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FACTORIAL THREEFOLD HYPERSURFACES

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Abstract

Let X be a hypersurface in $\mathbb{P}^4$ of degree d that has at worst isolated ordinary double points. We prove that X is factorial in the case when X has at most $(d-1)^2-1$ singular points.

We assume that all varieties are projective, normal, and defined over $\mathbb{C}$.

1. Introduction

The Cayley–Bacharach theorem (see [8] and [11]), in its classical form, may be seen as a result about the number of independent linear conditions imposed on forms of a given degree by a certain finite subset of $\mathbb{P}^n$. The purpose of this paper is to prove the following result.

Theorem 1.1. *Let Σ be a finite subset of $\mathbb{P}^n$ for $n \geqslant 2$, let μ be a natural number such that*

- *the inequalities $\mu \geqslant 2$ and $|\Sigma| \leqslant \mu^2 - 1$ hold,*
- *and at most μk points in the set Σ lie on a curve in $\mathbb{P}^n$ of degree $k = 1, \ldots, \mu - 1$.*

Then Σ imposes independent linear conditions on forms of degree $2\mu - 3$.

Let X be a hypersurface in $\mathbb{P}^4$ of degree $d \geqslant 3$ that has at most isolated ordinary double points. Then X can be given by the equation

$$f\big(x, y, z, t, u\big) = 0 \subset \mathbb{P}^4 \cong \mathrm{Proj}\Big(\mathbb{C}\big[x, y, z, t, u\big]\Big),$$

where $f(x, y, z, t, u)$ is a homogeneous polynomial of degree d.

Remark 1.2. It follows from [1, Section 11], [13, Theorem IV.3.1] and [10, Proposition 3.3] that the following conditions are equivalent:

- every Weil divisor on the threefold X is Cartier;
- every surface $S \subset X$ is cut out on X by a hypersurface in $\mathbb{P}^4$;

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- the ring
$$\mathbb{C}[x, y, z, t, u] \Big/ \langle f(x, y, z, t, u) \rangle$$
is a unique factorization domain (cf. [13, Exercise IV.3.5]);
- the set $\mathrm{Sing}(X)$ imposes independent linear conditions on forms of degree $2d-5$.

We say that X is factorial if every Weil divisor on X is Cartier.

Example 1.3. Suppose that X is given by
$$xg(x, y, z, t, u) + yh(x, y, z, t, u) = 0 \subset \mathbb{P}^4 \cong \mathrm{Proj}\Big(\mathbb{C}[x, y, z, t, u]\Big),$$
where g and h are general homogeneous polynomials of degree $d-1$. Then

- the threefold X has at worst isolated ordinary double points,
- the equality $|\mathrm{Sing}(X)| = (d-1)^2$ holds, but X is not factorial.

The assertion of Theorem 1.1 implies the following result (cf. [7], [3], and [5]).

Theorem 1.4. *Suppose that* $|\mathrm{Sing}(X)| < (d-1)^2$. *Then* X *is factorial.*

Proof. The set $\mathrm{Sing}(X)$ is a set-theoretic intersection of hypersurfaces of degree $d-1$. Now

- the inequalities $d-1 \geqslant 2$ and $|\,\mathrm{Sing}(X)| \leqslant (d-1)^2 - 1$ hold,
- and at most $(n-1)k$ points in $\mathrm{Sing}(X)$ lie on a curve in $\mathbb{P}^4$ of degree $k = 1, \ldots, n-2$,

which immediately implies that the points of $\mathrm{Sing}(X)$ impose independent linear conditions on forms of degree $2d-5$ by Theorem 1.1. Thus, the threefold X is factorial. $\square$

The assertion of Theorem 1.4 has been proved in [4], [6], and [14] for $d \leqslant 11$.

Remark 1.5. Suppose that $d = 4$ and X is factorial. Then it follows from [15, Theorem 2] that the threefold X is non-rational, and the threefold X is not birational to a fibration by rational surfaces. But general determinantal quartic hypersurfaces in $\mathbb{P}^4$ are rational.

2. The proof

Let Σ be a finite subset of $\mathbb{P}^n$ for $n \geqslant 2$, and let μ be a natural number such that

- the inequalities $\mu \geqslant 2$ and $|\Sigma| \leqslant \mu^2 - 1$ hold.
- at most μk points in the set Σ lie on a curve in $\mathbb{P}^n$ of degree $k = 1, \ldots, \mu - 1$.

Suppose that Σ imposes dependent linear conditions on forms of degree $2\mu-3$.

Remark 2.1. The inequality $\mu \geqslant 3$ holds, because otherwise we have $\mu=2$, at most 3 points in the set Σ lie on a line, and $|\Sigma| \leqslant 3$. But Σ imposes dependent linear conditions on linear forms.

The following result follows from [2, Theorem 2] and [9, Corollary 4.3].

Theorem 2.2. *Let $P_1, \ldots, P_\delta \in \mathbb{P}^2$ be distinct points such that*

- *at most $k(\xi+3-k)-2$ points in $\{P_1, \ldots, P_\delta\}$ lie on a curve of degree $k \leqslant (\xi+3)/2$,*
- *the inequality*

$$\delta \leqslant \max\left\{\left\lfloor\frac{\xi+3}{2}\right\rfloor\left(\xi+3-\left\lfloor\frac{\xi+3}{2}\right\rfloor\right)-1, \left\lfloor\frac{\xi+3}{2}\right\rfloor^2\right\}$$

holds, where ξ is a natural number such that $\xi \geqslant 3$,

and let $\pi\colon Y \to \mathbb{P}^2$ be a blow up of the points $P_1, \ldots, P_\delta$. Then the linear system

$$\left|\pi^*\Big(\mathcal{O}_{\mathbb{P}^2}(\xi)\Big)-\sum_{i=1}^{\delta} E_i\right|$$

does not have base points, where E_i is the π-exceptional divisor such that $\pi(E_i)=P_i$.

We see that there is a point $P \in \Sigma$ such that every hypersurface in $\mathbb{P}^n$ of degree $2\mu-3$ that contains the set $\Sigma \setminus P$ must contain the point $P \in \Sigma$. Let us derive a contradiction.

Lemma 2.3. *The inequality $n \neq 2$ holds.*

Proof. Suppose that $n=2$. Let us prove that at most $k(2\mu-k)-2$ points in $\Sigma \setminus P$ can lie on a curve of degree $k \leqslant \mu$. It is enough to show that

$$k\big(2\mu-k\big)-2 \geqslant k\mu$$

for every $k \leqslant \mu$. We must prove this only for $k \geqslant 1$ such that

$$k\big(2\mu-k\big)-2 < \big|\Sigma \setminus P\big| \leqslant \mu^2-2,$$

because otherwise the condition that at most $k(2\mu-k)-2$ points in the set $\Sigma \setminus P$ can lie on a curve of degree k is vacuous. Therefore, we may assume that $k<\mu$.

We may assume that $k \neq 1$, because at most $\mu \leqslant 2\mu-3$ points of $\Sigma \setminus P$ lie on a line. Then

$$k\big(2\mu-k\big)-2 \geqslant k\mu \iff \mu > k,$$

which implies that at most $k(2\mu-k)-2$ points in $\Sigma \setminus P$ can lie on a curve in $\mathbb{P}^2$ of degree $k \leqslant \mu$.

Thus, it follows from Theorem 2.2 that there is a curve of degree $2\mu-3$ that contains all points of the set $\Sigma\setminus P$ and does not contain the point $P\in\Sigma$, which is a contradiction. $\square$

Moreover, we may assume that $n=3$ because of the following result.

Lemma 2.4. *Let Σ be a finite subset in $\mathbb{P}^n$, let μ be a natural number such that*

- *the inequalities $\mu\geqslant 2$ and $|\Sigma|\leqslant\mu^2-1$ hold,*
- *at most μk points in the set Σ lie on a curve in $\mathbb{P}^n$ of degree $k=1,\dots,\mu-1$,*

let $\Lambda\subseteq\Sigma$ be a subset, let $\mathcal{M}\subseteq|\mathcal{O}_{\mathbb{P}^n}(k)|$ be a linear subsystem that contains all hypersurfaces of degree k that pass through Λ, and let $\psi\colon\mathbb{P}^n\dashrightarrow\mathbb{P}^m$ be a general linear projection. Suppose that

- *the inequality $|\Lambda|\geqslant\mu k+1$ holds,*
- *the set $\psi(\Lambda)$ is contained in an irreducible reduced curve of degree k,*

and $n>m\geqslant 2$. Then $\mathcal{M}$ has no base curves, and either $m=2$, or $k>\mu$.

Proof. There are linear subspaces Ω and $\Pi\subset\mathbb{P}^n$ such that

$$\psi\colon\mathbb{P}^n\dashrightarrow\Pi\cong\mathbb{P}^m$$

is a projection from Ω, where $\dim(\Omega)=n-m-1$ and $\dim(\Pi)=m$.

Suppose that there is an irreducible curve $Z\subset\mathbb{P}^n$ that is contained in the base locus of the linear system $\mathcal{M}$. Put $\Xi=Z\cap\Lambda$. We may assume that $\psi|_Z$ is a birational morphism, and

$$\psi\big(Z\big)\cap\psi\Big(\Lambda\setminus\Xi\Big)=\varnothing,$$

because ψ is general. Then $\deg(\psi(Z))=\deg(Z)$. Similarly, we may assume that $m=2$.

Let $C\subset\Pi$ be an irreducible curve of degree k that contains $\psi(\Lambda)$, and let $W\subset\mathbb{P}^n$ be the cone over the curve C whose vertex is Ω. Then $W\in\mathcal{M}$, which implies that $Z\subset W$. We have

$$\psi\big(Z\big)=C,$$

which immediately implies that $\Xi=\Lambda$ and $\deg(Z)=k$. But $|Z\cap\Sigma|\leqslant\mu k$, which is a contradiction. Therefore, the linear system $\mathcal{M}$ does not have base curves.

Now we suppose that $m\geqslant 3$ and $k\leqslant\mu$. Let us show that this assumption leads to a contradiction. Without loss of generality, we may assume that $m=3$ and $n=4$.

Let $\mathcal{Y}$ be the set of all irreducible reduced surfaces in $\mathbb{P}^4$ of degree k that contain the set Λ, and let Υ be a subset of $\mathbb{P}^4$ that consists of all points that are contained in every surface of the set $\mathcal{Y}$. Then $\Lambda \subseteq \Upsilon$. Arguing as above, we see that Υ is a finite set.

Let $\mathcal{S}$ be the set of all surfaces in $\mathbb{P}^3$ of degree k such that

$$S \in \mathcal{S} \iff \exists\, Y \in \mathcal{Y} \text{ such that } \psi\big(Y\big) = S \text{ and } \psi\big|_Y \text{ is a birational morphism,}$$

and let $\Psi \subset \mathbb{P}^3$ be a subset such that Ψ consists of all points that are contained in every surface of the set $\mathcal{S}$. Then $\mathcal{S} \neq \varnothing$ and $\psi(\Lambda) \subseteq \psi(\Upsilon) \subseteq \Psi$.

It follows from the generality of the point Ω that Ψ is a finite set. But $\psi(\Lambda) \subseteq \Psi$ contains at least $\mu k+1 \geqslant k^2+1$ points that are contained in a curve of degree k, which is impossible, because Ψ is a set-theoretic intersection of surfaces of degree k. $\square$

Fix a sufficiently general hyperplane $\Pi \subset \mathbb{P}^3$. Let

$$\psi\colon \mathbb{P}^3 \dashrightarrow \Pi \cong \mathbb{P}^2$$

be a projection from a sufficiently general point $O \in \mathbb{P}^3$. Put $\Sigma' = \psi(\Sigma)$ and $P' = \psi(P)$.

Lemma 2.5. *There is a curve $C \subset \Pi$ of degree $k \leqslant \mu - 1$ such that $|C \cap \Sigma'| \geqslant \mu k + 1$.*

Proof. We suppose that at most μk points of the set Σ' are contained in a curve in Π of degree k for every $k \leqslant \mu - 1$. Then arguing as in the proof of Lemma 2.3, we obtain a curve

$$Z \subset \Pi \cong \mathbb{P}^2$$

of degree $2\mu - 3$ that contains the set $\Sigma' \setminus P'$ and does not pass through the point P'.

Let Y be the cone in $\mathbb{P}^3$ over the curve Z whose vertex is the point O. Then Y is a surface of degree $2\mu - 3$ that contains all points of the set $\Sigma \setminus P$ but does not contain the point $P \in \Sigma$. $\square$

It immediately follows from Lemma 2.4 that for the curve C in Lemma 2.5, one has $k \geqslant 2$.

Lemma 2.6. *Suppose that $|C \cap \Sigma'| \geqslant 9$. Then $k \geqslant 3$.*

Proof. Suppose that $k = 2$. Let $\Phi \subseteq \Sigma$ be a subset such that $|\Phi| \geqslant 9$, but $\psi(\Phi)$ is contained in the conic $C \subset \Pi$. Then C is irreducible by Lemma 2.4.

Let $\mathcal{D}$ be a linear system of quadric hypersurfaces in $\mathbb{P}^3$ containing Φ. Then $\mathcal{D}$ does not have base curves by Lemma 2.4. Let W be a cone in $\mathbb{P}^3$ over C

with vertex Ω. Then

$$8 = D_1 \cdot D_2 \cdot W \geqslant \sum_{\omega \in \Phi} \mathrm{mult}_{\omega}(D_1)\mathrm{mult}_{\omega}(D_2) \geqslant |\,\Phi| \geqslant 9,$$

where D_1 and D_2 are general divisors in the linear system $\mathcal{D}$. $\square$

We may assume that k is the smallest natural number such that at least $\mu k + 1$ points in Σ' lie on a curve of degree k. Then there is a non-empty disjoint union

$$\bigcup_{j=k}^{l} \bigcup_{i=1}^{c_j} \Lambda_j^i \subset \Sigma$$

such that $|\Lambda_j^i| \geqslant \mu j + 1$, all points of the the set $\psi(\Lambda_j^i)$ are contained in an irreducible reduced curve of degree j, and for every natural number ζ at most $\mu\zeta$ points of the subset

$$\psi\left(\Sigma \setminus \Big(\bigcup_{j=k}^{l} \bigcup_{i=1}^{c_j} \Lambda_j^i\Big)\right) \subsetneq \Sigma' \subset \Pi \cong \mathbb{P}^2$$

lie on a curve in Π of degree ζ. Put

$$\Lambda = \bigcup_{j=k}^{l} \bigcup_{i=1}^{c_j} \Lambda_j^i.$$

Let Ξ_j^i be the base locus of the linear subsystem of $|\mathcal{O}_{\mathbb{P}^3}(j)|$ that contains all surfaces passing through the set Λ_j^i. Then Ξ_j^i is a finite set by Lemma 2.4, and

$$\big|\Sigma \setminus \Lambda\big| \leqslant \mu \left(\mu - \sum_{i=k}^{l} c_i i\right) - 2. \tag{2.7}$$

Corollary 2.8. *The inequality* $\sum_{i=k}^{l} ic_i \leqslant \mu - 1$ *holds.*

Put $\Delta = \Sigma \cap (\bigcup_{j=k}^{l} \bigcup_{i=1}^{c_j} \Xi_j^i)$. Then $\Lambda \subseteq \Delta \subseteq \Sigma$.

Lemma 2.9. *The set* Δ *imposes independent linear conditions on forms of degree* $2\mu - 3$.

Proof. Let us consider the subset $\Delta \subset \mathbb{P}^3$ as a closed subscheme of $\mathbb{P}^3$, and let $\mathcal{I}_{\Delta}$ be the ideal sheaf of the subscheme Δ. Then there is an exact sequence

$$0 \longrightarrow \mathcal{I}_{\Delta} \otimes \mathcal{O}_{\mathbb{P}^3}\big(2\mu - 3\big) \longrightarrow \mathcal{O}_{\mathbb{P}^3}\big(2\mu - 3\big) \longrightarrow \mathcal{O}_{\Delta} \longrightarrow 0,$$

which implies that Δ imposes independent conditions on forms of degree $2\mu - 3$ if and only if

$$h^1\Big(\mathcal{I}_{\Delta} \otimes \mathcal{O}_{\mathbb{P}^3}\big(2\mu - 3\big)\Big) = 0.$$

Suppose $h^1(\mathcal{I}_{\Delta} \otimes \mathcal{O}_{\mathbb{P}^3}(2\mu - 3)) \neq 1$. Let us show that this assumption leads to a contradiction.

Let $\mathcal{M}$ be the linear subsystem of $|\mathcal{O}_{\mathbb{P}^3}(\mu-1)|$ that contains all surfaces that pass through all points of the set Δ. Then the base locus of $\mathcal{M}$ is zero-dimensional, because $\sum_{i=k}^{l} ic_i \leqslant \mu-1$ and

$$\Delta \subseteq \bigcup_{j=k}^{l}\bigcup_{i=1}^{c_j}\Xi_j^i,$$

but Ξ_j^i is zero-dimensional base locus of a linear subsystem of $|\mathcal{O}_{\mathbb{P}^3}(j)|$. Put

$$\Gamma = M_1\cdot M_2\cdot M_3,$$

where M_1, M_2, M_3 are general surfaces in the linear system $\mathcal{M}$. Then Γ is a zero-dimensional subscheme of $\mathbb{P}^3$, and Δ is a closed subscheme of the scheme Γ.

Let Υ be a closed subscheme of Γ such that

$$\mathcal{I}_{\Upsilon} = \mathrm{Ann}\Big(\mathcal{I}_{\Delta}/\mathcal{I}_{\Gamma}\Big),$$

where $\mathcal{I}_{\Upsilon}$ and $\mathcal{I}_{\Gamma}$ are the ideal sheaves of the subschemes Υ and Γ, respectively. Then

$$0\neq h^1\Big(\mathcal{O}_{\mathbb{P}^3}\big(2\mu-3\big)\otimes\mathcal{I}_{\Delta}\Big)=h^0\Big(\mathcal{O}_{\mathbb{P}^3}\big(\mu-4\big)\otimes\mathcal{I}_{\Upsilon}\Big)-h^0\Big(\mathcal{O}_{\mathbb{P}^3}\big(\mu-4\big)\otimes\mathcal{I}_{\Gamma}\Big)$$

by [8, Theorem 3] (see also [11]). Thus, there is a surface $F\in|\mathcal{O}_{\mathbb{P}^n}(\mu-4)\otimes\mathcal{I}_{\Upsilon}|$. Then

$$\big(\mu-4\big)\big(\mu-1\big)^2 = F\cdot M_1\cdot M_2 \geqslant h^0\big(\mathcal{O}_{\Upsilon}\big) = h^0\big(\mathcal{O}_{\Gamma}\big)-h^0\big(\mathcal{O}_{\Delta}\big) = \big(\mu-1\big)^3-\big|\Delta\big|,$$

which implies that $|\Delta|\geqslant 3(\mu-1)^2$. But $|\Delta|\leqslant|\Sigma|<\mu^2$, which is impossible, because $\mu\geqslant 3$. $\square$

We see that $\Delta\subsetneq\Sigma$. Put $\Gamma=\Sigma\setminus\Delta$ and $d=2\mu-3-\sum_{i=k}^{l} ic_i$.

Lemma 2.10. *The set Γ imposes dependent linear conditions on forms of degree d.*

Proof. Suppose that the points of the set Γ impose independent linear conditions on homogeneous polynomials of degree d. Let us show that this assumption leads to a contradiction.

The construction of Δ implies the existence of a homogeneous form H of degree $\sum_{i=k}^{l} ic_i$ that vanishes at all points of the set Δ and does not vanish at any point of Γ.

Suppose that $P\in\Delta$. Then there is a homogenous form F of degree $2\mu-3$ that vanishes at every point of the set $\Delta\setminus P$ and does not vanish at P by Lemma 2.9. Put

$$\Gamma = \Big\{Q_1,\ldots,Q_{\gamma}\Big\},$$

where $Q_1, \ldots, Q_\gamma$ are distinct points in Γ. Then there is a homogeneous form G_i of degree d that vanishes at every point in $\Gamma \setminus Q_i$ and does not vanish at the point Q_i. Then

$$F\big(Q_i\big) + \mu_i H G_i\big(Q_i\big) = 0$$

for some $\mu_i \in \mathbb{C}$, because $G_i(Q_i) \neq 0$. Then the homogenous form

$$F + \sum_{i=1}^{\gamma} \mu_i H G_i$$

vanishes on the set $\Sigma \setminus P$ and does not vanish at the point P, which is a contradiction.

We see that $P \in \Gamma$. Then there is a homogeneous form G of degree d that vanishes at every point in $\Gamma \setminus P$ and does not vanish at P. Then HG vanishes at every point of the set $\Sigma \setminus P$ and does not vanish at the point P, which is a contradiction. □

Put $\Gamma' = \psi(\Gamma)$. Let us check that Γ' and d satisfy the hypotheses of Theorem 2.2.

Lemma 2.11. *The inequality $d \geqslant 3$ holds.*

Proof. Suppose that $d \leqslant 2$. It follows from Corollary 2.8 that

$$2 \geqslant d = 2\mu - 3 - \sum_{i=k}^{l} ic_i \geqslant \mu - 2 \geqslant 1,$$

because $\mu \geqslant 3$ by Remark 2.1. Thus, we see that either $\mu = 3$ or $\mu = 4$.

Suppose that $\mu = 3$. Then it follows from the inequality (2.7) that

$$|\Gamma| \leqslant \big|\Sigma \setminus \Lambda\big| \leqslant \mu\left(\mu - \sum_{i=k}^{l} c_i i\right) - 2 \leqslant 3\big(3-k\big) - 2 \leqslant 1,$$

because $k \geqslant 2$ by Lemma 2.4. But $d \geqslant 1$. So, the set Γ imposes independent linear conditions on forms of degree $d \geqslant 1$, which is impossible by Lemma 2.10.

Thus, we see that $\mu = 4$. Then $k = 3$ by Lemma 2.6, which implies that

$$\big|\Gamma\big| \leqslant \big|\Sigma \setminus \Lambda\big| \leqslant 14 - 4\sum_{i=k}^{l} c_i i \leqslant 2,$$

which is impossible by Lemma 2.10, because $d \geqslant 1$. □

It follows from the inequality (2.7) that

$$|\Gamma'| = |\Gamma| \leqslant |\Sigma \setminus \Lambda| \leqslant \mu\left(\mu - \sum_{i=k}^{l} c_i i\right) - 2.$$

Then

$$\begin{aligned}\left|\Gamma'\right| &\leqslant \mu\left(\mu - \sum_{i=k}^{l} c_i i\right) - 2\\ &\leqslant \max\left\{\left\lfloor\frac{d+3}{2}\right\rfloor\left(d+3-\left\lfloor\frac{d+3}{2}\right\rfloor\right) - 1, \left\lfloor\frac{d+3}{2}\right\rfloor^2\right\},\end{aligned}$$

because $d = 2\mu - 3 - \sum_{i=k}^{l} c_i i$ and $\mu \geqslant 3$ (see Remark 2.1).

Lemma 2.12. *At most d points of the set Γ are contained in a line.*

Proof. Suppose that at least $d+1$ points of the set Γ are contained in some line. Then

$$\mu \geqslant d + 1 = 2\mu - 2 - \sum_{i=k}^{l} c_i i,$$

because at most μ points of Γ are contained in a line. It follows from Corollary 2.8 that

$$\mu - 1 \geqslant \sum_{i=k}^{l} c_i i \geqslant \mu - 2.$$

Suppose that $\sum_{i=k}^{l} c_i i = \mu - 2$. Then $|\Gamma| \leqslant 2\mu - 2$. So, the set Γ imposes independent linear conditions on forms of degree $d = \mu - 1$ by [12, Theorem 2], which is impossible by Lemma 2.10.

We see that $\sum_{i=k}^{l} c_i i = \mu - 1$. Then $|\Gamma| \leqslant \mu - 2 = d$, which is impossible by Lemma 2.10. ☐

Therefore, at most d points of the set Γ' lie on a line by Lemmas 2.12 and 2.4.

Lemma 2.13. *For every $t \leqslant (d+3)/2$, at most*

$$t\big(d + 3 - t\big) - 2$$

points of the set Γ' lie on a curve of degree t in $\Pi \cong \mathbb{P}^2$.

Proof. At most μt points of the set Γ' lie on a curve of degree t. It is enough to show that

$$t\big(d + 3 - t\big) - 2 \geqslant \mu t$$

for every $t \leqslant (d+3)/2$ such that $t > 1$ and $t(d+3-t) - 2 < |\Gamma'|$. But

$$t(d+3-t) - 2 \geqslant t\mu \iff \mu - \sum_{i=k}^{l} c_i i > t,$$

because $t > 1$. Thus, we may assume that $t(d+3-t) - 2 < |\Gamma'|$ and

$$\mu - \sum_{i=k}^{l} c_i i \leqslant t \leqslant \frac{d+3}{2}.$$

Let $g(x) = x(d+3-x) - 2$. Then

$$g(t) \geqslant g\Big(\mu - \sum_{i=k}^{l} c_i i\Big),$$

because $g(x)$ is increasing for $x < (d+3)/2$. Therefore, we have

$$\mu\Big(\mu - \sum_{i=k}^{l} ic_i\Big) - 2 \geqslant \big|\Gamma'\big| > g(t) \geqslant g\Big(\mu - \sum_{i=k}^{l} c_i i\Big) = \mu\Big(\mu - \sum_{i=k}^{l} ic_i\Big) - 2,$$

which is a contradiction. $\square$

Thus, the set Γ' imposes independent linear conditions on forms of degree d by Theorem 2.2, which implies that Γ also imposes independent linear conditions on forms of degree d, which is impossible by Lemma 2.10. The assertion of Theorem 1.1 is proved.

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