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Weighted Fano threefold hypersurfaces

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Abstract. We study birational transformations into elliptic fibrations and birational automorphisms of quasismooth anticanonically embedded weighted Fano 3-fold hypersurfaces with terminal singularities classified by A. R. Iano-Fletcher, J. Johnson, J. Kollár, and M. Reid.

1. Introduction

Let S be a smooth cubic surface in $\mathbb{P}^3$ defined over a perfect field k with Picard rank 1. For example, the equation $2x^3 + 3y^3 + 5z^3 + 7w^3 = 0$ defines such a cubic in $\mathrm{Proj}(\mathbb{Q}[x, y, z, w])$ (see [16] or [20]). The condition that the Picard rank is one simply means that every curve on S defined over k is cut by some hypersurface in $\mathbb{P}^3$. The surface S is proved to be birationally rigid and hence nonrational (see [15]).

Let P and Q be distinct k-rational points on the surface S. We then consider the projection $\phi : S \dashrightarrow \mathbb{P}^2$ from the point P. Because the map ϕ is a double cover generically over $\mathbb{P}^2$, it induces a birational involution α of the surface S that interchanges two points of a generic fiber of the rational map ϕ. Traditionally, the involution α is called a Geiser involution.

Meanwhile, we consider the line $L \subset \mathbb{P}^3$ passing through the points P and Q. Then the line L meets the surface S at another k-rational point O. For a sufficiently general hyperplane H in $\mathbb{P}^3$ passing through the line L, the intersection $H \cap S$ is a smooth elliptic curve E. Then the reflection of the elliptic curve E centered at the point O induces a birational involution β of the surface S that is called a Bertini involution.

Yu. Manin proved the group $\mathrm{Bir}(S)$ of birational automorphisms of the surface S is generated by the group $\mathrm{Aut}(S)$ of biregular automorphisms and Bertini and Geiser involutions of the surface S, more precisely, the sequence of groups

$$1 \to \Gamma_S \to \mathrm{Bir}(S) \to \mathrm{Aut}(S) \to 1$$

is exact, where Γ_S is the group generated by Bertini and Geiser involutions. Furthermore, he also described all the relations among these involutions (see [16]). These properties mentioned so far remain true for smooth del Pezzo surfaces of degrees 1 and 2 with Picard rank 1.

Moreover, on a smooth del Pezzo surface of degree 2, the group Γ_S is the free product of involutions. But in the case of degree 1, every birational automorphism is biregular (see [15]).

Smooth del Pezzo surfaces of degree 1, 2 and 3 are the only smooth del Pezzo surfaces that can be anticanonically embedded into weighted projective spaces as quasismooth hypersurfaces. Therefore, the properties described above can be naturally expected on anticanonically embedded quasismooth weighted Fano 3-fold hypersurfaces with terminal singularities. The first step in this direction is done in [10], where the birational superrigidity of smooth quartic 3-folds is proved.

Smooth quartic 3-folds are the first example of quasismooth anticanonically embedded weighted Fano 3-fold hypersurfaces with terminal singularities that were completely classified into 95 families by A. R. Iano-Fletcher, J. Johnson, J. Kollár, and M. Reid (see [9] and [11]) and which were studied quite extensively in [6] and [18].

Throughout this paper, we always let $X \subset \mathbb{P}(1, a_2, a_3, a_4, a_5)$ be a sufficiently general quasismooth anticanonically embedded Fano hypersurface of degree d and of type N with terminal singularities[1)], where the notation N is the entry number in Table 1 of Appendix.

The hypersurface X is proved to be rationally connected (see [22]) and birationally rigid (see [6]). Furthermore, it follows from [6] that the sequence of groups

$$1 \to \Gamma_X \to \mathrm{Bir}(X) \to \mathrm{Aut}(X) \to 1$$

is exact, where the group Γ_X is a subgroup of $\mathrm{Bir}(X)$ generated by a finite set of distinct birational involutions $\tau_1, \dots, \tau_\ell$ explicitly described in [6]. All the involutions here are either an elliptic involution or a quadratic involution. The former is a generalization of a Bertini involution and the latter is that of Geiser involution.

Even though the paper [6] describes the number of the birational involutions $\tau_1, \dots, \tau_\ell$ and their explicit constructions, the relations among them have been in question. We show that the group Γ_X has exactly one of the following group presentations:

$$\begin{aligned}
\mathbf{F}^0 &= \text{the trivial group},\\
\mathbf{F}^1 &= \langle \tau_1 \,|\, \tau_1^2 = 1 \rangle,\\
\mathbf{F}^2 &= \langle \tau_1, \tau_2 \,|\, \tau_1^2 = \tau_2^2 = 1 \rangle,\\
\mathbf{F}^3 &= \langle \tau_1, \tau_2, \tau_3 \,|\, \tau_1^2 = \tau_2^2 = \tau_3^2 = 1 \rangle,\\
\hat{\mathbf{F}}^3 &= \langle \tau_1, \tau_2, \tau_3 \,|\, \tau_1^2 = \tau_2^2 = \tau_3^2 = \tau_1\tau_2\tau_3\tau_1\tau_2\tau_3 = 1 \rangle,\\
\mathbf{F}^5 &= \langle \tau_1, \tau_2, \tau_3, \tau_4, \tau_5 \,|\, \tau_1^2 = \tau_2^2 = \tau_3^2 = \tau_4^2 = \tau_5^2 = 1 \rangle,
\end{aligned}$$

where the generator τ_i comes from an involution of X and the group operation from the composition of maps. When the group Γ_X is trivial, the 3-fold X is birationally superrigid.

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1) The weighted projective space $\mathrm{Proj}(\mathbb{F}[x_1, x_2, \dots, x_n])$ defined over an arbitrary field $\mathbb{F}$ with $\mathrm{wt}(x_i) = a_i$ is denoted by $\mathbb{P}_{\mathbb{F}}(a_1, a_2, \dots, a_n)$. The weights a_i are always assumed that $a_1 \leqq a_2 \leqq \cdots \leqq a_n$. When the field of definition is clear, we use simply the notation $\mathbb{P}(a_1, a_2, \dots, a_n)$ instead of $\mathbb{P}_{\mathbb{F}}(a_1, a_2, \dots, a_n)$.

Also, when X has a unique birational involution, the group Γ_X has the presentation $\mathbf{F}^1$ that is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Because the number of generators of Γ_X is completely determined in [6], in order to describe the group Γ_X, it is enough to find their relations for $\ell \geqq 2$. We prove the following result:

Theorem 1.1. *The group Γ_X has the group presentation as follows*:

- $\mathbf{F}^5$ *if* $N = 7$,
- $\hat{\mathbf{F}}^3$ *if* $N = 4, 9, 17, 27$,
- $\mathbf{F}^3$ *if* $N = 20$,
- $\mathbf{F}^2$ *if* $N = 5, 6, 12, 13, 15, 23, 25, 30, 31, 33, 36, 38, 40, 41, 42, 44, 58, 61, 68, 76$,
- $\mathbf{F}^1$ *if* $N = 2, 8, 16, 18, 24, 26, 32, 43, 45, 46, 47, 48, 54, 56, 60, 65, 69, 74, 79$,
- $\mathbf{F}^0$ *otherwise*.

This theorem with the results of [6] can be considered as a 3-fold analogue of Yu. Manin's results on smooth del Pezzo surfaces of degree $\leqq 3$.

The proof of Theorem 1.1 is based on the simple observation that except the cases $N = 7, 20, 60$, the involutions $\tau_1, \ldots, \tau_\ell$ are actually elliptic and induced by a single elliptic fibration. This shows that it is worth our while to study birational transformations of the hypersurface X into elliptic fibrations. In particular, it is an interesting question when the 3-fold X is birational to an elliptic fibration. We prove the following result:

Theorem 1.2. *The hypersurface $X \subset \mathbb{P}(1, a_2, a_3, a_4, a_5)$ can be birationally transformed into an elliptic fibration if and only if $N \notin \{3, 60, 75, 84, 87, 93\}$.*

We remark that the hypersurface X of $N = 3$ is the only smooth Fano 3-fold that is not birationally equivalent to an elliptic fibration. Many examples in the 95 families of weighted Fano 3-folds have not so many ways in which we can transform them into an elliptic fibration. Naturally, they make us expect that the hypersurface X, in almost all cases, has a single birational elliptic fibration structure (see Conjecture 2.15 and Proposition 2.16).

After the theorem above, it may be a next step to ask whether the hypersurface X can be birationally transformed to a K3 fibration or not. To this question we give an affirmative answer.

Proposition 1.3. *The hypersurface X is birationally equivalent to a* K3 *fibration.*

We should remark here that D. Ryder[2)] has studied birational transformations of the hypersurface X into K3 and elliptic fibrations in his Ph.D. thesis (see [18]). His thesis

2) After the early version of this paper, he announced a paper to reinforce his thesis. In his paper, he classified birational transformations into K3 and elliptic fibrations for the cases $N = 34, 75, 88, 90$ (see [19]).

applied the techniques of the papers [3] and [6] to classify all birational transformations of X into K3 and elliptic fibrations in the case $N = 5$. In addition, he constructed various kinds of birational transformations of the hypersurface X into K3 and elliptic fibrations and obtained partial results on the existence of submaximal singularities on the hypersurface X in many cases.

Meanwhile, as far as we know, arithmetical properties on quasismooth anticanonically embedded weighted Fano 3-fold hypersurfaces with terminal singularities have never been investigated. The papers [1], [2], and [8] give us a stimulating result that rational points are potentially dense[3)] on smooth Fano 3-folds possibly except double covers of $\mathbb{P}^3$ ramified along smooth sextic surfaces. In the case $N = 1$, the potential density of rational points on the hypersurface X is proved in [8]. The hypersurface X of $N = 2$ is birational to a double cover of $\mathbb{P}^3$ ramified along a sextic surface with 15 nodes, which implies the potential density of rational points (see [4]). Furthermore, we prove the following:

Proposition 1.4. *Suppose that X is defined over a number field. Then rational points are potentially dense on the hypersurface X for*

$$N = 1, 2, 4, 5, 6, 7, 9, 11, 12, 13, 15, 17, 19, 20, 23, 25, 27,$$

$$30, 31, 33, 36, 38, 40, 41, 42, 44, 58, 61, 68, 76.$$

It immediately follows from Theorem 1.1 that the group Γ_X is infinite if $\ell > 1$. In this case, the constructions of the involutions $\tau_1, \dots, \tau_\ell$ easily imply that the hypersurface X contains infinitely many rational surfaces, which implies Proposition 1.4 except the cases $N = 1, 2, 11, 19$.

Even though our main result is Theorem 1.1, for the convenience this paper starts with the problem on existence of birational transformations of the hypersurface X into elliptic fibrations. In Section 2, we prove Theorem 1.2 and classify birational transformations of the hypersurface X into elliptic fibrations in some cases. And then Proposition 1.3 is proved in Section 3. We prove Theorem 1.1 in Section 4. Finally, we complete the proof of Proposition 1.4 by proving the potential density of rational points on X in the cases $N = 11$ and $N = 19$.

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[3)] The set of rational points of a variety V defined over a number field $\mathbb{F}$ is called potentially dense if for some finite field extension $\mathbb{K}$ of the field $\mathbb{F}$ the set of $\mathbb{K}$-rational points of the variety V is Zariski dense.

2. Elliptic fibrations

In this section we prove Theorem 1.2. We start with the simple results below that are useful for this section.

Lemma 2.1. *Let Y be a variety and $\mathcal{M}$ be a linear system without fixed components on the variety Y. If the linear system $\mathcal{M}$ is not composed from a pencil, then there is no Zariski closed proper subset $\Sigma \subsetneqq Y$ such that* $\mathrm{Supp}(S_1) \cap \mathrm{Supp}(S_2) \subset \Sigma$, *where S_1 and S_2 are sufficiently general divisors of the linear system $\mathcal{M}$.*

Proof. Suppose there is a proper Zariski closed subset $\Sigma \subset Y$ such that the set-theoretic intersection of the sufficiently general divisors S_1 and S_2 of the linear system $\mathcal{M}$ is contained in the set Σ. Let $\rho : Y \dashrightarrow \mathbb{P}^n$ be the rational map induced by the linear system $\mathcal{M}$, where n is the dimension of the linear system $\mathcal{M}$. Then there is a commutative diagram

$$\begin{tikzcd} & W \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ Y \arrow[rr, dashed, "\rho"'] & & \mathbb{P}^n, \end{tikzcd}$$

where W is a smooth variety, α is a birational morphism, and β is a morphism. Let Z be the image of the morphism β. Then $\dim(Z) \geqq 2$ because $\mathcal{M}$ is not composed from a pencil.

Let Λ be a Zariski closed subvariety of the variety W such that the morphism

$$\alpha|_{W\setminus\Lambda} : W\setminus\Lambda \to Y\setminus\alpha(\Lambda)$$

is an isomorphism, and Δ be the union of the subset $\Lambda \subset W$ and the closure of the proper transform of the set $\Sigma\setminus\alpha(\Lambda)$ on W. Then Δ is a Zariski closed proper subset of W.

Let B_1 and B_2 be general hyperplane sections of the variety Z, and D_1 and D_2 be the proper transforms of the divisors B_1 and B_2 on the variety W respectively. Then $\alpha(D_1)$ and $\alpha(D_2)$ are general divisors of the linear system $\mathcal{M}$. Hence, in the set-theoretic sense we have

$$\emptyset \neq \beta^{-1}\bigl(\mathrm{Supp}(B_1) \cap \mathrm{Supp}(B_2)\bigr) = \mathrm{Supp}(D_1) \cap \mathrm{Supp}(D_2) \subset \Delta \subsetneqq W$$

because $\dim(Z) \geqq 2$. However, this set-theoretic identity is absurd. $\square$

The following result is implied by Lemma 0.3.3 in [13] and Lemma 2.1.

Corollary 2.2. *Let Y be a three-dimensional variety with canonical singularities. Suppose that a linear system $\mathcal{M}$ on Y without fixed components is not composed from a pencil. For sufficiently general surfaces S_1 and S_2 in the linear system $\mathcal{M}$ and a nef and big divisor D, the inequality $D \cdot S_1 \cdot S_2 > 0$ holds.*

In addition, the proof of Lemma 2.1 implies the following result.

Lemma 2.3. *Let Y be a variety. For linear systems $\mathcal{M}$ and $\mathcal{D}$ on Y without fixed components, if the linear system $\mathcal{M}$ is not composed from a pencil, then there is no Zariski closed* *proper subset* $\Sigma \subsetneqq Y$ *such that* $\operatorname{Supp}(S) \cap \operatorname{Supp}(D) \subset \Sigma$, *where* S *and* D *are general divisors of the linear system* $\mathscr{M}$ *and* $\mathscr{D}$, *respectively*.

Before we proceed, we first observe that the following hold:

- for $N = 1$, a general fiber of the projection of a smooth quartic 3-fold $X \subset \mathbb{P}^4$ from a line contained in X is a smooth elliptic curve;

- for $N = 2$, the 3-fold X is birational to a double cover of $\mathbb{P}^3$ ramified along a singular nodal sextic (see [4]), which is birationally equivalent to an elliptic fibration.

Lemma 2.4. *Suppose that* $N \notin \{1, 2, 3, 7, 11, 19, 60, 75, 84, 87, 93\}$. *Then a sufficiently general fiber of the natural projection* $X \dashrightarrow \mathbb{P}(1, a_2, a_3)$ *is a smooth elliptic curve.*

Proof. Let C be a general fiber of the projection $X \dashrightarrow \mathbb{P}(1, a_2, a_3)$. Then C is not a rational curve by [6] but C is a hypersurface of degree d in $\mathbb{P}(1, a_4, a_5) \cong \operatorname{Proj}(\mathbb{C}[x_1, x_4, x_5])$, where either $\lfloor d/a_4 \rfloor \leqq 3$ or $\lfloor d/a_4 \rfloor \leqq 4$ and $2a_5 \leqq d < 2a_5 + a_4$.

Let $V \subset \mathbb{P}(1, a_4, a_5)$ be the open subset given by $x_1 \neq 0$. Then $V \cong \mathbb{C}^2$ and the affine curve $V \cap C$ is either a cubic curve when $\lfloor d/a_4 \rfloor \leqq 3$ or a double cover of $\mathbb{C}$ ramified at most four points when $\lfloor d/a_4 \rfloor \leqq 4$ and $2a_5 \leqq d < 2a_5 + a_4$. Therefore, the curve C is elliptic. $\square$

Remark 2.5. If $N \notin \{2, 7, 20, 36, 60\}$, each involution τ_i generating the group Γ_X gives the commutative diagram

$$\begin{tikzcd} X \arrow[r, dashed, "\tau_i"] \arrow[d, dashed, "\psi"'] & X \arrow[d, dashed, "\psi"] \\ \mathbb{P}(1, a_2, a_3) \arrow[r, equal] & \mathbb{P}(1, a_2, a_3), \end{tikzcd}$$

where ψ is the natural projection.

Lemma 2.6. *Suppose that* $N \in \{7, 11, 19\}$. *Then* X *is birational to an elliptic fibration.*

Proof. We consider only the case $N = 19$ because in the other cases the proofs are similar.

When $N = 19$, the hypersurface X in $\mathbb{P}(1, 2, 3, 3, 4)$ can be given by the equation

$$x_5 f_8(x_1, x_2, x_3, x_4, x_5) + x_3 f_9(x_3, x_4) + x_2 f_{10}(x_1, x_2, x_3, x_4, x_5) + x_1 f_{11}(x_1, x_2, x_3, x_4, x_5) = 0,$$

where f_i is a quasi-homogeneous polynomial of degree i.

Let $\mathscr{H}$ be the pencil of surfaces on X cut by $\lambda x_1^2 + \mu x_2 = 0$ and $\mathscr{B}$ the pencil of surfaces cut on X by $\delta x_1^3 + \gamma x_3 = 0$, where $(\delta : \gamma) \in \mathbb{P}^1$ and $(\lambda : \mu) \in \mathbb{P}^1$. Then $\mathscr{H}$ and $\mathscr{B}$ give a map

$$\rho : X \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1,$$

which is defined in the outside of $\mathrm{Bs}(\mathscr{H}) \cup \mathrm{Bs}(\mathscr{B})$.

Let C be a general fiber of ρ. Then C is a hypersurface in

$$\mathbb{P}(1,3,4) \cong \mathrm{Proj}(\mathbb{C}[x_1, x_4, x_5])$$

containing the point $(0:1:0)$. Thus, the affine piece of the curve C given by $x_1 \neq 0$ is a cubic curve in $\mathbb{C}^2$, but C is not rational (see [6]). Hence, the fiber C is elliptic. $\square$

Therefore, we have obtained

Corollary 2.7. *If $N \notin \{3, 60, 75, 84, 87, 93\}$, then X is birational to an elliptic fibration.*

To complete the proof of Theorem 1.2, we need to show that the 3-fold X is not birationally equivalent to an elliptic fibration when $N \in \{3, 60, 75, 84, 87, 93\}$. However, the paper [3] shows that the 3-fold X of $N = 3$ is not birationally equivalent to an elliptic fibration. Therefore, it is enough to consider the cases of $N = 60, 75, 84, 87, 93$. Suppose that for these five cases there are a birational map $\rho : X \dashrightarrow V$ and a morphism $\nu : V \to \mathbb{P}^2$ such that V is smooth and a general fiber of the morphism ν is a smooth elliptic curve. We must show that these assumptions lead us to a contradiction.

Let $\mathscr{D} = \left|\nu^*\left(\mathscr{O}_{\mathbb{P}^2}(1)\right)\right|$ and $\mathscr{M} = \rho^{-1}(\mathscr{D})$. Then $\mathscr{M} \sim -nK_X$ for some natural number n because the group $\mathrm{Cl}(X)$ is generated by $-K_X$ (see [7]). An irreducible subvariety $Z \subsetneqq X$ is called a center of canonical singularities of $\left(X, \frac{1}{n}\mathscr{M}\right)$ if there is a birational morphism $f : W \to X$ and an f-exceptional divisor $E_1 \subset W$ such that

$$K_W + \frac{1}{n} f^{-1}(\mathscr{M}) \sim_{\mathbb{Q}} f^*\left(K_X + \frac{1}{n}\mathscr{M}\right) + \sum_{i=1}^{m} c_i E_i,$$

where E_i is an f-exceptional divisor, $c_1 \leqq 0$, and $f(E_1) = Z$. The exceptional divisor E_1 is called a submaximal singularity of the log pair $\left(X, \frac{1}{n}\mathscr{M}\right)$. The set of all centers of canonical singularities of the log pair $\left(X, \frac{1}{n}\mathscr{M}\right)$ is denoted by $\mathrm{CS}\left(X, \frac{1}{n}\mathscr{M}\right)$.

We first show that the set $\mathrm{CS}\left(X, \frac{1}{n}\mathscr{M}\right)$ is not empty. A member of the set, a priori, can be a smooth point, a singular point, or a curve on X. And then we show that all these cases are excluded, which gives us a contradiction.

In what follows, we may assume that the singularities of $\left(X, \frac{1}{n}\mathscr{M}\right)$ are canonical because X is birationally rigid by [6].

Proposition 2.8. *The singularities of $\left(X, \frac{1}{n}\mathscr{M}\right)$ are not terminal.*

Proof. Suppose that the singularities of $\left(X, \frac{1}{n}\mathscr{M}\right)$ are terminal. Then $(X, \epsilon\mathscr{M})$ is terminal and $K_X + \epsilon\mathscr{M}$ is ample for some rational number $\epsilon > \frac{1}{n}$. Consider the commutative diagram

$$\begin{tikzcd} & W \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & & \\ X \arrow[rr, dashed, "\rho"'] & & V \arrow[r, "\nu"'] & \mathbb{P}^2, \end{tikzcd}$$

where α and β are birational morphisms and W is smooth. Then we have

$$\alpha^*(K_X + \epsilon\mathscr{M}) + \sum_{j=1}^{m} a_j F_j \sim_{\mathbb{Q}} K_W + \epsilon\mathscr{H} \sim_{\mathbb{Q}} \beta^*(K_V + \epsilon\mathscr{D}) + \sum_{i=1}^{l} b_i G_i,$$

where G_i is a β-exceptional divisor, F_j is an α-exceptional divisor, a_j and b_i are rational numbers, and $\mathscr{H} = \alpha^{-1}(\mathscr{M})$. Let C be a general fiber of $\nu \circ \beta$. Then

$$0 < C \cdot \alpha^*(K_X + \epsilon\mathscr{M}) \leqq C \cdot \left(\alpha^*(K_X + \epsilon\mathscr{M}) + \sum_{j=1}^{m} a_j F_j\right) = \beta(C) \cdot (K_V + \epsilon\mathscr{D}) = 0$$

because C is an elliptic curve, while the divisor $\sum_{j=1}^{m} a_j F_j$ is effective by our assumption. □

Consequently, the set of centers of canonical singularities $\mathrm{CS}\left(X, \frac{1}{n}\mathscr{M}\right)$ is not empty. However, in the sequel we will show that it is empty.

Lemma 2.9. *The set* $\mathrm{CS}\left(X, \frac{1}{n}\mathscr{M}\right)$ *does not contain any smooth point of* X.

Proof. See [5], Theorem 3.1, and [6], Theorem 5.6.2. □

Lemma 2.10. *The set* $\mathrm{CS}\left(X, \frac{1}{n}\mathscr{M}\right)$ *contains no curves on* X.

Proof. See [18], Lemmas 3.2 and 3.5. □

Therefore, the nonempty set $\mathrm{CS}\left(X, \frac{1}{n}\mathscr{M}\right)$ can contain only singular points of X. In particular, there is a point $P \in \mathrm{Sing}(X)$ such that P is a center of canonical singularities of the log pair $\left(X, \frac{1}{n}\mathscr{M}\right)$. Let $\pi : Y \to X$ be the Kawamata blow up at the point P, E be the exceptional divisor of π, and $\mathscr{B} = \pi^{-1}(\mathscr{M})$. Then $\mathscr{B} \sim_{\mathbb{Q}} -nK_Y$ by [12].

Suppose that $-K_Y^3 < 0$. Let $\overline{\mathrm{NE}}(Y) \subset \mathbb{R}^2$ be the cone of effective curves of Y. Then the class of $-E \cdot E$ generates an extremal ray of the cone $\overline{\mathrm{NE}}(Y)$.

Lemma 2.11. *There are integer numbers* $b > 0$ *and* $c \geqq 0$ *such that* $-K_Y \cdot (-bK_Y + cE)$ *is numerically equivalent to an effective irreducible reduced curve*

$\Gamma \subset Y$ and generates an extremal ray of the cone $\overline{\mathrm{NE}}(Y)$ different from the ray generated by $-E \cdot E$.

Proof. See [6], Corollary 5.4.6. $\square$

Let S_1 and S_2 be two different surfaces in $\mathscr{B}$. Then $S_1 \cdot S_2 \in \overline{\mathrm{NE}}(Y)$ but

$$S_1 \cdot S_2 \equiv n^2 K_Y^2,$$

which implies that the class of $S_1 \cdot S_2$ generates the extremal ray of the cone $\overline{\mathrm{NE}}(Y)$ that contains the curve Γ. However, the support of every effective cycle $C \in \mathbb{R}^+\Gamma$ is contained in $\mathrm{Supp}(S_1 \cdot S_2)$ because $S_1 \cdot \Gamma < 0$ and $S_2 \cdot \Gamma < 0$. Similarly, we have $\mathrm{Supp}(S_1 \cdot S_2) = \Gamma$, which contradicts Lemma 2.1 because the linear system $\mathscr{M}$ is not composed from a pencil.

Corollary 2.12. *The inequality $-K_Y^3 \geqq 0$ holds.*

Corollary 2.13. *When $N = 75, 84, 87, 93$, the hypersurface X is not birationally equivalent to an elliptic fibration.*

Proof. The result immediately follows from the fact that the intersection number $-K_Y^3$ is indeed negative if $N = 75, 84, 87, 93$ (see [6]). $\square$

From now on we consider the case $N = 60$. First of all, we can conclude that the set $\mathrm{CS}\left(X, \frac{1}{n}\mathscr{M}\right)$ must consist of the unique singular point O of type $\frac{1}{9}(1, 4, 5)$ on X because the Kawamata blow ups at the other singular points again give us negative $-K_Y^3$ (see [6]). It should be pointed out that the hypersurface X can be birationally transformed into a Fano 3-fold with canonical singularities.

Let $\pi : Y \to X$ be the Kawamata blow up at the point O and $\mathscr{B}$ be the proper transform of the linear system $\mathscr{M}$ on the variety Y. Also let P and Q be the singular points of the variety Y contained in the exceptional divisor $E \cong \mathbb{P}(1, 4, 5)$ of the morphism π that are quotient singularities of types $\frac{1}{4}(1, 1, 3)$ and $\frac{1}{5}(1, 1, 4)$ respectively.

Lemma 2.14. *The set $\mathrm{CS}\left(Y, \frac{1}{n}\mathscr{B}\right)$ contains the point P.*

Proof. It follows from [12] that the equivalence $\mathscr{B} \sim_{\mathbb{Q}} -nK_Y$ holds. Therefore, we can use the same proof of Proposition 2.8 with nef and big $-K_Y$ to obtain $\mathrm{CS}\left(Y, \frac{1}{n}\mathscr{B}\right) \neq \emptyset$.

We first claim that $\mathrm{CS}\left(Y, \frac{1}{n}\mathscr{B}\right)$ contains at least one of the points P and Q. Let L be the curve on E corresponding to the unique curve of the linear system $|\mathcal{O}_{\mathbb{P}(1,4,5)}(1)|$. Then the curve L passes through the points P and Q. Since $\mathscr{B} \sim_{\mathbb{Q}} -nK_Y$ we obtain $\mathscr{B}|_E \sim_{\mathbb{Q}} nL$. Let Z be an element of the set $\mathrm{CS}\left(Y, \frac{1}{n}\mathscr{B}\right)$.

Suppose that Z be a smooth point of Y. It then implies $\mathrm{mult}_Z \mathscr{B} > n$. Let C be the curve on E corresponding to a general curve in the linear system $|\mathcal{O}_{\mathbb{P}(1,4,5)}(20)|$ passing through the point Z. The curve C cannot be contained in the base locus of the linear system $\mathscr{B}$. Therefore, we obtain a contradictory inequality

$$n = C \cdot \mathscr{B} \geqq \mathrm{mult}_Z(C)\,\mathrm{mult}_Z(\mathscr{B}) > n.$$

Suppose that Z be a curve. Then $\mathrm{mult}_Z(\mathscr{B}) \geqq n$. Let C be the curve on E corresponding to a general curve in the linear system $|\mathcal{O}_{\mathbb{P}(1,4,5)}(20)|$. We then have

$$n = C \cdot \mathscr{B} \geqq \mathrm{mult}_Z(\mathscr{B}) C \cdot Z \geqq nC \cdot Z,$$

which implies $C \cdot Z = 1$ on E. Hence, the curve Z must be the curve L.

It follows from [12] that if the curve L belongs to the set $\mathrm{CS}\left(Y, \frac{1}{n}\mathscr{B}\right)$, then a singular point of the threefold Y on the curve L also belongs to the set $\mathrm{CS}\left(Y, \frac{1}{n}\mathscr{B}\right)$. It proves our claim.

For now, we suppose that the set $\mathrm{CS}\left(Y, \frac{1}{n}\mathscr{B}\right)$ contains the point Q.

Let $\alpha : U \to Y$ be the Kawamata blow up at the point Q and $\mathscr{D}$ be the proper transform of the linear system $\mathscr{M}$ on the variety U. We then see that $\mathscr{D} \sim_{\mathbb{Q}} -nK_U$. The complete linear system $|-4K_U|$ is the proper transform of the pencil $|-4K_X|$, the base locus of which consists of a curve Z_U such that $\pi \circ \alpha(Z_U)$ is the base curve of the pencil $|-4K_X|$.

Let H be a sufficiently general surface of the pencil $|-4K_U|$. Then the equality

$$Z_U^2 = -K_U^3 = -\frac{1}{30}$$

holds on the surface H but $\mathscr{D}|_H \sim_{\mathbb{Q}} nZ$. Therefore, it follows that

$$\mathrm{Supp}(D) \cap \mathrm{Supp}(H) = \mathrm{Supp}(Z_U),$$

where D is a general surface of the linear system $\mathscr{D}$, which is impossible by Lemma 2.3.

Consequently, the set $\mathrm{CS}\left(Y, \frac{1}{n}\mathscr{B}\right)$ contains the point P. $\square$

The hypersurface X can be given by a quasihomogeneous equation of degree 24

$$x_5^2 x_4 + x_5 f_{15}(x_1, x_2, x_3, x_4) + f_{24}(x_1, x_2, x_3, x_4) = 0 \subset \mathbb{P}(1,4,5,6,9),$$

where $f_i(x_1, x_2, x_3, x_4)$ is a quasihomogeneous polynomial of degree i. Let D be a general surface of the linear system $|-5K_X|$ and S be the unique surface of the linear system $|-K_X|$. Then D is cut on X by the equation

$$\lambda x_1^5 + \delta x_1 x_2 + \mu x_3 = 0,$$

where $(\lambda : \delta : \mu) \in \mathbb{P}^2$, and S is cut by the equation $x_1 = 0$. Moreover, the base locus of the linear system $|-5K_X|$ consists of the single irreducible curve C that is cut on the hypersurface X by the equations $x_1 = x_3 = 0$. In particular, we have $D \cdot S = C$.

In a neighborhood of the point O the monomials x_1, x_2, and x_3 can be considered as weighted local coordinates on X such that $\mathrm{wt}(x_1) = 1$, $\mathrm{wt}(x_2) = 4$ and $\mathrm{wt}(x_3) = 5$. In a neighborhood of the point P the birational morphism π can be given by the equations

$$x_1 = \tilde{x}_1 \tilde{x}_2^{\frac{1}{9}}, \quad x_2 = \tilde{x}_2^{\frac{4}{9}}, \quad x_3 = \tilde{x}_3 \tilde{x}_2^{\frac{5}{9}},$$

where $\tilde{x}_1$, $\tilde{x}_2$ and $\tilde{x}_3$ are weighted local coordinates on the variety Y in a neighborhood of the point P such that $\mathrm{wt}(\tilde{x}_1) = 1$, $\mathrm{wt}(\tilde{x}_2) = 3$ and $\mathrm{wt}(\tilde{x}_3) = 1$. Let $\tilde{D}$, $\tilde{S}$, and $\tilde{C}$ be the proper transforms of the surface D, the surface S, and the curve C on the variety Y respectively, and E be the exceptional divisor of π. Then in a neighborhood of P the surface E is given by the equation $\tilde{x}_2 = 0$, the surface $\tilde{D}$ is given by the equation

$$\lambda \tilde{x}_1^5 + \delta \tilde{x}_1 + \mu \tilde{x}_3 = 0,$$

and the surface $\tilde{S}$ is given by the equation $\tilde{x}_1 = 0$. Hence, we see that

$$\tilde{D} \sim_{\mathbb{Q}} \pi^*(-5K_X) - \frac{5}{9}E \sim_{\mathbb{Q}} 5\tilde{S} \sim_{\mathbb{Q}} -5K_Y,$$

the curve $\tilde{C}$ is the intersection of the surfaces $\tilde{D}$ and $\tilde{S}$; the linear system $|-5K_Y|$ is the proper transform of $|-5K_X|$; the base locus of $|-5K_Y|$ consists of the curve $\tilde{C}$.

Let $\beta : W \to Y$ be the Kawamata blow up of the point P. And let $\overline{D}$, $\overline{S}$, and $\overline{C}$ be the proper transforms on the variety W of the surface D, the surface S, and the curve C respectively and F be the exceptional divisor of the morphism β. Then the surface F is the weighted projective space $\mathbb{P}(1,1,3)$ and in a neighborhood of the singular point of the surface F the birational morphism β can be given by the equations

$$\tilde{x}_1 = \bar{x}_1 \bar{x}_2^{\frac{1}{4}}, \quad \tilde{x}_2 = \bar{x}_2^{\frac{3}{4}}, \quad \tilde{x}_3 = \bar{x}_3 \bar{x}_2^{\frac{1}{4}},$$

where $\bar{x}_1$, $\bar{x}_2$ and $\bar{x}_3$ are weighted local coordinates on the variety W in a neighborhood of the singular point of F such that $\mathrm{wt}(\bar{x}_1) = 1$, $\mathrm{wt}(\bar{x}_2) = 2$ and $\mathrm{wt}(\bar{x}_3) = 1$. In particular, the exceptional divisor F is given by the equation $\bar{x}_2 = 0$, the surface $\overline{D}$ is given by the equation

$$\lambda \bar{x}_1^5 \bar{x}_2 + \delta \bar{x}_1 + \mu \bar{x}_3 = 0,$$

and the surface $\overline{S}$ is given by the equation $\bar{x}_1 = 0$. Therefore,

$$\overline{D} \sim_{\mathbb{Q}} \beta^*(\tilde{D}) - \frac{1}{4}F \sim_{\mathbb{Q}} (\pi \circ \beta)^*(-5K_X) - \frac{5}{9}\beta^*(E) - \frac{1}{4}F,$$

$$\overline{S} \sim_{\mathbb{Q}} \beta^*(\overline{S}) - \frac{1}{4}F \sim_{\mathbb{Q}} -K_W,$$

and the curve $\bar{C}$ is the intersection of the surfaces $\bar{D}$ and $\bar{S}$. Let $\mathscr{P}$ be the proper transform of the linear system $|-5K_X|$ on W. Then $\bar{D}$ is a general surface of $\mathscr{P}$, the base locus of the linear system $\mathscr{P}$ consists of the curve $\bar{C}$, and the equalities

$$\bar{D} \cdot \bar{C} = \bar{D} \cdot \bar{D} \cdot \bar{S} = \frac{1}{3}$$

hold. Thus, the divisor $\bar{D}$ is nef and big because $\bar{D}^3 = 2$.

Let B_1 and B_2 be general divisors of $\mathscr{D}$. Then

$$\bar{D} \cdot B_1 \cdot B_2 = \left(\beta^*(-5K_Y) - \frac{1}{4}F\right) \cdot \left(\beta^*(-nK_Y) - \frac{n}{4}F\right)^2 = 0,$$

which contradicts Lemma 2.2. Hence, we have proved Theorem 1.2.

One can easily check that the hypersurface X can be birationally transformed into elliptic fibrations in several distinct ways in the case when

$$N \in \Omega = \{1, 2, 7, 9, 11, 17, 19, 20, 26, 30, 36, 44, 49, 51, 64\}.$$

In other words, in the case when $N \in \Omega$ there are rational maps $\alpha : X \dashrightarrow \mathbb{P}^2$ and $\beta \dashrightarrow \mathbb{P}^2$ such that the normalizations of general fibers of α and β are elliptic curves but they cannot make the diagram

$$\begin{array}{ccc} X & \overset{\sigma}{\dashrightarrow} & X \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ \mathbb{P}^2 & \underset{\zeta}{\dashrightarrow} & \mathbb{P}^2, \end{array}$$

commute for any birational maps σ and ζ.

Conjecture 2.15. *Let $\rho : X \dashrightarrow \mathbb{P}^2$ be a rational map such that the normalization of a general fiber of ρ is an elliptic curve. Then there is a commutative diagram*

$$\begin{array}{ccc} & X & \\ {\scriptstyle\psi}\swarrow & & \searrow{\scriptstyle\rho} \\ \mathbb{P}(1, a_2, a_3) & \underset{\phi}{\dashrightarrow} & \mathbb{P}^2, \end{array}$$

if $N \notin \{3, 60, 75, 84, 87, 93\} \cup \Omega$, where ψ is the natural projection and ϕ is a birational map.

In the case $N = 5$, Conjecture 2.15 has been verified in [18].

Proposition 2.16. *Conjecture* 2.15 *holds for*

$$N = 14, 22, 28, 34, 37, 39, 52, 53, 57, 59, 66, 70, 72, 73, 78, 81, 86, 88, 89, 90, 92, 94, 95.$$

Proof. In the proof of Theorem 1.2, we see that there is a point $P \in \mathrm{Sing}(X)$ that belongs to $\mathrm{CS}\left(X, \frac{1}{n}\mathscr{M}\right)$. Let $\pi : Y \to X$ be the Kawamata blow up at the point P, E be the exceptional divisor of π, and $\mathscr{B}$ be the proper transform on Y of $\mathscr{M}$. Then $\mathscr{B} \sim_{\mathbb{Q}} -nK_Y$ by [12].

There is exactly one singular point Q of the hypersurface X, such that we have $-K_Y^3 = 0$ if $P = Q$, and $-K_Y^3 < 0$ if $P \neq Q$. In the case when $P \neq Q$ we can proceed as in the proof of Theorem 1.2 to derive a contradiction. Thus, we have $P = Q$.

The linear system $|-rK_Y|$ is free for some $r \in \mathbb{N}$ and induces a morphism

$$\phi : Y \to \mathbb{P}(1, a_2, a_3)$$

such that $\phi = \psi \circ \pi$. However, for a general surface $S \in \mathscr{B}$ and a general fiber C of the morphism ϕ we have $S \cdot C = 0$. Hence, $\mathscr{B}$ lies in the fibers of the elliptic fibration ϕ, which implies the claim. $\square$

Therefore, in many cases, the hypersurface X can be birationally transformed into an elliptic fibration in a unique way.

3. Fibrations into K3 surfaces

In this section, we prove Proposition 1.3. Before we proceed, we should remark here that X is not birational to a fibration into ruled surfaces because X is birationally rigid by [6].

Lemma 3.1. *Suppose that $N \in \{18, 22, 28\}$. Then X is birational to a* K3 *fibration.*

Proof. Let $\mathscr{H}$ be the pencil in $|-a_3K_X|$ of surfaces passing through the singular points of the hypersurface X of type $\frac{1}{a_3}(1, -1, 1)$. Then a general surface in $\mathscr{H}$ is a compactification of a quartic in $\mathbb{C}^3$, which implies that X is birational to a K3 fibration. $\square$

Suppose that $N \notin \{18, 22, 28\}$. Let $\psi : X \dashrightarrow \mathbb{P}^1$ be the map induced by the projection

$$\mathbb{P}(1, a_2, a_3, a_4, a_5) \dashrightarrow \mathbb{P}(1, a_2)$$

and S be a general fiber of ψ. Then the surface S is a hypersurface of degree d in $\mathbb{P}(1, a_3, a_4, a_5)$ that is not uniruled because X is birationally rigid by [6]. Therefore, we may assume in the following that $a_2 \neq 1$. Let us show that S is birational to a K3 surface.

Lemma 3.2. *Suppose that $\lfloor d/a_3 \rfloor \leqq 4$. Then S is birational to a* K3 *surface.*

Proof. The surface S is a compactification of a quartic in $\mathbb{C}^3$. $\square$

Lemma 3.3. *Suppose that $2a_5 + a_3 > d$ and $\lfloor d/a_3 \rfloor \leqq 6$. Then the surface S is birationally equivalent to a* K3 *surface.*

Proof. The surface S is a compactification of a double cover of $\mathbb{C}^2$ ramified along a sextic curve, which implies that S is birational to a K3 surface. □

Lemma 3.4. *Suppose that $2a_5 + 2a_3 > d$, $3a_5 > d$, and $d \leqq 5a_3$. Then the surface S is birationally equivalent to a* K3 *surface.*

Proof. The surface S is a compactification of a double cover of $\mathbb{C}^2 \setminus L$ ramified along a quintic curve, where L is a line in $\mathbb{C}^2$, which implies the statement. □

Consequently, we may consider the 3-fold X only when

$$N \in \{27, 33, 48, 55, 56, 58, 63, 65, 68, 72, 74, 79, 80, 83, 85, 89, 90, 91, 92, 94, 95\}.$$

Lemma 3.5. *Suppose that $N \notin \{27, 56, 65, 68, 83\}$. Then the surface S is birationally equivalent to a* K3 *surface.*

Proof. In the case $N = 91$, the rational map ψ is studied in [18], Example 2.5, which implies that the surface S is birational to a K3 surface. We use the same approach for the others. We consider only the case $N = 72$, because the proofs are similar in other cases.

Let X be a general hypersurface in $\mathbb{P}(1, 2, 3, 10, 15)$ of degree 30. Let Γ be the curve on the hypersurface X given by the equation $x_1 = x_2 = 0$ and $\mathscr{B}$ be the pencil of surfaces on the hypersurface X that are cut by the equations

$$\lambda x_1^2 + \mu x_2 = 0,$$

where $(\lambda : \mu) \in \mathbb{P}^1$. Then S belongs to $\mathscr{B}$, the curve C is the base locus of the pencil $\mathscr{B}$, and the projection ψ is the rational map given by $\mathscr{B}$. Moreover, it follows from the generality of the hypersurface X that the curve Γ is reduced, irreducible, and rational.

Let P be a singular point of X of type $\frac{1}{3}(1, 2, 1)$ and $\pi : V \to X$ be the Kawamata blow up at the point P with the exceptional divisor $E \cong \mathbb{P}(1, 1, 2)$. Let $\mathscr{M}$, $\hat{\Gamma}$, $\hat{S}$, and $\hat{Y}$ be the proper transforms on V of the pencil $\mathscr{B}$, the curve Γ, the fiber S, and the surface Y cut by the equation $x_1 = 0$ on the hypersurface X, respectively. Then

$$-4K_V^3 = \hat{S} \cdot \hat{\Gamma} < 0,$$

where $\hat{S} \in \mathscr{M}$, the curve $\hat{\Gamma}$ is the base locus of the pencil $\mathscr{M}$, and the equivalences

$$\hat{S} \sim 2\hat{Y} \sim -2K_V \sim_{\mathbb{Q}} \pi^*(-2K_X) - \frac{2}{3}E$$

hold (see [6], Proposition 3.4.6). The surface $\hat{Y}$ has canonical singularities.

Let $\overline{\mathrm{NE}}(V) \subset \mathbb{R}^2$ be the cone of effective curves of V. Then the class of $-E \cdot E$ generates one extremal ray of the cone $\overline{\mathrm{NE}}(V)$, while the curve $\hat{\Gamma}$ generates another extremal ray of the cone $\overline{\mathrm{NE}}(V)$ because $\hat{S} \cdot \hat{\Gamma} < 0$ and $\hat{\Gamma}$ is the only base curve of the pencil $\mathscr{M}$, which implies that the curve $\hat{\Gamma}$ is the only curve contained in the extremal ray generated by $\hat{\Gamma}$.

The log pair $(V, \hat{Y})$ has log terminal singularities by [14], Theorem 17.4, which implies that the singularities of $(V, \hat{Y})$ are canonical because $\hat{Y} \sim -K_V$. Hence, for a sufficiently small rational number $\epsilon > 1$ the singularities of the log pair $(V, \epsilon\hat{Y})$ are still log terminal but the inequality $(K_V + \epsilon\hat{Y}) \cdot \hat{\Gamma} < 0$ holds. There is a log flip $\alpha : V \dashrightarrow U$ along the curve $\hat{\Gamma}$ by [21].

Let $\mathscr{P} = \alpha(\mathscr{M})$, $\overline{Y} = \alpha(\hat{Y})$, $\overline{S} = \alpha(\hat{S})$, and $\overline{\Gamma}$ be the flipped curve on U, namely, a possibly reducible curve such that $V \backslash \hat{\Gamma} \cong U \backslash \overline{\Gamma}$. Then the surface $\overline{S}$ is a member of the pencil $\mathscr{P}$, the log pair $(U, \epsilon\overline{Y})$ has log terminal singularities, $\overline{S} \cdot \overline{\Gamma} = 2\overline{Y} \cdot \overline{\Gamma} < 0$, and the equivalences $-K_U \sim \overline{Y}$ and $\overline{S} \sim -2K_U$ hold. Therefore, the log pair $(U, \overline{Y})$ has canonical singularities. In particular, the singularities of the variety U are canonical.

Suppose $\mathrm{Bs}(\mathscr{P}) \neq \emptyset$. Then $\mathrm{Bs}(\mathscr{P})$ consists of a possibly reducible curve Z that is numerically equivalent to $\overline{\Gamma}$. Hence, every surface in $\mathscr{P}$ is nef. Let H be a general very ample divisor on V and $\overline{H} = \alpha(H)$. Then $\overline{H} \cdot Z < 0$, which implies $Z \subset \overline{H}$. The inequality

$$\overline{H} \cdot \overline{S}_1 \cdot \overline{S}_2 < 0$$

holds for general surfaces $\overline{S}_1$ and $\overline{S}_2$ in $\mathscr{P}$, which contradicts the numerical effectiveness of the surface S_2 because $\overline{H} \cdot \overline{S}_1$ is effective. Consequently, the pencil $\mathscr{P}$ has no base points, and hence the surface $\overline{S}$ has canonical singularities.

Let $\phi : U \to \mathbb{P}^1$ be the morphism given by the pencil $\mathscr{P}$. Then $\overline{S}$ is a sufficiently general fiber of ϕ and $2\overline{Y}$ is a fiber of ϕ. Moreover, we have $K_{\overline{S}} \sim 0$ by the adjunction formula because the equivalences $-K_U \sim \overline{Y}$ and $\overline{Y}|_{\overline{S}} \sim 0$ hold. Therefore, the surface $\overline{S}$ is either an abelian surface or a K3 surface.

Let $C = E \cap \hat{S}$. Then $\overline{S}$ contains $\alpha(C)$ because $C \neq \hat{\Gamma}$ and α is an isomorphism in the outside of $\hat{\Gamma}$. However, a component of C must be rational because C is a hypersurface of degree 2 in $\mathbb{P}(1,1,2)$, which implies that $\overline{S}$ cannot be an abelian surface. $\square$

Therefore, it is enough to check the cases $N \in \{27, 56, 65, 68, 83\}$ to conclude the proof of Proposition 1.3. We prove that S is birational to a K3 surface case by case.

Case $N = 27$ or 65. Because the methods for $N = 27$ and 65 are the same, we only consider the case $N = 27$.

The surface $S \subset \mathrm{Proj}(\mathbb{C}[x_1, x_3, x_4, x_5]) \cong \mathbb{P}(1,3,5,5)$ can be given by the equation

$$x_5^2 f_5(x_1, x_3, x_4) + x_5 f_{10}(x_1, x_3, x_4) + f_{15}(x_1, x_3, x_4) = 0,$$

where f_i is a quasi-homogeneous polynomial of degree i. Introducing a variable $y = x_5 f_5(x_1, x_3, x_4)$ of weight 10, we obtain the hypersurface

$$\tilde{S} \subset \mathbb{P}(1,3,5,10) \cong \mathrm{Proj}(\mathbb{C}[x_1, x_3, x_4, y])$$

of degree 20 given by the equation

$$y^2 + yf_{10}(x_1, x_3, x_4)f_5(x_1, x_3, x_4) + f_{15}(x_1, x_3, x_4)f_5(x_1, x_3, x_4) = 0$$

and birational to S. The surface $\tilde{S}$ is a compactification of a double cover of $\mathbb{C}^2$ ramified along a sextic curve. Therefore, the surface S is birational to a K3 surface.

Case $N = 56$. The surface S is a hypersurface of degree 24 in $\mathbb{P}(1,3,8,11)$ given by the equation

$$x_5^2x_1^2 + x_5x_1f_{12}(x_1, x_3, x_4) + f_{24}(x_1, x_3, x_4) = 0 \subset \mathrm{Proj}(\mathbb{C}[x_1, x_3, x_4, x_5]),$$

where f_i is a quasi-homogeneous polynomial of degree i. Introducing a new variable $y = x_1x_5$ of weight 12, we obtain the hypersurface $\tilde{S}$ of degree 24 in $\mathbb{P}(1,3,8,12)$ given by the equation

$$y^2 + yf_{12}(x_1, x_3, x_4) + f_{24}(x_1, x_3, x_4) = 0 \subset \mathrm{Proj}(\mathbb{C}[x_1, x_3, x_4, y])$$

which is birational to S. We have $K_{\tilde{S}} \sim 0$, which implies the claim.

Case $N = 68$. The surface S is a general quasismooth hypersurface of degree 28 in $\mathrm{Proj}(\mathbb{C}[x_1, x_3, x_4, x_5])$, where $\mathrm{wt}(x_1) = 1$, $\mathrm{wt}(x_3) = 4$, $\mathrm{wt}(x_4) = 7$, $\mathrm{wt}(x_5) = 14$. The surface S has a canonical singular point Q of type $\mathbb{A}_1$ and two singular points P_1 and P_2 of type $\frac{1}{7}(1,4)$.

Let $\mathscr{P}$ be the pencil of curves on S given by

$$\lambda x_1^4 + \mu x_3 = 0,$$

where $(\lambda : \mu) \in \mathbb{P}^1$. Then the pencil $\mathscr{P}$ gives a rational map $\phi : S \dashrightarrow \mathbb{P}^1$ whose general fiber is an elliptic curve. Let $\tau : Y \to S$ be the minimal resolution of singularities, Z be the proper transform on the surface Y of the irreducible curve that is cut on the surface S by the equation $x_1 = 0$, and $\psi = \phi \circ \tau$. Then ψ is a morphism and Z lies in a fiber of ψ.

Consider τ-exceptional curves E, $\hat{E}_1$, $\check{E}_1$, $\hat{E}_2$, and $\check{E}_2$, where $\tau(E) = Q$, $\tau(\hat{E}_i) = P_i$, $\tau(\check{E}_i) = P_i$, $\check{E}_i^2 = -4$, and $E^2 = \hat{E}_i^2 = -2$. Let L be the fiber of ψ over the point $\psi(Z)$. Then $Z \cong \tau(Z) \cong \mathbb{P}^1$, the curve Z is a component of L of multiplicity 4, the fiber L contains the curve E, and either the surface Y is a minimal model or $Z^2 = -1$. Taking into account all possibilities for the fiber L to be a blow up of a reducible fiber of minimal smooth elliptic fibration, we see that the equality $Z^2 = -1$ holds, the curves $\hat{E}_1$ and $\hat{E}_1$ are sections of the elliptic fibration ψ, but $\check{E}_1$ and $\check{E}_2$ are contained in the fiber L. On the other hand, the equivalences

$$K_Y \sim_{\mathbb{Q}} \tau^*\big(\mathcal{O}_{\mathbb{P}(1,4,7,14)}(2)|_S\big) - \frac{2}{7}\hat{E}_1 - \frac{4}{7}\check{E}_1 - \frac{2}{7}\hat{E}_2 - \frac{4}{7}\check{E}_2 \sim_{\mathbb{Q}} 2Z + E$$

hold. Let $\gamma : Y \to \overline{Y}$ be the contraction of the curves Z and E. Then $\overline{Y}$ is smooth, the curve $\gamma(L)$ is a fiber of type III of the relatively minimal elliptic fibration $\psi \circ \gamma^{-1}$, and the equivalence $K_{\overline{Y}} \sim 0$ holds. Therefore, the surface S is birational to a K3 surface.

Case $N = 83$. The surface S is a hypersurface of degree 36 in

$$\mathbb{P}(1, 4, 11, 18) \cong \mathrm{Proj}(\mathbb{C}[x_1, x_3, x_4, x_5]),$$

where $\mathrm{wt}(x_1) = 1$, $\mathrm{wt}(x_3) = 4$, $\mathrm{wt}(x_4) = 11$, and $\mathrm{wt}(x_5) = 18$. Therefore, the surface S has a canonical singular point Q of type $\mathbb{A}_1$ given by the equations $x_1 = x_4 = 0$ and an isolated singular point P at $(0 : 0 : 1 : 0)$. The surface S is not quasismooth at the point P which is not a rational singular point of S, a posteriori.

Let $\mathscr{P}$ be the pencil of curves on S given by the equations

$$\lambda x_1^4 + \mu x_3 = 0,$$

where $(\lambda : \mu) \in \mathbb{P}^1$, C be a general curve in $\mathscr{P}$, and $v : \hat{C} \to C$ be the normalization of the curve C. Then the base locus of the pencil $\mathscr{P}$ consists of the point P and $\mathscr{P}$ gives a rational map $\phi : S \dashrightarrow \mathbb{P}^1$ whose general fiber is C. On the other hand, the curve C is a hypersurface of degree 36 in $\mathbb{P}(1, 11, 18)$. Therefore, the curve $\hat{C}$ is an elliptic curve, and the birational map v is a bijection because C is a compactification of the affine curve

$$C \cap \{x_1 \neq 0\} \subset \mathbb{C}^2,$$

which is a double cover of $\mathbb{C}$ ramified at three points. In particular, we have $\kappa(S) \leqq 1$.

Let $\tau : Y \to S$ be the minimal resolution of singularities of S. Then we have an elliptic fibration $\psi : Y \to \mathbb{P}^1$ such that $\psi = \phi \circ \tau$. We can identify a general fiber of ψ with the curve $\hat{C}$ and the normalization v with the restriction $\tau|_{\hat{C}}$. Therefore, there is exactly one exceptional curve Z of the resolution τ not contained in a fiber of ψ. The curve Z must be a section of ψ.

Let F be the proper transform of the smooth rational curve in the pencil $\mathscr{P}$ that is given by the equation $x_1 = 0$, E be the exceptional curve of the morphism τ that is mapped to the point Q, and $E_1, \ldots, E_m$ be the exceptional curves of the birational morphism τ that are different from the curves Z and E. Then $\tau(E_i) = \tau(Z) = P$ and the union

$$F \cup E \cup E_1 \cup \cdots \cup E_m$$

lies in a single fiber L of ψ. Moreover, the smooth rational curve F is a component of the fiber L of multiplicity 4, the curve E is rational, and $E^2 = -2$. We have

$$K_Y \sim_{\mathbb{Q}} 2F + aE + \sum_{i=1}^{m} c_i E_i,$$

where a, b, c_i are rational numbers. The elliptic fibration ψ is not relatively minimal, but the curve F is the only curve in the fiber L whose self-intersection is -1.

Let $\xi : Y \to \overline{Y}$ be the birational morphism such that the surface $\overline{Y}$ is the minimal model of the surface Y and $\eta = \psi \circ \xi^{-1}$. Then $\eta : \overline{Y} \to \mathbb{P}^1$ is a relatively minimal elliptic fibration.

Let $\overline{L} = \xi(L)$. Then $K_{\overline{Y}} \sim_{\mathbb{Q}} \gamma \overline{L}$ for some rational number $\gamma \geqq 0$. Hence, we have

$$K_Y \sim_{\mathbb{Q}} \xi^*(\gamma \overline{L}) + \alpha F + \beta E + \sum_{i=1}^{m} \delta_i E_i,$$

where α, β, δ_i are non-negative integer numbers. Because the birational morphism ξ must contract the curves F and E, we see that $\alpha \geqq 2$, $\beta \geqq 1$. Also, the inequality $\delta_i \neq 0$ holds if and only if the curve E_i is contracted by ξ. Moreover, the equality $\alpha = 2$ implies that the only curves contracted by ξ are F and E. Hence, the inequality $\gamma \geqq 0$ and the equivalence

$$2F + aE + \sum_{i=1}^{m} c_i E_i \sim_{\mathbb{Q}} \xi^*(\gamma \overline{L}) + \alpha F + \beta E + \sum_{i=1}^{m} \delta_i E_i$$

imply that $\gamma = 0$, $\alpha = 2$, and $m > 0$. In particular, the surface $\overline{Y}$ is either a K3 surface or an Enriques surface. On the other hand, the only possible multiple fiber of the elliptic fibration η is the fiber $\overline{L}$, which implies that $\overline{Y}$ is a K3 surface.

Therefore, we have proved Theorem 1.3. In addition, we have shown that X is birational to a fibration whose general fiber is an elliptic K3 surface if $N \notin \{3, 60, 75, 87, 93\}$.

We conclude the section with one remark.

Remark 3.6. In the proof of Case $N = 83$ the equality $\alpha = 2$ and the fact that F is a component of L of multiplicity 4 imply that $\overline{L}$ is an elliptic fiber of type I_r^*, while the birational morphism ξ is the composition of the blow up at a point of the component of the fiber $\overline{L}$ of multiplicity 2 and the blow up at the intersection point of the proper transform of the component of multiplicity 2 with the exceptional curve on the first blow up.

It was pointed out to us by D. Stepanov that one can explicitly resolve the singularity of the surface S at the point P to prove that the surface S is birationally equivalent to a smooth K3 surface. Indeed, the surface S can be locally given near P by the equation

$$x^2 + y^3 + z^9 = 0 \subset \mathbb{C}^3/\mathbb{Z}_{11}(7,4,1)$$

where $P = (0,0,0)$.

Let σ_1 be the weighted blow up of $\mathbb{C}^3/\mathbb{Z}_{11}(7,4,1)$ at the singular point P with weights $\frac{1}{11}(10,3,1)$. Then the blown up variety is covered by 3 affine charts, the first chart is isomorphic to $\mathbb{C}^3/\mathbb{Z}_{10}(1,-3,-1)$, and in the first chart σ_1 is given by

$$x = x^{10/11}, \quad y = x^{3/11} y, \quad z = x^{1/11} z,$$

where we denote the coordinates on $\mathbb{C}^3/\mathbb{Z}_{10}(1,-3,-1)$ by the same letters x, y, z as the coordinates on $\mathbb{C}^3/\mathbb{Z}_{11}(7,4,1)$. The full transform of S is given by the equation

$$x^{20/11} + x^{9/11}y^3 + x^{9/11}z^9 = 0,$$

but the strict transform $\bar{S}$ of the surface S is given by the equation

$$x + y^3 + z^9 = 0 \subset \mathbb{C}^3/\mathbb{Z}_{10}(1, -3, -1),$$

and the exceptional divisor

$$x = 0 = y^3 + z^9 = \prod_{i=1}^{3}(y + \varepsilon^i z^9)$$

consists of 3 smooth rational curves $\bar{E}_1$, $\bar{E}_2$, $\bar{E}_3$ that intersect at the singular point $(0,0,0)$, where ε is a primitive cubic root of unity. Moreover, the surface $\bar{S}$ has quotient singularity of type $\frac{1}{10}(-3, -1)$ at the singular point $(0,0,0)$.

In the second chart that is isomorphic to $\mathbb{C}^3/\mathbb{Z}_3(-1, 2, -1)$, the strict transform of S is given by the equation $x^2y + 1 + z^9 = 0$, and in the third chart that is isomorphic to $\mathbb{C}^3$, the strict transform of the surface S is given by $x^2z + y^3 + 1 = 0$, which imply that they are nonsingular.

We have a surface $\bar{S}$ that is locally isomorphic to $\mathbb{C}^2/\mathbb{Z}_{10}(-3, -1)$ and we have 3 smooth rational curves on $\bar{S}$ given by the equation

$$\prod_{i=1}^{3}(x + \varepsilon^i y^3) = 0,$$

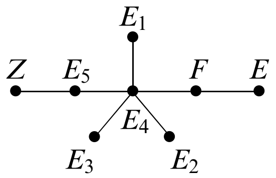
where x and y are local coordinates on $\mathbb{C}^2/\mathbb{Z}_{10}(-3, -1)$.

Let σ_2 be the weighted blow up of the surface $\bar{S}$ at the point $(0,0)$ with weights $\frac{1}{10}(1, 7)$. The blown up variety is covered by 2 charts. The first chart is $\mathbb{C}^2$ and it does not contain the strict transforms of the curve $\bar{E}_i$. The second chart is isomorphic to $\mathbb{C}^2/\mathbb{Z}_7(-1, 3)$ and in this chart the weighted blow up σ_2 is given by the formulas $x = y^{1/10}x$, $y = y^{7/10}$ but the strict transform of the curve $\bar{E}_i$ is given by the equation $x + \varepsilon^i y^2 = 0$, where the exceptional divisor $\bar{Z}$ of the weighted blow up σ_2 is given by $y = 0$.

Now let σ_3 be the weighted blow up at the origin of the last considered chart with weights $\frac{1}{7}(2, 1)$. In the first chart $\mathbb{C}^2/\mathbb{Z}_2(1, 1)$, the equation of the proper transform of the curve $\bar{E}_i$ is $1 + \varepsilon^i y^2 = 0$, the equation of the proper transform of $\bar{Z}$ is $y = 0$, and the exceptional divisor $\bar{E}_4$ of σ_3 is given by $x = 0$, but the second chart of σ_3 is nonsingular.

Let σ_4 be the blow up of $\mathbb{C}^2/\mathbb{Z}_2(1, 1)$ with weights $\frac{1}{2}(1, 1)$ and let $\bar{E}_5$ be the exceptional divisor of σ_4. Then σ_4 resolves the singularity of S in a neighborhood of the point P and after blowing up the point Q of S we get our minimal resolution $\tau : Y \to S$.

Let E_i and Z be the proper transforms of the irreducible curves $\bar{E}_i$ and $\bar{Z}$ on the non-singular surface Y, respectively. Then $E_4^2 = -4$, $Z^2 = E_{i\neq 4}^2 = E^2 = -2$, where



is the dual graph of the rational curves $Z, E_1, \dots, E_5$, F, and E. In particular, the fiber $\bar{L}$ is of type I_0^*.

4. Birational automorphisms

The group $\mathrm{Bir}(X)$ of birational automorphisms is generated by biregular automorphisms and a finite set of birational involutions $\tau_1, \dots, \tau_\ell$ that are described in [6]. To be precise, we have an exact sequence of groups

$$1 \to \Gamma_X \to \mathrm{Bir}(X) \to \mathrm{Aut}(X) \to 1,$$

where the group Γ_X is the subgroup of $\mathrm{Bir}(X)$ generated by a finite set of distinct birational involutions $\tau_1, \dots, \tau_\ell$.

In this section we describe the group Γ_X with group presentations. When the number ℓ of generators of Γ_X is 0, namely, the group Γ_X is trivial, $\mathrm{Bir}(X) = \mathrm{Aut}(X)$, and hence the 3-fold X is birationally superrigid. When the number ℓ of generators of Γ_X is 1, the group Γ_X is the group of order 2, i.e., $\mathbb{Z}/2\mathbb{Z}$. Therefore, we may assume that $\ell \geqq 2$ to prove Theorem 1.1. Throughout this section, a relation of involutions means one different from the trivial relation, i.e., $\tau_i^2 = 1$.

First of all, we present the following important observation:

Lemma 4.1. *Suppose that the set* $\mathrm{CS}(X, \lambda\mathcal{M})$ *contains at most one element, where* $\mathcal{M}$ *is a linear system without fixed components on* X *and* λ *is a positive rational number such that the divisor* $-(K_X + \lambda\mathcal{M})$ *is ample. Then there is no relation among* $\tau_1, \dots, \tau_\ell$.

Proof. See [17], Proposition 2.2 and Lemma 2.3. They show the condition implies a given birational automorphism is untwisted[4)] by the involutions $\tau_1, \dots, \tau_\ell$ in a unique way (see also [6]). □

[4)] Fix a very ample linear system $\mathcal{H}$ on X. Let $\phi : X \dashrightarrow X$ be a birational automorphism such that $\phi^{-1}(\mathcal{H}) \subset |-rK_X|$. We say that an involution τ of X untwists the map ϕ if $(\phi \circ \tau)^{-1}(\mathcal{H}) \subset |-r'K_X|$ for some $r' < r$. More generally, for a log pair $\left(X, \frac{1}{r}\mathcal{M}\right)$ with $\mathcal{M} \sim_{\mathbb{Q}} -K_X$ that is not terminal we also say that an involution τ of X untwists a maximal singularity of $\left(X, \frac{1}{r}\mathcal{M}\right)$ if $\tau(\mathcal{M}) \sim_{\mathbb{Q}} -r'K_X$ for some $r' < r$. For more generalized detail, refer to [6].

Note that the assumption $\ell \geqq 2$ implies that

$$N \in \{4, 5, 6, 7, 9, 12, 13, 15, 17, 20, 23, 25, 27, 30, 31, 33, 36, 38, 40, 41, 42, 44, 58, 61, 68, 76\}.$$

Lemma 4.2. *Suppose that $N \in \{6, 15, 23, 30, 36, 40, 41, 42, 44, 61, 68, 76\}$. Then Γ_X is the free product of two involutions τ_1 and τ_2.*

Proof. Suppose that $N = 36$. Then the hypersurface X is a sufficiently general hypersurface in $\mathbb{P}(1, 1, 4, 6, 7)$ of degree 18 with $-K_X^3 = 3/28$. It has three singular points, namely, the point P_1 that is a quotient singularity of type $\frac{1}{2}(1, 1, 1)$, the point P_2 that is a quotient singularity of type $\frac{1}{4}(1, 1, 3)$, and the point P_3 that is a quotient singularity of type $\frac{1}{7}(1, 1, 6)$.

Suppose that the group Γ_X is not the free product of the involutions τ_1 and τ_2. Then there is a linear system $\mathcal{M}$ without fixed components on the hypersurface X such that the set $\mathrm{CS}(X, \lambda\mathcal{M})$ contains at least two subvarieties of the hypersurface X, where λ is a positive rational number such that the divisor $-(K_X + \lambda\mathcal{M})$ is ample. Therefore, it follows from [6] that $\mathrm{CS}(X, \lambda\mathcal{M}) = \{P_2, P_3\}$.

The hypersurface X can be given by the quasihomogeneous equation of degree 18

$$x_3^3 x_4 + x_3^2 g(x_1, x_2, x_4, x_5) + x_3 h(x_1, x_2, x_4, x_5) + q(x_1, x_2, x_4, x_5) = 0 \subset \mathbb{P}(1, 1, 4, 6, 7),$$

where f, g, h, and q are quasihomogeneous polynomials. Then the point P_2 is located at $(0 : 0 : 1 : 0 : 0)$ and the point P_3 at $(0 : 0 : 0 : 0 : 1)$.

Let $\xi : X \dashrightarrow \mathbb{P}^7$ be the rational map that is given by the linear subsystem of the linear system $|-6K_X|$ consisting of the divisors

$$\mu x_4 + \sum_{i=0}^{6} \lambda_i x_1^i x_2^{6-i} = 0,$$

where $(\mu : \lambda_0 : \lambda_1 : \lambda_2 : \lambda_3 : \lambda_4 : \lambda_5 : \lambda_6) \in \mathbb{P}^7$. Then the rational map ξ is not defined at the points P_2 and P_3, the closure of the image of the rational map ξ is the surface $\mathbb{P}(1, 1, 6)$, and a general fiber of the map ξ is an elliptic curve. There is a commutative diagram

$$\begin{tikzcd}
 & W \arrow[dl, "\beta_3"'] \arrow[dr, "\beta_2"] \arrow[rrr, "\omega"] & & & \\
U_2 \arrow[dr, "\alpha_2"'] & & U_3 \arrow[dl, "\alpha_3"] & & \mathbb{P}(1,1,6), \\
 & X \arrow[urrr, dashed, "\xi"'] & & &
\end{tikzcd}$$

where α_2 is the Kawamata blow up at the singular point P_2, α_3 is the Kawamata blow up at the point P_3, β_2 is the Kawamata blow up at the point $\alpha_3^{-1}(P_2)$, β_3 is the Kawamata blow up at the point $\alpha_2^{-1}(P_3)$, and ω is an elliptic fibration.

Let S be the proper transform on the 3-fold W of a general surface of the linear system $\mathcal{M}$ and C be a general fiber of the fibration ω. The inequality $S \cdot C < 0$ follows from [12]. However, it is a contradiction because ω is an elliptic fibration.

Suppose that $N = 44$. Then X is a general hypersurface in $\mathbb{P}(1,2,5,6,7)$ of degree 20 with $-K_X^3 = 1/21$. The singularities of the hypersurface X consist of the points P_1, P_2, P_3 that are quotient singularities of type $\frac{1}{2}(1,1,1)$, the point P_4 that is a quotient singularity of type $\frac{1}{6}(1,1,5)$, and the point P_5 that is a quotient singularity of type $\frac{1}{7}(1,2,5)$. Moreover, there is a commutative diagram

$$\begin{tikzcd}
 & Y \arrow[dl, "\beta_5"'] \arrow[dr, "\beta_4"] \arrow[drr, "\eta"] & & \\
U_4 \arrow[dr, "\alpha_4"'] & & U_5 \arrow[dl, "\alpha_5"] & \mathbb{P}(1,2,5), \\
 & X \arrow[urr, dashed, "\psi"'] & &
\end{tikzcd}$$

where ψ is the natural projection, α_4 is the weighted blow up at the singular point P_4 with weights $(1,1,5)$, α_5 is the weighted blow up at the point P_5 with weights $(1,2,5)$, β_4 is the weighted blow up with weights $(1,1,5)$ at the point $\alpha_5^{-1}(P_4)$, β_5 is the weighted blow up with weights $(1,2,5)$ at the point $\alpha_4^{-1}(P_5)$, and η is an elliptic fibration. It follows from [6] that

$$\mathrm{CS}(X, \lambda\mathcal{M}) = \{P_4, P_5\},$$

and we can proceed as in the previous case to obtain a contradiction.

In the case when $N \in \{6, 15, 23, 30, 40, 41, 42, 61, 68, 76\}$ we can obtain a contradiction in the same way as in the case $N = 44$. $\square$

Lemma 4.3. *Suppose that* $N \in \{4, 9, 17, 27\}$. *Then* $\tau_1 \circ \tau_2 \circ \tau_3 = \tau_3 \circ \tau_2 \circ \tau_1$ *is the only relation among the birational involutions* τ_1, τ_2, *and* τ_3.

Proof. It follows from [6] that $\ell = 3$, $a_4 = a_5$, and $d = 3a_4$. A general fiber of the projection $\psi : X \dashrightarrow \mathbb{P}(1, a_2, a_3)$ is a smooth elliptic curve. Moreover, the hypersurface X has singular points P_1, P_2, P_3 of index a_4 which are the points of the indeterminacy of the map ψ.

Let $\pi : V \to X$ be the Kawamata blow up at the points P_1, P_2, P_3. We also let E_i be the exceptional divisor of π dominating P_i and $\phi = \psi \circ \pi$. Then π is a resolution of indeterminacy of the rational map ψ, the divisors E_1, E_2, E_3 are sections of ϕ, the equivalence

$$-K_V \sim_{\mathbb{Q}} \pi^*(-K_X) - \frac{1}{a_4}E_1 - \frac{1}{a_4}E_2 - \frac{1}{a_4}E_3$$

holds, the linear system $|-a_3a_4a_5K_V|$ is free and lies in the fibers of ϕ.

Let $\mathbb{F}$ be the field of rational functions on $\mathbb{P}(1, a_2, a_3)$ and C be a generic fiber of the elliptic fibration ϕ considered as an elliptic curve over $\mathbb{F}$. Then the section E_j of the elliptic fibration ϕ can be considered as an $\mathbb{F}$-rational point of the elliptic curve C.

One can show using Lemma 4.7 that $\mathbb{F}$-rational points E_1, E_2, E_3 are $\mathbb{Z}$-linearly independent in the group $\mathrm{Pic}(C)$.

By our construction, the curve C is a hypersurface of degree $3a_4$ in $\mathbb{P}(1, a_4, a_4) \cong \mathbb{P}^2$, which implies that the curve C can be naturally identified with a cubic curve in $\mathbb{P}^2$ such that the points E_1, E_2, E_3 lie on a single line in $\mathbb{P}^2$.

Let σ_i be the involution of the curve C that interchanges the fibers of the projection of the curve C from the point E_i. Then σ_i can also be considered as a birational involution of the 3-fold V such that

$$\sigma_i = \pi^{-1} \circ \tau_i \circ \pi \in \mathrm{Bir}(V).$$

Consider the curve C as a group scheme. Let Q_k be the point $(E_i + E_j)/2$ on the elliptic curve C, where $\{i, j\} = \{1, 2, 3\} \backslash \{k\}$. Then the involution σ_k is the reflection of the elliptic curve C at the point Q_k because the points E_1, E_2, E_3 are $\mathbb{Z}$-linearly independent, which implies that Q_1, Q_2, Q_3 are $\mathbb{Z}$-linearly independent and the compositions

$$\sigma_2 \circ \sigma_1 \circ \sigma_3, \quad \sigma_1 \circ \sigma_2 \circ \sigma_3, \quad \sigma_1 \circ \sigma_3 \circ \sigma_2$$

are reflections at E_1, E_2, E_3 respectively. Thus, we have the identity

$$\tau_1 \circ \tau_2 \circ \tau_3 = \tau_3 \circ \tau_2 \circ \tau_1,$$

which implies the similar identities that can be obtained from $\tau_1 \circ \tau_2 \circ \tau_3 = \tau_3 \circ \tau_2 \circ \tau_1$ by a permutation of the elements in the set $\{1, 2, 3\}$.

It follows from [6] that for any linear system $\mathscr{M}$ on the hypersurface X having no fixed components, the singularities of the log pair $\left(X, \frac{1}{r}\mathscr{M}\right)$ are canonical in the outside of the points P_1, P_2, P_3, where r is the natural number such that $\mathscr{M} \sim_{\mathbb{Q}} -rK_X$. Moreover, when the singularities of the log pair $\left(X, \frac{1}{r}\mathscr{M}\right)$ are not canonical at the point P_i, we have

$$\frac{1}{r}\mathscr{B} \sim_{\mathbb{Q}} \pi^*\left(\frac{1}{r}\mathscr{M}\right) - m_1E_1 - m_2E_2 - m_3E_3,$$

where $\mathscr{B}$ is the proper transform of $\mathscr{M}$ on V and $m_i > 1/a_4$. We have the inequality

$$m_1 + m_2 + m_3 \leqq \frac{3}{a_4},$$

which implies that the linear system $\mathscr{B}$ lies in the fibers of the elliptic fibration ϕ if the equality $m_1 + m_2 + m_3 = 3/a_4$ holds.

When the inequality $m_i > 1/a_4$ holds, the birational involution τ_i untwists the maximal singularity of the log pair $\left(X, \frac{1}{r}\mathcal{M}\right)$ at the point P_i, namely, the equivalence

$$\tau_i(\mathcal{M}) \sim_{\mathbb{Q}} -r' K_X$$

holds for some natural number $r' < r$. Similarly, the involution $\tau_i \circ \tau_k \circ \tau_j$ untwists the maximal singularities of the log pair $\left(X, \frac{1}{r}\mathcal{M}\right)$ at the points P_i and P_j simultaneously when the inequalities $m_i > 1/a_4$ and $m_j > 1/a_4$ hold for $i \neq j$, where $k \in \{1, 2, 3\} \setminus \{i, j\}$.

Now we can use the arguments of the proof of Theorem 7.8 in [16], Section V, to prove that the identity $\tau_1 \circ \tau_2 \circ \tau_3 = \tau_3 \circ \tau_2 \circ \tau_1$ is the only relation among our birational involutions τ_1, τ_2, and τ_3. However, it should be pointed out that the arguments of the proof of Theorem 7.8 in [16], Section V, are too sophisticated for our purposes[5]. □

Lemma 4.4. *Suppose that $N = 7$. Then there are no relations among $\tau_1, \dots, \tau_5$.*

Proof. The 3-fold X is a general hypersurface in $\mathbb{P}(1, 1, 2, 2, 3)$ of degree 8 which has singular points $P_1, \dots, P_4$ of type $\frac{1}{2}(1, 1, 1)$ and a singular point Q of type $\frac{1}{3}(1, 2, 1)$.

Let $\alpha_i : V_i \to X$ be the weighted blow up of X at the singular points P_i and Q with weights $\frac{1}{2}(1, 1, 1)$ and $\frac{1}{3}(1, 2, 1)$, respectively. Then

$$K_{V_i} \sim_{\mathbb{Q}} \alpha_i^*(K_X) + \frac{1}{2}E_i + \frac{1}{3}F_i,$$

where E_i and F_i are the exceptional divisors of the birational morphism α_i dominating the singular points P_i and Q, respectively. The linear system $|-2K_{V_i}|$ induces the morphism

$$\psi_i : V_i \to \mathbb{P}(1, 1, 2),$$

which is an elliptic fibration. Moreover, the divisor E_i is a 2-section of the fibration ψ_i, while the divisor F_i is a section of ψ_i. Up to relabelling, the birational involutions $\tau_1, \dots, \tau_5$ can be constructed as follows: the involution τ_i is induced by the reflection of a general fiber of the morphism ψ_i at the section F_i but the involution τ_5 is induced by the natural projection $X \dashrightarrow \mathbb{P}(1, 1, 2, 2)$.

---

[5] The following arguments are due to A. Borisov. Let W be a composition of σ_1, σ_2, σ_3 such that W is the identity map of the elliptic curve C and W does not contain squares of σ_i. Then we can show that W has even number of entries and each entry appears the same number of times in the even and the odd positions, and we can use the identity $\sigma_1 \circ \sigma_2 \circ \sigma_3 = \sigma_3 \circ \sigma_2 \circ \sigma_1$ to make σ_3 jump 2 spots left or right. Shifting the last σ_3 in the odd position in W that is followed not right away by σ_3 in the even position, we can collapse them and get a composition of σ_1, σ_2, σ_3 having a smaller number of entries. Therefore, the only relation among the involutions σ_1, σ_2, σ_3 is the identity $\sigma_1 \circ \sigma_2 \circ \sigma_3 = \sigma_3 \circ \sigma_2 \circ \sigma_1$.

Let $\mathcal{M}$ be a linear system on X without fixed components such that $\mathcal{M} \sim_{\mathbb{Q}} -rK_X$ for some natural number r. Then the singularities of the log pair $\left(X, \frac{1}{r}\mathcal{M}\right)$ are canonical in the outside of the points $P_1, \ldots, P_4$, Q due to [6].

Let $\mathcal{B}_i$ be the proper transform of $\mathcal{M}$ on V_i. Then

$$\mathcal{B}_i \sim_{\mathbb{Q}} \alpha_i^*(\mathcal{M}) - m_i E_i - m F_i,$$

where m_i and m are positive rational numbers. Moreover, the log pair $\left(X, \frac{1}{r}\mathcal{M}\right)$ is not canonical at the point P_i if and only if $m_i > r/2$. On the other hand, the singularities of the log pair $\left(X, \frac{1}{r}\mathcal{M}\right)$ are not canonical at Q if and only if $m > r/3$. Now intersecting the linear system $\mathcal{B}_i$ with a sufficiently general fiber of ψ_i, we see that

$$2m_i + m \leqq \frac{4r}{3}.$$

The equivalence $\tau_i(\mathcal{M}) \sim_{\mathbb{Q}} -r'K_X$ holds for some natural number r'. Moreover, the inequality $r' < r$ holds if $m_i > r/2$ when $i = 1, \ldots, 4$ or if $m > r/3$ when $i = 5$, namely, the involutions $\tau_1, \ldots, \tau_5$ untwist the maximal singularities of the log pair $\left(X, \frac{1}{r}\mathcal{M}\right)$.

In order to prove that the involutions $\tau_1, \ldots, \tau_5$ do not have any relation, it is enough to prove that the singularities of the log pair $\left(X, \frac{1}{r}\mathcal{M}\right)$ are not canonical at at most one point by Lemma 4.1. However, the inequality $2m_i + m \leqq 4r/3$ implies that the log pair $\left(X, \frac{1}{r}\mathcal{M}\right)$ is canonical at one of the singular points P_i and Q. To conclude the proof, therefore, we must show that for $i \neq j$ the singularities of the log pair $\left(X, \frac{1}{r}\mathcal{M}\right)$ are canonical at one of the points P_i and P_j.

Suppose that the log pair $\left(X, \frac{1}{r}\mathcal{M}\right)$ is not canonical at the points P_1 and P_2. Let S be a general surface in $|-K_X|$ and C be the base curve of $|-K_X|$. Then S is a K3 surface whose singular points are the singular points of X. Moreover, the point P_i is a singular point of type $\mathbb{A}_1$ on the surface S and the point Q is a singular point of type $\mathbb{A}_2$ on S.

The curve C is a smooth curve passing through the points $P_1, \ldots, P_4$, and Q. We have

$$\mathcal{M}|_S = \mathcal{P} + \mathrm{mult}_C(\mathcal{M})C,$$

where $\mathcal{P}$ is a linear system without fixed components. The inequality $\mathrm{mult}_C(\mathcal{M}) \leqq r$ holds; otherwise the log pair $\left(X, \frac{1}{r}\mathcal{M}\right)$ would not be canonical at the point Q by [12].

Let $\pi : Y \to S$ be the composition of blow ups of the singular points P_1 and P_2, G_i be the exceptional divisor of π dominating P_i, $\bar{C}$ be the proper transform on the surface Y

of the curve C, and $\mathscr{H}$ be the proper transform on the surface Y of the linear system $\mathscr{P}$. Then

$$\mathscr{H} + \mathrm{mult}_C(\mathscr{M})\bar{C} \sim_{\mathbb{Q}} \pi^*(-rK_X|_S) - \bar{m}_1 G_1 - \bar{m}_2 G_2,$$

where $\bar{m}_i \geqq m_i > r/2$. However, we have $\bar{C}^2 = -1/3$ on the surface Y and we see that

$$-\frac{r}{3} \leqq \big(\mathscr{H} + \mathrm{mult}_C(\mathscr{M})\bar{C}\big) \cdot \bar{C} \leqq \frac{2r}{3} - \bar{m}_1 - \bar{m}_2 < -\frac{r}{3},$$

which is a contradiction. $\square$

Lemma 4.5. *Suppose that $N = 20$. Then there are no relations among τ_1, τ_2, τ_3.*

Proof. We have a general hypersurface $X \subset \mathbb{P}(1,1,3,4,5)$ given by

$$x_3^4 f_1(x_1,x_2) + x_3^3 f_4(x_1,x_2,x_4) + x_3^2 f_7(x_1,x_2,x_4,x_5) + x_3 f_{10}(x_1,x_2,x_4,x_5) + f_{13}(x_1,x_2,x_4,x_5) = 0,$$

where f_i is a general quasihomogeneous polynomial of degree i. The 3-fold X has 3 singular points at $P = (0:0:1:0:0)$, $Q = (0:0:0:1:0)$, $O = (0:0:0:0:1)$ and a general fiber of the natural projection of X to $\mathbb{P}(1,1,3)$ is an elliptic curve. However, a general fiber of the natural projection of X to $\mathbb{P}(1,1,4)$ may not be an elliptic curve.

Let us take $t = x_3 f_1(x_1,x_2) + f_4(x_1,x_2,x_4)$ as a homogeneous variable of weight 4 instead of the homogeneous variable x_4. Then the hypersurface X is given by the equation

$$x_3^3 t + x_3^2 g_7(x_1,x_2,t,x_5) + x_3 g_{10}(x_1,x_2,t,x_5) + g_{13}(x_1,x_2,t,x_5) = 0,$$

where g_i is a sufficiently general quasihomogeneous polynomial of degree i. A general fiber of the natural projection of X to $\mathbb{P}(1,1,4)$ is an elliptic curve.

Up to relabelling, the involutions τ_1, τ_2, τ_3 can be constructed as follows:

- the birational involution τ_1 is induced by the reflection of a general fiber of the natural projection $X \dashrightarrow \mathbb{P}(1,1,4)$ at the point O;
- the birational involution τ_2 is induced by the reflection of a general fiber of the natural projection $X \dashrightarrow \mathbb{P}(1,1,3)$ at the point O;
- the birational involution τ_3 is induced by the reflection of a general fiber of the natural projection $X \dashrightarrow \mathbb{P}(1,1,3)$ at the point Q but the involution τ_3 is also induced by the natural projection $X \dashrightarrow \mathbb{P}(1,1,3,4)$.

Let $\mathscr{M}$ be a linear system on X without fixed components such that $\mathscr{M} \sim_{\mathbb{Q}} -rK_X$ for some natural number r. Then the singularities of the log pair $\left(X, \frac{1}{r}\mathscr{M}\right)$ are canonical in the outside of the points P, Q, O due to [6], and the equivalence $\tau_i(\mathcal{M}) \sim_{\mathbb{Q}} -r'K_X$ holds for some natural number $r' < r$ in the following cases:

- the log pair $\left(X, \frac{1}{r}\mathcal{M}\right)$ is not canonical at the point P and $i = 1$;
- the log pair $\left(X, \frac{1}{r}\mathcal{M}\right)$ is not canonical at the point Q and $i = 2$;
- the log pair $\left(X, \frac{1}{r}\mathcal{M}\right)$ is not canonical at the point O and $i = 3$.

In order to prove that the involutions τ_1, τ_2, τ_3 are not related by any relation, by Lemma 4.1 it is enough to show that the singularities of $\left(X, \frac{1}{r}\mathcal{M}\right)$ are not canonical at at most one point.

Suppose that $\left(X, \frac{1}{r}\mathcal{M}\right)$ is not canonical at the points P and O. Let $\alpha : V \to X$ be the Kawamata blow up at the points P and O. Then

$$K_V \sim_{\mathbb{Q}} \alpha^*(K_X) + \frac{1}{3}E + \frac{1}{5}F,$$

where E and F are the exceptional divisors of the birational morphism α dominating the singular points P and O, respectively. The linear system $|-4K_V|$ does not have base points and induces the morphism $\psi : V \to \mathbb{P}(1,1,4)$ which is an elliptic fibration. The divisor F is a section of ψ and the divisor E is a 2-section of ψ. Let $\mathcal{B}$ be the proper transform of the linear system $\mathcal{M}$ on the 3-fold V. Then

$$\mathcal{B} \sim_{\mathbb{Q}} \alpha^*(\mathcal{M}) - aE - bF,$$

where a and b are rational numbers such that $a > r/3$ and $b > r/5$. Intersecting the linear system $\mathcal{B}$ with a sufficiently general fiber of ψ, we see that

$$2a + b \leqq \frac{52r}{60},$$

which is impossible because $a > r/3$ and $b > r/5$.

We next suppose that the singularities of the log pair $\left(X, \frac{1}{r}\mathcal{M}\right)$ are not canonical at the singular points Q and O. Let $\gamma : W \to X$ be the composition of the weighted blow ups at the points Q and O with weights $(1,1,3)$ and $(1,1,4)$, respectively. Then

$$K_W \sim_{\mathbb{Q}} \gamma^*(K_X) + \frac{1}{4}G + \frac{1}{5}H,$$

where G and H are the γ-exceptional divisors dominating the singular points Q and O, respectively. Moreover, there is a commutative diagram

$$\begin{tikzcd} & W \arrow[dl, "\gamma"'] \arrow[dr, dashed, "\phi"] & \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,1,3), \end{tikzcd}$$

where ψ is the natural projection and ϕ is the rational map given by $|-3K_W|$.

Let $\mathscr{D}$ be the proper transform of $\mathscr{M}$ on W. Then

$$\mathscr{D} \sim_{\mathbb{Q}} \gamma^*(\mathscr{M}) - cG - dH,$$

where $c > r/4$ and $d > r/5$ by [12].

The natural projection ψ has a one-dimensional family of fibers that have a singularity at the singular point O. Let C be the proper transform on the variety W of a sufficiently general fiber of the projection ψ that is singular at the point O. Intersecting a general surface of the linear system $\mathscr{D}$ with the curve C, we obtain the inequality

$$c + 2d \leqq \frac{13r}{20},$$

which is impossible because $c > r/4$ and $d > r/5$.

Let S be a sufficiently general surface in the linear system $|-K_X|$ and L be the curve on the hypersurface X cut by the equations $x_1 = x_2 = 0$. Then S is a K3 surface whose singular points are the singular points of the hypersurface X. Moreover, one can easily show that the point P is a singular point of type $\mathbb{A}_2$ on the surface S, the point Q is a singular point of type $\mathbb{A}_3$ on the surface S, and the point O is a singular point of type $\mathbb{A}_4$ on the surface S. The curve L is a smooth rational curve passing through P, Q, and O. We have

$$\mathscr{M}|_S = \mathscr{P} + \mathrm{mult}_L(\mathscr{M})L,$$

where $\mathscr{P}$ is a linear system on S without fixed components. Moreover, it immediately follows from [12] that the inequality $\mathrm{mult}_L(\mathscr{M}) \leqq r$ holds because we already proved that the singularities of $\left(X, \frac{1}{r}\mathscr{M}\right)$ are canonical at least at one of the points P, Q, and O.

Finally, we suppose that the singularities of the log pair $\left(X, \frac{1}{r}\mathscr{M}\right)$ are not canonical at the singular points Q and P. Let $\pi : Y \to S$ be the composition of the weighted blow ups at the points P and Q that are induced by the Kawamata blow ups of the hypersurface X at the singular points P and Q. Then

$$\mathscr{H} + \mathrm{mult}_L(\mathscr{M})\bar{L} \sim_{\mathbb{Q}} \pi^*(-rK_X|_S) - m_1E_1 - m_2E_2,$$

where E_1 and E_2 are the π-exceptional divisors dominating P and Q, respectively, $\bar{L}$ is the proper transform on the surface Y of the curve L, $\mathscr{H}$ is the proper transform on Y of the linear system $\mathscr{P}$, m_1 and m_2 are rational numbers. Then $\bar{L}^2 = -1/30$, but it follows from the paper [12] that the inequalities $m_1 > r/3$ and $m_2 > r/4$ hold.

The curve $\bar{L}$ intersects the curves E_1 and E_2 at singular points of types $\mathbb{A}_1$ and $\mathbb{A}_2$ respectively. Therefore, the inequalities $\bar{L} \cdot E_1 \geqq 1/2$ and $\bar{L} \cdot E_2 \geqq 1/3$ hold. Consequently, we obtain

$$-\frac{r}{30} \leqq \big(\mathscr{H} + \mathrm{mult}_L(\mathscr{M})\bar{L}\big) \cdot \bar{L} \leqq \frac{13r}{60} - \frac{m_1}{2} - \frac{m_2}{3} < -\frac{r}{30},$$

which is a contradiction. □

Therefore, to conclude the proof of Theorem 1.1, it is enough to consider the cases

$$N \in \{5, 12, 13, 25, 31, 33, 38, 58\}.$$

In these cases the group Γ_X is generated by two involutions τ_1 and τ_2. We must show that the group Γ_X is the free product of the groups $\langle\tau_1\rangle$ and $\langle\tau_2\rangle$.

Perhaps, the simplest possible way to prove the required claim is to use the arguments of the proofs of Lemmas 4.4 and 4.5. For example, the arguments used during the elimination of the points Q and O in the proof of Lemma 4.5 immediately imply the required claim in the case $N = 5$. However, we choose an alternative approach.

Let $\psi : X \dashrightarrow \mathbb{P}(1, a_2, a_3)$ be the natural projection.

Lemma 4.6. *There are only finitely many reducible fibers of ψ.*

Proof. We consider only the case $N = 58$ because the other cases are similar. Then X is a sufficiently general hypersurface of degree 24 in $\mathbb{P}(1, 3, 4, 7, 10)$. It is enough to show that the fiber C of the projection ψ over a point $(p_1 : p_2 : p_3) \in \mathbb{P}(1, 3, 4)$ is irreducible if $p_1 \neq 0$ and $(p_1 : p_2 : p_3)$ belongs to the complement to a finite set.

By construction, the fiber C is a curve of degree $24/70$ in

$$\mathbb{P}(1, 7, 10) = \mathrm{Proj}(\mathbb{C}[x_1, x_4, x_5]),$$

where $\mathrm{wt}(x_1) = 1$, $\mathrm{wt}(x_4) = 7$, and $\mathrm{wt}(x_5) = 10$. If the curve C is reducible, it must contain a curve of degree $1/70$, $1/10$, or $1/7$. However, we have a unique curve of degree $1/70$ in $\mathbb{P}(1, 7, 10)$, namely, the curve defined by $x_1 = 0$. Hence, the fiber C cannot contain the curve of degree $1/70$ by the generality of the hypersurface X.

Let $\mathscr{X} = |\mathcal{O}_{\mathbb{P}(1,3,4,7,10)}(24)|$ and $\mathscr{C}_{1/7}$ be the set of curves in $\mathbb{P}(1, 3, 4, 7, 10)$ given by

$$\lambda x_1^3 + x_2 = \mu x_1^4 + x_3 = v_0 x_1^{10} + v_1 x_5 + v_2 x_1^3 x_4 = 0,$$

where $(v_0 : v_1 : v_2) \in \mathbb{P}^2$ and $(\lambda, \mu) \in \mathbb{C}^2$. Put

$$\Gamma = \{(X, C) \in \mathscr{X} \times \mathscr{C}_{1/7} \mid C \subset X\}$$

and consider the natural projections $f : \Gamma \to \mathscr{X}$ and $g : \Gamma \to \mathscr{C}_{1/7}$. Then the projection g is surjective, $\dim\big(g^{-1}(x_2 = x_3 = x_5 = 0)\big) = \dim(\mathscr{X}) - 4$, and $\dim(\mathscr{C}_{1/7}) = 4$. Thus, we have

$$\dim(\mathscr{X}) \geqq \dim(\Gamma),$$

which implies that X contains finitely many curves of degree $1/7$. Similarly, it is impossible to have infinitely many curves of degree $1/10$ on X. Therefore, the fiber C is irreducible whenever the point P is in the outside of the finitely many points in $\mathbb{P}(1,3,4)$ and not in the hyperplane $x_1 = 0$. Consequently, the statement for the case $N = 58$ is true. □

The rational map ψ is not defined at two distinct points of the hypersurface X, which we denote by P and Q. Let C be a very general fiber of the map ψ. Then C is a smooth elliptic curve passing through the points P and Q. Moreover, the following well known result implies that the divisor $P - Q$ is not a torsion in $\mathrm{Pic}(C)$.

Lemma 4.7. *Let $\tau : S \to \mathbb{P}^1$ be an elliptic fibration such that the surface S is normal and all fibers of the elliptic fibration τ are irreducible. Suppose that there are distinct disjoint sections C_1 and C_2 of the elliptic fibration τ such that $C_1^2 < 0$ and $C_2^2 < 0$. Then for a very general fiber L of the elliptic fibration τ the divisor $(C_1 - C_2)|_L$ is not a torsion in* $\mathrm{Pic}(L)$.

Proof. For every natural number n we have

$$n(C_1 - C_2)|_L \sim 0 \Rightarrow C_1 - C_2 \equiv \Sigma,$$

where Σ is a $\mathbb{Q}$-divisor on the surface S whose support is contained in the fibers of the elliptic fibration τ. On the other hand, because all fibers of τ are irreducible, the curves C_1, C_2, and L are linearly dependent in the group $\mathrm{Div}(S) \otimes \mathbb{Q}/\equiv$. However,

$$\begin{vmatrix} C_1^2 & C_1 \cdot C_2 & C_1 \cdot L \\ C_1 \cdot C_2 & C_2^2 & C_2 \cdot L \\ C_1 \cdot L & C_2 \cdot L & L^2 \end{vmatrix} = -C_1^2 - C_2^2 \neq 0,$$

which contradicts the linear dependence of the curves C_1, C_2, and L. □

The curve C is invariant under the action of the birational involutions τ_1 and τ_2. Moreover, up to relabelling the involutions τ_1 and τ_2 act on the elliptic curve C by reflections with respect to the points Q and P, respectively. Hence, the composition $\tau_1 \circ \tau_2$ acts on the smooth elliptic curve C by the translation by $2(P - Q)$. Therefore, the composition $(\tau_1 \circ \tau_2)^n$ never acts identically on the curve C for any natural number $n \neq 0$ because the divisor $P - Q$ on the curve C is not a torsion in $\mathrm{Pic}(C)$. Hence, the group Γ_X is the free product of the groups $\langle \tau_1 \rangle$ and $\langle \tau_2 \rangle$, which concludes the proof of Theorem 1.1.

5. Potential density

Suppose that the hypersurface X is defined over a number field $\mathbb{F}$. The purpose of this section is to complete the proof of Proposition 1.4 by proving the potential density of the set of rational points of the hypersurface X in the cases $N = 11$ and 19.

Lemma 5.1. *Suppose that $N = 19$. Then rational points on X are potentially dense.*

Proof. The 3-fold X is a general hypersurface in $\mathbb{P}_{\mathbb{F}}(1,2,3,3,4)$ given by the equation

$$\sum_{\substack{i,j,k,l,m\geqq 0\\ i+a_2j+a_3k+a_4l+a_5m=12}} a_{ijklm}x_1^ix_2^jx_3^kx_4^lx_5^m=0,$$

where $a_{ijklm}\in\mathbb{F}$ and we may assume that $a_{00040}=0$ and $a_{00003}=1$ possibly after replacing the field $\mathbb{F}$ by its finite extension. Let $P=(0:0:0:1:0)$. Then X has a cyclic quotient singularity of type $\frac{1}{3}(1,2,1)$ at the point P.

Let $\alpha: V\to X$ be the Kawamata blow up at P. Then the equality $-K_V^3=0$ holds, the linear system $|-6K_V|$ has no base point, and

$$K_V\sim_{\mathbb{Q}}\alpha^*(K_X)+\frac{1}{3}E,$$

where $E=\alpha^{-1}(P)\cong\mathbb{P}(1,1,2)$. Let $\psi: V\to\mathbb{P}(1,2,3)$ be the morphism given by the linear system $|-6K_V|$. Then ψ is an elliptic fibration (see the proof of Lemma 2.6).

The restriction $\psi|_E: E\to\mathbb{P}(1,2,3)$ is a triple cover, namely, the divisor E is a 3-section of the elliptic fibration ψ. In the case when $\psi|_E$ is branched at a point contained in a smooth fiber of ψ, the set of rational points on V is potentially dense (see [1]) because E is a rational surface. Therefore, it is enough to find a smooth fiber C of the fibration ψ such that the intersection $C\cap E$ consists of at most two points.

Let Z be the curve on X given by the equations $x_2=\lambda x_1^2$ and $x_3=\mu x_1^3$, where $\lambda,\mu\in\mathbb{F}$, and $\hat{Z}=\alpha^{-1}(Z)$. Then $\hat{Z}$ is a fiber of ψ. The intersection $\hat{Z}\cap E$ consists of three different points if and only if Z has an ordinary triple point at P. However, the curve Z has an ordinary triple point at the point P if and only if the homogeneous polynomial

$$f(x_1,x_5)=x_5^3+a_{10012}x_1x_5^2+x_5x_1^2(a_{20021}+\lambda a_{01021})+x_1^3(\mu a_{00130}+\lambda a_{10030}+a_{30030})$$

has three distinct roots. Now if we put

$$\lambda=\frac{a_{10012}^2-4a_{20021}}{4a_{01021}}\quad\text{and}\quad\mu=-\frac{\lambda a_{10030}+a_{30030}}{a_{00130}},$$

then the generality of the hypersurface X together with the Bertini theorem implies that the curve $\hat{Z}$ is smooth but the intersection $\hat{Z}\cap E$ consists of only two different points. □

To prove the potential density of the case $N=11$, we first consider a general surface in $|-K_X|$.

Lemma 5.2. *Let Y be a general surface in $|-K_X|$. Suppose that at least one singular point of Y is defined over the field $\mathbb{F}$. Then the set of $\mathbb{F}$-rational points of Y is Zariski dense.*

Proof. We have a hypersurface $Y\subset\mathbb{P}(1,2,2,5)$ which can be given by the equation

$$x_4^2 = x_1^2 f_4(x_2, x_3) + x_1^4 f_3(x_2, x_3) + x_1^6 f_2(x_2, x_3) + x_1^8 f_1(x_2, x_3) + x_1^{10} + x_3 g_4(x_2, x_3),$$

where f_i and g_i are general homogeneous polynomials of degree i.

Let P be the point $(0:1:0:0)$ and $\mathscr{H}$ be the pencil of curves on Y given by the equations $\lambda x_1^2 + \mu x_3 = 0$, $(\lambda : \mu) \in \mathbb{P}^1_{\mathbb{F}}$. Then Y has a singularity of type $\mathbb{A}_1$ at the point P which is a unique base point of the pencil $\mathscr{H}$.

Let C be the curve in $\mathscr{H}$ corresponding to the point $(\lambda : \mu) \in \mathbb{P}^1_{\mathbb{F}}$ and

$$f_4(x_2, x_3) = \sum_{i=0}^{4} \alpha_i x_2^i x_3^{4-i}, \quad g_4(x_2, x_3) = \sum_{i=0}^{4} \beta_i x_2^i x_3^{4-i},$$

where α_i and β_i are sufficiently general constants. Then the curve C has an ordinary double point at the point P when $(\lambda : \mu) \neq (1:0)$ and $(\lambda : \mu) \neq (\alpha_4 : \beta_4)$. Let F be the curve in the pencil $\mathscr{H}$ corresponding to the point $(\lambda : \mu) = (\alpha_4 : \beta_4)$ and L be the curve on the surface Y given by the equation $x_1 = 0$. Then F is smooth in the outside of P and has an ordinary cusp at P, while L is a smooth rational curve.

Let $\pi : W \to Y$ be the blow up at the point P, E be the π-exceptional divisor, and $\mathscr{B}$ be the proper transform of the pencil $\mathscr{H}$ on the surface W. Then $\mathscr{B}$ has no base point and induces an elliptic fibration $\psi : W \to \mathbb{P}^1$. The proper transform $\hat{F}$ of F by π is a smooth elliptic fiber of the fibration ψ. Moreover, the restriction $\pi|_E : E \to \mathbb{P}^1$ is a double cover branched at the point $\hat{F} \cap E$. Because the set of all $\mathbb{F}$-rational points of the curve E is Zariski dense, it follows from [1] that the set of $\mathbb{F}$-rational points of the surface S is Zariski dense. □

Because we may assume that the singular points of X are $\mathbb{F}$-rational by replacing $\mathbb{F}$ by its finite extension, one can easily prove the density of $\mathbb{F}$-rational points on X with the lemma above.

6. Appendix

The list of quasismooth anticanonically embedded weighted Fano 3-fold hypersurfaces is found in [9]. The completeness of the list is proved in [11].

Table 1. Weighted Fano hypersurfaces of degree d in $\mathbb{P}(1, a_2, a_3, a_4, a_5)$.

N	d	a_2	a_3	a_4	a_5	$-K_X^3$	Sing(X)	Γ_X
1	4	1	1	1	1	4	$\emptyset$	$\mathbf{F}^0$
2	5	1	1	1	2	5/2	$\frac{1}{2}(1,1,1)$	$\mathbf{F}^1$
3	6	1	1	1	3	2	$\emptyset$	$\mathbf{F}^0$
4	6	1	1	2	2	3/2	$3 \times \frac{1}{2}(1,1,2)$	$\hat{\mathbf{F}}^3$
5	7	1	1	2	3	7/6	$\frac{1}{2}(1,1,1)$, $\frac{1}{3}(1,1,2)$	$\mathbf{F}^2$
6	8	1	1	2	4	1	$2 \times \frac{1}{2}(1,1,1)$	$\mathbf{F}^2$

Table 1. Continued

N	d	a_2	a_3	a_4	a_5	$-K_X^3$	$\mathrm{Sing}(X)$	Γ_X
7	8	1	2	2	3	2/3	$4\times\frac{1}{2}(1,1,1), \frac{1}{3}(1,1,2)$	$\mathbf{F}^5$
8	9	1	1	3	4	3/4	$\frac{1}{4}(1,1,3)$	$\mathbf{F}^1$
9	9	1	2	3	3	1/2	$\frac{1}{2}(1,1,1), 3\times\frac{1}{3}(1,1,2)$	$\hat{\mathbf{F}}^3$
10	10	1	1	3	5	2/3	$\frac{1}{3}(1,1,2)$	$\mathbf{F}^0$
11	10	1	2	2	5	1/2	$5\times\frac{1}{2}(1,1,1)$	$\mathbf{F}^0$
12	10	1	2	3	4	5/12	$2\times\frac{1}{2}(1,1,1), \frac{1}{3}(1,1,2), \frac{1}{4}(1,1,3)$	$\mathbf{F}^2$
13	11	1	2	3	5	11/30	$\frac{1}{2}(1,1,1), \frac{1}{3}(1,1,2), \frac{1}{5}(1,2,3)$	$\mathbf{F}^2$
14	12	1	1	4	6	1/2	$\frac{1}{2}(1,1,1)$	$\mathbf{F}^0$
15	12	1	2	3	6	1/3	$2\times\frac{1}{2}(1,1,1), 2\times\frac{1}{3}(1,1,2)$	$\mathbf{F}^2$
16	12	1	2	4	5	3/10	$3\times\frac{1}{2}(1,1,1), \frac{1}{5}(1,1,4)$	$\mathbf{F}^1$
17	12	1	3	4	4	1/4	$3\times\frac{1}{4}(1,1,3)$	$\hat{\mathbf{F}}^3$
18	12	2	2	3	5	1/5	$6\times\frac{1}{2}(1,1,1), \frac{1}{5}(1,2,3)$	$\mathbf{F}^1$
19	12	2	3	3	4	1/6	$3\times\frac{1}{2}(1,1,1), 4\times\frac{1}{3}(1,1,2)$	$\mathbf{F}^0$
20	13	1	3	4	5	13/60	$\frac{1}{3}(1,1,2), \frac{1}{4}(1,1,3), \frac{1}{5}(1,1,4)$	$\mathbf{F}^3$
21	14	1	2	4	7	1/4	$3\times\frac{1}{2}(1,1,1), \frac{1}{4}(1,1,3)$	$\mathbf{F}^0$
22	14	2	2	3	7	1/6	$7\times\frac{1}{2}(1,1,1), \frac{1}{3}(1,1,2)$	$\mathbf{F}^0$
23	14	2	3	4	5	7/60	$3\times\frac{1}{2}(1,1,1), \frac{1}{3}(1,1,2), \frac{1}{4}(1,1,3), \frac{1}{5}(1,2,3)$	$\mathbf{F}^2$
24	15	1	2	5	7	3/14	$\frac{1}{2}(1,1,1), \frac{1}{7}(1,2,5)$	$\mathbf{F}^1$
25	15	1	3	4	7	5/28	$\frac{1}{4}(1,1,3), \frac{1}{7}(1,3,4)$	$\mathbf{F}^2$
26	15	1	3	5	6	1/6	$2\times\frac{1}{3}(1,1,2), \frac{1}{6}(1,1,5)$	$\mathbf{F}^1$
27	15	2	3	5	5	1/10	$\frac{1}{2}(1,1,1), 3\times\frac{1}{5}(1,2,3)$	$\hat{\mathbf{F}}^3$
28	15	3	3	4	5	1/12	$5\times\frac{1}{3}(1,1,2), \frac{1}{4}(1,1,3)$	$\mathbf{F}^0$
29	16	1	2	5	8	1/5	$2\times\frac{1}{2}(1,1,1), \frac{1}{5}(1,2,3)$	$\mathbf{F}^0$
30	16	1	3	4	8	1/6	$\frac{1}{3}(1,1,2), 2\times\frac{1}{4}(1,1,3)$	$\mathbf{F}^2$
31	16	1	4	5	6	2/15	$\frac{1}{2}(1,1,1), \frac{1}{5}(1,1,4), \frac{1}{6}(1,1,5)$	$\mathbf{F}^2$
32	16	2	3	4	7	2/21	$4\times\frac{1}{2}(1,1,1), \frac{1}{3}(1,1,2), \frac{1}{7}(1,3,4)$	$\mathbf{F}^1$
33	17	2	3	5	7	17/210	$\frac{1}{2}(1,1,1), \frac{1}{3}(1,1,2), \frac{1}{5}(1,2,3), \frac{1}{7}(1,2,5)$	$\mathbf{F}^2$
34	18	1	2	6	9	1/6	$3\times\frac{1}{2}(1,1,1), \frac{1}{3}(1,1,2)$	$\mathbf{F}^0$
35	18	1	3	5	9	2/15	$2\times\frac{1}{3}(1,1,2), \frac{1}{5}(1,1,4)$	$\mathbf{F}^0$
36	18	1	4	6	7	3/28	$\frac{1}{4}(1,1,3), \frac{1}{2}(1,1,1), \frac{1}{7}(1,1,6)$	$\mathbf{F}^2$

Table 1. Continued

N	d	a_2	a_3	a_4	a_5	$-K_X^3$	$\mathrm{Sing}(X)$	Γ_X
37	18	2	3	4	9	1/12	$4\times\frac{1}{2}(1,1,1)$, $2\times\frac{1}{3}(1,1,2)$, $\frac{1}{4}(1,1,3)$	$\mathbf{F}^0$
38	18	2	3	5	8	3/40	$2\times\frac{1}{2}(1,1,1)$, $\frac{1}{5}(1,2,3)$, $\frac{1}{8}(1,3,5)$	$\mathbf{F}^2$
39	18	3	4	5	6	1/20	$3\times\frac{1}{3}(1,1,2)$, $\frac{1}{4}(1,1,3)$, $\frac{1}{2}(1,1,1)$, $\frac{1}{5}(1,1,4)$	$\mathbf{F}^0$
40	19	3	4	5	7	19/420	$\frac{1}{3}(1,1,2)$, $\frac{1}{4}(1,1,3)$, $\frac{1}{5}(1,2,3)$, $\frac{1}{7}(1,3,4)$	$\mathbf{F}^2$
41	20	1	4	5	10	1/10	$\frac{1}{2}(1,1,1)$, $2\times\frac{1}{5}(1,1,4)$	$\mathbf{F}^2$
42	20	2	3	5	10	1/15	$2\times\frac{1}{2}(1,1,1)$, $\frac{1}{3}(1,1,2)$, $2\times\frac{1}{5}(1,2,3)$	$\mathbf{F}^2$
43	20	2	4	5	9	1/18	$5\times\frac{1}{2}(1,1,1)$, $\frac{1}{9}(1,4,5)$	$\mathbf{F}^1$
44	20	2	5	6	7	1/21	$3\times\frac{1}{2}(1,1,1)$, $\frac{1}{6}(1,1,5)$, $\frac{1}{7}(1,2,5)$	$\mathbf{F}^2$
45	20	3	4	5	8	1/24	$\frac{1}{3}(1,1,2)$, $2\times\frac{1}{4}(1,1,3)$, $\frac{1}{8}(1,3,5)$	$\mathbf{F}^1$
46	21	1	3	7	10	1/10	$\frac{1}{10}(1,3,7)$	$\mathbf{F}^1$
47	21	1	5	7	8	3/40	$\frac{1}{5}(1,2,3)$, $\frac{1}{8}(1,1,7)$	$\mathbf{F}^1$
48	21	2	3	7	9	1/18	$\frac{1}{2}(1,1,1)$, $2\times\frac{1}{3}(1,1,2)$, $\frac{1}{9}(1,2,7)$	$\mathbf{F}^1$
49	21	3	5	6	7	1/30	$3\times\frac{1}{3}(1,1,2)$, $\frac{1}{5}(1,2,3)$, $\frac{1}{6}(1,1,5)$	$\mathbf{F}^0$
50	22	1	3	7	11	2/21	$\frac{1}{3}(1,1,2)$, $\frac{1}{7}(1,3,4)$	$\mathbf{F}^0$
51	22	1	4	6	11	1/12	$\frac{1}{4}(1,1,3)$, $\frac{1}{2}(1,1,1)$, $\frac{1}{6}(1,1,5)$	$\mathbf{F}^0$
52	22	2	4	5	11	1/20	$5\times\frac{1}{2}(1,1,1)$, $\frac{1}{4}(1,1,3)$, $\frac{1}{5}(1,1,4)$	$\mathbf{F}^0$
53	24	1	3	8	12	1/12	$2\times\frac{1}{3}(1,1,2)$, $\frac{1}{4}(1,1,3)$	$\mathbf{F}^0$
54	24	1	6	8	9	1/18	$\frac{1}{2}(1,1,1)$, $\frac{1}{3}(1,1,2)$, $\frac{1}{9}(1,1,8)$	$\mathbf{F}^1$
55	24	2	3	7	12	1/21	$2\times\frac{1}{2}(1,1,1)$, $2\times\frac{1}{3}(1,1,2)$, $\frac{1}{7}(1,2,5)$	$\mathbf{F}^0$
56	24	2	3	8	11	1/22	$3\times\frac{1}{2}(1,1,1)$, $\frac{1}{11}(1,3,8)$	$\mathbf{F}^1$
57	24	3	4	5	12	1/30	$2\times\frac{1}{3}(1,1,2)$, $2\times\frac{1}{4}(1,1,3)$, $\frac{1}{5}(1,2,3)$	$\mathbf{F}^0$
58	24	3	4	7	10	1/35	$\frac{1}{2}(1,1,1)$, $\frac{1}{7}(1,3,4)$, $\frac{1}{10}(1,3,7)$	$\mathbf{F}^2$
59	24	3	6	7	8	1/42	$4\times\frac{1}{3}(1,1,2)$, $\frac{1}{2}(1,1,1)$, $\frac{1}{7}(1,1,6)$	$\mathbf{F}^0$
60	24	4	5	6	9	1/45	$2\times\frac{1}{2}(1,1,1)$, $\frac{1}{5}(1,1,4)$, $\frac{1}{3}(1,1,2)$, $\frac{1}{9}(1,4,5)$	$\mathbf{F}^1$
61	25	4	5	7	9	5/252	$\frac{1}{4}(1,1,3)$, $\frac{1}{7}(1,2,5)$, $\frac{1}{9}(1,4,5)$	$\mathbf{F}^2$
62	26	1	5	7	13	2/35	$\frac{1}{5}(1,2,3)$, $\frac{1}{7}(1,1,6)$	$\mathbf{F}^0$
63	26	2	3	8	13	1/24	$3\times\frac{1}{2}(1,1,1)$, $\frac{1}{3}(1,1,2)$, $\frac{1}{8}(1,3,5)$	$\mathbf{F}^0$
64	26	2	5	6	13	1/30	$4\times\frac{1}{2}(1,1,1)$, $\frac{1}{5}(1,2,3)$, $\frac{1}{6}(1,1,5)$	$\mathbf{F}^0$
65	27	2	5	9	11	3/110	$\frac{1}{2}(1,1,1)$, $\frac{1}{5}(1,1,4)$, $\frac{1}{11}(1,2,9)$	$\mathbf{F}^1$
66	27	5	6	7	9	1/70	$\frac{1}{5}(1,1,4)$, $\frac{1}{6}(1,1,5)$, $\frac{1}{3}(1,1,2)$, $\frac{1}{7}(1,2,5)$	$\mathbf{F}^0$

Table 1. Continued

N	d	a_2	a_3	a_4	a_5	$-K_X^3$	$\mathrm{Sing}(X)$	Γ_X
67	28	1	4	9	14	1/18	$\frac{1}{2}(1,1,1)$, $\frac{1}{9}(1,4,5)$	$\mathbf{F}^0$
68	28	3	4	7	14	1/42	$\frac{1}{3}(1,1,2)$, $\frac{1}{2}(1,1,1)$, $2\times\frac{1}{7}(1,3,4)$	$\mathbf{F}^2$
69	28	4	6	7	11	1/66	$2\times\frac{1}{2}(1,1,1)$, $\frac{1}{6}(1,1,5)$, $\frac{1}{11}(1,4,7)$	$\mathbf{F}^1$
70	30	1	4	10	15	1/20	$\frac{1}{4}(1,1,3)$, $\frac{1}{2}(1,1,1)$, $\frac{1}{5}(1,1,4)$	$\mathbf{F}^0$
71	30	1	6	8	15	1/24	$\frac{1}{2}(1,1,1)$, $\frac{1}{3}(1,1,2)$, $\frac{1}{8}(1,1,7)$	$\mathbf{F}^0$
72	30	2	3	10	15	1/30	$3\times\frac{1}{2}(1,1,1)$, $2\times\frac{1}{3}(1,1,2)$, $\frac{1}{5}(1,2,3)$	$\mathbf{F}^0$
73	30	2	6	7	15	1/42	$5\times\frac{1}{2}(1,1,1)$, $\frac{1}{3}(1,1,2)$, $\frac{1}{7}(1,1,6)$	$\mathbf{F}^0$
74	30	3	4	10	13	1/52	$\frac{1}{4}(1,1,3)$, $\frac{1}{2}(1,1,1)$, $\frac{1}{13}(1,3,10)$	$\mathbf{F}^1$
75	30	4	5	6	15	1/60	$\frac{1}{4}(1,1,3)$, $2\times\frac{1}{2}(1,1,1)$, $2\times\frac{1}{5}(1,1,4)$, $\frac{1}{3}(1,1,2)$	$\mathbf{F}^0$
76	30	5	6	8	11	1/88	$\frac{1}{2}(1,1,1)$, $\frac{1}{8}(1,3,5)$, $\frac{1}{11}(1,5,6)$	$\mathbf{F}^2$
77	32	2	5	9	16	1/45	$2\times\frac{1}{2}(1,1,1)$, $\frac{1}{5}(1,1,4)$, $\frac{1}{9}(1,2,7)$	$\mathbf{F}^0$
78	32	4	5	7	16	1/70	$2\times\frac{1}{4}(1,1,3)$, $\frac{1}{5}(1,1,4)$, $\frac{1}{7}(1,5,2)$	$\mathbf{F}^0$
79	33	3	5	11	14	1/70	$\frac{1}{5}(1,1,4)$, $\frac{1}{14}(1,3,11)$	$\mathbf{F}^1$
80	34	3	4	10	17	1/60	$\frac{1}{3}(1,1,2)$, $\frac{1}{4}(1,1,3)$, $\frac{1}{2}(1,1,1)$, $\frac{1}{10}(1,3,7)$	$\mathbf{F}^0$
81	34	4	6	7	17	1/84	$\frac{1}{4}(1,1,3)$, $2\times\frac{1}{2}(1,1,1)$, $\frac{1}{6}(1,1,5)$, $\frac{1}{7}(1,4,3)$	$\mathbf{F}^0$
82	36	1	5	12	18	1/30	$\frac{1}{5}(1,2,3)$, $\frac{1}{6}(1,1,5)$	$\mathbf{F}^0$
83	36	3	4	11	18	1/66	$2\times\frac{1}{3}(1,1,2)$, $\frac{1}{2}(1,1,1)$, $\frac{1}{11}(1,4,7)$	$\mathbf{F}^0$
84	36	7	8	9	12	1/168	$\frac{1}{7}(1,2,5)$, $\frac{1}{8}(1,1,7)$, $\frac{1}{4}(1,1,3)$, $\frac{1}{3}(1,1,2)$	$\mathbf{F}^0$
85	38	3	5	11	19	2/165	$\frac{1}{3}(1,1,2)$, $\frac{1}{5}(1,1,4)$, $\frac{1}{11}(1,3,8)$	$\mathbf{F}^0$
86	38	5	6	8	19	1/120	$\frac{1}{5}(1,1,4)$, $\frac{1}{6}(1,1,5)$, $\frac{1}{2}(1,1,1)$, $\frac{1}{8}(1,3,5)$	$\mathbf{F}^0$
87	40	5	7	8	20	1/140	$2\times\frac{1}{5}(1,2,3)$, $\frac{1}{7}(1,1,6)$, $\frac{1}{4}(1,1,3)$	$\mathbf{F}^0$
88	42	1	6	14	21	1/42	$\frac{1}{2}(1,1,1)$, $\frac{1}{3}(1,1,2)$, $\frac{1}{7}(1,1,6)$	$\mathbf{F}^0$
89	42	2	5	14	21	1/70	$3\times\frac{1}{2}(1,1,1)$, $\frac{1}{5}(1,1,4)$, $\frac{1}{7}(1,2,5)$	$\mathbf{F}^0$
90	42	3	4	14	21	1/84	$2\times\frac{1}{3}(1,1,2)$, $\frac{1}{4}(1,1,3)$, $\frac{1}{2}(1,1,1)$, $\frac{1}{7}(1,3,4)$	$\mathbf{F}^0$
91	44	4	5	13	22	1/130	$\frac{1}{2}(1,1,1)$, $\frac{1}{5}(1,2,3)$, $\frac{1}{13}(1,4,9)$	$\mathbf{F}^0$
92	48	3	5	16	24	1/120	$2\times\frac{1}{3}(1,1,2)$, $\frac{1}{5}(1,1,4)$, $\frac{1}{8}(1,3,5)$	$\mathbf{F}^0$
93	50	7	8	10	25	1/280	$\frac{1}{7}(1,3,4)$, $\frac{1}{8}(1,1,7)$, $\frac{1}{2}(1,1,1)$, $\frac{1}{5}(1,2,3)$	$\mathbf{F}^0$
94	54	4	5	18	27	1/180	$\frac{1}{4}(1,1,3)$, $\frac{1}{2}(1,1,1)$, $\frac{1}{5}(1,2,3)$, $\frac{1}{9}(1,4,5)$	$\mathbf{F}^0$
95	66	5	6	22	33	1/330	$\frac{1}{5}(1,2,3)$, $\frac{1}{2}(1,1,1)$, $\frac{1}{3}(1,1,2)$, $\frac{1}{11}(1,5,6)$	$\mathbf{F}^0$

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