularity [19,26,27], and non-conjugacy of finite subgroups in Cremona groups [12], Moreover, a combination of results about global and local log canonical thresholds of del Pezzo surfaces helped to answer an old standing open question in affine geometry [21], and make a first step towards Gizatullin's conjecture about automorphisms of the affine complements to Fano hypersurfaces [15].

About a decade ago, it was realized [24] that global log canonical thresholds (glcts) are the algebraic analogues of Tian's alpha invariants [62] that are central in the study of Kähler–Einstein (KE) metrics. This paved the way to using a wide range of algebraic tools to prove existence of KE metrics via Tian's theorem that stipulates that an estimate on Tian's invariant guarantees the existence of such a metric [11–13,16,18,20,25]. While this has been arguably the most fruitful method for finding new KE metrics, a sticking point with this approach has been that Tian's theorem only provides a sufficient condition for the existence of KE metrics. Recently, this has been remedied by Fujita–Odaka [39] that introduced a new invariant, that we refer to as the basis log canonical threshold (blct) reminiscent of the global log canonical threshold (the blct has also been referred to as the delta invariant and the stability threshold in the literature, see Definition 2.5 below for detailed references). The advantage of estimating this modified threshold is that it provides a necessary and sufficient condition for K-stability, which in turn is equivalent to the existence of KE metrics [28,65]. In fact, in an appendix we show that the algebraic invariant blct coincides with an analytic invariant studied by Tian almost thirty years ago, namely, the greatest Ricci lower bound invariant.

However, while there are many techniques for estimating glcts, at the moment rather little is known about how to actually estimate blcts. Recently, an important first step in this direction was taken by Park–Won [52] who developed algebraic methods for estimating blcts in dimension 2. First, they explicitly computed blct of $\mathbb{P}^2$ similar to what was done later for all toric Fano varieties by Blum–Jonsson [4]. Then Park and Won used the key fact that every two-dimensional Fano variety different from $\mathbb{P}^1 \times \mathbb{P}^1$ can be (non-canonically) obtained from $\mathbb{P}^2$ by blowing up ≤ 8 points in general position. This allowed them to use their toric computations to estimate blcts from above. The Park–Won approach is quite different from computations of glcts in the literature [12] where all estimates are done using only intrinsic geometry of the surface. Moreover, it is not clear how to adapt toric-type computations as in [52] to our logarithmic setting which also includes a boundary divisor. Finally, it is also not clear how to adapt their method to higher dimensions, simply because blow-ups of projective space or other higher-dimensional toric variety very rarely have ample

anticanonical class. While we do not tackle this here, we believe that it should be possible to extend our method to dimension three.

In this article we develop intrinsic techniques for estimating blcts in dimension 2. The methods we develop are of a geometric nature and involve new criteria for log canonicity in terms of local intersection numbers, that we believe are of independent interest. As we show in a sequel, in the setting considered by Park–Won our methods yield stronger estimates with perhaps more geometric proofs. Moreover, in this article we use our new log canonicity criteria, coupled with vanishing order estimates for basis divisors to prove the log K-stability of an important family of logarithmic surfaces. This partially resolves a conjecture of two of us that we now turn to describe.

1.2 Kähler–Einstein edge metrics

Smooth Kähler–Einstein (KE) metrics have been studied for over 80 years, with intriguing relations to algebraic geometry emerging over the last 30 years. More recently, motivated by suggestions of Tsuji, Tian, and Donaldson, singular Kähler–Einstein metrics called Kähler–Einstein edge (KEE) metrics have been intensely studied, mainly as a tool for understanding smooth Kähler–Einstein metrics. KEE metrics are a natural generalization of KE metrics: they are smooth metrics on the complement of a divisor, and have a conical singularity of angle $2\pi\beta$ transverse to that 'complex edge' (i.e., the metric as being 'bent' at an angle $2\pi\beta$ along the divisor). They tie naturally to the study of log pairs in algebraic geometry. When $\beta = 1$, of course, a KEE metric is just an ordinary KE metric that extends smoothly across the divisor, and so understanding existence of KEE metrics as well as their asymptotics near the divisor [42] as well as the limit $\beta \to 1$ [28,65] has attracted much work; we refer to the survey [57] for a thorough discussion and many more references.

In [22], two of us initiated a program whose aim is to understand the behavior in the other extreme when the *cone angle β goes to zero* consisting of:

(a) Classifying all triples (X, D, β) satisfying the necessary cohomological condition (1.1) for *sufficiently small β*;
(b) Obtaining a condition equivalent to existence of KEE metrics for such triples;
(c) Understanding the limit, when such exists, of these KEE metrics as β tends to zero.

The cohomological condition alluded to in (a),

$$- K_X - (1 - \beta)D \text{ is } \mu \text{ times an ample class, for some } \mu \in \mathbb{R}, \qquad (1.1)$$

is also the necessary and sufficient condition for (b) if $\mu \leq 0$ [42, Theorem 2]; moreover, a classification (i.e., part (a)) is essentially impossible when $\mu \leq 0$ [33], [57, §8], and so we will restrict our attention in (a)–(b) exclusively to the case $\mu > 0$, that we have previously called the *asymptotically log Fano regime* [22, Definition 1.1].

Our previous work accomplished (a) in dimension 2, providing a complete classification [22, Theorem 2.1]. Furthermore, we also obtained the "necessary" portion of (b) [22,23], and this was extended to higher dimensions by Fujita [35]. The purpose

of this article is to complete the "sufficient" portion of (b) in dimension 2 in all but finitely many (in fact, all but 6) of the (infinite list of) cases classified in [22].

1.3 The Calabi problem for asymptotically log Fano varieties

A special class of asymptotically log Fano varieties is as follows. This is a special case of [22, Definition 1.1].

Definition 1.1 We say that a pair (X, D) consisting of a smooth projective variety X and a smooth irreducible divisor D on X is *asymptotically log Fano* if the divisor $-K_X - (1 - \beta)D$ is ample for sufficiently small $\beta \in (0, 1]$.

This definition contains the class of smooth Fano varieties ($D = 0$) as well as the classical notion of a smooth log Fano pair due to Maeda ($\beta = 0$) [50].

One can show using a result of Kawamata–Shokurov that if (X, D) is asymptotically log Fano then $|k(K_X + D)|$ (for some $k \in \mathbb{N}$) is free from base points and gives a morphism [22, §1]

$$\eta \colon X \to Z.$$

The following conjecture, posed in our earlier work, gives a rather complete picture concerning (b) when D is smooth.

Conjecture 1.2 [22, Conjecture 1.11] *Suppose that (X, D) is an asymptotically log Fano manifold with D smooth and irreducible. There exist KEE metrics with angle $2\pi\beta$ along D for all sufficiently small β if and only if η is not birational.*

This conjecture stipulates that the existence problem for KEE metrics in the small angle regime boils down to a simple birationality criterion. In fact, this amounts to computing a single intersection number, i.e., checking whether

$$(K_X + D)^n = 0.$$

This would be a rather far-reaching simplification as compared to checking the much harder condition of log K-stability that involves, in theory, computing the Futaki invariant of an infinite number of log test configurations, or else estimating the blct which involves, in theory, estimation of singularities of pairs that may occur after an unbounded number of blow-ups.

1.4 The Calabi problem for asymptotically log del Pezzo surfaces

Following [22,57] we will refer to understanding (b) as the Calabi problem for asymptotically log Fano varieties. This article makes an important step towards solving this problem in dimension 2, where Fano varieties are commonly called del Pezzo surfaces. To explain this, let us recall what is already known about this problem from our previous work.

1.4.1 The big case

The necessary direction of Conjecture 1.2 in dimension 2 is known as we now recall.

According to [22, Theorems 1.4, 2.1] asymptotically log del Pezzo pairs (X, D) (with D smooth and irreducible) for which $-K_X - D$ is big are as follows. Either (X, D) is one of the five Maeda pairs (i.e., with $-K_X - D$ ample):

- (I.1B) := $(\mathbb{P}^2$, smooth conic),
- (I.1C) := $(\mathbb{P}^2$, line),
- (I.2.n) := $(\mathbb{F}_n, Z_n)$ (where, for any $n \geq 0$, $\mathbb{F}_n$ is the Hirzebruch surface containing a curve Z_n whose self intersection is $-n$ and fiber F whose self intersection is 0),
- (I.3B) := $(\mathbb{F}_1$, smooth element of $|Z_1 + F|)$,
- (I.4C) := $(\mathbb{P}^1 \times \mathbb{P}^1$, smooth bi-degree $(1, 1)$ curve),

or else (X, D) is obtained from one of the five Maeda surfaces (X_M, D_M) as follows: X is the blow-up of X_M at any number of distinct points on D_M, and D is the proper transform of D_M.

According to Conjecture 1.2 none of these pairs should admit KEE metrics with small angles. This was verified in a unified manner using flop slope stability [23, Theorem 1.6], but can also be obtained as follows: for (I.1B) and (I.4C) [49, Example 3.12], for (I.1C) := $(\mathbb{P}^2$, line) and (I.2.n) as well as their blow-ups this is a consequence of the Matsushima theorem for edge metrics [22, Theorem 1.12, Proposition 7.1], for (I.3B) [23, Example 2.8], for the blow-ups of (I.1B), (I.3B) and (I.4C) [23, Proposition 5.2].

1.4.2 The non-big case

The harder sufficient direction of Conjecture 1.2 in dimension 2 is still partly open. Let us recall the state-of-the-art.

According to [22, Theorems 1.4, 2.1] asymptotically log del Pezzo pairs (X, D) (with D smooth) for which $(K_X + D)^2 = 0$ are as follows:

- (X, D) is del Pezzo with D a smooth anti-canonical curve,
- (I.3A) := $(\mathbb{F}_1$, smooth element of $|2(Z_1 + F)|)$,
- (I.4B) := $(\mathbb{P}^1 \times \mathbb{P}^1$, smooth bi-degree $(2, 1)$ curve),
- (I.9B.m) := (X, D) with X a blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at m distinct points on a smooth bi-degree $(2, 1)$ curve with no two of them on a single curve of bi-degree $(0, 1)$, and D is the proper transform of the bi-degree $(2, 1)$ curve.

According to Conjecture 1.2 all of these pairs should admit KEE metrics with small angles. The following cases are known: del Pezzo with a smooth anti-canonical curve [42, Corollary 1], (I.3A) [22, Proposition 7.5], (I.4B) [22, Proposition 7.4]. Thus, the only remaining cases are (I.9B.m), $m \geq 1$.

1.5 Main result

In this article, we treat all but finitely many of the remaining open cases (I.9B.m):

Theorem 1.3 *The log Fano pairs* (I.9B.m), $m \geq 7$ *are uniformly log K-stable for all sufficiently small* $\beta > 0$.

Indeed, recent results show that log K-stability implies the existence of a KEE metric on a given log Fano pair (the stability implies existence of weak solutions results of [1,66] combined with the regularity results of [40,42,51]). The finitely-many remaining cases (I.9B.m), $1 \leq m \leq 6$ require a different approach and will be discussed elsewhere (although we omit the details, the techniques of this article can also be used to show the cases (I.9B.5) and (I.9B.6) are log K-semistable).

In the course of the proof we develop new local intersection criteria for showing log-canonicity on a surface—see Sect. 3. We believe these are of substantial interest independently of their application to proving Theorem 1.3. Sections 4–5 are concerned with the proof of Theorem 1.3. In Sect. 4 we estimate the vanishing order of divisors on the logarithmic surfaces (I.9B.m), while in Sect. 5 we use these estimates together with the criteria of Sect. 3 to estimate the basis log canonical threshold of the logarithmic surfaces (I.9B.m). The article concludes with an appendix that identifies the basis log canonical threshold of Fano manifolds with Tian's greatest Ricci lower bound. After this paper was first posted, we were informed that Berman–Boucksom–Jonsson also obtained Theorem 5.7 independently and that Blum–Liu obtained a variant of Lemma 5.8 [5].

2 Preliminaries

In this section—except in the last lemma where we specialize to surfaces $(n = 2)$—we let X be a complex algebraic variety of complex dimension n.

2.1 Log pairs

Given a proper birational morphism $\pi : Y \to X$, we define the exceptional set of π to be the smallest subset $\mathrm{exc}(\pi) \subset Y$, such that $\pi : Y \setminus \mathrm{exc}(\pi) \to X \setminus \pi(\mathrm{exc}(\pi))$ is an isomorphism.

A log resolution of (X, Δ) is a proper birational morphism $\pi : Y \to X$ such that $\pi^{-1}(\Delta) \cup \{\mathrm{exc}(\pi)\}$ is divisor with simple normal crossing (snc) support in a smooth Y. Log resolutions exist for all the pairs we will consider in this article, by Hironaka's theorem.

Assume that $K_X + \Delta$ is a $\mathbb{Q}$-Cartier divisor. Given a log resolution of (X, Δ), write

$$\pi^{\star}(K_X + \Delta) = K_Y + \tilde{\Delta} + \sum e_i E_i,$$

where $\tilde{\Delta}$ denotes the proper transform of Δ, and where $\mathrm{exc}(\pi) = \cup E_i$, and E_i are irreducible codimension one subvarieties. Also, assume $\Delta = \sum \delta_i \Delta_i$, with Δ_i irreducible codimension one subvarieties, so $\tilde{\Delta} = \sum \delta_i \tilde{\Delta}_i$. Singularities of pairs can be measured as follows.

Definition 2.1 Let $Z \subset X$ be a subvariety. A pair (X, Δ) has at most log canonical (lc) singularities (or klt singularities, respectively) along Z if $e_i, \delta_j \leq 1$ for every i

(or if e_i, $\delta_j < 1$ for every i, respectively) such that $\pi(E_i) \cap Z \neq \emptyset$ and every j such that $\Delta_j \cap Z \neq \emptyset$.

On a normal variety, an effective $\mathbb{Q}$-divisor D is a formal linear combination with coefficients in $\mathbb{Q}_+$ of prime divisors. Thus, given such a D and a prime divisor F, one has $D = aF + \Delta$, for some $a \in \mathbb{Q}_+$ and Δ is an effective $\mathbb{Q}$-divisor with $F \not\subset \mathrm{supp}\Delta$. The number a is called the *vanishing order of D along F*, denoted

$$\mathrm{ord}_F D.$$

2.2 Log canonical thresholds

Definition 2.2 Let $Z \subset X$ be a subvariety and let Δ be Cartier $\mathbb{Q}$-divisor on X. The log canonical threshold of the pair (X, Δ) along Z is

$$\mathrm{lct}_Z(X, \Delta) := \sup\{\lambda \,:\, (X, \lambda\Delta) \text{ is log-canonical along } Z\}.$$

Set $\mathrm{lct}(X, \Delta) := \mathrm{lct}_X(X, \Delta)$.

Let (X, B) be a klt log pair. Let D be an effective $\mathbb{Q}$-Cartier divisor on the variety X. Recall that the log canonical threshold of the boundary D is the number

$$\mathrm{lct}(X, B; D) = \sup\{c \,:\, \text{the pair } (X, B + cD) \text{ is log canonical}\}.$$

Let H be an ample $\mathbb{Q}$-divisor on X, and let $[H]$ be the class of the divisor H in $\mathrm{Pic}(X) \otimes \mathbb{Q}$.

A fact we will use over and over again is that the property of being lc or klt is preserved under blow-ups, and therefore can be checked either upstairs on Y or downstairs on X [43, Lemma 3.10]. When $n = 2$ this becomes quite concrete: let $\pi : \tilde{S} \to S$ be the blow up of a point $p \in S$ and let $E := \pi^{-1}(p)$. Denote by $\tilde{\Delta}$ the proper transform of Δ under π. Then the log pair (S, Δ) is lc/klt at p if and only if $(\tilde{S}, \tilde{\Delta} + (\mathrm{mult}_P \Delta - 1)E)$ is lc/klt along E [12, Remark 2.6].

Definition 2.3 Let B be an effective divisor in X. The global log canonical threshold of the pair (X, B) with respect to $[H]$ is

$$\mathrm{glct}(X, B, [H]) := \sup\{c > 0 \,:\, (X, B + cD) \text{ is lc for every } D \sim_{\mathbb{Q}} H\}.$$

2.3 The basis log canonical threshold

In this part we collect some known results about a basis-type invariant for log pairs due to Fujita–Odaka [39], see also [4,29].

Let L be an ample $\mathbb{Q}$-divisor in X. For any $k \in \mathbb{N}$ such that kL is Cartier, let

$$d_k := \dim_{\mathbb{C}} H^0(X, kL) > 0.$$

In this article, whenever we mention multiples kL of L
we will always assume (implicitly) that k is such an integer.

Definition 2.4 We say that $D \sim_{\mathbb{Q}} L$ is a basis divisor if for some $k \in \mathbb{N}$,

$$D = \frac{1}{kd_k} \sum_{i=0}^{d_k} (s_i),$$

where $s_1, \ldots, s_{d_k}$ is a basis of $H^0(X, kL)$, and where (s_i) is the divisor cut out by s_i. We also say that D is the k-basis divisor associated to the basis $\{s_i\}_{i=1}^{d_k}$.

The following definition is due to Fujita–Odaka [39, Definition 0.2], extended to the logarithmic setting by Fujita [38, Definition 5.4] (Fujita's definition can be shown to equal a logarithmic version of the original definition of Fujita–Odaka, see [4,29]) who denoted it $\delta(X, B)$, and Codogni–Patakfalvi [29, Definition 4.3] who denoted it $\delta(X, B; L)$. It roughly amounts to replacing "D effective $\mathbb{Q}$-divisor" by "D basis divisor" in Definition 2.3. So, it yields an invariant larger than glct, albeit one that is significantly more difficult to compute.

Definition 2.5 Let (X, B) be a klt log pair. The basis log canonical threshold of the pair (X, B) with respect to L is

$$\mathrm{blct}_\infty(X, B, L) := \limsup_k \mathrm{blct}_k(X, B, L),$$

where $\mathrm{blct}_k(X, B, L) := \sup\{c > 0 : (X, B+cD) \text{ is lc for any } k\text{-basis divisor } D \sim_{\mathbb{Q}} L\}$.

Estimating the invariant $\mathrm{blct}_\infty(X, B, L)$ is of interest since it coincides with an analytic invariant related to Ricci curvature (see Theorem 5.7 below) and also due to the following theorem that follows from the work of Fujita–Odaka [39], Fujita [37], Li [48], Blum–Jonsson [4] (see also [29, Corollary 4.8]). For the precise definition of uniform log K-stability, we refer the reader to [7, Definition 8.1].

Theorem 2.6 *The triple* $(X, \Delta, -K_X - \Delta)$ *is uniformly log K-stable if* $\mathrm{blct}_\infty(X, \Delta, -K_X - \Delta) > 1$.

In fact, the converse also holds, due to the references cited above, but we only need the somewhat easier direction stated.

2.4 Volume estimates on the order of vanishing

In most of this subsection we follow closely [39]. Estimating log canonical thresholds naturally involves estimating from above orders of vanishing along divisors, oftentimes upstairs on a resolution of the original log pair (recall Sect. 2.2). A crude upper bound on $\mathrm{ord}_F(\pi^\star D)$ is the pseudoeffective threshold of the divisor $\pi^\star L$ with respect to the curve F,

$$\tau(\pi^\star L, F) = \sup\{\lambda : \pi^\star L - \lambda F \text{ is effective}\}, \tag{2.1}$$

since $D \sim_{\mathbb{Q}} L$ and $\pi^{\star}D = \mathrm{ord}_F(\pi^{\star}D)F + \Delta$, with Δ an effective $\mathbb{Q}$-divisor, whose support does not contain the curve F. A better estimate can be obtained by using a quantized version of the pseudoeffective threshold,

$$\tau_k(\pi^{\star}L, F) := \max\{x \in \mathbb{N} \ : \ H^0(Y, k\pi^{\star}L - xF) \neq 0\}.$$

When no confusion arises we will often abbreviate these two invariants by τ and τ_k. Note that,

$$\limsup_k \tau_k(\pi^{\star}L, F)/k = \tau(\pi^{\star}L, F), \tag{2.2}$$

as trivially $\tau_k/k \leq \tau$ for every k (let $s \in H^0(Y, k\pi^{\star}L - xF)$, then $(s)/k \sim_{\mathbb{Q}} \pi^{\star}L - \frac{x}{k}F$ is effective), while if $\pi^{\star}L - \frac{x}{F}$ is big, then $H^0(Y, i_k\pi^{\star}L - i_kxF) \neq 0$ for a increasing sequence of integers $\{i_k\}$ (rememeber we are working with $\mathbb{Q}$-divisors) so $\tau_{i_k} \geq i_k x$, i.e., $\limsup_k \tau_k/k \geq x$, and now let $x \to \tau$.

Lemma 2.7 *Let $\pi : Y \to X$ be a log resolution of (X, Δ), and let F be a prime divisor in Y. Let $D \sim_{\mathbb{Q}} L$ be a k-basis divisor. Then*

$$\mathrm{ord}_F(\pi^{\star}D) \leq \frac{1}{kd_k} \sum_{b=1}^{\tau_k(\pi^{\star}L,F)} h^0(Y, k\pi^{\star}L - bF),$$

and equality is attained for an appropriate choice of basis.

Proof For completeness, we provide the proof that can be easily extracted from [39, Lemma 2.2]. Fix Y and F as in the statement. Filter $H^0(Y, k\pi^{\star}L)$ in increasing order of vanishing along F,

$$H^0(Y, k\pi^{\star}L) \supseteq H^0(Y, k\pi^{\star}L - F) \supseteq \ldots \supseteq H^0(Y, k\pi^{\star}L - \tau_k F)$$
$$\supset H^0(Y, k\pi^{\star}L - \tau_k F - F) = \{0\}.$$

Now, fix a basis $s_1, \ldots, s_{d_k}$ of $H^0(X, kL)$, and let $D \sim_{\mathbb{Q}} L$ be the associated k-basis divisor (recall Definition 2.4). For each $b \in \{0, \ldots, \tau_k + 1\}$ suppose that exactly $i(b)$ of the sections $s_1 \circ \pi, \ldots, s_{d_k} \circ \pi$ are elements in $H^0(Y, k\pi^{\star}L - bF)$. Note that $i(0) = h^0(X, kL)$ and $i(\tau_k + 1) = 0$, and denoting

$$h^0(Y, k\pi^{\star}L - bF) := \dim H^0(Y, k\pi^{\star}L - bF),$$

of course we have $i(b) \leq h^0(Y, k\pi^{\star}L - bF)$. Then,

$$\mathrm{ord}_F(\pi^{\star}D) = \frac{\sum_{b=1}^{\tau_k} i(b)}{kd_k} \leq \frac{\sum_{b=1}^{\tau_k} h^0(Y, k\pi^{\star}L - bF)}{kd_k}.$$

So we get

$$\mathrm{ord}_F(\pi^\star D) \leq \frac{\sum_{b=1}^{\tau_k} h^0(kL - bF)}{kd_k}.$$

Moreover, we may choose a basis $\tilde{s}_1, \ldots, \tilde{s}_{d_k}$ of $H^0(X, kL)$ as follows to obtain for the associated k-basis divisor $\tilde{D}$,

$$\mathrm{ord}_F(\pi^\star \tilde{D}) = \frac{\sum_{b=1}^{\tau_k} h^0(kL - bF)}{kd_k},$$

as follows: let $\tilde{s}_1, \ldots, \tilde{s}_{h^0(Y, k\pi^\star L - k\tau_k F)}$ be a basis for $H^0(Y, k\pi^\star L - \tau_k F)$; thus, $i(\tau_k(F)) = h^0(Y, k\pi^\star L - \tau_k F)$. Next, choose the following $h^0(Y, k\pi^\star L - \tau_k F + F) - i(\tau_k)$ $\tilde{s}_i$'s to complete the sections from the first step to a basis for $H^0(Y, k\pi^\star L - \tau_k F + F)$. Thus, $i(\tau_k - 1) = h^0(Y, k\pi^\star L - \tau_k F + F)$. By induction, we see that $i(b) = h^0(Y, k\pi^\star L - bF)$ for each b, as desired. $\qquad\square$

Asymptotically, we may estimate the sum in Lemma 2.7 using volumes. Let us first recall some basic facts about volumes, following [44,45].

Definition 2.8 Let D be a Cartier divisor on X. The volume of D is defined by

$$\mathrm{vol}(D) := \limsup_k \frac{h^0(X, kD)}{k^n/n!}.$$

In fact, one may replace the limsup by a limit [44, Example 11.4.7], and by rescaling and continuity [44, Corollary 2.2.45] $\mathrm{vol}(D)$ makes sense for any $\mathbb{R}$-Cartier divisor D. Also, the volume function is invariant under pull-back by a birational morphism, i.e., $\mathrm{vol}(\pi^\star D) = \mathrm{vol}(D)$. Finally,

when the divisor D is nef (i.e., a limit of ample divisors) then $\mathrm{vol}(D) = D^n$.

$$(2.3)$$

Corollary 2.9 *Let* $\pi : Y \to X$ *be a log resolution of* (X, Δ), *and let* F *be a prime divisor in* Y. *Let* $D \sim_{\mathbb{Q}} L$ *be a* k-*basis divisor. Then,*

$$ord_F(\pi^\star D) \leq \frac{1}{L^n} \int_0^{\tau(\pi^\star L, F)} \mathrm{vol}(\pi^\star L - xF)dx + \epsilon_k,$$

with $\lim_k \epsilon_k = 0$.

Proof This result is probably standard (see, e.g., [36, Lemma 4.7]), but since it plays an important rôle in this article let us sketch a proof. By Riemann–Roch asymptotics, $L^n/n! = d_k/k^n + O(1/k)$ [44, 1.4.41]. Thus,

$$\frac{k^n}{n!d_k} = \frac{1}{L^n} + O(1/k)$$

Define a decreasing step function by

$$f_k(x) := \frac{h^0(Y, k\pi^\star L - \lfloor kx \rfloor F)}{k^n/n!}, \qquad x \in [0, \infty).$$

Then by Okounkov body theory for filtrated linear series [6, Lemma 1.6], [45, Theorem 2.13], [7, Theorem 5.3], $f_k(x) = \mathrm{vol}(\pi^\star L - xF) + \epsilon_k$, with $\lim_k \epsilon_k = 0$; thus,

$$\frac{k^n f_k(x)}{n!d_k} = \frac{\mathrm{vol}(\pi^\star L - xF)}{L^n} + \epsilon_k.$$

In other words, as $k \to \infty$, the function $\frac{k^n f_k(x)}{n!d_k}$ converges pointwise to $\frac{\mathrm{vol}(\pi^\star L - xF)}{L^n}$ for $x \in [0, \infty)$. Finally, using Lemma 2.7 and dominated convergence, we see that

$$\mathrm{ord}_F(\pi^\star D) \leq \frac{\sum_{b=1}^{\tau_k(\pi^\star L, F)} h^0(Y, k\pi^\star L - bF)}{kd_k}$$

$$= \int_0^{\frac{\tau_k(\pi^\star L, F)}{k}} \frac{k^n f_k(x)}{n!d_k} dx + \epsilon_k$$

$$= \frac{1}{L^n} \int_0^{\tau(\pi^\star L, F)} \mathrm{vol}(\pi^\star L - xF)dx + \epsilon_k,$$

(ϵ_k can change from line to line as long as $\lim_k \epsilon_k = 0$) where we used (2.2) (although it is actually enough to use $\lim\sup_k \tau_k(\pi^\star L, F)/k \geq \tau(\pi^\star L, F)$ as $\mathrm{vol}(\pi^\star L - xF) = 0$ for $x > \tau(\pi^\star L, F)$, i.e., it is enough to integrate until $\tau(\pi^\star L, F)$). □

The following lemma is handy when computing the volume of divisors on a surface.

Lemma 2.10 *Let B be a $\mathbb{Q}$-Cartier divisor on a surface S and let Z an irreducible curve in S with $Z^2 < 0$ and $B.Z \leq 0$. Then,*

$$\mathrm{vol}(B) = \mathrm{vol}\left(B - \frac{B.Z}{Z^2}Z\right).$$

Proof Take $k \in \mathbb{N}$ so that kB is Cartier and let $D \in |kB|$. Decompose, $D = Z \, \mathrm{ord}_Z D + \Delta$. Then,

$$kB.Z = D.Z = Z^2 \, \mathrm{ord}_Z D + \Delta.Z \geq Z^2 \, \mathrm{ord}_Z D,$$

and as $Z^2 < 0$ this yields

$$\mathrm{ord}_Z D \geq k\frac{B.Z}{Z^2},$$

and the right hand side is non-negative as $B.Z \leq 0$. Since D was any element of $|kB|$, we have shown that

$$h^0(S, kB) = h^0\Big(S, kB - k\frac{B.Z}{Z^2}Z\Big).$$

By Definition 2.8 we are done. □

3 Local intersection estimates on surfaces

In this section we derive new criteria for log canonicity in terms of local intersection estimates.

Let $\mathcal{O}_p$ be the local ring of germs of holomorphic functions defined in some neighborhood of p.

Definition 3.1 Let C_1 and C_2 be two irreducible curves on a surface S. Suppose that C_1 and C_2 intersect at a smooth point $p \in S$. Then the local intersection number of C_1 and C_2 at the point p is defined by

$$(C_1.C_2)_p := \dim_{\mathbb{C}} \mathcal{O}_p/(f_1, f_2),$$

where f_1 and f_2 are local defining functions of C_1 and C_2 around the point p.

Definition 3.1 extends to $\mathbb{R}$-divisors by linearity. For instance, say we have a curve C and a $\mathbb{R}$-divisor Ω meeting at the point p. We decompose Ω as $\Omega = \sum_i a_i Z_i$, where Z_i's are distinct prime divisors and $a_i \in \mathbb{R}$. Then,

$$(C.\Omega)_p := \sum_i a_i (C.Z_i)_p,$$

where $(C.Z_i)_p = 0$ if Z_i does not pass through the point p. A useful fact we will use often is that under a blow-up the local intersection number changes as follows,

$$(\tilde{C}.\tilde{\Omega})_q \leq (C.\Omega)_p - \mathrm{mult}_p \, \Omega, \qquad \text{with equality if } C \text{ is smooth at } p, \quad (3.1)$$

where $\tilde{C}$ and $\tilde{\Omega}$ are the proper transforms of C and Ω, where p is the blown-up point on C, and where q is any point on the intersection of the exceptional divisor and $\tilde{C}$. The classical inversion of adjunction on surfaces has the following well-known consequence [43], [14, Theorem 7].

Lemma 3.2 *Let C be an irreducible curve on a surface S, and let p be a smooth point in both C and S. Let $a \in \mathbb{Q} \cap [0, 1]$, and let Ω be an effective $\mathbb{Q}$-divisor on S with $C \not\subset \mathrm{supp}\Omega$. If*

$$(C.\Omega)_p \leq 1,$$

then $(S, aC + \Omega)$ is log canonical at p.

Lemma 3.2 can be improved by taking into account the parameter a as well as the vanishing order of Ω. Throughout this section we set

$$m := \mathrm{mult}_p \, \Omega.$$

Proposition 3.3 *Let C be an irreducible curve on a surface S, and let p be a smooth point in both C and S. Let $a \in \mathbb{Q} \cap [0, 1]$, and let Ω be an effective $\mathbb{Q}$-divisor on S with $C \not\subset \mathrm{supp}\Omega$. Suppose that*

$$(C.\Omega)_p \le 2 - a, \quad \text{if } m \le 1,$$

Then $(S, aC + \Omega)$ is log canonical at p. (However, if $m > 1$ we do not claim anything.)

Proof Suppose $m \le 1$. Let $\pi : \tilde{S} \to S$ be the blow-up at the point p, with exceptional curve $\pi^{-1}(p) = E$, and let $\tilde{C}$ and $\tilde{\Omega}$ denote the proper transforms of C and Ω. Then the log pair $(S, aC + \Omega)$ lifts to $(\tilde{S}, a\tilde{C} + \tilde{\Omega} + (a + m - 1)E)$. Since $a, m \le 1$ by assumption, $a + m - 1 \le 1$ so the latter pair is lc at a general point of E. It remains to check lc at the intersection points of E with $\tilde{\Omega}$ and $\tilde{C}$. First, let $q \in (E \cap \tilde{\Omega})\backslash\tilde{C}$. Then, $(E.\tilde{\Omega})_q \le E.\tilde{\Omega} = m \le 1$, so by Lemma 3.2 our log pair is lc at q. Second, let $\{q\} = E \cap \tilde{C}$. Again, by Lemma 3.2, it suffices to check that $\left(\tilde{C}.(\tilde{\Omega} + (a + m - 1)E)\right)_q \le 1$, and since $(\tilde{C}.\tilde{\Omega})_q = (C.\Omega)_p - m$ (by (3.1)) and $(\tilde{C}.E)_q = 1$ this amounts to $(C.\Omega)_p + a - 1 \le 1$, precisely our assumption. Thus, $(\tilde{S}, a\tilde{C} + \tilde{\Omega} + (a + m - 1)E)$ is lc along E, equivalently $(S, aC + \Omega)$ is lc at p. $\qquad\square$

We continue with a new local inequality incorporating also an additional "boundary curve".

Theorem 3.4 *Let B and C be irreducible curves on a surface S intersect transversally at a point p that is smooth in B, C and S. Let $a, b \in \mathbb{Q} \cap [0, 1)$, and let Ω be an effective $\mathbb{Q}$-divisor on S with $B, C \not\subset \mathrm{supp}\Omega$. Suppose that*

$$
(B.\Omega)_p \\
\le \frac{m}{(m-b)_+}(1-a) - b \quad \text{if } m \in (0, 1] \text{ and either } a + (C.\Omega)_p - b \le 1 \text{ or } a + m \le 1.
$$

Then $(S, (1 - b)B + aC + \Omega)$ is log canonical at p. (However, if $m > 1$ we do not claim anything.)

Note that the case $m = 0$ is trivial. Here, $(x)_+ := \max\{x, 0\}$. Thus, when $m \le b$ and either $a + (C.\Omega)_p - b \le 1$ or $a + m \le 1$, we are not assuming anything on $(B.\Omega)_p$.

Proof The case $b = 0$ follows from Lemma 3.2.
Suppose then $m \le 1$ and $b > 0$. We will use an inductive argument. Let $\pi : S_2 \to S$ be the blow-up at the point p, with exceptional curve $\pi^{-1}(p) = E_1$, and let B_2, C_2, Ω_2 denote the corresponding proper transforms of B, C, Ω. Then the log pair $(S, (1 - b)B + aC + \Omega)$ lifts to $(S_2, (1 - b)B_2 + aC_2 + \Omega_2 + (a + m - b)E_1)$.

Let

$$\{p_2\} := B_2 \cap E_1, \qquad \{q_C\} := C_2 \cap E_1.$$

First, let $q \in (E_1 \cap \Omega_2)\setminus\{q_C, p_2\}$. Then, $(E_1.\Omega_2)_q \leq m \leq 1$, so by Lemma 3.2 our log pair is lc at q.

Second, let us consider q_C. Lemma 3.2 applied to C_2 and E_1 yield lc at q_C if either

$$1 \geq \big(C_2.((1-b)B_2 + \Omega_2 + (a+m-b)E_1)\big)_{q_C} = (C_2.\Omega_2)_{q_C} + a + m - b,$$

(note C_2 and B_2 do not intersect at q_C) or

$$1 \geq \big(E_1.((1-b)B_2 + \Omega_2)\big)_{q_C} = a + (E_1.\Omega_2)_{q_C}$$

(note E_1 and B_2 do not intersect at q_C). Since $(C_2.\Omega_2)_{q_C} = (C.\Omega)_p - m$ by (3.1) the first inequality holds if $a + (C.\Omega)_p - b \leq 1$. Since $(E_1.\Omega_2)_{q_C} \leq m$ the second inequality holds if $a + m \leq 1$. Thus, by our assumptions, one of these must hold, so our pair is lc at q_C.

It remains to consider p_2. Lemma 3.2 yields lc at p_2 if

$$1 \geq \big(E_1.((1-b)B_2 + aC_2 + \Omega_2)\big)_{p_2} = 1 - b + (E_1.\Omega_2)_{p_2},$$

i.e., if $(E_1.\Omega_2)_{p_2} \leq b$, so in particular if $(E_1.\Omega_2)_{p_2} \leq m \leq b$. We are therefore done, unless

$$m > b, \tag{3.2}$$

which we henceforth assume.

Set

$$m_2 := \mathrm{mult}_{p_2} \Omega_2.$$

At this point, we can already set up an inductive argument to conclude the proof; instead, for the sake of clarity, let us carry through most of the first step in induction before switching to the general step. To start the induction, let us verify that the pair

$$(S_2, (1-b)B_2 + aC_2 + \Omega_2 + (a+m-b)E_1),$$

or, equivalently (as we are working at p_2, away from C_2),

$$(S_2, (1-b)B_2 + \Omega_2 + (a+m-b)E_1)$$

satisfies the assumptions of the Theorem. So E_1 is our new "C", B_2 is our new "B", Ω_2 is our new "Ω", and the new "a" is

$$a_2 := a + m - b > a.$$

First, note that $a_2 \in [0, 1)$, actually even $a_2 \leq 1 - b$ (recall $b > 0$ throughout): if $a + m \leq 1$ this is obvious, and if $a + (C.\Omega)_p - b \leq 1$ then as $m \leq (C.\Omega)_p$ we are also done. Note also that $m_2 \leq m \leq 1$ since multiplicities cannot increase under blow-ups. So, it remains to check that

$$a_2 + (E_1.\Omega_2)_{p_2} - b \leq 1 \quad \text{or} \quad a_2 + m_2 \leq 1, \tag{3.3}$$

and that

$$(B_2.\Omega_2)_{p_2} \leq \frac{m_2}{m_2 - b}(1 - a_2) - b. \tag{3.4}$$

Let us first check (3.3). The key is to use the estimate on $(B.\Omega)_p$ in the statement, as we will see shortly. In fact, we will prove that a statement stronger than (3.3) holds:

$$a_2 + m - b \leq 1 \quad \text{or} \quad a_2 - m + (B.\Omega)_p \leq 1. \tag{3.5}$$

The first inequality is stronger since $(E_1.\Omega_2)_q \leq m$, while the second inequality is stronger since $(B.\Omega)_p = (B_2.\Omega_2)_{p_2} + m \geq m_2 + m$. Now, the first inequality in (3.5) can be written as $a + m - 2b + m \leq 1$ or $2 \leq \frac{1-a}{m-b}$, while the second inequality can be written as $1 - a + b \geq m + m_2 \geq (B.\Omega)_p$ or $2 \geq \frac{(B.\Omega)_p - (1-a) + b}{b}$. By our assumption $\frac{m}{m-b}(1 - a) - b \geq (B.\Omega)_p$, so we can write yet stronger inequalities:

$$2 \leq \frac{1 - a}{m - b} \quad \text{or} \quad 2 \geq \frac{\frac{m}{m-b}(1 - a) - b - (1 - a) + b}{b} = \frac{1 - a}{m - b}, \tag{3.6}$$

which it trivially true, concluding the proof of (3.3). It remains to check (3.4). Let us prove this as part of an inductive argument (that will yield (3.4) as the first step in the induction, i.e., by setting $k = 2$ below).

Then we are in exactly the same setting as before, and we may blow-up S_2 at p_2 as all the conditions of Theorem 3.4 are satisfied for $(S_2, (1 - b)B_2 + \Omega_2 + (a + m - b)E_1)$ at $p_2 = B_2 \cap E_1$. Let $k \in \mathbb{N}$. By induction on the number of blow-ups, assume that we have performed $k - 1$ blow-ups at $p_1 := p, p_2, \ldots, p_{k-1}$ with exceptional divisors $E_1, \ldots, E_{k-1}$, where B_i is the proper transform of B_{i-1} (and $B_1 := B$, $\Omega_1 := \Omega, C_1 := C$) and where $p_k = E_{k-1} \cap B_k$, and that at for each $i = 0, \ldots, k - 1$ we have (set $m_1 := m$)

$$m_{i+1} > b, \tag{3.7}$$

and (set $a_1 = a$)

$$a_{i+1} = a_i + m_i - b, \tag{3.8}$$

and

$$(B_i.\Omega_i)_{p_i} \leq \frac{m_i}{m_i - b}(1 - a_i) - b,$$

and that

$$\text{either } a_i + (C_i.\Omega_i)_{p_i} - b \leq 1 \text{ or } a_i + m_i \leq 1.$$

We need to show that

$$(B_k.\Omega_k)_{p_k} \leq \frac{m_k}{m_k - b}(1 - a_k) - b \quad \text{if either } a_k + (C_k.\Omega_k)_{p_k} - b \leq 1$$
$$\text{or } a_k + m_k \leq 1$$

First, note that indeed either $a_k + (C_k.\Omega_k)_{p_k} - b \leq 1$ or $a_k + m_k \leq 1$: this is checked in each step just as we did for $k = 2$ with the number "2" in (3.6) being replaced by k (we omit the details). Note that $(B_k.\Omega_k)_{p_k} = (B_{k-1}.\Omega_{k-1})_{p_{k-1}} - m_{k-1}$, so it suffices to show that

$$(B_{k-1}.\Omega_{k-1})_{p_{k-1}} - m_{k-1} \leq \frac{m_k}{m_k - b}(1 - a_k) - b.$$

Since $m_k \leq m_{k-1}$ and since

$$\frac{x}{x - b} \text{ is decreasing in } x \in (b, \infty), \tag{3.9}$$

it suffices to show (recall (3.7)) that $(B_{k-1}.\Omega_{k-1})_{p_{k-1}} - m_{k-1} \leq \frac{m_{k-1}}{m_{k-1}-b}(1 - a_k) - b$, i.e.,

$$(B_{k-1}.\Omega_{k-1})_{p_{k-1}} - m_{k-1} \leq \frac{m_{k-1}}{m_{k-1} - b}(1 - a_k + m_{k-1} - b) - b$$
$$= \frac{m_{k-1}}{m_{k-1} - b}(1 - a_{k-1} - b) - b,$$

so we are done, by induction, if we can show that an infinite number of blow-ups is impossible. This is indeed so, since $(B_k.\Omega_k)_{p_k} = (B_{k-1}.\Omega_{k-1})_{p_{k-1}} - m_{k-1}$, and by induction $m_{i+1} > b$, so after at most $N := \lceil (B.\Omega)_p/b \rceil$ blow-ups we would have $m_{N+1} < b$ and then the pair would be lc at p_{N+1} by our original argument (just before (3.2)) using Lemma 3.2 with no further need to blow-up.   □

Most of the technical assumptions in Theorem 3.4 can actually be removed, to yield the following elegant and very useful criterion.

Corollary 3.5 *Let B and C be irreducible curves on a surface S intersect transversally at a point p that is smooth in B, C and S. Let $a, b \in \mathbb{Q} \cap [0, 1]$, and let Ω be an effective $\mathbb{Q}$-divisor on S with $B, C \not\subset \text{supp}\Omega$. Suppose that*

$$(B.\Omega)_p \le \frac{(C.\Omega)_p}{((C.\Omega)_p - b)_+}(1 - a) - b \quad \text{if } m \in (0, 1].$$

Then $(S, (1 - b)B + aC + \Omega)$ is log canonical at p. (However, if $m > 1$ we do not claim anything.)

Note that the case $m = 0$, i.e., $(C.\Omega)_p = 0$ is trivial since then also $(B.\Omega)_p = 0$ and $(S, (1 - b)B + aC + \Omega)$ is lc at p iff $(S, (1 - b)B + aC)$ is which is true as $1 - b, a \le 1$.

Proof If $b = 0$ then the assumption is $(B.\Omega)_p \le 1 - a$ so we are done by Proposition 3.3. Suppose now that $b > 0$ and $m \le 1$.

First, $m \le (C.\Omega)_p$. Thus, by (3.9),

$$(B.\Omega)_p \le \frac{m}{(m - b)_+}(1 - a) - b.$$

Thus, if

$$\text{either } a + (C.\Omega)_p - b \le 1 \text{ or } a + m \le 1, \tag{3.10}$$

Theorem 3.4 is applicable (the cases $b = 1$ or $a = 1$ are handled separately by Proposition 3.3 since for either one we may take $B = 0$) and we are done. We claim that always holds. First, if $(C.\Omega)_p \le b$ the first inequality in (3.10) automatically holds. So, suppose $((C.\Omega)_p - b)_+ = (C.\Omega)_p - b > 0$. Since $(B.\Omega)_p \ge m$,

$$m \le (B.\Omega)_p \le \frac{(C.\Omega)_p}{(C.\Omega)_p - b}(1 - a) - b = 1 - a - b + \frac{b(1 - a)}{(C.\Omega)_p - b},$$

i.e., $a + m \le 1 - b\frac{a + (C.\Omega)_p - b - 1}{(C.\Omega)_p - b}$, implying (3.10). $\qquad\square$

4 Vanishing order estimates

For the remainder of the article we will concentrate on the proof of Theorem 1.3. Set,

$$\overline{S} := \mathbb{P}^1 \times \mathbb{P}^1, \qquad \overline{C} := \text{a smooth curve of bi-degree } (1, 2) \subset \overline{S}.$$

Denote by $\overline{F}$ a general line of bi-degree $(1, 0)$ and by $\overline{G}$ a general line of bi-degree $(0, 1)$.

Note that, the curve $\overline{C}$ is, by definition, cut out by a bi-degree $(1, 2)$ polynomial. To be more precise, let $([s : t], [x : y])$ be the bi-homogeneous coordinate system on $\overline{S}$. Then $\overline{C}$ is cut out by some polynomial $F(s, t, x, y)$ that is homogeneous with degree 1 in s, t variables and homogeneous with degree 2 in x, y variables. Up to a coordinate change, we may assume $F(s, t, x, y) = sy^2 - tx^2$, for simplicity. (Indeed, we may assume $F = sP(x, y) + tQ(x, y)$. If both P and Q are of a linear polynomial squared, we are done. Assume that at least one of them is not a square. Apply a coordinate change

to x, y so that $F = Csxy + tQ(x, y)$ with $Q(x, y) = x^2 + axy + y^2$, and let $x \to x + y$, $y \to x - y$, so in the new coordinates $F = Cs(x^2 - y^2) + t((2 + a)x^2 + (2 - a)y^2)$. Finally, apply a linear coordinate change to s, t.)

The linear system $|F|$ contains exactly two curves that are tangent to $\overline{C}$. Denote them by

$$\overline{F}_0, \overline{F}_\infty,$$

and let

$$\overline{p}_0 := \overline{F}_0 \cap \overline{C}, \qquad \overline{p}_\infty := \overline{F}_\infty \cap \overline{C}.$$

In $([s : t], [x : y])$ coordinates, one simply has $\overline{F}_0 = \{s = 0\}$, $\overline{F}_\infty = \{t = 0\}$ and $p_0 = ([0 : 1], [0; 1])$ and $p_\infty = ([1 : 0], [1 : 0])$. Let $\overline{F}_1, \ldots, \overline{F}_r$ be distinct bi-degree $(1, 0)$ curves in $\overline{S}$ that are all different from the curves $\overline{F}_0$ and $\overline{F}_\infty$. Then each intersection $\overline{F}_i \cap \overline{C}$ consists of two points. For each $i = 1, \ldots r$, let

$$\overline{p}_i \in \overline{F}_i \cap \overline{C}$$

be one of these two points.

Let

$$I := \{i_1, \ldots, i_r\} \subset \{0, 1, \ldots, r, \infty\},$$

let $\pi : S \to \overline{S}$ be the blow-up of $\overline{S}$ at the r points $\{\overline{p}_i\}_{i \in I} \subset \{\overline{p}_0, \overline{p}_1, \ldots, \overline{p}_r, \overline{p}_\infty\}$, and denote by

$$E_j := \pi^{-1}(\overline{p}_j), \qquad j \in I,$$

the exceptional curves of π. To be precise, we note that we are blowing-up r of the $r + 2$ points $\{\overline{p}_0, \overline{p}_1, \ldots, \overline{p}_r, \overline{p}_\infty\}$. Denote by

$$F_0, F_1, \ldots, F_r, F_\infty$$

the proper transform on the surface S of the curves $\overline{F}_0, \overline{F}_1, \ldots, \overline{F}_r, \overline{F}_\infty$ (note that exactly r of these are -1-curves and the remaining two are 0-curves). Let C be the proper transform of the curve $\overline{C}$, so

$$C = \pi^\star \overline{C} - \sum_{j \in I} E_j \sim \pi^\star(\overline{F} + 2\overline{G}) - \sum_{j \in I} E_j.$$

Let

$$K_\beta := K_S + (1 - \beta)C.$$

Then, as $-K_{\overline{S}} = 2\overline{F} + 2\overline{G}$, and $K_S = \pi^\star K_{\overline{S}} + \sum_{j\in I} E_j$,

$$-K_\beta \sim_\mathbb{Q} 2\pi^\star \overline{F} + 2\pi^\star \overline{G} - \sum_{j\in I} E_j - (1-\beta)(\pi^\star \overline{C} - \sum_{j\in I} E_j) \sim_\mathbb{Q} \pi^\star \overline{F} + \beta C,$$

$$(4.1)$$

thus (S, C) is asymptotically log Fano, more precisely [22, (4.2)],

$$-K_\beta \text{ is ample for } 0 < \beta < \tfrac{2}{r-4}. \tag{4.2}$$

Below, we will always assume

$$r \geq 7. \tag{4.3}$$

To prove Theorem 1.3, we will show in Sect. 5 that for some $\lambda > 1$ (independent of k) and for any k-basis divisor D, the log pair

$$\big(S, (1-\beta)C + \lambda D\big) \tag{4.4}$$

has lc singularities for sufficiently small $\beta > 0$, and sufficiently large k. To do this, in the present section we obtain explicit estimates on the order of vanishing of basis divisors. This involves estimating integrals appearing in Corollary 2.9, which in turn involves elementary computations of Seshadri constants and pseudoeffective thresholds.

For the estimate on the order of vanishing, we require the Seshadri constant and the pseudoeffective threshold (recall (2.1)). Let us recall the definition of the former. Let X be a smooth projective variety, L an ample $\mathbb{Q}$-divisor on it, and Z a prime divisor on it. The Seshadri constant of (X, Z) with respect to L is defined to be

$$\sigma(Z, L) = \sup \big\{c > 0 : L - cZ \text{ is ample}\big\}. \tag{4.5}$$

We start by computing $\tau(-K_\beta, Z)$.

Lemma 4.1 *Let* $0 < \beta < \tfrac{2}{r-4}$. *One has,*

$$\tau(-K_\beta, Z) = \begin{cases} 1 & \text{if } Z \text{ be an irreducible curve in } |\pi^\star \overline{F}|, \\ \beta & \text{if } Z = C, \\ 1 & \text{if } Z \in \cup_{i\in I}\{E_i, F_i\}. \end{cases} \tag{4.6}$$

Proof If Z is an irreducible curve in $|\pi^\star \overline{F}|$, as $-K_\beta \sim_\mathbb{Q} \pi^\star \overline{F} + \beta C$ and $\pi^\star \overline{F}$ is effective (has 0 self-intersection) and C has zero volume (as $C^2 = 4 - r < 0$), we must have $\tau(-K_\beta, Z) = 1$.

If $Z = C$, $-K_\beta - xC \sim_\mathbb{Q} \pi^\star \overline{F} + (\beta - x)C$. For $x = \beta$ we get $\text{vol}(-K_\beta - \beta C) = \text{vol}(\pi^\star \overline{F}) = (\pi^\star \overline{F})^2 = 0$, so $\tau(-K_\beta, C) = \beta$.

If $Z = E_1$, say, we claim $\mathrm{vol}(-K_\beta - E_1) = 0$, i.e., $\tau(-K_\beta, E_1) \leq 1$. Since $-K_\beta - xE_1 \sim_\mathbb{Q} \pi^\star \overline{F} + \beta C - xE_1 \sim_\mathbb{Q} F_1 + (1-x)E_1 + \beta C$ is effective for $x \in [0, 1]$, we would thus have $\tau(-K_\beta, E_1) = 1$. To prove the claim, Lemma 2.10 implies that $\mathrm{vol}(-K_\beta - E_1) = \mathrm{vol}(-K_\beta - E_1 - (1-\beta)F_1) = \mathrm{vol}(\beta(F_1 + C)) = \beta^2 \mathrm{vol}(F_1 + C) = 0$ by Claim 4.2 below. If $Z = F_1$, say, the computations are similar. By Remark 4.3, we are done. □

Claim 4.2 *Let $i \in I$. Then,* $\mathrm{vol}(F_i + C) = 0$.

Proof Suppose $r > 5$. Repeated application of Lemma 2.10 implies that $\mathrm{vol}(F_i + C) = \mathrm{vol}(F_i + C - \frac{r-5}{r-4}C) = \mathrm{vol}\big(F_i + C - \frac{r-5}{r-4}C - (1 - \frac{1}{r-4})F_i\big) = \ldots = \mathrm{vol}(a_j F_i + b_j C)$. The sequences $\{a_j, b_j\}$ are decreasing so let $a := \lim a_j$, $b := \lim b_j$. This process will stop if $aF_i + bC$ is nef. But that implies (by intersecting with F_i and C) that $a, b \geq 0$ and $-a + b \geq 0$ and $a + (4 - r)b \geq 0$; adding up and using that $r > 5$ we see that $b = 0$ which then implies $a = 0$, so the claim follows. □

Remark 4.3 As we just saw above, the computations depend only on the intersection-theoretic properties of Z, so they are exactly the same if Z is any element of $\cup_{i \in I}\{E_i, F_i\}$ (in particular also if $\{0, \infty\} \cap I \neq \emptyset$).

By (4.2), $-K_\beta$ is ample for small β, so it makes sense $\sigma(-K_\beta, Z)$.

Lemma 4.4 *Let $0 < \beta < \frac{2}{r-3}$. One has,*

$$\sigma(-K_\beta, Z) = \begin{cases} 1 - \beta(r-4)/2 & \text{if } Z \text{ be an irreducible curve in } |\pi^\star \overline{F}|, \\ \beta & \text{if } Z = C, \\ \beta & \text{if } Z \in \cup_{i \in I}\{E_i, F_i\}. \end{cases} \quad (4.7)$$

Proof For the first two cases we can actually assume $0 < \beta < \frac{2}{r-4}$. In the first case, using (4.1),

$$(-K_\beta - xZ).C = (1-x)\pi^\star\overline{F}.C + \beta C^2 = 2(1-x) - \beta(r-4),$$

(while $(-K_\beta - xZ).\pi^\star\overline{F} = 2\beta > 0$), i.e., $\sigma(-K_\beta, Z) = 1 - \beta(r-4)/2$. In the second case,

$$(-K_\beta - xC).C = (\pi^\star\overline{F} + (\beta - x)C).C = 2 - (x - \beta)(r - 4),$$

while

$$(-K_\beta - xC).\pi^\star\overline{F} = (\pi^\star\overline{F} + (\beta - x)C).\pi^\star\overline{F} = 2(\beta - x),$$

so $\sigma(-K_\beta, C) = \beta$. In the third case, say $1 \in I$, observe that $-K_\beta - xE_1 \sim_\mathbb{Q} F_1 + (1-x)E_1 + \beta C$, thus to check nefness it suffices to take $x < 1$ and check intersection with F_1, E_1 and C. Now,

$$(-K_\beta - xE_1).C = (F_1 + (1-x)E_1 + \beta C).C = 1 + 1 - x + \beta(4 - r),$$

(which is where we need $\beta < 2/(r-3)$), while

$$(-K_\beta - xE_1).F_1 = (F_1 + (1-x)E_1 + \beta C).F_1 = -1 + 1 - x + \beta = \beta - x,$$

and

$$(-K_\beta - xE_1).E_1 = (F_1 + (1-x)E_1 + \beta C).E_1 = 1 - (1-x) + \beta = \beta + x,$$

so $\sigma(-K_\beta, E_1) = \beta$. If $Z = F_1$, say, the computations are similar. By Remark 4.3, we are done. $\square$

Lemma 4.5 Let $0 < \beta < \frac{2}{r-3}$ and let D be a k-basis divisor. One has,

$$\operatorname{ord}_Z D \le \begin{cases} \dfrac{1}{2} - \dfrac{\beta(r-4)}{8} + O(\beta^2) + \epsilon_k, & \text{if } Z \text{ be an irreducible curve in } |\pi^\star \overline{F}|, \\[2ex] \dfrac{\beta}{2} + O(\beta^2) + \epsilon_k, & \text{if } Z = C, \\[2ex] \dfrac{1}{2} - \dfrac{\beta(r-6)}{8} + O(\beta^2) + \epsilon_k, & \text{if } Z \in \cup_{i\in I}\{E_i, F_i\}, \end{cases} \tag{4.8}$$

with $\lim_k \epsilon_k = 0$. In the first and third cases ϵ_k does not depend on the choice of Z.

Proof The result follows from Lemmas 4.1 and 4.4 and Corollary 2.9 by estimating the volume integral

$$\begin{aligned} \frac{1}{K_\beta^2} \int_0^\tau \operatorname{vol}(-K_\beta - xZ)dx &= \frac{1}{K_\beta^2} \int_0^\sigma (K_\beta + xZ)^2 dx + \frac{1}{K_\beta^2} \int_\sigma^\tau \operatorname{vol}(-K_\beta - xZ)dx \\ &= \frac{1}{K_\beta^2} \int_0^\sigma (K_\beta^2 + 2xZ.K_\beta + x^2 Z^2)dx \\ &\quad + \frac{1}{K_\beta^2} \int_\sigma^\tau \operatorname{vol}(-K_\beta - xZ)dx \\ &= \sigma + \frac{1}{K_\beta^2}\left[\frac{Z^2}{3}\sigma^3 \right. \\ &\quad \left. + Z.K_\beta\sigma^2 + \int_\sigma^\tau \operatorname{vol}(-K_\beta - xZ)dx\right]. \end{aligned} \tag{4.9}$$

We will also use the estimate

$$\int_\sigma^\tau \operatorname{vol}(-K_\beta - xZ)dx \le (\tau - \sigma)(K_\beta + \sigma Z)^2 = (\tau - \sigma)(K_\beta^2 + 2\sigma Z.K_\beta + \sigma^2 Z^2). \tag{4.10}$$

First, let Z be an irreducible curve in $|\pi^\star \overline{F}|$. Then $Z^2 = 0$, and

$$Z.K_\beta = -2\beta, \quad K_\beta^2 = 4\beta - \beta^2(r-4),$$

thus

$$K_\beta^2 + 2\sigma Z.K_\beta + \sigma^2 Z^2 = 4\beta - \beta^2(r-4) - (2 - (r-4)\beta)2\beta = O(\beta^2),$$

and as $\tau - \sigma = O(\beta)$, we get (4.10)$= O(\beta^3)$. Thus, by Lemmas 4.1 and 4.4,

$$
\begin{aligned}
\frac{1}{K_\beta^2} & \int_0^\tau \mathrm{vol}(-K_\beta - x\pi^\star \overline{F})dx \\
& \leq \sigma + \frac{-2\beta\sigma^2 + O(\beta^3)}{4\beta - \beta^2(r-4)} \\
& = 1 - \frac{r-4}{2}\beta + \frac{-2\beta\sigma^2 + O(\beta^3)}{4\beta - \beta^2(r-4)} = \frac{1}{2} - \frac{r-4}{8}\beta + O(\beta^2).
\end{aligned}
$$

Second, let $Z = C$. Then $\sigma = \tau = \beta$, i.e., (4.10)$= 0$. Thus,

$$\frac{1}{K_\beta^2} \int_0^\tau \mathrm{vol}(-K_\beta - xC)dx = \beta + \frac{\frac{4-r}{3}\beta^3 + (-2 + \beta(r-4))\beta^2}{4\beta - \beta^2(r-4)} = \frac{\beta}{2} + O(\beta^2).$$

Third, say $1 \in I$ and let $Z = E_1$ (the proof for $Z = F_1$ is identical as E_1 and F_1 play symmetric roles in the computations). Then, as $\sigma = -E_1.K_\beta = \beta$,

$$
\begin{aligned}
\frac{1}{K_\beta^2} & \int_0^\tau \mathrm{vol}(-K_\beta - xZ)dx \\
& = \beta + \frac{1}{4\beta - \beta^2(r-4)}\left[O(\beta^3) + \int_\beta^1 \mathrm{vol}(-K_\beta - xZ)dx\right] \qquad (4.11) \\
& = \beta + O(\beta^2) + \frac{1}{4\beta - \beta^2(r-4)}\int_\beta^1 \mathrm{vol}(-K_\beta - xZ)dx.
\end{aligned}
$$

The remaining integral can be simplified using Lemma 2.10. Indeed, $(-K_\beta - xZ).F_1 = \beta - x$, so

$$\mathrm{vol}(-K_\beta - xZ) = \mathrm{vol}(-K_\beta - xZ - (x-\beta)F_1), \quad x \in (\beta, 1).$$

The divisor on the right hand side is nef for

$$x \leq \sigma' := \sigma(-K_\beta + \beta F_1, Z + F_1) = 1 + \beta(5-r)/2$$

since $-K_\beta - xZ - (x-\beta)F_1 \sim_\mathbb{Q} (1-x)E_1 + (1-x+\beta)F_1 + \beta C$ and this intersects non-negatively with E_1 and F_1 while intersecting with C gives $2 - 2x + \beta(5-r)$. Similarly to (4.10), we estimate

$$\int_{\sigma'}^1 \mathrm{vol}(-K_\beta - xZ)dx = \int_{\sigma'}^1 \mathrm{vol}\big(-K_\beta + \beta F_1 - x(Z + F_1)\big)dx$$

$$\le \beta\frac{r-5}{2}\Big((K_\beta - \beta F_1)^2 + 2\sigma'(Z + F_1).(K_\beta - \beta F_1)\Big)$$

$$\le \beta\frac{r-5}{2}\Big(4\beta + \beta^2(5 - r) - 4\beta\sigma'\Big) = O(\beta^3),$$

as $(Z + F_1)^2 = 0$ and $(K_\beta - \beta F_1)^2 = 4\beta + \beta^2(5 - r)$, $(Z + F_1).(K_\beta - \beta F_1) = -2\beta$.
Next,

$$\int_\sigma^{\sigma'} \mathrm{vol}(-K_\beta - xZ)dx = \int_\sigma^{\sigma'} \mathrm{vol}\big(-K_\beta + \beta F_1 - x(Z + F_1)\big)dx$$

$$= \int_0^{\sigma'} - \int_0^\sigma \mathrm{vol}\big(-K_\beta + \beta F_1 - x(Z + F_1)\big)dx,$$

and we can compute each integral as in (4.9), namely,

$$\int_\sigma^{\sigma'} \mathrm{vol}(-K_\beta - xZ)dx = \sigma'(K_\beta - \beta F_1)^2 + \frac{(Z + F_1)^2}{3}(\sigma')^3$$

$$+ (Z + F_1).(K_\beta - \beta F_1)(\sigma')^2$$

$$- \sigma(K_\beta - \beta F_1)^2 - \frac{(Z + F_1)^2}{3}\sigma^3$$

$$- (Z + F_1).(K_\beta - \beta F_1)\sigma^2$$

$$= \Big(1 + \frac{3-r}{2}\beta\Big)\Big(4\beta + \beta^2(5 - r)\Big) - 2\beta((\sigma')^2 - \sigma^2)$$

$$= \Big(1 + \frac{3-r}{2}\beta\Big)\Big(4\beta + \beta^2(5 - r)\Big)$$

$$- 2\beta\Big(1 + \beta(5 - r) + O(\beta^2)\Big)$$

$$= 2\beta + \beta^2(5 - r + 6 - 2r + 2r - 10) + O(\beta^3)$$

$$= 2\beta + \beta^2(1 - r) + O(\beta^3).$$

Altogether, we have shown

$$\frac{1}{K_\beta^2}\int_0^\tau \mathrm{vol}(-K_\beta - xZ)dx$$

$$= \beta + O(\beta^3) + \frac{1}{4\beta + \beta^2(4 - r)}\Big[\int_\sigma^\tau \mathrm{vol}(-K_\beta - xZ)dx\Big]$$

$$\le \beta + O(\beta^3) + \frac{1}{4\beta + \beta^2(4 - r)}\Big[2\beta + \beta^2(1 - r)$$

$$+ O(\beta^3) + O(\beta^3)\Big]$$

$$= \frac{1}{4\beta + \beta^2(4-r)}\left[2\beta + \beta^2(5-r) + O(\beta^2)\right] + O(\beta^2)$$

$$\leq \frac{1}{2} - \frac{\beta(r-6)}{8} + O(\beta^2),$$

as desired. By Remark 4.3, we are done. Note that from our proof it follows that ϵ_k only depends on the class of Z. □

5 Basis log canonical thresholds

We use the same notation as in Sect. 4. The purpose of this section is to prove Theorem 1.3 by showing that for all sufficiently large k and some $\lambda > 1$ (independent of k) and for any k-basis divisor $D \sim_{\mathbb{Q}} -K_\beta = -K_S - (1-\beta)C$, the log pair

$$\big(S, (1-\beta)C + \lambda D\big) \tag{5.1}$$

has log canonical singularities for sufficiently small $\beta > 0$ (independent of k).

Let us fix such β, k, D, and set

$$\lambda := 1 + \frac{\beta}{100}. \tag{5.2}$$

We split the argument into several lemmas.

Claim 5.1 *The pair* (5.1) *is lc at* $S\backslash(C \cup_{i\in I} \{E_i, F_i\})$ *for all sufficiently large k.*

Proof Let $p \in S\backslash(C \cup_{i\in I} \{E_i, F_i\})$. Thus, $\pi(p) \notin \{\overline{p}_i\}_{i\in I}$, so if we let ℓ be the $(1, 0)$-curve passing through $\pi(p)$ then $Z := \pi^{-1}(\ell) \in |\pi^\star \overline{F}|$ is a smooth irreducible curve passing through p. As $p \notin C$, the pair (5.1) is lc at p if and only if the pair $\big(S, \lambda D\big)$ is. Write $\lambda D = \lambda Z \operatorname{ord}_Z D + \Delta$. Then

$$(Z.\Delta)_p \leq Z.\Delta = Z.(\lambda D - \lambda Z \operatorname{ord}_Z D) = Z.(\lambda(Z + \beta C) - \lambda Z \operatorname{ord}_Z D) = 2\beta\lambda \leq 1,$$

for β small so we are done by Lemma 3.2 (using $\operatorname{ord}_Z D \leq 1/2$ (by Lemma 4.5) it follows that $\lambda \operatorname{ord}_Z D \leq 1$ so Lemma 3.2 is applicable)). □

Claim 5.2 *The pair* (5.1) *is lc at* $\cup_{i\in I}\{E_i, F_i\}\backslash C$ *for all sufficiently large k.*

Proof Say $1 \in I$ and let $p \in E_1\backslash C$. Again, it suffices to show the pair $\big(S, \lambda D\big)$ is lc at p. Decompose λD as $\lambda D = \Delta + \lambda E_1 \operatorname{ord}_{E_1} D$. Then,

$$(E_1.\Delta)_p \leq E_1.\Delta = E_1.(\lambda D - \lambda E_1 \operatorname{ord}_{E_1} D)$$
$$= E_1.(\lambda(E_1 + F_1 + \beta C) - \lambda E_1 \operatorname{ord}_{E_1} D) = \lambda(\beta + \operatorname{ord}_{E_1} D) \leq 1,$$

for β small by Lemma 4.5 and (5.2), so we are done by Lemma 3.2 (once again we know that $\lambda \operatorname{ord}_{E_1} D \leq 1$ by Lemma 4.5). □

Claim 5.3 *The pair* (5.1) *is lc at* $C \setminus (\{p_0, p_\infty\} \cup \bigcup_{i \in I}\{E_i, F_i\})$ *for all sufficiently large k.*

Proof Let $p \in C \setminus (\{p_0, p_\infty\} \cup \bigcup_{i \in I}\{E_i, F_i\})$. As in the proof of Claim 5.1, let $Z \in |\pi^\star \overline{F}|$ be a smooth irreducible curve passing through p. Write

$$\lambda D + (1 - \beta)C = \lambda Z \operatorname{ord}_Z D + (1 - \beta + \lambda \operatorname{ord}_C D)C + \Omega.$$

As Z intersects C transversally at p (as $p \in \{p_0, p_\infty\}$), by Corollary 3.5 it suffices to show that

$$\operatorname{mult}_p \Omega \leq 1 \tag{5.3}$$

and that

$$C.\Omega \leq \frac{(Z.\Omega)_p}{((Z.\Omega)_p - \beta + \lambda \operatorname{ord}_C D)_+}(1 - \lambda \operatorname{ord}_Z D) - \beta + \lambda \operatorname{ord}_C D.$$

For (5.3), note that by (5.4) $\operatorname{mult}_p \Omega \leq (Z.\Omega)_p = O(\beta)$. Now,

$$\begin{aligned}
C.\Omega &= C.\big(\lambda D - \lambda Z \operatorname{ord}_Z D - \lambda C \operatorname{ord}_C D\big) \\
&= C.\big(\lambda(\pi^\star \overline{F} + \beta C) - \lambda Z \operatorname{ord}_Z D - \lambda C \operatorname{ord}_C D\big) \\
&= \lambda(2 + \beta(4 - r) - 2 \operatorname{ord}_Z D - (4 - r)\operatorname{ord}_C D)
\end{aligned}$$

by Lemma 4.5, while

$$\begin{aligned}
(Z.\Omega)_p &\leq Z.\Omega = Z.\big(\lambda D - \lambda Z \operatorname{ord}_Z D - \lambda C \operatorname{ord}_C D\big) \\
&= 2\lambda(\beta - \operatorname{ord}_C D).
\end{aligned} \tag{5.4}$$

As $\lambda > 1$, this is larger than $\beta - \lambda \operatorname{ord}_C D$, and using (3.9), it suffices to show

$$2\lambda\left(1 - \operatorname{ord}_Z D + \frac{r - 4}{2}(\operatorname{ord}_C D - \beta)\right) \leq \frac{2\lambda(\beta - \operatorname{ord}_C D)}{\beta(2\lambda - 1) - \lambda \operatorname{ord}_C D}(1 - \lambda \operatorname{ord}_Z D) \\ - \beta + \lambda \operatorname{ord}_C D,$$

i.e.,

$$1 - \operatorname{ord}_Z D + \frac{r - 5}{2}(\operatorname{ord}_C D - \beta) \leq \frac{\beta - \operatorname{ord}_C D}{\beta(2\lambda - 1) - \lambda \operatorname{ord}_C D}(1 - \lambda \operatorname{ord}_Z D) + \frac{\lambda - 1}{2\lambda}\beta,$$

i.e.,

$$\begin{aligned}
\operatorname{ord}_Z D &\left(\frac{\lambda(\beta - \operatorname{ord}_C D)}{\beta(2\lambda - 1) - \lambda \operatorname{ord}_C D} - 1\right) + 1 + \frac{r - 5}{2}(\operatorname{ord}_C D - \beta) \\
&\leq \frac{\beta - \operatorname{ord}_C D}{\beta(2\lambda - 1) - \lambda \operatorname{ord}_C D} + \frac{\lambda - 1}{2\lambda}\beta,
\end{aligned}$$

i.e.,

$$\operatorname{ord}_Z D \frac{\beta(1-\lambda)}{\beta(2\lambda-1) - \lambda \operatorname{ord}_C D} + 1 + \frac{r-5}{2} (\operatorname{ord}_C D - \beta)$$
$$\leq \frac{\beta - \operatorname{ord}_C D}{\beta(2\lambda-1) - \lambda \operatorname{ord}_C D} + \frac{\lambda-1}{2\lambda} \beta.$$

The first term is negative while the last is positive, and since $\operatorname{ord}_C D - \beta < 0$ and $r - 5 \geq 2$, suffices to show

$$1 \leq (\beta - \operatorname{ord}_C D) \left(\frac{1}{\lambda(\beta - \operatorname{ord}_C D) + \beta(\lambda-1)} + 1 \right)$$

i.e.,

$$\lambda(\beta - \operatorname{ord}_C D) + \beta(\lambda-1) \leq (\beta - \operatorname{ord}_C D)\Big(1 + \lambda(\beta - \operatorname{ord}_C D) + \beta(\lambda-1)\Big)$$

i.e.,

$$(\beta - \operatorname{ord}_C D)\big((\lambda-1)(1-\beta) - \lambda(\beta - \operatorname{ord}_C D)\big) + \beta(\lambda-1) \leq 0 \qquad (5.5)$$

By Lemma 4.5 and (5.2), for β sufficiently small,

$$(\beta - \operatorname{ord}_C D)\big((\lambda-1)(1-\beta) - \lambda(\beta - \operatorname{ord}_C D)\big) + \beta(\lambda-1)$$
$$\leq (\beta - \operatorname{ord}_C D)\big((\lambda-1)(1-\beta) - \lambda\beta/3\big) + \beta(\lambda-1)$$
$$\leq -\frac{\beta}{4}(\beta - \operatorname{ord}_C D) + \beta(\lambda-1) \leq -\frac{\beta^2}{12} + \beta(\lambda-1) \leq 0,$$

proving (5.5). □

Claim 5.4 *The pair* (5.1) *is lc at* $\{C \cap E_1, \ldots, C \cap E_r, C \cap F_1, \ldots, C \cap F_r\}$ *for all sufficiently large k.*

Proof We prove the result for $p = C \cap E_1$ (the proof for other cases is similar). Write

$$\lambda D + (1 - \beta)C = \lambda E_1 \operatorname{ord}_{E_1} D + \lambda F_1 \operatorname{ord}_{F_1} D + (1 - \beta + \lambda \operatorname{ord}_C D)C + \Omega.$$

Note that E_1 intersects C transversally at p and F_1 does not pass through p. We have

$$\begin{cases} \lambda\beta - \lambda \operatorname{ord}_C D + \lambda \operatorname{ord}_{E_1} D - \lambda \operatorname{ord}_{F_1} D = E_1.\Omega \geq \operatorname{mult}_p \Omega, \\ \lambda\beta - \lambda \operatorname{ord}_C D - \lambda \operatorname{ord}_{E_1} D + \lambda \operatorname{ord}_{F_1} D = F_1.\Omega \geq 0. \end{cases}$$

From these two inequalities we get $\operatorname{mult}_p \Omega \leq 2\lambda(\beta - \operatorname{ord}_C D) = O(\beta)$ and

$$\lambda \operatorname{ord}_{E_1} D - \lambda \operatorname{ord}_{F_1} D \leq \lambda(\beta - \operatorname{ord}_C D). \qquad (5.6)$$

In particular, $\mathrm{mult}_p \Omega \leq 1$. Then by Corollary 3.5 if suffices to show

$$C.\Omega \leq \frac{(E_1.\Omega)_p}{((E_1.\Omega)_p - \beta + \lambda \operatorname{ord}_C D)_+}(1 - \lambda \operatorname{ord}_{E_1} D) - \beta + \lambda \operatorname{ord}_C D.$$

Now, using (5.6),

$$\begin{aligned}
C.\Omega &= C.\big(\lambda D - \lambda E_1 \operatorname{ord}_{E_1} D - \lambda F_1 \operatorname{ord}_{F_1} D - \lambda C \operatorname{ord}_C D\big) \\
&= C.\big(\lambda(\pi^\star \overline{F} + \beta C) - \lambda E_1 \operatorname{ord}_{E_1} D - \lambda F_1 \operatorname{ord}_{F_1} D - \lambda C \operatorname{ord}_C D\big) \\
&= \lambda(2 + \beta(4 - r) - \operatorname{ord}_{E_1} D - \operatorname{ord}_{F_1} D - (4 - r)\operatorname{ord}_C D) \\
&\leq \lambda(2 + \beta(5 - r) - 2\operatorname{ord}_{E_1} D - (5 - r)\operatorname{ord}_C D)
\end{aligned}$$

and

$$\begin{aligned}
(E_1.\Omega)_p &\leq E_1.\Omega = E_1.\big(\lambda D - \lambda E_1 \operatorname{ord}_{E_1} D - \lambda F_1 \operatorname{ord}_{F_1} D - \lambda C \operatorname{ord}_C D\big) \\
&= \lambda\beta - \lambda \operatorname{ord}_C D + \lambda \operatorname{ord}_{E_1} D - \lambda \operatorname{ord}_{F_1} D \leq 2\lambda(\beta - \operatorname{ord}_C D).
\end{aligned}$$

As $\lambda > 1$, this is larger than $\beta - \lambda \operatorname{ord}_C D$, and using (3.9), it suffices to show

$$2\lambda\left(1 - \operatorname{ord}_{E_1} D + \frac{r-5}{2}(\operatorname{ord}_C D - \beta)\right) \leq \frac{2\lambda(\beta - \operatorname{ord}_C D)}{\beta(2\lambda - 1) - \lambda \operatorname{ord}_C D}(1 - \lambda \operatorname{ord}_{E_1} D) \\ - \beta + \lambda \operatorname{ord}_C D,$$

i.e.,

$$\begin{aligned}
&1 - \operatorname{ord}_{E_1} D + \frac{r-6}{2}(\operatorname{ord}_C D - \beta) \\
&\leq \frac{\beta - \operatorname{ord}_C D}{\beta(2\lambda - 1) - \lambda \operatorname{ord}_C D}(1 - \lambda \operatorname{ord}_{E_1} D) + \frac{\lambda - 1}{2\lambda}\beta,
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\operatorname{ord}_{E_1} D\left(\frac{\lambda(\beta - \operatorname{ord}_C D)}{\beta(2\lambda - 1) - \lambda \operatorname{ord}_C D} - 1\right) + 1 + \frac{r-6}{2}(\operatorname{ord}_C D - \beta) \\
&\leq \frac{\beta - \operatorname{ord}_C D}{\beta(2\lambda - 1) - \lambda \operatorname{ord}_C D} + \frac{\lambda - 1}{2\lambda}\beta,
\end{aligned}$$

i.e.

$$\begin{aligned}
&\operatorname{ord}_{E_1} D\frac{\beta(1 - \lambda)}{\beta(2\lambda - 1) - \lambda \operatorname{ord}_C D} + 1 + \frac{r-6}{2}(\operatorname{ord}_C D - \beta) \\
&\leq \frac{\beta - \operatorname{ord}_C D}{\beta(2\lambda - 1) - \lambda \operatorname{ord}_C D} + \frac{\lambda - 1}{2\lambda}\beta.
\end{aligned}$$

The first term is negative while the last is positive, so it suffices to show

$$1 - \frac{\beta - \operatorname{ord}_C D}{\beta(2\lambda - 1) - \lambda \operatorname{ord}_C D} \leq \frac{r - 6}{2}(\beta - \operatorname{ord}_C D),$$

i.e.,

$$(\lambda - 1)\frac{2\beta - \operatorname{ord}_C D}{\beta(2\lambda - 1) - \lambda \operatorname{ord}_C D} \leq \frac{r - 6}{2}(\beta - \operatorname{ord}_C D).$$

Using Lemma 4.5, (5.2) and (4.3) it suffices to show (for β small)

$$\frac{\beta}{100} \cdot \frac{2\beta}{\beta - \frac{2}{3}\beta} \leq \frac{1}{2} \cdot (\beta - \frac{2}{3}\beta),$$

i.e., $3\beta/50 \leq \beta/6$, so we are done. $\square$

To finish the proof of Theorem 1.3, it remains to show that the pair (5.1) is lc at $p = C \cap F_0$ (for $C \cap F_\infty$ the proof is similar). Then there are two cases to consider, since the argument depends on whether the point $\overline{p_0}$ is blown up or not. The more difficult case is the following:

Claim 5.5 *Suppose that $\overline{p_0}$ is not blown up, then the pair (5.1) is lc at $p = C \cap F_0$ for all sufficiently large k.*

Proof In this case, the curve F_0 intersects C tangentially at p. Write

$$\lambda D = \lambda F_0 \operatorname{ord}_{F_0} D + \lambda C \operatorname{ord}_C D + \Omega.$$

We put

$$m = \operatorname{mult}_p \Omega.$$

Since $2\beta\lambda = \lambda D.F_0 = 2\lambda \operatorname{ord}_C D + \Omega.F_0 \geq 2\lambda \operatorname{ord}_C D + m$, we get

$$m \leq 2\lambda(\beta - \operatorname{ord}_C D). \tag{5.7}$$

Let $g : \tilde{S} \to S$ be the blow-up of the point p, and let G be the exceptional curve of g. We let $\tilde{C}$, $\tilde{F}_0$ and $\tilde{\Omega}$ be the proper transform of C, F_0 and Ω respectively on the surface $\tilde{S}$. We put

$$\tilde{p} = \tilde{C} \cap G, \quad \tilde{m} = \operatorname{mult}_{\tilde{p}} \tilde{\Omega}.$$

Note that G, $\tilde{C}$ and $\tilde{F}_0$ are three smooth curves intersecting pairwise transversally at $\tilde{p}$.

To show the pair (5.1) is lc at p, it suffices to show the pair

$$(\tilde{S}, (1 - \beta + \lambda \operatorname{ord}_C D)\tilde{C} + \lambda \tilde{F}_0 \operatorname{ord}_{F_0} D + \tilde{\Omega} + (\lambda \operatorname{ord}_{F_0} D + m - \beta + \lambda \operatorname{ord}_C D)G)$$

is lc at any point $q \in G$.

First, suppose that $q \neq \tilde{p}$. We then need to prove that the pair

$$(\tilde{S}, \tilde{\Omega} + (\lambda \operatorname{ord}_{F_0} D + m - \beta + \lambda \operatorname{ord}_C D)G)$$

is lc at q. Note that $(\lambda \operatorname{ord}_{F_0} D + m - \beta + \lambda \operatorname{ord}_C D) \leq 1$ by Lemma 4.5 and (5.7), so we may apply Lemma 3.2 at q and it suffices to prove

$$G.\tilde{\Omega} \leq 1,$$

which is true since $G.\tilde{\Omega} = m \leq 1$ (recall (5.7)).

To finish the proof, it then suffices to show that the pair

$$(\tilde{S}, (1 - \beta + \lambda \operatorname{ord}_C D)\tilde{C} + \lambda \tilde{F}_0 \operatorname{ord}_{F_0} D + \tilde{\Omega} + (\lambda \operatorname{ord}_{F_0} D + m - \beta + \lambda \operatorname{ord}_C D)G)$$

is lc at $\tilde{p}$.

Let $h : \hat{S} \to \tilde{S}$ be the blow up of $\tilde{p}$ and let H be the exceptional curve of h. We let $\hat{C}$, $\hat{F}_0$, $\hat{G}$ and $\hat{\Omega}$ be the proper transform of $\tilde{C}$, $\tilde{F}_0$, G and $\tilde{\Omega}$ respectively on the surface $\hat{S}$. Then $\hat{C}$, $\hat{F}_0$ and $\hat{G}$ intersect transversally with H at three different points. Also notice that

$$2\lambda(\beta - \operatorname{ord}_C D) - m - \tilde{m} = \hat{F}_0.\hat{\Omega} \geq 0,$$

so we get

$$m + \tilde{m} \leq 2\lambda(\beta - \operatorname{ord}_C D). \tag{5.8}$$

Using $\tilde{m} \leq m$, we have

$$\tilde{m} \leq \lambda(\beta - \operatorname{ord}_C D). \tag{5.9}$$

And also, using Lemma 4.5, (5.8) and (4.3) we have (for small β)

$$(2\lambda \operatorname{ord}_{F_0} D + m + \tilde{m} - 2\beta + 2\lambda \operatorname{ord}_C D) \leq 1. \tag{5.10}$$

Now to finish the proof, it is enough to show that the log pair $(\hat{S}, (1 - \beta + \lambda \operatorname{ord}_C D)\hat{C} + \lambda \hat{F}_0 \operatorname{ord}_{F_0} D + \hat{\Omega} + (\lambda \operatorname{ord}_{F_0} D + m - \beta + \lambda \operatorname{ord}_C D)\hat{G} + (2\lambda \operatorname{ord}_{F_0} D + m + \tilde{m} - 2\beta + 2\lambda \operatorname{ord}_C D)H)$ is lc at any point $o \in H$.

First, suppose that $o \notin \hat{C} \cup \hat{F}_0 \cup \hat{G}$. Then we need to show

$$(\hat{S}, \hat{\Omega} + (2\lambda \operatorname{ord}_{F_0} D + m + \tilde{m} - 2\beta + 2\lambda \operatorname{ord}_C D)H))$$

is lc at o. By (5.10) and Lemma 3.2, it is enough to show $H.\hat{\Omega} \leq 1$, but $H.\hat{\Omega} = \tilde{m} \leq 1$ (recall (5.9)).

Second, suppose that $o = H \cap \hat{F}_0$. Then we need to show

$$(\hat{S}, \lambda \hat{F}_0 \operatorname{ord}_{F_0} D + \hat{\Omega} + (2\lambda \operatorname{ord}_{F_0} D + m + \tilde{m} - 2\beta + 2\lambda \operatorname{ord}_C D)H)$$

is lc at o. By 5.10 and Lemma 3.2, it is enough to show

$$H.(\lambda \hat{F}_0 \operatorname{ord}_{F_0} D + \hat{\Omega}) \leq 1,$$

i.e., $\lambda \operatorname{ord}_{F_0} D + \tilde{m} \leq 1$, and this follows from Lemma 4.5 and (5.9).

Third, suppose that $o = H \cap \hat{G}$. Then we need to show

$$(\hat{S}, \hat{\Omega} + (\lambda \operatorname{ord}_{F_0} D + m - \beta + \lambda \operatorname{ord}_C D)\hat{G}$$
$$+ (2\lambda \operatorname{ord}_{F_0} D + m + \tilde{m} - 2\beta + 2\lambda \operatorname{ord}_C D)H)$$

is lc at o. By (5.10) and Lemma 3.2, it is enough to show

$$H.(\hat{\Omega} + \lambda \operatorname{ord}_{F_0} D + m - \beta + \lambda \operatorname{ord}_C D)\hat{G}) \leq 1,$$

i.e., $\lambda \operatorname{ord}_{F_0} D + m + \tilde{m} - \beta + \lambda \operatorname{ord}_C D \leq 1$, which holds by Lemma 4.5 and (5.8).

Hence, to conclude the proof, it suffices to show the pair

$$(\hat{S}, (1 - \beta + \lambda \operatorname{ord}_C D)\hat{C} + \hat{\Omega} + (2\lambda \operatorname{ord}_{F_0} D + m + \tilde{m} - 2\beta + 2\lambda \operatorname{ord}_C D)H)$$

is lc at $o = H \cap \hat{C}$. Note that $\operatorname{mult}_o \hat{\Omega} \leq \tilde{m} \leq 1$, so by Corollary 3.5, it suffices to show

$$(\hat{C}.\hat{\Omega})_o \leq \frac{(H.\hat{\Omega})_o}{((H.\hat{\Omega})_o - (\beta - \lambda \operatorname{ord}_C D))_+}(1 - (2\lambda \operatorname{ord}_{F_0} D + m + \tilde{m}$$
$$- 2\beta + 2\lambda \operatorname{ord}_C D)) - (\beta - \lambda \operatorname{ord}_C D).$$

Now using Lemma 4.5, (4.3), (5.2) and (5.8), it is clear that, for β sufficiently small,

$$(2\lambda \operatorname{ord}_{F_0} D + m + \tilde{m} - 2\beta + 2\lambda \operatorname{ord}_C D) \leq 1 - \frac{\beta}{10}.$$

Also, $(\hat{C}.\hat{\Omega})_o \leq C.\Omega \leq 3$, and using (5.9), $(H.\hat{\Omega})_o \leq H.\hat{\Omega} = \tilde{m} \leq \lambda(\beta - \operatorname{ord}_C D)$. So by (3.9) it suffices to show

$$3 + (\beta - \lambda \operatorname{ord}_C D) \leq \frac{\lambda(\beta - \operatorname{ord}_C D)}{\lambda(\beta - \operatorname{ord}_C D) - (\beta - \lambda \operatorname{ord}_C D)} \cdot \frac{\beta}{10},$$

i.e., $3 + (\beta - \lambda \operatorname{ord}_C D) \leq \frac{\lambda(\beta - \operatorname{ord}_C D)}{(\lambda - 1)\beta} \cdot \frac{\beta}{10}$. Using Lemma 4.5 and (5.2), it is enough show $3 + \beta \leq \frac{\beta - \frac{2}{3}\beta}{\beta^2/100} \cdot \frac{\beta}{10}$, i.e., $3 + \beta \leq \frac{10}{3}$, concluding the proof. $\qquad \square$

Claim 5.6 *Suppose that $\overline{p_0}$ is blown up, then the pair (5.1) is lc at $p = C \cap F_0$ for all sufficiently large k.*

Proof In this case, both E_0 and F_0 meet C transversally at p. Write

$$\lambda D = \lambda E_0 \operatorname{ord}_{E_0} D + \lambda F_0 \operatorname{ord}_{F_0} D + \lambda C \operatorname{ord}_C D + \Omega.$$

Put $m = \operatorname{mult}_p \Omega$. We have

$$\begin{cases} \lambda\beta + \lambda \operatorname{ord}_{E_0} D - \lambda \operatorname{ord}_{F_0} D - \lambda \operatorname{ord}_C D = \Omega.E_0 \geq m, \\ \lambda\beta - \lambda \operatorname{ord}_{E_0} D + \lambda \operatorname{ord}_{F_0} D - \lambda \operatorname{ord}_C D = \Omega.F_0 \geq m. \end{cases}$$

Summing up, we get

$$m \leq \lambda(\beta - \operatorname{ord}_C D). \tag{5.11}$$

Then by Lemma 4.5, $r \geq 7$ and (5.2), we clearly have (for β small enough)

$$(\lambda \operatorname{ord}_{E_0} D + \lambda \operatorname{ord}_{F_0} D + m - \beta + \lambda \operatorname{ord}_C D) \leq 1. \tag{5.12}$$

Let $g : \tilde{S} \to S$ be the blow up of p and let G be the exceptional curve of h. We let $\tilde{C}$, $\tilde{E}_0$, $\tilde{F}_0$ and $\tilde{\Omega}$ be the proper transform of C, E_0, F_0 and Ω respectively on the surface $\hat{S}$. Then $\tilde{C}$, $\tilde{E}_0$ and $\tilde{F}_0$ intersect transversally with H at three different points.

To show the pair (5.1) is lc at p, it is enough to show the pair

$$(\tilde{S}, (1 - \beta + \lambda \operatorname{ord}_C D)\tilde{C} + \lambda\tilde{E}_0 \operatorname{ord}_{E_0} D + \lambda\tilde{F}_0 \operatorname{ord}_{F_0} D + \tilde{\Omega}$$
$$+ (\lambda \operatorname{ord}_{E_0} D + \lambda \operatorname{ord}_{F_0} D + m - \beta + \lambda \operatorname{ord}_C D)G)$$

is lc at any point $q \in G$.

First suppose that $q \notin \tilde{C} \cup \tilde{E}_0 \cup \tilde{F}_0$, then we need to show the pair

$$(\tilde{S}, \tilde{\Omega} + (\lambda \operatorname{ord}_{E_0} D + \lambda \operatorname{ord}_{F_0} D + m - \beta + \lambda \operatorname{ord}_C D)G)$$

is lc at q. Using (5.12) and Lemma 3.2, it suffices to show $G.\tilde{\Omega} \leq 1$, but $G.\tilde{\Omega} = m \leq 1$ (recall (5.11)).

Second, suppose that $q = G \cap \tilde{E}_0$. Then we need to show the pair

$$(\tilde{S}, \lambda\tilde{E}_0 \operatorname{ord}_{E_0} D + \tilde{\Omega} + (\lambda \operatorname{ord}_{E_0} D + \lambda \operatorname{ord}_{F_0} D + m - \beta + \lambda \operatorname{ord}_C D)G)$$

is lc at q. Using (5.12) and Lemma 3.2, it suffices to show

$$G.(\lambda\tilde{E}_0 \operatorname{ord}_{E_0} D + \tilde{\Omega}) \leq 1,$$

i.e., $\lambda \operatorname{ord}_{E_0} D + m \leq 1$, which is true by Lemma 4.5, (5.2) and (5.11). The proof for $q = G \cap \tilde{F}_0$ is similar.

So to finish the proof, it suffices to show the pair

$$(\tilde{S}, (1 - \beta + \lambda \operatorname{ord}_C D)\tilde{C} + \tilde{\Omega} + (\lambda \operatorname{ord}_{E_0} D + \lambda \operatorname{ord}_{F_0} D + m - \beta + \lambda \operatorname{ord}_C D)G)$$

is lc at $q = H \cap \tilde{C}$. Since $\operatorname{mult}_q \tilde{\Omega} \leq m \leq 1$ by (5.11), then by Corollary 3.5, it suffices to show

$$(\tilde{C}.\tilde{\Omega})_q \leq \frac{(G.\tilde{\Omega})_q}{((G.\tilde{\Omega})_q - (\beta - \lambda \operatorname{ord}_C D))_+}(1 - (\lambda \operatorname{ord}_{E_0} D + \lambda \operatorname{ord}_{F_0} D$$
$$+ m - \beta + \lambda \operatorname{ord}_C D)) - (\beta - \lambda \operatorname{ord}_C D).$$

Now using Lemma 4.5, (4.3), (5.2) and (5.11), we have (for β small enough)

$$(\lambda \operatorname{ord}_{E_0} D + \lambda \operatorname{ord}_{F_0} D + m - \beta + \lambda \operatorname{ord}_C D) \leq 1 - \frac{\beta}{10}.$$

Also we have $(\tilde{C}.\tilde{\Omega})_q \leq C.\Omega \leq 3$, and by (5.11), $(G.\tilde{\Omega})_q \leq G.\tilde{\Omega} = m \leq \lambda(\beta - \operatorname{ord}_C D)$. So by (3.9) it suffices to show

$$3 + (\beta - \lambda \operatorname{ord}_C D) \leq \frac{\lambda(\beta - \operatorname{ord}_C D)}{\lambda(\beta - \operatorname{ord}_C D) - (\beta - \lambda \operatorname{ord}_C D)} \cdot \frac{\beta}{10},$$

i.e., $3 + (\beta - \lambda \operatorname{ord}_C D) \leq \frac{\lambda(\beta - \operatorname{ord}_C D)}{(\lambda - 1)\beta} \cdot \frac{\beta}{10}$ Using Lemma 4.5 and (5.2), it is enough show $3 + \beta \leq \frac{\beta - \frac{2}{3}\beta}{\beta^2/100} \cdot \frac{\beta}{10}$, i.e., $3 + \beta \leq \frac{10}{3}$, concluding the proof. $\qquad\square$

Claims 5.1–5.6 and Definition 2.5 imply that there exists $b := b(r)$ such that for all rational $\beta \in (0, b)$ we have $\operatorname{blct}_k(S, (1 - \beta)C, -K_S - (1 - \beta)C) \geq 1 + \frac{\beta}{100}$ for all sufficiently large k. Thus, $\operatorname{blct}_\infty(S, (1 - \beta)C, -K_S - (1 - \beta)C) \geq 1 + \frac{\beta}{100}$. Hence, Theorem 1.3 follows from Theorem 2.6.

Appendix: Blct and the greatest Ricci lower bound

Let X be a Fano manifold. A well-known result of Demailly states $\operatorname{glct}(X, -K_X) = \alpha(X)$, i.e., the global log canonical threshold coincides with Tian's α-invariant [24]. Here we show that the basis log canonical threshold of Fujita–Odaka coincides with Tian's β-invariant. Recall the definition of the latter

$$\beta(X) := \sup\{\, b : \operatorname{Ric} \omega \geq b\omega, [\omega] = c_1(X)\},$$

where $\operatorname{Ric} \omega := -\sqrt{-1}/2\pi \cdot \partial\bar{\partial} \log \det(g_{i\bar{j}})$ denotes the Ricci form of $\omega = \sqrt{-1}/2\pi \cdot g_{i\bar{j}}(z)dz^i \wedge d\bar{z}^j$. This invariant was the topic of Tian's article [63] although it was not explicitly defined there, but was first explicitly defined by one of us in [55, (32)], [56, Problem 3.1] and was later further studied by Székelyhidi [61], Li [46], Song–Wang [60], and Cable [9].

Theorem 5.7 *On a Fano manifold X, $\beta(X) = \min\{\mathrm{blct}_\infty(X, -K_X), 1\}$.*

Some special cases of this are known. First, the result is inspired by the work of Blum–Jonsson who derived this identity in the special toric case by directly computing $\mathrm{blct}_\infty(X, -K_X)$ [4, Corollary 7.19] and observing it coincides with Li's formula for $\beta(X)$ for toric X [46]. Second, Theorem 5.7 is known if $(X, -K_X)$ is semistable in an algebraic/analytic sense. Indeed, Li [48] showed that $\beta(X) = 1$ if and only if the Mabuchi energy is bounded below solving a problem posed by one of us [56, Problem 3.1]. He also showed, using [28,65], that this happens if and only if $(X, -K_X)$ is K-semistable. On the other hand, by Fujita–Odaka and Blum–Jonsson [4,39] $(X, -K_X)$ is K-semistable if and only if $\mathrm{blct}_\infty(X, -K_X) \geq 1$. Below we give a short proof of Theorem 5.7 in the remaining case, i.e., when $(X, -K_X)$ is K-unstable, namely, when $\beta(X) \in (0, 1)$ ($\beta(X)$ is positive by the Calabi–Yau theorem), so Theorem 5.7 reduces to the formula

$$\mathrm{blct}_\infty(X, -K_X) = \beta(X). \tag{5.13}$$

Our strategy will be to use the scaling property $b^{-1}\mathrm{blct}_\infty(X, -K_X) = \mathrm{blct}_\infty(X, -bK_X)$ for $0 < b \in \mathbb{Q}$ [4, Remark 4.5] and show $\mathrm{blct}_\infty(X, -bK_X) \geq 1$ for $b \in (0, \beta(X)) \cap \mathbb{Q}$ and $\mathrm{blct}_\infty(X, -bK_X) \leq 1$ for $b \in (\beta(X), 1) \cap \mathbb{Q}$.

The proof makes use of KEE metrics (see Sect. 1.2 and [57] for background). In the edge setting, one has an analogue of $\beta(X)$ due to Donaldson [34] and Li–Sun [48]: for all $m \in \mathbb{N}$ sufficiently large, choose $\Delta_m \in |-mK_X|$ a smooth divisor (exists by Bertini's theorem), denote by $[\Delta_m]$ the current of integration along Δ_m, and set

$$\beta(X, \Delta_m/m) := \sup\{b > 0 : \mathrm{Ric}\,\omega = b\omega + (1-b)[\Delta_m]/m, [\omega] = c_1(X)\}.$$

It is known that [60] (see also [47, Corollary 2.4])

$$\lim_m \beta(X, \Delta_m/m)) = \beta(X). \tag{5.14}$$

In particular, fixing any $b \in (0, \beta(X)) \cap \mathbb{Q}$, there is $m_0 \in \mathbb{N}$ such that $\beta(X, \Delta_m/m)) > b$ for all $m \geq m_0$, and by definition and there are KEE metrics with Ricci curvature b and with angles $2\pi(1-b)/m$ along Δ_m (here we use [49, Theorem 1.1] that guarantees the interval of such values of b is connected) and therefore $(X, (1-b)\Delta_m/m, -bK_X)$ are log K-semistable [49, Corollary 1.12]. Thus by [29, Corollary 4.8], we have

$$\mathrm{blct}_\infty(X, (1-b)\Delta_m/m, -bK_X) \geq 1.$$

By definition, $\mathrm{blct}_\infty(X, -bK_X) \geq \mathrm{blct}_\infty(X, (1-b)\Delta_m/m, -bK_X)$. Thus, $\mathrm{blct}_\infty(X, -bK_X) \geq 1$ for each $b \in (0, \beta(X)) \cap \mathbb{Q}$, so $\mathrm{blct}_\infty(X, -K_X) \geq \beta(X)$.

For the other direction of (5.13) we make use an algebraic counterpart of (5.14):

Lemma 5.8 *For $b \in (0, 1) \cap \mathbb{Q}$, $\lim_m \mathrm{blct}_\infty(X, (1-b)\Delta_m/m, -bK_X) = \mathrm{blct}_\infty(X, -bK_X)$.*

Proof As just noted, one direction follows from the definitions.

For the reverse direction, first fix k, and then let $c \in \big(0, \mathrm{blct}_k(X, -bK_X)\big)$. Let $D \sim_{\mathbb{Q}} -bK_X$ be a k-basis divisor, so (X, cD) is lc. Observe that, of course, (X, Δ_m) is lc (as Δ_m is smooth). Recall that if (X, A) and (X, B) are lc then so is $(X, (1 - \delta)A + \delta B)$ for any $\delta \in (0, 1)$ [12, Remark 2.1]. Thus $(X, (1 - \delta)cD + \delta \Delta_m)$ is lc. Put $\delta = (1 - b)/m$ to obtain that

$$\mathrm{blct}_k(X, (1 - b)\Delta_m/m, -bK_X) \geq (1 - O(1/m)) \, \mathrm{blct}_k(X, -bK_X).$$

Now let first k and then m tend to infinity to conclude. □

Thus, suppose that $\mathrm{blct}_\infty(X, -bK_X) > 1$ for some $b \in (\beta(X), 1) \cap \mathbb{Q}$. By Lemma 5.8 $\mathrm{blct}_\infty\big(X, (1 - b)\Delta_m/m, -bK_X\big) > 1$ and $(X, (1 - b)\Delta_m/m, -bK_X)$ is uniformly log K-stable for all sufficiently large m [29, Corollary 4.8]. So it follows from [28,65] (see also [66]) that there exists a KEE metric associated to this triple, i.e., that $\beta(X, \Delta_m/m) \geq b$, contradicting (5.14). Thus, $\mathrm{blct}_\infty(X, -bK_X) \leq 1$, i.e., $\mathrm{blct}_\infty(X, -K_X) \leq b$ for all $b \in (\beta(X), 1) \cap \mathbb{Q}$. This concludes the proof of (5.13) and hence of Theorem 5.7.

Remark 5.9 In the last paragraph one may also use [8, Corollary 2.11] to obtain the polarized pair $\big(X, -bK_X\big)$ is K-semistable in the adjoint sense, hence twisted K-semistable in the sense of [32] (see [7, Proposition 8.2]). So [31, Proposition 10] guarantees that for some $b \in (\beta(X), 1)$, we can find two Kähler forms ω, α cohomologous to $c_1(X)$ such that $\mathrm{Ric}\,\omega = b\omega + (1 - b)\alpha$, that is again a contradiction.

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