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Finite collineation groups and birational rigidity

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Abstract
We classify finite groups G in $\mathrm{PGL}_4(\mathbb{C})$ such that $\mathbb{P}^3$ is G-birationally rigid.

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1 Introduction

The projective space $\mathbb{P}^3$ is one of the most basic objects in geometry. Its biregular and birational symmetries have attracted attention of many mathematicians. Among them was Hans Frederick Blichfeldt, who classified in [3] all finite subgroups of

$$\mathrm{Aut}\big(\mathbb{P}^3\big) \cong \mathrm{PGL}_4\big(\mathbb{C}\big)$$

back in 1905. In 1917, he wrote his *magnum opus* [4], which gave a detailed coverage of classification of finite subgroups in $\mathrm{PGL}_3(\mathbb{C})$ and $\mathrm{PGL}_4(\mathbb{C})$. This book became a standard reference for the theory of finite collineation groups. We want to celebrate its centennial by enriching this classical theory with modern technique of birational geometry.

Finite subgroups of $\mathrm{PGL}_4(\mathbb{C})$ are traditionally divided into two major types: transitive and intransitive. In geometric language, transitive groups do not fix any point in $\mathbb{P}^3$ and do not leave any line invariant. Intransitive groups are those that are not transitive. Transitive groups are further subdivided into imprimitive and primitive groups.

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Imprimitive groups either leave a union of two skew lines invariant or have an orbit of length 4 (or both). All other finite subgroups in $\mathrm{PGL}_4(\mathbb{C})$ are said to be primitive.

The structure of the group $\mathrm{Bir}(\mathbb{P}^3)$, known as the space Cremona group, is way more complicated than that of $\mathrm{PGL}_4(\mathbb{C})$. In particular, the classification of all finite subgroups in $\mathrm{Bir}(\mathbb{P}^3)$ seems to be out of reach at the moment. There are just several sporadic known results in this direction. These include the classification of all finite *simple* subgroups in $\mathrm{Bir}(\mathbb{P}^3)$ in [36], and some boundedness properties in [38]. The goal of this paper is to understand how the finite subgroups of $\mathrm{PGL}_4(\mathbb{C})$ described by Blichfeldt behave as subgroups of the larger group $\mathrm{Bir}(\mathbb{P}^3)$.

Given a finite subgroup G in $\mathrm{PGL}_4(\mathbb{C})$, the most natural birational problem is to describe all G-birational maps from $\mathbb{P}^3$ to other three-dimensional G-varieties. Minimal Model Program implies that to solve this problem it is enough to describe G-birational maps from $\mathbb{P}^3$ to *G-Mori fibre spaces* (see [10, Definition 1.1.5]). In this paper we aim to describe all finite subgroups $G \subset \mathrm{PGL}_4(\mathbb{C})$ such that the projective space $\mathbb{P}^3$ cannot be G-birationally transformed into other G-Mori fiber spaces. In this case, we say that $\mathbb{P}^3$ is *G-birationally rigid* (see [10, Definition 3.1.1]). This simply means that the following three conditions are satisfied:

(1) There is no G-rational map $\mathbb{P}^3 \dashrightarrow S$ whose general fiber is a rational curve.
(2) There is no G-rational map $\mathbb{P}^3 \dashrightarrow \mathbb{P}^1$ whose general fiber is a rational surface.
(3) There is no G-birational map $\mathbb{P}^3 \dashrightarrow X$ such that X is a Fano threefold with terminal singularities, the G-invariant class group of the threefold X is of rank 1 and X is not G-isomorphic to $\mathbb{P}^3$ (equipped with the same action of the group G).

The main result of our paper is the following theorem.

Theorem 1.1 *Let G be a finite subgroup in $\mathrm{PGL}_4(\mathbb{C})$. Then $\mathbb{P}^3$ is G-birationally rigid if and only if G is a primitive group that is not isomorphic to $\mathfrak{A}_5$ or $\mathfrak{S}_5$.*

It is a three-dimensional analogue of the following recent result of Sakovics.

Theorem 1.2 [40] *Let G be a finite subgroup in $\mathrm{PGL}_3(\mathbb{C})$. Then $\mathbb{P}^2$ is G-birationally rigid if and only if G is a transitive subgroup that is not isomorphic to $\mathfrak{A}_4$ or $\mathfrak{S}_4$.*

Let us describe the plan of the proof of Theorem 1.1. First, we show that if G is a finite intransitive or imprimitive subgroup of $\mathrm{PGL}_4(\mathbb{C})$, then $\mathbb{P}^3$ is not G-birationally rigid. This is done in Corollary 2.4. Thus, we are left with the case of finite primitive groups. Up to conjugation, there are just 30 such groups. They are listed in [4, Chapter VII]. For the reader's convenience, in Appendix A we present these groups together with (some) inclusions between them, including those that are necessary for the proof of Theorem 1.1. However, we point out that we will not need the whole classification for the proof, but only the information about the few smallest groups.

We prove that $\mathbb{P}^3$ is not G-birationally rigid if G is a primitive subgroup isomorphic to $\mathfrak{A}_5$ or $\mathfrak{S}_5$, see Corollary 2.10. This completes the proof of the "only if" part of Theorem 1.1. Thus, to prove Theorem 1.1, we have to show that $\mathbb{P}^3$ is G-birationally rigid for the remaining finite primitive subgroups G in $\mathrm{PGL}_4(\mathbb{C})$. These can be described as follows (as usual, we denote by $\boldsymbol{\mu}_n$ the cyclic group of order n).

- Primitive groups that leave a quadric surface invariant.
- Primitive subgroups in $\mathrm{PGL}_4(\mathbb{C})$ that contain the imprimitive group $\boldsymbol{\mu}_2^4$ as a normal subgroup. Some of them, like the group $\mathfrak{A}_4 \times \mathfrak{A}_4$, also leave a quadric surface invariant, so that this description is not exclusive (see the discussion in [4, §125]).
- The Klein simple group $\mathrm{PSL}_2(\mathbf{F}_7)$.
- The groups $\mathfrak{A}_6$, $\mathfrak{S}_6$, and $\mathfrak{A}_7$.
- The simple group of order 25920, which is isomorphic to $\mathrm{PSp}_4(\mathbf{F}_3)$.

Every primitive subgroup in $\mathrm{PGL}_4(\mathbb{C})$ that leaves a quadric surface invariant (and is not isomorphic to $\mathfrak{A}_5$ and $\mathfrak{S}_5$) contains a subgroup isomorphic to $\mathfrak{A}_4 \times \mathfrak{A}_4$. Likewise, every primitive subgroup that contains the imprimitive group $\boldsymbol{\mu}_2^4$ and does not leave any quadric surface invariant has a subgroup isomorphic to $\boldsymbol{\mu}_2^4 \rtimes \boldsymbol{\mu}_5$. Both of them contain the imprimitive subgroup $\boldsymbol{\mu}_2^4$ as a normal subgroup, so that they have similar group-theoretical properties. In Sect. 6, we prove that $\mathbb{P}^3$ is G-birationally rigid if G is isomorphic either to $\mathfrak{A}_4 \times \mathfrak{A}_4$ or to $\boldsymbol{\mu}_2^4 \rtimes \boldsymbol{\mu}_5$, see Corollaries 6.7 and 6.9. If $G \cong \mathfrak{A}_6$, then $\mathbb{P}^3$ is G-birationally rigid by [8, Theorem 1.24]. If $G \cong \mathrm{PSL}_2(\mathbf{F}_7)$, then $\mathbb{P}^3$ is G-birationally rigid by [7, Theorem 1.9].

Therefore, to prove Theorem 1.1, it remains to consider primitive subgroups in $\mathrm{PGL}_4(\mathbb{C})$ that contain one of the following three groups as a proper subgroup: $\mathfrak{A}_4 \times \mathfrak{A}_4$, $\boldsymbol{\mu}_2^4 \rtimes \boldsymbol{\mu}_5$, or $\mathfrak{A}_6$. We warn the reader that in general one cannot automatically conclude that $\mathbb{P}^3$ is G-birationally rigid if it is H-birationally rigid for a subgroup $H \subset G$, cf. Remark 6.5. However, in our particular cases, this is indeed true. We show this in Sect. 6, see Corollary 6.7 for groups containing $\mathfrak{A}_4 \times \mathfrak{A}_4$, Corollary 6.9 and Lemma 6.10 for groups containing $\boldsymbol{\mu}_2^4 \rtimes \boldsymbol{\mu}_5$, and Lemma 6.4 for groups containing $\mathfrak{A}_6$.

In addition to Theorem 1.1, we also provide a classification of finite subgroups $G \subset \mathrm{PGL}_4(\mathbb{C})$ such that $\mathbb{P}^3$ is G-birationally super-rigid. Recall from [10, Definition 3.1.1] that $\mathbb{P}^3$ is said to be G-birationally super-rigid if it is G-birationally rigid and

$$\mathrm{Bir}^G\big(\mathbb{P}^3\big) = \mathrm{Aut}^G\big(\mathbb{P}^3\big),$$

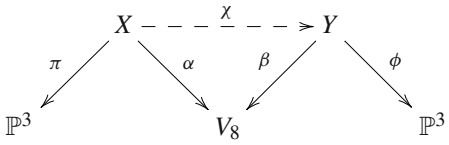
where $\mathrm{Bir}^G(\mathbb{P}^3)$ and $\mathrm{Aut}^G(\mathbb{P}^3)$ are the normalizers of the group G in $\mathrm{Bir}(\mathbb{P}^3)$ and $\mathrm{Aut}(\mathbb{P}^3)$, respectively. We prove the following

Theorem 1.3 *Let G be a finite subgroup in $\mathrm{PGL}_4(\mathbb{C})$. Then $\mathbb{P}^3$ is G-birationally super-rigid if and only if G is a primitive group that is isomorphic neither to $\mathfrak{A}_5$, nor to $\mathfrak{S}_5$, nor to $\mathrm{PSL}_2(\mathbf{F}_7)$, nor to $\mathfrak{A}_6$, nor to $\boldsymbol{\mu}_2^4 \rtimes \boldsymbol{\mu}_5$, nor to $\boldsymbol{\mu}_2^4 \rtimes \mathrm{D}_{10}$.*

Looking at Blichfeldt's classification of finite primitive subgroups in $\mathrm{PGL}_4(\mathbb{C})$, we conclude that $\mathbb{P}^3$ is G-birationally rigid for 23 subgroups G, and is G-birationally super-rigid for 19 subgroups G.

Let us describe the structure of the paper. In Sect. 2, we show that if G is a finite intransitive or imprimitive subgroup of $\mathrm{PGL}_4(\mathbb{C})$, or G is a primitive subgroup isomorphic to $\mathfrak{A}_5$ or $\mathfrak{S}_5$, then $\mathbb{P}^3$ is not G-birationally rigid. In Sect. 3, we present basic facts about the imprimitive group $\boldsymbol{\mu}_2^4$. In Sect. 4, we study equivariant geometry of $\mathbb{P}^3$ acted on by a primitive group $\mathfrak{A}_4 \times \mathfrak{A}_4$. Similarly, in Sect. 5, we do the same for the

group $\boldsymbol{\mu}_2^4 \rtimes \boldsymbol{\mu}_5$. In particular, we construct a G-commutative diagram



Here G is a primitive subgroup $\boldsymbol{\mu}_2^4 \rtimes \boldsymbol{\mu}_5$ in $\mathrm{PGL}_4(\mathbb{C})$, the morphisms π and ϕ are blow ups of G-invariant smooth irreducible curves of degree 8 and genus 5, the variety V_8 is a complete intersection of three quadrics in $\mathbb{P}^6$ that has isolated terminal Gorenstein singularities, the morphisms α and β are small birational contractions, and χ is a composition of flops. Finally, in Sect. 6, we prove our Theorem 1.1 using the results obtained in Sects. 4 and 5. In Appendix A, we present the finite primitive subgroups of $\mathrm{PGL}_4(\mathbb{C})$ from [4, Chapter VII] together with necessary inclusions between.

Throughout the paper we denote by $\boldsymbol{\mu}_n$ the cyclic group of order n. If G is a group and V is a variety acted on by G, we say that V is *G-irreducible* if G acts transitively on the set of irreducible components of V. If Z is a subvariety of V, we will sometimes abuse terminology and refer to the union of the images $g(Z)$, $g \in G$, as the *G-orbit* of Z. By a G-commutative diagram we will mean a commutative diagram of G-rational maps, see [10, p. xix]. All varieties are assumed to be projective and defined over the field of complex numbers.

2 Birational non-rigidity

In this section we prove that if $\mathbb{P}^3$ is G-birationally rigid, then G is a primitive group that is not isomorphic to $\mathfrak{A}_5$ or $\mathfrak{S}_5$. We start with

Proposition 2.1 *Suppose that G is a transitive subgroup in $\mathrm{PGL}_4(\mathbb{C})$ such that there exists a G-orbit in $\mathbb{P}^3$ of length 4. Then there exists a G-birational map $\mathbb{P}^3 \dashrightarrow V_{24}$ such that V_{24} is a toric Fano threefold, one has $-K_{V_{24}}^3 = 24$, the singular locus of V_{24} consists of 8 quotient singularities of type $\frac{1}{2}(1,1,1)$. In particular, the singularities of V_{24} are terminal and $\mathbb{Q}$-factorial. Moreover, one has $\mathrm{Pic}(V_{24}) \cong \mathbb{Z}^3$. Furthermore, if there is no G-invariant pair of lines in $\mathbb{P}^3$, then $\mathrm{Pic}(V_{24})^G \cong \mathbb{Z}$.*

Proof By assumption, there exists a G-orbit in $\mathbb{P}^3$ of length 4. Denote the points of this G-orbit by P_1, P_2, P_3 and P_4. Without loss of generality, we may assume that

$$P_1 = [1:0:0:0], \quad P_2 = [0:1:0:0], \quad P_3 = [0:0:1:0], \quad P_4 = [0:0:0:1].$$

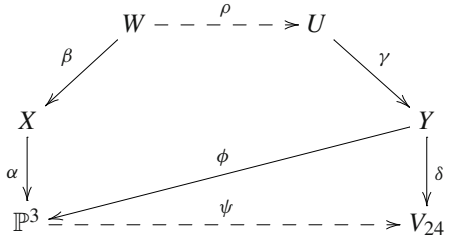
For every $1 \leqslant i < j \leqslant 4$, let L_{ij} be the line in $\mathbb{P}^3$ that passes though P_i and P_j. Let $\mathcal{M}$ be the linear system consisting of all sextic surfaces in $\mathbb{P}^3$ that are singular along every line L_{ij}. Then the surfaces in $\mathcal{M}$ are given by

$$a_0x^2y^2z^2 + a_1x^2y^2w^2 + a_2x^2z^2w^2 + a_3y^2z^2w^2 + a_4x^3yzw + a_5x^2y^2zw + a_6x^2yz^2w + a_7x^2yzw^2 + a_8xy^3zw + a_9xy^2z^2w + a_{10}xy^2zw^2 + a_{11}xyz^3w + a_{12}xyz^2w^2 + a_{13}xyzw^3 = 0 \tag{2.2}$$

for $[a_0 : \ldots : a_{13}] \in \mathbb{P}^{13}$. The linear system $\mathcal{M}$ gives a rational map $\psi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^{13}$. Note that ψ is well-defined away from the lines L_{12}, L_{13}, L_{14}, L_{23}, L_{24} and L_{34}. Moreover, the image of the map ψ is three-dimensional: one can express the ratios $\frac{x^2}{w^2}$ $\frac{y^2}{w^2}$ and $\frac{z^2}{w^2}$ as rational functions of the monomials $x^2y^2z^2$, $x^2y^2w^2$, $x^2z^2w^2$ and $y^2z^2w^2$. Denote the image of ψ by V_{24}. By construction, the threefold V_{24} is toric.

Let Π_1, Π_2, Π_3 and Π_4 be the planes in $\mathbb{P}^3$ that are given by $x = 0$, $y = 0$, $z = 0$ and $w = 0$, respectively. Then ψ contracts these planes to 4 different points.

To resolve the indeterminacy of the rational map ψ, let $\alpha : X \to \mathbb{P}^3$ be the blow up of the points P_1, P_2, P_3 and P_4. Denote by $\widetilde{L}_{ij}$ the proper transform of the line L_{ij} on the threefold X. Let $\beta : W \to X$ be the blow up of the curves $\widetilde{L}_{12}$, $\widetilde{L}_{13}$, $\widetilde{L}_{14}$, $\widetilde{L}_{23}$, $\widetilde{L}_{24}$ and $\widetilde{L}_{34}$. Then there exists a G-commutative diagram:



(2.3)

such that ρ is a composition of 12 Atiyah flops, and γ, δ and ϕ are birational morphisms that we are about to describe.

To describe ρ, denote by $\widetilde{\Pi}_i$, $1 \leqslant i \leqslant 4$, the proper transform on X of the plane Π_i. Then each $\widetilde{\Pi}_i$ is a smooth del Pezzo surface of degree 6. Denote by E_i the exceptional divisor of the birational morphism α that is mapped to the point P_i. Let ℓ_{ij} be the intersection curve $E_i \cap \widetilde{\Pi}_j$, and let $\overline{\ell}_{ij}$ be its proper transform on the threefold W. This gives us 12 disjoint curves $\overline{\ell}_{ij}$, $1 \leqslant i \neq j \leqslant 4$. The map ρ is a composition of Atiyah flops in these curves.

To describe γ and δ, denote by $\overline{E}_i$ and $\overline{\Pi}_i$, $1 \leqslant i \leqslant 4$, the proper transforms on W of the surfaces E_i and Π_i, respectively. Then all of them are smooth del Pezzo surfaces of degree 6. Moreover, the curves flopped by ρ are pairwise disjoint (-1)-curves on these surfaces. Denote by $\widehat{E}_i$ and $\widehat{\Pi}_i$ the proper transforms on U of the surfaces $\overline{E}_i$ and $\overline{\Pi}_i$, respectively. Then the surfaces $\widehat{E}_1$, $\widehat{E}_2$, $\widehat{E}_3$, $\widehat{E}_4$, $\widehat{\Pi}_1$, $\widehat{\Pi}_2$, $\widehat{\Pi}_3$ and $\widehat{\Pi}_4$ are pairwise disjoint, all of them are isomorphic to $\mathbb{P}^2$, and their normal bundles in U are $\mathcal{O}_{\mathbb{P}^2}(-2)$. The map γ is the contraction of the surfaces $\widehat{E}_1$, $\widehat{E}_2$, $\widehat{E}_3$ and $\widehat{E}_4$ to 4 quotient singular points of type $\frac{1}{2}(1, 1, 1)$. Similarly, the map δ is the contraction of the surfaces $\gamma(\widehat{\Pi}_1)$, $\gamma(\widehat{\Pi}_2)$, $\gamma(\widehat{\Pi}_3)$ and $\gamma(\widehat{\Pi}_4)$. Thus, the threefold V_{24} has exactly 8 singular points, and each of them is a quotient singularity of type $\frac{1}{2}(1, 1, 1)$. In particular, the singularities of V_{24} are terminal and $\mathbb{Q}$-factorial.

To describe ϕ, denote by $\widehat{F}_{12}$, $\widehat{F}_{13}$, $\widehat{F}_{14}$, $\widehat{F}_{23}$, $\widehat{F}_{24}$ and $\widehat{F}_{34}$ the proper transforms on Y of the β-exceptional surfaces that are mapped to the curves $\widetilde{L}_{12}$, $\widetilde{L}_{13}$, $\widetilde{L}_{14}$, $\widetilde{L}_{23}$, $\widetilde{L}_{24}$ and $\widetilde{L}_{34}$, respectively. Then ϕ is the contraction of these six surfaces to the lines L_{12}, L_{13}, L_{14}, L_{23}, L_{24} and L_{34}, respectively. By [27, Theorem 4.9], the morphism ϕ is the *symbolic blow up* of the reducible curve $L_{12} + L_{13} + L_{14} + L_{23} + L_{24} + L_{34}$, see also [37, §6.1].

By construction, the threefold V_{24} is toric Fano threefold, one has $-K^3_{V_{24}} = 24$ and $\mathrm{Pic}(V_{24}) \cong \mathbb{Z}^3$. If there is no G-invariant pair of lines in $\mathbb{P}^3$, then the group G permutes the six lines L_{ij} transitively. This gives $\mathrm{Pic}(V_{24})^G \cong \mathbb{Z}$. □

In the notation of [5], the threefold V_{24} in Proposition 2.1 is the terminal toric Fano threefold №47. One can show that a general hyperplane section of the threefold $V_{24} \subset \mathbb{P}^{13}$ is a smooth Enriques surface. Of course, this is well-known: the Eq. (2.2) was found by Enriques to construct what is now known as an Enriques surface. The variety V_{24} is one of the Fano–Enriques threefolds classified by Bayle in [1]. To be precise, the threefold V_{24} is a quotient of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ by an involution τ that acts as

$$\Big(\big[x_1 : y_1\big], \big[x_2 : y_2\big], \big[x_3 : y_3\big]\Big) \mapsto \Big(\big[y_1 : x_1\big], \big[y_2 : x_2\big], \big[y_3 : x_3\big]\Big).$$

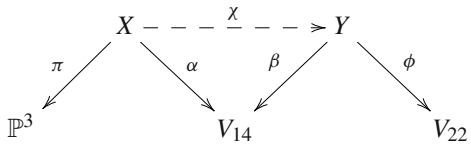
Corollary 2.4 *Suppose that G is a finite intransitive or imprimitive subgroup in $\mathrm{PGL}_4(\mathbb{C})$. Then $\mathbb{P}^3$ is not G-birationally rigid.*

Proof If G fixes a point in $\mathbb{P}^3$, then the linear projection from this point $\mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ is a G-rational dominant map whose general fiber is $\mathbb{P}^1$. Likewise, if there is a G-invariant line in $\mathbb{P}^3$, then the linear projection from this line $\mathbb{P}^3 \dashrightarrow \mathbb{P}^1$ is a G-rational dominant map whose general fiber is $\mathbb{P}^2$. Thus, if G is intransitive, then $\mathbb{P}^3$ is not G-birationally rigid.

To complete the proof, we may assume that G is imprimitive. If there is a G-invariant union of two skew lines in $\mathbb{P}^3$, then the blow up of $\mathbb{P}^3$ along them is a G-equivariant $\mathbb{P}^1$-bundle over $\mathbb{P}^1 \times \mathbb{P}^1$. Likewise, if there exists a G-orbit in $\mathbb{P}^3$ that consists of 4 points, but there is no G-invariant pair of skew lines, then $\mathbb{P}^3$ is G-birational to a toric Fano threefold V_{24} with terminal $\mathbb{Q}$-factorial singularities such that the G-invariant class group of V_{24} is of rank 1 by Proposition 2.1. Therefore, by definition, $\mathbb{P}^3$ is not G-birationally rigid in this case as well. □

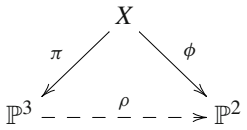
Thus, we are left with the case of primitive subgroups in $\mathrm{PGL}_4(\mathbb{C})$ isomorphic to $\mathfrak{A}_5$ or $\mathfrak{S}_5$. Up to conjugation, there are two primitive subgroups in $\mathrm{PGL}_4(\mathbb{C})$ isomorphic to $\mathfrak{A}_5$. One of them leaves a quadric surface invariant. Its action on $\mathbb{P}^3$ comes from the irreducible four-dimensional representation of the icosahedral group. Another one preserves a twisted cubic curve, so that its action on $\mathbb{P}^3$ comes from an irreducible four-dimensional representation of the binary icosahedral group. Similarly, there are two primitive subgroups in $\mathrm{PGL}_4(\mathbb{C})$ isomorphic to $\mathfrak{S}_5$. One of them preserves a quadric surface, and its action on $\mathbb{P}^3$ comes from an irreducible four-dimensional representation of the group $\mathfrak{S}_5$. Another one leaves invariant a pair of disjoint twisted cubic curves. Its action on $\mathbb{P}^3$ comes from an irreducible four-dimensional representation of a central extension of the group $\mathfrak{S}_5$. In all these cases, the projective space $\mathbb{P}^3$ is not G-birationally rigid.

Example 2.5 ([42, 2.13.1], [35, §2], [10, Example 1.2.8]) Let G be a primitive finite subgroup $\mathfrak{A}_5$ in $\mathrm{PGL}_4(\mathbb{C})$ such that there exists a G-invariant quadric surface in $\mathbb{P}^3$. Then there is a unique G-invariant smooth cubic surface in $\mathbb{P}^3$, which is known as the *Clebsch cubic surface*. Denote this cubic surface by S. Then S contains a G-invariant irreducible smooth rational sextic curve. Denote it by $\mathcal{C}$. The surface S also contains a G-invariant curve that consists of six disjoint lines, which are bi-tangents of the curve $\mathcal{C}$. Denote this curve by $\mathcal{L}_6$. Let $\pi\colon X \to \mathbb{P}^3$ be the blow-up of the rational sextic curve $\mathcal{C}$. Then there exists a G-commutative diagram (that is, a commutative diagram of G-rational maps, see [10, p. xix] for terminology)



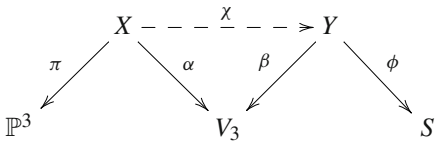
Here V_{14} is a Fano threefold with six isolated ordinary double points such that $-K^3_{V_{14}} = 14$, and V_{22} is a smooth Fano threefold such that $-K^3_{V_{22}} = 22$ and $\mathrm{Aut}(V_{22}) \cong \mathrm{PSL}_2(\mathbb{C})$, which is known as the Mukai–Umemura threefold (see [34]). The morphism α is the contraction of the proper transforms of the irreducible components of the curve $\mathcal{L}_6$, the rational map χ is a composition of Atiyah flops in these 6 curves, the morphism β is a flopping contraction, and ϕ is a contraction of the proper transform of the cubic surface S to a smooth point of the threefold V_{22}.

Example 2.6 [41, Application 1] Let G be a primitive finite subgroup $\mathfrak{A}_5$ in $\mathrm{PGL}_4(\mathbb{C})$ such that $\mathbb{P}^3$ contains a G-invariant twisted cubic curve $\mathcal{C}$. Let $\pi\colon X \to \mathbb{P}^3$ be the blow-up of the curve $\mathcal{C}$. Then there exists a G-commutative diagram



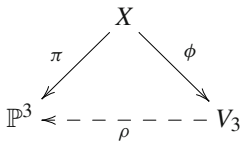
Here ϕ is a $\mathbb{P}^1$-bundle whose fibers are proper transforms of secants of the curve $\mathcal{C}$, and the map ρ is given by the linear system of quadric surfaces that contain $\mathcal{C}$.

Example 2.7 [35, Proposition 4.7] Let G be a primitive finite subgroup $\mathfrak{S}_5$ in $\mathrm{PGL}_4(\mathbb{C})$ such that there exists a G-invariant quadric surface in $\mathbb{P}^3$. Then $\mathbb{P}^3$ contains a G-orbit Σ_5 of length 5. Let $\pi\colon X \to \mathbb{P}^3$ be the blow up of this orbit. Then there exists a G-commutative diagram



Here α is the contraction of the proper transforms of the 10 lines in $\mathbb{P}^3$ passing through pairs of points in Σ_5, the rational map χ is a composition of Atiyah flops in these 10 curves, the morphism β is a flopping contraction, the morphism ϕ is a $\mathbb{P}^1$-bundle, the variety V_3 is the Segre cubic hypersurface in $\mathbb{P}^4$, and S is the smooth del Pezzo surface of degree 5.

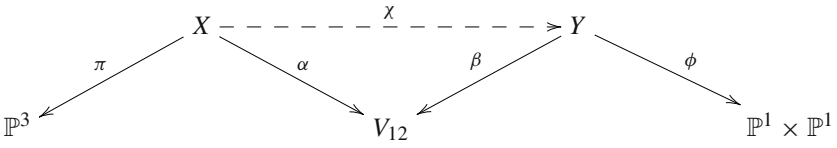
Example 2.8 Similarly to Example 2.7, let G be a primitive finite subgroup $\mathfrak{S}_5$ in $\mathrm{PGL}_4(\mathbb{C})$ such that there exists a G-invariant quadric surface in $\mathbb{P}^3$. Then such quadric surface is smooth and unique. Denote it by Q. There is a unique G-invariant smooth cubic surface in $\mathbb{P}^3$, which is the Clebsch cubic surface. The complete intersection of this cubic surface and the quadric surface Q is a G-invariant irreducible smooth sextic curve of genus 4, which is known as the *Bring curve*. Let $\pi : X \to \mathbb{P}^3$ be the blow up of this curve. Then there exists a G-commutative diagram



Here V_3 is a cubic hypersurface in $\mathbb{P}^4$ that has one isolated ordinary double point, the morphism ϕ is a contraction of the proper transform of the quadric Q to the singular point of the cubic V_3, and ρ is a linear projection from this point.

The case of a primitive subgroup $\mathfrak{S}_5$ in $\mathrm{PGL}_4(\mathbb{C})$ that does not leave invariant any quadric surfaces is a little bit more tricky.

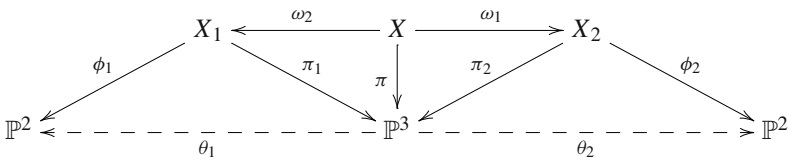
Proposition 2.9 *Suppose that G is a primitive subgroup in $\mathrm{PGL}_4(\mathbb{C})$ such that $G \cong \mathfrak{S}_5$, and G leaves invariant a pair of disjoint twisted cubic curves. Denote them by C_1 and C_2. Let $\pi : X \to \mathbb{P}^3$ be the blow up of the curves C_1 and C_2. Then there is a G-commutative diagram*



Here V_{12} is a divisor of bi-degree $(2, 2)$ in $\mathbb{P}^2 \times \mathbb{P}^2$ with ten isolated terminal Gorenstein singularities, α is a small birational morphism that contracts proper transforms of ten common secants of the curves C_1 and C_2, the map χ is a composition of ten flops, the morphism β is a flopping contraction, the morphism ϕ is a $\mathbb{P}^1$-bundle, and the action of the group G on $\mathbb{P}^1 \times \mathbb{P}^1$ is faithful.

Proof Denote by Γ the subgroup in G isomorphic to $\mathfrak{A}_5$. Let $\pi_1 : X_1 \to \mathbb{P}^3$ be the blow up of the curve C_1, and let $\pi_2 : X_2 \to \mathbb{P}^3$ be the blow up of the curve C_2. Then

X_1 and X_2 are isomorphic Fano threefolds. Moreover, it follows from Example 2.6 that there is a Γ-commutative diagram



Here ϕ_1 and ϕ_2 are $\mathbb{P}^1$-bundles, θ_1 and θ_2 are rational maps that are given by the linear systems of quadrics passing through C_1 and C_2, respectively, and ω_1 and ω_2 are blow ups of proper transforms of the curves C_1 and C_2, respectively. Note that the action of Γ on $\mathbb{P}^2$ is faithful.

We claim that there are exactly 10 lines in $\mathbb{P}^3$ that are common secants of both twisted cubic curves C_1 and C_2 (cf. [46, §2] or [19, Remark 2]). Indeed, denote by E_1^1 the π_1-exceptional surface. Then the morphism ϕ_1 is given by the linear system $|\pi_1^*(\mathcal{O}_{\mathbb{P}^2}(2)) - E_1^1|$, and its fibers are proper transforms of the secant lines of the curve C_1. Denote by $\overline{C}_2$ the proper transform of the curve C_2 on the threefold X_1. Since $\overline{C}_2$ is Γ-invariant, the curve $\phi_1(\overline{C}_2)$ is also Γ-invariant. By [10, Lemma 5.3.1], there are no Γ-invariant irreducible curves in $\mathbb{P}^2$ of degree 1 and 3. Thus we see that either $\phi_1(\overline{C}_2)$ is an irreducible conic, or $\phi_1(\overline{C}_2)$ is an irreducible curve of degree 6. The former case is clearly impossible. Indeed, if $\phi_1(\overline{C}_2)$ is a conic, then the induced morphism $\overline{C}_2 \to \phi_1(\overline{C}_2)$ is a triple cover, so that the curve C_2 has infinitely many 3-secant lines, which are also secants of the curve C_1. However, twisted cubic curves in $\mathbb{P}^2$ do not have 3-secant lines at all, because they are cut out by quadrics. Therefore, we see that $\phi_1(\overline{C}_2)$ is an irreducible curve of degree 6, so that the induced morphism $\overline{C}_2 \to \phi_1(\overline{C}_2)$ is birational. By [10, Lemma 5.3.1], the sextic curve $\phi_1(\overline{C}_2)$ is contained in the pencil that consists of all Γ-invariant curves of degree 6 in $\mathbb{P}^2$. The curve $\phi_1(\overline{C}_2)$ must be singular, since the curve C_2 is rational. All singular curves in this pencil are described in [10, Remark 6.1.5]. Using this description, we deduce that $\phi_1(\overline{C}_2)$ has 10 double points (cf. [19, Remark 5]), and these singular points form one Γ-orbit in $\mathbb{P}^2$. Note that the singular points of $\phi_1(\overline{C}_2)$ are ordinary double points. Indeed, since the arithmetic genus of the curve $\phi_1(\overline{C}_2)$ equals 10, its singular points are either ordinary double points or ordinary cusps. In the latter case, the curve $\overline{C}_2$ would have a Γ-orbit of length 10, which is impossible by [10, Lemma 5.1.5]. Thus, every singular point of the curve $\phi_1(\overline{C}_2)$ is an ordinary double point. The fibers of ϕ_1 over the singular points of the curve $\phi_1(\overline{C}_2)$ are exactly the proper transforms of the common secants of both curves C_1 and C_2. Vice versa, if ℓ is a common secant of the curves C_1 and C_2, then $\theta_1(\ell)$ must be a singular point of the curve $\phi_1(\overline{C}_2)$. Hence, we conclude that there are exactly 10 lines in $\mathbb{P}^3$ that are secants of both C_1 and C_2, and the group Γ acts transitively on them.

Denote by E_1^1 the π_1-exceptional surface, and denote by E_2^2 the π_2-exceptional surface. Then

$$-K_X \sim \omega_2^*\Big(\pi_1^*\big(\mathcal{O}_{\mathbb{P}^2}(2)\big) - E_1^1\Big) + \omega_1^*\Big(\pi_2^*\big(\mathcal{O}_{\mathbb{P}^2}(2)\big) - E_2^2\Big).$$

Moreover, both divisors $\pi_1^*(\mathcal{O}_{\mathbb{P}^2}(2)) - E_1^1$ and $\pi_2^*(\mathcal{O}_{\mathbb{P}^2}(2)) - E_2^2$ are nef. In particular, the divisor $-K_X$ is nef and big, since $-K_X^3 = 12$.

Denote the 10 common secants of both C_1 and C_2 by $L_1, \dots, L_{10}$, and denote their proper transforms on X by $\widetilde{L}_1, \dots, \widetilde{L}_{10}$, respectively. Then $-K_X$ intersects these curves trivially. Moreover, the curves $\widetilde{L}_1, \dots, \widetilde{L}_{10}$ are the only irreducible curves in X that have trivial intersection with $-K_X$. Indeed, let C be an irreducible curve in the threefold X such that $-K_X \cdot C = 0$. Then

$$\omega_2(C) \cdot \left(\pi_1^*\big(\mathcal{O}_{\mathbb{P}^2}(2)\big) - E_1^1\right) = \omega_1(C) \cdot \left(\pi_2^*\big(\mathcal{O}_{\mathbb{P}^2}(2)\big) - E_2^2\right) = 0,$$

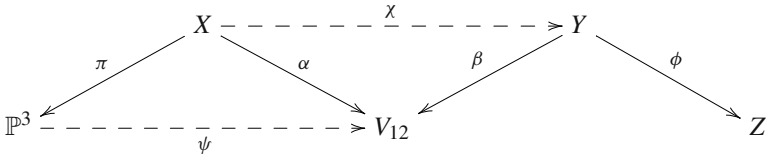
and $\pi(C)$ is an irreducible curve. Since ϕ_1 is given by the linear system $|\pi_1^*(\mathcal{O}_{\mathbb{P}^2}(2)) - E_1^1|$, we see that $\omega_2(C)$ is contracted by ϕ_1 to a point. Similarly, we see that $\omega_1(C)$ is contracted by ϕ_2 to a point. Hence, the curve $\pi(C)$ is a common secant of both curves C_1 and C_2, which implies that C is one of the curves $\widetilde{L}_1, \dots, \widetilde{L}_{10}$.

Consider the morphism

$$\alpha = \pi_1 \circ \omega_2 = \pi_2 \circ \omega_1.$$

We already know that the image of α is three-dimensional. Now we claim that α is birational. Indeed, choose a general fiber of α, and suppose that it contains at least two different points, say A and A'. Then $\omega_1(A) = \omega_1(A')$, so that $\pi(A)$ and $\pi(A')$ lie on a secant of the twisted cubic C_2. Similarly, we have $\omega_2(A) = \omega_2(A')$, so that $\pi(A)$ and $\pi(A')$ lie on a secant of C_1. On the other hand, since the fiber of α was chosen to be general, we can assume that the points $\pi(A)$ and $\pi(A')$ are different points of $\mathbb{P}^3$. Hence they lie on one of the lines $L_1, \dots, L_{10}$, which gives a contradiction.

We have constructed the morphism $\alpha\colon X \to V_{12}$, where V_{12} is a divisor of bi-degree $(2, 2)$ in $\mathbb{P}^2 \times \mathbb{P}^2$. The variety V_{12} is an anticanonical model of the threefold X, and the morphism α is given by the linear system $|-K_X|$. The threefold V_{12} has 10 isolated terminal Gorenstein singular points, which are the images of the curves $\widetilde{L}_1, \dots, \widetilde{L}_{10}$. Thus, there exists a G-commutative diagram



Here ψ is a birational map that is given by the linear system of quartic surfaces in $\mathbb{P}^3$ that contain both curves C_1 and C_2, and χ is a composition of flops in the curves $\widetilde{L}_1, \dots, \widetilde{L}_{10}$. The map β is a flopping contraction, and the morphism ϕ is a G-extremal contraction such that the divisor $-K_Y$ is ϕ-ample. Now arguing as in [17,42] and using [32, Theorem 3.3] together with [15, Theorem 4], we see that ϕ is a $\mathbb{P}^1$-bundle, and $Z \cong \mathbb{P}^1 \times \mathbb{P}^1$. Finally, using the fact that our $\mathbb{P}^3$ contains exactly two Γ-orbits of length 12 by [9, Lemma 3.2], we conclude that the group G acts faithfully on Z. □

In fact, one can show that the variety V_{12} in Proposition 2.9 has isolated ordinary double points, and χ is a composition of Atiyah flops. Moreover, one can show that a general fiber of the rational map $\phi \circ \chi \circ \pi^{-1}$ is a twisted cubic curve Z such that both intersections $Z \cap C_1$ and $Z \cap C_2$ consist of 5 points.

Corollary 2.10 *Let G be a subgroup in $\mathrm{PGL}_4(\mathbb{C})$ such that $G \cong \mathfrak{A}_5$ or $G \cong \mathfrak{S}_5$. Then $\mathbb{P}^3$ is not G-birationally rigid.*

Proof If G is not primitive, the assertion follows from Corollary 2.4. If G is primitive, the assertion follows from Examples 2.5, 2.6, 2.7 and Proposition 2.9. Alternatively, instead of using Example 2.7, we can use Example 2.8. □

3 The Heisenberg group

Let $\varphi\colon \mathrm{SL}_4(\mathbb{C}) \to \mathrm{PGL}_4(\mathbb{C})$ be the natural projection. For every group $G \subset \mathrm{SL}_4(\mathbb{C})$, let us denote by $\overline{G}$ the group $\varphi(G)$. Similarly, for every element $g \in \mathrm{SL}_4(\mathbb{C})$, we denote by $\overline{g}$ its image $\varphi(g)$. Let

$$S_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$
$$T_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{3.1}$$

and denote by H the subgroup in $\mathrm{SL}_4(\mathbb{C})$ that is generated by the matrices S_1, S_2, T_1, T_2. Then the center of H coincides with the commutator subgroup of H and consists of $\pm I_4$, where I_4 is the identity matrix. Since $|\mathrm{H}| = 32$, we have $\overline{\mathrm{H}} \cong \boldsymbol{\mu}_2^4$. We say that H is the Heisenberg group. The group $\overline{\mathrm{H}}$ is a transitive imprimitive subgroup of $\mathrm{PGL}_4(\mathbb{C})$.

Lemma 3.2 *The following assertions hold.*

(i) *For every non-trivial element $g \in \overline{\mathrm{H}}$, the locus of fixed points of g consists of two skew lines L_g and L'_g in $\mathbb{P}^3$.*

(ii) *For two distinct non-trivial elements g and h in $\overline{\mathrm{H}}$, one has*

$$\left\{L_g, L'_g\right\} \cap \left\{L_h, L'_h\right\} = \varnothing.$$

(iii) *No subgroup of order 8 in $\overline{\mathrm{H}}$ has a fixed point in $\mathbb{P}^3$.*

Proof Assertions (i) and (ii) are easy to check by direct computations (cf. [21, §2.2]). Assertion (iii) follows from the transitivity of the subgroup $\overline{\mathrm{H}}$. □

Lemma 3.3 *Let E be a rational or an elliptic curve. Then $\overline{\mathrm{H}}$ cannot act faithfully on E.*

Proof If E is rational, the assertion follows from the classification of finite subgroups of the group $\mathrm{PGL}_2(\mathbb{C})$. Thus, we suppose that E is a (smooth) elliptic curve with a faithful action of $\overline{\mathrm{H}}$. Then there is an exact sequence of groups

$$1 \to E(\mathbb{C}) \to \mathrm{Aut}(E) \to \boldsymbol{\mu}_r \to 1,$$

where $E(\mathbb{C})$ is the group of points of E acting on E by translations, and $r \leqslant 6$. We see that $\overline{\mathrm{H}} \cap E(\mathbb{C})$ contains a subgroup isomorphic to $\boldsymbol{\mu}_2^3$. This is impossible, because there are just four 2-torsion points in $E(\mathbb{C})$. □

Denote by $\mathbf{N}$ the normalizer of the group H in $\mathrm{SL}_4(\mathbb{C})$, and let

$$S = \frac{1+i}{\sqrt{2}}\begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad T = \frac{1+i}{2}\begin{pmatrix} -i & 0 & 0 & i \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -i & i & 0 \end{pmatrix}, \tag{3.4}$$

where $i = \sqrt{-1}$. One can check that both S and T are contained in $\mathbf{N}$, and $S^4 = T^5 = -I_4$. Let

$$A = \frac{1+i}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & i & -i & 0 \\ i & 0 & 0 & -i \end{pmatrix}, \quad B = \frac{1+i}{2}\begin{pmatrix} -1 & i & 0 & 0 \\ 1 & i & 1 & 0 \\ 0 & 0 & i & -1 \\ 0 & 0 & -i & -1 \end{pmatrix}.$$

Then A and B are contained in $\mathbf{N}$, because $A = T^4 \cdot S \cdot T \cdot S$ and

$$B = S \cdot T^4 \cdot S \cdot T \cdot S \cdot T^2 \cdot S \cdot T^3 \cdot S \cdot T \cdot S \cdot T^3 \cdot S \cdot T^3 \cdot S.$$

Similarly, let

$$R = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i & 0 & 0 \\ i & 1 & 0 & 0 \\ 0 & 0 & i & 1 \\ 0 & 0 & -1 & -i \end{pmatrix}.$$

Then R is also contained in $\mathbf{N}$ (see [4, §124]).

It follows from [4, §124] that S and T together with H and $\pm i I_4$ generate the group $\mathbf{N}$. Let $\mathbb{H} = \langle \mathrm{H}, \pm i I_4 \rangle$. Then $\overline{\mathbb{H}} = \overline{\mathrm{H}}$, and it follows from [4, §124] that there is an exact sequence

$$1 \longrightarrow \mathbb{H} \longrightarrow \mathbf{N} \longrightarrow \mathfrak{S}_6 \longrightarrow 1. \tag{3.5}$$

Let $\mathcal{U}_4$ be the vector space of all $\mathbb{H}$-invariant homogeneous polynomials of degree 4 in $\mathbb{C}[x, y, z, w]$. Recall from [21, §2.3] that $\mathcal{U}_4$ is generated by the polynomials

$$x^4 + y^4 + z^4 + w^4, \quad x^2y^2 + z^2w^2, \quad x^2z^2 + y^2w^2, \quad x^2w^2 + y^2z^2, \quad xyzw.$$

This vector space is $\mathbf{N}$-invariant. Write

$$\begin{aligned}
t_0 &= \frac{x^4 + y^4 + z^4 + w^4}{3} - 2\big(x^2y^2 + z^2w^2\big) - 2\big(x^2z^2 + y^2w^2\big) - 2\big(x^2w^2 + y^2z^2\big),\\
t_1 &= \frac{x^4 + y^4 + z^4 + w^4}{3} - 2\big(x^2y^2 + z^2w^2\big) + 2\big(x^2z^2 + y^2w^2\big) + 2\big(x^2w^2 + y^2z^2\big),\\
t_2 &= \frac{x^4 + y^4 + z^4 + w^4}{3} + 2\big(x^2y^2 + z^2w^2\big) - 2\big(x^2z^2 + y^2w^2\big) + 2\big(x^2w^2 + y^2z^2\big),\\
t_3 &= \frac{x^4 + y^4 + z^4 + w^4}{3} + 2\big(x^2y^2 + z^2w^2\big) + 2\big(x^2z^2 + y^2w^2\big) - 2\big(x^2w^2 + y^2z^2\big),\\
t_4 &= -\frac{2}{3}\big(x^4 + y^4 + z^4 + w^4\big) + 8xyzw,\\
t_5 &= -\frac{2}{3}\big(x^4 + y^4 + z^4 + w^4\big) - 8xyzw.
\end{aligned} \tag{3.6}$$

Note that $t_0 + t_1 + t_2 + t_3 + t_4 + t_5 = 0$, and $t_0, t_1, t_2, t_3, t_3, t_4, t_5$ generate $\mathcal{U}_4$.

Remark 3.7 The representation of the group $\mathbf{N}/\mathbb{H} \cong \mathfrak{S}_6$ in the vector space $\mathcal{U}_4$ is the standard five-dimensional representation twisted by the sign character. Indeed, the matrix T acts on the polynomials $t_0, \dots, t_5$ as the cycle $(t_0\, t_4\, t_2\, t_5\, t_1)$, and the matrix $-S$ acts on $t_0, \dots, t_5$ as the permutation $(t_0\, t_1)(t_2\, t_3)(t_4\, t_5)$. Observe also that the matrix A acts on them as the permutation $(t_0\, t_5\, t_2)(t_1\, t_3\, t_4)$. Similarly, the matrix B acts as the permutation $(t_0\, t_2\, t_5)(t_1\, t_3\, t_4)$.

Let G_{80} be the subgroup in $\mathrm{SL}_4(\mathbb{C})$ that is generated by $\mathbb{H}$ and the matrix T, and let $\overline{G}_{80}$ be its image in $\mathrm{PGL}_4(\mathbb{C})$. Then

$$\overline{G}_{80} \cong \boldsymbol{\mu}_2^4 \rtimes \boldsymbol{\mu}_5$$

by Remark 3.7. Similarly, let G_{320} be the subgroup in $\mathrm{SL}_4(\mathbb{C})$ that is generated by $\mathbb{H}$ and the matrices T and R, and let $\overline{G}_{320}$ be its image in $\mathrm{PGL}_4(\mathbb{C})$. Then there is a non-split exact sequence of groups

$$1 \longrightarrow \overline{\mathrm{H}} \longrightarrow \overline{G}_{320} \longrightarrow \boldsymbol{\mu}_5 \rtimes \boldsymbol{\mu}_4 \longrightarrow 1,$$

see [4, §124]. Moreover, it follows from (3.5) that there exists an exact sequence of groups

$$1 \longrightarrow \mathbb{H} \longrightarrow G_{320} \longrightarrow \boldsymbol{\mu}_5 \rtimes \boldsymbol{\mu}_4 \longrightarrow 1.$$

Let G_{160} be the subgroup in $\mathrm{SL}_4(\mathbb{C})$ that is generated by $\mathbb{H}$ and the matrices T and R^2, and let $\overline{G}_{160}$ be its image in $\mathrm{PGL}_4(\mathbb{C})$. Then

$$\overline{G}_{160} \cong \boldsymbol{\mu}_2^4 \rtimes \big(\boldsymbol{\mu}_5 \rtimes \boldsymbol{\mu}_2\big) \cong \boldsymbol{\mu}_2^4 \rtimes \mathrm{D}_{10},$$

where D_{10} is the dihedral group of order 10, see [4, §124]. There exists an exact sequence of groups

$$1 \longrightarrow \mathbb{H} \longrightarrow G_{160} \longrightarrow \mathrm{D}_{10} \longrightarrow 1.$$

Furthermore, the group $\overline{G}_{80}$ is a normal subgroup in both $\overline{G}_{160}$ and $\overline{G}_{320}$, and the group $\overline{G}_{160}$ is a normal subgroup in $\overline{G}_{320}$. Let G_{144} be the subgroup in $\mathrm{SL}_4(\mathbb{C})$ that is generated by the subgroup $\mathbb{H}$ together with the matrices A and B, and let $\overline{G}_{144}$ be its image in $\mathrm{PGL}_4(\mathbb{C})$. Then G_{144} is a finite subgroup of order 576, and there exists an exact sequence of groups

$$1 \longrightarrow \mathbb{H} \longrightarrow G_{144} \longrightarrow \boldsymbol{\mu}_3 \times \boldsymbol{\mu}_3 \longrightarrow 1.$$

Moreover, one has

$$\overline{G}_{144} \cong \boldsymbol{\mu}_2^4 \rtimes \big(\boldsymbol{\mu}_3 \times \boldsymbol{\mu}_3\big) \cong \mathfrak{A}_4 \times \mathfrak{A}_4$$

by Remark 3.7. The groups $\overline{G}_{80}$, $\overline{G}_{160}$, $\overline{G}_{320}$, $\overline{G}_{144}$ are primitive subgroups in $\mathrm{PGL}_4(\mathbb{C})$.

It is well known that $\mathbb{P}^3$ contains exactly ten $\overline{\mathrm{H}}$-invariant quadrics (see [21, §2.4]). The groups $\overline{G}_{80}$, $\overline{G}_{160}$, $\overline{G}_{320}$ and $\overline{G}_{144}$ naturally act on them, because $\overline{\mathrm{H}}$ is a normal subgroup of these four groups. For instance, the group $\overline{G}_{80}$ splits them into two $\overline{G}$-orbits. One of them consists of the five quadrics

$$\begin{aligned} &x^2+y^2+z^2+w^2=0, \quad xw+yz=0, \quad xz+yw=0,\\ &x^2+y^2-z^2-w^2=0, \quad x^2-y^2-z^2+w^2=0. \end{aligned} \tag{3.8}$$

Let us denote them by $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$, $\mathcal{Q}_4$ and $\mathcal{Q}_5$, respectively. The second $\overline{G}_{80}$-orbit consists of the five quadrics

$$x^2-y^2+z^2-w^2=0, \quad xy+zw=0, \quad xy-zw=0, \quad xz-yw=0, \quad xw-yz=0. \tag{3.9}$$

We will denote them by $\mathcal{Q}_6$, $\mathcal{Q}_7$, $\mathcal{Q}_8$, $\mathcal{Q}_9$ and $\mathcal{Q}_{10}$, respectively. Observe that all these ten quadrics are smooth. Similarly, the only $\overline{G}_{144}$-invariant quadric in $\mathbb{P}^3$ is the quadric $\mathcal{Q}_3$, i.e. the quadric given by $xz+yw=0$. In fact, we can say more.

Lemma 3.10 *There are no $\overline{G}_{80}$-invariant surfaces in $\mathbb{P}^3$ of degree 1, 2 and 3. Similarly, there are no $\overline{G}_{144}$-invariant surfaces in $\mathbb{P}^3$ of degree 1 and 3. Finally, there is unique $\overline{G}_{144}$-invariant quadric surface in $\mathbb{P}^3$.*

Proof This follows from explicit computations of symmetric powers of corresponding four-dimensional representations of the groups G_{80} and G_{144}. We used [44] to perform these computations. □

Recall from Lemma 3.2(i) that for every non-trivial element $g \in \overline{\mathrm{H}}$, there exist two skew lines L_g and L'_g in $\mathbb{P}^3$ that are pointwise fixed by g. The stabilizer of each of these lines in $\overline{\mathrm{H}}$ is $\boldsymbol{\mu}_2^3$, and the kernel of its action on any of these lines is a subgroup $\boldsymbol{\mu}_2$ generated by g. In total, there are 30 lines in $\mathbb{P}^3$ that are pointwise fixed by elements in $\overline{\mathrm{H}}$. Denote the set of these lines by $\mathbb{L}_{30}$. Let us describe these lines explicitly.

The group $\overline{\mathrm{H}} \cong \boldsymbol{\mu}_2^4$ acts on $\mathcal{Q}_1 \cong \mathbb{P}^1 \times \mathbb{P}^1$ naturally. This means the following. The group $\boldsymbol{\mu}_2^2$ acts on $\mathbb{P}^1$ in a standard way: in appropriate homogeneous coordinates one of its generators acts as $[u : v] \mapsto [v : u]$, and the other by $[u : v] \mapsto [-u : v]$. The product action of the group $\boldsymbol{\mu}_2^2 \times \boldsymbol{\mu}_2^2$ on $\mathbb{P}^1 \times \mathbb{P}^1$ gives the action of the group $\overline{\mathrm{H}}$ on $\mathcal{Q}_1$.

Since every subgroup $\boldsymbol{\mu}_2$ in $\boldsymbol{\mu}_2^2$ fixes exactly two points in $\mathbb{P}^1$, and there are three such subgroups in $\boldsymbol{\mu}_2^2$, we see that there are exactly six pairs of skew lines in $\mathcal{Q}_1$ that are $\overline{\mathrm{H}}$-invariant. The smooth quadric surface $\mathcal{Q}_1$ contains two one-parameter families of lines. Three of the above pairs of skew lines lie in one family of lines on $\mathcal{Q}_1$, and the remaining three pairs lie in another family of lines on $\mathcal{Q}_1$. Let ℓ be the line in $\mathbb{P}^3$ that passes through the points $[0 : i : 1 : 0]$ and $[1 : 0 : 0 : -i]$, and let $\check{\ell}$ be the line in $\mathbb{P}^3$ that passes through the points $[0 : -i : 1 : 0]$ and $[1 : 0 : 0 : i]$. Then ℓ and $\check{\ell}$ are skew lines contained in $\mathcal{Q}_1$. Moreover, both of them are pointwise fixed by $\overline{T}_1 \circ \overline{S}_2 \circ \overline{S}_1$ (see (3.1)), so that ℓ and $\check{\ell}$ are two lines in $\mathbb{L}_{30}$. Applying powers of $\overline{T}$ to them, we obtain 8 more lines in $\mathbb{L}_{30}$. Here $\overline{T}$ is an element of the group $\overline{G}$ of order 5 defined in (3.4). To be precise, let

$$\begin{aligned} &\ell_1 = \ell, \quad \ell_2 = \overline{T}(\ell_1), \quad \ell_3 = \overline{T}(\ell_2), \quad \ell_4 = \overline{T}(\ell_3), \quad \ell_5 = \overline{T}(\ell_4), \\ &\check{\ell}_1 = \check{\ell}, \quad \check{\ell}_2 = \overline{T}(\check{\ell}_1), \quad \check{\ell}_3 = \overline{T}(\check{\ell}_2), \quad \check{\ell}_4 = \overline{T}(\check{\ell}_3), \quad \check{\ell}_5 = \overline{T}(\check{\ell}_4). \end{aligned} \tag{3.11}$$

Then the lines in (3.11) are contained in $\mathbb{L}_{30}$. Similarly, let ℓ' be the line in $\mathbb{P}^3$ that passes through the points $[1 : 0 : -i : 0]$ and $[0 : 1 : 0 : -i]$, let $\check{\ell}'$ be the line in $\mathbb{P}^3$ that passes through the points $[1 : 0 : i : 0]$ and $[0 : 1 : 0 : i]$, and let

$$\begin{aligned} &\ell'_1 = \ell', \quad \ell'_2 = \overline{T}(\ell'_1), \quad \ell'_3 = \overline{T}(\ell'_2), \quad \ell'_4 = \overline{T}(\ell'_3), \quad \ell'_5 = \overline{T}(\ell'_4), \\ &\check{\ell}'_1 = \check{\ell}', \quad \check{\ell}'_2 = \overline{T}(\check{\ell}'_1), \quad \check{\ell}'_3 = \overline{T}(\check{\ell}'_2), \quad \check{\ell}'_4 = \overline{T}(\check{\ell}'_3), \quad \check{\ell}'_5 = \overline{T}(\check{\ell}'_4). \end{aligned} \tag{3.12}$$

Then the lines in (3.12) are contained in $\mathbb{L}_{30}$. Finally, let ℓ'' be the line in $\mathbb{P}^3$ that passes through the points $[1 : i : 0 : 0]$ and $[0 : 0 : 1 : -i]$, let $\check{\ell}''$ be the line in $\mathbb{P}^3$ that passes through the points $[1 : -i : 0 : 0]$ and $[0 : 0 : 1 : i]$, and let

$$\begin{aligned} &\ell''_1 = \ell'', \quad \ell''_2 = \overline{T}(\ell''_1), \quad \ell''_3 = \overline{T}(\ell''_2), \quad \ell''_4 = \overline{T}(\ell''_3), \quad \ell''_5 = \overline{T}(\ell''_4), \\ &\check{\ell}''_1 = \check{\ell}'', \quad \check{\ell}''_2 = \overline{T}(\check{\ell}''_1), \quad \check{\ell}''_3 = \overline{T}(\check{\ell}''_2), \quad \check{\ell}''_4 = \overline{T}(\check{\ell}''_3), \quad \check{\ell}''_5 = \overline{T}(\check{\ell}''_4). \end{aligned} \tag{3.13}$$

Then the lines in (3.13) are the remaining 10 lines in $\mathbb{L}_{30}$. By construction, the lines ℓ, $\check{\ell}$, ℓ', $\check{\ell}'$, ℓ'' and $\check{\ell}''$ are contained in the quadric surface $\mathcal{Q}_1$.

Table 1 Ten $\overline{\mathrm{H}}$-invariant quadrics and thirty lines in them

	$\mathcal{Q}_1$	$\mathcal{Q}_2$	$\mathcal{Q}_3$	$\mathcal{Q}_4$	$\mathcal{Q}_5$	$\mathcal{Q}_6$	$\mathcal{Q}_7$	$\mathcal{Q}_8$	$\mathcal{Q}_9$	$\mathcal{Q}_{10}$
ℓ_1	+	−	−	−	+	−	+	−	+	−
ℓ_2	−	−	−	+	+	−	−	+	−	+
ℓ_3	−	−	+	+	−	+	+	−	−	−
ℓ_4	−	+	+	−	−	−	−	−	+	+
ℓ_5	+	+	−	−	−	+	−	+	−	−
$\check{\ell}_1$	+	−	−	−	+	−	+	−	+	−
$\check{\ell}_2$	−	−	−	+	+	−	−	+	−	+
$\check{\ell}_3$	−	−	+	+	−	+	+	−	−	−
$\check{\ell}_4$	−	+	+	−	−	−	−	−	+	+
$\check{\ell}_5$	+	+	−	−	−	+	−	+	−	−
ℓ'_1	+	−	−	−	−	+	+	−	−	+
ℓ'_2	−	−	−	−	+	+	−	−	+	+
ℓ'_3	−	−	−	+	−	+	−	+	+	−
ℓ'_4	−	−	+	−	−	−	+	+	+	−
ℓ'_5	−	+	−	−	−	−	+	+	−	+
$\check{\ell}'_1$	+	−	−	−	−	+	+	−	−	+
$\check{\ell}'_2$	−	−	−	−	+	+	−	−	+	+
$\check{\ell}'_3$	−	−	−	+	−	+	−	+	+	−
$\check{\ell}'_4$	−	−	+	−	−	−	+	+	+	−
$\check{\ell}'_5$	−	+	−	−	−	−	+	+	−	+
ℓ''_1	+	+	−	+	−	−	−	−	+	−
ℓ''_2	+	−	+	−	+	−	−	+	−	−
ℓ''_3	−	+	−	+	+	−	+	−	−	−
ℓ''_4	+	−	+	+	−	−	−	−	−	+
ℓ''_5	−	+	+	−	+	+	−	−	−	−
$\check{\ell}''_1$	+	+	−	+	−	−	−	−	+	−
$\check{\ell}''_2$	+	−	+	−	+	−	−	+	−	−
$\check{\ell}''_3$	−	+	−	+	+	−	+	−	−	−
$\check{\ell}''_4$	+	−	+	+	−	−	−	−	−	+
$\check{\ell}''_5$	−	+	+	−	+	+	−	−	−	−

Lemma 3.14 *Each line among the* 30 *lines in* $\mathbb{L}_{30}$ *is contained in exactly* 4 *of the quadric surfaces* $\mathcal{Q}_1, \ldots, \mathcal{Q}_{10}$. *Each of these quadrics contains exactly* 12 *such lines. For every quadric, these* 12 *lines contained in it are described by Table* 1. *Every two quadrics among* $\mathcal{Q}_1, \ldots, \mathcal{Q}_{10}$ *intersect in a quadruple of lines in* $\mathbb{L}_{30}$.

Proof Direct computations. The last assertion can be also deduced from Lemma 3.3. □

Let us describe the intersection points of the 30 lines in $\mathbb{L}_{30}$. Namely, let Σ_{20} be the subset in $\mathbb{P}^3$ that consists of the 20 points

$$[i:0:0:1],[0:i:1:0],[1:0:1:0],[0:-1:0:1],[1:i:-i:1],$$
$$[1:-i:i:1],[0:0:-1:1],[0:0:1:1],[-1:i:-i:1],[1:i:i:1],$$
$$[-i:0:0:1],[0:-i:1:0],[1:0:1:0],[0:1:0:1],[-1:i:i:1],$$
$$[1:-i:-i:1],[-1:1:0:0],[1:1:0:0],[-1:-i:i:1],[1:-i:-i:1]. \tag{3.15}$$

Similarly, let Σ'_{20} be the subset in $\mathbb{P}^3$ that consists of the 20 points

$$[-i:i:-1:1],[i:0:1:0],[i:i:1:1],[0:i:0:1],[-1:1:-1:1],$$
$$[i:-i:-1:1],[1:-1:-1:1],[-1:0:0:1],[-1:-1:1:1],[0:-1:1:0],$$
$$[1:0:0:0],[1:0:0:1],[0:0:0:1],[-i:0:1:0],[0:0:1:0],$$
$$[-i:-i:1:1],[0:-i:0:1],[1:1:1:1],[0:1:1:0],[0:1:0:0]. \tag{3.16}$$

Finally, let Σ''_{20} be the subset in $\mathbb{P}^3$ that consists of the 20 points

$$[i:-1:i:1],[0:0:i:1],[-i:1:i:1],[-i:1:0:0],[-i:-1:i:1],$$
$$[-i:i:1:1],[-i:1:-i:1],[i:-i:-1:1],[1:-1:1:1],[i:-1:-i:1],$$
$$[1:-1:-1:1],[i:1:0:0],[i:1:i:1],[i:-i:1:1],[-1:1:1:1],$$
$$[i:1:-i:1],[0:0:-i:1],[i:-1:-i:1],[i:i:-1:1],[1:1:-1:1]. \tag{3.17}$$

Explicit computations show that Σ_{20}, Σ'_{20} and Σ''_{20} are $\overline{G}_{80}$-orbits. This also follows from Lemma 5.13.

Lemma 3.18 *The set $\Sigma_{20}\cup\Sigma'_{20}\cup\Sigma''_{20}$ contains all intersection points of the lines from $\mathbb{L}_{30}$. Moreover, for every point $P\in\Sigma_{20}\cup\Sigma'_{20}\cup\Sigma''_{20}$ there are exactly three lines from $\mathbb{L}_{30}$ passing through P.*

Proof Direct computations. □

Now we are going to describe the $\overline{G}_{80}$-invariant quartic surfaces and the $\overline{G}_{144}$-invariant quartic surfaces.

Lemma 3.19 *There are exactly five $\overline{G}_{80}$-invariant irreducible quartic surfaces in $\mathbb{P}^3$. Similarly, there are exactly four $\overline{G}_{144}$-invariant irreducible quartic surfaces in $\mathbb{P}^3$.*

Proof It follows from Remark 3.7 that $\mathcal{U}_4$ splits as a sum of distinct one-dimensional representations of G_{80} and G_{144}. Now the assertion follows from Lemma 3.10. □

Up to scaling, all homogeneous semi-invariants of the group G_{144} of degree 4 in $\mathbb{C}[x, y, z, w]$ are

$$\begin{cases} p_0 = -\frac{1}{4}\Big(t_0 + t_2 + t_5\Big), \\ p_1 = -\Big(\xi_3 t_0 + t_2 + \xi_3^2 t_5\Big), \\ p_2 = (\xi_3 + 1)\Big(t_0 + \xi_3^2 t_5 + \xi_3 t_2\Big), \\ p_3 = (\xi_3 + 1)\Big(t_1 + \xi_3^2 t_4 + \xi_3 t_3\Big), \\ p_4 = -\Big(\xi_3 t_1 + t_3 + \xi_3^2 t_4\Big). \end{cases} \tag{3.20}$$

Here ξ_3 is a primitive cubic root of unity, and $t_0, \dots, t_5$ are the polynomials defined in (3.6). Observe that

$$p_0 = (wy + xz)^2$$

is the only homogeneous invariant of degree 4 of the group G_{144}. The remaining four polynomials p_1, p_2, p_3 and p_4 are semi-invariants, and they are irreducible. Thus, the four irreducible G_{144}-invariant quartic surfaces in $\mathbb{P}^3$ are given by $p_1 = 0$, $p_2 = 0$, $p_3 = 0$ and $p_4 = 0$. We will study properties of these surfaces in Sect. 4.

The eigenvectors of the matrix T on the vector space of all $\mathbb{H}$-invariant homogeneous quartic polynomials in $\mathbb{C}[x, y, z, w]$ are the polynomials

$$\begin{cases} q_0 = t_0 + t_1 + t_2 + t_3 + t_4, \\ q_1 = \xi_5^4 t_4 + \xi_5^3 t_2 + \xi_5^2 t_5 + \xi_5 t_1 + t_0, \\ q_2 = \xi_5^4 t_5 + \xi_5^3 t_4 + \xi_5^2 t_1 + \xi_5 t_2 + t_0, \\ q_3 = \xi_5^4 t_2 + \xi_5^3 t_1 + \xi_5^2 t_4 + \xi_5 t_5 + t_0, \\ q_4 = \xi_5^4 t_1 + \xi_5^3 t_5 + \xi_5^2 t_2 + \xi_5 t_4 + t_0. \end{cases} \tag{3.21}$$

Here ξ_5 is a primitive fifth root of unity. Note that q_i is the eigenvector of T that corresponds to the eigenvalue ξ_5^i. The five $\overline{G}_{80}$-invariant quartic surfaces in $\mathbb{P}^3$ are the quartic surfaces $q_0 = 0$, $q_1 = 0$, $q_2 = 0$, $q_3 = 0$ and $q_4 = 0$. We will study properties of these surfaces in Sect. 5. The polynomial

$$q_0 = \frac{1}{2}\Big(w^4 + y^4 + z^4 + x^4\Big) + 3x^2y^2 + 3z^2w^2 + 3x^2z^2 + 3y^2w^2 - 3x^2w^2 - 3y^2z^2.$$

is the only G_{80}-invariant homogeneous quartic polynomial in $\mathbb{C}[x, y, z, w]$.

Lemma 3.22 *There are exactly four $\overline{G}_{80}$-invariant surfaces in $\mathbb{P}^3$ of degree 6. Moreover, they are irreducible and reduced.*

Proof One can find all irreducible representations of the group G_{80}. They can be described as follows: 10 one-dimensional representations, 10 four-dimensional representations, and 6 five-dimensional representations. Similarly, one can compute the

sixth symmetric power of every four-dimensional representation. These computations imply that the sixth symmetric power of every four-dimensional representations contains exactly 4 different one-dimensional subrepresentations. Geometrically, this means that there are exactly four $\overline{G}_{80}$-invariant surfaces in $\mathbb{P}^3$ of degree 6. Since $\mathbb{P}^3$ does not contain $\overline{G}_{80}$-invariant surfaces of degree 1, 2 and 3 by Lemma 3.10, each $\overline{G}_{80}$-invariant surface of degree 6 must be irreducible and reduced. We used [44] to perform these computations. □

One can find explicit equations of the $\overline{G}_{80}$-invariant sextic surfaces in $\mathbb{P}^3$. However, we decided not to do this to keep the paper shorter.

4 The group of order 144

Let us use notation and assumptions of Sect. 3. In this section, we present basic facts about $\overline{G}_{144}$-orbits, $\overline{G}_{144}$-invariant curves and $\overline{G}_{144}$-invariant surfaces in $\mathbb{P}^3$.

Let us start with studying basic group-theoretic properties of the group $\overline{G}_{144}$. Recall that $\overline{G}_{144}$ is isomorphic to

$$\mathfrak{A}_4 \times \mathfrak{A}_4 \cong \boldsymbol{\mu}_2^4 \rtimes \big(\boldsymbol{\mu}_3 \times \boldsymbol{\mu}_3\big).$$

Note that the isomorphism $\overline{G}_{144} \cong \mathfrak{A}_4 \times \mathfrak{A}_4$ is uniquely defined up to permutations of the factors and automorphisms of the group $\mathfrak{A}_4$. We will fix such an isomorphism until the end of this section. Denote by $\mathfrak{A}_4^{\Delta}$ the *diagonal* subgroup in $\overline{G}_{144}$, i.e. the subgroup that consists of the elements (g, g), where $g \in \mathfrak{A}_4$. Similarly, denote by $\mathfrak{A}_4^{\nabla}$ the *twisted diagonal* subgroup in $\overline{G}_{144}$, i.e. the subgroup that consists of the elements $(g, \sigma(g))$, where $g \in \mathfrak{A}_4$, and σ is an outer automorphism of $\mathfrak{A}_4$ that is given by conjugation with a fixed odd element of $\mathfrak{S}_4$. By a *basic* subgroup in $\overline{G}_{144}$, we will mean any subgroup contained in one of the factors. Similarly, we say that a subgroup in $\overline{G}_{144}$ is of *product type* if it is a product of two basic subgroups.

Lemma 4.1 *Let Γ be a subgroup in $\overline{G}_{144}$. Then the following assertions hold.*

(o) *If Γ is cyclic, then $|\Gamma| \in \{1, 2, 3, 6\}$.*
(i) *The group $\overline{\mathrm{H}}$ contains all 2-subgroups of the group $\overline{G}_{144}$.*
(ii) *If $|\Gamma|$ is divisible by 16, then Γ contains the subgroup $\overline{\mathrm{H}}$.*
(iii) *If $|\Gamma| = 9$, then $\Gamma \cong \boldsymbol{\mu}_3^2$ and Γ is of product type.*
(iv) *One has $|\Gamma| \neq 18$.*
(v) *If $|\Gamma| = 6$, then $\Gamma \cong \boldsymbol{\mu}_6 \cong \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_3$, and Γ is of product type. Moreover, there are just two such subgroups up to conjugation. Furthermore, every subgroup in $\overline{G}_{144}$ of order 12 that contains Γ is isomorphic to $\boldsymbol{\mu}_2^2 \times \boldsymbol{\mu}_3$ and is of product type.*
(vi) *If $|\Gamma| = 12$ and Γ is not conjugate to $\mathfrak{A}_4^{\Delta}$ and $\mathfrak{A}_4^{\nabla}$, then the group Γ contains a basic subgroup $\boldsymbol{\mu}_2^2$.*
(vii) *If $\Gamma \neq \overline{G}_{144}$, then $|\Gamma| \leqslant 48$.*
(viii) *If $|\Gamma| > 16$, then there is a surjective homomorphism $\Gamma \to \mathfrak{A}_4$.*

Proof Assertion (o) is obvious. The subgroup $\overline{\mathrm{H}}$ is the normal Sylow 2-subgroup of the group $\overline{G}_{144}$. This implies assertion (i). Assertion (ii) follows from (i) and Sylow theorems. Assertion (iii) also follows from Sylow theorems.

Let $\pi_1\colon \overline{G}_{144} \to \mathfrak{A}_4$ and $\pi_2\colon \overline{G}_{144} \to \mathfrak{A}_4$ be the two projections to the factors. If $|\Gamma| = 18$, then the subgroups $\pi_1(\Gamma)$ and $\pi_2(\Gamma)$ are proper subgroups, so that at least one of them has order greater than 4, which is impossible. This proves assertion (iv).

Similarly, if $|\Gamma| = 6$, then, up to permutation, the groups $\pi_1(\Gamma)$ and $\pi_2(\Gamma)$ are isomorphic to $\boldsymbol{\mu}_2$ and $\boldsymbol{\mu}_3$, respectively. In this case, we have $\Gamma \cong \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_3 \cong \boldsymbol{\mu}_6$, and the conjugacy class of Γ is determined by the choice of π_i such that $\pi_i(\Gamma) \cong \boldsymbol{\mu}_2$. Let Υ be a subgroup in $\overline{G}_{144}$ of order 12 that contains Γ. We may assume that $\pi_1(\Gamma) \cong \boldsymbol{\mu}_2$ and $\pi_2(\Gamma) \cong \boldsymbol{\mu}_3$. Then $\pi_2(\Upsilon) \cong \boldsymbol{\mu}_3$, because $\boldsymbol{\mu}_3$ is a maximal proper subgroup in $\mathfrak{A}_4$. This easily implies assertion (v).

To prove assertion (vi), we suppose that $|\Gamma| = 12$. If $\pi_1(\Gamma) = \mathfrak{A}_4$ and $\pi_2(\Gamma) = \mathfrak{A}_4$, then Γ is conjugate to either $\mathfrak{A}_4^{\Delta}$ or $\mathfrak{A}_4^{\nabla}$. Thus, we may assume that $\pi_1(\Gamma) \neq \mathfrak{A}_4$. Then the kernel of $\pi_1|_{\Gamma}$ contains a basic subgroup $\boldsymbol{\mu}_2^2$.

To prove assertion (vii), we have to show that the index of Γ in $\overline{G}_{144}$ is not 2. Suppose that it is. Then Γ is normal. Thus, the intersection of Γ with each factor in $\overline{G}_{144} \cong \mathfrak{A}_4 \times \mathfrak{A}_4$ is either $\mathfrak{A}_4$ or a subgroup of index 2 in $\mathfrak{A}_4$. Since $\mathfrak{A}_4$ does not contain subgroups of index 2, we see that Γ contains both factors, so it is the whole group $\overline{G}_{144}$, which is absurd.

To prove assertion (viii), we suppose that $|\Gamma| > 16$. If $\pi_1(\Gamma) = \mathfrak{A}_4$ or $\pi_2(\Gamma) = \mathfrak{A}_4$, then we are done. Otherwise, both $\pi_1(\Gamma)$ and $\pi_2(\Gamma)$ are proper subgroups of $\mathfrak{A}_4$, so that their orders are at most 4. This is impossible, because $|\Gamma| > 16$. □

To study $\overline{G}_{144}$-invariant curves in $\mathbb{P}^3$, we need the following result, which is a simple consequence of the Riemann–Hurwitz formula.

Lemma 4.2 *Let C be a smooth irreducible curve of genus $g \leqslant 13$ with a faithful action of the group $\overline{G}_{144}$. Then $g \in \{8, 13\}$, and the curve C is not hyperelliptic. Furthermore, if $\Omega \subset C$ is a $\overline{G}_{144}$-orbit, then $|\Omega| \in \{24, 48, 72, 144\}$. Finally, the possible numbers a_i of $\overline{G}_{144}$-orbits of length $i \in \{24, 48, 72\}$ in C are contained in the following table:*

g	8	13	13
a_{24}	0	1	2
a_{48}	0	2	0
a_{72}	3	0	1

Proof The assertion about the lengths of $\overline{G}_{144}$-orbits follows from Lemma 4.1(o), since the stabilizers in $\overline{G}_{144}$ of points in C are cyclic (see [10, Lemma 5.1.4]). By Lemma 3.3, the group $\overline{G}_{144}$ cannot act faithfully on $\mathbb{P}^1$ and on a smooth elliptic curve, so that $g \geqslant 2$.

Suppose that the curve C is hyperelliptic. Since the group $\overline{G}_{144}$ does not contain normal subgroups of order 2, it does not contain the hyperelliptic involution of C,

and we obtain a faithful action of the group $\overline{G}_{144}$ on $\mathbb{P}^1$. The latter is impossible by Lemma 3.3.

Let $\widehat{C} = C/\overline{G}_{144}$. Then $\widehat{C}$ is a smooth curve. Let $\hat{g}$ be the genus of the curve $\widehat{C}$. Then the Riemann–Hurwitz formula gives

$$2g - 2 = 144\big(2\hat{g} - 2\big) + 72a_{72} + 96a_{48} + 120a_{24}.$$

Since $a_k \geqslant 0$ and $g \leqslant 13$, one has $\hat{g} = 0$, so that

$$2g - 2 = -288 + 72a_{72} + 96a_{48} + 120a_{24},$$

Going through the possible values of g, and solving this equation case by case we obtain the required result. □

Denote by $\mathcal{Q}$ the smooth quadric in $\mathbb{P}^3$ that is given by

$$wy + xz = 0,$$

which is the quadric $\mathcal{Q}_3$ in the notation of Sect. 3. Then $\mathcal{Q}$ is $\overline{G}_{144}$-invariant. We will see in Lemmas 4.10 and 4.16 that this quadric contains most of $\overline{G}_{144}$-orbits of small length in $\mathbb{P}^3$ and most of $\overline{G}_{144}$-irreducible curves in $\mathbb{P}^3$ that have small degree.

Observe that the action of the group $\overline{G}_{144} \cong \mathfrak{A}_4 \times \mathfrak{A}_4$ on the surface $\mathcal{Q} \cong \mathbb{P}^1 \times \mathbb{P}^1$ is just the product action given by the natural action of the group $\mathfrak{A}_4$ on $\mathbb{P}^1$ induced by a two-dimensional irreducible representation of the binary tetrahedral group $2.\mathfrak{A}_4$. Denote this representation by $\mathbb{V}_2$. Then there is a $\overline{G}_{144}$-equivariant identification

$$\mathbb{P}^3 \cong \mathbb{P}\Big(\mathbb{W}_4\Big), \tag{4.3}$$

where $\mathbb{W}_4$ is the four-dimensional representation of the group $2.\mathfrak{A}_4 \times 2.\mathfrak{A}_4$ in the vector space of 2×2-matrices such that the first (respectively, the second) factor

$$2.\mathfrak{A}_4 \subset \mathrm{GL}(\mathbb{V}_2) \cong \mathrm{GL}_2(\mathbb{C})$$

acts by left (respectively, right) multiplications. Then the points of the quadric $\mathcal{Q}$ correspond to 2×2-matrices of rank 1 in $\mathbb{W}_4$.

Since $\mathbb{P}^1$ contains two $\mathfrak{A}_4$-orbits of length 4 and one orbit of length 6, this gives four $\overline{G}_{144}$-irreducible curves in $\mathcal{Q}$ that are unions of 4 disjoint lines, and two $\overline{G}_{144}$-irreducible curves in $\mathcal{Q}$ that are unions of 6 disjoint lines. Denote the former four curves by $\mathcal{L}_4^1$, $\mathcal{L}_4^2$, $\mathcal{L}_4^3$ and $\mathcal{L}_4^4$, and denote the latter two curves by $\mathcal{L}_6^1$ and $\mathcal{L}_6^2$. Without loss of generality, we may assume that

$$\mathcal{L}_4^1 \cap \mathcal{L}_4^2 = \mathcal{L}_4^1 \cap \mathcal{L}_6^1 = \mathcal{L}_4^3 \cap \mathcal{L}_4^4 = \mathcal{L}_4^3 \cap \mathcal{L}_6^2 = \varnothing.$$

In other words, on $\mathcal{Q} \cong \mathbb{P}^1 \times \mathbb{P}^1$, the curves $\mathcal{L}_4^1$ and $\mathcal{L}_4^2$ have bi-degree $(4, 0)$, while the curves $\mathcal{L}_4^3$ and $\mathcal{L}_4^4$ have bi-degree $(0, 4)$. Similarly, the curve $\mathcal{L}_6^1$ has bi-degree $(6, 0)$,

and the curve $\mathcal{L}_6^2$ has bi-degree $(0, 6)$. Let

$$\Sigma_{16}^1 = \mathcal{L}_4^1 \cap \mathcal{L}_4^3, \quad \Sigma_{16}^2 = \mathcal{L}_4^1 \cap \mathcal{L}_4^4, \quad \Sigma_{16}^3 = \mathcal{L}_4^2 \cap \mathcal{L}_4^3, \quad \Sigma_{16}^4 = \mathcal{L}_4^2 \cap \mathcal{L}_4^4.$$

Then Σ_{16}^1, Σ_{16}^2, Σ_{16}^3 and Σ_{16}^4 are $\overline{G}_{144}$-orbits of length 16. Similarly, let

$$\Sigma_{24}^1 = \mathcal{L}_4^1 \cap \mathcal{L}_6^2, \quad \Sigma_{24}^2 = \mathcal{L}_4^2 \cap \mathcal{L}_6^2, \quad \Sigma_{24}^3 = \mathcal{L}_4^3 \cap \mathcal{L}_6^1, \quad \Sigma_{24}^4 = \mathcal{L}_4^4 \cap \mathcal{L}_6^1.$$

Then the subsets Σ_{24}^1, Σ_{24}^2, Σ_{24}^3, Σ_{24}^4 are $\overline{G}_{144}$-orbits of length 24. Finally, let $\Sigma_{36} = \mathcal{L}_6^1 \cap \mathcal{L}_6^2$. Then the subset Σ_{36} is a $\overline{G}_{144}$-orbit of length 36. We summarize the intersections of the curves $\mathcal{L}_4^1$, $\mathcal{L}_4^2$, $\mathcal{L}_4^3$, $\mathcal{L}_4^4$, $\mathcal{L}_6^1$ and $\mathcal{L}_6^2$ in the following table:

	$\mathcal{L}_4^1$	$\mathcal{L}_4^2$	$\mathcal{L}_4^3$	$\mathcal{L}_4^4$	$\mathcal{L}_6^1$	$\mathcal{L}_6^2$
$\mathcal{L}_4^1$	$\mathcal{L}_4^1$	$\varnothing$	Σ_{16}^1	Σ_{16}^2	$\varnothing$	Σ_{24}^1
$\mathcal{L}_4^2$	$\varnothing$	$\mathcal{L}_4^2$	Σ_{16}^3	Σ_{16}^4	$\varnothing$	Σ_{24}^2
$\mathcal{L}_4^3$	Σ_{16}^1	Σ_{16}^3	$\mathcal{L}_4^3$	$\varnothing$	Σ_{24}^3	$\varnothing$
$\mathcal{L}_4^4$	Σ_{16}^2	Σ_{16}^4	$\varnothing$	$\mathcal{L}_4^4$	Σ_{24}^4	$\varnothing$
$\mathcal{L}_6^1$	$\varnothing$	$\varnothing$	Σ_{24}^3	Σ_{24}^4	$\mathcal{L}_6^1$	Σ_{36}
$\mathcal{L}_6^2$	Σ_{24}^1	Σ_{24}^2	$\varnothing$	$\varnothing$	Σ_{36}	$\mathcal{L}_6^2$

Lemma 4.4 *Let Σ be a $\overline{G}_{144}$-orbit in $\mathcal{Q}$ such that $|\Sigma| \leqslant 36$. Then Σ is one of the $\overline{G}_{144}$-orbits Σ_{16}^1, Σ_{16}^2, Σ_{16}^3, Σ_{16}^4, Σ_{24}^1, Σ_{24}^2, Σ_{24}^3, Σ_{24}^4, Σ_{36}.*

Proof This follows from the fact that $\mathbb{P}^1$ contains exactly two $\mathfrak{A}_4$-orbits of length 4, exactly one orbit of length 6, and the remaining $\mathfrak{A}_4$-orbits in $\mathbb{P}^1$ are of length 12. □

By construction, the smooth quadric $\mathcal{Q}$ contains the curves $\mathcal{L}_4^1$, $\mathcal{L}_4^2$, $\mathcal{L}_4^3$, $\mathcal{L}_4^4$, $\mathcal{L}_6^1$ and $\mathcal{L}_6^2$. The same construction gives infinitely many $\overline{G}_{144}$-irreducible curves in $\mathcal{Q}$ that are unions of 12 disjoint lines. Moreover, we have

Lemma 4.5 *Let C be a $\overline{G}_{144}$-irreducible curve in $\mathcal{Q}$ such that $\deg(C) \leqslant 23$. Then either C is a disjoint union of 12 lines, or C is one of the curves $\mathcal{L}_4^1$, $\mathcal{L}_4^2$, $\mathcal{L}_4^3$, $\mathcal{L}_4^4$, $\mathcal{L}_6^1$, $\mathcal{L}_6^2$.*

Proof The curve C is a curve of bi-degree (m, n) on $\mathcal{Q} \cong \mathbb{P}^1 \times \mathbb{P}^1$ such that $m + n \leqslant 23$. The required assertion can be checked by a direct computation using the identification (4.3), however, we prefer to provide a more geometric proof.

Without loss of generality, we may assume that $m \leqslant n$. Then $m \leqslant 11$. Suppose that C is none of the curves $\mathcal{L}_4^1$, $\mathcal{L}_4^2$, $\mathcal{L}_4^3$, $\mathcal{L}_4^4$, $\mathcal{L}_6^1$, $\mathcal{L}_6^2$. Suppose also that C is not a disjoint union of 12 lines. Then $m \geqslant 1$. Let us seek for a contradiction.

Let P be a point in C that is not contained in the curves $\mathcal{L}_4^1$, $\mathcal{L}_4^2$, $\mathcal{L}_4^3$, $\mathcal{L}_4^4$, $\mathcal{L}_6^1$, $\mathcal{L}_6^2$. Let L be the line in $\mathcal{Q}$ that is a curve of bi-degree $(0, 1)$ that passes through P. Then L is not an irreducible component of the curve C, since C is $\overline{G}_{144}$-irreducible. Let Γ be

the subgroup $\mathfrak{A}_4 \times \mathrm{Id}$ in $\overline{G}_{144}$ (so that Γ is one of the two basic subgroups $\mathfrak{A}_4$). Then L and C are Γ-invariant, and Γ acts faithfully on the line L. Hence, the intersection $L \cap C$ is a union of Γ-orbits of length $L \cdot C = m \leqslant 11$. On the other hand, all such orbits are contained in the union $\mathcal{L}_4^1 \cup \mathcal{L}_4^2 \cup \mathcal{L}_4^3 \cup \mathcal{L}_4^4 \cup \mathcal{L}_6^1 \cup \mathcal{L}_6^2$ by Lemma 4.4, which gives us a contradiction. □

Corollary 4.6 *Let C be a $\overline{G}_{144}$-invariant curve in $\mathbb{P}^3$ such that $\deg(C) < 8$. Then C is one of the curves $\mathcal{L}_4^1$, $\mathcal{L}_4^2$, $\mathcal{L}_4^3$, $\mathcal{L}_4^4$, $\mathcal{L}_6^1$, $\mathcal{L}_6^2$.*

Proof If C is not contained in the quadric $\mathcal{Q}$, then

$$|C \cap \mathcal{Q}| \leqslant C \cdot \mathcal{Q} = 2\deg(C) \leqslant 14,$$

which is impossible by Lemma 4.4. □

Corollary 4.7 *Let C be a $\overline{G}_{144}$-irreducible curve in $\mathcal{Q}$, let $\mathcal{D}$ be a (non-empty) mobile $\overline{G}_{144}$-invariant linear system on $\mathbb{P}^3$, let D be a general surface in $\mathcal{D}$, and let n be a positive integer such that $\mathcal{D} \sim \mathcal{O}_{\mathbb{P}^3}(n)$. Write*

$$D\big|_{\mathcal{Q}} = mC + \Delta,$$

where m is a non-negative integer, and Δ is an effective divisor on $\mathcal{Q}$ whose support does not contain irreducible components of the curve C. Then $m \leqslant \frac{n}{4}$.

Proof Suppose that $m > \frac{n}{4}$. The curve C is a divisor of bi-degree (a, b) on $\mathcal{Q} \cong \mathbb{P}^1 \times \mathbb{P}^1$. But $mC + \Delta$ is a divisor of bi-degree (n, n), so that $a < 4$ and $b < 4$, which is impossible by Lemma 4.5. □

Corollary 4.8 *Let C be a $\overline{G}_{144}$-irreducible curve in $\mathcal{Q}$, let $\mathcal{D}$ be a (non-empty) mobile $\overline{G}_{144}$-invariant linear system on $\mathbb{P}^3$, and let n be a positive integer such that $\mathcal{D} \sim \mathcal{O}_{\mathbb{P}^3}(n)$. Then $\mathrm{mult}_C(\mathcal{D}) \leqslant \frac{n}{4}$.*

Denote by Σ_{12} the subset in $\mathbb{P}^3$ that consists of the 12 points

$$\begin{gathered}[0:1:0:1], [0:-1:0:1], [1:i:i:1], [1:-i:-i:1],\\ [1:-i:i:-1], [1:i:-i:-1],\\ [1:-1:i:-i], [1:-1:-i:i], [1:1:i:i], [1:1:-i:-i],\\ [1:0:1:0], [1:0:-1:0].\end{gathered}$$

Similarly, denote by Σ'_{12} the subset in $\mathbb{P}^3$ that consists of the 12 points

$$\begin{gathered}[1:1:1:1], [1:-1:-1:1], [1:i:1:-i], [1:-i:1:i],\\ [1:i:-1:i], [1:-i:-1:-i],\\ [1:0:i:0], [1:0:-i:0], [0:i:0:1], [0:-i:0:1],\\ [1:1:-1:-1], [1:-1:1:-1].\end{gathered}$$

Explicit computations show that Σ_{12} and Σ'_{12} are $\overline{G}_{144}$-orbits. This also follows from Lemma 4.12 below.

Lemma 4.9 *The subsets Σ_{12} and Σ'_{12} are the only $\overline{G}_{144}$-orbits in $\mathbb{P}^3$ of length 12.*

Proof Let Υ be a basic subgroup $\boldsymbol{\mu}_2^2 \subset \overline{G}_{144}$. Recall that the action of the group Υ on $\mathcal{Q} \cong \mathbb{P}^1 \times \mathbb{P}^1$ is a product action, with trivial group acting on one of the factors. Let $\widetilde{\Upsilon}$ be the preimage of Υ via the canonical projection $2.\mathfrak{A}_4 \to \mathfrak{A}_4$. Then the restriction of the two-dimensional representation $\mathbb{V}_2$ to $\widetilde{\Upsilon}$ is an irreducible two-dimensional representations of the group $\widetilde{\Upsilon}$. The restriction of the four-dimensional representation $\mathbb{W}_4$ of the group $2.\mathfrak{A}_4 \times 2.\mathfrak{A}_4$ to the group $\widetilde{\Upsilon}$ splits as sum of two isomorphic irreducible two-dimensional representations. Thus, Υ does not have fixed points in $\mathbb{P}^3$.

Let Σ be a $\overline{G}_{144}$-orbit in $\mathbb{P}^3$ of length 12, and let Γ be the stabilizer in $\overline{G}_{144}$ of a point in Σ. Then $|\Gamma| = 12$. We have seen that Γ does not contain Υ. Thus, it follows from Lemma 4.1(vi) that Γ is conjugate to either $\mathfrak{A}_4^{\Delta}$ or $\mathfrak{A}_4^{\nabla}$. Using the identification (4.3), we see that each of the latter groups has a unique fixed point in $\mathbb{P}^3$. Since we already know two $\overline{G}_{144}$-orbits in $\mathbb{P}^3$ of length 12, namely Σ_{12} and Σ'_{12}, the assertion follows. □

Now we are ready to describe $\overline{G}_{144}$-orbits in $\mathbb{P}^3$ of small length.

Lemma 4.10 *Let Σ be a $\overline{G}_{144}$-orbit in $\mathbb{P}^3$ such that $|\Sigma| \leqslant 35$. Then Σ is one of the $\overline{G}_{144}$-orbits Σ_{12}, Σ'_{12}, Σ^1_{16}, Σ^2_{16}, Σ^3_{16}, Σ^4_{16}, Σ^1_{24}, Σ^2_{24}, Σ^3_{24}, Σ^4_{24}.*

Proof One has $|\Sigma| > 4$, because the group $\overline{G}_{144}$ is primitive. This implies that

$$\big|\Sigma\big| \in \big\{6, 8, 9, 12, 16, 18, 24\big\}.$$

Let Γ be a stabilizer in $\overline{G}_{144}$ of a point in Σ. If $|\Sigma| \in \{6, 9, 18\}$, then $|\Gamma|$ is divisible by 8, so that Γ contains a subgroup $\boldsymbol{\mu}_2^3 \subset \overline{\mathrm{H}}$ by Sylow theorem and Lemma 4.1(i). This is impossible by Lemma 3.2(iii). If $|\Sigma| = 8$, then $|\Gamma| = 18$, which is impossible by Lemma 4.1(iv). If $|\Sigma| = 12$, then the required assertion follows from Lemma 4.9.

Suppose that $|\Sigma| = 16$. Then $|\Gamma| = 9$, so that $\Gamma \cong \boldsymbol{\mu}_3^2$ by Lemma 4.1(iii). Recall that the action of the group $\Gamma \cong \boldsymbol{\mu}_3 \times \boldsymbol{\mu}_3$ on $\mathcal{Q} \cong \mathbb{P}^1 \times \mathbb{P}^1$ is a product action. Let $\widetilde{\boldsymbol{\mu}}_3$ be the preimage of $\boldsymbol{\mu}_3$ via the canonical projection $2.\mathfrak{A}_4 \to \mathfrak{A}_4$. Then the restriction of the two-dimensional representation $\mathbb{V}_2$ to $\widetilde{\boldsymbol{\mu}}_3$ splits as a sum of two distinct one-dimensional representations of the group $\widetilde{\boldsymbol{\mu}}_3$. The restriction of the four-dimensional representation $\mathbb{W}_4$ to the group $\widetilde{\boldsymbol{\mu}}_3 \times \widetilde{\boldsymbol{\mu}}_3$ splits as sum of four pairwise non-isomorphic one-dimensional subrepresentations, which are generated by 2×2-matrices of rank 1. Since matrices of rank 1 correspond to the points of the quadric $\mathcal{Q}$, we see that $\Sigma \subset \mathcal{Q}$, so that Σ is one of the $\overline{G}_{144}$-orbits Σ^1_{16}, Σ^2_{16}, Σ^3_{16}, Σ^4_{16} by Lemma 4.4.

Suppose that $|\Sigma| = 24$. Then $\Gamma \cong \boldsymbol{\mu}_6 \cong \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_3$ by Lemma 4.1(v). Now arguing as in the previous case, we see that Σ is one of the $\overline{G}_{144}$-orbits Σ^1_{24}, Σ^2_{24}, Σ^3_{24}, Σ^4_{24}. □

Let p_1, p_2, p_3 and p_4 be the quartic polynomials in (3.20). Denote by S_1, S_2, S_3, S_4 the quartic surfaces $\mathbb{P}^3$ that are given by $p_1 = 0$, $p_2 = 0$, $p_3 = 0$ and $p_4 = 0$, respectively. Then S_1, S_2, S_3, S_4 are all $\overline{G}_{144}$-invariant irreducible quartic surfaces in $\mathbb{P}^3$ by Lemma 3.19. Moreover, explicit computations imply that

$$\Sigma_{12} \in S_1 \cap S_2, \quad \Sigma'_{12} \in S_3 \cap S_4, \quad \Sigma_{12} \not\subset S_3 \cup S_4, \quad \Sigma'_{12} \not\subset S_1 \cup S_2. \tag{4.11}$$

In fact, we can say more.

Lemma 4.12 *One has* $\mathrm{Sing}(S_1) = \mathrm{Sing}(S_2) = \Sigma_{12}$ *and* $\mathrm{Sing}(S_3) = \mathrm{Sing}(S_4) = \Sigma'_{12}$. *Moreover, all singular points of the surfaces* S_1, S_2, S_3 *and* S_4 *are ordinary double points.*

Proof It is enough to show the required assertions for the surface S_1, since the proof is identical in the remaining cases. We already know that S_1 is irreducible. Taking a general plane section of the surface S_1, we see that either S_1 has isolated singularities, or $\mathrm{Sing}(S_1)$ contains a $\overline{G}_{144}$-invariant curve of degree at most 3. The latter is impossible by Corollary 4.6. Thus, the surface S_1 has at most isolated singularities.

To see that S_1 is singular at every point of the $\overline{G}_{144}$-orbit Σ_{12}, one can simply take partial derivatives of the polynomial p_1, and plug the point $[0 : 1 : 0 : 1]$ into them. Alternatively, one can use the fact that the stabilizer in $\overline{G}_{144}$ of every point in Σ_{12} is isomorphic to $\mathfrak{A}_4$. Since the group $\mathfrak{A}_4$ does not have faithful two-dimensional representations, the surface S_1 must be singular at every point of Σ_{12} by [10, Lemma 4.4.1].

Thus, we see that S_1 has isolated singularities and $\Sigma_{12} \subset \mathrm{Sing}(S_1)$. Moreover, the surface S_1 cannot have more that two non-Du Val singular points by [45, Theorem 1]. Thus, the surface S_1 has at most Du Val singularities by Lemma 4.10.

Let $f\colon \widetilde{S}_1 \to S_1$ be the minimal resolution of singularities of the surface S_1. Then $\widetilde{S}_1$ is a smooth $K3$ surface. Then

$$\mathrm{rk}\,\mathrm{Pic}\big(\widetilde{S}_1\big) \geqslant \mathrm{rk}\,\mathrm{Pic}\big(S_1\big) + \big|\mathrm{Sing}(S_1)\big| = \mathrm{rk}\,\mathrm{Pic}\big(S_1\big) + 12 + \big|\mathrm{Sing}(S_1)\backslash\Sigma_{12}\big|.$$

On the other hand, the rank of the Picard group of a smooth $K3$ surface is at most 20, so that $\mathrm{Sing}(S_1) = \Sigma_{12}$, since S_1 does not contain $\overline{G}_{144}$-orbits of length less than 12 by Lemma 4.10. Moreover, the same arguments imply that every singular point of the surface S_1 is an ordinary double point, since otherwise we would have

$$20 \geqslant \mathrm{rk}\,\mathrm{Pic}\big(\widetilde{S}_1\big) \geqslant \mathrm{rk}\,\mathrm{Pic}\big(S_1\big) + 2\big|\mathrm{Sing}(S_1)\big| = \mathrm{rk}\,\mathrm{Pic}\big(S_1\big) + 24 \geqslant 25,$$

which is absurd. □

Corollary 4.13 *Let* Σ *be one of the* $\overline{G}_{144}$*-orbits* Σ_{12} *or* Σ'_{12} *and let* $\mathcal{M}$ *be a linear system that consists of cubic surfaces in* $\mathbb{P}^3$ *containing* Σ. *Then* Σ *is the base locus of the linear system* $\mathcal{M}$.

Proof By Lemma 4.12, we have $\Sigma = \mathrm{Sing}(S_i)$ for some $i \in \{1, 2, 3, 4\}$. Recall that the surface S_i is given by the equation $p_i(x, y, z, w) = 0$, where p_i is the quartic polynomial in (3.20). Thus, the linear system $\mathcal{M}$ contains the linear system that consists of cubic surfaces

$$\lambda_0\frac{\partial p_i}{\partial x} + \lambda_1\frac{\partial p_i}{\partial y} + \lambda_2\frac{\partial p_i}{\partial z} + \lambda_3\frac{\partial p_i}{\partial w} = 0,$$

where $[\lambda_0 : \lambda_1 : \lambda_2 : \lambda_3] \in \mathbb{P}^3$. The base locus of the latter system is $\mathrm{Sing}(S_i) = \Sigma$. □

Remark 4.14 Let H be the plane section of the quadric $\mathcal{Q}$. Then there is an exact sequences of G-representations

$$0 \longrightarrow H^0\Big(\mathcal{O}_{\mathbb{P}^3}(2)\Big) \longrightarrow H^0\Big(\mathcal{O}_{\mathbb{P}^3}(4)\Big) \longrightarrow H^0\Big(\mathcal{O}_{\mathcal{Q}}(4H)\Big) \longrightarrow 0.$$

On the other hand, the divisors $\mathcal{L}_4^1+\mathcal{L}_4^3$, $\mathcal{L}_4^1+\mathcal{L}_4^4$, $\mathcal{L}_4^2+\mathcal{L}_4^3$, $\mathcal{L}_4^2+\mathcal{L}_4^4$ are contained in the linear system $|\mathcal{O}_{\mathcal{Q}}(4H)|$. Thus, each intersection among $S_1\cap\mathcal{Q}$, $S_2\cap\mathcal{Q}$, $S_3\cap\mathcal{Q}$ and $S_4\cap\mathcal{Q}$ is one of the unions $\mathcal{L}_4^1\cup\mathcal{L}_4^3$, $\mathcal{L}_4^1\cup\mathcal{L}_4^4$, $\mathcal{L}_4^2\cup\mathcal{L}_4^3$ and $\mathcal{L}_4^2\cup\mathcal{L}_4^4$.

Using this remark, we see that for every $\overline{G}_{144}$-orbit among Σ_{16}^1, Σ_{16}^2, Σ_{16}^3 and Σ_{16}^4, there exists exactly one quartic surface among S_1, S_2, S_3 and S_4 that does not contain it. This implies

Lemma 4.15 *Let C be a $\overline{G}_{144}$-invariant curve in $\mathbb{P}^3$ of degree less than 12. Then $C\subset\mathcal{Q}$.*

Proof We may assume that C is $\overline{G}_{144}$-irreducible. Suppose that C is not contained in the quadric $\mathcal{Q}$. Let us seek for a contradiction. Observe that

$$C\cdot\mathcal{Q}=2\cdot\deg(C)\leqslant 22.$$

Thus, it follows from Lemma 4.4 that $\deg(C)=8$, and the intersection $C\cap\mathcal{Q}$ is one of the orbits Σ_{16}^1, Σ_{16}^2, Σ_{16}^3 or Σ_{16}^4. Then the curve C does not contains other $\overline{G}_{144}$-orbits of length 16 in $\mathbb{P}^3$, because all of them lie in $\mathcal{Q}$ by Lemma 4.10. Let S_k be the surface among S_1, S_2, S_3 and S_4 that does not contain $C\cap\mathcal{Q}$. Then S_k does not contain the curve C. On the other hand, we have

$$C\cdot S_k=4\cdot\deg(C)=32.$$

Hence, the intersection $C\cap S_k$ is one of the $\overline{G}_{144}$-orbits Σ_{16}^1, Σ_{16}^2, Σ_{16}^3, Σ_{16}^4 by Lemma 4.10. This is impossible, since S_k does not contain the $\overline{G}_{144}$-orbit $C\cap\mathcal{Q}$, and C does not contain other $\overline{G}_{144}$-orbits of length 16. □

Lemma 4.16 *Let C be a reducible $\overline{G}_{144}$-irreducible curve in $\mathbb{P}^3$ of degree less than 16. Then $C\subset\mathcal{Q}$.*

Proof Suppose that C is not contained in the quadric $\mathcal{Q}$. Then $\deg(C)\geqslant 12$ by Lemma 4.15. Moreover, since

$$30\geqslant 2\cdot\deg(C)=C\cdot\mathcal{Q},$$

we see that $\deg(C)=12$ by Lemma 4.4.

Let $C=C_1+\cdots+C_r$, where C_i is an irreducible curve of degree d. Denote by Γ the stabilizer in $\overline{G}_{144}$ of the curve C_1. Since $rd=\deg(C)=12$ and $\overline{G}_{144}$ does not have subgroups of index 2 by Lemma 4.1(vii), we see that either $r=3$ and $d=4$, or $r=4$ and $d=3$, or $r=6$ and $d=2$, or $r=12$ and $d=1$. If $d\neq 1$, then C_1

is not contained in a plane, because there are no $\overline{G}_{144}$-orbits in $\mathbb{P}^3$ of length at most 6 by Lemma 4.10. In particular, the curve C_1 is not a conic, so that $r \neq 6$. We also conclude that Γ acts faithfully on C_1 and on its normalization in the case when $d \neq 1$. Thus, the curve C is either a union of 3 irreducible quartic curves, or a union of 4 cubic curves, or a union of 12 lines. Let us deal with each of these cases one by one.

Suppose that $r = 3$ and $d = 4$. Then $|\Gamma| = 48$, so that Γ contains the subgroup $\overline{\mathrm{H}}$ by Lemma 4.1(ii). On the other hand, the curve C_1 must be either rational or elliptic, because $d = 4$ and C_1 is not contained in a plane. This is impossible by Lemma 3.3.

Suppose that $r = 4$ and $d = 3$. Then C_1 is a twisted cubic curve. One has $|\Gamma| = 36$, so that Γ has a surjective homomorphism to $\mathfrak{A}_4$ by Lemma 4.1(viii). This is impossible by the classification of finite subgroups in $\mathrm{PGL}_2(\mathbb{C})$.

We see that $r = 12$ and $d = 1$, so that C_1 is a line and $|\Gamma| = 12$. Since $C \cdot \mathcal{Q} = 24$, we conclude from Lemma 4.4 that $C \cap \mathcal{Q}$ is one of the $\overline{G}_{144}$-orbits Σ^1_{24}, Σ^2_{24}, Σ^3_{24} or Σ^4_{24}. Without loss of generality, we may assume that $C \cap \mathcal{Q} = \Sigma^1_{24}$. We see that every irreducible component of the curve C contains exactly two points in Σ^1_{24}, because every line C_i can intersect $\mathcal{Q}$ by at most two points. Write $C_1 \cap \Sigma^1_{24} = \{P, Q\}$, and denote the stabilizer of the point P in $\overline{G}_{144}$ by Γ_P. Then $|\Gamma_P| = 6$, so that $\Gamma_P \cong \boldsymbol{\mu}_6$ by Lemma 4.1(v). On the other hand, the stabilizer of this point in Γ has index at most 2. This means that this stabilizer is the group Γ_P, so that $\Gamma_P \subset \Gamma$. Now using Lemma 4.1(v) again, we see that $\Gamma \cong \boldsymbol{\mu}_2^2 \times \boldsymbol{\mu}_3$ and Γ is of product type. Pick an element $\gamma \in \Gamma$ such that $\gamma \notin \Gamma_P$ and the order of γ is two. Recall that

$$\Sigma^1_{24} = \mathcal{L}^1_4 \cap \mathcal{L}^2_6.$$

Then γ preserves the irreducible component of $\mathcal{L}^1_4$ that contains the point P. Denote it by L. This means that $Q = \gamma(P) \in L$, so that the lines L and C_1 coincide, which is impossible, since C_1 is not contained in $\mathcal{Q}$ by assumption. □

Let us describe the intersections $S_i \cap S_j$, $1 \leqslant i < j \leqslant 4$.

Lemma 4.17 *Both $S_1 \cap S_2$ and $S_3 \cap S_4$ are $\overline{G}_{144}$-irreducible curves that are unions of 16 lines.*

Proof It is enough to show that $S_1 \cap S_2$ is a $\overline{G}_{144}$-irreducible curve that is a union of 16 lines, since the proof is identical in the remaining case. Write

$$S_2\big|_{S_1} = \sum_{i=1}^{r} n_i Z_i,$$

where each Z_i is a $\overline{G}_{144}$-irreducible curve, each n_i is a positive integer, and $r \geqslant 1$. We may assume that $\deg(Z_i) \leqslant \deg(Z_j)$ for $i \leqslant j$. Then

$$16 = \sum_{i=1}^{r} n_i \deg(Z_i).$$

Note that the intersection $S_1 \cap S_2$ does not contain the curve $\mathcal{L}_6^1$ and the curve $\mathcal{L}_6^2$, simply because S_1 and S_2 do not contain them. Indeed, the restrictions $S_1|_{\mathcal{Q}}$ and $S_2|_{\mathcal{Q}}$ are divisors of bi-degree (4, 4) on the quadric $\mathcal{Q} \cong \mathbb{P}^1 \times \mathbb{P}^1$, while both curves $\mathcal{L}_6^1$ and $\mathcal{L}_6^2$ are divisors on $\mathcal{Q}$ of bi-degree (6, 0) and (0, 6), respectively. Moreover, it follows from Remark 4.14 that the intersection $S_1 \cap S_2$ cannot contain two curves among $\mathcal{L}_4^1$, $\mathcal{L}_4^2$, $\mathcal{L}_4^3$ and $\mathcal{L}_4^4$. Thus, it follows from Lemmas 4.5 and 4.15 that one of the following cases holds:

- either $r = 1$ and $n_1 = 1$,
- or $r = 2$, $n_1 = n_2 = 1$, $\deg(Z_2) = 12$, and Z_1 is one of the curves $\mathcal{L}_4^1$, $\mathcal{L}_4^2$, $\mathcal{L}_4^3$, $\mathcal{L}_4^4$.

Let $f\colon \widetilde{S}_1 \to S_1$ be the minimal resolution of singularities of the surface S_1. Then the action of the group $\overline{G}_{144}$ lifts to the surface $\widetilde{S}_1$. Denote by $E_1, \dots, E_{12}$ the exceptional curves of the birational morphism f. Let $E = E_1 + \cdots + E_{12}$, and let H be a plane section of the surface S_1. Denote by $\widetilde{Z}_i$ the proper transform of the curve Z_i on the surface $\widetilde{S}_1$. Then

$$\sum_{i=1}^{r} Z_i \sim f^*(4H) - mE$$

for some none-negative integer m. Moreover, we have $m \geqslant 2$, since S_2 is singular at every point of Σ_{12}. On the other hand, since $\widetilde{S}_1$ is a smooth $K3$ surface, we have

$$\widetilde{Z}_i^2 \geqslant -2 \cdot \deg(Z_i),$$

because the curve $\widetilde{Z}_i$ cannot consist of more than $\deg(Z_i)$ irreducible components.

If $r = 2$, then $\widetilde{Z}_1^2 = -8$ and $\widetilde{Z}_1 \cdot E = 0$, since Z_1 is a disjoint union of 4 lines that does not contain Σ_{12}. In this case, we have

$$-24 = -2 \cdot \deg\big(Z_2\big) \leqslant \widetilde{Z}_2^2 = \Big(f^*(4H) - mE - \widetilde{Z}_1\Big)^2 = 64 - 24m^2 - 40 \leqslant -72,$$

which is absurd.

Thus, we see that $r = 1$, so that $S_1 \cap S_2 = Z_1$ is a $\overline{G}_{144}$-irreducible curve of degree 16. This gives

$$-32 = -2 \cdot \deg\big(Z_1\big) \leqslant \widetilde{Z}_1^2 = \Big(f^*(4H) - mE\Big)^2 = 64 - 24m^2 \leqslant -32,$$

so that $\widetilde{Z}_1^2 = -2 \cdot \deg(Z_1)$. This easily implies that Z_1 is a union of $\deg(Z_1) = 16$ lines, cf. the proof of [10, Lemma 10.1.1]. □

As we already mentioned in Remark 4.14, each of the intersections $S_1 \cap \mathcal{Q}$, $S_2 \cap \mathcal{Q}$, $S_3 \cap \mathcal{Q}$ and $S_4 \cap \mathcal{Q}$ is one of the unions $\mathcal{L}_4^1 \cup \mathcal{L}_4^3$, $\mathcal{L}_4^1 \cup \mathcal{L}_4^4$, $\mathcal{L}_4^2 \cup \mathcal{L}_4^3$ or $\mathcal{L}_4^2 \cup \mathcal{L}_4^4$. On the other hand, we just proved that the intersections $S_1 \cap S_2$ and $S_3 \cap S_4$ are $\overline{G}_{144}$-irreducible

curves. Thus, without loss of generality, we may assume that

$$S_1 \cap \mathcal{Q} = \mathcal{L}_4^2 \cup \mathcal{L}_4^4, \quad S_2 \cap \mathcal{Q} = \mathcal{L}_4^1 \cup \mathcal{L}_4^3, \quad S_3 \cap \mathcal{Q} = \mathcal{L}_4^1 \cup \mathcal{L}_4^4, \quad S_4 \cap \mathcal{Q} = \mathcal{L}_4^2 \cup \mathcal{L}_4^3. \tag{4.18}$$

Using this, we can determine which $\overline{G}_{144}$-orbit Σ_{16}^1, Σ_{16}^2, Σ_{16}^3, Σ_{16}^4 is contained in which surface S_1, S_2, S_3 and S_4. This is summarized in the following table.

	S_1	S_2	S_3	S_4
Σ_{16}^1	−	+	+	+
Σ_{16}^2	+	+	+	−
Σ_{16}^3	+	+	−	+
Σ_{16}^4	+	−	+	+

Observe that the intersection $S_1 \cap S_3$ contains the curve $\mathcal{L}_4^4$, the intersection $S_1 \cap S_4$ contains the curve $\mathcal{L}_4^2$, the intersection $S_2 \cap S_3$ contains the curve $\mathcal{L}_4^1$, and the intersection $S_2 \cap S_4$ contains the curve $\mathcal{L}_4^3$. This gives

Lemma 4.19 *One has*

$$S_1 \cap S_3 = \mathcal{L}_4^4 \cup C_{12}^4, \quad S_1 \cap S_4 = \mathcal{L}_4^2 \cup C_{12}^2, \quad S_2 \cap S_3 = \mathcal{L}_4^1 \cup C_{12}^1, \quad S_2 \cap S_4 = \mathcal{L}_4^3 \cup C_{12}^3,$$

where C_{12}^1, C_{12}^2, C_{12}^3 and C_{12}^4 are distinct $\overline{G}_{144}$-invariant irreducible smooth curves in $\mathbb{P}^3$ of degree 12 and genus 13.

Proof It is enough to prove that $S_1 \cap S_3 = \mathcal{L}_4^4 \cup C_{12}^4$, where C_{12}^4 is a $\overline{G}_{144}$-invariant irreducible smooth curve of degree 12 and genus 13. We see from (4.18) that S_1 and S_3 contain the curve $\mathcal{L}_4^4$. Thus, we write

$$S_3\big|_{S_1} = m\mathcal{L}_4^4 + \sum_{i=1}^{r} n_i Z_i,$$

where m is a positive integer, each Z_i is a $\overline{G}_{144}$-irreducible curve that is different from $\mathcal{L}_4^4$, each n_i is a positive integer, and $r \geqslant 0$. Then

$$16 = 4m + \sum_{i=1}^{r} n_i \deg(Z_i).$$

Note that r is actually positive, since $S_3|_{S_1}$ is an ample divisor. If $m \geqslant 2$ or $r \geqslant 2$, then $\deg(Z_i) < 12$ for each curve Z_i, so that Z_i is contained in $\mathcal{Q}$ by Lemma 4.15, which contradicts (4.18). Thus, we see that $m = 1$ and $r = 1$. Similarly, we obtain $n_1 = 1$.

For simplicity, denote $Z = Z_1$. Let H be a plane section of the surface S_1. Then

$$S_3\big|_{S_1} = \mathcal{L}_4^4 + Z \sim 4H.$$

Note that the curve Z is contained in the smooth locus of S_1, because $S_1 \cap S_3$ does not contain Σ_{12} by (4.11). Thus, we have

$$-8 + \mathcal{L}_4^4 \cdot Z = \mathcal{L}_4^4 \cdot \left(\mathcal{L}_4^4 + Z\right) = 4H \cdot \mathcal{L}_4^4 = 16,$$

so that $\mathcal{L}_4^4 \cdot Z = 24$ on the surface S_1. This gives

$$Z^2 + 24 = Z^2 + \mathcal{L}_4^4 \cdot Z = Z \cdot \left(\mathcal{L}_4^4 + Z\right) = 4H \cdot Z = 48,$$

so that $Z^2 = 24$.

If Z is reducible, then Z is contained in the quadric $\mathcal{Q}$ by Lemma 4.16, which is impossible, because $S_1 \cap \mathcal{Q} = \mathcal{L}_4^2 \cup \mathcal{L}_4^4$ by (4.18). Thus, we see that Z is irreducible. Then it has arithmetic genus 13. If Z is singular, then it has at least 12 singular points by Lemma 4.10. In this case, the genus of its normalization is at most 1, which contradicts Lemma 4.2. Therefore, we see that the curve Z is smooth. □

Corollary 4.20 *Let C be one of the curves C_{12}^1, C_{12}^2, C_{12}^3, C_{12}^4, and let $\mathcal{M}$ be the linear system of surfaces in $\mathbb{P}^3$ of degree 6 that contains C. Then the base locus of $\mathcal{M}$ consists of the curve C.*

Proof Without loss of generality, we may assume that $C = C_{12}^4$. Then C is contained in S_1 and

$$S_3\big|_{S_1} = \mathcal{L}_4^4 + C.$$

by Lemma 4.19. Note that both curves $\mathcal{L}_4^4$ and C are contained in the smooth locus of the surface S_1 by (4.11) and Lemma 4.12. Let H be a plane section of S_1, and let $D = 2H + \mathcal{L}_4^4$. Then $D^2 = 24$ and D is nef, since $D \cdot \mathcal{L}_4^4 = 0$. Using the Riemann–Roch formula and Nadel vanishing theorem (see [30, Theorem 9.4.8]), we see that $h^0(\mathcal{O}_{S_1}(D)) = 14$. This implies that the linear system $|D|$ does not have fixed curves. Indeed, if it does, then its fixed part must be the curve $\mathcal{L}_4^4$, so that

$$14 = h^0\Big(\mathcal{O}_{S_1}\big(D\big)\Big) = h^0\Big(\mathcal{O}_{S_1}\big(2H\big)\Big) = 10,$$

which is absurd. Thus $|D|$ does not have base points by [39, Corollary 3.2]. Therefore, the linear system $|6H - C|$ is base point free. Since S_1 is projectively normal, we immediately obtain the required assertions. □

Corollary 4.21 *Let $\mathcal{D}$ be a (non-empty) mobile $\overline{G}_{144}$-invariant linear system on $\mathbb{P}^3$, let n be a positive integer such that $\mathcal{D} \sim \mathcal{O}_{\mathbb{P}^3}(n)$, and let C be one of the curves C_{12}^1, C_{12}^2, C_{12}^3 or C_{12}^4. Then* $\mathrm{mult}_C(\mathcal{D}) \leqslant \frac{n}{4}$.

Proof Without loss of generality, we may assume that $C = C_{12}^4$. Then C is contained in the surface S_1 by Lemma 4.19, and $S_3|_{S_1} = \mathcal{L}_4^4 + C$. Let D be a general surface in

$\mathcal{D}$. Then

$$D|_{S_1} = aC + b\mathcal{L}_4^4 + \Delta,$$

where a and b are non-negative integers such that $a \geqslant \mathrm{mult}_C(\mathcal{D})$, and Δ is an effective divisor on S_1 whose support contains neither C nor any irreducible component of the curve $\mathcal{L}_4^4$. Then

$$aC + b\mathcal{L}_4^4 + \Delta \sim_{\mathbb{Q}} \frac{n}{4}\Big(\mathcal{L}_4^4 + C\Big).$$

If $a > \frac{n}{4}$, then $b < \frac{n}{4}$, since we have

$$\Big(\frac{n}{4} - b\Big)\mathcal{L}_4^4 \sim_{\mathbb{Q}} \Big(a - \frac{n}{4}\Big)C + \Delta.$$

Therefore, if $a > \frac{n}{4}$, then

$$0 > -8\Big(\frac{n}{4} - b\Big) = \Big(\frac{n}{4} - b\Big)\mathcal{L}_4^4 \cdot \mathcal{L}_4^4 = \Big(a - \frac{n}{4}\Big)\mathcal{L}_4^4 \cdot C + \mathcal{L}_4^4 \cdot \Delta \geqslant 0,$$

which is absurd. This shows that $\mathrm{mult}_C(\mathcal{D}) \leqslant a \leqslant \frac{n}{4}$ as required. □

Now we are ready to prove.

Theorem 4.22 *Let C be $\overline{G}_{144}$-irreducible curve in $\mathbb{P}^3$ such that* $\deg(C) \leqslant 15$. *Then one of the following cases holds:*

- $\deg(C) = 4$, *and C is one of the curves $\mathcal{L}_4^1$, $\mathcal{L}_4^2$, $\mathcal{L}_4^3$, $\mathcal{L}_4^4$;*
- $\deg(C) = 6$, *and C is one of the curves $\mathcal{L}_6^1$ or $\mathcal{L}_6^2$;*
- $\deg(C) = 12$, *and C is one of the smooth irreducible curves C_{12}^1, C_{12}^2, C_{12}^3, C_{12}^4;*
- $\deg(C) = 12$, *and C is a union of* 12 *disjoint lines contained in the quadric $\mathcal{Q}$.*

Proof By Lemma 4.5, we may assume that C is not contained in the quadric $\mathcal{Q}$. Then $\deg(C) \geqslant 12$ by Lemma 4.15, and C is irreducible by Lemma 4.16. Moreover, since

$$30 \geqslant 2 \cdot \deg(C) = C \cdot \mathcal{Q},$$

we see that $\deg(C) = 12$ by Lemma 4.4. Let us show that C is one of the smooth irreducible curves C_{12}^1, C_{12}^2, C_{12}^3 or C_{12}^4.

Since $C \cdot \mathcal{Q} = 24$, it follows from Lemma 4.4 that $C \cap \mathcal{Q}$ is a $\overline{G}_{144}$-orbit of length 24, and C intersects $\mathcal{Q}$ transversally at the points of $C \cap \mathcal{Q}$. In particular, the curve C is smooth at these points. Recall from Lemma 4.10 that all $\overline{G}_{144}$-orbits of length 24 in $\mathbb{P}^3$ and all $\overline{G}_{144}$-orbits of length 16 in $\mathbb{P}^3$ are contained in $\mathcal{Q}$. Thus, the curve C does not contain $\overline{G}_{144}$-orbits of length 24 that are different from $C \cap \mathcal{Q}$, and it does not contain $\overline{G}_{144}$-orbits of length 16. Without loss of generality, we may assume that

$$C \cap \mathcal{Q} = \Sigma_{24}^1 = \mathcal{L}_4^1 \cap \mathcal{L}_6^2$$

Recall that Σ_{24}^1 is contained in $S_2 \cap S_3$, because $S_2 \cap \mathcal{Q} = \mathcal{L}_4^1 \cup \mathcal{L}_4^3$ and $S_3 \cap \mathcal{Q} = \mathcal{L}_4^1 \cup \mathcal{L}_4^4$, see (4.18). Since $S_2 \cdot C = S_3 \cdot C = 48$, it follows from Lemma 4.10, (4.11) and Lemma 4.12 that one of the following cases holds:

- the curve C is contained in one of the surfaces S_2 and S_3,
- both S_2 and S_3 are tangent to C at the points of Σ_{24}^1, and $S_2 \cap C = S_3 \cap C = \Sigma_{24}^1$,
- $S_2 \cap C = \Sigma_{24}^1 \cup \Sigma_{12}$ or $S_3 \cap C = \Sigma_{24}^1 \cup \Sigma_{12}'$.

Suppose that $S_2 \cap C = \Sigma_{24}^1 \cup \Sigma_{12}$. Since C is irreducible, it must be singular at every point of Σ_{12} by Lemma 4.2. Then

$$48 = S_2 \cdot C \geqslant |\Sigma_{24}^1| \mathrm{mult}_{\Sigma_{24}^1}(S_2) + |\Sigma_{12}| \mathrm{mult}_{\Sigma_{12}}(S_2) \mathrm{mult}_{\Sigma_{12}}(C) \geqslant 72,$$

which is absurd. Thus, we have $S_2 \cap C \neq \Sigma_{24}^1 \cup \Sigma_{12}$. Similarly, we see that the intersection $S_3 \cap C$ is not the union $\Sigma_{24}^1 \cup \Sigma_{12}'$.

Suppose that the surfaces S_2 and S_3 are tangent to the curve C at the points of Σ_{24}^1, and

$$S_2 \cap C = S_3 \cap C = \Sigma_{24}^1.$$

Let $\mathcal{P}$ be the pencil generated by S_2 and S_3. Then $\mathcal{P}$ is $\overline{G}_{144}$-invariant. Moreover, every surface in this pencil either contains C or is tangent to the curve C at the points of Σ_{24}^1 (here we include the case when the surface is singular at some points of Σ_{24}^1). This implies that there exists a surface S in $\mathcal{P}$ that contains C. Indeed, let P be a point in C that is not contained in Σ_{24}^1, and let S be a surface in $\mathcal{P}$ that passes through P. If C is not contained in S, then we obtain contradictory inequalities

$$48 = S \cdot C \geqslant \sum_{Q \in \Sigma_{24}^1} \mathrm{mult}_Q\Big(S \cdot C\Big) + \mathrm{mult}_P\Big(S \cdot C\Big) \geqslant 2|\Sigma_{24}^1| + 1 = 49,$$

because S is tangent to C at the points of Σ_{24}^1. Thus, we see that the curve C is contained in S. Moreover, if S is not $\overline{G}_{144}$-invariant, then there exists $\bar{g} \in \overline{G}_{144}$ such that $\bar{g}(S) \neq S$, so that $\bar{g}(S) \in \mathcal{P}$, and the surfaces S and $\bar{g}(S)$ generate the whole pencil $\mathcal{P}$. This implies that C is contained in $S_2 \cap S_3$, which is not the case by assumption.

We are left with the case when C is contained in one of the surfaces S_2 and S_3. Without loss of generality, we may assume that $C \subset S_2$. Let us show that the curve C is also contained in S_3. This would imply that $C = C_{12}^1$ by Lemma 4.19.

Let $f\colon \widetilde{S}_2 \to S_2$ be the minimal resolution of singularities of the surface S_2. Then the action of the group $\overline{G}_{144}$ lifts to the surface $\widetilde{S}_2$. Denote by $E_1, \ldots, E_{12}$ the f-exceptional curves. Let $E = E_1 + \cdots + E_{12}$, and let H be a plane section of the surface S_2. Denote by $\widetilde{C}$ the proper transform of the curve C on the surface $\widetilde{S}_2$. Recall from Lemma 4.19 that $S_2 \cap S_3 = \mathcal{L}_4^1 \cup C_{12}^1$ and $S_2 \cap S_4 = \mathcal{L}_4^3 \cup C_{12}^3$. Denote by $\widetilde{C}_{12}^1$ and $\widetilde{C}_{12}^3$ the proper transforms on the surface $\widetilde{S}_2$ of the curves C_{12}^1 and C_{12}^3, respectively. Similarly, denote by $\widetilde{\mathcal{L}}_4^1$ and $\widetilde{\mathcal{L}}_4^3$ the proper transforms on the surface $\widetilde{S}_2$ of the curves $\mathcal{L}_4^1$ and $\mathcal{L}_4^3$, respectively. Since

$$\mathrm{Sing}(S_2) = \Sigma_{12} \not\subset S_3 \cup S_4$$

by (4.11) and Lemma 4.12, we have

$$\widetilde{\mathcal{L}}_4^1 + \widetilde{C}_{12}^1 \sim \widetilde{\mathcal{L}}_4^3 + \widetilde{C}_{12}^3 \sim f^*(4H).$$

Moreover, we have $\widetilde{\mathcal{L}}_4^1 + \widetilde{\mathcal{L}}_4^3 \sim f^*(2H)$, so that $\widetilde{C} \cdot (\widetilde{\mathcal{L}}_4^1 + \widetilde{\mathcal{L}}_4^3) = 24$. Using Lemma 4.4 and keeping in mind that the curve C contains the $\overline{G}_{144}$-orbit $\Sigma_{24}^1 = \mathcal{L}_4^1 \cap \mathcal{L}_6^2$, we conclude that $\widetilde{C} \cdot \widetilde{\mathcal{L}}_4^1 = 24$ and $\widetilde{C} \cdot \widetilde{\mathcal{L}}_4^3 = 0$. Thus

$$48 = \widetilde{C} \cdot \left(\widetilde{\mathcal{L}}_4^1 + \widetilde{C}_{12}^1\right) = 24 + \widetilde{C} \cdot \widetilde{C}_{12}^1,$$

so that $\widetilde{C} \cdot \widetilde{C}_{12}^1 = 24$. Therefore, it follows from Hodge index theorem that

$$\begin{vmatrix} \widetilde{C}^2 & \widetilde{C} \cdot \widetilde{C}_{12}^1 \\ \widetilde{C} \cdot \widetilde{C}_{12}^1 & \widetilde{C}_{12}^1 \cdot \widetilde{C}_{12}^1 \end{vmatrix} = \begin{vmatrix} \widetilde{C}^2 & 24 \\ 24 & 24 \end{vmatrix} \leqslant 0.$$

This implies that either $\widetilde{C}^2 \leqslant 22$ or $\widetilde{C}^2 = 24$. Therefore, the arithmetic genus of the curve $\widetilde{C}$ is at most

$$\frac{\widetilde{C}^2}{2} + 1 \leqslant 13.$$

If $\widetilde{C}$ is singular, then it has at least 12 singular points by Lemma 4.10. In this case, the genus of its normalization is at most 1, which contradicts Lemma 4.2. Thus, we see that the curve $\widetilde{C}$ is a smooth curve of genus $\frac{\widetilde{C}^2}{2} + 1$. Therefore, it follows from Lemma 4.2 that $\widetilde{C}^2 = 24$, because $\widetilde{C}$ contains $\overline{G}_{144}$-orbits of length 24, for instance, the $\overline{G}_{144}$-orbit $\widetilde{C} \cap \widetilde{\mathcal{L}}_4^1$. Thus, it follows from Hodge index theorem that the curves $\widetilde{C}$ and $\widetilde{C}_{12}^1$ are numerically equivalent on $\widetilde{S}_2$, so that $\widetilde{C} \sim \widetilde{C}_{12}^1$. Hence $\widetilde{C} \cdot E = 0$, so that C does not contain the singular locus of the surface S_2, which implies that

$$C + \mathcal{L}_4^1 \sim 4H.$$

Since S_2 is projectively normal, we see that $C = C_{12}^1$ by Lemma 4.19. □

5 The group of order 80

Let us use notation and assumptions of Sect. 3. In this section, we present basic facts about $\overline{G}_{80}$-orbits, $\overline{G}_{80}$-invariant curves and $\overline{G}_{80}$-invariant surfaces in $\mathbb{P}^3$. Let us start with the following very easy group-theoretic result.

Lemma 5.1 *Let Γ be a proper non-trivial subgroup of the group $\overline{G}_{80}$. Then Γ is isomorphic to one of the following groups: $\boldsymbol{\mu}_2$, $\boldsymbol{\mu}_2^2$, $\boldsymbol{\mu}_2^3$, $\boldsymbol{\mu}_2^4$, or $\boldsymbol{\mu}_5$. Moreover, a subgroup of $\overline{G}_{80}$ isomorphic to $\boldsymbol{\mu}_2^4$ is unique, and a subgroup isomorphic to $\boldsymbol{\mu}_5$ is unique up to conjugation. Furthermore, the only non-trivial proper normal subgroup of $\overline{G}_{80}$ is $\overline{\mathrm{H}} \cong \boldsymbol{\mu}_2^4$.*

Proof Sylow theorems imply that the order of every non-trivial element of $\overline{G}_{80}$ equals either 2 or 5. If every non-trivial element of Γ has order 2, then Γ is contained in the Sylow 2-subgroup of $\overline{G}_{80}$, which is exactly $\overline{\mathrm{H}}$. If every non-trivial element of Γ has order 5, then $\Gamma \cong \boldsymbol{\mu}_5$. The uniqueness assertions for subgroups isomorphic to $\boldsymbol{\mu}_2^4$ and $\boldsymbol{\mu}_5$ also follow from Sylow theorems.

Suppose that Γ contains both elements of order 2 and elements of order 5. Then it is isomorphic to $\Gamma' \rtimes \boldsymbol{\mu}_5$, where Γ' is a non-trivial proper subgroup of the group $\overline{\mathrm{H}}$ that is invariant under the action of

$$\overline{G}_{80}/\overline{\mathrm{H}} \cong \boldsymbol{\mu}_5 \subset \mathrm{GL}_4\big(\mathbf{F}_2\big)$$

on $\overline{\mathrm{H}}$. However, the group $\boldsymbol{\mu}_5$ is not a subgroup of $\mathrm{GL}_k(\mathbf{F}_2)$ for $k \leqslant 3$. This implies that the action of $\boldsymbol{\mu}_5$ on $\mathbf{F}_2^4$ is irreducible, so that $\overline{\mathrm{H}}$ has no non-trivial proper $\boldsymbol{\mu}_5$-invariant subgroups. Therefore, we see that $\Gamma = \overline{G}_{80}$.

Now we suppose that Γ is a non-trivial normal subgroup of $\overline{G}_{80}$. Then Γ is not isomorphic to $\boldsymbol{\mu}_5$. Also we know from the above arguments that it cannot be a proper subgroup of $\boldsymbol{\mu}_2^4$. Thus, if $\Gamma \ncong \boldsymbol{\mu}_2^4$, then Γ must contain both elements of order 2 and elements of order 5. This implies that $\Gamma = \overline{G}_{80}$. □

Corollary 5.2 *Let $\overline{G}$ be a finite subgroup in $\mathrm{PGL}_4(\mathbb{C})$ that is isomorphic to $\overline{G}_{80}$. Then $\overline{G}$ is conjugate to $\overline{G}_{80}$.*

Proof It follows from [4, Chapter VII] that $\overline{G}$ is conjugate to $\overline{G}_{80}$ provided that $\overline{G}$ is primitive. Suppose $\overline{G}$ is not primitive. Let us seek for a contradiction. To start with, suppose that $\overline{G}$ is transitive. Then either there exists a $\overline{G}$-orbit of length 4, or there exists a $\overline{G}$-invariant pair of skew lines. In the former case, we have a non-trivial homomorphism $\overline{G} \to \mathfrak{S}_4$. In the latter case, we have a non-trivial homomorphism $\overline{G} \to \mathfrak{S}_2$. None of this is possible by Lemma 5.1.

Thus, we see that $\overline{G}$ is intransitive. Then either there exists a $\overline{G}$-invariant point P, or there exist a $\overline{G}$-invariant line L. In the former case, the group $\overline{G}$ acts faithfully on the tangent space $T_P(\mathbb{P}^3) \cong \mathbb{C}^3$ by [10, Lemma 4.4.1], which is impossible, because $\mathrm{GL}_3(\mathbb{C})$ does not contain subgroups isomorphic to $\boldsymbol{\mu}_2^4$. In the latter case, we obtain a homomorphism

$$\overline{G} \to \mathrm{Aut}\big(L\big) \cong \mathrm{PGL}_2\big(\mathbb{C}\big).$$

We conclude from Lemma 5.1 and the classification of finite subgroups in $\mathrm{PGL}_2(\mathbb{C})$ that the kernel of this homomorphism contains the subgroup in $\overline{G}$ isomorphic to $\boldsymbol{\mu}_2^4$, so that this subgroup must fix every point in L, which is again impossible by [10, Lemma 4.4.1]. □

To study $\overline{G}_{80}$-invariant curves in $\mathbb{P}^3$, we need the following result, which follows from the Riemann–Hurwitz formula.

Lemma 5.3 *Let C be a smooth irreducible curve of genus $g \leqslant 19$ with a faithful action of the group $\overline{G}_{80}$. Then $g \in \{5, 13, 17\}$, and C is not hyperelliptic. Furthermore, if $\Omega \subset C$ is a $\overline{G}_{80}$-orbit, then $|\Omega| \in \{16, 40, 80\}$. Finally, the possible numbers a_i of $\overline{G}_{80}$-orbits of length $i \in \{16, 40\}$ in C are contained in the following table:*

g	5	13	17
a_{16}	2	1	3
a_{40}	1	3	0

Proof The assertion about the lengths of $\overline{G}_{80}$-orbits follows from Lemma 5.1, since the stabilizers in $\overline{G}_{80}$ of points in C must be cyclic (see [10, Lemma 5.1.4]). Moreover, the group $\overline{G}_{80}$ cannot act faithfully on $\mathbb{P}^1$ and on a smooth elliptic curve by Lemma 3.3. Hence, one has $g \geqslant 2$.

Suppose that the curve C is hyperelliptic. Since the group $\overline{G}_{80}$ does not contain normal subgroups of order 2 by Lemma 5.1, it does not contain the hyperelliptic involution of C, and we obtain a faithful action of the group $\overline{G}_{80}$ on $\mathbb{P}^1$. We already proved that this is impossible by Lemma 3.3.

Let $\widehat{C} = C/\overline{G}_{80}$. Then $\widehat{C}$ is a smooth curve. Let $\hat{g}$ be the genus of the curve $\widehat{C}$. Then the Riemann–Hurwitz formula gives

$$2g - 2 = 80\big(2\hat{g} - 2\big) + 40a_{40} + 64a_{16}.$$

Since $a_k \geqslant 0$ and $g \leqslant 19$, one has $\hat{g} = 0$, so that

$$2g - 2 = -160 + 40a_{40} + 64a_{16},$$

Going through the possible values of g, and solving this equation case by case we obtain the required result. □

Lemma 5.4 *There are exactly four $\overline{G}_{80}$-orbits of length* 16 *in* $\mathbb{P}^3$.

Proof A subgroup $\boldsymbol{\mu}_5 \subset \overline{G}_{80}$ has a fixed point in $\mathbb{P}^3$, so that the length of its $\overline{G}_{80}$-orbit is 16 by Lemma 5.1. This shows that there is at least one $\overline{G}_{80}$-orbit of length 16 in $\mathbb{P}^3$.

Now let Γ be a stabilizer of a point in a $\overline{G}_{80}$-orbit of length 16. Then $\Gamma \cong \boldsymbol{\mu}_5$. By Lemma 5.1, the subgroup Γ is conjugate to the subgroup generated by the image T in $\mathrm{PGL}_4(\mathbb{C})$. On the other hand, the eigenvalues of the matrix T are four distinct primitive fifth roots of unity. Since the normalizer of Γ in $\overline{G}_{80}$ coincides with Γ by Lemma 5.1, we conclude that there are exactly four $\overline{G}_{80}$-orbits of length 16 in $\mathbb{P}^3$. □

Let us denote the $\overline{G}_{80}$-orbits of length 16 in $\mathbb{P}^3$ by Σ^1_{16}, Σ^2_{16}, Σ^3_{16} and Σ^4_{16}. Computing the null-spaces of the matrices $T - \xi_5^i I_4$, $1 \leqslant i \leqslant 4$, one can find these orbits explicitly. In particular, we may assume that

$$\begin{aligned}
&\big[(-1+i)\xi_5^3 + (-1+i)\xi_5^2 - \xi_5 - 1 : 1 + (1+i)\xi_5^2 + \xi_5 : 1\\
&\quad +(1+i)\xi_5^3 + \xi_5 : \xi_5 - 1\big] \in \Sigma^1_{16},\\
&\big[(1+i)\xi_5^3 + (1-i)\xi_5^2 + i\xi_5 + 1 : -2\xi_5^3 + (-1-i)\xi_5^2 - \xi_5 - 2 - i : (-1+i)\\
&\quad \xi_5^3 + i\xi_5 + i : \xi_5 - 1\big] \in \Sigma^2_{16},
\end{aligned}$$

$$\big[-2i\xi_5^3+(1-i)\xi_5^2-i\xi_5+1-2i : (1-i)\xi_5^3-(1+i)\xi_5^2+\xi_5-i : (1-i)\xi_5^3-i\xi_5-i : \xi_5-1\big] \in \Sigma_{16}^3,$$

and

$$\big[(-1+i)\xi_5^2+i\xi_5+i : (1+i)\xi_5^3+(1+i)\xi_5^2+i\xi_5+i : -1+(-1-i)\xi_5^3-\xi_5 : \xi_5-1\big] \in \Sigma_{16}^4.$$

Recall from Sect. 3, that there are three $\overline{G}_{80}$-orbits Σ_{20}, Σ'_{20} and Σ''_{20} of length 20 in $\mathbb{P}^3$ described by (3.15), (3.16) and (3.17), respectively.

Lemma 5.5 *The orbits Σ_{20}, Σ'_{20} and Σ''_{20} are the only $\overline{G}_{80}$-orbits of length 20 in $\mathbb{P}^3$.*

Proof Let Σ be a $\overline{G}_{80}$-orbit of length 20, let P be a point in Σ, and let Γ be the stabilizer in $\overline{G}_{80}$ of the point P. Then $\Gamma \cong \boldsymbol{\mu}_2^2$ by Lemma 5.1. Let γ_1, γ_2 and γ_3 be three non-trivial elements in Γ. Then the locus of fixed points of each of them consists of two skew lines in $\mathbb{P}^3$ by Lemma 3.2(i). One of the lines from each pair must pass through the point P, so that $P \in \Sigma_{20} \cup \Sigma'_{20} \cup \Sigma''_{20}$ by Lemma 3.18. □

To describe $\overline{G}_{80}$-orbits in $\mathbb{P}^3$ of length 40, recall from Lemma 3.2(i) that there are 30 lines in $\mathbb{P}^3$ such that each of them is pointwise fixed by some non-trivial element in $\overline{\mathrm{H}}$. These lines have been described in (3.11), (3.12) and (3.13). Let

$$\mathcal{L}_{10} = \sum_{i=1}^{5} \ell_i + \sum_{i=1}^{5} \check{\ell}_i.$$

Similarly, let

$$\mathcal{L}'_{10} = \sum_{i=1}^{5} \ell'_i + \sum_{i=1}^{5} \check{\ell}'_i.$$

Finally, we let

$$\mathcal{L}''_{10} = \sum_{i=1}^{5} \ell''_i + \sum_{i=1}^{5} \check{\ell}''_i.$$

Then $\mathcal{L}_{10}$, $\mathcal{L}'_{10}$, $\mathcal{L}''_{10}$ are $\overline{G}_{80}$-irreducible curves by construction.

Lemma 5.6 *The curve $\mathcal{L}_{10}$ is a disjoint union of ten lines, $\mathcal{L}'_{10}$ and $\mathcal{L}''_{10}$ are nodal unions of ten lines,* $\mathrm{Sing}(\mathcal{L}'_{10}) = \Sigma'_{20} \not\subset \mathcal{L}''_{10}$ *and* $\mathrm{Sing}(\mathcal{L}''_{10}) = \Sigma''_{20} \not\subset \mathcal{L}'_{10}$. *Moreover, one has*

$$\mathcal{L}_{10} \cap \mathcal{L}'_{10} \cap \mathcal{L}''_{10} = \mathcal{L}'_{10} \cap \mathcal{L}''_{10} = \Sigma_{20}, \quad \mathcal{L}_{10} \cap \mathcal{L}'_{10} = \Sigma_{20} \cup \Sigma'_{20},$$
$$\mathcal{L}_{10} \cap \mathcal{L}''_{10} = \Sigma_{20} \cup \Sigma''_{20},$$

and $\Sigma_{20} \cup \Sigma'_{20} \cup \Sigma''_{20} \subset \mathcal{L}_{10}$. *Furthermore, the* $\overline{G}_{80}$*-orbit of every point in*

$$\Big(\mathcal{L}_{10} \cup \mathcal{L}'_{10} \cup \mathcal{L}''_{10}\Big) \setminus \Big(\Sigma_{20} \cup \Sigma'_{20} \cup \Sigma''_{20}\Big)$$

consists of 40 *points, and every* $\overline{G}_{80}$*-orbit in* $\mathbb{P}^3$ *of length* 40 *is contained in* $\mathcal{L}_{10} \cup \mathcal{L}'_{10} \cup \mathcal{L}''_{10}$.

Proof It is straightforward to check using explicit equations of the lines of $\mathcal{L}_{10}$ that they are pairwise disjoint. Similarly, we see that $\mathcal{L}_{10}$ contains Σ_{20}, Σ'_{20} and Σ''_{20}. Likewise, the curve $\mathcal{L}'_{10}$ contains Σ_{20} and Σ'_{20}, and it does not contain Σ''_{20}. Moreover, for every point in Σ'_{20}, there are exactly two among the ten lines of $\mathcal{L}'_{10}$ that pass through this point, and all these ten lines are pairwise disjoint away from Σ'_{20}. This means that the curve $\mathcal{L}'_{10}$ is nodal and $\mathrm{Sing}(\mathcal{L}'_{10}) = \Sigma'_{20}$. In fact, one can also check that

$$\ell'_i \cap \ell'_j \neq \varnothing \iff \check{\ell}'_i \cap \check{\ell}'_j \neq \varnothing \iff \ell'_i \cap \check{\ell}'_j \neq \varnothing \iff j - i = \pm 1 \bmod 5.$$

Similarly, we see that $\mathcal{L}''_{10}$ contains $\Sigma_{20} \cup \Sigma''_{20}$, and it does not contain Σ'_{20}. Furthermore, for every point in Σ''_{20}, there are exactly two lines among the ten lines of $\mathcal{L}''_{10}$ that pass through this point, the lines of $\mathcal{L}''_{10}$ do not intersect each other in other points, and

$$\ell''_i \cap \ell''_j \neq \varnothing \iff \check{\ell}''_i \cap \check{\ell}''_j \neq \varnothing \iff \ell''_i \cap \check{\ell}''_j \neq \varnothing \iff j - i = \pm 2 \bmod 5.$$

In particular, the curve $\mathcal{L}''_{10}$ is nodal, and $\mathrm{Sing}(\mathcal{L}''_{10}) = \Sigma''_{20}$.

Explicit computations show that

$$\mathcal{L}'_{10} \cap \mathcal{L}''_{10} = \Sigma_{20}, \quad \mathcal{L}_{10} \cap \mathcal{L}'_{10} = \Sigma_{20} \cup \Sigma'_{20}, \quad \mathcal{L}_{10} \cap \mathcal{L}''_{10} = \Sigma_{20} \cup \Sigma''_{20}.$$

By Lemma 3.2(i), the curves $\mathcal{L}_{10}$, $\mathcal{L}'_{10}$ and $\mathcal{L}''_{10}$ contain all $\overline{G}_{80}$-orbits in $\mathbb{P}^3$ of length 40. Vice versa, let P be a point in $\mathcal{L}_{10} \cup \mathcal{L}'_{10} \cup \mathcal{L}''_{10}$ such that $P \notin \Sigma_{20} \cup \Sigma'_{20} \cup \Sigma''_{20}$, and let Σ be its $\overline{G}_{80}$-orbit. Then $|\Sigma| \neq 80$ by construction. Thus, we have $|\Sigma| = 40$ by Lemmas 5.1 and 5.5. □

Corollary 5.7 *Let* Σ *be a* $\overline{G}_{80}$*-orbit in* $\mathbb{P}^3$ *such that* $|\Sigma| < |\overline{G}_{80}| = 80$. *Then one of the following possibilities holds:*

- *either* $|\Sigma| = 40$ *and* $\Sigma \subset \mathcal{L}_{10} \cup \mathcal{L}'_{10} \cup \mathcal{L}''_{10}$,
- *or* $|\Sigma| = 20$ *and* Σ *is one of the* $\overline{G}_{80}$*-orbits* Σ_{20}, Σ'_{20}, Σ''_{20},
- *or* $|\Sigma| = 16$ *and* Σ *is one of the* $\overline{G}_{80}$*-orbits* Σ^1_{16}, Σ^2_{16}, Σ^3_{16}, Σ^4_{16}.

Proof Let Γ be a stabilizer of a point in Σ. If $\Gamma \cong \boldsymbol{\mu}_2$, then $|\Sigma| = 40$, and the assertion follows from Lemma 5.6. If $\Gamma \cong \boldsymbol{\mu}_2^2$, then $|\Sigma| = 20$, and the assertion follows from Lemma 5.5. If $\Gamma \cong \boldsymbol{\mu}_5$, then $|\Sigma| = 16$, and the assertion follows from Lemma 5.4. Otherwise, the group Γ contains a subgroup $\boldsymbol{\mu}_2^3 \subset \overline{\mathrm{H}}$ by Lemma 5.1, which is impossible by Lemma 3.2(iii). □

Now we are ready to prove.

Lemma 5.8 *Let C be a reducible $\overline{G}_{80}$-invariant curve in $\mathbb{P}^3$ such that $\deg(C) \leqslant 15$. Then C is one of the curves $\mathcal{L}_{10}$, $\mathcal{L}'_{10}$ or $\mathcal{L}''_{10}$.*

Proof We may assume that the curve C is $\overline{G}_{80}$-irreducible. Write $C = C_1 + \cdots + C_r$, where each C_i is an irreducible curve. Then $r \geqslant 2$ by assumption. Let d be the degree of the curve C_1. Then

$$15 \geqslant \deg(C) = dr.$$

Let Γ be the stabilizer in $\overline{G}_{80}$ of the curve C_1. Then there exists an exact sequence of groups

$$1 \to \Gamma_{C_1} \to \Gamma \to \mathrm{Aut}(C_1),$$

where Γ_{C_1} fixes the curve C_1 pointwise. By Corollary 5.7, there are only finitely many $\overline{G}_{80}$-orbits in $\mathbb{P}^3$ of length at most 20. Thus, we see that $|\Gamma_{C_1}| < 4$, so that either Γ_{C_1} is trivial or $\Gamma_{C_1} \cong \boldsymbol{\mu}_2$.

By Lemma 5.1, the group Γ is one of the groups $\boldsymbol{\mu}_2$, $\boldsymbol{\mu}_2^2$, $\boldsymbol{\mu}_2^3$, $\boldsymbol{\mu}_2^4$ or $\boldsymbol{\mu}_5$. Since

$$r|\Gamma| = |\overline{G}_{80}| = 80$$

and $15 > dr$, we see that either $r = 5$ or 10.

Suppose that $r = 5$. Then $\Gamma \cong \boldsymbol{\mu}_2^4$ and $\Gamma/\Gamma_{C_1} \supset \boldsymbol{\mu}_2^3$. Hence, the curve C_1 is not rational, because the group $\boldsymbol{\mu}_2^3$ does not act faithfully on $\mathbb{P}^1$. Since $d \leqslant 3$, the curve C_1 is a smooth plane cubic curve. Hence there is a $\overline{G}_{80}$-invariant set of 5 planes in $\mathbb{P}^3$, which implies that there exists a $\overline{G}_{80}$-orbit in $\mathbb{P}^3$ of length 5. The latter is impossible by Corollary 5.7.

Thus, we see that $r = 10$, which implies that $d = 1$, so that C_1 is a line. In this case $\Gamma \cong \boldsymbol{\mu}_2^3$ and $\Gamma_{C_1} \cong \boldsymbol{\mu}_2$, because $\boldsymbol{\mu}_2^3$ cannot act faithfully on $\mathbb{P}^1$. Then C_1 is one of two lines that is pointwise fixed by Γ_{C_1}, cf. Lemma 3.2(i). Now it follows from Lemma 5.6 that C_1 is an irreducible component of one of the curves $\mathcal{L}_{10}$, $\mathcal{L}'_{10}$, $\mathcal{L}''_{10}$. □

Corollary 5.9 *There are no $\overline{G}_{80}$-invariant curves in $\mathbb{P}^3$ of degree less than 5.*

Proof Let C be a $\overline{G}_{80}$-invariant curve in $\mathbb{P}^3$ of degree less 5. Then C is irreducible by Lemma 5.8. Thus the genus of its normalization is at most 3, which is impossible by Lemma 5.3. □

Using Lemma 3.14, we obtain the following two technical results.

Lemma 5.10 *Let $\mathcal{M}$ (respectively, $\mathcal{M}'$, $\mathcal{M}''$) be the linear system of surfaces in $\mathbb{P}^3$ of degree 6 (respectively, 4, 4) that contains $\mathcal{L}_{10}$ (respectively, $\mathcal{L}'_{10}$, $\mathcal{L}''_{10}$). Then the base locus of $\mathcal{M}$ (respectively, $\mathcal{M}'$, $\mathcal{M}''$) does not contain curves except for $\mathcal{L}_{10}$ (respectively, $\mathcal{L}'_{10}$, $\mathcal{L}''_{10}$).*

Proof Recall from Sect. 3, that $\mathbb{P}^3$ contains ten $\overline{\mathrm{H}}$-invariant quadrics $\mathcal{Q}_i$, where $1 \leqslant i \leqslant 10$, which are defined by Eqs. (3.8) and (3.9). For every $i < j < k$, define the reducible sextic surface

$$T_{i,j,k} = \mathcal{Q}_i + \mathcal{Q}_j + \mathcal{Q}_k.$$

By Lemma 3.14, the curve $\mathcal{L}_{10}$ is contained in $T_{1,2,4}$, $T_{3,5,6}$, $T_{2,4,5}$, $T_{2,3,5}$, $T_{7,8,9}$, $T_{1,6,10}$ and $T_{6,9,10}$, so that these surfaces are contained in the linear system $\mathcal{M}$. Using Lemma 3.14, one can check that

$$T_{1,2,4} \cap T_{3,5,6} = \mathcal{L}_{10} \cup \ell'_1 \cup \check{\ell}'_1 \cup \ell'_3 \cup \check{\ell}'_3 \cup \ell''_2 \cup \check{\ell}''_2 \cup \ell''_3 \cup \check{\ell}''_3 \cup \ell''_4 \cup \check{\ell}''_4 \cup \ell''_5 \cup \check{\ell}''_5.$$

On the other hand, the sextic surface $T_{2,4,5}$ does not contain the lines ℓ'_1 and $\check{\ell}'_1$, and the sextic surface $T_{2,3,5}$ does not contain the lines ℓ'_3 and $\check{\ell}'_3$. This also follows from Lemma 3.14. Similarly, the sextic surface $T_{7,8,9}$ does not contain the lines ℓ''_4, $\check{\ell}''_4$, ℓ''_5 and $\check{\ell}''_5$, the sextic surface $T_{1,6,10}$ does not contain the lines ℓ''_3 and $\check{\ell}''_3$, and the sextic surface $T_{6,9,10}$ does not contain the lines ℓ''_2 and $\check{\ell}''_2$. Thus, the intersection of the above seven sextic surfaces does not contain other curves except for $\mathcal{L}_{10}$, so that the same holds for the base locus of $\mathcal{M}$.

For every $i < j$, define the reducible quartic surface

$$R_{i,j} = \mathcal{Q}_i + \mathcal{Q}_j.$$

By Lemma 3.14, the curve $\mathcal{L}'_{10}$ is contained in $R_{6,7}$, $R_{8,10}$ and $R_{9,10}$, so that these surfaces are contained in the linear system $\mathcal{M}'$. Using Lemma 3.14 one more time, we see that

$$R_{6,7} \cap R_{8,10} = \mathcal{L}'_{10} \cup \ell_5 \cup \check{\ell}_5.$$

However, the surface $R_{9,10}$ does not contain the lines ℓ_5 and $\check{\ell}_5$. This shows that the base locus of $\mathcal{M}'$ does not contain curves except for $\mathcal{L}'_{10}$.

Similarly, the quartics $R_{1,2}$, $R_{3,4}$ and $R_{4,5}$ contain the curve $\mathcal{L}''_{10}$. Using Lemma 3.14, one can check that

$$R_{1,2} \cap R_{3,4} = \mathcal{L}''_{10} \cup \ell_4 \cup \check{\ell}_4.$$

However, the surface $R_{4,5}$ does not contain the lines ℓ_4 and $\check{\ell}_5$, which implies that the base locus of $\mathcal{M}''$ does not contain curves except for $\mathcal{L}''_{10}$. □

Lemma 5.11 *Let $\mathcal{D}$ be a (non-empty) mobile $\overline{G}_{80}$-invariant linear system on $\mathbb{P}^3$, and let n be a positive integer such that $\mathcal{D} \sim \mathcal{O}_{\mathbb{P}^3}(n)$. Then*

$$\max\left\{\mathrm{mult}_{\mathcal{L}_{10}}(\mathcal{D}), \mathrm{mult}_{\mathcal{L}'_{10}}(\mathcal{D}), \mathrm{mult}_{\mathcal{L}''_{10}}(\mathcal{D})\right\} \leqslant \frac{n}{4}.$$

Proof Let us use the notation of the proof of Lemma 5.6. Let D be a general surface in $\mathcal{D}$. Since the lines ℓ_1, ℓ_5, $\check{\ell}_1$ and $\check{\ell}_5$ are contained in the quadric surface $\mathcal{Q}_1$ by Lemma 3.14, we have

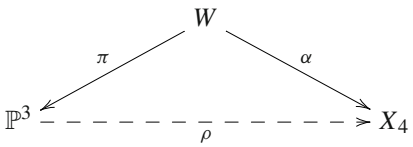
$$D|_{\mathcal{Q}_1} = m\big(\ell_1 + \ell_5 + \check{\ell}_1 + \check{\ell}_5\big) + \Delta,$$

where m is a non-negative integer such that $m \geqslant \mathrm{mult}_{\mathcal{L}_{10}}(\mathcal{D})$, and Δ is an effective divisor on $\mathcal{Q}_1$ whose support does not contain the lines ℓ_1, ℓ_5, $\check{\ell}_1$ and $\check{\ell}_5$. By Lemma 5.6, the lines ℓ_1, ℓ_5, $\check{\ell}_1$ and $\check{\ell}_5$ are disjoint. Thus, we may assume that these lines are divisors of bi-degree $(1, 0)$ on $\mathcal{Q}_1 \cong \mathbb{P}^1 \times \mathbb{P}^1$. Let ℓ be a general line in $\mathcal{Q}_1$ that is a divisor of bi-degree $(0, 1)$. Then

$$n = D \cdot \ell = m\big(\ell_1 + \ell_5 + \check{\ell}_1 + \check{\ell}_5\big) \cdot \ell + \Delta \cdot \ell \geqslant m\big(\ell_1 + \ell_5 + \check{\ell}_1 + \check{\ell}_5\big) \cdot \ell = 4m,$$

so that $\mathrm{mult}_{\mathcal{L}_{10}}(\mathcal{D}) \leqslant m \leqslant \frac{n}{4}$ as required. Similarly, it follows from Lemma 3.14 that the quadric $\mathcal{Q}_1$ also contains the lines ℓ''_1, ℓ''_2, ℓ''_4, $\check{\ell}''_1$, $\check{\ell}''_2$ and $\check{\ell}''_4$. Moreover, it follows from the proof of Lemma 5.6 that the four lines ℓ''_1, ℓ''_2, $\check{\ell}''_1$ and $\check{\ell}''_2$ are disjoint. Arguing as in the previous case, we see that $\mathrm{mult}_{\mathcal{L}''_{10}}(\mathcal{D}) \leqslant \frac{n}{4}$. Finally, observe that it also follows from Lemma 3.14 that the quadric $\mathcal{Q}_6$ contains the lines ℓ'_1, ℓ'_2, ℓ'_3, $\check{\ell}'_1$, $\check{\ell}'_2$ and $\check{\ell}'_3$. Furthermore, in the proof of Lemma 5.6 we established that the four lines ℓ'_1, ℓ'_3, $\check{\ell}'_1$ and $\check{\ell}'_3$ are disjoint, which implies that $\mathrm{mult}_{\mathcal{L}'_{10}}(\mathcal{D}) \leqslant \frac{n}{4}$. □

Remark 5.12 As we mentioned in the proof of Lemma 5.6, the lines ℓ'_1, ℓ'_2, ℓ'_3, ℓ'_4 and ℓ'_5 form a "wheel" in $\mathbb{P}^3$, the lines $\check{\ell}'_1$, $\check{\ell}'_2$, $\check{\ell}'_3$, $\check{\ell}'_4$ and $\check{\ell}'_5$ also form a "wheel", and these two wheels intersect by ten points in Σ'_{20}. Using Lemma 5.10, one can check that there exists a $\overline{G}_{80}$-commutative diagram



Here π is the blow up of the curve $\mathcal{L}'_{10}$, the morphism α is given by the linear system $|-K_W|$, the map ρ is given by be the linear system of quartic surfaces in $\mathbb{P}^3$ that contain $\mathcal{L}'_{10}$, and X_4 is a quartic threefold in $\mathbb{P}^4$. This configuration of ten lines in $\mathbb{P}^3$ has been studied by Todd [43] under generality assumption (and without group action). He constructed the same commutative diagram with α being a small birational morphism. However, the position of our 10 lines is not general from this point of view. Namely, it follows from the proof of Lemma 5.11 that the map α contracts the proper transforms of the quadrics $\mathcal{Q}_6$, $\mathcal{Q}_7$, $\mathcal{Q}_8$, $\mathcal{Q}_9$ and $\mathcal{Q}_{10}$ (so that our $\overline{G}_{80}$-commutative diagram can be called a *bad* $\overline{G}_{80}$-*Sarkisov link* similarly to the terminology of [14]). One can consider this as a geometrical meaning of Lemma 5.11. Similarly, the lines of $\mathcal{L}''_{10}$ form the same configuration in $\mathbb{P}^3$, which results in a similar $\overline{G}_{80}$-commutative diagram; the

only difference in this case is that α contracts the proper transforms of the quadrics $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$, $\mathcal{Q}_4$ and $\mathcal{Q}_5$.

Denote by S_0, S_1, S_2, S_3 and S_4 the quartic surfaces in $\mathbb{P}^3$ that are given by the equations $q_0 = 0$, $q_1 = 0$, $q_2 = 0$, $q_3 = 0$ and $q_4 = 0$, respectively. Here q_0, q_1, q_2, q_3 and q_4 are quartic polynomials in (3.21). Then S_0, S_1, S_2, S_3 and S_4 are the only $\overline{G}_{80}$-invariant quartic surfaces in $\mathbb{P}^3$ by Lemma 3.10.

Lemma 5.13 *The polynomials q_0, q_1, q_2, q_3 and q_4 are irreducible. Moreover, the surface S_0 is smooth, and the surfaces S_1, S_2, S_3 and S_4 have isolated ordinary double points. Furthermore, we have*

$$\mathrm{Sing}(S_1) = \Sigma_{16}^1, \quad \mathrm{Sing}(S_2) = \Sigma_{16}^2, \quad \mathrm{Sing}(S_3) = \Sigma_{16}^3, \quad \mathrm{Sing}(S_4) = \Sigma_{16}^4.$$

Proof We can deduce all required assertions using the explicit formulas of the polynomials q_0, q_1, q_2, q_3 and q_4. However, most of these assertions follow easily from general facts. Namely, the irreducibility of the polynomials q_0, q_1, q_2, q_3 and q_4 follows from Lemma 3.10. Now taking the general plane section of the surface S_i, we see that either it has isolated singularities, or its singular locus contains a $\overline{G}_{80}$-curve of degree less than 4. The latter is impossible by Corollary 5.9. Thus, the surfaces S_0, S_1, S_2, S_3 and S_4 have isolated singularities.

By [45, Theorem 1], the surface S_i cannot have more that two non-Du Val singular points. Thus, the surface S_i has at most Du Val singularities by Corollary 5.7. Then its minimal resolution of singularities is a smooth $K3$ surface. Now using Corollary 5.7 and the fact that the rank of the Picard group of a smooth $K3$ surface is at most 20, we see that either S_i is smooth, or S_i has isolated ordinary double points, and its singular locus is one of the $\overline{G}_{80}$-orbits Σ_{16}^1, Σ_{16}^2, Σ_{16}^3 or Σ_{16}^4. Taking partial derivatives of the polynomial q_0 at the points of the set $\Sigma_{16}^1 \cup \Sigma_{16}^2 \cup \Sigma_{16}^3 \cup \Sigma_{16}^4$, we see that S_0 is smooth. Similarly, we see that $\mathrm{Sing}(S_i) = \Sigma_{16}^i$ for $1 \leqslant i \leqslant 4$. □

Corollary 5.14 *Let Σ be one of the $\overline{G}_{80}$-orbits Σ_{16}^1, Σ_{16}^2, Σ_{16}^3, Σ_{16}^4, and let $\mathcal{M}$ be the linear system that consists of cubic surfaces in $\mathbb{P}^3$ containing Σ. Then Σ is the base locus of the linear system $\mathcal{M}$.*

Proof See the proof of Corollary 4.13. □

The $\overline{G}_{80}$-orbits Σ_{20}, Σ_{20}' and Σ_{20}'' are not contained in any of the quartic surfaces S_0, S_1, S_2, S_3 and S_4. By Lemma 5.6 and Corollary 5.7, this shows that each curve among $\mathcal{L}_{10}$, $\mathcal{L}_{10}'$, $\mathcal{L}_{10}''$ intersects each surface among S_0, S_1, S_2, S_3 and S_4 transversally in a $\overline{G}_{80}$-orbit of length 40. Furthermore, by Lemma 5.6, the curves $\mathcal{L}_{10}$, $\mathcal{L}_{10}'$ and $\mathcal{L}_{10}''$ contain all $\overline{G}_{80}$-orbits of length 40, and they are disjoint away from the $\overline{G}_{80}$-orbits of length 20. Thus, we conclude that the surfaces S_0, S_1, S_2, S_3 and S_4 contain exactly three $\overline{G}_{80}$-orbits of length 40. Observe also that S_0 contains all $\overline{G}_{80}$-orbits Σ_{16}^1, Σ_{16}^2, Σ_{16}^3, Σ_{16}^4. This gives the following

Corollary 5.15 *The group $\mathrm{Pic}(S_0)^{\overline{G}_{80}}$ is generated by the plane section of S_0.*

Table 2 Incidence between S_i and Σ_{16}^j

	S_1	S_2	S_3	S_4
Σ_{16}^1	$\mathrm{Sing}(S_1)$	$+$	$-$	$+$
Σ_{16}^2	$-$	$\mathrm{Sing}(S_2)$	$+$	$+$
Σ_{16}^3	$+$	$+$	$\mathrm{Sing}(S_3)$	$-$
Σ_{16}^4	$+$	$-$	$+$	$\mathrm{Sing}(S_4)$

Proof By [33, Lemma 2.1], the action of the group $\overline{G}_{80}$ on the $K3$ surface S_0 is symplectic. Thus, the surface $S_0/\overline{G}_{80}$ is a singular $K3$ surface. By construction, it has four Du Val singular points of type $\mathbb{A}_4$ and three Du Val singular points of type $\mathbb{A}_1$. The minimal resolution of singularities of this surface is a smooth $K3$ surface. Since the rank of the Picard group of a smooth $K3$ surfaces cannot exceed 20, we see that the rank of the Picard group of the surface $S_0/\overline{G}_{80}$ must be 1. This shows that the rank of the group $\mathrm{Pic}(S_0)^{\overline{G}_{80}}$ is also 1. Since $\mathrm{Pic}(S_0)$ has no torsion and the intersection form of curves on the surface S_0 is even, we see that the group $\mathrm{Pic}(S_0)^{\overline{G}_{80}}$ is generated by the plane section of S_0. □

Table 2 shows which $\overline{G}_{80}$-orbit Σ_{16}^1, Σ_{16}^2, Σ_{16}^3 and Σ_{16}^4 is contained in which surface S_1, S_2, S_3 and S_4.

In particular, we see that $\Sigma_{16}^j \subset S_i$ if and only if $j \neq 2i \bmod 5$.

Now we are ready to prove.

Theorem 5.16 *Let C be a $\overline{G}_{80}$-irreducible curve in $\mathbb{P}^3$ such that $\deg(C) \leqslant 15$. Then $C \not\subset S_0$ and one of the following cases holds:*

- $\deg(C) = 10$, *and C is one of the curves $\mathcal{L}_{10}$, $\mathcal{L}'_{10}$, $\mathcal{L}''_{10}$,*
- $\deg(C) = 12$, *and C is an irreducible smooth curve of genus 5 that is contained in exactly one surface among S_1, S_2, S_3, S_4,*
- $\deg(C) = 8$, *and C is an irreducible smooth curve of genus 5 such that either $C = S_1 \cap S_4$ and C contains $\mathrm{Sing}(S_1) \cup \mathrm{Sing}(S_1)$, or $C = S_2 \cap S_3$ and C contains $\mathrm{Sing}(S_2) \cup \mathrm{Sing}(S_3)$.*

Proof Recall from Corollary 5.15 that the group $\mathrm{Pic}(S_0)^{\overline{G}_{80}}$ is generated by the plane section H_{S_0} of the surface S_0. Since there is an exact sequence of $\overline{G}_{80}$-representations

$$0 \longrightarrow H^0\Big(\mathcal{O}_{\mathbb{P}^3}(n-4)\Big) \longrightarrow H^0\Big(\mathcal{O}_{\mathbb{P}^3}(n)\Big) \longrightarrow H^0\Big(\mathcal{O}_{S_0}(nH_{S_0})\Big) \longrightarrow 0$$

for every $n \geqslant 1$, we see that S_0 does not contain C by Lemma 3.10.

By Lemma 5.8, we may assume that C is irreducible. Let $d = \deg(C)$. Then

$$4d = S_0 \cdot C = 16a + 40b$$

for some non-negative integers a and b, because Σ_{20}, Σ'_{20} and Σ''_{20} are not contained in S_0. Keeping in mind that $d \geqslant 5$ by Corollary 5.9, we have the following possibilities:

d	a	b
8	2	0
10	0	1
12	3	0
14	1	1

Suppose that $d = 14$. Since $a = 1$ and S_0 contains all $\overline{G}_{80}$-orbits of length 16 in $\mathbb{P}^3$, we see that the curve C contains a unique $\overline{G}_{80}$-orbit of length 16. Without loss of generality, we may assume that C contains Σ^1_{16}. Then $C \not\subset S_3$, because S_3 does not contain Σ^1_{16}, see Table 2. By Corollary 5.7, this implies that

$$56 = 4d = S_3 \cdot C$$

is divisible by 40, which is absurd. This shows that $d \neq 14$, so that $d \leqslant 12$.

Pick four general points O_1, O_2, O_3 and O_4 in the curve C. Since C is irreducible, their stabilizers in $\overline{\mathrm{H}}$ are trivial. In particular, the $\overline{\mathrm{H}}$-orbits of these points consist of 16 different points. On the other hand, the vector space generated by the quartic polynomials

$$q_0(x, y, z, w),\ q_1(x, y, z, w),\ q_2(x, y, z, w),\ q_3(x, y, z, w),\ q_4(x, y, z, w),$$

contains all H-invariant polynomials of degree 4. There exists $[\lambda_0 : \lambda_1 : \lambda_2 : \lambda_3 : \lambda_4] \in \mathbb{P}^4$ such that the polynomial

$$\begin{aligned}\lambda_0 q_0(x, y, z, w) + \lambda_1 q_1(x, y, z, w) + \lambda_2 q_2(x, y, z, w)\\ +\lambda_3 q_3(x, y, z, w) + \lambda_4 q_4(x, y, z, w)\end{aligned}$$

vanishes at the four points O_1, O_2, O_3 and O_4. The latter polynomial defines a (possibly reducible or non-reduced) $\overline{\mathrm{H}}$-invariant quartic surface S in $\mathbb{P}^3$. Then S contains O_1, O_2, O_3 and O_4 together with their $\overline{\mathrm{H}}$-orbits. This implies that

$$4d = S \cdot C \geqslant 64$$

provided that the curve C is not contained in S. Since $d \leqslant 12$, we see that $C \subset S$. Thus, the vector space of all H-invariant quartic polynomials in $\mathbb{C}[x, y, z, w]$ that vanish along C is at least one-dimensional. Since H is a normal subgroup in G_{80}, the group

$$G_{80}/\mathrm{H} \cong \boldsymbol{\mu}_5$$

acts on this space. Keeping in mind that $\boldsymbol{\mu}_5$ is abelian, we see that there exists at least one G_{80}-invariant quartic polynomial in $\mathbb{C}[x, y, z, w]$ that vanishes along the curve C. Moreover, if $d \leqslant 10$, then the rank-nullity theorem implies that the vector space consisting of all H-invariant quartic polynomials in $\mathbb{C}[x, y, z, w]$ that vanish

along C is at least two-dimensional, because we can repeat the same arguments with three general points in C instead of four. Therefore, if $d \leqslant 10$, then there exists at least two linearly independent G_{80}-invariant quartic polynomials in $\mathbb{C}[x, y, z, w]$ that vanish at the curve C. Now using Lemma 5.13, we see that C is contained in one of the surfaces S_1, S_2, S_3 or S_4. Moreover, if $d \leqslant 10$, then C is contained in at least two surfaces among S_1, S_2, S_3, S_4.

Let $t \in \{1, 2, 3, 4\}$ be such that C is contained in S_t. Denote by H the plane section of the surface S_t. If $d \neq 12$, then there is $k \in \{1, 2, 3, 4\}$ such that C is contained in the surfaces S_k and $k \neq t$. In this case, we have

$$S_k|_{S_t} = C + Z,$$

where Z is a $\overline{G}_{80}$-invariant effective divisor such that $H \cdot Z = 16 - d \leqslant 8$. On the other hand, we already proved earlier that $\mathbb{P}^3$ does not contain $\overline{G}_{80}$-invariant curves of degree less than 8. This shows that $d \neq 10$, so that either $d = 8$ or $d = 12$.

Recall that Σ_{16}^t is the singular locus of the surface S_t. Let $f\colon \widetilde{S}_t \to S_t$ be the minimal resolution of singularities of the surface S_t. Then the action of the group $\overline{G}_{80}$ lifts to the surface $\widetilde{S}_t$, because f is the minimal resolution of singularities. Denote by $E_1, \ldots, E_{16}$ the f-exceptional curves, and denote by $\widetilde{C}$ the proper transform of the curve C on the surface $\widetilde{S}_t$. Let $E = E_1 + \cdots + E_{16}$ and $\widetilde{H} = f^*(H)$. Then

$$\widetilde{C} \sim_{\mathbb{Q}} f^*(C) - \frac{m}{2}E$$

for some non-negative integer number m. By Hodge index theorem, one has

$$0 \geqslant \begin{vmatrix} \widetilde{H}^2 & \widetilde{H} \cdot (\widetilde{C} + \frac{m}{2}E) \\ \widetilde{H} \cdot (\widetilde{C} + \frac{m}{2}E) & (\widetilde{C} + \frac{m}{2}E)^2 \end{vmatrix} = \begin{vmatrix} 4 & d \\ d & (\widetilde{C} + \frac{m}{2}E)^2 \end{vmatrix} = 4(\widetilde{C}^2 + 8m^2) - d^2.$$

This implies

$$\widetilde{C}^2 \leqslant \frac{d^2}{4} - 8m^2.$$

Let g be the genus of the normalization of the curve $\widetilde{C}$, and let $p_a(\widetilde{C})$ be its arithmetic genus. Then

$$\begin{aligned} g &\leqslant p_a(\widetilde{C}) - \big|\mathrm{Sing}(\widetilde{C})\big| = \frac{\widetilde{C}^2}{2} + 1 - \big|\mathrm{Sing}(\widetilde{C})\big| \\ &\leqslant \frac{d^2}{8} - 4m^2 + 1 - \big|\mathrm{Sing}(\widetilde{C})\big| \\ &\leqslant \frac{144}{8} - 4m^2 + 1 - \big|\mathrm{Sing}(\widetilde{C})\big| = 19 - 4m^2 - \big|\mathrm{Sing}(\widetilde{C})\big|. \end{aligned}$$

On the other hand, it follows from Lemma 5.3 that $g \geqslant 5$. Thus, as $\mathbb{P}^3$ does not have $\overline{G}_{80}$-orbits of length less than 16 by Corollary 5.7, we see that the curve $\widetilde{C}$ is smooth,

one has

$$g = p_a(\widetilde{C}) \leqslant 19 - 4m^2,$$

and $m \leqslant 1$. In particular, since $m \leqslant 1$, we see that the curve C is also smooth. Moreover, since $g \leqslant 19$, we have $g \in \{5, 13, 17\}$ by Lemma 5.3.

Suppose that $g = 17$. Then C contains three $\overline{G}_{80}$-orbits of length 16 by Lemma 5.3. Thus, the curve C contains every $\overline{G}_{80}$-orbit Σ_{16}^i except for Σ_{16}^j such that $j = 2t \bmod 5$, because these are the only $\overline{G}_{80}$-orbits of length 16 contained in the surface S_t. Since C contains $\Sigma_{16}^t = \mathrm{Sing}(S_t)$, we have $m \geqslant 1$, which implies that $m = 1$, because we proved already that $m \leqslant 1$. Since we also proved that $g \leqslant 19 - 4m^2$, we have

$$17 = g \leqslant 19 - 4m^2 = 15,$$

which is absurd. This shows that $g \neq 17$.

Suppose that $g = 13$. Then C contains one $\overline{G}_{80}$-orbit of length 16 by Lemma 5.3. Recall that there exists $r \in \{1, 2, 3, 4\}$ such that the surface S_r that does not contain this $\overline{G}_{80}$-orbit. Then $C \not\subset S_r$, so that $S_r \cdot C$ must be a multiple of 40 by Lemma 5.3. Since

$$S_r \cdot C = 4d,$$

we see that d is divisible by 10. This is impossible, because we already proved that either $d = 8$ or $d = 12$. This shows that $g \neq 13$.

We have seen that C is a smooth curve of degree $d \in \{8, 12\}$ and genus 5 that is contained in S_t such that $t \neq 0$. In particular, we have seen that there are no $\overline{G}_{80}$-invariant curves in $\mathbb{P}^3$ of degree less than 8. Thus, if $d = 12$, then S_t is the only surface among S_1, S_2, S_3 and S_4 that contains C. Therefore, to complete the proof of the theorem, we may assume that $d = 8$. We have to show that either $C = S_1 \cap S_4$ and C contains $\mathrm{Sing}(S_1) \cup \mathrm{Sing}(S_1)$, or $C = S_2 \cap S_3$ and C contains $\mathrm{Sing}(S_2) \cup \mathrm{Sing}(S_3)$.

Recall that C is also contained in a quartic surface S_k such that $k \neq t$. Then

$$4H \sim S_k|_{S_t} = C + Z, \tag{5.17}$$

where Z is a $\overline{G}_{80}$-invariant effective divisor such that $H \cdot Z = 8$. Since we already proved that $\mathbb{P}^3$ does not contain $\overline{G}_{80}$-invariant curves of degree less than 8, we see that Z is an irreducible curve of degree 8. Moreover, we already proved that every $\overline{G}_{80}$-invariant curve in $\mathbb{P}^3$ of degree 8 is a smooth curve of genus 5. In particular, we see that Z is a smooth curve of genus 5. Note that, a priori, we may have the case when $Z = C$ (and it will turn out that this is exactly the case).

Suppose that both curves C and Z do not contain the singular locus of the surface S_t. Then $C^2 = Z^2 = 8$ by the adjunction formula, which gives

$$32 = 4H \cdot C = C \cdot (C + Z) = C^2 + C \cdot Z = 8 + C \cdot Z,$$

which implies that $C \cdot Z = 24$. This is impossible. Indeed, if we have $C = Z$, then $C \cdot Z = C^2 = 8$. Similarly, if $C \neq Z$, then

$$C \cdot Z = 16\alpha + 40\beta$$

for some non-negative integers α and β, because C only contains $\overline{G}_{80}$-orbits of length 16, 40 and 80 by Lemma 5.3. Thus, we see that $C \cdot Z \neq 24$. This shows that either C or Z contains $\Sigma_{16}^t = \mathrm{Sing}(S_t)$.

If $\Sigma_{16}^t \subset C$, then $\Sigma_{16}^t \subset Z$ by (5.17), because both C and Z are smooth. Similarly, if $\Sigma_{16}^t \subset Z$, then $\Sigma_{16}^t \subset C$. Thus, we see that $\Sigma_{16}^t \subset C \cap Z$. Applying the same arguments to the surface S_k, we see that

$$\mathrm{Sing}(S_k) = \Sigma_{16}^k \subset C \cap Z,$$

which implies that $k + t = 5$ (see Table 2).

Denote by $\widetilde{Z}$ the proper transform of the curve Z on the surface $\widetilde{S}_t$. Then

$$\widetilde{C} + \widetilde{Z} \sim f^*(4H) - E.$$

This follows from the fact that $C + Z$ is a Cartier divisor at every point of Σ_{16}^t, because both C and Z are smooth and $\Sigma_{16}^t \subset C \cap Z$. As above, the adjunction formula gives $\widetilde{C}^2 = \widetilde{Z}^2 = 8$. Then

$$8 + \widetilde{C} \cdot \widetilde{Z} = \widetilde{C}^2 + \widetilde{C} \cdot \widetilde{Z} = \widetilde{C} \cdot \big(\widetilde{C} + \widetilde{Z}\big) = \widetilde{C} \cdot \big(f^*(4H) - E\big) = 32 - \widetilde{C} \cdot E = 16,$$

so that $\widetilde{C} \cdot \widetilde{Z} = 8$. Keeping in mind that $\widetilde{C}$ only contains $\overline{G}_{80}$-orbits of length 16, 40, 80, we see that $\widetilde{C} = \widetilde{Z}$. Then $C = Z$, so that $S_k \cap S_t = C$ and $k + t = 5$ as required. This completes the proof of Theorem 5.16. □

It should be pointed out that Theorem 5.16 does not assert that the projective space $\mathbb{P}^3$ contains $\overline{G}_{80}$-irreducible curves of degree 8 and 12.

Remark 5.18 The $\overline{G}_{80}$-invariant curves of degree 8 in $\mathbb{P}^3$ do exist. In fact, there are exactly two of them. Indeed, one can show that the ideal in $\mathbb{C}[x, y, z, w]$ generated by the polynomials

$$q_1(x, y, z, w), \quad q_4(x, y, z, w), \quad \frac{\partial q_1}{\partial x}\frac{\partial q_4}{\partial y} - \frac{\partial q_4}{\partial x}\frac{\partial q_1}{\partial y}, \frac{\partial q_1}{\partial x}\frac{\partial q_4}{\partial z} - \frac{\partial q_4}{\partial x}\frac{\partial q_1}{\partial z},$$
$$\frac{\partial q_1}{\partial x}\frac{\partial q_4}{\partial w} - \frac{\partial q_4}{\partial x}\frac{\partial q_1}{\partial w}, \quad \frac{\partial q_1}{\partial y}\frac{\partial q_4}{\partial z} - \frac{\partial q_4}{\partial y}\frac{\partial q_1}{\partial z}, \frac{\partial q_1}{\partial y}\frac{\partial q_4}{\partial w} - \frac{\partial q_4}{\partial y}\frac{\partial q_1}{\partial w}, \frac{\partial q_1}{\partial z}\frac{\partial q_4}{\partial w} - \frac{\partial q_4}{\partial z}\frac{\partial q_1}{\partial w}$$

defines a one-dimensional subscheme in $\mathbb{P}^3$. This shows that the scheme-theoretic intersection $S_1 \cdot S_4$ is not reduced. By Theorem 5.16, this implies that the set-theoretic intersection $S_1 \cap S_4$ is a $\overline{G}_{80}$-invariant irreducible smooth curve of genus 5. Similarly,

we see that $S_2 \cap S_3$ is another $\overline{G}_{80}$-invariant irreducible smooth curve of genus 5. Note that the curves $S_1 \cap S_4$ and $S_2 \cap S_3$ are projectively equivalent, since

$$\overline{R}(S_1) = S_2, \quad \overline{R}(S_2) = S_4, \quad \overline{R}(S_4) = S_3, \quad \overline{R}(S_3) = S_1.$$

This also implies that both of these curves are $\overline{G}_{160}$-invariant, and they are swapped by the group $\overline{G}_{320}$. In particular, there exists a smooth irreducible curve of genus 5 with an action of the group $\overline{G}_{160} \cong \boldsymbol{\mu}_2^4 \rtimes \mathrm{D}_{10}$. This curve is well-known: it is a complete intersection in $\mathbb{P}^4$ that is given by

$$\begin{cases} x_0^2 + x_1^2 + x_3^2 + x_4^2 + x_5^2 = 0, \\ x_0^2 + \xi_5 x_1^2 + \xi_5^2 x_3^2 + \xi_5^3 x_4^2 + \xi_5^4 x_5^2 = 0, \\ \xi_5^4 x_0^2 + \xi_5^3 x_1^2 + \xi_5^2 x_3^2 + \xi_5 x_4^2 + x_5^2 = 0, \end{cases} \tag{5.19}$$

where ξ_5 is a primitive fifth root of unity. This was proved by Joseph McKelvey in [31], who also proved that $\boldsymbol{\mu}_2^4 \rtimes \mathrm{D}_{10}$ is the full automorphism group of the curve (5.19), see also [22,29]. The curve given by (5.19) is a special case of the so-called Humbert curves, which have been discovered by Humbert in [23], and have been studied by Edge in [20].

Remark 5.20 Let C be a $\overline{G}_{80}$-invariant curve of degree 8 in $\mathbb{P}^3$. Then C is a smooth curve of genus 5 by Theorem 5.16. Thus, it contains unique $\overline{G}_{80}$-orbit of length 40 by Lemma 5.3. We claim that this orbit is the intersection $\mathcal{L}_{10} \cap C$. To show this, let us describe the intersections $\ell_1 \cap S_1$, $\ell_1 \cap S_2$, $\ell_1 \cap S_3$ and $\ell_1 \cap S_4$, where ℓ_1 is the irreducible component of the curve $\mathcal{L}_{10}$ defined in (3.11). Note that ℓ_1 is the line in $\mathbb{P}^3$ that passes through the points $[0 : i : 1 : 0]$ and $[1 : 0 : 0 : -i]$, so that is parameterized by

$$\big[\mu : \lambda i : \lambda : -\mu i\big],$$

where $[\lambda : \mu] \in \mathbb{P}^1$. To find the intersections $\ell_1 \cap S_1$, $\ell_1 \cap S_2$, $\ell_1 \cap S_3$ and $\ell_1 \cap S_4$, we may assume that $\lambda = 1$, because $[0 : i : 1 : 0]$ is not contained in $S_1 \cup S_2 \cup S_3 \cup S_4$. Substituting the point $[\mu : i : 1 : -\mu i]$ into the polynomials q_1, q_2, q_3, q_4 defined in (3.21), we see that both intersections $\ell_1 \cap S_1$ and $\ell_1 \cap S_4$ are given by

$$\mu^4 - (4\xi_5^3 + 4\xi_5^2 + 2)\mu^2 + 1 = 0,$$

and both intersections $\ell_1 \cap S_2$ and $\ell_1 \cap S_3$ are given by

$$\mu^4 + (4\xi_5^3 + 4\xi_5^2 + 2)\mu^2 + 1 = 0.$$

Thus, we see that $\ell_1 \cap S_1 = \ell_1 \cap S_4$ and $\ell_1 \cap S_2 = \ell_1 \cap S_3$. Since C is one of the curves $S_1 \cap S_4$ or $S_2 \cap S_3$ by Theorem 5.16, the intersection $\mathcal{L}_{10} \cap C$ is the unique $\overline{G}_{80}$-orbit in C of length 40.

We will show later that $\mathbb{P}^3$ contains $\overline{G}_{80}$-invariant curves of degree 12 (see Lemma 5.26).

Remark 5.21 Let C be a $\overline{G}_{80}$-invariant curve in $\mathbb{P}^3$ of degree 12. By Theorem 5.16, the curve C is smooth, its genus is 5, and $C \subset S_t$, where S_t is one of the surfaces S_1, S_2, S_3 and S_4. By Remark 5.18, the surface S_t contains an irreducible $\overline{G}_{80}$-invariant curve Z such that $2Z = S_k|_{S_t}$, where $k + t = 5$. If C does not contain $\mathrm{Sing}(S_t)$, then $Z \cdot C = 24$, which is impossible, because the only $\overline{G}_{80}$-orbits contained in C are of lengths 16, 40 and 80 by Lemma 5.3. Thus, we see that C contains the singular locus of the surface S_t.

The first consequence of Theorem 5.16 is

Corollary 5.22 *Let C be an irreducible $\overline{G}_{80}$-invariant curve in $\mathbb{P}^3$ of degree 12, let $\mathcal{D}$ be a mobile $\overline{G}_{80}$-invariant linear system on $\mathbb{P}^3$, let n be a positive integer such that $\mathcal{D} \sim \mathcal{O}_{\mathbb{P}^3}(n)$. Then $\mathrm{mult}_C(\mathcal{D}) \leqslant \frac{n}{4}$.*

Proof By Theorem 5.16, the curve C is a smooth curve of genus 5 that is contained in a surface among S_1, S_2, S_3, S_4. Denote this surface by S, and denote by H its general plane section. Let D be a general surface in $\mathcal{D}$. Then

$$D\big|_S = mC + \Delta \sim nH,$$

where m is a non-negative integer such that $m \geqslant \mathrm{mult}_C(\mathcal{D})$, and Δ is an effective divisor on S such that $C \not\subset \mathrm{Supp}(\Delta)$.

Recall from Lemma 5.13, that $\mathrm{Sing}(S)$ is one of the $\overline{G}_{80}$-orbits Σ_{16}^1, Σ_{16}^2, Σ_{16}^3, Σ_{16}^4, and S has isolated ordinary double points. Let $f : \widetilde{S} \to S$ be the minimal resolution of singularities of the surface S. Then the action of the group $\overline{G}_{80}$ lifts to the surface $\widetilde{S}$. Denote by $E_1, \ldots, E_{16}$ the f-exceptional curves, denote by $\widetilde{C}$ the proper transform of the curve C on the surface $\widetilde{S}$, and denote by $\widetilde{\Delta}$ the proper transform of the divisor Δ on the surface $\widetilde{S}$. Let $E = E_1 + \cdots + E_{16}$. Then $E^2 = -32$ and

$$m\widetilde{C} + \widetilde{\Delta} \sim_{\mathbb{Q}} f^*(nH) - \epsilon E$$

for some non-negative rational number ϵ. Moreover, the curve $\widetilde{C}$ is a smooth irreducible curve of genus 5, so that $\widetilde{C}^2 = 8$ by the adjunction formula.

By Remark 5.21, the curve C contains $\mathrm{Sing}(S)$. Then $\widetilde{C} \cdot E = 16$. Let T be the divisor $f^*(6H) - \widetilde{C} - E$. Then $T \cdot E = 16$ and $T^2 = 8$, so that

$$\begin{aligned} h^0\big(\mathcal{O}_{\widetilde{S}}(T)\big) &= 6 + h^1\big(\mathcal{O}_{\widetilde{S}}(T)\big) - h^2\big(\mathcal{O}_{\widetilde{S}}(T)\big) \\ &= 6 + h^1\big(\mathcal{O}_{\widetilde{S}}(T)\big) + h^0\big(\mathcal{O}_{\widetilde{S}}(-T)\big) = 6 + h^1\big(\mathcal{O}_{\widetilde{S}}(T)\big) \geqslant 6 \end{aligned}$$

by the Riemann–Roch formula and Serre duality, since $T \cdot f^*(H) = 12 > 0$.

We claim that the linear system $|T|$ does not have base curves. Indeed, suppose it does. Then $T \sim F + M$, where F is the fixed part of the linear system $|T|$, and M is its mobile part. Then $M \cdot f^*(H) \geqslant 3$, because S is not uniruled. This gives

$$0 \leqslant f(F) \cdot H = F \cdot f^*(H) = 12 - M \cdot f^*(H) \leqslant 9.$$

Since F is $\overline{G}_{80}$-invariant, it follows from Theorem 5.16 that either F is contracted by f, or $f(F)$ is an irreducible smooth curve of degree 8 and genus 5. In the former case we have $F = rE$ for some integer $r \geqslant 1$, so that

$$M^2 = \Big(T - rE\Big)^2 = 6 - 2rT \cdot E - 32r^2 \leqslant 6 - 32r^2 < 0,$$

which is absurd. Thus, we see that $f(F)$ is an irreducible smooth curve of degree 8 and genus 5. Denote its proper transform on $\widetilde{S}$ by $\widetilde{C}_8$. Then $\widetilde{C}_8 \subset \mathrm{Supp}(F)$ and $\widetilde{C}_8^2 = 8$. Hence, it follows from the Riemann–Roch formula and Serre duality that

$$\begin{aligned} h^0\big(\mathcal{O}_{\widetilde{S}}(F)\big) &\geqslant h^0\big(\mathcal{O}_{\widetilde{S}}(\widetilde{C}_8)\big) = 6 + h^1\big(\mathcal{O}_{\widetilde{S}}(\widetilde{C}_8)\big) - h^2\big(\mathcal{O}_{\widetilde{S}}(\widetilde{C}_8)\big) \\ &= 6 + h^1\big(\mathcal{O}_{\widetilde{S}}(\widetilde{C}_8)\big) + h^0\big(\mathcal{O}_{\widetilde{S}}(-\widetilde{C}_8)\big) = 6 + h^1\big(\mathcal{O}_{\widetilde{S}}(\widetilde{C}_8)\big) \geqslant 6, \end{aligned}$$

which is also absurd, since F is the fixed part of the linear system $|T|$. Thus, we see that the linear system $|T|$ does not have base curves.

Let Z be a general curve in $|T|$. Then $Z \cdot E = 16$ and

$$Z \cdot \widetilde{C} = \widetilde{C} \cdot \big(6f^*(H) - \widetilde{C} - E\big) = 48.$$

On the other hand, the divisor Z is nef, so that $Z \cdot \widetilde{\Delta} \geqslant 0$. Therefore, we have

$$0 \leqslant Z \cdot \widetilde{\Delta} = Z \cdot \Big(f^*(nH) - m\widetilde{C} - \epsilon E\Big) = 12n - mZ \cdot \widetilde{C} - \epsilon Z \cdot E \leqslant 12n - 48m,$$

which implies that $\mathrm{mult}_C(\mathcal{D}) \leqslant m \leqslant \frac{n}{4}$. □

Corollary 5.23 *Let C be a $\overline{G}_{80}$-invariant curve of degree 12 in $\mathbb{P}^3$, and let $\mathcal{M}$ be the linear system of surfaces in $\mathbb{P}^3$ of degree 6 that contains C. Then the base locus of $\mathcal{M}$ does not have curves that are different from C.*

Proof The required assertion easily follows from the proof of Corollary 5.22. Namely, let us use notation of the proof of Corollary 5.22. Then the linear system $|f^*(6H) - \widetilde{C} - E|$ is free from base curves. This was shown in the proof of Corollary 5.22. In particular, the linear system $|6H - C|$ does not have base points. Since the quartic surface S is projectively normal, we immediately obtain the required assertions. □

Now let us prove that $\overline{G}_{80}$-invariant curves of degree 12 in $\mathbb{P}^3$ do exist.

Lemma 5.24 *Each surface among S_1, S_2, S_3, S_4 contains at least one $\overline{G}_{80}$-invariant curve of degree 12.*

Proof Let C be a smooth curve of genus 5 that admits a faithful action of the group $\overline{G}_{80}$. Then it follows from [18, Proposition 2.2] that there exists an exact sequence of groups

$$0 \longrightarrow \mathrm{Hom}\big(\overline{G}_{80}, \mathbb{C}^*\big) \longrightarrow \mathrm{Pic}\big(\overline{G}_{80}, C\big) \longrightarrow \mathrm{Pic}\big(C\big)^{\overline{G}_{80}} \longrightarrow H^2\big(\overline{G}_{80}, \mathbb{C}^*\big) \longrightarrow 0, \tag{5.25}$$

where $\mathrm{Pic}(\overline{G}_{80}, C)$ stands for the group of $\overline{G}_{80}$-equivariant line bundles on C. One can compute that $\mathrm{Hom}(\overline{G}_{80}, \mathbb{C}^*) \cong \boldsymbol{\mu}_5$ and $H^2(\overline{G}_{80}, \mathbb{C}^*) \cong \boldsymbol{\mu}_2^2$. Moreover, one has $C/\overline{G}_{80} \cong \mathbb{P}^1$, so that it follows from [18, (2.2)] that

$$\mathrm{Pic}\big(\overline{G}_{80}, C\big) \cong \mathbb{Z} \oplus \boldsymbol{\mu}_5.$$

Furthermore, let κ be a positive generator of the free part of $\mathrm{Pic}(\overline{G}_{80}, C)$. Then it follows from [18, (2.1)] that $\deg(\kappa) = 8$, so that we may assume that $\kappa = K_C$. Therefore, there is an exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathrm{Pic}(C)^{\overline{G}_{80}} \longrightarrow \boldsymbol{\mu}_2^2 \longrightarrow 0.$$

Thus, either $\mathrm{Pic}(C)^{\overline{G}_{80}} \cong \mathbb{Z} \oplus \boldsymbol{\mu}_2$ or $\mathrm{Pic}(C)^{\overline{G}_{80}} \cong \mathbb{Z} \oplus \boldsymbol{\mu}_2^2$. In both cases, there exists a $\overline{G}_{80}$-invariant 2-torsion line bundle on C. In the former case, such a 2-torsion is unique, and there exists a $\overline{G}_{80}$-invariant line bundle $\theta \in \mathrm{Pic}(C)$ such that $2\theta \sim K_C$, i.e. the divisor θ is a $\overline{G}_{80}$-invariant theta-characteristic. In the latter case, there are exactly three non-trivial $\overline{G}_{80}$-invariant 2-torsion line bundles on C, and there are no $\overline{G}_{80}$-invariant theta-characteristics.

Let D be a 2-torsion divisor on C such that its class in $\mathrm{Pic}(C)$ is $\overline{G}_{80}$-invariant. Then

$$h^0\big(\mathcal{O}_C(K_C + D)\big) = h^0\big(\mathcal{O}_C(-D)\big) + 4 = 4$$

by the Riemann–Roch formula. Moreover, the linear system $|K_C + D|$ does not have base points, because the degree of the divisor $K_C + D$ is 8, and C does not contain $\overline{G}_{80}$-orbits of length less than 16 by Lemma 5.3. So, the linear system $|K_C + D|$ gives a $\overline{G}_{80}$-equivariant morphism

$$\sigma\colon C \to \mathbb{P}\Big(H^0(\mathcal{O}_C(K_C + D))^\vee\Big).$$

By Corollary 5.2, we can identify the action of the group $\overline{G}_{80}$ on $\mathbb{P}(H^0(\mathcal{O}_C(K_C + D))^\vee)$ with the action of the group $\overline{G}_{80}$ on $\mathbb{P}^3$ introduced in Sect. 3. Note that $\sigma(C)$ is a $\overline{G}_{80}$-invariant curve of degree at most $\deg(K_C + D) = 8$. Thus, the degree of the curve $\sigma(C)$ is 8 by Theorem 5.16. In particular, this gives another proof that $\mathbb{P}^3$ contains a $\overline{G}_{80}$-invariant curve of degree 8 (see Remark 5.18). Note that Theorem 5.16 also implies that the curve $\sigma(C)$ is smooth, and either $\sigma(C) = S_1 \cap S_4$ or $\sigma(C) = S_2 \cap S_3$. Hence the morphism σ induces an isomorphism $C \cong \sigma(C)$. Since the curves $S_1 \cap S_4$ and $S_2 \cap S_3$ are projectively equivalent by Remark 5.18, we see that the class of the divisor D in $\mathrm{Pic}(C)$ is uniquely determined.

We see that there exists a unique $\overline{G}_{80}$-invariant 2-torsion line bundle on C, which implies that there exists a $\overline{G}_{80}$-invariant theta-characteristic $\theta \in \mathrm{Pic}(C)$. In fact, there are exactly two such theta-characteristics: one is θ, and another one is $\theta + D$. Arguing as above, we see that both linear systems $|K_C + \theta|$ and $|K_C + D + \theta|$ are free from base points and define $\overline{G}_{80}$-equivariant morphisms $C \to \mathbb{P}^7$, where $\mathbb{P}^7$ are the

projectivizations of $H^0(\mathcal{O}_C(K_C+\theta))^\vee$ and $H^0(\mathcal{O}_C(K_C+D+\theta))^\vee$, which are eight-dimensional representations of central extensions of the group $\overline{G}_{80}$ given by the images of $K_C+\theta$ and $K_C+D+\theta$ in

$$H^2(\overline{G}_{80},\mathbb{C}^*)\cong\boldsymbol{\mu}_2^2$$

in (5.25), respectively. One can show that these eight-dimensional representations split as sums of two four-dimensional representations, so that there exists a $\overline{G}_{80}$-equivariant map ς from C to a three-dimensional projective space that is given by some $\overline{G}_{80}$-invariant three-dimensional linear subsystem $\mathcal{P}$ in $|K_C+\theta|$ or $|K_C+D+\theta|$. By Corollary 5.2, we can identify the action of the group $\overline{G}_{80}$ on the target space of ς with the action of the group $\overline{G}_{80}$ on $\mathbb{P}^3$ introduced in Sect. 3. Since the degree of both divisors $K_C+\theta$ and $K_C+D+\theta$ is 12, the linear system $\mathcal{P}$ does not have base points. Thus, the curve $\varsigma(C)$ is a $\overline{G}_{80}$-invariant curve whose degree divides

$$\deg(K_C+\theta)=\deg(K_C+D+\theta)=12.$$

Hence, $\varsigma(C)$ is a smooth curve of degree 12 and genus 5 by Theorem 5.16.

By Theorem 5.16, the curve $\varsigma(C)$ is contained in a unique quartic surface among S_1, S_2, S_3 and S_4. Let $\overline{R}$ be the element in $\mathrm{PGL}_4(\mathbb{C})$ defined in Sect. 3, cf. Remark 5.18. Then

$$\overline{R}(S_1)=S_2,\quad \overline{R}(S_2)=S_4,\quad \overline{R}(S_4)=S_3,\quad \overline{R}(S_3)=S_1,$$

so that the cyclic group generated by $\overline{R}$ acts transitively on the set $\{S_1,S_2,S_3,S_4\}$. Since this group also acts transitively on the set

$$\Big\{\varsigma(C),\overline{R}(\varsigma(C)),\overline{R}^2(\varsigma(C)),\overline{R}^3(\varsigma(C))\Big\},$$

we conclude that each surface among S_1, S_2, S_3, S_4 contains a unique curve among $\varsigma(C)$, $\overline{R}(\varsigma(C))$, $\overline{R}^2(\varsigma(C))$ and $\overline{R}^3(\varsigma(C))$. □

Now we are going to describe all $\overline{G}_{80}$-invariant curves of degree 12 in $\mathbb{P}^3$.

Lemma 5.26 *There are exactly eight $\overline{G}_{80}$-invariant curves of degree 12 in $\mathbb{P}^3$. Moreover, each quartic surface among S_1, S_2, S_3, S_4 contains exactly two such curves. Furthermore, the union of these two curves is cut out on the quartic surface by a $\overline{G}_{80}$-invariant sextic surface in $\mathbb{P}^3$.*

Proof By Theorem 5.16, it is enough to show that S_1 contains exactly two $\overline{G}_{80}$-invariant curves of degree 12, and the union of these two curves is cut out by a $\overline{G}_{80}$-invariant sextic surface in $\mathbb{P}^3$. In the remaining cases, the proof is the same.

By Lemma 5.24, the surface S_1 contains a $\overline{G}_{80}$-invariant curve of degree 12. Denote this curve by C_{12}, and consider the exact sequence of G_{80}-representations

$$0 \longrightarrow H^0\big(\mathcal{O}_{\mathbb{P}^3}(6) \otimes \mathcal{I}_{C_{12}}\big) \longrightarrow H^0\big(\mathcal{O}_{\mathbb{P}^3}(6)\big) \longrightarrow H^0\big(\mathcal{O}_{C_{12}}(9K_C)\big) \longrightarrow H^1\big(\mathcal{O}_{\mathbb{P}^3}(6) \otimes \mathcal{I}_{C_{12}}\big) \longrightarrow 0,$$

where $\mathcal{I}_{C_{12}}$ is the ideal sheaf of the curve C_{12}. By Lemma 3.22, the vector space $H^0(\mathcal{O}_{\mathbb{P}^3}(6))$ contains exactly four one-dimensional subrepresentations of G_{80}, and $H^0(\mathcal{O}_{C_{12}}(9K_C))$ contains exactly three one-dimensional subrepresentations of the group G_{80}, which correspond to three $\overline{G}_{80}$-invariant effective divisors on C of degree 72. Hence, there is a $\overline{G}_{80}$-invariant sextic surface R in $\mathbb{P}^3$ that contains the curve C_{12}. By Lemma 3.22, the surface R is irreducible and reduced, so that Theorem 5.16 implies that

$$R\big|_{S_1} = C_{12} + Z_{12},$$

where Z_{12} is an irreducible smooth $\overline{G}_{80}$-invariant curve of degree 12 and genus 5.

By Remark 5.21, both curves C_{12} and Z_{12} contain the singular locus of the surface S_1. Let $f\colon \widetilde{S}_1 \to S_1$ be the minimal resolution of singularities of the surface S_1. Then the action of the group $\overline{G}_{80}$ lifts to the surface $\widetilde{S}_1$. Denote by $E_1, \ldots, E_{16}$ the exceptional curves of the birational morphism f. Denote by $\widetilde{C}_{12}$ and $\widetilde{Z}_{12}$ the proper transforms of the curves C_{12} and Z_{12} on the surface $\widetilde{S}_1$, respectively. Let $E = E_1 + \cdots + E_{16}$ and let H be a plane section of the surface S_1. Then

$$\widetilde{C}_{12} + \widetilde{Z}_{12} \sim f^*(6H) - E,$$

because both curves C_{12} and Z_{12} are smooth. Then $\widetilde{C}_{12} \cdot \widetilde{Z}_{12} = 48$, which implies, in particular, that $\widetilde{C}_{12} \neq \widetilde{Z}_{12}$, because $\widetilde{C}_{12} \cdot \widetilde{C}_{12} = \widetilde{Z}_{12} \cdot \widetilde{Z}_{12} = 8$ by the adjunction formula.

To complete the proof, we have to show that S_1 does not contain $\overline{G}_{80}$-invariant curves of degree 12 that are distinct from C_{12} and Z_{12}. Suppose that this is not the case, i.e. the surface S_1 contains a $\overline{G}_{80}$-invariant curve Z of degree 12 such that Z is distinct from the curves C_{12} and Z_{12}. Let us seek for a contradiction.

Let $\widetilde{Z}$ be the proper transform of the curve Z on $\widetilde{S}_1$ via f. Then $\widetilde{Z} \cdot (\widetilde{C}_{12} + \widetilde{Z}_{12}) = 56$. Since the only $\overline{G}_{80}$-orbits in $\widetilde{Z}$ are of lengths 16, 40 and 80 by Lemma 5.3, we see that either $\widetilde{Z} \cdot \widetilde{C}_{12} = 16$ and $\widetilde{Z} \cdot \widetilde{Z}_{12} = 40$, or $\widetilde{Z} \cdot \widetilde{C}_{12} = 40$ and $\widetilde{Z} \cdot \widetilde{Z}_{12} = 16$. Without loss of generality, we may assume that $\widetilde{Z} \cdot \widetilde{C}_{12} = 16$ and $\widetilde{Z} \cdot \widetilde{Z}_{12} = 40$. Then

$$\Big(\widetilde{C}_{12} + f^*(2H) - \widetilde{Z}\Big)^2 = 0.$$

Thus, it follows from the Riemann–Roch formula and Serre duality that

$$h^0\Big(\mathcal{O}_{\widetilde{S}}\big(\widetilde{C}_{12} + f^*(2H) - \widetilde{Z}\big)\Big) = 2 + h^1\Big(\mathcal{O}_{\widetilde{S}}\big(\widetilde{C}_{12} + f^*(2H) - \widetilde{Z}\big)\Big) - h^2\Big(\mathcal{O}_{\widetilde{S}}\big(\widetilde{C}_{12} + f^*(2H) - \widetilde{Z}\big)\Big)$$

$$= 2 + h^1\Big(\mathcal{O}_{\widetilde{S}}\big(\widetilde{C}_{12} + f^*(2H) - \widetilde{Z}\big)\Big) - h^0\Big(\mathcal{O}_{\widetilde{S}}\big(\widetilde{Z} - \widetilde{C}_{12} - f^*(2H)\big)\Big)$$
$$= 2 + h^1\Big(\mathcal{O}_{\widetilde{S}}\big(\widetilde{C}_{12} + f^*(2H) - \widetilde{Z}\big)\Big) \geqslant 2.$$

Hence, the linear system $|\widetilde{C}_{12} + f^*(2H) - \widetilde{Z}|$ is at least a pencil. Moreover, it does not contain base curves. Indeed, suppose it does. Then

$$\widetilde{C}_{12} + f^*(2H) - \widetilde{Z} \sim F + M,$$

where F is the fixed part of the linear system $|\widetilde{C}_{12} + f^*(2H) - \widetilde{Z}|$, and M is its mobile part. Then

$$0 \leqslant f(F) \cdot H = F \cdot f^*(H) = 8 - M \cdot f^*(H) < 8.$$

By Theorem 5.16, this implies that F is contracted by f, so that $F = mE$ for some $m \geqslant 1$. Then

$$M^2 = \Big(\widetilde{C}_{12} + f^*(2H) - \widetilde{Z} - mE\Big)^2 = -32m^2,$$

which is absurd. This shows that the linear system $|\widetilde{C}_{12} + f^*(2H) - \widetilde{Z}|$ is free from base curves. Since

$$\big(\widetilde{C}_{12} + f^*(2H) - \widetilde{Z}\big)^2 = 0,$$

we see that it has no base points, so that it is composed of a base point free pencil. This pencil gives a $\overline{G}_{80}$-equivariant morphism $\upsilon\colon \widetilde{S} \to \mathbb{P}^1$, whose general fibers are smooth elliptic curves (by the adjunction formula). This is impossible, because $\overline{\mathrm{H}}$ is the only non-trivial proper normal subgroup of the group $\overline{G}_{80}$ by Lemma 5.1, and the group $\overline{\mathrm{H}}$ does not act faithfully on rational and elliptic curves by Lemma 3.3. This completes the proof of Lemma 5.26. □

Recall from Remark 5.18 that there are two $\overline{G}_{80}$-invariant curves of degree 8 in $\mathbb{P}^3$. These curves are just the intersections $S_1 \cap S_4$ or $S_2 \cap S_3$ taken with the reduced structure. Furthermore, these curves are smooth curves of genus 5 by Theorem 5.16.

Proposition 5.27 (cf. [2,16,17]) *Let C be a $\overline{G}_{80}$-invariant curve of degree 8 in $\mathbb{P}^3$, and let $\pi : X \to \mathbb{P}^3$ be a blow up of the curve C. Then there exists a $\overline{G}_{80}$-commutative diagram*

$$\begin{array}{ccccc} X & & \overset{\iota}{- - - - -} & & X \\ & \searrow^{\alpha} & & \swarrow^{\alpha} & \\ \pi\downarrow & & V_8 \xrightarrow{\upsilon} V_8 & & \downarrow\pi \\ \mathbb{P}^3 & & \underset{\tau}{- - - - -} & & \mathbb{P}^3 \end{array} \tag{5.28}$$

where V_8 is the Fano threefold with terminal Gorenstein singularities such that $-K_{V_8}^3 = 8$, the map α is a flopping contraction, the map τ is a birational map, υ is an isomorphism, and ι is a composition of flops that acts on $\mathrm{Pic}(X)$ *in the following way:*

$$\begin{cases} \iota^*\big(E\big) \sim 24\pi^*(H) - 7E, \\ \iota^*\big(\pi^*(H)\big) \sim 7\pi^*(H) - 2E, \end{cases}$$

where H is a plane in $\mathbb{P}^3$, and E is the π-exceptional divisor. Moreover, the diagram (5.28) *is also $\overline{G}_{160}$-commutative.*

Proof Without loss of generality, we may assume that $C = S_1 \cap S_4$. Denote by $\widetilde{S}_1$ and $\widetilde{S}_4$ the proper transforms of the surfaces S_1 and S_4 on the threefold X, respectively. Recall from Theorem 5.16 that C contains singular loci of the surfaces S_1 and S_4. This implies that $\widetilde{S}_1$ and $\widetilde{S}_4$ are smooth, because the singularities of the surfaces S_1 and S_4 are isolated ordinary double points by Lemma 5.13. Furthermore, we have

$$\widetilde{S}_1 \sim \widetilde{S}_4 \sim -K_X \sim \pi^*(4H) - E.$$

Since $-K_X^3 = 8$, we see that the intersection $\widetilde{S}_1 \cap \widetilde{S}_4$ is a $\overline{G}_{80}$-invariant smooth curve isomorphic to C. Let us denote this curve by $\widetilde{C}$. Note that $\widetilde{C}$ is the scheme-theoretic intersection of the surfaces $\widetilde{S}_1$ and $\widetilde{S}_4$, and

$$\widetilde{S}_1 \cdot \widetilde{C} = \widetilde{S}_4 \cdot \widetilde{C} = -K_X^3 = 8,$$

so that $-K_X$ is nef and big. By the Base Point Free Theorem (see [24, Theorem 1.3.6]), the linear system $|-nK_X|$ gives a birational morphism $\alpha\colon X \to V_8$ for some $n \gg 0$, where V_8 is the Fano threefold with canonical Gorenstein singularities such that $-K_{V_8}^3 = 8$.

Let us show that the birational morphism α is small (cf. [25, Theorems 4.9 and 4.11]). Suppose that α is not small. Then there exists a prime divisor G that is contracted by α. One has $(-K_X)^2 \cdot G = 0$. On the other hand, we have $G \sim \pi^*(aH) - bE$ for some positive integers a and b. Then $a = 3b$, since

$$0 = (-K_X)^2 \cdot G = \Big(\pi^*(4H) - E\Big)^2 \cdot \Big(\pi^*(aH) - bE\Big) = 8(a - 3b).$$

If $\alpha(G)$ is a point, then

$$0 = -K_X \cdot G^2 = 4(a^2 - 4ab + 8b^2) = 20b^2,$$

which is absurd. Thus, we see that $\alpha(G)$ is a curve, so that the intersection $G \cap \widetilde{S}_1$ contains a curve. Then

$$0 = (-K_X)^2 \cdot G = G|_{\widetilde{S}_1} \cdot \widetilde{S}_2|_{\widetilde{S}_1} = G|_{\widetilde{S}_1} \cdot \widetilde{C},$$

which implies that $\widetilde{C}$ is not an ample divisor on the surface $\widetilde{S}_1$. On the other hand, arguing as in the proof of Corollary 5.22, one can show that the linear system $|2\widetilde{C} - \pi^*(H)|_{\widetilde{S}_1}|$ is free from base points, which implies the ampleness of the divisor

$$\pi^*(H)|_{\widetilde{S}_1} + \left(2\widetilde{C} - \pi^*(H)|_{\widetilde{S}_1}\right) \sim 2\widetilde{C}.$$

The obtained contradiction implies that the birational morphism α is small.

Since α is a small birational morphism, the singularities of the threefold V_8 are terminal. Moreover, there exists a $\overline{G}_{80}$-commutative diagram

$$\begin{array}{ccccccc} & & X & \overset{\chi}{- - - - - - - - \dashrightarrow} & Y & & \\ & \overset{\pi}{\swarrow} & & \overset{\alpha}{\searrow} & \overset{\beta}{\swarrow} & \overset{\phi}{\searrow} & \\ \mathbb{P}^3 & & & & V_8 & & Z \end{array} \tag{5.29}$$

such that χ is a composition of flops, β is a small birational morphism, and ϕ is an extremal contraction such that $-K_Y$ is ϕ-ample. Observe that the threefold Y is smooth, because the threefold X is smooth and χ is a composition of flops. Now arguing as in the proof of [16, Proposition 4.4], we see that the morphism ϕ is birational, one has $Z \cong \mathbb{P}^3$, and ϕ is a blow up of a smooth curve of degree 8 and genus 5. Moreover, this proof also implies that

$$\widetilde{D} \sim 24\pi^*(H) - 7E, \tag{5.30}$$

where $\widetilde{D}$ is the proper transform on X of the ϕ-exceptional surface.

By Corollary 5.2, we may assume that the diagram (5.29) is $\overline{G}_{80}$-commutative. Then ϕ is a blow up of one of the two $\overline{G}_{80}$-invariant smooth curves of degree 8 and genus 5. By Remark 5.18, the group $\overline{G}_{320}$ swaps these curves. Thus, composing the map ϕ with an appropriate element in $\overline{G}_{320}$, we may assume that ϕ is a blow up of the curve C. Therefore, we can identify X with Y and ϕ with π. This gives the $\overline{G}_{80}$-commutative diagram (5.28) with

$$\iota = \chi, \quad \tau = \pi \circ \iota \circ \pi^{-1}, \quad \upsilon = \alpha \circ \iota \circ \alpha^{-1}.$$

Since V_8 is an anticanonical model of the threefold X, the map υ must be biregular. The action of ι on $\mathrm{Pic}(X)$ follows from (5.30), since $\iota^*(K_X) \sim K_X$. The diagram (5.28) also $\overline{G}_{160}$-commutative, because $\mathrm{Aut}(C) \cong \overline{G}_{160}$ by Remark 5.18. □

Using Remark 5.20, one can show that the morphism α in Proposition 5.27 contracts the proper transforms of the curve $\mathcal{L}_{10}$. Using this, one can prove that V_8 in Proposition 5.27 is a complete intersection of three quadrics in $\mathbb{P}^6$ that has 10 isolated ordinary double points.

6 Birational rigidity

Let G be a finite subgroup in $\mathrm{PGL}_4(\mathbb{C})$. In this section we prove Theorems 1.1 and 1.3, which are the main results of our paper. To do this, we will use the following well-known result, which goes back to the classical works of Noether, Fano, and Iskovskikh.

Theorem 6.1 [10, Theorem 3.3.1] *Let Γ be a subset in $\mathrm{Bir}^G(\mathbb{P}^3)$. Suppose that for every non-empty G-invariant mobile linear system $\mathcal{D}$ on $\mathbb{P}^3$, there exists $\gamma \in \Gamma$ such that the log pair $(\mathbb{P}^3, \frac{4}{n}\gamma(\mathcal{D}))$ has canonical singularities, where n be a positive integer that is defined by $\gamma(\mathcal{D}) \sim \mathcal{O}_{\mathbb{P}^3}(n)$. Then $\mathbb{P}^3$ is G-birationally rigid, and the group $\mathrm{Bir}^G(\mathbb{P}^3)$ is generated by Γ and $\mathrm{Aut}^G(\mathbb{P}^3)$.*

Corollary 6.2 *Suppose that for every non-empty G-invariant mobile linear system $\mathcal{D}$ on $\mathbb{P}^3$, the log pair $(\mathbb{P}^3, \frac{4}{n}\mathcal{D})$ has canonical singularities, where n be a positive integer that is defined by $\mathcal{D} \sim \mathcal{O}_{\mathbb{P}^3}(n)$. Then $\mathbb{P}^3$ is G-birationally super-rigid.*

The following simple fact is useful for the proof of G-birational rigidity of $\mathbb{P}^3$.

Lemma 6.3 *Let $\mathcal{D}$ be a non-empty G-invariant mobile linear system on $\mathbb{P}^3$, and let Z be a G-irreducible curve in $\mathbb{P}^3$. Then*

$$\mathrm{mult}_Z(\mathcal{D}) \leqslant \frac{n}{\sqrt{\deg(Z)}},$$

where n is a positive integer such that $\mathcal{D} \sim \mathcal{O}_{\mathbb{P}^3}(n)$.

Proof Let D_1 and D_2 be two general surfaces in $\mathcal{D}$. Then

$$D_1 \cdot D_2 = mZ + \Delta,$$

where m is a positive integer such that $m \geqslant \mathrm{mult}_Z^2(\mathcal{D})$, and Δ is an effective one-cycle on $\mathbb{P}^3$ such that $\mathrm{Supp}(\Delta)$ does not contain irreducible components of the curve Z. Let H be a general plane in $\mathbb{P}^3$. Then

$$n^2 = H \cdot D_1 \cdot D_2 = H \cdot \Big(mZ + \Delta\Big) \geqslant m \cdot \deg(Z) \geqslant \mathrm{mult}_Z^2(\mathcal{D}) \cdot \deg(Z),$$

which implies the required inequality. □

To illustrate Lemma 6.3 and Corollary 6.2, let us prove.

Lemma 6.4 *Suppose that G contains a proper subgroup isomorphic to $\mathfrak{A}_6$. Then $\mathbb{P}^3$ is G-birationally super-rigid.*

Proof Let $\mathcal{D}$ be a non-empty G-invariant mobile linear system on $\mathbb{P}^3$. Then $\mathcal{D} \sim \mathcal{O}_{\mathbb{P}^3}(n)$ for some positive integer n. By Corollary 6.2, to complete the proof, we have to show that the singularities of the log pair $(\mathbb{P}^3, \frac{4}{n}\mathcal{D})$ are canonical. By assumption,

the group G contains a subgroup isomorphic to $\mathfrak{A}_6$. Denote it by Θ. It follows from [8, §4] that there are two Θ-irreducible curves $\mathcal{L}_6$ and $\mathcal{L}_6'$ in $\mathbb{P}^3$ such that $\mathcal{L}_6 \cap \mathcal{L}_6' = \varnothing$, and each of them is a disjoint union of 6 lines. Moreover, it follows from [8, Theorem 4.3] that the log pair $(\mathbb{P}^3, \frac{4}{n}\mathcal{D})$ has canonical singularities provided that

$$\max\left\{\mathrm{mult}_{\mathcal{L}_6}(\mathcal{D}), \mathrm{mult}_{\mathcal{L}_6'}(\mathcal{D})\right\} \leqslant \frac{n}{4}.$$

On the other hand, it follows from the proof of [8, Theorem 4.9] that

$$\min\left\{\mathrm{mult}_{\mathcal{L}_6}(\mathcal{D}), \mathrm{mult}_{\mathcal{L}_6'}(\mathcal{D})\right\} \leqslant \frac{n}{4}.$$

Note that Θ is a primitive subgroup in $\mathrm{PGL}_4(\mathbb{C})$. In particular, it does not have orbits of length at most 4, and it does not leave invariant any line or a pair of lines in $\mathbb{P}^3$. This implies that G also enjoys these properties, so that it is primitive as well. Thus, it follows from Blichfeldt's classification [4, Chapter VII] that G is one of the following groups: $\mathfrak{S}_6$, $\mathfrak{A}_7$, or a simple group of order 25920. If $G \cong \mathfrak{S}_6$, then the log pair $(\mathbb{P}^3, \frac{4}{n}\mathcal{D})$ has canonical singularities, because G permutes the curves $\mathcal{L}_6$ and $\mathcal{L}_6'$ in this case, so that

$$\mathrm{mult}_{\mathcal{L}_6}(\mathcal{D}) = \mathrm{mult}_{\mathcal{L}_6'}(\mathcal{D}).$$

Similarly, if G is a simple group of order 25920, then the log pair $(\mathbb{P}^3, \frac{4}{n}\mathcal{D})$ also has canonical singularities, because this group contains $\mathfrak{S}_6$ as a subgroup (see [11, p. 26]).

To complete the proof, we may assume that $G \cong \mathfrak{A}_7$. Let Z be a G-orbit of the curve $\mathcal{L}_6$. Then $\deg(Z) = 42$, so that $\mathrm{mult}_{\mathcal{L}_6}(\mathcal{D}) \leqslant \frac{n}{4}$ by Lemma 6.3. Similarly, we see that $\mathrm{mult}_{\mathcal{L}_6'}(\mathcal{D}) \leqslant \frac{n}{4}$. Thus, it follows from [8, Theorem 4.3] that the log pair $(\mathbb{P}^3, \frac{4}{n}\mathcal{D})$ has canonical singularities. □

Remark 6.5 Let Θ be a subgroup of the group G. If $\mathbb{P}^3$ is Θ-birationally super-rigid, then it is also G-birationally super-rigid. This follows from Corollary 6.2 and equivariant Minimal Model Program (see [6, Corollary 1.4.3]). However, if $\mathbb{P}^3$ is Θ-birationally rigid, then we cannot immediately conclude that $\mathbb{P}^3$ is G-birationally rigid. A priori, the projective space $\mathbb{P}^3$ can be G-birational to a Fano threefold X with terminal singularities such that the G-invariant class group of the threefold X is of rank 1, but the Θ-invariant class group is not of rank 1. We refer the reader to [26] for a discussion of a similar problem for the action of Galois groups.

Now we will state two technical propositions and use them to prove Theorems 1.1 and 1.3. After this, we will present the proofs of these propositions. Let $\overline{G}_{144}$ be the primitive subgroup in $\mathrm{PGL}_4(\mathbb{C})$ constructed in Sect. 3 and dealt with in Sect. 4.

Proposition 6.6 *Let $\mathcal{D}$ be a non-empty $\overline{G}_{144}$-invariant mobile linear system on $\mathbb{P}^3$, and let n be a positive integer such that $\mathcal{D} \sim \mathcal{O}_{\mathbb{P}^3}(n)$. Then the log pair $(\mathbb{P}^3, \frac{4}{n}\mathcal{D})$ has canonical singularities.*

Proposition 6.6 and Corollary 6.2 imply the following:

Corollary 6.7 *Suppose that G contains $\overline{G}_{144}$. Then $\mathbb{P}^3$ is G-birationally super-rigid.*

Similarly, let $\overline{G}_{80}$ and $\overline{G}_{160}$ be primitive subgroups in $\mathrm{PGL}_4(\mathbb{C})$ constructed in Sect. 3 and dealt with in Sect. 5. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two distinct $\overline{G}_{80}$-invariant irreducible smooth curves of degree 8 and genus 5 in $\mathbb{P}^3$ constructed in Sect. 5.

Proposition 6.8 *Let $\mathcal{D}$ be a non-empty $\overline{G}_{80}$-invariant mobile linear system on $\mathbb{P}^3$, and let n be a positive integer such that $\mathcal{D} \sim \mathcal{O}_{\mathbb{P}^3}(n)$. Suppose that*

$$\max\Big\{\mathrm{mult}_{\mathcal{C}_1}\big(\mathcal{D}\big), \mathrm{mult}_{\mathcal{C}_2}\big(\mathcal{D}\big)\Big\} \leqslant \frac{n}{4}.$$

Then the log pair $(\mathbb{P}^3, \frac{4}{n}\mathcal{D})$ has canonical singularities.

Proposition 6.8 and Theorem 6.1 imply the following:

Corollary 6.9 *Suppose that $G = \overline{G}_{80}$ or $G = \overline{G}_{160}$. Then $\mathbb{P}^3$ is G-birationally rigid.*

Proof Let Γ be a subgroup in $\mathrm{Bir}^G(\mathbb{P}^3)$ that is generated by the birational maps constructed in Proposition 5.27. Let $\mathcal{D}$ be a non-empty G-invariant mobile linear system on $\mathbb{P}^3$. For every $\sigma \in \Gamma$, let n_σ be a positive integer such that $\sigma(\mathcal{D}) \sim \mathcal{O}_{\mathbb{P}^3}(n_\sigma)$. Then there is $\gamma \in \Gamma$ such that

$$n_\gamma = \min\Big\{n_\sigma \;\Big|\; \sigma \in \Gamma\Big\}.$$

Let $n = n_\gamma$ and $\mathcal{M} = \gamma(\mathcal{D})$. By Theorem 6.1, to prove the required assertion, it is enough to show that the log pair $(\mathbb{P}^3, \frac{4}{n}\mathcal{M})$ has canonical singularities. Suppose that this is not true. By Proposition 6.8, either $\mathrm{mult}_{\mathcal{C}_1}(\mathcal{M}) > \frac{n}{4}$ or $\mathrm{mult}_{\mathcal{C}_2}(\mathcal{M}) > \frac{n}{4}$. In the former case, we let $C = \mathcal{C}_1$. In the latter case, we let $C = \mathcal{C}_2$. Let τ be the birational map constructed in Proposition 5.27 starting from the curve C. Then $\tau \in \Gamma$ and

$$\tau\big(\mathcal{M}\big) \sim \mathcal{O}_{\mathbb{P}^3}\Big(7n - 24\mathrm{mult}_C\big(\mathcal{M}\big)\Big)$$

by Proposition 5.27. This contradicts the minimality of n, since $7n - 24\mathrm{mult}_C(\mathcal{D}) < n$. □

To prove Theorems 1.1 and 1.3, we need the last auxiliary result.

Lemma 6.10 *Suppose that the group G contains $\overline{G}_{80}$, and that G does not coincide with $\overline{G}_{80}$ and $\overline{G}_{160}$. Then $\mathbb{P}^3$ is G-birationally super-rigid.*

Proof Let $\mathcal{D}$ be a non-empty G-invariant mobile linear system on $\mathbb{P}^3$. Then $\mathcal{D} \sim \mathcal{O}_{\mathbb{P}^3}(n)$ for some positive integer n. By Corollary 6.2, to complete the proof, we have to show that the singularities of the log pair $(\mathbb{P}^3, \frac{4}{n}\mathcal{D})$ are canonical. This follows from Proposition 6.8 provided that

$$\max\Big\{\mathrm{mult}_{\mathcal{C}_1}\big(\mathcal{D}\big), \mathrm{mult}_{\mathcal{C}_2}\big(\mathcal{D}\big)\Big\} \leqslant \frac{n}{4},$$

because $\mathcal{D}$ is also $\overline{G}_{80}$-invariant. Let us prove the latter inequality.

Let Z be a G-orbit of the curve $\mathcal{C}_1$. Recall from Sect. 5 that $\mathcal{C}_1$ is $\overline{G}_{160}$-invariant. Moreover, it follows from Remark 5.18 that $Z \neq \mathcal{C}_1$. Thus, we have $\deg(Z) \geqslant 2\deg(\mathcal{C}_1) = 16$, so that $\mathrm{mult}_{\mathcal{C}_1}(\mathcal{D}) \leqslant \frac{n}{4}$ by Lemma 6.3. Similarly, we see that $\mathrm{mult}_{\mathcal{C}_2}(\mathcal{D}) \leqslant \frac{n}{4}$. Therefore, the log pair $(\mathbb{P}^3, \frac{4}{n}\mathcal{D})$ has canonical singularities by Proposition 6.8. □

Now we are ready to prove the main results of our paper.

Proof (Proof of Theorems 1.1 and 1.3) By Corollary 2.4, we may assume that G is a primitive subgroup in $\mathrm{PGL}_4(\mathbb{C})$. By Corollary 2.10, we may assume that G is not isomorphic to $\mathfrak{A}_5$ and $\mathfrak{S}_5$. For the remaining groups, we use the classification provided by Blichfeldt in [4, Chapter VII], see Appendix A.

If $G \cong \mathrm{PSL}_2(\mathbf{F}_7)$, then $\mathbb{P}^3$ is G-birationally rigid and is not G-birationally super-rigid by [7, Theorem 1.9]. Similarly, if $G \cong \mathfrak{A}_6$, then $\mathbb{P}^3$ is G-birationally rigid and is not G-birationally super-rigid by [8, Theorem 1.24]. Moreover, if G contains $\mathfrak{A}_6$ as a proper subgroup, then $\mathbb{P}^3$ is G-birationally super-rigid by Lemma 6.4.

Recall from [4, Chapter VII] that up to conjugation all remaining finite primitive subgroups of $\mathrm{PGL}_4(\mathbb{C})$ belongs to one of the following two classes. The first class consists of primitive groups that leave a smooth quadric surface in $\mathbb{P}^3$ invariant and contain $\overline{G}_{144}$ (see [4, §121]). The second class consists of primitive groups containing an imprimitive normal subgroup $\boldsymbol{\mu}_2^4$ (see [4, §124]). These classes overlap. For instance, the group $\overline{G}_{144} \cong \boldsymbol{\mu}_2^4 \rtimes (\boldsymbol{\mu}_3 \times \boldsymbol{\mu}_3)$ itself is contained in both of them (see [4, §125]). However, if G is in the second class and is not in the first class, then G must contain the subgroup $\overline{G}_{80}$ (see [4, §124]). Therefore, to complete the proof we may assume that G contains either $\overline{G}_{144}$ or $\overline{G}_{80}$.

If the group G contains $\overline{G}_{144}$ as a subgroup, then the projective space $\mathbb{P}^3$ is G-birationally super-rigid by Corollary 6.7. If $G = \overline{G}_{80}$ or $G = \overline{G}_{160}$, then $\mathbb{P}^3$ is G-birationally rigid by Corollary 6.9, and it is not G-birationally super-rigid by Proposition 5.27. Finally, if G contains $\overline{G}_{80}$ and neither $G = \overline{G}_{80}$ nor $G = \overline{G}_{160}$, then $\mathbb{P}^3$ is G-birationally super-rigid by Lemma 6.10. □

Thus, to complete the proofs of our main results, it is enough to prove Propositions 6.6 and 6.8. This is what we are going to do in the remaining part of this section. Namely, suppose that either $G = \overline{G}_{144}$ or $G = \overline{G}_{80}$. Let $\mathcal{D}$ be a G-invariant mobile linear system on $\mathbb{P}^3$. Then $\mathcal{D} \sim \mathcal{O}_{\mathbb{P}^3}(n)$ for a positive integer n. If $G = \overline{G}_{80}$, we also suppose that

$$\max\left\{\mathrm{mult}_{\mathcal{C}_1}(\mathcal{D}), \mathrm{mult}_{\mathcal{C}_2}(\mathcal{D})\right\} \leqslant \frac{n}{4}. \tag{6.11}$$

To prove Propositions 6.6 and 6.8, we have to show that the singularities of the log pair $(\mathbb{P}^3, \frac{4}{n}\mathcal{D})$ are canonical. Suppose that this is not true. Let us show that this assumption leads to a contradiction.

Lemma 6.12 *Let C be a G-irreducible curve in $\mathbb{P}^3$. Then* $\mathrm{mult}_C(\mathcal{D}) \leqslant \frac{n}{4}$.

Proof By Lemma 6.3, we may assume that $\deg(Z) \leqslant 15$. If $G = \overline{G}_{144}$, the assertion follows from Theorem 4.22 together with Corollaries 4.8 and 4.21. If $G = \overline{G}_{80}$, the

assertion follows from the classification of $\overline{G}_{80}$-invariant curves of small degree done in Theorem 5.16 together with Lemma 5.11, Corollary 5.22, and our assumption (6.11) on curves of degree 8. □

Thus, the log pair $(\mathbb{P}^3, \frac{4}{n}\mathcal{D})$ is canonical away from finitely many points in $\mathbb{P}^3$ (see, for instance, [13, Proposition 5.14]). On the other hand, this log pair is not canonical at some point $P \in \mathbb{P}^3$ by assumption. Denote by Σ the G-orbit of the point P. Let D_1 and D_2 be two sufficiently general surfaces in $\mathcal{D}$. Then

$$\operatorname{mult}_O\Big(D_1 \cdot D_2\Big) > \frac{n^2}{4} \tag{6.13}$$

for every point $O \in \Sigma$ by [12, Corollary 3.4].

For every $r \geqslant 2$, denote by $\mathcal{M}_r$ the linear system of surfaces of degree r in $\mathbb{P}^3$ that contain Σ. If $\mathcal{M}_r$ is not empty, let M_r be a general surface in $\mathcal{M}_r$. In this case (6.13) implies that

$$rn^2 = M_r \cdot D_1 \cdot D_2 \geqslant \sum_{O \in \Sigma} \operatorname{mult}_O\Big(D_1 \cdot D_2\Big) > \frac{n^2}{4}|\Sigma|$$

provided that M_r does not contain any irreducible component of the one-cycle $D_1 \cdot D_2$. Thus, if the base locus of the linear system $\mathcal{M}_r$ does not contain curves, then (6.13) implies the inequality

$$|\Sigma| < 4r. \tag{6.14}$$

We expect that the base locus of the linear system $\mathcal{M}_r$ contains no curves for $r = \lfloor \frac{|\Sigma|}{4} \rfloor$. Unfortunately, we failed to prove this, so instead we use an alternative approach that utilizes results obtained in Sects. 4 and 5. Let us start with

Corollary 6.15 *If $G = \overline{G}_{144}$, then $|\Sigma| \geqslant 16$. Similarly, if $G = \overline{G}_{80}$, then $|\Sigma| \geqslant 20$.*

Proof If $G = \overline{G}_{144}$ and $|\Sigma| < 16$, then $|\Sigma| = 12$ by Lemma 4.10. If $G = \overline{G}_{80}$ and $|\Sigma| < 20$, then $|\Sigma| = 16$ by Lemma 5.7. Thus, the base locus of the linear system $\mathcal{M}_3$ contains no curves by Corollaries 4.13, 5.7 and 5.14. This gives a contradiction with (6.14). □

Recall from Lemma 3.10 that there is unique $\overline{G}_{144}$-invariant quadric surface in $\mathbb{P}^3$.

Lemma 6.16 *Suppose that $G = \overline{G}_{144}$. Let $\mathcal{Q}$ be the unique G-invariant quadric surface in $\mathbb{P}^3$. Then Σ is not contained in $\mathcal{Q}$.*

Proof Suppose that Σ is contained in $\mathcal{Q}$. Then the log pair $(\mathbb{P}^3, \mathcal{Q} + \frac{4}{n}\mathcal{D})$ is not log canonical at every point of Σ. Thus, the log pair $(\mathcal{Q}, \frac{4}{n}\mathcal{D}|_{\mathcal{Q}})$ is not log canonical at every point of Σ by Inversion of Adjunction (see [28, Theorem 5.50]). Hence, there is $\mu < \frac{4}{n}$ such that the log pair $(\mathcal{Q}, \mu\mathcal{D}|_{\mathcal{Q}})$ is not Kawamata log terminal. By Corollary 4.7, this log pair is Kawamata log terminal away from finitely many points in $\mathcal{Q}$.

Let $\mathcal{I}$ be the multiplier ideal sheaf of the log pair $(\mathcal{Q}, \mu\mathcal{D}|_{\mathcal{Q}})$. Then the ideal $\mathcal{I}$ defines a (non-empty) zero-dimensional subscheme $\mathcal{L}$ of the quadric $\mathcal{Q}$ such that the support of the subscheme $\mathcal{L}$ contains Σ. Moreover, this subscheme is G-invariant, because $\mathcal{D}$ is G-invariant. Let H be a plane section of the quadric $\mathcal{Q}$. Then

$$h^1\Big(\mathcal{I} \otimes \mathcal{O}_{\mathcal{Q}}\big(2H\big)\Big) = 0$$

by Nadel vanishing theorem (see [30, Theorem 9.4.8]). Now using the exact sequence of sheaves

$$0 \longrightarrow \mathcal{I} \otimes \mathcal{O}_{\mathcal{Q}}\big(2H\big) \longrightarrow \mathcal{O}_{\mathcal{Q}}\big(2H\big) \longrightarrow \mathcal{O}_{\mathcal{L}} \otimes \mathcal{O}_{\mathcal{Q}}\big(2H\big) \longrightarrow 0,$$

we see that

$$\big|\Sigma\big| \leqslant h^0\big(\mathcal{O}_{\mathcal{L}}\big) = h^0\Big(\mathcal{O}_{\mathcal{L}} \otimes \mathcal{O}_{\mathcal{Q}}\big(2H\big)\Big) \leqslant h^0\Big(\mathcal{O}_{\mathcal{Q}}\big(2H\big)\Big) = 9,$$

which is impossible by Lemma 4.4. □

Lemma 6.17 *Let C be a G-irreducible curve in $\mathbb{P}^3$ such that $\deg(C) \leqslant 15$. Then $\Sigma \not\subset C$.*

Proof Suppose that $\Sigma \subset C$. Let us seek for a contradiction. We have

$$D_1 \cdot D_2 = mC + \Delta,$$

where m is a non-negative integer, and Δ is an effective one-cycle on $\mathbb{P}^3$ such that $\mathrm{Supp}(\Delta)$ does not contain irreducible components of the curve C. Hence, it follows from (6.13) that

$$\mathrm{mult}_O\big(\Delta\big) > \frac{n^2}{4} - m \cdot \mathrm{mult}_O\big(C\big) \tag{6.18}$$

for every point $O \in \Sigma$. Let H be a plane in $\mathbb{P}^3$. Thus

$$n^2 = H \cdot D_1 \cdot D_2 = H \cdot \Big(mC + \Delta\Big) = m \cdot \deg(C) + H \cdot \Delta \geqslant m \cdot \deg(C). \tag{6.19}$$

For every $r \geqslant 2$, denote by $\mathcal{B}_r$ the linear system that consists of surfaces of degree r in $\mathbb{P}^3$ containing C. If the base locus of $\mathcal{B}_r$ does not contain G-irreducible curves different from C, then it follows from (6.18) that

$$\begin{aligned} r\Big(n^2 - m \cdot \deg(C)\Big) &= B_r \cdot \Big(D_1 \cdot D_2 - mC\Big) \\ &= B_r \cdot \Delta \geqslant \sum_{O \in \Sigma} \mathrm{mult}_O\big(\Delta\big) > |\Sigma|\Big(\frac{n^2}{4} - m \cdot \mathrm{mult}_O\big(C\big)\Big), \end{aligned} \tag{6.20}$$

where B_r is a sufficiently general surface in the linear system $\mathcal{B}_r$.

Suppose that $G = \overline{G}_{144}$. By Lemma 6.16 and Theorem 4.22, the curve C is a smooth irreducible curve of degree 12 and genus 13. In particular, we have $|\Sigma| \geqslant 24$ by Lemma 4.2. Moreover, it follows from Corollary 4.20 that the base locus of the linear system $\mathcal{B}_6$ does not contain curves different from C. Thus, it follows from (6.20) that

$$6(n^2 - 12m) > 24\left(\frac{n^2}{4} - m\right),$$

so that $0 > 48m$, which is absurd.

We see that $G = \overline{G}_{80}$. Let us use the notation of Sect. 5. By Theorem 5.16, one of the following cases holds:

- $\deg(C) = 8$, and C is an irreducible smooth curve of genus 5;
- $\deg(C) = 12$, and C is an irreducible smooth curve of genus 5;
- $\deg(C) = 10$, and C is one of the curves $\mathcal{L}_{10}, \mathcal{L}'_{10}, \mathcal{L}''_{10}$.

Note that in the former two cases one has $|\Sigma| \geqslant 40$ by Lemma 5.3 and Corollary 6.15. Also, we know from (6.19) that $n^2 \geqslant m \cdot \deg(C) > 4m$. Let us consider the above possibilities case by case.

If $\deg(C) = 8$, then the base locus of $\mathcal{B}_4$ does not contain G-irreducible curves different from C by Theorem 5.16. Hence (6.20) gives

$$4n^2 - 32m = 4\left(n^2 - 8m\right) > |\Sigma|\left(\frac{n^2}{4} - m\right) \geqslant 40\left(\frac{n^2}{4} - m\right) = 10n^2 - 40m.$$

This implies that $6n^2 < 8m$. On the other hand, one has $8m \leqslant n^2$ by (6.19), which is absurd.

If $\deg(C) = 12$, then the base locus of $\mathcal{B}_6$ does not contain curves different from C by Corollary 5.23. Hence (6.20) gives

$$6n^2 - 72m = 6\left(n^2 - 12m\right) > |\Sigma|\left(\frac{n^2}{4} - m\right) \geqslant 40\left(\frac{n^2}{4} - m\right) = 10n^2 - 40m,$$

which is absurd.

Thus, we see that C is one of the curves $\mathcal{L}_{10}, \mathcal{L}'_{10}$ or $\mathcal{L}''_{10}$. In particular, by (6.19) we have $n^2 \geqslant 10m > 8m$.

If $C = \mathcal{L}'_{10}$ or $C = \mathcal{L}''_{10}$, then the base locus of $\mathcal{B}_4$ does not contain G-irreducible curves different from C by Lemma 5.10, so that Lemma 5.6 and (6.20) give

$$4n^2 - 40m = 4\left(n^2 - 10m\right) > |\Sigma|\left(\frac{n^2}{4} - 2m\right) \geqslant 20\left(\frac{n^2}{4} - 2m\right) = 5n^2 - 40m,$$

which is absurd. Thus, neither $C = \mathcal{L}'_{10}$ nor $C = \mathcal{L}''_{10}$ (that is, Σ is not contained in $\mathcal{L}'_{10}$ and in $\mathcal{L}''_{10}$). We see that $C = \mathcal{L}_{10}$.

By Lemma 5.10, the base locus of $\mathcal{B}_6$ does not contain G-irreducible curves different from C. If $|\Sigma| \neq 20$, then Lemma 5.6 implies that $|\Sigma| \geqslant 40$, so that (6.20) gives

$$6n^2 - 60m = 6\Big(n^2 - 10m\Big) > |\Sigma|\left(\frac{n^2}{4} - m\right) \geqslant 40\left(\frac{n^2}{4} - m\right) = 10n^2 - 40m,$$

which is absurd. We see that $C = \mathcal{L}_{10}$ and $|\Sigma| = 20$. Then $\Sigma \subset \mathcal{L}'_{10} \cup \mathcal{L}''_{10}$ by Lemma 5.6. However, we just proved that Σ is not contained in $\mathcal{L}'_{10}$ and $\mathcal{L}''_{10}$. □

Corollary 6.21 *One has* $|\Sigma| > 35$.

Proof Assume that $|\Sigma| \leqslant 35$. Let us seek for a contradiction.

Suppose that $G = \overline{G}_{144}$. Then Σ is not contained in $\mathcal{Q}$ by Lemma 6.16. Thus, Σ is one of two $\overline{G}_{144}$-orbits of length 12 by Lemma 4.10. This contradicts Corollary 6.15.

Thus, we see that $G = \overline{G}_{80}$. Then $|\Sigma| \geqslant 20$ by Corollary 6.15, so that $|\Sigma| = 20$. Hence, Σ is contained in one of the curves $\mathcal{L}_{10}$, $\mathcal{L}'_{10}$ or $\mathcal{L}''_{10}$ by Lemmas 5.5 and 5.6, which is impossible by Lemma 6.17. □

Now we are going to use the *multiplication by two trick* that was introduced in [7,8]. Recall that $P \in \Sigma$ is a center of non-canonical singularities of the log pair $(\mathbb{P}^3, \frac{4}{n}\mathcal{D})$, so that $\mathrm{mult}_P(\mathcal{D}) > \frac{n}{4}$ (see [13, Proposition 5.14]). This immediately implies that P is a center of non-log canonical singularities of the log pair $(\mathbb{P}^3, \frac{8}{n}\mathcal{D})$. In particular, the log pair $(\mathbb{P}^3, \frac{8}{n}\mathcal{D})$ is not log canonical at P. Thus, there is a positive rational number $\mu < \frac{8}{n}$ such that $(\mathbb{P}^3, \mu\mathcal{D})$ is log canonical at the point P, and it is not Kawamata log terminal at the point P. Of course, the same holds for every point in Σ, because Σ is the G-orbit of the point P, and the linear system $\mathcal{D}$ is G-invariant.

Lemma 6.22 *The log pair* $(\mathbb{P}^3, \mu\mathcal{D})$ *is Kawamata log terminal in a punctured neighborhood of every point in* Σ.

Proof Suppose that the required assertion is not true. Then there exists a G-irreducible curve C in $\mathbb{P}^3$ such that $\Sigma \subset C$ and $(\mathbb{P}^3, \mu\mathcal{D})$ is not Kawamata log terminal in every point of the curve C. Let D_1 and D_2 be two general surfaces in $\mathcal{D}$. Then

$$D_1 \cdot D_1 = mC + \Delta,$$

where m is a positive integer, and Δ is an effective one-cycle on $\mathbb{P}^3$ such that $\mathrm{Supp}(\Delta)$ does not contain irreducible components of the curve C. By [12, Theorem 3.1], one has

$$m \geqslant \frac{4}{\mu^2} > \frac{n^2}{16}.$$

Let H be a general plane in $\mathbb{P}^3$. Then

$$n^2 = H \cdot D_1 \cdot D_2 = H \cdot \Big(mC + \Delta\Big) = m \cdot \deg(C) + H \cdot \Delta$$

$$\geqslant m \cdot \deg(C) > \frac{n^2}{16} \cdot \deg(C),$$

so that $\deg(C) \leqslant 15$, which is impossible by Lemma 6.17. □

Let $\mathcal{I}$ be the multiplier ideal sheaf of the log pair $(\mathbb{P}^3, \mu\mathcal{D})$. Then the ideal $\mathcal{I}$ is not trivial, since $(\mathbb{P}^3, \mu\mathcal{D})$ is not Kawamata log terminal. Thus, it defines a (non-empty) subscheme $\mathcal{L}$ of $\mathbb{P}^3$ such that the support of the subscheme $\mathcal{L}$ contains Σ. Since $\mathcal{D}$ is mobile, the subscheme $\mathcal{L}$ has no two-dimensional components. Moreover, the subscheme $\mathcal{L}$ is reduced in a neighborhood of every point in Σ, because the log pair $(\mathbb{P}^3, \mu\mathcal{D})$ is log canonical at every point of Σ. Furthermore, every point of Σ is an isolated irreducible component of the subscheme $\mathcal{L}$ by Lemma 6.22. On the other hand, it follows from Nadel vanishing theorem (see [30, Theorem 9.4.8]) that

$$h^1\Big(\mathcal{I} \otimes \mathcal{O}_{\mathbb{P}^3}(4)\Big) = 0.$$

Now using the exact sequence of sheaves

$$0 \longrightarrow \mathcal{I} \otimes \mathcal{O}_{\mathbb{P}^3}(4) \longrightarrow \mathcal{O}_{\mathbb{P}^3}(4) \longrightarrow \mathcal{O}_{\mathcal{L}} \otimes \mathcal{O}_{\mathbb{P}^3}(4) \longrightarrow 0,$$

we see that

$$|\Sigma| \leqslant h^0\Big(\mathcal{O}_{\mathcal{L}} \otimes \mathcal{O}_{\mathbb{P}^3}(4)\Big) \leqslant h^0\Big(\mathcal{O}_{\mathbb{P}^3}(4)\Big) = 35,$$

which is impossible by Corollary 6.21. The obtained contradiction completes the proof of Propositions 6.6 and 6.8. This in turns completes the proof of Theorems 1.1 and 1.3.

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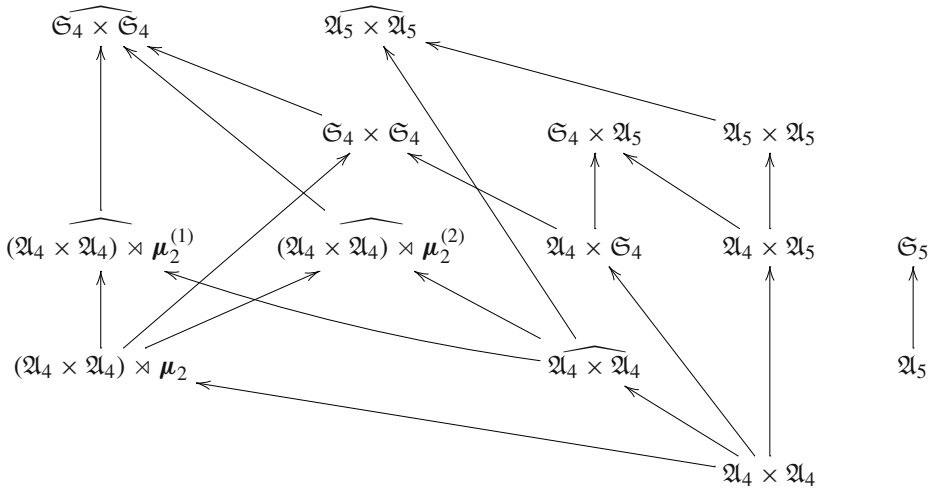
Appendix A: Finite primitive groups

In diagrams (A.1), (A.2), and (A.3) below, we present all the finite primitive subgroups of $\mathrm{PGL}_4(\mathbb{C})$ together with some inclusions between them, including those that are necessary for the proof of Theorem 1.1. We emphasize that we will not need the whole classification for the proof, but only the information about the few smallest groups.

By $\boldsymbol{\mu}_n$ we denote the cyclic group of order n. The inclusions are depicted by arrows going from a smaller group to a larger one. We point out that there are

indeed several additional inclusions (apart from those that are obtained merely by composing those inclusions that we list). For instance, the group $\mathfrak{A}_4 \times \mathfrak{A}_4$ from diagram (A.1) and the group $\mathfrak{A}_6$ from diagram (A.3) are both subgroups of $\boldsymbol{\mu}_2^4 \rtimes \mathfrak{A}_6$ from diagram (A.2).

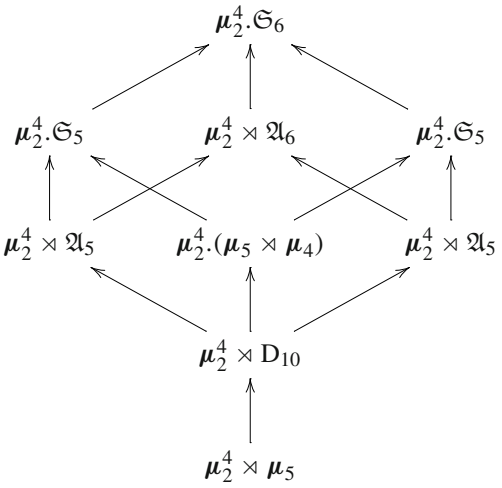
In diagram (A.1), we list all finite primitive subgroups of $\mathrm{PGL}_4(\mathbb{C})$ that leave invariant a quadric surface. The larger "connected component" of the diagram contains the groups 1°–12° described in [4, §§121, 122]; the smaller "connected component" contains the group (B) described in [4, §102] and the group (H) described in [4, §119]. We denote by $\widehat{G \times G}$ the group generated by $G \times G$ and an involution that exchanges the factors of $G \times G$. This group is isomorphic to $(G \times G) \rtimes \boldsymbol{\mu}_2$. However, we reserve the notation $(\mathfrak{A}_4 \times \mathfrak{A}_4) \rtimes \boldsymbol{\mu}_2$ not for a group of the latter type, but for a different subgroup of $\mathrm{PGL}_4(\mathbb{C})$ generated by $\mathfrak{A}_4$ and an element of order 2 that *does not* exchange the factors; this element can be thought of as a product of two elementary transpositions in the factors of $\mathfrak{S}_4 \times \mathfrak{S}_4 \supset \mathfrak{A}_4 \times \mathfrak{A}_4$. We refer the reader to [4, §121] for details. The above group is contained as a subgroup of index two in two other primitive subgroups of $\mathrm{PGL}_4(\mathbb{C})$ preserving a quadric surface so that in both cases an additional element of order 2 exchanges the factors of $\mathfrak{A}_4 \times \mathfrak{A}_4 \subset \widehat{(\mathfrak{A}_4 \times \mathfrak{A}_4) \rtimes \boldsymbol{\mu}_2}$, see [4, §122] for details. We denote these two groups by $\widehat{(\mathfrak{A}_4 \times \mathfrak{A}_4) \rtimes \boldsymbol{\mu}_2^{(1)}}$ and $\widehat{(\mathfrak{A}_4 \times \mathfrak{A}_4) \rtimes \boldsymbol{\mu}_2^{(2)}}$.



(A.1)

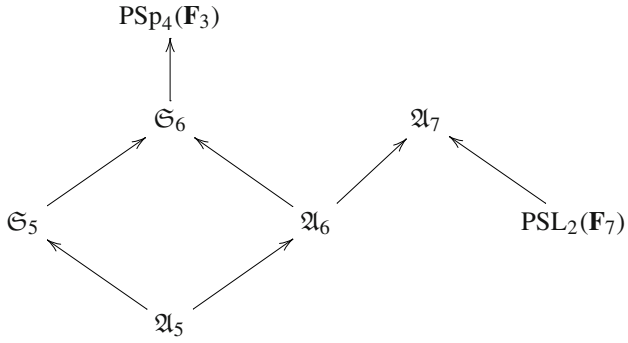
In diagram (A.2), we list all finite primitive subgroups of $\mathrm{PGL}_4(\mathbb{C})$ that contain a subgroup isomorphic to $\boldsymbol{\mu}_2^4$ except for those that leave invariant a quadric surface. These are the groups 13°–21° described in [4, §124]. We denote by D_{10} the dihedral group of order 10. By $F.G$ we mean a non-split extension of a group G by a group F.

(A.2)



Finally, in diagram (A.3), we list the remaining finite primitive subgroups of $\mathrm{PGL}_4(\mathbb{C})$. These are the groups (A), (C), (D), (E), and (F) described in [4, §102], and the groups (G) and (K) described in [4, §119].

(A.3)



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