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ON FACTORIALITY OF NODAL THREEFOLDS

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Abstract

We prove the $\mathbb{Q}$-factoriality of a nodal hypersurface in $\mathbb{P}^4$ of degree n with at most $\frac{(n-1)^2}{4}$ nodes and the $\mathbb{Q}$-factoriality of a double cover of $\mathbb{P}^3$ branched over a nodal surface of degree $2r$ with at most $\frac{(2r-1)r}{3}$ nodes.

1. Introduction

Nodal 3-folds[1] arise naturally in many different topics of algebraic geometry. For example, the non-rationality of many smooth rationally connected 3-folds is proved via the degeneration to nodal 3-folds (see [10], [5]). However, the geometry can be very different in smooth and nodal cases: every surface in a smooth hypersurface in $\mathbb{P}^4$ is a complete intersection by the Lefschetz theorem, which is not the case if the hypersurface is nodal; the birational automorphisms of a smooth quartic 3-fold in $\mathbb{P}^4$ form a finite group consisting of projective automorphisms (see [20]), but for any non-smooth nodal quartic 3-fold this group is always infinite (see [24]). The simplest examples of nodal 3-folds are nodal hypersurfaces in $\mathbb{P}^4$ and double covers of $\mathbb{P}^3$ branched over a nodal surface. The latter are called double solids (see [9]).

For a given nodal 3-fold X, one of the substantial questions is whether X is $\mathbb{Q}$-factorial[2] or not. The global topological condition $\operatorname{rk} H^2(X,\mathbb{Z}) = \operatorname{rk} H_4(X,\mathbb{Z})$ is equivalent to the $\mathbb{Q}$-factoriality of X when it is a hypersurface or a double solid. On the other hand, a three-dimensional ordinary double point admits two small resolutions that differ by a simple flop (see [31]). Thus

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All varieties are assumed to be projective, normal, and defined over $\mathbb{C}$.

[1] A 3-fold is called nodal if all its singular points are ordinary double points.

[2] A variety is called $\mathbb{Q}$-factorial if a multiple of every Weil divisor on the variety is a Cartier divisor.

a nodal 3-fold with k nodes has 2^k small resolutions. In particular, the $\mathbb{Q}$-factoriality of a nodal 3-fold implies that it has no projective small resolutions.

Remark 1. The $\mathbb{Q}$-factoriality of a nodal 3-fold imposes strong geometrical restrictions on its birational geometry. For example, $\mathbb{Q}$-factorial nodal quartic 3-folds and nodal sextic double solids are non-rational, but there are rational non-$\mathbb{Q}$-factorial ones (see [24], [7]).

Consider a double cover $\pi : X \to \mathbb{P}^3$ branched over a nodal hypersurface $S \subset \mathbb{P}^3$ of degree $2r$ and a nodal hypersurface $V \subset \mathbb{P}^4$ of degree n. The proof of the following result is due to [9], [31], [15], [13].

Proposition 2. *The 3-folds X and V are $\mathbb{Q}$-factorial if and only if their nodes impose independent linear conditions on homogeneous forms of degree $3r-4$ and $2n-5$ respectively.*

In particular, X and V are $\mathbb{Q}$-factorial if $|\mathrm{Sing}(X)| \leq 3r-3$ and $|\mathrm{Sing}(V)| \leq 2n-4$ respectively. The $\mathbb{Q}$-factoriality of X and V implies

$$\mathrm{Cl}(X) \otimes \mathbb{Q} \cong \mathrm{Pic}(X) \otimes \mathbb{Q} \cong \mathrm{Cl}(V) \otimes \mathbb{Q} \cong \mathrm{Pic}(V) \otimes \mathbb{Q} \cong \mathbb{Q}$$

due to the Lefschetz theorem and [9]. Moreover, the groups $\mathrm{Pic}(X)$ and $\mathrm{Pic}(V)$ have no torsion due to the Lefschetz theorem and [9]. On the other hand, the local class group of an ordinary double point is $\mathbb{Z}$ (see [25]). Therefore, the groups $\mathrm{Cl}(X)$ and $\mathrm{Cl}(V)$ have no torsion as well. Hence, the $\mathbb{Q}$-factoriality of X and V is equivalent to the following two conditions respectively:

- $\mathrm{Cl}(X)$ and $\mathrm{Pic}(X)$ are generated by $\pi^*(H)$, where H is a hyperplane in $\mathbb{P}^3$;
- $\mathrm{Cl}(V)$ and $\mathrm{Pic}(V)$ are generated by the class of a hyperplane section.

The main purpose of this paper is to prove the following two results.

Theorem 3. *Suppose that $|\mathrm{Sing}(X)| \leq \frac{(2r-1)r}{3}$. Then X is $\mathbb{Q}$-factorial.*

Theorem 4. *Suppose that $|\mathrm{Sing}(V)| \leq \frac{(n-1)^2}{4}$. Then V is $\mathbb{Q}$-factorial.*

The bounds in Theorems 3 and 4 may not be sharp in general. For example, in the case $r = 3$ the 3-fold X is $\mathbb{Q}$-factorial if $|\mathrm{Sing}(X)| \leq 14$ due to [7], and in the case $n = 4$ the 3-fold V is $\mathbb{Q}$-factorial if $|\mathrm{Sing}(V)| \leq 8$ due to [5].

Example 5. Consider a hypersurface $X \subset \mathbb{P}(1^4, r)$ given by the equation

$$u^2 = g_r^2(x, y, z, t) + h_1(x, y, z, t) f_{2r-1}(x, y, z, t) \subset \mathbb{P}(1^4, r) \cong \mathrm{Proj}(\mathbb{C}[x, y, z, t, u]),$$

where g_i, h_i, and f_i are sufficiently general polynomials of degree i. Let $\pi : X \to \mathbb{P}^3$ be a restriction of the natural projection $\mathbb{P}(1^4, r) \dashrightarrow \mathbb{P}^3$, induced by an embedding of the graded algebras $\mathbb{C}[x_0, \ldots, x_{2n}] \subset \mathbb{C}[x_0, \ldots, x_{2n}, y]$. Then $\pi : X \to \mathbb{P}^3$ is a double cover branched over a nodal hypersurface

$g_r^2 + h_1 f_{2r-1} = 0$ of degree $2r$ and $|\mathrm{Sing}(X)| = (2r-1)r$; the 3-fold X is not $\mathbb{Q}$-factorial, i.e., the divisor $h_1 = 0$ splits into 2 non-$\mathbb{Q}$-Cartier divisors.

Example 6. Let $V \subset \mathbb{P}^4$ be a hypersurface,

$$xg_{n-1}(x,y,z,t,w) + yf_{n-1}(x,y,z,t,w) \subset \mathbb{P}^4 \cong \mathrm{Proj}(\mathbb{C}[x,y,z,t,w]),$$

where g_{n-1} and f_{n-1} are general polynomials of degree $n-1$. Then V is nodal and contains the plane $x = y = 0$. Hence, the 3-fold V is not $\mathbb{Q}$-factorial and $|\mathrm{Sing}(V)| = (n-1)^2$.

Therefore, asymptotically the bounds in Theorems 3 and 4 are not very far from being sharp. On the other hand, the following result is proved in [8].

Proposition 7. *Every smooth surface on V is a Cartier divisor if* $\mathrm{Sing}(V) < (n-1)^2$.

We expect the following to be true.

Conjecture 8. The 3-fold X is $\mathbb{Q}$-factorial whenever the inequality $|\mathrm{Sing}(X)| < (2r-1)r$ holds; the 3-fold V is $\mathbb{Q}$-factorial whenever the inequality $|\mathrm{Sing}(V)| < (n-1)^2$ holds.

The claim of Conjecture 8 is proved only for $r \leq 3$ and $n \leq 4$ (see [16], [7], [5]), but for many r and n the bounds in Theorems 3 and 4 can be improved. For example, we prove the following result.

Proposition 9. *Suppose that the equalities $r = 4$ and $n = 5$ hold.*[3] *Then X is $\mathbb{Q}$-factorial whenever $|\mathrm{Sing}(X)| < 25$, and the 3-fold V is $\mathbb{Q}$-factorial whenever $|\mathrm{Sing}(V)| < 14$.*

The following result is proved in [8].

Theorem 10. *Suppose that the subset $\mathrm{Sing}(V) \subset \mathbb{P}^4$ is a set-theoretic intersection of hypersurfaces of degree $l < \frac{n}{2}$ and $|\mathrm{Sing}(V)| < \frac{(n-2l)(n-1)^2}{n}$. Then V is $\mathbb{Q}$-factorial.*

The saturated ideal of a set of k points in general position in $\mathbb{P}^4$ is generated by polynomials of degree at most $\frac{n}{4}$ when $k < (n-1)^2$ and $n > 72$ by [17]. Therefore, Theorem 10 implies the $\mathbb{Q}$-factoriality of V having less than $\frac{1}{2}(n-1)^2$ nodes in assumption that the nodes of V are in general position. However, the latter condition implies the $\mathbb{Q}$-factoriality of V due to Proposition 2. We prove the following generalization of Theorem 10.

Theorem 11. *Let $\mathcal{H} \subset |\mathcal{O}_{\mathbb{P}^3}(k)|$ and $\mathcal{D} \subset |\mathcal{O}_{\mathbb{P}^4}(l)|$ be linear subsystems of hypersurfaces vanishing at $\mathrm{Sing}(S)$ and $\mathrm{Sing}(V)$ respectively. Put $\hat{\mathcal{H}} = \mathcal{H}|_S$ and $\hat{\mathcal{D}} = \mathcal{D}|_V$. Suppose that inequalities $k < r$ and $l < \frac{n}{2}$ hold. Then $\dim(\mathrm{Bs}(\hat{\mathcal{H}})) = 0$ implies the $\mathbb{Q}$-factoriality of the 3-fold X, and $\dim(\mathrm{Bs}(\hat{\mathcal{D}})) = 0$ implies the $\mathbb{Q}$-factoriality of the 3-fold V.*

[3] Namely, the 3-folds X and V are nodal Calabi-Yau 3-folds.

Corollary 12. *Suppose* $\mathrm{Sing}(S) \subset \mathbb{P}^3$ *and* $\mathrm{Sing}(V) \subset \mathbb{P}^4$ *are set-theoretic intersections of hypersurfaces of degree* $k < r$ *and* $l < \frac{n}{2}$ *respectively. Then* X *and* V *are* $\mathbb{Q}$*-factorial.*

From the point of view of birational geometry the most important application of Theorems 3 and 4 is the $\mathbb{Q}$-factoriality condition for a nodal quartic 3-fold and a sextic double solid, i.e., the cases $r = 3$ and $n = 4$ respectively, because in these cases the $\mathbb{Q}$-factoriality implies the non-rationality (see [24], [7]). However, it is possible to apply Theorems 3 and 4 to certain higher-dimensional problems in birational algebraic geometry.

Theorem 13. *Let* $\tau : U \to \mathbb{P}^s$ *be a double cover branched over a hypersurface* F *of degree* $2r$ *and* D *be a hyperplane in* $\mathbb{P}^s$ *such that* $D_1 \cap \cdots \cap D_{s-3}$ *is a* $\mathbb{Q}$*-factorial nodal* 3*-fold, where* D_i *is a general divisor in* $|\tau^*(D)|$*. Then* $\mathrm{Cl}(U)$ *and* $\mathrm{Pic}(U)$ *are generated by* $\tau^*(D)$*.*

Theorem 14. *Let* $W \subset \mathbb{P}^r$ *be a hypersurface of degree* n *such that* $H_1 \cap \cdots \cap H_{r-4}$ *is a* $\mathbb{Q}$*-factorial nodal* 3*-fold, where* H_i *is a general enough hyperplane section of* W*. Then the groups* $\mathrm{Cl}(W)$ *and* $\mathrm{Pic}(W)$ *are generated by the class of a hyperplane section of* $W \subset \mathbb{P}^r$*.*

A priori Theorems 13 and 14 can be used to prove the non-rationality of certain singular hypersurfaces of degree r in $\mathbb{P}^r$ and double covers of $\mathbb{P}^s$ branched over singular hypersurfaces of degree $2s$ (see [6]). In the latter case the application of Theorem 13 can be explicit. For example, we prove the following result.

Proposition 15. *Let* $\xi : Y \to \mathbb{P}^4$ *be a double cover branched over a hypersurface* $F \subset \mathbb{P}^4$ *of degree* 8 *such that* F *is smooth outside of a smooth curve* $C \subset F$*, the singularity of the hypersurface* F *in a sufficiently general point of* C *is locally isomorphic to the singularity*

$$x_1^2 + x_2^2 + x_3^2 = 0 \subset \mathbb{C}^4 \cong \mathrm{Spec}(\mathbb{C}[x_1, x_2, x_3, x_4]),$$

the singularities of F *in other points of* C *are locally isomorphic to the singularity*

$$x_1^2 + x_2^2 + x_3^2 x_4 = 0 \subset \mathbb{C}^4 \cong \mathrm{Spec}(\mathbb{C}[x_1, x_2, x_3, x_4]),$$

and a general 3*-fold in the linear system* $|-K_Y|$ *is* $\mathbb{Q}$*-factorial. Then* Y *is a birationally rigid*[4] *terminal* $\mathbb{Q}$*-factorial Fano* 4*-fold with* $\mathrm{Pic}(Y) \cong \mathbb{Z}$ *and* $\mathrm{Bir}(Y)$ *is a finite group consisting of biregular automorphisms. In particular, the* 4*-fold* Y *is non-rational.*

Example 16. Let $Y \subset \mathbb{P}(1^5, 4)$ be a hypersurface

$$u^2 = \sum_{i=1}^{3} f_i(x, y, z, t, w) g_i^2(x, y, z, t, w) \subset \mathbb{P}(1^5, 4) \cong \mathrm{Proj}(\mathbb{C}[x, y, z, t, w, u]),$$

[4]Namely, the 4-fold Y is a unique Mori fibration birational to Y (see [12]).

where f_i and g_i are sufficiently general non-constant homogeneous polynomials such that $\deg(f_i) + 2\deg(g_i) = 8$. Then the natural projection $\mathbb{P}(1^5, 4) \dashrightarrow \mathbb{P}^4$ induces a double cover $\tau : Y \to \mathbb{P}^4$ branched over a hypersurface $F \subset \mathbb{P}^4$, whose equation is $\sum_{i=1}^{3} f_i g_i^2 = 0$ and which is smooth outside of a curve $g_1 = g_2 = g_3 = 0$. Therefore, the 4-fold X is not rational due to Proposition 15 and Theorems 3 and 11.

How many nodes can X and V have? The best known upper bounds (see [29]) are the following: $|\mathrm{Sing}(X)| \leq \mathrm{A}_3(2r)$ and $|\mathrm{Sing}(V)| \leq \mathrm{A}_4(n)$, where $\mathrm{A}_i(j)$ is the Arnold number, a number of points $(a_1, \ldots, a_i) \subset \mathbb{Z}^i$ such that $(i-2)\frac{j}{2} + 1 < \sum_{t=1}^{i} a_t \leq \frac{ij}{2}$ and $a_t \in (0, j)$, which implies $|\mathrm{Sing}(X)| \leq 68$ and 180 when $r = 3$ and 4, and $|\mathrm{Sing}(V)| \leq 45$ and 135 when $n = 4$ and 5 respectively. This bound is sharp for $n = 4$ (see [21]). There is a sharp bound $|\mathrm{Sing}(X)| \leq 65$ in the case $r = 3$ (see [2], [19], [30]). However, there are no known example of a nodal quintic in $\mathbb{P}^4$ having more than 130 nodes (see [28]).

2. Preliminaries

Let X be a variety and B_X be a boundary[5] on X, i.e., $B_X = \sum_{i=1}^{k} a_i B_i$, where B_i is a prime divisor on X and $a_i \in \mathbb{Q}$ (see [22]). The log pair (X, B_X) is called movable when every component B_i is a linear system on X such that the base locus of B_i has codimension at least 2 (see [12], [4]). We assume that K_X and B_X are $\mathbb{Q}$-Cartier divisors.

Definition 17. A log pair (V, B^V) is a log pull-back of the log pair (X, B_X) with respect to a birational morphism $f : V \to X$ if $B^V = f^{-1}(B_X) - \sum_{i=1}^{n} a(X, B_X, E_i)E_i$ such that the equivalence $K_V + B^V \sim_{\mathbb{Q}} f^*(K_X + B_X)$ holds, where E_i is an f-exceptional divisor and $a(X, B_X, E_i) \in \mathbb{Q}$. The number $a(X, B_X, E_i)$ is called a discrepancy of (X, B_X) in the f-exceptional divisor E_i.

Definition 18. A birational morphism $f : V \to X$ is called a log resolution of the log pair (X, B_X) if the variety V is smooth and the union of all proper transforms of the divisors B_i and all f-exceptional divisors forms a divisor with simple normal crossing.

Definition 19. A proper irreducible subvariety $Y \subset X$ is called a center of log canonical singularities of the log pair (X, B_X) if there are a birational morphism $f : V \to X$ together with a not necessarily f-exceptional divisor $E \subset V$ such that E is contained in the support of the effective part of the

[5] Usually boundaries are assumed to be effective (see [22]), but we do not assume this.

divisor $\lfloor B^V \rfloor$ and $f(E) = Y$. The set of all the centers of log canonical singularities of the log pair (X, B_X) is denoted by $\mathbb{LCS}(X, B_X)$.

Definition 20. For a log resolution $f : V \to X$ of (X, B_X) the subscheme $\mathcal{L}(X, B_X)$ associated to the ideal sheaf $\mathcal{I}(X, B_X) = f_*(\mathcal{O}_V(\lceil -B^V \rceil))$ is called a log canonical singularity subscheme of the log pair (X, B_X).

The support of the log canonical singularity subscheme $\mathcal{L}(X, B_X)$ is a union of all elements in the set $\mathbb{LCS}(X, B_X)$. The following result is due to [27] (see [23], [1], [4]).

Theorem 21. *Suppose that B_X is effective and for some nef and big divisor H on X the divisor $D = K_X + B_X + H$ is Cartier. Then*

$$H^i(X, \mathcal{I}(X, B_X) \otimes \mathcal{O}_X(D)) = 0 \textit{ for } i > 0.$$

Consider the following application of Theorem 21.

Lemma 22. *Let $\Sigma \subset \mathbb{P}^n$ be a finite subset, and $\mathcal{M}$ be a linear system of hypersurfaces of degree k passing through all points of the set Σ. Suppose that the base locus of the linear system $\mathcal{M}$ is zero-dimensional. Then the points of the set Σ impose independent linear conditions on the homogeneous forms on $\mathbb{P}^n$ of degree $n(k-1)$.*

Proof. Let $\Lambda \subset \mathbb{P}^n$ be a base locus of the linear system $\mathcal{M}$. Then $\Sigma \subseteq \Lambda$ and Λ is a finite subset in $\mathbb{P}^n$. Now consider sufficiently general different divisors $H_1, \ldots, H_s$ in the linear system $\mathcal{M}$ for $s \gg 0$. Let $X = \mathbb{P}^n$ and $B_X = \frac{n}{s}\sum_{i=1}^{s} H_i$. Then $\mathrm{Supp}(\mathcal{L}(X, B_X)) = \Lambda$.

To prove the claim it is enough to prove that for every point $P \in \Sigma$ there is a hypersurface in $\mathbb{P}^n$ of degree $n(k-1)$ that passes through all the points in the set $\Sigma \backslash P$ and does not pass through the point P. Let $\Sigma \backslash P = \{P_1, \ldots, P_k\}$, where P_i is a point of $X = \mathbb{P}^n$, and let $f : V \to X$ be a blowup at the points of $\Sigma \setminus P$. Then

$$K_V + (B_V + \sum_{i=1}^{k}(\mathrm{mult}_{P_i}(B_X) - n)E_i) + f^*(H) = f^*(n(k-1)H) - \sum_{i=1}^{k} E_i,$$

where $E_i = f^{-1}(P_i)$, $B_V = f^{-1}(B_X)$ and H is a hyperplane in $\mathbb{P}^n$. By construction we have $\mathrm{mult}_{P_i}(B_X) = n\mathrm{mult}_{P_i}(\mathcal{M}) \geq n$ and $\hat{B}_V = B_V + \sum_{i=1}^{k}(\mathrm{mult}_{P_i}(B_X) - n)E_i$ is effective.

Let $\bar{P} = f^{-1}(P)$. Then $\bar{P} \in \mathbb{LCS}(V, \hat{B}_V)$ and $\bar{P}$ is an isolated center of log canonical singularities of the log pair $(V, \hat{B}_V)$, because in the neighborhood of the point P the birational morphism $f : V \to X$ is an isomorphism. On the other hand, the map

$$H^0(\mathcal{O}_V(f^*(n(k-1)H) - \sum_{i=1}^{k} E_i)) \to H^0(\mathcal{O}_{\mathcal{L}(V, \hat{B}_V)} \otimes \mathcal{O}_V(f^*(n(k-1)H) - \sum_{i=1}^{k} E_i))$$

is surjective by Theorem 21. However, in the neighborhood of the point $\bar{P}$ the support of the subscheme $\mathcal{L}(V, \hat{B}_V)$ consists just of the point $\bar{P}$. The latter implies the existence of a divisor $D \in |f^*(n(k-1)H) - \sum_{i=1}^{k} E_i|$ that does not pass through $\bar{P}$. Thus, $f(D)$ is a hypersurface in $\mathbb{P}^n$ of degree $n(k-1)$ that passes through the points of $\Sigma \setminus P$ and does not pass through the point $P \in \Sigma$. □

Actually, arguing as in the proof of Lemma 22 we can prove Theorem 11.

Proof of Theorem 11. We have a double cover $\pi : X \to \mathbb{P}^3$ branched over a nodal hypersurface $S \subset \mathbb{P}^3$ of degree $2r$, a linear subsystem $\mathcal{H} \subset |\mathcal{O}_{\mathbb{P}^3}(k)|$ of hypersurfaces vanishing at $\mathrm{Sing}(S)$ for $k < r$ such that $\dim(\mathrm{Bs}(\hat{\mathcal{H}})) = 0$, where $\hat{\mathcal{H}} = \mathcal{H}|_S$. We must show that the nodes of S impose independent linear conditions on homogeneous forms of degree $3r-4$ due to Proposition 2. Suppose that $\dim(\mathrm{Bs}(\mathcal{H})) = 0$. Then Lemma 22 implies that the nodes of S impose independent linear conditions on homogeneous forms of degree $3r-4$, which proves Corollary 12. In the general case we can repeat the proof of Lemma 22 replacing $\frac{3}{s}\sum_{i=1}^{s} H_i$ by $S + \frac{1}{s}\sum_{i=1}^{s} H_i$. The proof of the $\mathbb{Q}$-factoriality of the nodal hypersurface $V \subset \mathbb{P}^4$ is similar. □

Definition 23. A proper irreducible subvariety $Y \subset X$ is called a center of canonical singularities of (X, B_X) if there is a birational morphism $f : W \to X$ and an f-exceptional divisor $E \subset W$ such that the discrepancy $a(X, B_X, E) \leq 0$ and $f(E) = Y$. The set of all centers of canonical singularities of the log pair (X, B_X) is denoted by $\mathbb{CS}(X, B_X)$.

The following result is a corollary of Theorem 17.6 in [23].

Proposition 24. *Let H be an effective Cartier divisor on X and $Z \in \mathbb{CS}(X, B_X)$. Suppose that X and H are smooth in the generic point of Z, $Z \subset H$, $H \not\subset \mathrm{Supp}(B_X)$ and B_X is an effective boundary. Then $\mathbb{LCS}(H, B_X|_H) \neq \emptyset$.*

The following result is Corollary 7.3 in [26] (see [20], [12]).

Theorem 25. *Suppose that X is smooth, $\dim(X) \geq 3$, the boundary B_X is effective and movable, and the set $\mathbb{CS}(X, B_X)$ contains a closed point $O \in X$. Then $\mathrm{mult}_O(B_X^2) \geq 4$ and the equality implies $\mathrm{mult}_O(B_X) = 2$ and $\dim(X) = 3$.*

The following result is implied by Theorem 3.10 in [12] and Proposition 24.

Theorem 26. *Suppose that $\dim(X) \geq 3$, B_X is effective, and the set $\mathbb{CS}(X, B_X)$ contains an ordinary double point O of X. Then the equality $\mathrm{mult}_O(B_X) \geq 1$ holds;*[6] *moreover, the equality $\mathrm{mult}_O(B_X) = 1$ implies that $\dim(X) = 3$.*

The following result is an easy modification of Theorem 26.

[6]The rational number $\mathrm{mult}_O(B_X)$ is defined by the equivalence $f^*(B_X) \sim_{\mathbb{Q}} f^{-1}(B_X) + \mathrm{mult}_O(B_X)E$, where $f : W \to X$ is a blowup of O and E is an f-exceptional divisor.

Proposition 27. *Suppose that* $\dim(X) = 3$, B_X *is effective, and the set* $\mathbb{CS}(X, B_X)$ *contains an isolated singular point* O *of the variety* X, *which is locally isomorphic to the singularity* $y^3 = \sum_{i=1}^{3} x_i^2$. *Then the inequality* $\mathrm{mult}_O(B_X) \geq \frac{1}{2}$ *holds.*

Proof. The 3-fold W is smooth, E is isomorphic to a cone in $\mathbb{P}^3$ over a smooth conic, the restriction $-E|_E$ is rationally equivalent to a hyperplane section of $E \subset \mathbb{P}^3$, and

$$K_W + B_W \sim_{\mathbb{Q}} f^*(K_X + B_X) + (1 - \mathrm{mult}_O(B_X))E,$$

where $B_W = f^{-1}(B_X)$. Suppose that $\mathrm{mult}_O(B_X) < \frac{1}{2}$. Then

$$\mathbb{CS}(W, B_W) \subset \mathbb{CS}(W, B_W + (\mathrm{mult}_O(B_X) - 1)E),$$

because $\mathrm{mult}_O(B_X) - 1 < 0$. However, the log pair $(W, B_W + (\mathrm{mult}_O(B_X) - 1)E)$ is a log pull-back of (X, B_X) and $O \in \mathbb{CS}(X, B_X)$. Therefore, there is a proper irreducible subvariety $Z \subset E$ such that $Z \in \mathbb{CS}(W, B_W)$. Hence, $\mathbb{LCS}(E, B_W|_E) \neq \emptyset$ by Proposition 24.

Let $B_E = B_W|_E$. Then $\mathbb{LCS}(E, B_E)$ does not contains curves on E, because otherwise the intersection of B_E with the ruling of E is greater than $\frac{1}{2}$, which is impossible due to our assumption $\mathrm{mult}_O(B_X) < \frac{1}{2}$. Therefore, $\dim(\mathrm{Supp}(\mathcal{L}(E, B_E))) = 0$.

Let H be a hyperplane section of $E \subset \mathbb{P}^3$. Then

$$K_E + B_E + (1 - \mathrm{mult}_O(B_X))H \sim_{\mathbb{Q}} -H$$

and $H^0(\mathcal{O}_E(-H)) = 0$. On the other hand, the sequence of groups

$$H^0(\mathcal{O}_E(-H)) \to H^0(\mathcal{O}_{\mathcal{L}(E,B_E)}) \to H^1(E, \mathcal{I}(E, B_E) \otimes \mathcal{O}_E(-H))$$

is exact and $H^1(E, \mathcal{I}(E, B_E) \otimes \mathcal{O}_E(-H)) = 0$ by Theorem 21. Therefore, the latter implies the vanishing of $H^0(\mathcal{O}_{\mathcal{L}(E,B_E)})$, which contradicts $\mathbb{LCS}(E, B_E) \neq \emptyset$. □

The following result is due to [11] (see [26], [4]).

Theorem 28. *Let* X *be a Fano variety with* $\mathrm{Pic}(X) \cong \mathbb{Z}$ *with terminal* $\mathbb{Q}$*-factorial singularities such that either* X *is not birationally rigid or* $\mathrm{Bir}(X) \neq \mathrm{Aut}(X)$. *Then there is a linear system* $\mathcal{M}$ *on* X *whose base locus has codimension at least* 2 *such that the singularities of the log pair* $(X, \mu\mathcal{M})$ *are not canonical, where* $\mu \in \mathbb{Q}_{>0}$ *such that* $\mu\mathcal{M} \sim_{\mathbb{Q}} -K_X$.

The following result is due to [3].

Theorem 29. *Let* $\pi : Y \to \mathbb{P}^2$ *be the blowup at points* $P_1, \ldots, P_s$ *on* $\mathbb{P}^2$, $s \leq \frac{d^2+9d+10}{6}$, *such that at most* $k(d + 3 - k) - 2$ *of the points* P_i *lie on a curve of degree* $k \leq \frac{d+3}{2}$, *where* $d \geq 3$ *is a natural number. Then* $|\pi^*(\mathcal{O}_{\mathbb{P}^2}(d)) - \sum_{i=1}^{s} E_i|$ *is free, where* $E_i = \pi^{-1}(P_i)$.

Corollary 30. *Let $\Sigma \subset \mathbb{P}^2$ be a finite subset such that the inequality $|\Sigma| \le \frac{d^2+9d+16}{6}$ holds and at most $k(d+3-k)-2$ points of Σ lie on a curve of degree $k \le \frac{d+3}{2}$, where $d \ge 3$ is a natural number. Then for every point $P \in \Sigma$ there is a curve $C \subset \mathbb{P}^2$ of degree d that passes through all the points in $\Sigma \setminus P$ and does not pass through the point P.*

In the case $d = 3$ the claim of Theorem 29 is nothing but the freeness of the anticanonical linear system of a weak del Pezzo surface of degree $9-s \ge 2$ (see [14]).

3. Double solids

In this section we prove Theorem 3. Let $\pi : X \to \mathbb{P}^3$ be a double cover branched over a nodal hypersurface $S \subset \mathbb{P}^3$ of degree $2r$ such that $|\mathrm{Sing}(S)| \le \frac{(2r-1)r}{3}$. We must show that the nodes of $S \subset \mathbb{P}^3$ impose independent linear conditions on homogeneous forms of degree $3r-4$ on $\mathbb{P}^3$ due to Proposition 2. Moreover, we may assume $r \ge 3$, because in the case $r \le 2$ the required claim is trivial.

Definition 31. The points of a subset $\Gamma \subset \mathbb{P}^s$ satisfy the property ∇ if at most $t(2r-1)$ points of the set Γ can lie on a curve in $\mathbb{P}^s$ of degree $t \in \mathbb{N}$.

Let $\Sigma = \mathrm{Sing}(S) \subset \mathbb{P}^3$.

Proposition 32. *The points of the subset $\Sigma \subset \mathbb{P}^3$ satisfy the property ∇.*

Proof. Let $F(x_0, x_1, x_2, x_3) = 0$ be a homogeneous equation of degree $2r$ that defines $S \subset \mathbb{P}^3$, where $(x_0 : x_1 : x_2 : x_3)$ are homogeneous coordinates on $\mathbb{P}^3$. Consider the linear system

$$\mathcal{L} = \left| \sum_{i=0}^{3} \lambda_i \frac{\partial F}{\partial x_i} = 0 \right| \subset |\mathcal{O}_{\mathbb{P}^3}(2r-1)|,$$

where $\lambda_i \in \mathbb{C}$. The base locus of $\mathcal{L}$ consists of singular points of S. A curve in $\mathbb{P}^3$ of degree t intersects a generic member of $\mathcal{L}$ at most $(2r-1)t$ times, which implies the claim. □

Fix a point $P \in \Sigma$. To prove that the points of $\Sigma \subset \mathbb{P}^3$ impose independent linear conditions on homogeneous forms of degree $3r-4$ it is enough to construct a hypersurface in $\mathbb{P}^3$ of degree $3r-4$ that passes through $\Sigma \setminus P$ and does not pass through $P \in \Sigma$.

Lemma 33. *Suppose $\Sigma \subset \Pi$ for some hyperplane $\Pi \subset \mathbb{P}^3$. Then there is a hypersurface in $\mathbb{P}^3$ of degree $3r-4$ that passes through $\Sigma \setminus P$ and does not pass through $P \in \Sigma$.*

Proof. Let us apply Corollary 30 to $\Sigma \subset \Pi$ and $d = 3r-4 \ge 5$. We must check that all the conditions of Corollary 30 are satisfied, which is easy but not obvious. First of all,

$$|\Sigma| \leq \frac{(2r-1)r}{3} \Rightarrow |\Sigma| \leq \frac{d^2+9d+16}{6}$$

and at most $d = 3r-4$ points of Σ can lie on a line in Π because $r \geq 3$ and the points of the subset $\Sigma \subset \Pi$ satisfy the property ∇ due to Proposition 32.

Now we must prove that at most $t(3r-1-t)-2$ points of Σ can lie on a curve of degree $t \leq \frac{3r-1}{2}$. The case $t = 1$ is already done. Moreover, at most $t(2r-1)$ points of the set Σ can lie on a curve of degree t by Proposition 32. Thus, we must show that

$$t(3r-1-t)-2 \geq t(2r-1)$$

for all $t \leq \frac{3r-1}{2}$. Moreover, we must prove the latter inequality only for such $t > 1$ that the inequality $t(3r-1-t)-2 < |\Sigma|$ holds, because otherwise the corresponding condition on the points of the set Σ is vacuous. Moreover, we have

$$t(3r-1-t)-2 \geq t(2r-1) \iff r > t,$$

because $t > 1$. Suppose that the inequality $r \leq t$ holds for some natural number t such that $t \leq \frac{3r-1}{2}$ and $t(3r-1-t)-2 < |\Sigma|$. Let $g(x) = x(3r-1-x)-2$. Then $g(x)$ is increasing for $x < \frac{3r-1}{2}$. Thus, we have $g(t) \geq g(r)$, because $\frac{3r-1}{2} \geq t \geq r$. Hence,

$$\frac{(2r-1)r}{3} \geq |\Sigma| > g(t) \geq g(r) = r(2r-1)-2,$$

which is impossible when $r \geq 3$.

Therefore, there is a curve $C \subset \Pi$ of degree $3r-4$ that passes through $\Sigma \setminus P$ and does not pass through P by Corollary 30. Let $Y \subset \mathbb{P}^3$ be a sufficiently general cone over the curve $C \subset \Pi \cong \mathbb{P}^2$. Then $Y \subset \mathbb{P}^3$ is a hypersurface of degree $3r-4$ that passes through all the points of the set $\Sigma \setminus P$ and does not pass through the point $P \in \Sigma$. □

Take a sufficiently general hyperplane $\Pi \subset \mathbb{P}^3$. Let $\psi : \mathbb{P}^3 \dashrightarrow \Pi$ be a projection from a sufficiently general point $O \in \mathbb{P}^3$, $\Sigma' = \psi(\Sigma) \subset \Pi \cong \mathbb{P}^2$ and $\hat{P} = \psi(P) \in \Sigma'$.

Lemma 34. *Suppose that the points of $\Sigma' \subset \Pi$ satisfy the property ∇. Then there is a hypersurface in $\mathbb{P}^3$ of degree $3r-4$ containing $\Sigma \setminus P$ and not passing through P.*

Proof. Arguing as in the proof of Lemma 33 we obtain a curve $C \subset \Pi$ of degree $3r-4$ that passes through $\Sigma' \setminus \hat{P}$ and does not pass through $\hat{P}$. Let $Y \subset \mathbb{P}^3$ be a cone over the curve C with the vertex O. Then $Y \subset \mathbb{P}^3$ is a hypersurface of degree $3r-4$ that passes through $\Sigma \setminus P$ and does not pass through the point $P \in \Sigma$. □

Perhaps the points of the set $\Sigma' \subset \Pi$ always satisfy the property ∇, but we are unable to prove it. We may assume that the points of $\Sigma' \subset \Pi$ do not satisfy the property ∇.

Definition 35. The points of a subset $\Gamma \subset \mathbb{P}^s$ satisfy the property ∇_k if at most $i(2r-1)$ points of the set Γ can lie on a curve in $\mathbb{P}^s$ of degree $i \in \mathbb{N}$ for all $i \leq k$.

Therefore, there is a smallest $k \in \mathbb{N}$ such that the points of $\Sigma' \subset \Pi$ do not satisfy the property ∇_k, i.e., there is a subset $\Lambda_k^1 \subset \Sigma$ such that $|\Lambda_k^1| > k(2r-1)$ and all points of

$$\tilde{\Lambda}_k^1 = \psi(\Lambda_k^1) \subset \Sigma' \subset \Pi \cong \mathbb{P}^2$$

lie on a curve $C \subset \Pi$ of degree k. Moreover, the curve C is irreducible and reduced due to the minimality of k. In the case when the points of the subset $\Sigma' \setminus \tilde{\Lambda}_k^1 \subset \Pi$ do not satisfy the property ∇_k we can find a subset $\Lambda_k^2 \subset \Sigma \setminus \Lambda_k^1$ such that $|\Lambda_k^2| > k(2r-1)$ and all the points of the set $\tilde{\Lambda}_k^2 = \psi(\Lambda_k^2)$ lie on an irreducible curve of degree k. Thus, we can iterate this construction c_k times and get $c_k > 0$ disjoint subsets

$$\Lambda_k^i \subset \Sigma \setminus \bigcup_{j=1}^{i-1} \Lambda_k^j \subsetneq \Sigma$$

such that $|\Lambda_k^i| > k(2r-1)$, all the points of the subset $\tilde{\Lambda}_k^i = \psi(\Lambda_k^i) \subset \Sigma'$ lie on an irreducible reduced curve on Π of degree k, and all the points of the subset

$$\Sigma' \setminus \bigcup_{i=1}^{c_k} \tilde{\Lambda}_k^i \subset \Pi \cong \mathbb{P}^2$$

satisfy the property ∇_k. Now we can repeat this construction for the property ∇_{k+1} and find $c_{k+1} \geq 0$ disjoint subsets

$$\Lambda_{k+1}^i \subset (\Sigma \setminus \bigcup_{i=1}^{c_k} \Lambda_k^i) \setminus \bigcup_{j=1}^{i-1} \Lambda_{k+1}^j \subset \Sigma \setminus \bigcup_{i=1}^{c_k} \Lambda_k^i \subsetneq \Sigma$$

such that $|\Lambda_{k+1}^i| > (k+1)(2r-1)$, the points of $\tilde{\Lambda}_{k+1}^i = \psi(\Lambda_{k+1}^i) \subset \Sigma'$ lie on an irreducible reduced curve on Π of degree $k+1$, and the points of the subset

$$\Sigma' \setminus \bigcup_{j=k}^{k+1} \bigcup_{i=1}^{c_j} \tilde{\Lambda}_j^i \subsetneq \Sigma' \subset \Pi \cong \mathbb{P}^2$$

satisfy the property ∇_{k+1}. Now we can iterate this construction for $\nabla_{k+2}, \ldots, \nabla_l$ and get disjoint subsets $\Lambda_j^i \subset \Sigma$ for $j = k, \ldots, l \geq k$ such that

$|\Lambda_j^i| > j(2r-1)$, all the points of the subset $\tilde{\Lambda}_j^i = \psi(\Lambda_j^i) \subset \Sigma'$ lie on an irreducible reduced curve of degree j in Π, and all the points of the subset

$$\bar{\Sigma} = \Sigma' \setminus \bigcup_{j=k}^{l} \bigcup_{i=1}^{c_j} \tilde{\Lambda}_j^i \subsetneq \Sigma' \subset \Pi \cong \mathbb{P}^2$$

satisfy the property ∇, where $c_j \geq 0$ is the number of subsets $\tilde{\Lambda}_j^i$. The subset $\Lambda_k^1 \subset \Sigma$ is non-empty, i.e., $c_k > 0$, but every subset $\Lambda_j^i \subset \Sigma$ can be empty when $j \neq k$ or $i \neq 1$, and the subset $\bar{\Sigma} \subset \Sigma'$ can be empty as well. Nevertheless, we always have the inequality

$$(36) \qquad |\bar{\Sigma}| < \frac{(2r-1)r}{3} - \sum_{i=k}^{l} c_i(2r-1)i = \frac{(2r-1)}{3}\Big(r - 3\sum_{i=k}^{l} ic_i\Big).$$

Corollary 37. *The inequality* $\sum_{i=k}^{l} ic_i < \frac{r}{3}$ *holds.*

In particular, $\Lambda_j^i \neq \emptyset$ implies $j < \frac{r}{3}$.

Lemma 38. *Suppose that* $\Lambda_j^i \neq \emptyset$. *Let* $\mathcal{M}$ *be a linear system of hypersurfaces of degree* j *in* $\mathbb{P}^3$ *passing through all the points in* Λ_j^i. *Then the base locus of* $\mathcal{M}$ *is zero-dimensional.*

Proof. By the construction of the set Λ_j^i all the points of the subset

$$\tilde{\Lambda}_j^i = \psi(\Lambda_j^i) \subset \Sigma' \subset \Pi \cong \mathbb{P}^2$$

lie on an irreducible reduced curve $C \subset \Pi$ of degree j. Let $Y \subset \mathbb{P}^3$ be a cone over C with the vertex O. Then Y is a hypersurface in $\mathbb{P}^3$ of degree j that contains all the points of the set Λ_j^i. Therefore, $Y \in \mathcal{M}$.

Suppose that the base locus of the linear system $\mathcal{M}$ contains an irreducible reduced curve $Z \subset \mathbb{P}^3$. Then $Z \subset Y$ and $\psi(Z) = C$. Moreover, $\Lambda_j^i \subset Z$, because $\Lambda_j^i \not\subset Z$ implies that $\tilde{\Lambda}_j^i \not\subset C$ due to the generality of ψ. Finally, the restriction $\psi|_Z : Z \to C$ is a birational morphism, because the projection ψ is general. Hence, $\deg(Z) = j$ and Z contains at least $|\Lambda_j^i| > j(2r-1)$ points of Σ. The latter contradicts Proposition 32. □

Corollary 39. *The inequality* $k \geq 2$ *holds.*

For every $\Lambda_j^i \neq \emptyset$ let $\Xi_j^i \subset \mathbb{P}^3$ be a base locus of the linear system of hypersurfaces of degree j in $\mathbb{P}^3$ passing through all the points in Λ_j^i. For $\Lambda_j^i = \emptyset$ put $\Xi_j^i = \emptyset$. Then Ξ_j^i is a finite set by Lemma 38 and $\Lambda_j^i \subseteq \Xi_j^i$ by construction.

Lemma 40. *Suppose that* $\Xi_j^i \neq \emptyset$. *Then the points of the subset* $\Xi_j^i \subset \mathbb{P}^3$ *impose independent linear conditions on homogeneous forms on* $\mathbb{P}^3$ *of degree* $3(j-1)$.

Proof. The claim follows from Lemma 22. □

Corollary 41. *Suppose that $\Lambda_j^i \neq \emptyset$. Then the points of the subset $\Lambda_j^i \subset \mathbb{P}^3$ impose independent linear conditions on homogeneous forms on $\mathbb{P}^3$ of degree $3(j-1)$.*

Lemma 42. *Suppose that $\bar{\Sigma} = \emptyset$. Then there is a hypersurface in $\mathbb{P}^3$ of degree $3r-4$ containing $\Sigma \setminus P$ and not passing through the point P.*

Proof. The set Σ is a disjoint union $\bigcup_{j=k}^{l}\bigcup_{i=1}^{c_j}\Lambda_j^i$, and there is a unique set Λ_a^b containing the point P. In particular, $P \in \Xi_a^b$. On the other hand, the union $\bigcup_{j=k}^{l}\bigcup_{i=1}^{c_j}\Xi_j^i$ is not necessarily disjoint. Thus, a priori the point P can be contained in many sets Ξ_j^i.

For every $\Xi_j^i \neq \emptyset$ containing P there is a hypersurface of degree $3(j-1)$ that passes through $\Xi_j^i \setminus P$ and does not pass through P by Lemma 40. For every $\Xi_j^i \neq \emptyset$ not containing the point P there is a hypersurface of degree j that passes through Ξ_j^i and does not pass through the point P by the definition of the set Ξ_j^i. Moreover, $j < 3(j-1)$, because $k \geq 2$ by Corollary 39. Therefore, for every $\Xi_j^i \neq \emptyset$ there is a hypersurface $F_i^j \subset \mathbb{P}^3$ of degree $3(j-1)$ that passes through $\Xi_j^i \setminus P$ and does not pass through the point P. Let

$$F = \bigcup_{j=k}^{l}\bigcup_{i=1}^{c_j} F_j^i \subset \mathbb{P}^3$$

be a possibly reducible hypersurface of degree $\sum_{i=k}^{l} 3(i-1)c_i$. Then F passes through all the points of the set $\Sigma \setminus P$ and does not pass through the point P. Moreover, we have

$$\deg(F) = \sum_{i=k}^{l} 3(i-1)c_i < \sum_{i=k}^{l} 3ic_i < r < 3r-4$$

by Corollary 37, which implies the claim. □

Let $\hat{\Sigma} = \bigcup_{j=k}^{l}\bigcup_{i=1}^{c_j}\Lambda_j^i$ and $\check{\Sigma} = \Sigma \setminus \hat{\Sigma}$. Then $\Sigma = \hat{\Sigma} \cup \check{\Sigma}$ and $\psi(\check{\Sigma}) = \bar{\Sigma} \subset \Pi$. Therefore, we proved Theorem 3 in the extreme cases: $\hat{\Sigma} = \emptyset$ and $\check{\Sigma} = \emptyset$. Now we must combine the proofs of the Lemmas 34 and 42 to prove Theorem 3 in the case when $\hat{\Sigma} \neq \emptyset$ and $\check{\Sigma} \neq \emptyset$.

Remark 43. Arguing as in the proof of Lemma 42 we obtain a hypersurface $F \subset \mathbb{P}^3$ of degree $\sum_{i=k}^{l} 3(i-1)c_i$ that passes through all the points of the subset $\hat{\Sigma} \setminus P \subsetneq \Sigma$ and does not pass through the point $P \in \Sigma$.

Put $d = 3r - 4 - \sum_{i=k}^{l} 3(i-1)c_i$. Let us check that the subset $\bar{\Sigma} \subset \Pi \cong \mathbb{P}^2$ and the number d satisfy all the hypotheses of Theorem 29. We may assume that $\emptyset \neq \hat{\Sigma} \subsetneq \Sigma$.

Lemma 44. *The inequality $d \geq 6$ holds.*

Proof. The claim is implied by Corollary 37 and $c_k \geq 1$. □

Lemma 45. *The inequality* $|\bar{\Sigma}| \leq \frac{d^2+9d+10}{6}$ *holds.*

Proof. To prove the claim it is enough to show that

$$2(2r-1)(r-3\sum_{i=k}^{l} ic_i) \leq (3r-4-\sum_{i=k}^{l} 3(i-1)c_i)^2+9(3r-4-\sum_{i=k}^{l} 3(i-1)c_i)+10,$$

because $|\bar{\Sigma}| < \frac{(2r-1)}{3}(r-3\sum_{i=k}^{l} ic_i)$ by the inequality 36. However, we have

$$(3r-4-\sum_{i=k}^{l} 3(i-1)c_i)^2+9(3r-4-\sum_{i=k}^{l} 3(i-1)c_i)+10$$
$$> (2r-4+3c_k)^2+9(2r-4+3c_k)+10,$$

because $c_k \geq 1$ and $\sum_{i=k}^{l} 3ic_i < r$ by Corollary 37. Thus, we have

$$(2r-4+3c_k)^2+9(2r-4+3c_k)+10 \geq (2r-1)^2+9(2r-1)+10 = 4r^2+14r+2,$$

which implies $4r^2+14r+2 > 4r^2-2r > 2(2r-1)(r-3\sum_{i=k}^{l} ic_i)$. □

Lemma 46. *At most* $t(d+3-t)-2$ *points of* $\bar{\Sigma}$ *lie on a curve in* $\mathbb{P}^2$ *of degree* $t \leq \frac{d+3}{2}$.

Proof. In the case $t=1$ the claim is implied by Proposition 32, Corollary 37 and the inequality $c_k \geq 1$. Hence, we may assume that $t > 1$.

The points of the subset $\bar{\Sigma} \subset \mathbb{P}^2$ satisfy the property ∇. Thus, at most $(2r-1)t$ of the points of $\bar{\Sigma}$ lie on a curve in $\mathbb{P}^2$ of degree t. Therefore, to conclude the proof it is enough to show that $t(d+3-t)-2 \geq (2r-1)t$ for all $t \leq \frac{d+3}{2}$. Moreover, it is enough to prove the latter inequality only for $t > 1$ such that $t(d+3-t)-2 < |\bar{\Sigma}|$, because otherwise the corresponding condition on the points of the set $\bar{\Sigma}$ is vacuous.

Now we have

$$t(d+3-t)-2 \geq t(2r-1) \iff t(r-\sum_{i=k}^{l} 3(i-1)c_i - t) \geq 2$$
$$\iff r-\sum_{i=k}^{l} 3(i-1)c_i > t,$$

because $t > 1$. We may assume that the inequalities $t(d+3-t)-2 < |\bar{\Sigma}|$ and

$$r-\sum_{i=k}^{l} 3(i-1)c_i \leq t \leq \frac{d+3}{2}$$

hold. Let $g(x) = x(d+3-x)-2$. Then $g(x)$ is increasing for $x < \frac{d+3}{2}$. Therefore, the inequality $g(t) \geq g(r - \sum_{i=k}^{l} 3(i-1)c_i)$ holds. Hence, we have

$$\frac{(2r-1)}{3}(r - 3\sum_{i=k}^{l} ic_i) > |\bar{\Sigma}| > g(t) \geq (r - \sum_{i=k}^{l} 3(i-1)c_i)(2r-1) - 2$$

and $(2r-1)(6\sum_{i=k}^{l} ic_i - 2r) + 6 - 9\sum_{i=k}^{l} c_i(2r-1) > 0$. Now we have

$$\begin{aligned}(2r-1)(6\sum_{i=k}^{l} ic_i - 2r) + 6 - 9\sum_{i=k}^{l} c_i(2r-1) &< 6 - 9\sum_{i=k}^{l} c_i(2r-1) \\ &< 6 - 9c_k(2r-1) < 0,\end{aligned}$$

because $\sum_{i=k}^{l} 3ic_i < r$ by Corollary 37. The obtained contradiction implies the claim. □

Therefore, we can apply Theorem 29 to the blowup of the hyperplane Π at the points of the set $\bar{\Sigma} \setminus \hat{P} \subset \Pi$ due to Lemmas 44, 45 and 46. The application of Theorem 29 gives a curve $C \subset \Pi \cong \mathbb{P}^2$ of degree $3r - 4 - \sum_{i=k}^{l} 3(i-1)c_i$ that passes through all the points of the set $\bar{\Sigma} \setminus \hat{P}$ and does not pass through the point $\hat{P} = \psi(P)$. It should be pointed out that the subset $\bar{\Sigma} \subset \Sigma'$ may not contain $\hat{P} \in \Sigma'$. Namely, $\hat{P} \in \bar{\Sigma}$ if and only if $P \in \check{\Sigma}$.

Let $G \subset \mathbb{P}^3$ be a cone over the curve C with the vertex O, where $O \in \mathbb{P}^3$ is the center of the projection $\psi : \mathbb{P}^3 \dashrightarrow \Pi$. Then G is a hypersurface of degree $3r - 4 - \sum_{i=k}^{l} 3(i-1)c_i$ that passes through the points of $\check{\Sigma} \setminus P$ and does not pass through P. On the other hand, we already have the hypersurface $F \subset \mathbb{P}^3$ of degree $\sum_{i=k}^{l} 3(i-1)c_i$ that passes through the points of $\hat{\Sigma} \setminus P$ and does not pass through P. Therefore, $F \cup G \subset \mathbb{P}^3$ is a hypersurface of degree $3r - 4$ that passes through all the points of the set $\Sigma \setminus P$ and does not pass through the point $P \in \Sigma$. Hence, we proved Theorem 3.

4. Hypersurfaces in $\mathbb{P}^4$

In this section we prove Theorem 4. Let $V \subset \mathbb{P}^4$ be a nodal hypersurface of degree n such that $|\mathrm{Sing}(V)| \leq \frac{(n-1)^2}{4}$. In order to prove Theorem 4 it is enough to show that the nodes of the hypersurface V impose independent linear conditions on homogeneous forms of degree $2n-5$ on $\mathbb{P}^4$ due to Proposition 2. Moreover, we may always assume that $n \geq 4$, because in the case $n \leq 3$ the required claim is trivial.

Definition 47. The points of a subset $\Gamma \subset \mathbb{P}^r$ satisfy the property ★ if at most $k(n-1)$ points of the set Γ can lie on a curve in $\mathbb{P}^r$ of degree $k \in \mathbb{N}$.

Let $\Sigma = \mathrm{Sing}(V) \subset \mathbb{P}^4$. Then arguing as in the proof of Proposition 32 we obtain the following result.

Proposition 48. *The points of the subset $\Sigma \subset \mathbb{P}^4$ satisfy the property $\bigstar$.*

Fix a point $P \in \Sigma$. To prove that the points of $\Sigma \subset \mathbb{P}^4$ impose independent linear conditions on homogeneous forms on $\mathbb{P}^4$ of degree $2n-5$ it is enough to construct a hypersurface in $\mathbb{P}^4$ of degree $2n-5$ that passes through the points of the set $\Sigma \setminus P$ and does not pass through $P \in \Sigma$. Arguing as in the proof of Lemma 33 we obtain the following result.

Lemma 49. *Suppose that the subset $\Sigma \subset \mathbb{P}^4$ is contained in some two-dimensional linear subspace $\Pi \subset \mathbb{P}^4$. Then there is a hypersurface in $\mathbb{P}^4$ of degree $2n-5$ that passes through the points of the set $\Sigma \setminus P$ and does not pass through the point $P \in \Sigma$.*

Fix a general two-dimensional linear subspace $\Pi \subset \mathbb{P}^4$. Let $\psi : \mathbb{P}^4 \dashrightarrow \Pi$ be a projection from a general line $L \subset \mathbb{P}^4$, $\Sigma' = \psi(\Sigma)$ and $\hat{P} = \psi(P)$. Then $\psi|_{\Sigma} : \Sigma \to \Sigma'$ is a bijection.

Lemma 50. *Suppose that the points in $\Sigma' \subset \Pi$ satisfy the property $\bigstar$. Then there is a hypersurface in $\mathbb{P}^4$ of degree $2n-5$ containing $\Sigma \setminus P$ and not passing through $P \in \Sigma$.*

Proof. Arguing as in the proof of Lemma 33 we prove the existence of a curve $C \subset \Pi$ of degree $2n-5$ that passes through $\Sigma' \setminus \hat{P}$ and does not pass through $\hat{P}$. Let $Y \subset \mathbb{P}^4$ be a three-dimensional cone over the curve C with the vertex $L \subset \mathbb{P}^4$. Then $Y \subset \mathbb{P}^4$ is the required hypersurface. □

We may assume that the points of $\Sigma' \subset \Pi$ do not satisfy the property $\bigstar$. Arguing as in the proof of Theorem 3 we can construct disjoint subsets $\Lambda_j^i \subset \Sigma$ for $j = r, \ldots, l \geq r$ such that the inequality $|\Lambda_j^i| > j(n-1)$ holds, all the points of the subset $\tilde{\Lambda}_j^i = \psi(\Lambda_j^i) \subset \Sigma'$ lie on an irreducible reduced curve in $\Pi \cong \mathbb{P}^2$ of degree j, and all the points in the subset

$$\bar{\Sigma} = \Sigma' \setminus \bigcup_{j=r}^{l} \bigcup_{i=1}^{c_j} \tilde{\Lambda}_j^i \subsetneq \Sigma' \subset \Pi \cong \mathbb{P}^2$$

satisfy the property $\bigstar$, where $c_j \geq 0$ is a number of subsets $\tilde{\Lambda}_j^i$ and $c_r > 0$. In particular,

$$0 \leq |\bar{\Sigma}| < \frac{(n-1)^2}{4} - \sum_{i=r}^{l} c_i (n-1) i = \frac{n-1}{4}\left(n - 1 - 4\sum_{i=r}^{l} i c_i\right). \tag{51}$$

Corollary 52. *The inequality $\sum_{i=r}^{l} i c_i < \frac{n-1}{4}$ holds.*

For every $\Lambda_j^i \neq \emptyset$ let $\Xi_j^i \subset \mathbb{P}^4$ be a base locus of the linear system of hypersurfaces of degree j in $\mathbb{P}^4$ passing through all the points in Λ_j^i; otherwise put $\Xi_j^i = \emptyset$. Then Ξ_j^i is a finite set (see the proof of Lemma 38) and, in particular, $r \geq 2$. Moreover, $\Lambda_j^i \subseteq \Xi_j^i$ by definition of $\Xi_j^i \subset \mathbb{P}^4$. Therefore, the points of the set $\Xi_j^i \subset \mathbb{P}^4$ impose independent linear conditions on the homogeneous forms on $\mathbb{P}^4$ of degree $4(j-1)$ by Lemma 22. In particular, the points of the set Λ_j^i impose independent linear conditions on the homogeneous forms on $\mathbb{P}^4$ of degree $4(j-1)$.

Let $\hat{\Sigma} = \bigcup_{j=r}^{l} \bigcup_{i=1}^{c_j} \Lambda_j^i$ and $\check{\Sigma} = \Sigma \setminus \hat{\Sigma}$. Then $\Sigma = \hat{\Sigma} \cup \check{\Sigma}$ and $\psi(\check{\Sigma}) = \bar{\Sigma} \subset \Pi$. Then arguing as in the proof of Lemma 42 we obtain a hypersurface in $\mathbb{P}^4$ of degree $2n-5$ containing all points in $\Sigma \setminus P$ and not passing through P in the case when $\bar{\Sigma} = \emptyset$. Actually, arguing as in the proof of Lemma 42 we prove the existence of a hypersurface $F \subset \mathbb{P}^4$ of degree $\sum_{i=r}^{l} 4(i-1)c_i$ that passes through all the points of the subset $\hat{\Sigma} \setminus P \subsetneq \Sigma$ and does not pass through the point $P \in \Sigma$. Put $d = 2n - 5 - \sum_{i=r}^{l} 4(i-1)c_i$. Let us check that the subset $\bar{\Sigma} \subset \Pi$ and the number d satisfy all hypotheses of Theorem 29. We may assume $\hat{\Sigma} \neq \emptyset$ and $\check{\Sigma} \neq \emptyset$.

Lemma 53. *The inequality $d \geq 5$ holds.*

Proof. We have $\sum_{i=r}^{l} 4ic_i < n-1$ by Corollary 52. Thus, $d > n-4+4c_r \geq n \geq 4$. □

Lemma 54. *The inequality $|\bar{\Sigma}| \leq \frac{d^2+9d+10}{6}$ holds.*

Proof. Suppose that $|\bar{\Sigma}| > \frac{d^2+9d+10}{6}$. Then

$$3(n-1)(n-1-4\sum_{i=r}^{l} ic_i) > 2(2n-5-\sum_{i=r}^{l} 4(i-1)c_i)^2 + 18(2n-5-\sum_{i=r}^{l} 4(i-1)c_i) + 20,$$

because $|\bar{\Sigma}| < \frac{n-1}{4}(n-1-4\sum_{i=r}^{l} ic_i)$. Let $A = \sum_{i=r}^{l} ic_i$ and $B = \sum_{i=r}^{l} c_i$. Then

$$3(n-1)^2 - 12(n-1)A > 2(2n-1)^2 - 16A(2n-1) + 32A^2 + 18(2n-1) - 72A + 20,$$

because $B \geq c_r \geq 1$. Thus, for $n \geq 4$ we have

$$3(n-1)^2 > 8n^2 + 28n + 4 + 32A^2 - A(20n+68) > 5n^2 + 12n + 23 > 3(n-1)^2,$$

because $A < \frac{n-1}{4}$ by Corollary 52. □

Lemma 55. *At most $k(d+3-k)-2$ points of $\bar{\Sigma}$ lie on a curve in $\mathbb{P}^2$ of degree $k \leq \frac{d+3}{2}$.*

Proof. The case $k=1$ follows from Corollary 52 and $c_r \geq 1$. Therefore, we may assume that $k > 1$. The points of $\bar{\Sigma} \subset \mathbb{P}^2$ satisfy the property ★. So, at most $k(n-1)$ of the points of $\bar{\Sigma}$ lie on a curve of degree k. To conclude the proof it is enough to prove that

$$k(d+3-k) - 2 \geq k(n-1)$$

for all $k \leq \frac{d+3}{2}$. Moreover, it is enough to prove the latter inequality only for such natural numbers $k > 1$ that the inequality $k(d+3-k) - 2 < |\bar{\Sigma}|$ holds, because otherwise the corresponding condition on the points of the set $\bar{\Sigma}$ is vacuous.

The inequality $k(d+3-k) - 2 \geq k(n-1)$ holds if and only if $n - 1 - \sum_{i=r}^{l} 4(i-1)c_i > k$, because $k > 1$. Thus, we may assume that the inequalities $k(d+3-k) - 2 < |\bar{\Sigma}|$ and

$$n - 1 - \sum_{i=r}^{l} 4(i-1)c_i \leq k \leq \frac{d+3}{2}$$

hold. Let $g(x) = x(d+3-x) - 2$. Then $g(x)$ is increasing for $x < \frac{d+3}{2}$. Thus, we have

$$\frac{(n-1)}{4}(n - 1 - 4\sum_{i=r}^{l} ic_i) > |\bar{\Sigma}| > g(k) \geq g(n - 1 - \sum_{i=r}^{l} 4(i-1)c_i).$$

Let $A = \sum_{i=r}^{l} ic_i$ and $B = \sum_{i=r}^{l} c_i$. Then the inequality

$$\frac{(n-1)}{4}(n-1-4A) > 4(n-1-4A+4B)(n-1) - 2$$

holds. Therefore, we have

$$n - 1 - 4A > 4(n-1) - 16A + 16B - 1 > 4(n-1) - 16A,$$

because $B \geq c_r \geq 1$. Thus, $4A > n-1$, but $A < \frac{n-1}{4}$ by Corollary 52. $\square$

Now we can apply Corollary 30 to get a curve $C \subset \Pi$ of degree $2n - 5 - \sum_{i=r}^{l} 4(i-1)c_i$ that passes through the points of the subset $\bar{\Sigma} \setminus \hat{P} \subset \Pi \cong \mathbb{P}^2$ and does not pass through the point $\hat{P} \subset \Sigma'$. Let $G \subset \mathbb{P}^4$ be a cone over C with the vertex in the center L of the projection $\psi : \mathbb{P}^4 \dashrightarrow \Pi$. Then $G \subset \mathbb{P}^4$ is a hypersurface of degree $2n - 5 - \sum_{i=r}^{l} 4(i-1)c_i$ that passes through $\check{\Sigma} \setminus P$ and does not pass through P. However, we already have the hypersurface $F \subset \mathbb{P}^4$ of degree $\sum_{i=r}^{l} 4(i-1)c_i$ that passes through $\hat{\Sigma} \setminus P$ and does not pass through P. Therefore, $F \cup G \subset \mathbb{P}^4$ is a hypersurface of degree $2n - 5$ that passes through $\Sigma \setminus P$ and does not pass through $P \in \Sigma$. Thus, Theorem 4 is proved.

5. Calabi-Yau 3-folds

In this section we prove Proposition 9. Let $\pi : X \to \mathbb{P}^3$ be a double cover branched over a nodal hypersurface $S \subset \mathbb{P}^3$ of degree 8 such that $|\mathrm{Sing}(S)| \leq 25$, and let $V \subset \mathbb{P}^4$ be a nodal hypersurface of degree 5 such that $|\mathrm{Sing}(V)| \leq 14$. Due to Proposition 2 it is enough to prove that the nodes of the surface

$S \subset \mathbb{P}^3$ impose independent linear conditions on homogeneous forms of degree 8 on $\mathbb{P}^3$ and the nodes of the hypersurface $V \subset \mathbb{P}^4$ impose independent linear conditions on homogeneous forms of degree 5 on $\mathbb{P}^4$.

Let $\Sigma = \mathrm{Sing}(S) \subset \mathbb{P}^3$ and $\Lambda = \mathrm{Sing}(V) \subset \mathbb{P}^4$. Arguing as in the proof of Proposition 32 we see that no more than $7k$ points of Σ and no more than $4k$ points of Λ can lie on a curve of degree $k = 1, 2, 3$. Let us fix a point $P \in \Sigma$ and a point $Q \in \Lambda$. To prove Proposition 9 we must construct a hypersurface in $\mathbb{P}^3$ of degree 8 that passes through $\Sigma \setminus P$ and does not pass through P and a hypersurface in $\mathbb{P}^4$ of degree 5 that passes through $\Lambda \setminus Q$ and does not pass through the point Q.

Take general two-dimensional linear subspaces $\Pi \subset \mathbb{P}^3$ and $\Omega \subset \mathbb{P}^4$. Let $\psi : \mathbb{P}^3 \dashrightarrow \Pi$ be a projection from a general point $P \in \mathbb{P}^3$, and $\xi : \mathbb{P}^4 \dashrightarrow \Omega$ be a projection from a general line $L \subset \mathbb{P}^4$. Put $\Sigma' = \psi(\Sigma)$, $\hat{P} = \psi(P)$, $\Lambda' = \xi(\Lambda)$ and $\hat{Q} = \xi(Q)$. Then no more than 7 points of the subset $\Sigma' \subset \Pi$ and no more than 5 points of the subset $\Lambda' \subset \Omega$ can lie on a line (see the proof of Lemma 38).

Lemma 56. *No more than* 14 *points of the subset* $\Sigma' \subset \Pi$ *and no more than* 10 *points of the subset* $\Lambda' \subset \Omega$ *can lie on a conic.*

Proof. Let $\Phi \subset \Lambda$ be a subset with $|\Phi| > 10$. Consider the projection ξ as a composition of a projection $\alpha : \mathbb{P}^4 \dashrightarrow \mathbb{P}^3$ from some point $A \in L$ and a projection $\beta : \mathbb{P}^3 \dashrightarrow \Omega$ from the point $B = \alpha(L)$. The generality in the choice of the line L implies the generality of the projections α and β. We claim that the points of the sets $\alpha(\Phi)$ and $\xi(\Phi)$ do not lie on a conic in $\mathbb{P}^3$ and $\Omega \cong \mathbb{P}^2$ respectively.

Suppose that the points of $\alpha(\Phi)$ lie on a conic $C \subset \mathbb{P}^3$. Then conic C is irreducible. Let $\mathcal{D}$ be a linear system of quadric hypersurfaces in $\mathbb{P}^4$ passing through the points of Φ. As in the proof of Lemma 38 we see that the base locus of $\mathcal{D}$ is zero-dimensional, because the points of $\Phi \subset \mathbb{P}^4$ do not lie on a conic in $\mathbb{P}^4$. Take a cone $W \subset \mathbb{P}^4$ over the conic C with the vertex A. Then $\Phi \subset W$. Moreover, we have $\Phi \subset \mathrm{Bs}(\mathcal{D}|_W)$ and $\mathcal{D}|_W$ has no base components. Let D_1 and D_2 be general curves in $\mathcal{D}|_W$. Then

$$8 = D_1 \cdot D_2 \geq \sum_{\omega \in \Phi} \mathrm{mult}_\omega(D_1)\mathrm{mult}_\omega(D_2) \geq |\Phi| > 10,$$

which is a contradiction. Therefore, the points of $\alpha(\Phi)$ do not lie on a conic in $\mathbb{P}^3$.

Suppose that the points of $\xi(\Phi)$ lie on a conic $C \subset \Pi$. Then we can repeat the previous arguments to get a contradiction. The rest of the claim can be proved in a similar way. □

Now we can apply Corollary 30 to the subset $\Lambda' \setminus \hat{Q} \subset \mathbb{P}^2$ and point $\hat{Q}$ to prove the existence of a hypersurface in $\mathbb{P}^4$ of degree 5 that passes through $\Lambda \setminus Q$ and does not pass through the point $Q \in \Lambda$ (see the proof of Theorem 4). Similarly, in the case when at most 22 points of the subset $\Sigma' \subset \Pi$ can lie on a cubic curve in $\Pi \cong \mathbb{P}^2$ we can construct a hypersurface in $\mathbb{P}^3$ of degree 8 that passes through the points of the set $\Sigma \setminus P$ and does not pass through the point $P \in \Sigma$.

Lemma 57. *Suppose that there is a subset $\Upsilon \subset \Sigma$ such that $|\Upsilon| > 22$ and all the points of the set $\psi(\Upsilon)$ lie on a cubic curve in $\Pi \cong \mathbb{P}^2$. Then there is a hypersurface in $\mathbb{P}^3$ of degree 8 that passes through the points of $\Sigma \setminus P$ and does not pass through the point P.*

Proof. Let $\mathcal{H}$ be a linear system of cubic hypersurfaces in $\mathbb{P}^3$ passing through the points of the set Υ. Then the base locus of $\mathcal{H}$ is zero-dimensional by Lemma 38.

Suppose $P \in \Upsilon$. Then there is a hypersurface $F \subset \mathbb{P}^3$ of degree 6 that passes through the points of $\Upsilon \setminus P$ and does not pass through the point P by Lemma 22. On the other hand, the subset $\Sigma \setminus \Upsilon \subset \mathbb{P}^3$ contains at most 2 points. Hence, there is a quadric $G \subset \mathbb{P}^3$ that passes through the points of $\Sigma \setminus \Upsilon$ and does not pass through P. Thus, $F \cup G$ is the required hypersurface.

In the case when $P \notin \Upsilon$ and $P \in \mathrm{Bs}(\mathcal{H})$ we can repeat every step of the proof of the previous case. In the case when $P \notin \Upsilon$ and $P \notin \mathrm{Bs}(\mathcal{H})$ there is a cubic hypersurface in $\mathbb{P}^3$ that passes through the points of Υ and does not pass through the point P, which easily implies the existence of the required hypersurface. □

Hence, Proposition 9 is proved.

6. Non-isolated singularities

In this section we prove Theorem 13, but we omit the proof of Theorem 14, because it is similar. Let $\tau : U \to \mathbb{P}^s$ be a double cover branched over a hypersurface $F \subset \mathbb{P}^s$ of degree $2r$ such that $D_1 \cap \cdots \cap D_{s-3}$ is a $\mathbb{Q}$-factorial nodal 3-fold, where D_i is a general divisor in $|\tau^*(\mathcal{O}_{\mathbb{P}^s}(1))|$ and $s \geq 4$. Let D be a general divisor in $|\tau^*(\mathcal{O}_{\mathbb{P}^s}(1))|$. We must show that the group $\mathrm{Cl}(U)$ is generated by D. Note that U is normal.

Lemma 58. *The group $H^1(\mathcal{O}_U(-nD))$ for $n > 0$ vanishes.*

Proof. In the case when the singularities of the variety U are mild enough the claim is implied by the Kawamata-Viehweg vanishing (see [22]). In general let us prove the claim by induction on s. Suppose that $s = 4$. Then we have an exact sequence of sheaves

$$0 \to \mathcal{O}_U(-(n+1)D) \to \mathcal{O}_U(-nD) \to \mathcal{O}_D(-nD) \to 0$$

for any $n \in \mathbb{Z}$. Therefore, we have an exact sequence of the cohomology groups

$$0 \to H^1(\mathcal{O}_U(-(n+1)D)) \to H^1(\mathcal{O}_U(-nD)) \to H^1(\mathcal{O}_D(-nD)) \to \cdots$$

for $n > 0$. However, the 3-fold D is nodal by assumption. Thus, the group $H^1(\mathcal{O}_D(-nD))$ vanishes by the Kawamata-Viehweg vanishing. Hence, we have

$$H^1(\mathcal{O}_U(-D)) \cong H^1(\mathcal{O}_U(-2D)) \cong \cdots \cong H^1(\mathcal{O}_U(-nD))$$

for every $n > 0$. On the other hand, the group $H^1(\mathcal{O}_U(-nD))$ vanishes for $n \gg 0$ by the lemma of Enriques-Severi-Zariski (see [32]).

Suppose that $s > 4$. Then we have an exact sequence of sheaves

$$0 \to \mathcal{O}_U(-(n+1)D) \to \mathcal{O}_U(-nD) \to \mathcal{O}_D(-nD) \to 0$$

for any $n \in \mathbb{N}$. Therefore, we have an exact sequence of the cohomology groups

$$0 \to H^1(\mathcal{O}_U(-(n+1)D)) \to H^1(\mathcal{O}_U(-nD)) \to H^1(\mathcal{O}_D(-nD)) \to \cdots$$

for $n > 0$. However, the group $H^1(\mathcal{O}_D(-nD))$ vanishes by the induction. Hence,

$$H^1(\mathcal{O}_U(-D)) \cong H^1(\mathcal{O}_U(-2D)) \cong \cdots \cong H^1(\mathcal{O}_U(-nD))$$

for $n > 0$, but $H^1(\mathcal{O}_U(-nD)) = 0$ for $n \gg 0$ by the lemma of Enriques-Severi-Zariski. □

Consider a Weil divisor G on U. Let us prove by induction on s that $G \sim kD$ for some $k \in \mathbb{Z}$. Suppose that $s = 4$. Then the 3-fold D is nodal and $\mathbb{Q}$-factorial by assumption. Moreover, the group $\mathrm{Cl}(D)$ is generated by the class of the divisor $R|_D$, where R is a general divisor in $|D|$. Thus, there is an integer k such that we have the equivalence $G|_D \sim kR|_D$. Let $\Delta = G - kR$. We may assume that $\Delta \not\sim 0$.

The sequence of sheaves

$$0 \to \mathcal{O}_U(\Delta) \otimes \mathcal{O}_U(-D) \to \mathcal{O}_U(\Delta) \to \mathcal{O}_D \to 0$$

is exact, because $\mathcal{O}_U(\Delta)$ is locally free in the neighborhood of D.

Every section $\eta \in H^0(\mathcal{O}_U(\Delta) \otimes \mathcal{O}_U(-D))$ gives an effective Weil divisor S different from the divisor D, because the divisor D is the pull-back of a sufficiently general hyperplane on $\mathbb{P}^s$. Thus, the divisor $S \cap D$ is effective and

$S\cap D\sim -D|_D$, which is impossible. Hence, we have $H^0(\mathcal{O}_U(\Delta)\otimes\mathcal{O}_U(-D))=0$. Therefore, the sequence

$$0\to H^0(\mathcal{O}_U(\Delta))\to H^0(\mathcal{O}_D)\to H^1(\mathcal{O}_U(\Delta)\otimes\mathcal{O}_U(-D))$$

is exact.

Lemma 59. *The group $H^1(\mathcal{O}_U(\Delta)\otimes\mathcal{O}_U(-nD))$ vanishes for every $n>0$.*

Proof. The sheaf $\mathcal{O}_U(\Delta)$ is reflexive (see [18]). Thus, there is an exact sequence of sheaves

$$0\to\mathcal{O}_U(\Delta)\to\mathcal{E}\to\mathcal{F}\to 0$$

where $\mathcal{E}$ is a locally free sheaf and $\mathcal{F}$ is a torsion free sheaf. Hence, the sequence of groups

$$H^0(\mathcal{F}\otimes\mathcal{O}_U(-nD))\to H^1(\mathcal{O}_D(\Delta)\otimes\mathcal{O}_D(-nD))\to H^1(\mathcal{E}\otimes\mathcal{O}_U(-nD))$$

is exact. However, for $n\gg 0$ the cohomology group $H^0(\mathcal{F}\otimes\mathcal{O}_U(-nD))$ vanishes because the sheaf $\mathcal{F}$ is torsion free, and the cohomology group $H^1(\mathcal{E}\otimes\mathcal{O}_U(-nD))$ vanishes by the lemma of Enriques-Severi-Zariski. Therefore, $H^1(\mathcal{O}_U(\Delta)\otimes\mathcal{O}_U(-nD))=0$ for $n\gg 0$.

Now consider an exact sequence of sheaves

$$0\to\mathcal{O}_U(\Delta)\otimes\mathcal{O}_U(-(n+1)D)\to\mathcal{O}_U(\Delta)\otimes\mathcal{O}_U(-nD)\to\mathcal{O}_D(-nD)\to 0$$

and the induced sequence of cohomology groups

$$0\to H^1(\mathcal{O}_U(\Delta)\otimes\mathcal{O}_U(-(n+1)D))\to H^1(\mathcal{O}_U(\Delta)\otimes\mathcal{O}_U(-nD))\\ \to H^1(\mathcal{O}_D(-nD))\to\cdots$$

for $n>0$. Then the group $H^1(\mathcal{O}_D(-nD))$ vanishes by Lemma 58. Hence, we have

$$H^1(\mathcal{O}_U(\Delta)\otimes\mathcal{O}_U(-D))\cong H^1(\mathcal{O}_U(\Delta)\otimes\mathcal{O}_U(-2D))\cong\cdots\\ \cong H^1(\mathcal{O}_U(\Delta)\otimes\mathcal{O}_U(-nD))$$

for $n>0$, but we already proved that $H^1(\mathcal{O}_U(-nD))$ vanishes for $n\gg 0$. $\square$

Therefore, $H^0(\mathcal{O}_U(\Delta))\cong\mathbb{C}$. Similarly $H^0(\mathcal{O}_U(-\Delta))\cong\mathbb{C}$. Thus, the Weil divisor Δ is rationally equivalent to zero and $G\sim kD$ in the case $s=4$, which contradicts our assumption $\Delta\not\sim 0$. Thus, the case $s=4$ is done.

Suppose that $s>4$. By the induction we may assume that the group $\mathrm{Cl}(D)$ is generated by the class of the divisor $R|_D$, where R is a general divisor in $|D|$. Thus, there is an integer k such that $G|_D\sim kR|_D$. Put $\Delta=G-kR$. Then the sequence of sheaves

$$0\to\mathcal{O}_U(\Delta)\otimes\mathcal{O}_U(-D))\to\mathcal{O}_U(\Delta)\to\mathcal{O}_D\to 0$$

is exact, because $\mathcal{O}_U(\Delta)$ is locally free in the neighborhood of D. Therefore, the sequence

$$0 \to H^0(\mathcal{O}_U(\Delta)) \to H^0(\mathcal{O}_D) \to H^1(\mathcal{O}_U(\Delta) \otimes \mathcal{O}_U(-D))$$

is exact. However, the proof of Lemma 59 holds for $s > 4$. Thus, the cohomology group $H^1(\mathcal{O}_U(\Delta) \otimes \mathcal{O}_U(-D))$ vanishes. Hence, $H^0(\mathcal{O}_U(\Delta)) \cong \mathbb{C}$. The same arguments prove that $H^0(\mathcal{O}_U(-\Delta)) \cong \mathbb{C}$. Therefore, the Weil divisor Δ is rationally equivalent to zero and $G \sim kD$. Thus, we proved Theorem 13.

7. Birational rigidity

In this section we prove Proposition 15. Let $\xi : Y \to \mathbb{P}^4$ be a double cover branched over a hypersurface $F \subset \mathbb{P}^4$ of degree 8 such that the hypersurface F is smooth outside of a smooth curve $C \subset F$, the singularity of the hypersurface F in a sufficiently general point of the curve C is locally isomorphic to the singularity

$$x_1^2 + x_2^2 + x_3^2 = 0 \subset \mathbb{C}^4 \cong \mathrm{Spec}(\mathbb{C}[x_1, x_2, x_3, x_4]),$$

the singularities of F in other points of C are locally isomorphic to the singularity

$$x_1^2 + x_2^2 + x_3^2 x_4 = 0 \subset \mathbb{C}^4 \cong \mathrm{Spec}(\mathbb{C}[x_1, x_2, x_3, x_4]),$$

and a general 3-fold in $|-K_Y|$ is $\mathbb{Q}$-factorial. Then Y is a Fano 4-fold with terminal singularities and $-K_Y \sim \xi^*(\mathcal{O}_{\mathbb{P}^4}(1))$. Moreover, $\mathrm{Cl}(Y)$ and $\mathrm{Pic}(Y)$ are generated by the divisor $-K_Y$ by Theorem 13. Hence, Y is a Mori fibration (see [22]). We must prove that the 4-fold Y is a unique Mori fibration birational to Y and $\mathrm{Bir}(Y) = \mathrm{Aut}(Y)$. It is well known that the latter implies the finiteness of the group $\mathrm{Bir}(Y)$.

Suppose that either Y is not birationally rigid or $\mathrm{Bir}(Y) \neq \mathrm{Aut}(Y)$. Then Theorem 28 implies the existence of a linear system $\mathcal{M}$ on Y such that $\mathcal{M}$ has no fixed components and the singularities of $(X, \frac{1}{n}\mathcal{M})$ are not canonical, where $\mathcal{M} \sim -nK_Y$. Thus, there is a rational number $\mu < \frac{1}{n}$ such that $(X, \mu\mathcal{M})$ is not canonical, i.e., $\mathbb{CS}(Y, \mu\mathcal{M}) \neq \emptyset$.

Let Z be an element of the set $\mathbb{CS}(Y, \mu\mathcal{M})$. Then $\mathrm{mult}_Z(\mathcal{M}) > n$.

Lemma 60. *The subvariety $Z \subset Y$ is not a smooth point of Y.*

Proof. Suppose Z is a smooth point of Y. Then $\mathrm{mult}_Z(\mathcal{M}^2) > 4n^2$ by Theorem 25 and

$$2n^2 = \mathcal{M}^2 \cdot H_1 \cdot H_2 \geq \mathrm{mult}_Z(\mathcal{M}^2)\mathrm{mult}_Z(H_1)\mathrm{mult}_Z(H_2) > 4n^2$$

for general divisors H_1 and H_2 in $|-K_Y|$ containing Z, which is a contradiction. □

Lemma 61. *The subvariety $Z \subset Y$ is not a singular point of Y.*

Proof. Let $\xi(Z) = O$. Then O is a singular point of the hypersurface $F \subset \mathbb{P}^4$. Therefore, the point O is contained in the curve $C \subset F$ by assumption. There are two possible cases, i.e., either the singularity of F in the point O is locally isomorphic to the singularity

$$x_1^2 + x_2^2 + x_3^2 = 0 \subset \mathbb{C}^4 \cong \mathrm{Spec}(\mathbb{C}[x_1, x_2, x_3, x_4]),$$

or the singularity of F in the point O is locally isomorphic to the singularity

$$x_1^2 + x_2^2 + x_3^2 x_4 = 0 \subset \mathbb{C}^4 \cong \mathrm{Spec}(\mathbb{C}[x_1, x_2, x_3, x_4]),$$

where $x_1 = x_2 = x_3$ are local equations of the curve $C \subset F$. Let us call the former case ordinary and the latter case non-ordinary.

Let X be a sufficiently general divisor in the linear system $|-K_Y|$ passing through the point Z. Then the double cover ξ induces the double cover $\tau : X \to \mathbb{P}^3$ ramified along an octic surface. The singularities of $X \setminus Z$ are ordinary double points. Moreover, Z is an ordinary double point of X in the ordinary case. In the non-ordinary case the singularity of the 3-fold X at the point Z is locally isomorphic to

$$x_1^2 + x_2^2 + x_3^2 + x_4^3 = 0 \subset \mathbb{C}^4 \cong \mathrm{Spec}(\mathbb{C}[x_1, x_2, x_3, x_4]).$$

Let $\mathcal{D} = \mathcal{M}|_X$ and $H = -K_Y|_X$. Then $\mathcal{D}$ has no fixed components, $\mathcal{D} \sim nH$ and we have $Z \in \mathbb{LCS}(X, \mu\mathcal{D})$ by Proposition 24. In particular, $Z \in \mathbb{CS}(X, \mu\mathcal{D})$.

Let $f : V \to X$ be a blowup of Z, $E = f^{-1}(Z)$ and $\mathcal{H}$ be a proper transform of the linear system $\mathcal{D}$ on V. Then V is smooth in the neighborhood of E and E is isomorphic to a quadric surface in $\mathbb{P}^3$. In the ordinary case E is smooth. In the non-ordinary case the quadric surface E has one singular point $P \in E$, i.e., the surface E is isomorphic to a quadric cone in $\mathbb{P}^3$. Note that $K_V \sim E$.

Let $\mathrm{mult}_Z(\mathcal{D}) \in \mathbb{N}$ such that $\mathcal{H} \sim f^*(nH) - \mathrm{mult}_Z(\mathcal{D})E$. Then $\mathrm{mult}_Z(\mathcal{D}) > n$ in the ordinary case by Theorem 26. On the other hand, in the non-ordinary case we have the inequality $\mathrm{mult}_Z(\mathcal{D}) > \frac{n}{2}$ due to Proposition 27.

By construction the linear system $|f^*(H) - E|$ is free and gives a morphism $\psi : V \to \mathbb{P}^2$ such that $\psi = \phi \circ \tau \circ f$, where $\phi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ is a projection from the point O. Moreover, the restriction $\psi|_E : E \to \mathbb{P}^2$ is a double cover. Let L be a sufficiently general fiber of the morphism ψ. Then L is a smooth curve of genus 2 and $L \cdot E = L \cdot f^*(H) = 2$. Thus,

$$L \cdot \mathcal{H} = L \cdot f^*(nH) - \mathrm{mult}_Z(\mathcal{D}) L \cdot E = 2n - 2\mathrm{mult}_Z(\mathcal{D}) \geq 0,$$

because $\mathcal{H}$ has no base components. Hence, $\mathrm{mult}_Z(\mathcal{D}) \leq n$. In particular, the ordinary case is impossible and it remains to eliminate the non-ordinary case.

The inequalities $\mathrm{mult}_Z(\mathcal{D}) \leq n$ and $\mu < \frac{1}{n}$, the equivalence

$$K_V + \mu\mathcal{H} \sim f^*(K_X + \mu\mathcal{D}) + (1 - \mu\mathrm{mult}_Z(\mathcal{D}))E$$

and $Z \in \mathbb{CS}(X, \mu\mathcal{D})$ imply the existence of a proper irreducible subvariety $S \subset E$ such that $S \in \mathbb{CS}(V, \mu\mathcal{H} + (\mu\mathrm{mult}_Z(\mathcal{D}) - 1)E)$. In particular, $S \in \mathbb{CS}(V, \mu\mathcal{H})$.

Suppose that S is a curve. Then $\mathrm{mult}_S(\mathcal{H}) > n$. Let L_ω be a fiber of ψ passing through a general point $\omega \in S$. Then L_ω describes a divisor in V when we vary ω on S. Hence,

$$\begin{aligned} L_\omega \cdot \mathcal{H} = L_\omega \cdot f^*(nH) - \mathrm{mult}_Z(\mathcal{D}) L_\omega \cdot E = 2n - 2\mathrm{mult}_Z(\mathcal{D}) \\ \geq \mathrm{mult}_\omega(L_\omega)\mathrm{mult}_S(\mathcal{H}) > n, \end{aligned}$$

which contradicts the inequality $\mathrm{mult}_Z(\mathcal{D}) > \frac{n}{2}$.

Therefore, S is a point on E. Then $\mathrm{mult}_S(\mathcal{H}) > n$ and $\mathrm{mult}_S(\mathcal{H}^2) > 4n^2$ by Theorem 25, because S is smooth on V. It is easy to see that the point S is not a vertex P of the quadric cone E, because the numerical intersection of a general ruling of E with a general divisor in $\mathcal{H}$ is equal to $\mathrm{mult}_Z(\mathcal{D}) \leq n$. Let Γ be a fiber of the morphism ψ that passes through the point S, and let D be a general divisor in the linear system $|f^*(H) - E|$ that passes through the point S. Then $\Gamma \subset D$. Note that Γ may be reducible and singular, but we always have the inequality $\mathrm{mult}_S(\Gamma) \leq 2$, because $\tau \circ f(\Gamma)$ is a line passing through the point O and $\tau|_{f(\Gamma)}$ is a double cover.

Suppose that Γ is irreducible. Let $\mathcal{H}^2 = \lambda\Gamma + T$, where $\lambda \in \mathbb{N}$ and T is a one-cycle such that $\Gamma \not\subset \mathrm{Supp}(T)$. Then the inequalities

$$\mathrm{mult}_S(T) > 4n^2 - \lambda\mathrm{mult}_S(\Gamma) \geq 4n^2 - 2\lambda$$

hold. On the other hand, the inequalities

$$\mathrm{mult}_S(T) \leq \mathrm{mult}_S(T)\mathrm{mult}_S(D) \leq T \cdot D = \mathcal{H}^2 \cdot D = 2n^2 - \mathrm{mult}_Z^2(\mathcal{D}) < \frac{7}{4}n^2$$

hold. Thus, we have $\lambda > \frac{9}{8}n^2$. Let $\tilde{D}$ be a general divisor in $|f^*(H)|$. Then

$$2n^2 = \tilde{D} \cdot \mathcal{H}^2 \geq \lambda\Gamma \cdot \tilde{D} = 2\lambda > \frac{9}{4}n^2,$$

which is a contradiction.

Therefore, the fiber Γ is reducible. Then $\Gamma = \Gamma_1 \cup \Gamma_2$, where Γ_i is a smooth rational curve such that $\tau \circ f(\Gamma_1) = \tau \circ f(\Gamma_2)$ is a line in $\mathbb{P}^3$ containing point O. Let

$$\mathcal{H}^2 = \lambda_1\Gamma_1 + \lambda_2\Gamma_2 + T,$$

where $\lambda_i \in \mathbb{N}$ and T is a one-cycle such that $\Gamma_i \not\subset \mathrm{Supp}(T)$. Then the inequalities

$$\frac{7}{4}n^2 > 2n^2 - \mathrm{mult}_Z^2(\mathcal{D}) \geq T \cdot D \geq \mathrm{mult}_S(T) > 4n^2 - \lambda_1 - \lambda_2$$

hold. Thus, $\lambda_1 + \lambda_2 > \frac{9}{4}n^2$. Hence, we have

$$2n^2 = \tilde{D} \cdot \mathcal{H}^2 \geq \lambda_1 \Gamma_1 \cdot \tilde{D} + \lambda_2 \Gamma_2 \cdot \tilde{D} = \lambda_1 + \lambda_2 > \frac{9}{4}n^2$$

for a general divisor $\tilde{D} \in |f^*(H)|$, which is a contradiction. □

Lemma 62. *The subvariety $Z \subset Y$ is not a curve.*

Proof. Suppose Z is a curve. Let X be a general divisor in $|-K_Y|$ and P be a point in the intersection $Z \cap X$. Then X is a nodal Calabi-Yau 3-fold. The point P is smooth on the 3-fold X if and only if $Z \not\subset \mathrm{Sing}(X)$. In the case $Z \subset \mathrm{Sing}(X)$ the point P is an ordinary double point on X. Moreover, $P \in \mathbb{CS}(X, \mu\mathcal{D})$, where $\mathcal{D} = \mathcal{M}|_X$. In the case when the point P is smooth on X we can proceed as in the proof of Lemma 60 to get a contradiction. In the case when the point P is an ordinary double point on X we can proceed as in the proof of Lemma 61 to get a contradiction. □

Lemma 63. *The subvariety $Z \subset Y$ is not a surface.*

Proof. Suppose Z is a surface. Then $\mathrm{mult}_Z(\mathcal{M}) > n$. Let V be a general divisor in the linear system $|-K_Y|$, $S = Z \cap V$ and $\mathcal{D} = \mathcal{M}|_V$. Then V is a nodal Calabi-Yau 3-fold, the linear system $\mathcal{D}$ has no base components, $S \subset V$ is an irreducible reduced curve and $\mathrm{mult}_S(\mathcal{D}) > n$. The double cover ξ induces a double cover $\tau : V \to \mathbb{P}^3$ ramified along a nodal hypersurface $G \subset \mathbb{P}^3$ of degree 8.

Take a sufficiently general divisor H in $|\tau^*(\mathcal{O}_{\mathbb{P}^3}(1))|$. Then

$$2n^2 = \mathcal{D}^2 \cdot H \geq \mathrm{mult}_S^2(\mathcal{D}) S \cdot H > n^2 S \cdot H,$$

which implies $S \cdot H = 1$. Hence, $\tau(S)$ is a line in $\mathbb{P}^3$ and $\tau|_S$ is an isomorphism.

Suppose that $\tau(S) \not\subset G$. Then there is a smooth rational curve $\tilde{S} \subset V$ such that $S \neq \tilde{S}$ and $\tau(S) = \tau(\tilde{S})$. Take a sufficiently general surface $D \in |\tau^*(\mathcal{O}_{\mathbb{P}^3}(1))|$ passing through the curve S. Then D is smooth outside of $S \cap \tilde{S}$. Moreover, the surface D is smooth in every point of $S \cap \tilde{S}$ that is smooth on V, and D has an ordinary double point in every point of $S \cap \tilde{S}$ that is an ordinary double point on V. On the other hand, at most 4 nodes of the hypersurface $G \subset \mathbb{P}^3$ can lie on the line $\tau(S)$, i.e., $|\mathrm{Sing}(D)| \leq 4$. The sub-adjunction formula (see [22], [23]) implies

$$(K_D + \tilde{S})|_{\tilde{S}} = K_{\tilde{S}} + \mathrm{Diff}_{\tilde{S}}(0)$$

and $\deg(\mathrm{Diff}_{\tilde{S}}(0)) = \frac{k}{2}$, where $k = |\mathrm{Sing}(D)|$. Thus, the self-intersection $\tilde{S}^2$ is negative on the surface D, because $K_D \cdot \tilde{S} = 1$. Put $\mathcal{H} = \mathcal{D}|_D$. A priori the linear system $\mathcal{H}$ can have a base component. However, the generality in the choice of D implies

$$\mathcal{H} = \mathrm{mult}_S(\mathcal{D})S + \mathrm{mult}_{\tilde{S}}(\mathcal{D})\tilde{S} + \mathcal{B}$$

where $\mathcal{B}$ is a linear system on D having no base components. Moreover, the equivalence

$$(n - \mathrm{mult}_{\tilde{S}}(\mathcal{D}))\tilde{S} \sim_{\mathbb{Q}} (\mathrm{mult}_S(\mathcal{D}) - n)S + \mathcal{B}$$

holds, because $\tilde{S} + S \sim D|_D$ and $\mathcal{H} \sim nD|_D$. Therefore, the inequality $\tilde{S}^2 < 0$ implies the inequality $\mathrm{mult}_{\tilde{S}}(\mathcal{D}) > n$. Take a general divisor H in $|\tau^*(\mathcal{O}_{\mathbb{P}^3}(1))|$. Then

$$2n^2 = \mathcal{D}^2 \cdot H \geq \mathrm{mult}^2_S(\mathcal{D})S \cdot H + \mathrm{mult}^2_{\tilde{S}}(\mathcal{D})\tilde{S} \cdot H > n^2 S \cdot H + n^2 \tilde{S} \cdot H = 2n^2,$$

which is a contradiction.

Therefore, we have $\tau(S) \subset G$. Let O be a general point on $\tau(S)$ and Π be a hyperplane in $\mathbb{P}^3$ that is tangent to G at the point O. Consider a sufficiently general line $L \subset \Pi$ passing through O. Let $\hat{L} = \tau^{-1}(L)$ and $\hat{O} = \tau^{-1}(O)$. Then $\hat{L}$ is singular at $\hat{O}$. Therefore, the curve $\hat{L}$ is contained in the base locus of the linear system $\mathcal{D}$, because otherwise

$$2n = \hat{L} \cdot \mathcal{D} \geq \mathrm{mult}_{\hat{O}}(\hat{L})\mathrm{mult}_{\hat{O}}(\mathcal{D}) \geq 2\mathrm{mult}_S(\mathcal{D}) > 2n,$$

which is impossible. On the other hand, the curve $\hat{L}$ describes a divisor in V when we vary the line L in Π. The latter is impossible, because $\mathcal{D}$ has no base components. □

Therefore, Proposition 15 is proved.

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