ON THE RATIONALITY OF NON-GORENSTEIN $\mathbb{Q}$-FANO 3-FOLDS WITH AN INTEGER FANO INDEX

Ivan Cheltsov

All varieties are defined over $\mathbb{C}$ and projective. Notations and notions are contained in [KMM]. I express my thanks to professors V.A. Iskovskikh and Yu.G. Prokhorov for their kind attitude to me.

§1. Introduction.

1.1. The main purpose of this article is to prove the following:
Theorem: Let X be a non-Gorenstein $\mathbb{Q}$-Fano 3-fold with canonical singularities, such that $-K_X \equiv H$, where H is a Cartier divisor. If $H^3 \geq 10$, then X is rational.
1.2. The main application of 1.1 is the following:
Corollary: Let X be a normal 3-fold and H be a hyperplane section of X, such that H is an Enriques surface with Du Val singularities. Then either X is a cone over H or X is rational.
Proof: It follows from [Ch] that X is non-Gorenstein, $-2K_X \sim 2H$ and either X is a cone over H or X has canonical singularities. In the last case $H^3 \geq 10$ (see [CD]) and X is rational by 1.1. $\square$
1.3. Remark: In 1.2, we can assume that X is even non-normal, but the properties of H imply that X is reduced and irreducible. In that case, it follows from [KMM, 7-2-4] that X has canonical Gorenstein singularities along H. Thus, we can take the normalization of X and repeat proof of 1.2 using 1.1.
1.4. Biregular classification of 3-folds containing Enriques surface as a hyperplane section was considered by G. Fano in [F], where he "proved" that they are rational and even "classified" them, but his paper is incorrect. A. Conte and J.P. Murre renovated G. Fano's ideas in [CM] conjecturing unproved facts.
1.5. From the modern point of view, 1.4 can be generalized in the following way:
Problem: To classify all normal 3-folds containing Enriques surface with Du Val singularities as an ample Cartier divisor.
1.6. Definition: Let X be a 3-fold with canonical singularities, such that $-K_X$ is ample, i.e. $\mathbb{Q}$-Fano 3-fold with canonical singularities. Fano index of X is the maximal number λ, such that $-K_X \equiv \lambda H$, where H is a Cartier divisor. λ is a well-defined positive rational number.
1.7. In [Ch] I proved the following:
Theorem: Let X be a normal 3-fold and H be an ample Cartier divisor on X, such that H is an Enriques surface with Du Val singularities. Then either X is a generalized cone

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over H or X is a non-Gorenstein $\mathbb{Q}$-Fano 3-fold with canonical singularities and an integer Fano index.

1.8. In [P] Yu.G. Prokhorov proved the following:

Theorem: Let X be a non-Gorenstein $\mathbb{Q}$-Fano 3-fold with canonical singularities, such that $-K_X \equiv H$, where H is a Cartier divisor. Then the general element of $|H|$ is an Enriques surface with Du Val singularities.

1.9. Remark: It follows from 1.7 and 1.8 that, except generalized cones, 1.5 is equivalent to the classification of non-Gorenstein $\mathbb{Q}$-Fano 3-fold with canonical singularities and an integer Fano index. In the case of 3-folds with terminal cyclic quotient singularities the last problem was solved in [B] and [S].

§2. Non-Gorenstein $\mathbb{Q}$-Fano 3-folds with an integer Fano index.

2.1. Lemma: In 1.8, one of the following holds:

1. $\mathrm{Bs}|H| = \emptyset$;
2. $\mathrm{Bs}|H| = \{p_1, p_2\}$, where p_1 and p_2 are simple base points of $|H|$, in particular, X and the general member of $|H|$ are both smooth at p_1 and p_2;
3. $\mathrm{Bs}|H| = \{p\}$ and $H^3 = 2$.

Moreover, $-2K_X \sim 2H$ and $\dim|H| = \frac{1}{2}H^3 + 1$.

Proof: See [P, 3.3] and [Ch].

2.2. Remark: In 1.8, suppose that $H^3 \geq 4$. If $\mathrm{Bs}|H| \neq \emptyset$, then $\phi_{|H|}$ is a rational map of degree two, which is not defined only in p_1 and p_2. In that case $|H|$ is called hyperelliptic. If $\mathrm{Bs}|H| = \emptyset$, then, by [CD, 4.5.1, 4.6.3 and 4.7.1], one of the following holds:

1. $\phi_{|H|}$ is a birational morphism;
2. $\phi_{|H|}$ is a morphism of degree two and $H^3 = 6$ or 8;
3. $H^3 = 4$ and $\phi_{|H|} : X \to \mathbb{P}^3$ is a morphism of degree four;

Moreover, in the last case X is a quotient of an intersection of three quadrics in $\mathbb{P}^6$ by an involution with a finite number of fixed points (see [CD, 4.9.2] and [I, 3.5]).

2.3. Lemma: In 2.2, let $|H_O|$ be a linear system of elements of $|H|$ containing O, where O is a non-Gorenstein point of X. Then $|H_O|$ is not composed from the pencil and has no fixed components.

Proof: It is easy to show that $\dim(\phi_{|H|}(X)) = 3$, so $|H_O|$ is not composed from the pencil. Suppose that $|H_O|$ has a base component E, then $\dim(\phi_{|H|}(E)) = 0$. Intersection of E and the general member of $|H|$ is a curve, hence $\phi_{|H|}(|H|)$ contains $\phi_{|H|}(E)$, which is a contradiction. $\square$

§3. Linear system with maximal singularity.

3.1. In 2.3, $K_X + |H_O|$ (or pair $(K_X, |H_O|)$) is not canonical in the sense of [A] or, in other terms, linear system $|H_O|$ has maximal singularity in the sense of V.A. Iskovskikh. Indeed, in the neighbourhood of O elements of $|H_O|$ are non-Gorenstein and, in particular, have singularities worse than Du Val. The latter implies that $K_X + |H_O|$ is not canonical.

3.2. In 3.1, let $f : \tilde{X} \to X$ be a terminal modification for $K_X + |H_O|$ (see [A]) and $|\tilde{H}_O| = f^{-1}(|H_O|)$. Then $K_{\tilde{X}} + |\tilde{H}_O| \equiv -B$, where B is a non-zero effective f-exceptional divisor. After $(K_{\tilde{X}}+|\tilde{H}_O|)$-Minimal Model Program we will obtain $(K_{\hat{X}}+|\hat{H}_O|)$-negative extremal fibration $g:\hat{X}\to Z$, where $|\hat{H}_O|$ is a strict transform of $|H_O|$ on $\hat{X}$.

§4. The main case.

4.1. Lemma: In 3.2, suppose that $|\hat{H}_O|$ is not contained in the fibers of g. Then X is rational.
Proof: It is a well-known fact, see, for example, [A, 4.3]. $\square$
4.2. Lemma: In 3.2, suppose that $\dim(Z)=1$. Then X is rational.
Proof: It follows from 2.3 and 4.1. $\square$
4.3. Lemma: In 3.2, suppose that $\dim(Z)=0$ and $H^3\geq 10$. Then X is rational.
Proof: $\hat{X}$ is a $\mathbb{Q}$-Fano 3-fold with $\mathbb{Q}$-factorial terminal singularities, $\mathrm{Pic}(\hat{X})=\mathbb{Z}$. Terminality of $K_{\hat{X}}+|\hat{H}_O|$ implies that $|\hat{H}_O|$ can have only isolated base points, which are smooth on both $\hat{X}$ and the general element of $|\hat{H}_O|$ (see [A, 1.22]). The last statement, Bertini theorem and the adjunction formula imply that the general element of $|\hat{H}_O|$ is a smooth del Pezzo surface and a Cartier divisor on $\hat{X}$. It follows from the classification of such pairs $(\hat{X},|\hat{H}_O|)$ (see [CF, 1.5]) that $\hat{X}$ is rational, with the possible exception of $(\hat{X},|\hat{H}_O|)$ falling into one of the following cases:

1. $(X_6\subset\mathbb{P}(1,1,2,2,3),|X_6\cap\{T_3=0\}|)$;
2. $(X_6\subset\mathbb{P}(1,1,1,2,3),|X_6\cap\{T_0=0\}|)$;
3. $(X_4\subset\mathbb{P}(1,1,1,1,2),|X_4\cap\{T_0=0\}|)$;
4. $(X_3\subset\mathbb{P}^4,|X_3\cap\{T_0=0\}|)$.

Note that the above linear systems may be non-complete. Thus, $|\hat{H}_O|$ is a linear subsystem of complete linear system of dimension 4, 2, 3, 4 in cases (1), (2), (3), (4), correspondingly, but $\dim|\hat{H}_O|=\dim|H_O|=\frac{1}{2}H^3\geq 5$. $\square$
4.4. Lemma: In 2.2, suppose that $\phi_{|H|}$ is birational and $H^3\geq 10$. Then X is rational.
Proof: By 4.1, 4.2 and 4.3, we may assume that g is a conic bundle and $|\hat{H}_O|$ is contained in the fibers of g. This implies that $\dim(\phi_{|H_O|}(X))=2$ and $\phi_{|H|}(X)$ is a cone with vertex $\phi_{|H|}(O)$. Thus, the strict transform of the general element of $|H|$ on $\hat{X}$ is a rational section of g and Z is birationally isomorphic to Enriques surface, but Z should be rational (see, for example, [A, 4.3]). $\square$
4.5. Remark: Actually, 4.4 is enough to prove 1.2, even with assumption of 1.3.
4.6. Remark: Similar arguments imply that if X is a $\mathbb{Q}$-Fano 3-fold with canonical singularities, $\dim(\phi_{|-K_X|}(X))=3$ and the general element of $|-K_X|$ has singularities worse than Du Val, then either X is rational or pair $(X,|-K_X|)$ is birationally equivalent to one of the four pairs obtained in the proof of 4.3 (see also [A, 4.3]).
4.7. Remark: In classical terms the above construction is the "projection from the non-Gorenstein point", which is the analogue of the famous "double projection" of G. Fano and V.A. Iskovskikh (see [I]).

§5. The hyperelliptic case.

5.1. Suppose that 1.1 is wrong, then, in 2.2 and 3.2, the following holds:

1. $g:\hat{X}\to Z$ is a conic bundle and $|\hat{H}_O|$ is contained in the fibers of g;

(2) $|H|$ is hyperelliptic and $\phi_{|H|}(X)$ is a cone with vertex $\phi_{|H|}(O)$.

5.2. Lemma: In 5.1, let $\pi : \bar{X} \to X$ be a resolution of the base locus of $|H|$, i.e. the blow up of two smooth points p_1 and p_2 on X. If $\{l\}$ is a family of rulings of $\phi_{|H|}(X)$ and $\{\bar{C}\} = \pi^{-1}\{l\}$, then:

(1) $H\{\bar{C}\} = 2$;
(2) the general element of $\{\bar{C}\}$ is an irreducible rational curve, whose strict transform on $\hat{X}$ is the general fiber of g;
(3) $\pi^{-1}(p_j)\{\bar{C}\} = 0$ $(j = 1,\ 2)$.

Note that we will identify a family of curves with its general element when we intersect the latter with a divisor.

Proof: It follows from 5.1 that the strict transform of the general element of $\{\bar{C}\}$ on $\hat{X}$ is contained in the fibers of g. Consider the following diagram:

$$\bar{X} \overset{\alpha}{\leftarrow} \check{X} \overset{\beta}{\rightarrow} \hat{X} \overset{g}{\rightarrow} Z,$$

where $\check{X}$ is smooth, α and β are birational morphisms. $-K_{\check{X}} \equiv \alpha^*(H) - E$, where E is an effective α-exceptional divisor. $2 \leq -K_{\check{X}}\alpha^{-1}(\{\bar{C}\}) \leq H\{\bar{C}\} \leq 2$. Therefore, $2 = -K_{\check{X}}\alpha^{-1}(\{\bar{C}\}) = H\{\bar{C}\}$, which implies all the assertions. $\square$

5.3. Lemma: In 5.2, let $\mathrm{bs}\{\bar{C}\} \in \bar{X}$ be a union of centers of all divisors of $\hat{X}$ that are exceptional on $\bar{X}$ and not contained in the fibers of g. Then $\mathrm{bs}\{\bar{C}\}$ is a finite set of isolated non-terminal points[1] on $\bar{X}$, $\mathrm{bs}\{\bar{C}\} \in \mathrm{Bs}|\bar{H}_O|$ and $\phi_{\bar{H}}(\mathrm{bs}\{\bar{C}\}) = \phi_{|H|}(O)$, where $|\bar{H}| = \pi^{-1}(|H|)$ and $|\bar{H}_O| = \pi^{-1}(|H_O|)$.

Proof: The two last assertions are obvious. To prove the first one suppose that $\mathrm{bs}\{\bar{C}\}$ contains an irreducible curve V. Then V should intersect $\pi^{-1}(p_1)$ or $\pi^{-1}(p_2)$. Therefore $\bar{X}$ is smooth in the general point of V, which contradicts the inequality in the proof of 5.2. In the same way, we obtain that $\mathrm{bs}\{\bar{C}\}$ contains only isolated non-terminal points. $\square$

5.4. Proof of 1.1: In 5.3, it follows from [CD, 4.5.1, 4.5.2 and 4.5.3] that:

(1) $\phi_{|\bar{H}|}(\bar{X})$ is a cone over $\mathbb{F}_i$, where $i = 0,\ 1$ or 2;
(2) if s_∞ and e are the exceptional section and ruling of $\mathbb{F}_i$, then $\phi_{|\bar{H}|}(\bar{X})$ is the contraction δ of the exceptional section M of $\gamma : \mathbb{P}(\mathcal{O}_{\mathbb{F}_i} \oplus \mathcal{O}_{\mathbb{F}_i}(h)) \to \mathbb{F}_i$, where $h \sim s_\infty + (i+k)e$ and $k = \frac{1}{4}(H^3 - 2i - 2)$;
(3) the ramification divisor of $\phi_{|\bar{H}|}$ consists of $\phi_{\bar{H}}(\pi^{-1}(p_j))$ $(j = 1,\ 2)$ and divisor $\delta(D)$, where $D \sim rM + \gamma^*(4s_\infty + (4+2i)e)$.

We may assume that D does not contain M. 5.2 implies that $\delta^{-1}(\phi_{|\bar{H}|}(\pi^{-1}(p_j))$ $(j = 1,\ 2)$ is contained in the fibers of γ. Note that $r \geq 1$, because otherwise $\phi^{-1}_{|\bar{H}|}(\{l\})$ splits, which is impossible by 5.2. Moreover, $r \geq 2$, because $r = 1$ implies that g has a section and X is rational. Let s_0 be the general element in $|s_\infty + ie|$. Then $\gamma^{-1}(s_0) \cong \mathbb{F}_{i+k}$. Let s'_∞ and e' be the exceptional section and ruling of $\gamma^{-1}(s_0)$. Then $D|_{\gamma^{-1}(s_0)} \sim rs'_\infty + (4+2i)e'$ and

[1] Point $\bar{x} \in \bar{X}$ is isolated non-terminal if the desingularisation of $\bar{X}$ contains an exceptional divisor with zero discrepancy, whose image on $\bar{X}$ is $\bar{x}$.

$D|_{\gamma^{-1}(s_0)}$ does not contain s'_∞. Latter implies that $4+2i \geq r(i+k)$, $r=2$ and only four cases are possible:

(1) $i=0,\ k=2,\ H^3=10$;
(2) $i=1,\ k=2,\ H^3=12$;
(3) $i=2,\ k=2,\ H^3=14$;
(4) $i=2,\ k=1,\ H^3=10$.

In these cases we will use an improved version of the previous arguments. $\phi_{|\bar{H}|}$ contracts two divisors B_j to the lines b_j contained in $\phi_{\bar{H}}(\pi^{-1}(p_j))$ $(j=1,\ 2)$, the general fiber of $\phi_{|\bar{H}|}|_{B_j}$ is the elliptic curve (see [CD, 4.5.1]). It can be deduced from [Ch] that $K_X \sim -H+\pi(B_1-B_2)$. Note that at least one $\pi(B_j)$ $(j=1,\ 2)$ should contain O. We claim that $\phi_{|H|}(O) \in b_j$ $(j=1,\ 2)$. Suppose b_1 conta ins $\phi_{|H|}(O)$, but b_2 does not. Then $\pi(B_j)$ $(j=1,\ 2)$ is 2-Cartier and we may even assume that B_2 and $\pi(B_2)$ are Cartier, because otherwise K_X is not Cartier in the points where $\pi(B_2)$ is not Cartier and we can apply all the previous arguments to non-Gorenstein point of X different from O, whose image under $\phi_{|H|}$ is not the vertex of $\phi_{|H|}(X)$. Let $T=\phi^{-1}_{|\bar{H}|}(\delta(\gamma^{-1}(e))$, then $\pi(T)\sim 2\pi(B_2)$ and T is Cartier. $\pi(T)\pi(B_2)=\emptyset$, because $\pi(T)$ can intersect $\pi(B_2)$ only in p_2. Similarly, $T\pi^{-1}(p_1)=\emptyset$ and T does not contain $\phi^{-1}_{|\bar{H}|}(\phi_{|H|}(O))$, hence $|T|$ is a free linear system. The latter implies that $bs\{C\}$ contains some curve in $\phi^{-1}_{|\bar{H}|}(\phi_{|H|}(O))$, which is impossible by 5.3. Therefore, b_j is a line in $\phi_{\bar{H}}(\pi^{-1}(p_j))$ $(j=1,\ 2)$ containing $\phi_{|H|}(O)$. It follows from [CD, 4.5.2 and 4.5.3] that $\delta(D)$ has singularities along b_j $(j=1,\ 2)$. Now we are ready to apply the arguments about $D|_{\gamma^{-1}(s)}$ to the above cases, where s is a moveable smooth curve in $\mathbb{F}_i$. In the first case, let $s\sim s_\infty+e$, which contains $\phi_{|\bar{H}|}(\bar{H})b_j$ $(j=1,\ 2)$, where we identify $\bar{H}$ with a general enough element in $|\bar{H}|$ and $\mathbb{F}_i$. In the second case, let $s\sim s_0$, which contains $\phi_{|\bar{H}|}(\bar{H})b_j$ $(j=1$ or $j=2)$. In the last two cases, let $s\sim s_0$, which contains $\phi_{|\bar{H}|}(\bar{H})b_j$ $(j=1,\ 2)$. Inequalities similar to the ones we used in the first part of the proof give us a contradiction. $\square$

§6. Addendum: the case of a small degree.

6.1. In this chapter we will prove the following:
Theorem: In 2.3, suppose that X has no isolated non-terminal non-Gorenstein points or $\phi_{|H|}$ is birational. Then only the following possibilities exist:

(1) X is rational;
(2) pair $(X,|H_O|)$ is birationally equivalent to one of the four pairs obtained in the proof of 4.3;
(3) X falls into case (3) of 2.2.

6.2. Remark: In 2.3, the non-rationality of X and 2.2 imply that either X falls into case (3) of 2.2 or we can apply to X the arguments of the previous chapters except 5.4, because we used there only non-rationality of X and $\deg(\phi_H)=1$ or 2.
6.3. Lemma: Let S be an irreducible surface and $\{D\}$ be a family of curves on S, such that two general elements of $\{D\}$ have non-empty intersection and have no common components. Then the general element of $\{D\}$ is connected.

Proof: We may assume that S is normal. Let $\gamma : \tilde{S} \to S$ be the desingulrisation of S and $f^*(\{D\})$ be the numerical inverse image of the general element of $\{D\}$ on $\tilde{S}$. The general element of $f^*(\{D\})$ is nef and big on $\tilde{S}$, which implies that $f^*(\{D\})$ is connected. Thus, the general element of $f(f^*(\{D\}))$ is connected. $\square$

6.4. Proof of 6.1: 5.3 implies that $\mathrm{bs}\{C\}$ does not contain non-Gorenstein points. Let $\sigma : Y \to X$ be a canonical cover. Then the general element of $\sigma^{-1}(\{C\})$ is a disjoint union of two rational curves. The general element of $\sigma^{-1}(|H_O|)$ is irreducible and we can apply 6.3 to $\sigma^{-1}(|H_O|)$ and the subfamily of $\sigma^{-1}(\{C\})$ consisting of curves contained in $\sigma^{-1}(|H_O|)$. $\square$

6.5. Remark: In 2.2, suppose X is not rational and does not fall into case (3). Then either X is birationally equivalent to one of the four 3-folds obtained in the proof of 4.3 or $\dim(\phi_{|-K_X|}(X)) = 2$, the general element of $|-K_X|$ has singularities worse than Du Val and, in notations of 3.2, the strict transform of $|-K_X|$ on $\hat{X}$ is contained in the fibers of g. Indeed, restricting $|-K_X|$ on the general element of $|H|$ we obtain that $|-K_X|$ has no fixed components and $\dim(\phi_{|-K_X|}(X)) \geq 2$. Suppose that $\dim(\phi_{|-K_X|}(X)) = 3$, then the general element of $|-K_X|$ has Du Val singularities, because otherwise either X is rational or pair $(X, |-K_X|)$ is birationally equivalent to one of the four pairs obtained in the proof of 4.3 (see 4.6). Thus, X has no isolated non-terminal non-Gorenstein points and we can apply 6.1. Therefore, $\dim(\phi_{|-K_X|}(X)) = 2$ and, as above, we obtain that the general element of $|-K_X|$ has singularities worse than Du Val. Let $\tau : X' \to X$ be a terminal modification for $K_X + |-K_X|$. Then

$$-K_{X'} \sim \tau^{-1}(|-K_X|) + F,\ 2 = -K_{X'}\tau^{-1}(\{C\}) = (\tau^{-1}(|-K_X|) + F)\tau^{-1}(\{C\}),$$

where F is a non-zero effective τ-exceptional divisor. The arguments in the proof of 6.1 imply that $F\tau^{-1}(\{C\}) > 0$. If $\tau^{-1}(|-K_X|)\tau^{-1}(\{C\}) = 1$, then g has a section and X is rational. Thus, $\tau^{-1}(|-K_X|)\tau^{-1}(\{C\}) = 0$, which implies that the strict transform of $|-K_X|$ on $\hat{X}$ is contained in the fibers of g.

§7. Applications: the case of terminal cyclic quotient singularities.

7.1. In this chapter, X and H are as in 2.2, and X has terminal cyclic quotient singularities. The latter implies that X is a quotient of a smooth Fano 3-fold by an involution with eight fixed points, which is the canonical cover of X (see [S]).

7.2. Corollary: In 7.1, suppose that the canonical cover of X is one of the following 3-folds:

(1) an intersection of three divisors of type $(1,1)$ in $\mathbb{P}^3 \times \mathbb{P}^3$ $(H^3 = 10)$;
(2) $\mathbb{P}^1 \times S_4$ $(H^3 = 12)$;
(3) a divisor of type $(1,1,1,1)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ $(H^3 = 12)$;
(4) a blow up of the cone over a smooth qudric in $\mathbb{P}^3$ with the center in a disjoint union of the vertex and an elliptic curve $(H^3 = 14)$;
(5) an intersection of two quadrics in $\mathbb{P}^5$ $(H^3 = 16)$;
(6) $\mathbb{P}^1 \times S_6$ $(H^3 = 18)$;
(7) $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ $(H^3 = 24)$;

where S_j is the dell Pezzo surface of degree j. Then X is rational.
7.3. Remark: In 7.1, suppose that the canonical cover of X is one of the following 3-folds:

(1) $\mathbb{P}^1 \times S_2$ ($H^3 = 6$);
(2) a blow up of an intersection of two quadrics in $\mathbb{P}^5$ with the center in an elliptic curve, which is an intersection of two hyperplane sections ($H^3 = 8$).

Then X is a blow up of (2) of 7.2 in the first case and (4) of 7.2 in the second, hence X is rational.
7.4. Corollary: In 7.1, suppose that the canonical cover of X is a double cover of $\mathbb{P}^3$ ramified in a sextic ($H^3 = 8$). Then either X is rational or pair $(X, |H_O|)$ is birationally isomorphic to $(X_6 \subset \mathbb{P}(1,1,2,2,3), |X_6 \cap \{T_3 = 0\}|)$ with terminal $\mathbb{Q}$-factorial singularities.
Proof: By 6.1, it is enough to show that pair $(X, |H_O|)$ is not birationally isomorphic to $(X_3 \subset \mathbb{P}^4, |X_3 \cap \{T_0 = 0\}|)$. By [B, 6.1.2], $\phi_{|H|}$ is a morphism of degree two and $\phi_{|H|}(X)$ is an intersection of two quadrics, which implies the statement. □
7.5. Remark: In 7.1, suppose that the canonical cover of X is a blow up of a double cover of $\mathbb{P}^3$ ramified in a sextic with the center in an elliptic curve, which is an intersection of two elements of the half-anticanonical linear system ($H^3 = 4$). Then X is a blow up of 3-fold from 7.4. Thus, either X is rational or pair $(X, |H_O|)$ is birationally isomorphic to $(X_6 \subset \mathbb{P}(1,1,2,2,3), |X_6 \cap \{T_3 = 0\}|)$ with terminal $\mathbb{Q}$-factorial singularities.
7.6. Enriques 3-fold: In 7.1, suppose that the canonical cover of X is a double cover of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ ramified in a divisor of type $(1,1,1)$ ($H^3 = 6$), then $\phi_{|H|}$ is birational morphism to the non-normal 3-fold of degree six. It is well-known that:

(1) $\phi_H(X)$ can be represented in $\mathbb{P}^3$ by equation

$$x_1x_2x_3x_4(x_0^2 + x_0\sum_{i=1}^4 a_ix_i + \sum_{i,j=1}^4 b_{ij}x_ix_j)+$$
$$+c_1x_2^2x_3^2x_4^2 + c_2x_1^2x_3^2x_4^2 + c_3x_1^2x_2^2x_4^2 + c_4x_1^2x_2^2x_3^2 = 0,$$

and if $x_0 = 0$, then we obtain the classical representation of Enriques surfaces as a sextic passing doubly through the edges of the tetrahedron;
(2) $\phi_H(X)$ has six double planes given by $x_i = x_j = 0$ $(1 \le i < j \le 4)$, four triple lines given by $x_i = x_j = x_k$ $(1 \le i < j < k \le 4)$ and intersecting in one ordinary quadruple point of $\phi_H(X)$;
(3) each triple line contains two non-ordinary quadruple points, which are the images of eight non-Gorenstein points of X.

Projection from every non-triple line on $\phi_H(X)$ containing the images of two non-Gorenstein points of X is a conic bundle structure[2] on X. Moreover, it is easy to see that all conic bundle structures on X given by linear subsystems of $|H|$ can be obtained in such a way. In [É] S.Yu. Éndryushka and in [BV] L.P. Botta and A. Verra independently proved that the general X is birationally equivalent to the standart conic bundle with a smooth degeneration curve of genus five, whose primian is not a jacobian of a curve and is isomorphic to the intermediate jacobian of X. Thus, the general X is not rational and, by

[2]The conic bundle structure is a rational map, which is birationally equivalent to a conic bundle.

6.1, pair $(X, |H_O|)$ is birationally isomorphic to one of the following pairs with terminal $\mathbb{Q}$-factorial singularities:

(1) $(X_6 \subset \mathbb{P}(1,1,2,2,3), |X_6 \cap \{T_3 = 0\}|)$;
(2) $(X_3 \subset \mathbb{P}^4, |X_3 \cap \{T_0 = 0\}|)$;
(3) $(X_4 \subset \mathbb{P}(1,1,1,1,2), |X_4 \cap \{T_0 = 0\}|)$.

Actually, only the third case occurs. Indeed, in the first case, $\phi_{|X_6 \cap \{T_3=0\}|}$ is a morphism of degree two to the singular quadric X_2 in $\mathbb{P}^4$. $|H_O|$ should correspond to the projection of X_2 to $\mathbb{P}^3$ from a smooth point of X_2 not contained in the singular locus of the ramification divisor of $\phi_{|X_6 \cap \{T_3=0\}|}$. Latter implies that the normalization of the general element of $\{H_O^2\}$ is a connected curve of genus two, but it is not very hard to show that the normalization of the general element of $\{H_O^2\}$ is a connected curve of genus one, i.e. it is a contradiction. In the second case, the cubic should be smooth. Hence its intermediate jacobian $J(X_3)$ is a primian of standard conic bundle with a smooth degeneration curve of genus six. Moreover, $J(X_3)$ is irreducible as a principally polarized abelian variety and coincides with its Griffiths' component, which implies that the second case is impossible. Thus, the pair $(X, |H_O|)$ is birationally isomorphic to $(X_4 \subset \mathbb{P}(1,1,1,1,2), |X_4 \cap \{T_0 = 0\}|)$. $\phi_{|X_4 \cap \{T_0=0\}|} : X_4 \to \mathbb{P}^3$ is a morphism of degree two ramified in quartic Q with isolated singular points, which correspond to the singular points of X. The strict transform of $|H_O|$ on X_4 is the complete linear system $|-\frac{1}{2}K_{X_4}|$. The above arguments about conic bundle structures on X given by linear subsystems of $|H_O|$ imply that Q has exactly six singular points, which are the images under rational map $\phi_{|H_O|}$ of non-Gorenstein points of X not contained in one triple line with O. Moreover, Q contains three lines l_j $(j = 1,\ 2,\ 3)$, which are the images under rational map $\phi_{|H_O|}$ of double planes of $\phi_{|H|}(X)$ passing through $\phi_{|H|}(O)$. Each l_j contains two singular points of Q and $\cap_{j=1}^{3} l_j$ is a smooth point on Q, which is the image under rational map $\phi_{|H_O|}$ of the triple line containing $\phi_{|H|}(O)$.

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