Adv. Geom. **8** (2008), 441–450
DOI 10.1515/ADVGEOM.2008.028

Advances in Geometry

Nonrational del Pezzo fibrations

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(Communicated by A. Sommese)

Abstract. Let X be a general divisor in $|3M + nL|$ on the rational scroll $\mathrm{Proj}(\bigoplus_{i=1}^{4} \mathcal{O}_{\mathbb{P}^1}(d_i))$, where d_i and n are integers, M is the tautological line bundle, L is a fibre of the natural projection to $\mathbb{P}^1$, and $d_1 \geqslant \cdots \geqslant d_4 = 0$. We prove that X is rational $\iff$ $d_1 = 0$ and $n = 1$.

1 Introduction

The rationality problem for threefolds[1] splits in three cases: conic bundles, del Pezzo fibrations, and Fano threefolds. The cases of conic bundles and Fano threefolds are well studied.

Let $\psi\colon X \to \mathbb{P}^1$ be a fibration into del Pezzo surfaces of degree $k \geqslant 1$ such that X is smooth and $\mathrm{rk}\,\mathrm{Pic}(X) = 2$. Then X is rational if $k \geqslant 5$. The following result is due to [1] and [12].

Theorem 1.1. *Suppose that fibres of ψ are normal and $k = 4$. Then X is rational if and only if*

$$\chi(X) \in \{0, -8, -4\},$$

where $\chi(X)$ is the topological Euler characteristic.

The following result is due to [8].

Theorem 1.2. *Suppose that $K_X^2 \notin \mathrm{Int}\,\overline{\mathrm{NE}}(X)$ and $k \leqslant 2$. Then X is nonrational.*

In the case when $k \leqslant 2$ and $K_X^2 \in \mathrm{Int}\,\overline{\mathrm{NE}}(X)$, the threefold X belongs to finitely many deformation families, whose general members are nonrational (see [13], [7], [5], Proposition 1.5).

*The author would like to thank A. Corti, M. Grinenko, V. Iskovskikh, V. Shokurov for fruitful conversations.
[1] All varieties are assumed to be projective, normal, and defined over $\mathbb{C}$.

Suppose that $k=3$. Then X is a divisor in the linear system $|3M+nL|$ on the scroll

$$\mathrm{Proj}\Big(\bigoplus_{i=1}^{4}\mathcal{O}_{\mathbb{P}^1}(d_i)\Big),$$

where n and d_i are integers, M is the tautological line bundle, and L is a fibre of the natural projection to $\mathbb{P}^1$. Suppose that $d_1 \geqslant d_2 \geqslant d_3 \geqslant d_4 = 0$.

Suppose that X is a general[2] divisor in $|3M+nL|$. The following result is due to [8].

Theorem 1.3. *Suppose that* $K_X^2 \notin \mathrm{Int}\,\overline{\mathrm{NE}}(X)$. *Then* X *is nonrational.*

It follows from [4], [11], [2], [13], [3], [7] that X is nonrational when $(d_1,d_2,d_3,n) \in \{(0,0,0,2),(1,0,0,0),(2,1,1,-2),(1,1,1,-1)\}$.

We prove the following result in Section 3.

Theorem 1.4. *The threefold* X *is rational* $\iff$ $d_1 = 0$ *and* $n = 1$.

Therefore, the threefold X is nonrational if $\chi(X) \neq -14$. Indeed, we have

$$\chi(X) = -4K_X^3 - 54 = -4\big(18-6(d_1+d_2+d_3)-8n\big)-54 = 18-24(d_1+d_2+d_3)-32n,$$

and $\chi(X)=-14$ implies $(d_1,d_2,d_3,n)=(0,0,0,1)$ or $(d_1,d_2,d_3,n)=(2,1,1,-2)$.

The inequality $5n \geqslant 12-3(d_1+d_2+d_3)$ holds when $K_X^2 \notin \mathrm{Int}\,\overline{\mathrm{NE}}(X)$. For $n<0$, the inequality

$$5n \geqslant 12-3(d_1+d_2+d_3)$$

implies that $K_X^2 \notin \mathrm{Int}\,\overline{\mathrm{NE}}(X)$ (see Lemma 36 in [3]). Hence, the threefold X does not belong to finitely many deformation families in the case when $K_X^2 \in \mathrm{Int}\,\overline{\mathrm{NE}}(X)$ (see Section 2).

Let us illustrate our methods by proving the following result.

Proposition 1.5. *Let* X *be double cover of the scroll*

$$\mathrm{Proj}\big(\mathcal{O}_{\mathbb{P}^1}(2)\oplus\mathcal{O}_{\mathbb{P}^1}(2)\oplus\mathcal{O}_{\mathbb{P}^1}\big)$$

that is branched over a general[3] *divisor* $D \in |4M-2L|$, *where* M *is the tautological line bundle, and* L *is a fibre of the natural projection to* $\mathbb{P}^1$. *Then* X *is nonrational.*

Proof. Put $V = \mathrm{Proj}(\mathcal{O}_{\mathbb{P}^1}(2)\oplus\mathcal{O}_{\mathbb{P}^1}(2)\oplus\mathcal{O}_{\mathbb{P}^1})$. The divisor D is given by the equation

$$\alpha_6x_1^4+\alpha_6^1x_1^3x_2+\alpha_4x_1^3x_3+\alpha_6^2x_1^2x_2^2+\alpha_4^1x_1^2x_2x_3+\alpha_2x_1^2x_3^2+\alpha_6^3x_1x_2^3+ \\ +\alpha_4^2x_1x_2^2x_3+\alpha_2^1x_1x_2x_3^2+\alpha_0x_1x_3^3+\alpha_6^4x_2^4+\alpha_4^3x_2^3x_3+\alpha_2^2x_2^2x_3^2+\alpha_0^1x_2x_3^3=0$$

in bihomogeneous coordinates on V (see § 2.2 in [10]), where $\alpha_d^i = \alpha_d^i(t_1,t_2)$ is a sufficiently general homogeneous polynomial of degree $d \geqslant 0$. Let

$$\chi\colon Y \longrightarrow \mathrm{Proj}\big(\mathcal{O}_{\mathbb{P}^1}(2)\oplus\mathcal{O}_{\mathbb{P}^1}(2)\oplus\mathcal{O}_{\mathbb{P}^1}\big)$$

[2,3] A complement to a countable union of Zariski closed subsets.

be a double cover branched over a divisor $\Delta \subset V$ that is given by the same bihomogeneous equation as of divisor D with the only exception that $\alpha_0 = \alpha_0^1 = 0$. Then Y is singular, because the divisor Δ is singular along the curve $Y_3 \subset V$ that is given by the equations $x_1 = x_2 = 0$.

The Bertini theorem implies the smoothness of Δ outside of the curve Y_3. Let C be a curve on the threefold Y such that $\chi(C) = Y_3$. Then the threefold Y has singularities of type $\mathbb{A}_1 \times \mathbb{C}$ at general point of the curve C. We may assume that the system

$$\alpha_2(t_1, t_2) = \alpha_2^1(t_1, t_2) = \alpha_2^2(t_1, t_2) = 0$$

has no non-trivial solutions. Then Y has singularities of type $\mathbb{A}_1 \times \mathbb{C}$ at every point of C.

Let $\alpha\colon \tilde{V} \to V$ be the blow up of Y_3, and $\beta\colon \tilde{Y} \to Y$ be the blow up of C. Then the diagram

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\tilde{\chi}} & \tilde{V} \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\alpha} \\ Y & \xrightarrow{\chi} & V \end{array}$$

commutes, where $\tilde{\chi}\colon \tilde{Y} \to \tilde{V}$ is a double cover. The threefold $\tilde{Y}$ is smooth.

Let E be the exceptional divisor of α, and $\tilde{\Delta}$ be the proper transform of Δ via α. Then

$$\tilde{\Delta} \sim \alpha^*(4M - 2L) - 2E,$$

hence $\tilde{\Delta}$ is nef and big, because the pencil $|\alpha^*(M - 2L) - E|$ does not have base points. The morphism $\tilde{\chi}$ is branched over $\tilde{\Delta}$. Then $\mathrm{rk}\,\mathrm{Pic}(\tilde{Y}) = 3$ by Theorem 2 in [9].

The linear system $|g^*(M - L) - E|$ does not have base points and gives a $\mathbb{P}^1$-bundle

$$\tau\colon \tilde{V} \longrightarrow \mathrm{Proj}\,\big(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(2)\big) \cong \mathbb{F}_0,$$

which induces a conic bundle $\tilde{\tau} = \tau \circ \tilde{\chi}\colon \tilde{Y} \to \mathbb{F}_0$. Let $Y_2 \subset V$ be the subscroll given by $x_1 = 0$, and S be a proper transform of Y_2 via α. Then

$$Y_2 \cong \mathrm{Proj}\,\big(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}\big) \cong \mathbb{F}_2,$$

and $S \cong Y_2$. But τ maps S to the section of $\mathbb{F}_0$ that has trivial self-intersection.

Let $\tilde{S}$ be a surface in $\tilde{Y}$ such that $\tilde{\chi}(\tilde{S}) = S$, and $Z \subset \tilde{Y}$ be a general fibre of the natural projection to $\mathbb{P}^1$. Then $-K_Z$ is nef and big and $K_Z^2 = 2$. But the morphism

$$\alpha \circ \tilde{\chi}|_{\tilde{S}}\colon \tilde{S} \longrightarrow Y_2$$

is a double cover branched over a divisor that is cut out by the equation

$$\alpha_6^4(t_0, t_1)x_2^2 + \alpha_4^3(t_0, t_1)x_2x_3 + \alpha_2^2(t_0, t_1)x_3^2 = 0.$$

Let $\Xi \subset \mathbb{F}_0$ be a degeneration divisor of the conic bundle $\tilde{\tau}$. Then

$$\Xi \sim \lambda\tilde{\tau}(\tilde{S}) + \mu\tilde{\tau}(Z),$$

where λ and μ are integers. But $\lambda = 6$, because $K_Z^2 = 2$. We have $\tilde{\tau}(\tilde{S}) \not\subset \Xi$. Then

$$\mu = \tilde{\tau}(\tilde{S}) \cdot \Xi = 8 - K_{\tilde{S}}^2,$$

because μ is the number of reducible fibres of the conic bundle $\tilde{\tau}|_{\tilde{S}}$. These fibers are given by

$$\big(\alpha_4^3(t_0, t_1)\big)^2 = 4\alpha_2^2(t_0, t_1)\alpha_6^4(t_0, t_1),$$

which implies that $\mu = \tilde{\tau}(\tilde{S}) \cdot \Xi = 8$. Then $\tilde{Y}$ is nonruled by Theorem 10.2 in [11], which implies the nonrationality of the threefold X by Theorem 1.8.3 in § IV of the book [6]. □

2 Preliminaries

All results of this section follow from [3]. Take a scroll

$$V = \operatorname{Proj}\Big(\bigoplus_{i=1}^{4} \mathcal{O}_{\mathbb{P}^1}(d_i)\Big),$$

where d_i is an integer, and $d_1 \geqslant d_2 \geqslant d_3 \geqslant d_4 = 0$. Let M and L be the tautological line bundle and a fibre of the natural projection to $\mathbb{P}^1$, respectively. Then $\operatorname{Pic}(V) = \mathbb{Z}M \oplus \mathbb{Z}L$.

Let $(t_1 : t_2; x_1 : x_2 : x_3 : x_k)$ be bihomogeneous coordinates on V such that $x_i = 0$ defines a divisor in $|M - d_iL|$, and L is given by $t_1 = 0$. Then $|aM + bL|$ is spanned by divisors

$$c_{i_1 i_2 i_3 i_4}(t_1, t_2)x_1^{i_1}x_2^{i_2}x_3^{i_3}x_k^{i_4} = 0,$$

where $\sum_{j=1}^{4} i_j = a$ and $c_{i_1 i_2 i_3 i_4}(t_1, t_2)$ is a homogeneous polynomial of degree $b + \sum_{j=1}^{4} i_j d_j$. Let $Y_j \subseteq V$ be a subscroll $x_1 = \cdots = x_{j-1} = 0$. The following result holds (see § 2.8 in [10]).

Corollary 2.1. *Take $D \in |aM + bL|$ and $q \in \mathbb{N}$, where a and b are integers. Then*

$$\operatorname{mult}_{Y_j}(D) \geqslant q \iff ad_j + b + (d_1 - d_j)(q - 1) < 0.$$

Let X be a general[4] divisor in $|3M + nL|$, where n is an integer.

Lemma 2.2. *Suppose X is smooth and $\operatorname{rk}\operatorname{Pic}(X) = 2$. Then $d_1 \geqslant -n$ and $3d_3 \geqslant -n$.*

Proof. We see that $Y_2 \not\subset X$. Then $Y_3 \not\subset X$, because $\operatorname{rk}\operatorname{Pic}(X) = 2$. But $\operatorname{mult}_{Y_4}(X) \leqslant 1$, because the threefold X is smooth. The assertion of Corollary 2.1 concludes the proof. □

Lemma 2.3. *Suppose X is smooth and $\operatorname{rk}\operatorname{Pic}(X) = 2$. Then we have either $d_1 = -n$ or $d_2 \geqslant -n$.*

[4] A complement to a Zariski closed subset in moduli.

Proof. Suppose that $r = d_1 + n > 0$ and $d_2 < -n$. Then X can be given by the equation

$$\sum_{\substack{i,j,k\geqslant 0\\ i+j+k=2}} \gamma_{ijk}(t_0,t_2)x_1^i x_2^j x_3^k x_4 = \alpha_r(t_1,t_2)x_1x_4^2 + \sum_{\substack{i,j,k\geqslant 0\\ i+j+k=3}} \beta_{ijk}(t_0,t_2)x_1^i x_2^j x_3^k,$$

where $\alpha_r(t_1,t_2)$ is a homogeneous polynomial of degree r, β_{ijk} and γ_{ijk} are homogeneous polynomial of degree $n + id_1 + jd_2 + kd_3$. Then every point of the intersection

$$x_1 = x_2 = x_3 = \alpha_r(t_1,t_2) = 0$$

must be singular on the threefold X, which is a contradiction. □

Lemma 2.4. *Suppose X is smooth, $d_2 = d_3$, $n < 0$ and $\operatorname{rk}\operatorname{Pic}(X) = 2$. Then $3d_3 \neq -n$.*

Proof. Suppose that $3d_3 = -n$. Then X can be given by the the bihomogeneous equation

$$\sum_{\substack{j,k,l\geqslant 0\\ i+j+k=2}} \gamma_{jkl}(t_0,t_2)x_1 x_2^j x_3^k x_4^l = f_3(x_2,x_3) + \alpha_r(t_0,t_2)x_1^3 + \sum_{\substack{j,k,l\geqslant 0\\ j+k+l=1}} \beta_{jkl}(t_0,t_2)x_1^2 x_2^j x_3^k x_4^l,$$

where $f_3(x_2,x_3)$ is a homogeneous polynomial of degree 3, β_{jkl} and γ_{jkl} are homogeneous polynomial of degree $n + 2d_1 + jd_2 + kd_3$ and $n + d_1 + jd_2 + kd_3$ respectively, α_r is a homogeneous polynomial of degree $r = 3d_1 + n$. The threefold X contains 3 subscrolls given by the equations $x_1 = f_3(x_2,x_3) = 0$, which is impossible, because $\operatorname{rk}\operatorname{Pic}(X) = 2$. □

The following result follows from Lemmas 2.2, 2.3 and 2.4.

Lemma 2.5. *The threefold X is smooth and $\operatorname{rk}\operatorname{Pic}(X) = 2$ whenever*

(1) *in the case when $d_1 = 0$, the inequality $n > 0$ holds,*
(2) *either $d_1 = -n$ and $3d_3 \geqslant -n$, or $d_1 > -n$, $d_2 \geqslant -n$ and $3d_3 \geqslant -n$,*
(3) *in the case when $d_2 = d_3$ and $n < 0$, the inequality $3d_3 > -n$ holds.*

Proof. Suppose that all these conditions are satisfied. We must show that X is smooth, because the equality $\operatorname{rk}\operatorname{Pic}(X) = 2$ holds by Proposition 32 in [3].

The linear system $|3M + nL|$ does not have base points if $n \geqslant 0$. So, the threefold X is smooth by the Bertini theorem in the case $n \geqslant 0$. Therefore, we may assume that $n < 0$.

The base locus of $|3M + nL|$ consists of the curve Y_4, which implies that X is smooth outside of the curve Y_4 and in a general point of Y_4 by the Bertini theorem and Corollary 2.1, respectively.

In the case when $d_1 = -n$ and $d_2 < -n$, the bihomogeneous equation of the threefold X is

$$\sum_{\substack{i,j,k\geqslant 0\\ i+j+k=2}} \gamma_{ijk}(t_0,t_2)x_1^i x_2^j x_3^k x_4 = \alpha_0 x_1 x_4^2 + \sum_{\substack{i,j,k\geqslant 0\\ i+j+k=3}} \beta_{ijk}(t_0,t_2)x_1^i x_2^j x_3^k,$$

where β_{ijk} and γ_{ijk} are homogeneous polynomials of degree $n + id_1 + jd_2 + kd_3$ and α_0 is a nonzero constant. The curve Y_4 is given by $x_1 = x_2 = x_3 = 0$, which implies that X is smooth.

In the case when $d_1 > -n$ and $d_2 \geqslant -n$, the bihomogeneous equation of X is

$$\sum_{\substack{i,j,k\geqslant 0\\ i+j+k=2}} \gamma_{ijk}(t_0,t_2)x_1^i x_2^j x_3^k x_4 = \sum_{i=1}^{3} \alpha_i(t_0,t_2)x_i x_4^2 + \sum_{\substack{i,j,k\geqslant 0\\ i+j+k=3}} \beta_{ijk}(t_0,t_2)x_1^i x_2^j x_3^k,$$

where α_i is a homogeneous polynomial of degree $d_i + n$, and β_{ijk} and γ_{ijk} are homogeneous polynomials of degree $n + id_1 + jd_2 + kd_3$. Therefore, either $\alpha_1 x_1 x_4^2$ or $\alpha_2 x_2 x_4^2$ does not vanish at any given point of the curve Y_4, which implies that X is smooth. □

Thus, there is an infinite series of quadruples (d_1, d_2, d_3, n) such that the threefold X is smooth, the equality $\operatorname{rk}\operatorname{Pic}(X) = 2$ holds, the inequality $5n < 12 - 3(d_1 + d_2 + d_3)$ holds and $n < 0$.

3 Nonrationality

We use the notation of Section 2. Let X be a general[5] divisor in $|3M + nL|$, and suppose that the threefold X is smooth, $\operatorname{rk}\operatorname{Pic}(X) = 2$, and X is rational. Let us show that $d_1 = 0$ and $n = 1$.

The threefold X is given by a bihomogeneous equation

$$\sum_{l=0}^{3} \alpha_i(t_0,t_2)x_3^i x_4^{3-i} + x_1 F(t_0,t_1,x_1,x_2,x_3,x_4) + x_2 G(t_0,t_1,x_1,x_2,x_3,x_4) = 0,$$

where α_i is a general homogeneous polynomial of degree $n + id_3$, and F and G stand for

$$\sum_{\substack{i,j,k,l\geqslant 0\\ i+j+k+l=2}} \beta_{ijkl}(t_0,t_2)x_1^i x_2^j x_3^k x_4^l \quad \text{and} \quad \sum_{\substack{i,j,k,l\geqslant 0\\ i+j+k+l=2}} \gamma_{ijkl}(t_0,t_2)x_1^i x_2^j x_3^k x_4^l$$

respectively, where β_{ijkl} is a general homogeneous polynomial of degree $n + (i+1)d_1 + jd_2 + kd_3$, and γ_{ijkl} is a general homogeneous polynomial of degree $n + id_1 + (j + 1)d_2 + kd_3$.

Let Y be a threefold given by $x_1 F + x_2 G = 0$. Then $Y_3 \subset Y$, where Y_3 is given by $x_1 = x_2 = 0$.

Lemma 3.1. *The threefold Y has $2d_1 + 2d_2 + 4d_3 + 4n > 0$ isolated ordinary double points.*

Proof. The threefold Y is singular exactly at the points of V where

$$x_1 = x_2 = F(t_0,t_1,x_1,x_2,x_3,x_4) = G(t_0,t_1,x_1,x_2,x_3,x_4) = 0$$

by the Bertini theorem. But $Y_3 \cong \operatorname{Proj}(\mathcal{O}_{\mathbb{P}^1}(d_3) \oplus \mathcal{O}_{\mathbb{P}^1}) \cong \mathbb{F}_{d_3}$, where $(t_0 : t_1; x_3 : x_4)$ can be considered as natural bihomogeneous coordinates on the surface Y_3.

[5] A complement to a countable union of Zariski closed subsets.

Let C and Z be the curves on Y_3 that are cut out by the equations $F = 0$ and $G = 0$, respectively. Then C and Z are given by the equations

$$\sum_{\substack{k,l\geqslant 0\\ k+l=2}} \beta_{kl}(t_0,t_2)x_3^k x_4^l = 0 \quad \text{and} \quad \sum_{\substack{k,l\geqslant 0\\ k+l=2}} \gamma_{kl}(t_0,t_2)x_3^k x_4^l = 0$$

respectively, where $\beta_{kl} = \beta_{00kl}$ and $\gamma_{kl} = \gamma_{00kl}$. The degrees of β_{kl} and γ_{kl} are $n + d_1 + kd_3$ and $n + d_2 + kd_3$, respectively.

Let O be a point of the scroll V such that the set

$$x_1 = x_2 = F(t_0,t_1,x_1,x_2,x_3,x_4) = G(t_0,t_1,x_1,x_2,x_3,x_4) = 0$$

contains the point O. Then $O \in C \cap Z$ and $O \in \operatorname{Sing}(Y)$. It is easy to see that O is an isolated ordinary double point of the threefold Y in the case when the curves C and Z are smooth and intersect each other transversally at the point O.

Put $\bar{M} = M|_{Y_3}$ and $\bar{L} = L|_{Y_3}$. Then $C \in |2\bar{M} + (n + d_1)\bar{L}|$ and $Z \in |2\bar{M} + (n + d_2)\bar{L}|$. But

$$\big|2\bar{M} + (n + d_1)\bar{L}\big|$$

does not have base points, because $d_1 + n \geqslant 0$ by Lemma 2.2. So, the curve C is smooth.

The linear system $|2\bar{M} + (n + d_2)\bar{L}|$ may have base components, and Z may not be reduced or irreducible. We have to show that C intersects Z transversally at smooth points of Z, because

$$\big|C \cap Z\big| = C \cdot Z = 2d_1 + 2d_2 + 4d_3 + 4n,$$

where $2d_1 + 2d_2 + 4d_3 + 4n > 0$ by Lemmas 2.2, 2.3 and 2.4.

Suppose that $d_1 > -n$. Then $d_2 \geqslant -n$ by Lemma 2.3. We see that $|2\bar{M} + (n+d_2)\bar{L}|$ does not have base points. Then Z is smooth and C intersects Z transversally at every point of $C \cap Z$.

We may assume that $d_1 = -n$. Let $Y_4 \subset Y_3$ be a curve given by $x_3 = 0$. Then

$$C \cap Y_4 = \emptyset,$$

and either the linear system $|2\bar{M} + (n+d_2)\bar{L}|$ does not have base points, or the base locus of the linear system $|2\bar{M} + (n + d_2)\bar{L}|$ consist of the curve Y_4. However, we have

$$C \cap Z \subset Y_3 \setminus Y_4,$$

which implies that C intersects the curve Z transversally at smooth points of Z. □

Let $\pi\colon \tilde{V} \to V$ be the blow up of Y_3, and $\tilde{Y}$ be and the proper transforms of Y via π. Then

$$\tilde{Y} \sim \pi^*(3M + nL) - E,$$

where E is and exceptional divisor of π. The threefold $\tilde{Y}$ is smooth.

Lemma 3.2. *The equality* $\operatorname{rk}\operatorname{Pic}(\tilde{Y}) = 3$ *holds.*

Proof. The linear system $|\pi^*(M-d_2L)-E|$ does not have base points. Thus, the divisor

$$\tilde{Y} \sim \pi^*(3M+nL)-E$$

is nef and big when $n \geqslant 0$ by Lemmas 2.2, 2.3 and 2.4. Hence, the equality $\operatorname{rk}\operatorname{Pic}(\tilde{Y})=3$ holds in the case when $n \geqslant 0$ by Theorem 2 in [9]. So, we may assume that $n<0$.

Let $\omega\colon \tilde{Y} \to \mathbb{P}^1$ be the natural projection and S be the generic fibre of ω, which is considered as a surface defined over the function field $\mathbb{C}(t)$. Then S is a smooth cubic surface in $\mathbb{P}^3$, which contains a line in $\mathbb{P}^3$ defined over the field $\mathbb{C}(t)$, because $Y_3 \subset Y$. Then $\operatorname{rk}\operatorname{Pic}(S) \geqslant 2$.

To conclude the proof we must prove that $\operatorname{rk}\operatorname{Pic}(S)=2$, because there is an exact sequence

$$0 \longrightarrow \mathbb{Z}\big[\pi^*(L)\big] \longrightarrow \operatorname{Pic}(\tilde{Y}) \longrightarrow \operatorname{Pic}(S) \longrightarrow 0,$$

because every fibre of τ is reduced and irreducible (see the proof of Proposition 32 in [3]).

Let $\breve{S}$ be an example of the surface S that is given by the equation

$$x(q(t)x^2+p(t)w^2)+y(r(t)y^2+s(t)z^2)=0 \subset \operatorname{Proj}\big(\mathbb{C}[x,y,z,t]\big),$$

where $q(t)$, $p(t)$, $r(t)$, $s(t)$ are polynomials such that the inequalities

$$\deg(q(t))>0, \quad \deg(p(t))\geqslant 0, \quad \deg(r(t))>0, \quad \deg(q(t))\geqslant 0$$

hold. The existence of the surface $\breve{S}$ follows from the equation of the threefold Y.

Let $\mathbb{K}$ be an algebraic closure of the field $\mathbb{C}(t)$, let L be a line $x=y=0$, and let

$$\gamma\colon \breve{S} \to \mathbb{P}^1$$

be a projection from L. Then γ is a conic bundle defined over $\mathbb{C}(t)$. But γ has five geometrically reducible fibres F_1, F_2, F_3, F_4, F_5 defined over $\mathbb{F}$ such that

- $F_i=\tilde{F}_i \cup \bar{F}_i$, where $\tilde{F}_i$ and $\bar{F}_i$ are geometrically irreducible curves,
- the curve $L \cup F_i$ is cut out on the surface $\breve{S}$ by the equation

$$y=\varepsilon^i \sqrt[3]{q(t)/r(t)}x,$$

  where $\varepsilon=-(1+\sqrt{-3})/2$ and $i \in \{1,2,3\}$,
- the curve $F_4 \cup L$ is cut out on the surface $\breve{S}$ by the equation $x=0$,
- the curve $F_5 \cup L$ is cut out on the surface $\breve{S}$ by the equation $y=0$.

The group $\operatorname{Gal}(\mathbb{K}/\mathbb{C}(t))$ acts naturally on the set

$$\Sigma=\big\{\tilde{F}_1,\tilde{F}_2,\tilde{F}_3,\tilde{F}_4,\tilde{F}_5,\bar{F}_1,\bar{F}_2,\bar{F}_3,\bar{F}_4,\bar{F}_5\big\},$$

because the conic bundle γ is defined over $\mathbb{C}(t)$. The inequality $\operatorname{rk}\operatorname{Pic}(\breve{S})>2$ implies the existence of a subset $\Gamma \subsetneq \Sigma$ consisting of disjoint curves such that $\Gamma \subsetneq \Sigma$ is $\operatorname{Gal}(\mathbb{K}/\mathbb{C}(t))$-invariant.

The action of $\operatorname{Gal}(\mathbb{K}/\mathbb{C}(t))$ on the set Σ is easy to calculate explicitly. Putting

$$\Delta=\big\{\tilde{F}_1,\tilde{F}_2,\tilde{F}_3,\bar{F}_1,\bar{F}_2,\bar{F}_3\big\}, \quad \Lambda=\big\{\tilde{F}_4,\bar{F}_4\big\}, \quad \Xi=\big\{\tilde{F}_5,\bar{F}_5\big\},$$

we see that the group $\mathrm{Gal}(\mathbb{K}/\mathbb{C}(t))$ acts transitively on each subset Λ, Ξ, Δ, because we may assume that $q(t)$, $p(t)$, $r(t)$, $s(t)$ are sufficiently general. But each subset Λ, Ξ, Δ does not consist of disjoint curves. Hence, the equality $\mathrm{rk}\,\mathrm{Pic}(\breve{S}) = 2$ holds, which implies that $\mathrm{rk}\,\mathrm{Pic}(S) = 2$. □

The linear system $|\pi^*(M - d_2L) - E|$ does not have base points and induces a $\mathbb{P}^2$-bundle

$$\tau\colon \tilde{V} \longrightarrow \mathrm{Proj}\,\big(\mathcal{O}_{\mathbb{P}^1}(d_1) \oplus \mathcal{O}_{\mathbb{P}^1}(d_2)\big) \cong \mathbb{F}_r,$$

where $r = d_1 - d_2$. Let l be a fibre of the natural projection $\mathbb{F}_r \to \mathbb{P}^1$, and s_0 be an irreducible curve on the surface $\mathbb{F}_r$ such that $s_0^2 = r$, and s_0 is a section of the projection $\mathbb{F}_r \to \mathbb{P}^1$. Then

$$\pi^*(M - d_2L) - E \sim \tau^*(s_0)$$

and $\pi^*(L) \sim \tau^*(l)$. The morphism τ induces a conic bundle $\tilde{\tau} = \tau|_{\tilde{Y}}\colon \tilde{Y} \to \mathbb{F}_r$.

Let Δ be the degeneration divisor of the conic bundle $\tilde{\tau}$. Then

$$\Delta \sim 5s_\infty + \mu l,$$

where μ is a natural number, and s_∞ is the exceptional section of the surface $\mathbb{F}_r$.

Let S be a surface in $\tilde{Y}$ and B be a threefold in $\tilde{V}$ dominating the curve s_0. Then

$$B \cong \mathrm{Proj}\,\big(\mathcal{O}_{\mathbb{P}^1}(d_1) \oplus \mathcal{O}_{\mathbb{P}^1}(d_3) \oplus \mathcal{O}_{\mathbb{P}^1}\big)$$

and $\pi(B) \cong B$. But $\pi(B) \cap Y = \pi(S) \cup Y_3$.

The surface Y_3 is cut out on $\pi(B)$ by the equation $x_1 = 0$, where $\pi(B) \in |M - d_2L|$. We have

$$S \sim 2T + (d_1 + n)F,$$

where T is a tautological line bundle on B, and F is a fibre of the projection $B \to \mathbb{P}^1$. Then

$$K_S^2 = -5d_1 + 2d_3 - 4d_2 - 3n + 8$$

and $\mu = s_0 \cdot \Delta = 5d_1 - 2d_3 + 4d_2 + 3n$.

It follows from the equivalence $2K_{\mathbb{F}_r} + \Delta \sim s_\infty + (3d_1 - 2d_3 + 6d_2 + 3n - 4)l$ that

$$\big|2K_{\mathbb{F}_r} + \Delta\big| \neq \emptyset \iff 3d_1 - 2d_3 + 6d_2 + 3n \geqslant 4,$$

which implies that Y is nonrational by Theorem 10.2 in [11] if $3d_1 - 2d_3 + 6d_2 + 3n \geqslant 4$.

The threefold Y is nonruled if and only if it is nonrational, because the threefold Y is rationally connected. So, the threefold X is nonrational by Theorem 1.8.3 in § IV of the book [6] whenever

$$3d_1 - 2d_3 + 6d_2 + 3n \geqslant 4,$$

which implies that $3d_1 - 2d_3 + 6d_2 + 3n < 4$, because we assume that X is rational.

We see that either $d_1 = 0$ and $n = 1$ or $d_1 = 1$ and $d_2 = n = 0$ by Lemmas 2.2, 2.3 and 2.4, but the threefold X is birational to a smooth cubic threefold in the case when $d_1 = 1$ and $d_2 = n = 0$, which is nonrational by [4]. Then $d_1 = 0$ and $n = 1$. The assertion of Theorem 1.4 is proved.

References

[1] V. A. Alekseev, On conditions for the rationality of three-folds with a pencil of del Pezzo surfaces of degree 4. *Mat. Zametki* **41** (1987), 724–730, 766. MR898133 (88j:14048) Zbl 0623.14019

[2] F. Bardelli, Polarized mixed Hodge structures: on irrationality of threefolds via degeneration. *Ann. Mat. Pura Appl.* (4) **137** (1984), 287–369. MR772264 (86m:14030) Zbl 0579.14033

[3] G. Brown, A. Corti, F. Zucconi, Birational geometry of 3-fold Mori fibre spaces. In: *The Fano Conference*, 235–275, Univ. Torino, Turin 2004. MR2112578 (2005k:14031) Zbl 1063.14019

[4] C. H. Clemens, P. A. Griffiths, The intermediate Jacobian of the cubic threefold. *Ann. of Math.* (2) **95** (1972), 281–356. MR0302652 (46 #1796) Zbl 0245.14010

[5] M. M. Grinenko, Fibrations into del Pezzo surfaces. *Uspekhi Mat. Nauk* **61** (2006), no. 2(368), 67–112. MR2261543 (2007g:14009) Zbl 1124.14020

[6] J. Kollár, *Rational curves on algebraic varieties*. Springer 1996. MR1440180 (98c:14001) Zbl 0877.14012

[7] V. V. Przhiyalkovskiĭ, I. A. Chel'tsov, K. A. Shramov, Hyperelliptic and trigonal Fano threefolds. *Izv. Ross. Akad. Nauk Ser. Mat.* **69** (2005), 145–204. MR2136260 (2006e:14059) Zbl 1081.14059

[8] A. V. Pukhlikov, Birational automorphisms of three-dimensional algebraic varieties with a pencil of del Pezzo surfaces. *Izv. Ross. Akad. Nauk Ser. Mat.* **62** (1998), 123–164. MR1622258 (99f:14016) Zbl 0948.14008

[9] G. V. Ravindra, V. Srinivas, The Grothendieck-Lefschetz theorem for normal projective varieties. *J. Algebraic Geom.* **15** (2006), 563–590. MR2219849 (2006m:14008) Zbl 1123.14004

[10] M. Reid, Chapters on algebraic surfaces. In: *Complex algebraic geometry* (*Park City, UT,* 1993), volume 3 of *IAS/Park City Math. Ser.*, 3–159, Amer. Math. Soc. 1997. MR1442522 (98d:14049) Zbl 0910.14016

[11] V. V. Shokurov, Prym varieties: theory and applications. *Izv. Akad. Nauk SSSR Ser. Mat.* **47** (1983), 785–855. MR712095 (85e:14040) Zbl 0572.14025

[12] K. A. Shramov, On the rationality of nonsingular threefolds with a pencil of del Pezzo surfaces of degree 4. *Mat. Sb.* **197** (2006), 133–144. MR2230135 (2007h:14015)

[13] C. Voisin, Sur la jacobienne intermédiaire du double solide d'indice deux. *Duke Math. J.* **57** (1988), 629–646. MR962523 (90f:14029) Zbl 0698.14049

Received 23 January, 2007; revised 25 June, 2007

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