ERRATA TO "THE CALABI PROBLEM FOR FANO THREEFOLDS"

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Abstract. This document contains errata to the book [1].

1. The list of errata

1.1. About "§5.21, Family № 4.3" (Pages 343–344). Let C be the curve of degree $(1,1,2)$ in $\mathbb{P}^1_{x_0,x_1}\times\mathbb{P}^1_{y_0,y_1}\times\mathbb{P}^1_{z_0,z_1}$ given by

$$\begin{cases} x_0y_1-x_1y_0=0,\\ x_0^2z_1+x_1^2z_0=0.\end{cases}$$

Then C is smooth and irreducible. (*We used wrong defining equation of the curve C in* [1], *so that our proof was incorrect.*) Let $\pi\colon X\to\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$ be the blow up of the curve C. Then X is the unique smooth Fano 3-fold № 4.3. Let G be the subgroup of $\mathrm{Aut}\,(\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1)$ generated by

$$\begin{aligned}\alpha&\colon \big([x_0:x_1],[y_0:y_1],[z_0:z_1]\big)\mapsto\big([x_1:x_0],[y_1:y_0],[z_1:z_0]\big),\\ \beta&\colon \big([x_0:x_1],[y_0:y_1],[z_0:z_1]\big)\mapsto\big([y_0:y_1],[x_0:x_1],[z_0:z_1]\big),\\ \gamma_\epsilon&\colon \big([x_0:x_1],[y_0:y_1],[z_0:z_1]\big)\mapsto\big([x_0:\epsilon x_1],[y_0:\epsilon y_1],[z_0:\epsilon^2 z_1]\big),\end{aligned}$$

where $\epsilon\in\mathbb{C}^*$. Then $G\cong(\mathbb{G}_m\rtimes\boldsymbol{\mu}_2)\times\boldsymbol{\mu}_2$, and C is G-invariant, so that the G-action lifts to the threefold X. Let R_C be the G-invariant surface $\{x_0y_1-x_1y_0=0\}\subset\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$, let R be its proper transform via π on the threefold X, let E be the π-exceptional surface, let us set $\mathcal{C}:=E|_R\subset R$, and let $H_i=(\mathrm{pr}_i\circ\pi)^*(\mathcal{O}_{\mathbb{P}^1}(1))$, where $\mathrm{pr}_i\colon\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1\to\mathbb{P}^1$ is the ith-projection. Let T_C be the G-invariant surface $\{x_0y_0z_1+x_1y_1z_0=0\}\subset\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$, let T be its proper transform via π on the threefold X. Then

$$-K_X\sim 2H_1+2H_2+2H_3-E,\quad R\sim H_1+H_2-E,\quad T\sim H_1+H_2+H_3-E.$$

Moreover, we have:

Lemma 1.1 (cf. [1, Lemma 5.109])**.** *The following assertions holds:*

1. *both $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$ and X do not contain G-fixed points,*
2. *if Z is a G-invariant curve in X, then $H_i\cdot Z\geqslant 2$ for **some** $i\in\{1,2,3\}$,*
3. *if Z is a G-invariant irreducible curve in R, then $Z-\mathcal{C}$ is pseudo-effective on R,*
4. *if D is a G-invariant prime divisor on X with $-K_X-D$ big, then **either** $D=R$ **or** $D=T$.*

Proof. The first three assertions follow from the study of the G-action on $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$. The remaining assertion immediately follows from the description of the cone of effective divisors of X, which is given in [2]. □

The surface T_C is a smooth del Pezzo surface of degree 6. Let $\mathbf{e}_x^1, \mathbf{e}_x^2, \mathbf{e}_y^1, \mathbf{e}_y^2, \mathbf{e}_z^1, \mathbf{e}_z^2 \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be the curves defined by

$$\begin{aligned}\mathbf{e}_x^1 &:= (y_0 = z_0 = 0), \mathbf{e}_x^2 := (y_1 = z_1 = 0),\\ \mathbf{e}_y^1 &:= (x_1 = z_1 = 0), \mathbf{e}_y^2 := (x_0 = z_0 = 0),\\ \mathbf{e}_z^1 &:= (x_0 = y_1 = 0), \mathbf{e}_z^2 := (x_1 = y_0 = 0).\end{aligned}$$

Then the curves form the set of (-1)-curves in T_C. Note that $\mathrm{Pic}(T_C) = \mathbb{Z}\mathbf{e}_z^1 \oplus \mathbb{Z}\mathbf{e}_x^2 \oplus \mathbb{Z}\mathbf{e}_y^2 \oplus \mathbb{Z}\mathbf{e}_z^2$. Moreover, the curve $C \subset T_C$ satisfies that $C \sim 2\mathbf{e}_z^1 + \mathbf{e}_x^2 + \mathbf{e}_y^2$.

Lemma 1.2. *For any G-invariant irreducible curve Z_C in T_C, the divisor $Z_C - C$ on T_C is pseudo-effective.*

Proof. Obviously, $Z_C \neq \mathbf{e}_x^1, \dots, \mathbf{e}_z^2$. Set $a, b, c, g \in \mathbb{Z}$ with $Z_C \sim a\mathbf{e}_z^1 + b\mathbf{e}_x^2 + c\mathbf{e}_y^2 - g\mathbf{e}_z^2$. Since $0 \leq (Z_C \cdot \mathbf{e}_x^1) = (Z_C \cdot \mathbf{e}_x^2) = \left(Z_C \cdot \mathbf{e}_y^1\right) = \left(Z_C \cdot \mathbf{e}_y^2\right)$, we have $c = b \geq g$ and $a = 2b - g$. Moreover, since $0 \leq (Z_C \cdot \mathbf{e}_z^1) = (Z_C \cdot \mathbf{e}_z^2)$, we have $g \geq 0$. Thus $Z_C \sim bC - g\left(\mathbf{e}_z^1 + \mathbf{e}_X^2\right)$ with $0 \leq g \leq b$. We note that $g < b$. Indeed, if $g = b$, then $Z_C \sim b\left(\mathbf{e}_x^2 + \mathbf{e}_y^1\right)$. Since Z_C is irreducible, this must implies that $b = 1$. However, there is no G-invariant member in $\left|\mathbf{e}_x^2 + \mathbf{e}_y^1\right|$, a contradiction. Since $Z_C - (b-g)C \sim g\left(\mathbf{e}_x^2 + \mathbf{e}_y^1\right)$, we get the assertion. □

In the remaining part, we will prove that X is K-polystable. As usual, we will use notations introduced in [1]. We start with:

Lemma 1.3 (cf. [1, Lemma 5.110])**.** *Let Z be a G-invariant irreducible curve in R. Then $S(W^R_{\bullet,\bullet}; Z) < 1$.*

Proof. Let $-K_X - xR = P(x) + N(x)$ be the Zariski decomposition, where $x \in \mathbb{R}_{\geq}0$ such that $-K_X - xR$ is pseudo-effective. As in [2], we have

$$P(x) = \begin{cases} -K_X - xR & \text{if } x \in [0,1], \\ -K_X - xR - (x-1)E & \text{if } x \in [1,2], \end{cases}$$

$$N(x) = \begin{cases} 0 & \text{if } x \in [0,1], \\ (x-1)E & \text{if } x \in [1,2]. \end{cases}$$

In particular, we have

$$P(x)|_R \sim_{\mathbb{R}} \begin{cases} \mathcal{O}(2, 1+x) & \text{if } x \in [0,1], \\ \mathcal{O}(4-2x, 2) & \text{if } x \in [1,2]. \end{cases}$$

Thus, by Lemma 1.1, we have

$$\begin{aligned} S\big(W^R_{\bullet,\bullet}; Z\big) &\leq S\big(W^R_{\bullet,\bullet}; \mathcal{C}\big) \\ &= \frac{3}{30}\bigg(\int_0^1 \int_0^\infty \mathrm{vol}_R\left(\mathcal{O}(2, 1+x) - y\mathcal{O}(2,1)\right) dy dx \\ &\quad + \int_1^2 \Big(4(x-1)(4-2x) + \int_0^\infty \mathrm{vol}_R\left(\mathcal{O}(4-2x, 2) - y\mathcal{O}(2,1)\right) dy\Big) dx\bigg) \\ &= \frac{29}{60} < 1. \end{aligned}$$

where we used [1, Corollary 1.110]. □

We also need the following:

Lemma 1.4. *Let Z be a G-invariant irreducible curve in T. Then $S(W^T_{\bullet,\bullet}; Z) < 1$.*

Proof. Set $\mathcal{C}' := E|_T \subset T$. Let $-K_X - xT = P(x) + N(x)$ be the Zariski decomposition, where $x \in \mathbb{R}_{\geq} 0$ such that $-K_X - xT$ is pseudo-effective. As in [2], we have

$$P(x) = \begin{cases} -K_X - xT & \text{if } x \in [0,1], \\ -K_X - xT - (x-1)E & \text{if } x \in [1,2], \end{cases}$$

$$N(x) = \begin{cases} 0 & \text{if } x \in [0,1], \\ (x-1)E & \text{if } x \in [1,2]. \end{cases}$$

In particular, we have

$$P(x)|_T \sim_{\mathbb{R}} \begin{cases} (4-x)\mathbf{e}^1_z + (3-x)\mathbf{e}^2_x + (3-x)\mathbf{e}^2_y + (x-2)\mathbf{e}^2_z & \text{if } x \in [0,1], \\ (6-3x)\mathbf{e}^1_z + (4-2x)\mathbf{e}^2_x + (4-2x)\mathbf{e}^2_y + (x-2)\mathbf{e}^2_z & \text{if } x \in [1,2]. \end{cases}$$

Thus, by [1, Corollary 1.110], we have

$$\begin{aligned} & S\big(W^T_{\bullet,\bullet}; \mathcal{C}'\big) \\ = \ & \frac{3}{30}\bigg(\int_0^1 \int_0^\infty \mathrm{vol}_T\left((4-x-2y)\mathbf{e}^1_z + (3-x-y)\mathbf{e}^2_x + (3-x-y)\mathbf{e}^2_y + (x-2)\mathbf{e}^2_z\right) dydx \\ & + \int_1^2 \Big(6(x-1)(2-x)^2 \\ & + \int_0^\infty \mathrm{vol}_T\left((6-3x-2y)\mathbf{e}^1_z + (4-2x-y)\mathbf{e}^2_x + (4-2x-y)\mathbf{e}^2_y + (x-2)\mathbf{e}^2_z\right) dy\Big) dx\bigg) \\ = \ & \frac{29}{60} < 1. \end{aligned}$$

Moreover, by Lemma 1.2, we have $S\big(W^T_{\bullet,\bullet}; Z\big) \leq S\big(W^T_{\bullet,\bullet}; \mathcal{C}'\big)$. Thus the assertion follows. □

Now, we are ready to prove that X is K-polystable. Take any G-invariant prime divisor F over X, set $Z := c_X(F) \subset X$, and let $\eta_Z \in X$ be the generic point of Z. By [1, Theorem 1.22], it is enough to show that $A_X(F) > S_X(F)$. By Lemma 1.1, Z is either a curve or a surface. We may assume that Z is a curve by [2]. If $Z \subset R \cup T$, then, by [1, Theorem 1.101], [2] and Lemmas 1.3 and 1.4, we have $A_X(F) > S_X(F)$. Thus we may further assume that $Z \not\subset R \cup T$.

Assume that $\alpha_{G,\eta_Z}(X) < \frac{3}{4}$. By [1, Lemma 1.42], there exists $\lambda \in \left(0, \frac{3}{4}\right) \cap \mathbb{Q}$ and a G-invariant and effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ on X such that $Z \subset \mathrm{Nklt}(X, \lambda D)$ holds. By Lemma 1.1, since $Z \not\subset R \cup T$, the Z is a one-dimensional irreducible component of $\mathrm{Nklt}(X, \lambda D)$. However, by [1, Corollary A.12], we must have $H_i \cdot Z \leq 1$ for any $i \in \{1, 2, 3\}$. This contradicts to Lemma 1.1. Thus we get the inequality $\alpha_{G,\eta_Z}(X) \geq \frac{3}{4}$. Therefore, by [1, Lemma 1.45], we have the inequality $A_X(F) > S_X(F)$. As a consequence, our X is K-polystable.

1.2. **About "The table № 2.26" (Page 359).** In the published version of [1, §6], there is a typo in the big table. More precisely, for № 2.26, the following is correct:

№	$-K_X^3$	$h^{1,2}$	Brief description	$\mathrm{Aut}^0(X)$	**K-ps**	**K-ss**	Sections
2.26	34	0	blow up of $V_5 \subset \mathbb{P}^6$ along line	$\mathbb{G}_a \rtimes \mathbb{G}_m$ $\mathbb{G}_m$	No	$\exists$ No Yes $\star$	[1, §5.10]

References

[1] C. Araujo, A-M. Castravet, I. Cheltsov, K. Fujita, A-S. Kaloghiros, J. Martinez-Garcia, C. Shramov, H. Süss, N. Viswanathan, *The Calabi problem for Fano threefolds*, London Math. Soc. Lecture Note Ser., **485** Cambridge University Press, Cambridge, 2023. vii+441 pp.

[2] K. Fujita, *On K-stability and the volume functions of* $\mathbb{Q}$*-Fano varieties*, Proc. Lond. Math. Soc. **113** (2016), 541–582.

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