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Exceptional del Pezzo Hypersurfaces

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Abstract We compute global log canonical thresholds of a large class of quasismooth well-formed del Pezzo weighted hypersurfaces in $\mathbb{P}(a_0, a_1, a_2, a_3)$. As a corollary we obtain the existence of orbifold Kähler-Einstein metrics on many of them, and classify exceptional and weakly exceptional quasismooth well-formed del Pezzo weighted hypersurfaces in $\mathbb{P}(a_0, a_1, a_2, a_3)$.

Keywords Global log canonical threshold · Alpha-invariant of Tian · Del Pezzo orbifold · Weighted hypersurface · Kähler–Einstein metric · Exceptional Fano variety · Weakly exceptional Fano variety · Exceptional singularity · Weakly exceptional singularity

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All varieties are assumed to be complex, projective, and normal unless otherwise stated.

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1 Introduction

The multiplicity of a nonzero polynomial $f \in \mathbb{C}[z_1, \ldots, z_n]$ at a point $P \in \mathbb{C}^n$ can be defined by derivatives. Indeed, the multiplicity of f at the point P is the nonnegative integer

$$\mathrm{mult}_P(f) = \min\left\{ m \;\middle|\; \frac{\partial^m f}{\partial^{m_1} z_1 \partial^{m_2} z_2 \cdots \partial^{m_n} z_n}(P) \neq 0 \right\}.$$

On the other hand, we have a similar invariant that is defined by integrations. This invariant, which is called the complex singularity exponent of f at the point P, is given by

$$c_P(f) = \sup\left\{ c \;\middle|\; |f|^{-c} \text{ is locally } L^2 \text{ near the point } P \in \mathbb{C}^n \right\}.$$

In algebraic geometry this invariant is usually called a log canonical threshold. Let X be a variety with at most log canonical singularities, let $Z \subseteq X$ be a closed subvariety, and let D be an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on the variety X. Then the number

$$\mathrm{lct}_Z\big(X, D\big) = \sup\left\{ \lambda \in \mathbb{Q} \;\middle|\; \text{the log pair } \big(X, \lambda D\big) \text{ is log canonical along } Z \right\}$$

is called a log canonical threshold of D along Z. For simplicity, we put $\mathrm{lct}(X, D) = \mathrm{lct}_X(X, D)$. It follows from [13] that

$$\mathrm{lct}_P\Big(\mathbb{C}^n, \big(f = 0\big)\Big) = c_P\big(f\big).$$

Now we suppose that X is a Fano variety with at most log terminal singularities.

Definition 1.1 The global log canonical threshold of the Fano variety X is the number

$$\mathrm{lct}\big(X\big) = \inf\left\{ \mathrm{lct}\big(X, D\big) \;\middle|\; D \text{ is an effective } \mathbb{Q}\text{-divisor on } X \text{ with } D \sim_{\mathbb{Q}} -K_X \right\}.$$

The number $\mathrm{lct}(X)$ is an algebraic counterpart of the α-invariant of Tian (see [5, 22]).

Example 1.2 ([5]) Suppose that $\mathbb{P}(a_0, a_1, \ldots, a_n)$ is a well-formed (see [11, Definition 5.11]) weighted projective space with $a_0 \leqslant a_1 \leqslant \cdots \leqslant a_n$. Then $\mathrm{lct}(\mathbb{P}(a_0, a_1, \ldots, a_n)) = \frac{a_0}{\sum_{i=0}^n a_i}$.

Example 1.3 Let X be a general quasismooth well-formed (see [11, Definitions 6.3 and 6.9]) hypersurface in $\mathbb{P}(1, a_1, a_2, a_3, a_4)$ of degree $\sum_{i=1}^4 a_i$ with at most terminal singularities, where $a_1 \leqslant \cdots \leqslant a_4$. Then $\mathrm{lct}(X) = 1$ if $-K_X^3 \leqslant 1$ by [3].

So far we have not seen any single variety whose global log canonical threshold is irrational. In general, it is unknown whether global log canonical thresholds are rational numbers or not (cf. [24, Question 1]). Even for del Pezzo surfaces with log terminal singularities the rationality of their global log canonical thresholds is unknown. However, we expect more than this as follows:

Conjecture 1.4 There is an effective $\mathbb{Q}$-divisor D on the variety X such that it is $\mathbb{Q}$-linearly equivalent to $-K_X$ and $\mathrm{lct}(X) = \mathrm{lct}(X, D)$.

The following definition is due to [21] (cf. [17, 19]).

Definition 1.5 The Fano variety X is exceptional (resp. weakly exceptional, strongly exceptional) if for every effective $\mathbb{Q}$-divisor D on the variety X such that $D \sim_{\mathbb{Q}} -K_X$ and the pair (X, D) is log terminal (resp. $\mathrm{lct}(X) \geqslant 1$, $\mathrm{lct}(X) > 1$).

It is easy to see the implications

$$\text{strongly exceptional} \quad \Longrightarrow \quad \text{exceptional} \quad \Longrightarrow \quad \text{weakly exceptional.}$$

However, if Conjecture 1.4 holds for X, then we see that X is exceptional if and only if X is strongly exceptional. Exceptional del Pezzo surfaces, which are called del Pezzo surfaces without tigers in [14], lie in finitely many families (see [19, 21]). We expect that strongly exceptional Fano varieties enjoy very interesting geometrical properties (cf. [20, Theorem 3.3]).

The main motivation for this paper is that the global log canonical threshold turns out to play important roles both in birational geometry and in complex geometry. We have two significant applications of the global log canonical threshold of a Fano variety X. The first one is for the case when $\mathrm{lct}(X) \geqslant 1$. This inequality has serious applications to rationality problems for Fano varieties in birational geometry. The other is for the case when $\mathrm{lct}(X) > \frac{\dim(X)}{1+\dim(X)}$. This has important applications to Kähler-Einstein metrics on Fano varieties in complex geometry.

For a simple application of the first inequality, we can mention the following.

Example 1.6 ([3]) Let X_i be a threefold satisfying hypotheses of Example 1.3 with $\mathrm{lct}(X_i) = 1$ for each $i = 1, \ldots, r$. Then the variety $X_1 \times \cdots \times X_r$ is non-rational and

$$\mathrm{Bir}\Big(X_1 \times \ldots \times X_r\Big) = \Big\langle \prod_{i=1}^{r} \mathrm{Bir}(X_i),\ \mathrm{Aut}\Big(X_1 \times \cdots \times X_r\Big)\Big\rangle.$$

The following result that gives strong connection between global log canonical thresholds and Kähler-Einstein metrics was proved in [8, 18, 22] (see [5, Appendix A]).

Theorem 1.7 *Suppose that X is a Fano variety with at most quotient singularities. Then it admits an orbifold Kähler-Einstein metric if*

$$\mathrm{lct}\big(X\big) > \frac{\dim(X)}{\dim(X) + 1}.$$

There are many known obstructions for the existence of orbifold Kähler-Einstein metrics on Fano varieties with quotient singularities (see [9, 25]).

Example 1.8 ([10]) Let X be a quasismooth hypersurface in $\mathbb{P}(a_0, \dots, a_n)$ of degree $d < \sum_{i=0}^{n} a_i$, where $a_0 \leqslant \dots \leqslant a_n$. Suppose that X is well-formed and has a Kähler-Einstein metric. Then $\sum_{i=0}^{n} a_i \leqslant d + na_0$.

The problem of existence of Kähler-Einstein metrics on smooth del Pezzo surfaces is completely solved by [23] as follows:

Theorem 1.9 *If X is a smooth del Pezzo surface, then the following conditions are equivalent*:

- *the automorphism group* $\mathrm{Aut}(X)$ *is reductive*;
- *the surface X admits a Kähler-Einstein metric*;
- *the surface X is not a blow up of* $\mathbb{P}^2$ *at one or two points.*

Let X_d be a quasismooth and well-formed hypersurface in $\mathbb{P}(a_0, a_1, a_2, a_3)$ of degree d, where $a_0 \leqslant a_1 \leqslant a_2 \leqslant a_3$. Then the hypersurface X_d is given by a quasihomogeneous polynomial equation $f(x, y, z, t) = 0$ of degree d. The quasihomogeneous equation

$$f\big(x, y, z, t\big) = 0 \subset \mathbb{C}^4 \cong \mathrm{Spec}\Big(\mathbb{C}\big[x, y, z, t\big]\Big),$$

defines an isolated quasihomogeneous singularity (V, O) with the Milnor number $\prod_{i=0}^{n}(\frac{d}{a_i} - 1)$, where O is the origin of $\mathbb{C}^4$. It is well-known (see [13]) that the following conditions are equivalent:

- the inequality $d \leqslant \sum_{i=0}^{3} a_i - 1$ holds;
- the surface X_d is a del Pezzo surface;
- the singularity (V, O) is rational;
- the singularity (V, O) is canonical.

Blowing up $\mathbb{C}^4$ at the origin O with weights (a_0, a_1, a_2, a_3), we get a purely log terminal blow up of the singularity (V, O) (see [19]). It follows from [19, Proposition 4.5.5] that the following conditions are equivalent:

- the surface X_d is exceptional (weakly exceptional, respectively);
- the singularity (V, O) is exceptional[1] (weakly exceptional, respectively).

From now on we suppose that $d \leqslant \sum_{i=0}^{3} a_i - 1$. Then X_d is a del Pezzo surface. Put $I = \sum_{i=0}^{3} a_i - d$. In the case $I = 1$ the set of possible values of (a_0, a_1, a_2, a_3, d) is found in [12]. The global log canonical thresholds of such del Pezzo surfaces have been considered either implicitly or explicitly in [1, 4, 8, 12]. For example, the papers [1, 8] and [12] give us lower bounds for global log canonical thresholds of singular del Pezzo surfaces with $I = 1$. Meanwhile, all possible values of the global log canonical thresholds of smooth del Pezzo surfaces are found in the paper [4]. However, for singular del Pezzo surfaces, the exact values of global log canonical thresholds have not been considered seriously.

A singular del Pezzo hypersurface $X_d \subset \mathbb{P}(a_0, a_1, a_2, a_3)$ must satisfy exclusively one of the following properties:

[1] For notions of exceptional and weakly exceptional singularities see [21] and [19].

(1) $2I \geqslant 3a_0$.
(2) $2I < 3a_0$ and $(a_0, a_1, a_2, a_3, d) = (I - k, I + k, a, a + k, 2a + k + I)$ for some non-negative integer $k < I$ and some positive integer $a \geqslant I + k$.
(3) $2I < 3a_0$ but $(a_0, a_1, a_2, a_3, d) \neq (I - k, I + k, a, a + k, 2a + k + I)$ for some non-negative integer $k < I$ and some positive integer $a \geqslant I + k$.

For the first two cases one can check that $\mathrm{lct}(X_d) \leqslant \frac{2}{3}$ (see [2, 6]). All the values of (a_0, a_1, a_2, a_3, d) such that the hypersurface X_d is singular and satisfies the last condition are listed in Sect. 6. These values are found in [2] and [6]. The completeness of this list is proved in [6] by using [26].

We already know the global log canonical thresholds of smooth del Pezzo surfaces (see [4]). For del Pezzo surfaces satisfying one of the first two conditions, their global log canonical thresholds are relatively too small to enjoy the condition of Theorem 1.7. However, the global log canonical thresholds of del Pezzo surfaces satisfying the last condition have not been investigated sufficiently. In the present paper we compute all of them and obtain the following result.

Theorem 1.10 *Let X_d be a singular quasismooth well-formed del Pezzo surface in the weighted projective space* $\mathrm{Proj}(\mathbb{C}[x, y, z, t])$ *with weights* $\mathrm{wt}(x) = a_0 \leqslant \mathrm{wt}(y) = a_1 \leqslant \mathrm{wt}(z) = a_2 \leqslant \mathrm{wt}(t) = a_3$ *such that* $2I < 3a_0$ *but* $(a_0, a_1, a_2, a_3, d) \neq (I - k, I + k, a, a + k, 2a + k + I)$ *for some non-negative integer $k < I$ and some positive integer $a \geqslant I + k$, where $I = \sum_{i=0}^{3} a_i - d$. Then if $a_0 \neq a_1$, then*

$$\mathrm{lct}(X_d) = \min\left\{\mathrm{lct}\Big(X_d, \frac{I}{a_0}C_x\Big),\ \mathrm{lct}\Big(X_d, \frac{I}{a_1}C_y\Big),\ \mathrm{lct}\Big(X_d, \frac{I}{a_2}C_z\Big)\right\},$$

where C_x (resp. C_y, C_z) is the divisor on X_d defined by $x = 0$ (resp. $y = 0$, $z = 0$). If $a_0 = a_1$, then

$$\mathrm{lct}(X_d) = \mathrm{lct}\Big(X_d, \frac{I}{a_0}C\Big),$$

where C is a reducible divisor in $|\mathcal{O}_{X_d}(a_0)|$.

In particular, we obtain the value of $\mathrm{lct}(X_d)$ for every del Pezzo surface X_d listed in Sect. 6. As a result, we obtain the following corollaries.

Corollary 1.11 *The following assertions are equivalent*:

- *the surface X_d is exceptional*;
- $\mathrm{lct}(X_d) > 1$;
- *the quintuple (a_0, a_1, a_2, a_3, d) lies in the set*

$$\{(2,3,5,9,18), (3,3,5,5,15), (3,5,7,11,25), (3,5,7,14,28),$$
$$(3,5,11,18,36), (5,14,17,21,56), (5,19,27,31,81), (5,19,27,50,100),$$
$$(7,11,27,37,81), (7,11,27,44,88), (9,15,17,20,60), (9,15,23,23,69),$$
$$(11,29,39,49,127), (11,49,69,128,256), (13,23,35,57,127),$$

$$(13,35,81,128,256), (3,4,5,10,20), (3,4,10,15,30), (5,13,19,22,57),$$
$$(5,13,19,35,70), (6,9,10,13,36), (7,8,19,25,57), (7,8,19,32,64),$$
$$(9,12,13,16,48), (9,12,19,19,57), (9,19,24,31,81), (10,19,35,43,105),$$
$$(11,21,28,47,105), (11,25,32,41,107), (11,25,34,43,111),$$
$$(11,43,61,113,226), (13,18,45,61,135), (13,20,29,47,107),$$
$$(13,20,31,49,111), (13,31,71,113,226), (14,17,29,41,99),$$
$$(5,7,11,13,33), (5,7,11,20,40), (11,21,29,37,95), (11,37,53,98,196),$$
$$(13,17,27,41,95), (13,27,61,98,196), (15,19,43,74,148),$$
$$(9,11,12,17,45), (10,13,25,31,75), (11,17,20,27,71), (11,17,24,31,79),$$
$$(11,31,45,83,166), (13,14,19,29,71), (13,14,23,33,79),$$
$$(13,23,51,83,166), (11,13,19,25,63), (11,25,37,68,136),$$
$$(13,19,41,68,136), (11,19,29,53,106), (13,15,31,53,106),$$
$$(11,13,21,38,76)\}.$$

Corollary 1.12 *The following assertions are equivalent*:

- *the surface* X_d *is weakly exceptional and not exceptional*;
- $\mathrm{lct}(X_d) = 1$;
- *one of the following holds*
  - *the quintuple* (a_0, a_1, a_2, a_3, d) *lies in the set*

$$\{(2, 2n+1, 2n+1, 4n+1, 8n+4),$$
$$(3, 3n, 3n+1, 3n+1, 9n+3), (3, 3n+1, 3n+2, 3n+2, 9n+6),$$
$$(3, 3n+1, 3n+2, 6n+1, 12n+5), (3, 3n+1, 6n+1, 9n, 18n+3),$$
$$(3, 3n+1, 6n+1, 9n+3, 18n+6), (4, 2n+1, 4n+2, 6n+1, 12n+6),$$
$$(4, 2n+3, 2n+3, 4n+4, 8n+12), (6, 6n+3, 6n+5, 6n+5, 18n+15),$$
$$(6, 6n+5, 12n+8, 18n+9, 36n+24),$$
$$(6, 6n+5, 12n+8, 18n+15, 36n+30),$$
$$(8, 4n+5, 4n+7, 4n+9, 12n+23),$$
$$(9, 3n+8, 3n+11, 6n+13, 12n+35), (1,3,5,8,16), (2,3,4,7,14),$$
$$(5,6,8,9,24), (5,6,8,15,30)\}.$$

  where n *is a positive integer*,
  - $(a_0, a_1, a_2, a_3, d) = (1,1,2,3,6)$ *and the pencil* $|-K_X|$ *does not have cuspidal curves*,
  - $(a_0, a_1, a_2, a_3, d) = (1,2,3,5,10)$ *and* $C_x = \{x = 0\}$ *has an ordinary double point*,

– $(a_0, a_1, a_2, a_3, d) = (1, 3, 5, 7, 15)$ *and the defining equation of* X *contains* yzt,
– $(a_0, a_1, a_2, a_3, d) = (2, 3, 4, 5, 12)$ *and the defining equation of* X *contains* yzt.

Corollary 1.13 *In the notation and assumptions of Theorem 1.10, the surface* X_d *has an orbifold Kähler-Einstein metric with the following possible exceptions*: $X_{45} \subset \mathbb{P}(7, 10, 15, 19)$, $X_{81} \subset \mathbb{P}(7, 18, 27, 37)$, $X_{64} \subset \mathbb{P}(7, 15, 19, 32)$, $X_{82} \subset \mathbb{P}(7, 19, 25, 41)$, $X_{117} \subset \mathbb{P}(7, 26, 39, 55)$, $X_{15} \subset \mathbb{P}(1, 3, 5, 7)$ *whose defining equation does not contain* yzt, *and* $X_{12} \subset \mathbb{P}(2, 3, 4, 5)$ *whose defining equation does not contain* yzt.

Corollary 1.11 illustrates the fact that exceptional del Pezzo surfaces lie in finitely many families (see [19, 21]). On the other hand, Corollary 1.11 shows that weakly-exceptional del Pezzo surfaces do not enjoy this property. Note also that Corollary 1.11 follows from [15].

The plan of the paper is as follows. In Sect. 2 we recall the necessary background on the surfaces with quotient singularities. In Sect. 3 we briefly explain the pattern that is used to compute the global log-canonical thresholds of the surfaces X_d appearing in Theorem 1.10. In Sect. 4 we provide details of these computations for a sample of infinite series of such surfaces, and in Sect. 5 we do the same for a sample of sporadic cases, referring the reader to [7] and [6] for detailed computations in the remaining cases. In Sect. 6 we present the exact values of global log-canonical thresholds for the surfaces X_d appearing in Theorem 1.10.

2 Preliminaries

Let X be a surface with at most quotient singularities, i.e., a two-dimensional orbifold, let D be an effective $\mathbb{Q}$-divisor on X, and let $P \in X$ be a point that is a singularity of type $\frac{1}{r}(a, b)$. Then there is an orbifold chart $\pi : \tilde{U} \to U$ for some neighborhood $P \in U \subset X$ such that $\tilde{U}$ is smooth, and π is a cyclic cover of degree r that is unramified over $U \setminus P$. Put $D_U = D|_U$ and $D_{\tilde{U}} = \pi^{-1}(D_U)$. Let $\tilde{P} \in \tilde{U}$ be a point such that $\pi(\tilde{P}) = P$. Note that P is smooth if $r = 1$.

Lemma 2.1 *The log pair* (U, D_U) *is log canonical at* P *if and only if* $(\tilde{U}, D_{\tilde{U}})$ *is log canonical at* $\tilde{P}$.

Proof See [13]. □

We put $\mathrm{mult}_P(D) = \mathrm{mult}_{\tilde{P}}(D_{\tilde{U}})$, and refer to this quantity as the multiplicity of D at P. Let B be another effective $\mathbb{Q}$-divisor on X. Put $B_U = B|_U$ and $B_{\tilde{U}} = \pi^{-1}(B_U)$. Put

$$\mathrm{mult}_P\Big(D \cdot B\Big) = \mathrm{mult}_{\tilde{P}}\Big(D_{\tilde{U}} \cdot B_{\tilde{U}}\Big)$$

in the case when no component of B is contained in $\mathrm{Supp}(D)$. For every point $Q \in X$, let $r_Q \in \mathbb{Z}_{\geqslant 1}$ such that Q is a singular point of type $\frac{1}{r_Q}(a_Q, b_Q)$.

Lemma 2.2 *Suppose that no component of B is contained in* $\mathrm{Supp}(D)$. *Then*

$$B \cdot D = \sum_{Q \in X} \frac{\mathrm{mult}_Q(D \cdot B)}{r_Q} \geqslant \sum_{Q \in X} \frac{\mathrm{mult}_Q(D)\mathrm{mult}_Q(B)}{r_Q} \geqslant 0.$$

Proof This is an orbifold version of the usual Bezout theorem. □

Suppose that (X, D) is not log canonical at P.

Lemma 2.3 *The inequality* $\mathrm{mult}_P(D) > 1$ *holds.*

Proof The inequality $\mathrm{mult}_P(D) > 1$ follows from Lemma 2.1. □

Let C be a reduced irreducible curve on the surface X. Suppose that $P \in C \setminus \mathrm{Sing}(C)$. Put

$$D = mC + \Omega,$$

where $m \in \mathbb{Q}$ such that $m \geqslant 0$, and Ω is an effective $\mathbb{Q}$-divisor such that $C \not\subseteq \mathrm{Supp}(\Omega)$.

Lemma 2.4 *Suppose that $m \leqslant 1$. Then* $\mathrm{mult}_P(C \cdot \Omega) > 1$.

Proof Applying Lemma 2.1 and [5, Lemma 2.20], we get $\mathrm{mult}_P(C \cdot \Omega) > 1$. □

Lemma 2.5 *Suppose that $m \leqslant 1$. Then $C \cdot \Omega > 1/r$ and $r(C \cdot D - mC^2) > 1$.*

Proof The inequality $C \cdot \Omega > 1/r$ follows from Lemmas 2.2 and 2.4. Then

$$\frac{1}{r} < \Omega \cdot C = C \cdot (D - mC),$$

which gives $r(C \cdot D - mC^2) > 1$. □

Suppose that $B \sim_{\mathbb{Q}} D$, and (X, B) is log canonical at P.

Lemma 2.6 *There is an effective $\mathbb{Q}$-divisor D' on X such that $D' \sim_{\mathbb{Q}} B$, at least one irreducible component of B is not contained in the support of D', and (X, D') is not log canonical at the point P.*

Proof See [5, Remark 2.22]. □

Suppose, in addition, that X is a quasismooth well-formed hypersurface in $\mathbb{P} = \mathbb{P}(a_0, a_1, a_2, a_3)$ of degree d, and suppose that $D \sim_{\mathbb{Q}} \mathcal{O}_{\mathbb{P}}(I)|_X$ for some $I \in \mathbb{Z}_{\geqslant 1}$.

Lemma 2.7 *Let k be a positive integer. Suppose that $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(k))$ contains*

- *at least two different monomials of the form $x^{\alpha} y^{\beta}$,*

- *at least two different monomials of the form* $x^\gamma z^\delta$,

suppose that X *is smooth at* P, *and suppose that* $P \notin C_x$. *Then*

$$\mathrm{mult}_P\big(D\big) \leqslant \frac{Ikd}{a_0a_1a_2a_3}$$

if either $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(k))$ *contains at least two different monomials of the form* $x^\mu t^\nu$ *or the point* P *is not contained in a curve contracted by the projection* $\psi : X \dashrightarrow \mathbb{P}(a_0, a_1, a_2)$. *Here*, α, β, γ, δ, μ, *and* ν *are non-negative integers.*

Proof The first case follows from [1, Lemma 3.3]. Arguing as in the proof of [1, Corollary 3.4], we can also obtain the second case. □

Note that most of results of this section remain valid in much more general situations.

3 The Scheme of the Proof

We reserve the following notation that will be used throughout the paper:

- $\mathbb{P}(a_0, a_1, a_2, a_3)$ denotes the well-formed weighted projective space $\mathrm{Proj}(\mathbb{C}[x, y, z, t])$ with weights $\mathrm{wt}(x) = a_0$, $\mathrm{wt}(y) = a_1$, $\mathrm{wt}(z) = a_2$, $\mathrm{wt}(t) = a_3$, where we always assume the inequalities $a_0 \leqslant a_1 \leqslant a_2 \leqslant a_3$. We may use simply $\mathbb{P}$ instead of $\mathbb{P}(a_0, a_1, a_2, a_3)$ when this does not lead to confusion.
- X denotes a quasismooth and well-formed hypersurface in $\mathbb{P}(a_0, a_1, a_2, a_3)$ (see Definitions 6.3 and 6.9 in [11], respectively).
- O_x is the point in $\mathbb{P}(a_0, a_1, a_2, a_3)$ defined by $y = z = t = 0$. The points O_y, O_z, and O_t are defined in a similar way.
- C_x is the curve on X cut by the equation $x = 0$. The curves C_y, C_z, and C_t are defined in a similar way.
- L_{xy} is the one-dimensional stratum on $\mathbb{P}(a_0, a_1, a_2, a_3)$ defined by $x = y = 0$ and the other one-dimensional strata are labeled similarly.
- Let D be a divisor on X and $P \in X$. Choose an orbifold chart $\pi : \tilde{U} \to U$ for some neighborhood $P \in U \subset X$. We put $\mathrm{mult}_P(D) = \mathrm{mult}_Q(\pi^* D)$, where Q is a point on $\tilde{U}$ with $\pi(Q) = P$, and refer to this quantity as the multiplicity of D at P.

We have 83 families[2] of del Pezzo hypersurfaces in Sect. 6. Our computations to evaluate the global log canonical thresholds of these families are too huge. Moreover, these computations are based on the same methods. In the present section we explain the methods to compute the global log canonical thresholds of the del Pezzo hypersurfaces in Sect. 6. In the following sections, we show how to apply the methods to several families of del Pezzo hypersurfaces. These methods work for all the families of the del Pezzo hypersurfaces in Sect. 6. For details the reader is referred to [7]

[2]By family we mean either a one-parameter series (which actually gives rise to an infinite number of deformation families) or a sporadic case. We hope that this would not lead to confusion.

where 82 families have been dealt with, and to [6], where one infinite series has been treated.

Let $X \subset \mathbb{P}(a_0, a_1, a_2, a_3)$ be a del Pezzo surface of degree d in one of the 83 families. Set $I = a_0 + a_1 + a_2 + a_3 - d$. There are two exceptional cases where $a_0 = a_1$. The method for these two cases is a bit different from the other cases. Both cases will be individually dealt with (Lemmas 4.1 and 5.1).

If $a_0 \neq a_1$, then we will take steps as follows:

Step 1. Using Lemma 2.1, we compute the log canonical thresholds $\mathrm{lct}(X, \frac{I}{a_0}C_x)$, $\mathrm{lct}(X, \frac{I}{a_1}C_y)$, $\mathrm{lct}(X, \frac{I}{a_2}C_z)$, and $\mathrm{lct}(X, \frac{I}{a_3}C_t)$. Set

$$\lambda = \min\left\{\mathrm{lct}\left(X, \frac{I}{a_0}C_x\right), \mathrm{lct}\left(X, \frac{I}{a_1}C_y\right), \mathrm{lct}\left(X, \frac{I}{a_2}C_z\right), \mathrm{lct}\left(X, \frac{I}{a_3}C_t\right)\right\}.$$

Then the global log canonical threshold $\mathrm{lct}(X)$ is at most λ. In fact, the result of this article shows that λ can be attained by the minimum of the first three log canonical thresholds.

Step 2. We claim that the global log canonical threshold $\mathrm{lct}(X)$ is equal to λ. To prove this assertion, we suppose $\mathrm{lct}(X) < \lambda$. Then there is an effective $\mathbb{Q}$-divisor D equivalent to the anticanonical divisor $-K_X$ of X such that the log pair $(X, \lambda D)$ is not log canonical at some point $P \in X$. In particular, we obtain $\mathrm{mult}_P(\lambda D) > 1$ by Lemma 2.3.

Step 3. We show that the point P cannot be a singular point of X using the following methods.

Method 3.1 (Multiplicity) We may assume that a suitable irreducible component C of C_x, C_y, C_z, and C_t is not contained in the support of the divisor D. We derive a possible contradiction from the inequality

$$C \cdot D \geqslant \mathrm{mult}_P(C) \cdot \frac{\mathrm{mult}_P(D)}{r} > \frac{\mathrm{mult}_P(C)}{r\lambda},$$

where r is the index of the quotient singular point P. The last inequality follows from the assumption that $(X, \lambda D)$ is not log canonical at P. This method can be applied to exclude a smooth point.

Method 3.2 (Inversion of Adjunction) We consider a suitable irreducible curve C smooth at P. We then write $D = \mu C + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor whose support does not contain C. We check that $\lambda\mu \leqslant 1$. If so, then the log pair $(X, C + \lambda\Omega)$ is not log canonical at the point P either. By Lemma 2.5 we have

$$\lambda(D - \mu C) \cdot C = \lambda C \cdot \Omega > \frac{1}{r}.$$

We try to derive a contradiction from this inequality. The curve C is taken usually from an irreducible component of C_x, C_y, C_z, or C_t. This method can be applied to exclude a smooth point.

Method 3.3 (Weighted Blow Up) Sometimes we cannot exclude a singular point P only with the previous two methods. In such a case, we take a suitable weighted blow up $\pi : Y \to X$ at the point P. We can write

$$K_Y + D^Y \sim_{\mathbb{Q}} \pi^*(K_X + \lambda D),$$

where D^Y is the log pull-back of λD by π. Using Method 3.1 we obtain that D^Y is effective. Then we apply the previous two methods to the pair (Y, D^Y), or repeat this method until we get a contradictory inequality.

Step 4. We show that the point P cannot be a smooth point of X. To do so, we first apply Lemma 2.7. However, this method does not always work. If the method fails, then we try to find a suitable pencil $\mathcal{L}$ on X. The pencil has a member F which passes through the point P. We show that the pair $(X, \lambda F)$ is log canonical at the point P. Then, we may assume that the support of D does not contain at least one irreducible component of F. If the divisor D itself is irreducible, then we use Method 3.1 to exclude the point P. If F is reducible, then we use Method 3.2.

4 Infinite Series

Lemma 4.1 *Let X be a quasismooth hypersurface of degree* 12 *in* $\mathbb{P}(3, 3, 4, 4)$. *Then* $\mathrm{lct}(X) = 1$.

Proof The surface X can be defined by the quasihomogeneous equation

$$\prod_{i=1}^{4}(\alpha_i x + \beta_i y) = \prod_{j=1}^{3}(\gamma_j z + \delta_j t),$$

where $[\alpha_i : \beta_i]$ define four distinct points and $[\gamma_j : \delta_j]$ define three distinct points in $\mathbb{P}^1$.

Let P_i be the point in X given by $z = t = \alpha_i x + \beta_i y = 0$. These are singular points of X of type $\frac{1}{3}(1, 1)$. Let Q_j be the point in X that is given by $x = y = \gamma_j z + \delta_j t = 0$. Then each of them is a singular point of X of type $\frac{1}{4}(1, 1)$.

Let L_{ij} be the curve in X defined by $\alpha_i x + \beta_i y = \gamma_j z + \delta_j t = 0$, where $i = 1, \ldots, 4$ and $j = 1, \ldots, 3$.

The divisor C_i cut by the equation $\alpha_i x + \beta_i y = 0$ consists of three smooth curves L_{i1}, L_{i2}, L_{i3}. These divisors C_i, $i = 1, 2, 3, 4$, are the only reducible members in the linear system $|\mathcal{O}_X(3)|$. Meanwhile, the divisor B_j cut by $\gamma_j z + \delta_j t = 0$ consists of four smooth curves L_{1j}, L_{2j}, L_{3j}, L_{4j}. Note that $L_{i1} \cap L_{i2} \cap L_{i3} = \{P_i\}$ and $L_{1j} \cap L_{2j} \cap L_{3j} \cap L_{4j} = \{Q_j\}$. We have $L_{ij} \cdot L_{ik} = \frac{1}{3}$ and $L_{ji} \cdot L_{ki} = \frac{1}{4}$ if $k \neq j$. But $L_{ij}^2 = -\frac{5}{12}$.

Since $\mathrm{lct}(X, \frac{2}{3}C_i) = \mathrm{lct}(X, \frac{2}{4}B_j) = 1$, we have $\mathrm{lct}(X) \leqslant 1$.

Suppose that $\mathrm{lct}(X) < 1$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the pair (X, D) is not log canonical at some point P. For every $i = 1, \ldots, 4$, we may assume that the support of the divisor D does not contain at least one curve among L_{i1}, L_{i2}, L_{i3}. Suppose $L_{ik} \not\subset \mathrm{Supp}\,(D)$. Then the inequality

$$\mathrm{mult}_{P_i}(D) \leqslant 3D \cdot L_{ik} = \frac{1}{2}$$

implies that none of the points P_i can be the point P. For every $j = 1, 2, 3$, we may also assume that the support of the divisor D does not contain at least one curve among $L_{1j}, L_{2j}, L_{3j}, L_{4j}$. Suppose $L_{lj} \not\subset \mathrm{Supp}\,(D)$. Then the inequality

$$\mathrm{mult}_{Q_j}(D) \leqslant 4D \cdot L_{ik} = \frac{2}{3}$$

implies that none of the points Q_j can be the point P. Therefore, the point must be a smooth point of X.

Write $D = \mu L_{ij} + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor whose support does not contain L_{ij}. If $\mu > 0$, then we have $\mu L_{ij} \cdot L_{ik} \leqslant D \cdot L_{ik}$, and hence $\mu \leqslant \frac{1}{2}$. Since

$$\Omega \cdot L_{ij} = \frac{2 + 5\mu}{12} < 1,$$

Lemma 2.4 implies the point P cannot be on the curve L_{ij}. Consequently, $P \notin \bigcup_{i=1}^{4} \bigcup_{j=1}^{3} L_{ij}$.

There is a unique curve $C \subset X$ cut out by $\lambda x + \mu y = 0$, where $[\lambda : \mu] \in \mathbb{P}^1$, passing through the point P. Then the curve C is irreducible and quasismooth. Thus, we may assume that C is not contained in the support of D. Then

$$1 < \mathrm{mult}_P(D) \leqslant D \cdot C = \frac{1}{2}.$$

This is a contradiction. □

Lemma 4.2 *Let X be a quasismooth hypersurface of degree $9n + 3$ in $\mathbb{P}(3, 3n, 3n + 1, 3n + 1)$ for $n \geqslant 2$. Then $\mathrm{lct}(X) = 1$.*

Proof We may assume that the surface X is defined by the equation

$$xy(y - ax^n)(y - bx^n) + zt(z - ct) = 0,$$

where a, b, c are non-zero constants and $b \neq c$. The point O_y is a singular point of index $3n$ on X. The three points O_x, $P_a = [1 : a : 0 : 0]$, $P_b = [1 : b : 0 : 0]$ are singular points of index 3 on X. Also, X has three singular points O_z, O_t, $P_c = [0 : 0 : c : 1]$ of index $3n + 1$ on L_{xy}.

The curve C_x consists of three irreducible components L_{xz}, L_{xt}, and $L_c = \{x = z - ct = 0\}$. These three components intersect each other at O_y. It is easy to check that $\mathrm{lct}(X, \frac{2}{3}C_x) = 1$. Thus, $\mathrm{lct}(X) \leqslant 1$.

Suppose that $\mathrm{lct}(X) < 1$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair (X, D) is not log canonical at some point $P \in X$.

By Lemma 2.6 we may assume that at least one of the components of C_x is not contained in Supp (D). Then, the inequality

$$3nL_{xz} \cdot D = 3nL_{xt} \cdot D = 3nL_c \cdot D = \frac{2}{3n+1} < 1$$

implies that the point P cannot be the point O_y.

Put $D = \mu L_{xz} + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor whose support does not contain the curve L_{xz}. We claim that $\mu \leqslant \frac{2}{3n+1}$. Indeed, if the inequality fails, one of the curves L_{xt} and L_c is not contained in Supp (D). Then either

$$\frac{\mu}{3n} = \mu L_{xz} \cdot L_{xt} \leqslant D \cdot L_{xt} = \frac{2}{3n(3n+1)}, \quad \text{or}$$
$$\frac{\mu}{3n} = \mu L_{xz} \cdot L_c \leqslant D \cdot L_c = \frac{2}{3n(3n+1)}$$

holds. This is a contradiction. Note that

$$L_{xz}^2 = -\frac{6n-1}{3n(3n+1)}.$$

The inequality

$$\Omega \cdot L_{xz} = \frac{2+(6n-1)\mu}{3n(3n+1)} < \frac{1}{3n+1}$$

holds for all $n \geqslant 2$. Therefore, Lemma 2.5 implies the point P cannot belong to L_{xz}. By the same way, we can show that P cannot be located in either L_{xt} or L_c.

Let C be the curve on X cut out by the equation $z - \alpha t = 0$, where α is non-zero constant different from c. Then the curve C is quasismooth and hence $\mathrm{lct}(X, \frac{2}{3n+1}C) \geqslant 1$. Therefore, we may assume that the support of D does not contain the curve C. Then

$$\mathrm{mult}_{O_x}(D), \mathrm{mult}_{P_a}(D), \mathrm{mult}_{P_b}(D) \leqslant 3D \cdot C = \frac{2}{n} \leqslant 1$$

for $n \geqslant 2$. Therefore, P cannot be a singular point of X. Hence P is a smooth point of $X \setminus C_x$. However, applying Lemma 2.7, we get an absurd inequality

$$1 < \mathrm{mult}_P(D) \leqslant \frac{2(9n+3)^2}{3 \cdot 3n(3n+1)(3n+1)} \leqslant 1$$

for $n \geqslant 2$ since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(9n+3))$ contains x^{3n+1}, xy^3, and z^3. The obtained contradiction completes the proof. □

Lemma 4.3 *Let X be a quasismooth hypersurface of degree $36n+24$ in $\mathbb{P}(6, 6n+5, 12n+8, 18n+9)$ for $n \geqslant 1$. Then $\mathrm{lct}(X) = 1$.*

Proof We may assume that the surface X is defined by the equation

$$z^3 + y^3t + xt^2 - x^{6n+4} + ax^{2n+1}y^2z = 0,$$

where a is a constant. The only singularities of X are a singular point O_y of index $6n+5$, a singular point O_t of index $18n+9$, a singular point $Q=[1:0:0:1]$ of index 3, and a singular point $Q'=[1:0:1:0]$ of index 2.

The curve C_x is reduced and irreducible with $\mathrm{mult}_{O_t}(C_x)=3$. Clearly, $\mathrm{lct}(X, \frac{2}{3}C_x)=1$, and hence $\mathrm{lct}(X)\leqslant 1$. The curve C_y is quasismooth, and hence the log pair $(X, \frac{4}{6n+5}C_y)$ is log canonical.

Suppose that $\mathrm{lct}(X)<1$. Then there is an effective $\mathbb{Q}$-divisor $D\sim_{\mathbb{Q}} -K_X$ such that the log pair (X, D) is not log canonical at some point $P\in X$.

Since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(36n+30))$ contains x^{6n+5}, y^6, and z^3x, Lemma 2.7 implies

$$\mathrm{mult}_P(D)\leqslant \frac{4(36n+24)(36n+30)}{6(6n+5)(12n+8)(18n+9)}<1.$$

Therefore, the point P cannot be a smooth point in the outside of C_x.

By Lemma 2.6 we may assume that neither C_x nor C_y is contained in $\mathrm{Supp}\,(D)$. Then the inequality

$$3D\cdot C_y=\frac{2}{6n+3}\leqslant 1$$

implies that the point P is neither Q nor Q'. One the other hand, the inequality

$$(6n+5)D\cdot C_x=\frac{4}{6n+3}<1$$

shows that the point P can be neither a smooth point on C_x nor the point O_y. Therefore, it must be O_t. However, this is a contradiction since

$$\mathrm{mult}_{O_t}(D)=\frac{\mathrm{mult}_{O_t}(D)\mathrm{mult}_{O_t}(C_x)}{3}\leqslant\frac{18n+9}{3}D\cdot C_x=\frac{4}{6n+5}<1.$$

The obtained contradiction completes the proof. □

Lemma 4.4 *Let X be a quasismooth hypersurface of degree $12n+35$ in $\mathbb{P}(9, 3n+8, 3n+11, 6n+13)$ for $n\geqslant 1$. Then $\mathrm{lct}(X)=1$.*

Proof The surface X can be defined by the equation

$$z^2t+y^3z+xt^2+x^{n+3}y=0.$$

It is singular only at the points O_x, O_y, O_z, and O_t.

The curve C_x (resp. C_y, C_z, C_t) consists of two irreducible and reduced curves L_{xz} (resp. L_{yt}, L_{xz}, L_{yt}) and $R_x=\{x=zt+y^3=0\}$ (resp. $R_y=\{y=z^2+xt=0\}$, $R_z=\{z=t^2+x^{n+2}y=0\}$, $R_t=\{t=y^2z+x^{n+3}=0\}$). These two curves intersect at the point O_t (resp. O_x, O_y, O_z).

It follows from [16] that

$$\mathrm{lct}\left(X, \frac{2}{3}C_x\right)=1<\mathrm{lct}\left(X, \frac{6}{3n+8}C_y\right), \mathrm{lct}\left(X, \frac{6}{3n+11}C_z\right), \mathrm{lct}\left(X, \frac{6}{6n+13}C_t\right).$$

We have the following intersection numbers.

$$-L_{xz}\cdot K_X=\frac{6}{(3n+8)(6n+13)},\qquad -L_{yt}\cdot K_X=\frac{2}{3(3n+11)},$$

$$-R_x\cdot K_X=\frac{18}{(3n+11)(6n+13)},\qquad -R_y\cdot K_X=\frac{4}{3(6n+13)},$$

$$-R_z\cdot K_X=\frac{4}{3(3n+8)},\qquad -R_t\cdot K_X=\frac{6(n+3)}{(3n+8)(3n+11)},$$

$$L_{xz}\cdot R_x=\frac{3}{6n+13},\qquad L_{yt}\cdot R_y=\frac{2}{9},\qquad L_{xz}\cdot R_z=\frac{2}{3n+8},$$

$$L_{yt}\cdot R_t=\frac{n+3}{3n+11},\qquad L_{xz}^2=-\frac{9n+15}{(3n+8)(6n+13)},$$

$$L_{yt}^2=-\frac{3n+14}{9(3n+11)},\qquad R_x^2=-\frac{9n+6}{(3n+11)(6n+13)},$$

$$R_y^2=-\frac{6n+10}{9(6n+13)},\qquad R_z^2=\frac{6n+4}{9(3n+8)},\qquad R_t^2=\frac{(n+3)(3n+5)}{(3n+8)(3n+11)}.$$

Now we suppose that $\mathrm{lct}(X)<1$. Then there is an effective $\mathbb{Q}$-divisor $D\sim_{\mathbb{Q}}-K_X$ such that the log pair (X,D) is not log canonical at some point $P\in X$.

By Lemma 2.6 we may assume that $\mathrm{Supp}\,(D)$ does not contain both the curve L_{yt} and the curve R_y. Since these two curves intersect at the point O_x, the inequalities

$$L_{yt}\cdot D=\frac{2}{3(3n+11)}<\frac{1}{9},$$

$$R_y\cdot D=\frac{4}{3(6n+13)}<\frac{1}{9}$$

show that the point P cannot be the point O_x.

By Lemma 2.6 we may assume that $\mathrm{Supp}\,(D)$ does not contain both the curve L_{xz} and the curve R_z. Therefore, one of the following inequalities must hold:

$$\mathrm{mult}_{O_y}D\leqslant(3n+8)L_{xz}\cdot D=\frac{6}{6n+13}<1,$$

$$\mathrm{mult}_{O_y}D\leqslant\frac{3n+8}{2}R_z\cdot D=\frac{2}{3}.$$

Therefore, the point P cannot be the point O_y.

Suppose that $P=O_z$. If $L_{yt}\not\subset\mathrm{Supp}\,(D)$, then we get an absurd inequality

$$\frac{6}{9(3n+11)}=L_{yt}\cdot D>\frac{1}{3n+11}.$$

Therefore Supp (D) must contain the curve L_{yt}. By Lemma 2.6 we may assume that $M_t \not\subset$ Supp (D). Put $D = \mu L_{yt} + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor whose support does not contain the curve L_{yt}. Then

$$\frac{6(n+3)}{(3n+8)(3n+11)} = D \cdot R_t \geqslant \mu L_{yt} \cdot R_t + \frac{(\mathrm{mult}_P(D) - \mu)\mathrm{mult}_P(R_t)}{3n+11}$$
$$> \frac{\mu(n+3)}{3n+11} + \frac{2(1-\mu)}{3n+11},$$

and hence $\mu < \frac{2}{(3n+8)(n+1)}$. On the other hand, Lemma 2.5 shows

$$\frac{1}{3n+11} < \Omega \cdot L_{yt} = D \cdot L_{yt} - \mu L_{yt}^2 = \frac{6+\mu(3n+14)}{9(3n+11)}.$$

It implies $\frac{3}{3n+14} < \mu$. Consequently, the point P cannot be the point O_z.

Suppose that $P = O_t$. Since $L_{xz} \cdot D < \frac{1}{6n+13}$, the curve L_{xz} must be contained in Supp (D). Then, we may assume that $R_x \not\subset$ Supp (D). Put $D = \mu L_{xz} + \Omega$, assume that $R_x \not\subset$ Supp (D). Put $D = \mu L_{xz} + \Omega$, does not contain the curve L_{xz}. Then

$$\frac{18}{(3n+11)(6n+13)} = D \cdot R_x \geqslant \mu L_{xz} \cdot R_x + \frac{\mathrm{mult}_P(D) - \mu}{6n+13} > \frac{1+2\mu}{6n+13},$$

and hence $\mu < \frac{7-3n}{6n+22}$. However, Lemma 2.5 implies

$$\frac{1}{6n+13} < \Omega \cdot L_{xz} = D \cdot L_{xz} - \mu L_{xz}^2 = \frac{6+(9n+15)\mu}{(3n+8)(6n+13)},$$

and hence $\frac{3n+2}{9n+15} < \mu$. This is a contradiction. Therefore, the point P cannot be the point O_t.

Write $D = aL_{xz} + bR_x + \Delta$, where Δ is an effective $\mathbb{Q}$-divisor whose support contains neither L_{xz} nor R_x. Since the log pair (X, D) is log canonical at the point O_t, we have $0 \leqslant a, b \leqslant 1$. Then by Lemma 2.5 the following two inequalities

$$(bR_x + \Delta) \cdot L_{xz} = (D - aL_{xz}) \cdot L_{xz} = \frac{6+a(9n+15)}{(3n+8)(6n+13)} < 1,$$
$$(aL_{xz} + \Delta) \cdot R_x = (D - bR_x) \cdot R_x = \frac{18+b(9n+6)}{(3n+11)(6n+13)} < 1$$

show that $P \notin C_x$. By the same way, we can show $P \notin C_y \cup C_z \cup C_t$.

Consider the pencil $\mathcal{L}$ defined by the equations $\lambda xt + \mu z^2 = 0$, $[\lambda : \mu] \in \mathbb{P}^1$. Note that the curve L_{xz} is the only base component of the pencil $\mathcal{L}$. There is a unique divisor C_α in $\mathcal{L}$ passing through the point P. This divisor must be defined by an equation $xt + \alpha z^2 = 0$, where α is a non-zero constant, since the point P is located in the outside of $C_x \cup C_z \cup C_t$. Note that the curve C_t does not contain any component of C_α. Therefore, to see all the irreducible components of C_α, it is enough to see the affine curve

$$\left.\begin{cases} x+\alpha z^2=0 \\ z^2+y^3z+x+x^{n+3}y=0 \end{cases}\right\} \subset \mathbb{C}^3 \cong \mathrm{Spec}\Big(\mathbb{C}\big[x,y,z\big]\Big).$$

This is isomorphic to the plane affine curve defined by the equation

$$z\{(1-\alpha)z+y^3+(-\alpha)^{n+3}yz^{2n+5}\}=0\subset\mathbb{C}^2\cong\mathrm{Spec}\Big(\mathbb{C}\big[y,z\big]\Big).$$

Thus, if $\alpha\neq 1$, then the divisor C_α consists of two reduced and irreducible curves L_{xz} and Z_α. If $\alpha=1$, then it consists of three reduced and irreducible curves L_{xz}, R_y, R. Moreover, Z_α and R are smooth at the point P.

Suppose that $\alpha\neq 1$. Then we have

$$D\cdot Z_\alpha=\frac{2(24n+61)}{3(3n+8)(6n+13)}.$$

Since Z_α is different from R_x,

$$Z_\alpha^2=C_\alpha\cdot Z_\alpha-L_{xz}\cdot Z_\alpha\geqslant C_\alpha\cdot Z_\alpha-(L_{xz}+R_x)\cdot Z_\alpha=\frac{6n+13}{6}D\cdot Z_\alpha>0.$$

Put $D=\epsilon Z_\alpha+\Xi$, where Ξ is an effective $\mathbb{Q}$-divisor such that $Z_\alpha\not\subset\mathrm{Supp}(\Xi)$. Since the pair (X,D) is log canonical at the point O_t and the curve Z_α passes through the point O_t, we have $\epsilon\leqslant 1$. But

$$(D-\epsilon Z_\alpha)\cdot Z_\alpha\leqslant D\cdot Z_\alpha=\frac{2(24n+61)}{3(3n+8)(6n+13)}<1$$

and hence Lemma 2.5 implies that the point P cannot belong to the curve Z_α.

Suppose that $\alpha=1$. We have

$$D\cdot R=\frac{6(2n+5)}{(3n+8)(6n+13)}.$$

Since R is different from R_x and L_{yt},

$$\begin{aligned}R^2&=C_\alpha\cdot R-L_{xz}\cdot R-R_y\cdot R\geqslant C_\alpha\cdot R-(L_{xz}+R_x)\cdot R-(L_{yt}+R_y)\cdot R\\&=\frac{3n+5}{6}D\cdot D>0.\end{aligned}$$

Put $D=\epsilon_1R+\Xi'$, where Ξ' is an effective $\mathbb{Q}$-divisor such that $R\not\subset\mathrm{Supp}(\Xi')$. Since the curve R passes through the point O_t at which the pair (X,D) is log canonical, $\epsilon_1\leqslant 1$. Since

$$(D-\epsilon_1R)\cdot R\leqslant D\cdot R=\frac{6(2n+5)}{(3n+8)(6n+13)}<1.$$

Lemma 2.5 implies that the point P cannot belong to R. $\square$

5 Sporadic Cases

Lemma 5.1 *Let X be a quasismooth hypersurface of degree* 15 *in* $\mathbb{P}(3,3,5,5)$. *Then* $\mathrm{lct}(X)=2$.

Proof The surface X has five singular points $O_1,\dots,O_5$ of type $\frac{1}{3}(1,1)$. They are cut out by the equations $z=t=0$. The surface also has three singular points Q_1, Q_2, Q_3 of type $\frac{1}{5}(1,1)$. These three points are cut out by the equations $x=y=0$.

Let C_i be the curve in the pencil $|-3K_X|$ passing through the point O_i, where $i=1,\dots,5$. The curve C_i consists of three reduced and irreducible smooth rational curves

$$C_i=L_1^i+L_2^i+L_3^i.$$

The curve L_j^i contains the point Q_j. Furthermore, $L_1^i\cap L_2^i\cap L_3^i=\{O_i\}$. We see that

$$-K_X\cdot L_j^i=\frac{1}{15},\quad \left(L_j^i\right)^2=-\frac{7}{15},\quad L_j^i\cdot L_k^i=\frac{1}{3}$$

where $j\neq k$.

Note that $\mathrm{lct}(X,C_i)=\frac{2}{3}$. Thus $\mathrm{lct}(X)\leqslant 2$.

Suppose that $\mathrm{lct}(X)<2$. Then there is an effective $\mathbb{Q}$-divisor $D\sim_{\mathbb{Q}}-K_X$ such that the log pair $(X,2D)$ is not log canonical at some point $P\in X$. Then, $\mathrm{mult}_P(D)>\frac{1}{2}$.

Suppose that $P\notin C_1\cup C_2\cup C_3\cup C_4\cup C_5$. Then P is a smooth point of X. There is a unique curve $C\in|-3K_X|$ passing through point P. Then C is different from the curves $C_1,\dots,C_5$ and hence C is irreducible. Furthermore, the log pair (X,C) is log canonical. Thus, it follows from Lemma 2.6 that we may assume that $C\not\subseteq\mathrm{Supp}(D)$. Then we obtain an absurd inequality

$$\frac{1}{5}=D\cdot C\geqslant \mathrm{mult}_P\big(D\big)>\frac{1}{2},$$

since the log pair $(X,2D)$ is not log canonical at the point P. Therefore, $P\in C_1\cup C_2\cup C_3\cup C_4\cup C_5$. However, we may assume that $P\in C_1$ without loss of generality. Furthermore, by Lemma 2.6, we may assume that $L_i^1\not\subseteq\mathrm{Supp}(D)$ for some $i=1,2,3$.

Since

$$\frac{1}{5}=3D\cdot L_i^1\geqslant \mathrm{mult}_{O_1}\big(D\big),$$

the point P cannot be the point O_1.

Without loss of generality, we may assume that $P\in L_1^1$.

Let Z be the curve in the pencil $|-5K_X|$ passing through the point Q_1. Then

$$Z=Z_1+Z_2+Z_3+Z_4+Z_5,$$

where Z_i is a reduced and irreducible smooth rational curve. The curve Z_i contains the point O_i. Moreover, $Z_1\cap Z_2\cap Z_3\cap Z_4\cap Z_5=\{Q_1\}$. It is easy to check that

$\mathrm{lct}(X, Z) = \frac{2}{5}$. By Lemma 2.6, we may assume that $Z_k \not\subset \mathrm{Supp}(D)$ for some $k = 1, \ldots, 5$. Then

$$\frac{1}{3} = 5D \cdot Z_k \geqslant \mathrm{mult}_{Q_1}\big(D\big),$$

and hence the point P cannot be the point Q_1.

Thus, the point P is a smooth point on L_1^1. Put

$$D = mL_1^1 + \Omega,$$

where Ω is an effective $\mathbb{Q}$-divisor such that $L_1^1 \not\subset \mathrm{Supp}(\Omega)$. If $m \neq 0$, then

$$\frac{1}{15} = D \cdot L_i^1 = \big(mL_1^1 + \Omega\big) \cdot L_i^1 \geqslant mL_1^1 \cdot L_i^1 = \frac{m}{3},$$

and hence $m \leqslant \frac{1}{5}$. Then it follows from Lemma 2.5 that

$$\frac{1+7m}{15} = \big(D - mL_1^1\big) \cdot L_1^1 = \Omega \cdot L_1^1 > \frac{1}{2}.$$

This implies that $m > \frac{13}{14}$. But $m \leqslant \frac{1}{5}$. The obtained contradiction completes the proof. $\square$

Lemma 5.2 *Let X be a quasismooth hypersurface of degree* 127 *in* $\mathbb{P}(11, 29, 39, 49)$. *Then* $\mathrm{lct}(X) = \frac{33}{4}$.

Proof We may assume that the hypersurface X is defined by the equation

$$z^2t + yt^2 + xy^4 + x^8z = 0.$$

The singularities of X consist of a singular point of type $\frac{1}{11}(7, 5)$ at O_x, a singular point of type $\frac{1}{29}(1, 2)$ at O_y, a singular point of type $\frac{1}{39}(11, 29)$ at O_z, and a singular point of type $\frac{1}{49}(11, 39)$ at O_t.

The curve C_x (resp. C_y, C_z, C_t) consists of two irreducible curves L_{xt} (resp. L_{yz}, L_{yz}, L_{xt}) and $R_x = \{x = z^2 + yt = 0\}$ (resp. $R_y = \{y = x^8 + zt = 0\}$, $R_z = \{z = t^2 + xy^3 = 0\}$, $R_t = \{t = y^4 + x^7z = 0\}$). We can see that

$$L_{xt} \cap R_x = \{O_y\}, \qquad L_{yz} \cap R_y = \{O_t\},$$

$$L_{yz} \cap R_z = \{O_x\}, \qquad L_{xt} \cap R_t = \{O_z\}.$$

It is easy to check that $\mathrm{lct}(X, \frac{1}{11}C_x) = \frac{33}{4}$. The log pairs $(X, \frac{33}{4\cdot 29}C_y)$, $(X, \frac{33}{4\cdot 39}C_z)$ and $(X, \frac{33}{4\cdot 49}C_t)$ are log canonical.

Suppose that $\mathrm{lct}(X) < \frac{33}{4}$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \frac{33}{4}D)$ is not log canonical at some point $P \in X$.

By Lemma 2.6, we may assume that the support of D does not contain L_{xt} or R_x. Then one of the following two inequalities must hold:

$$\frac{4}{33} > \frac{1}{39} = 29L_{xt} \cdot D \geqslant \mathrm{mult}_{O_y}(D);$$

$$\frac{4}{33} > \frac{2}{49} = 29R_x \cdot D \geqslant \mathrm{mult}_{O_y}(D).$$

Therefore, the point P cannot be the point O_y. For the same reason, one of two inequalities

$$\frac{4}{33} > \frac{1}{49} = 11L_{yz} \cdot D \geqslant \mathrm{mult}_{O_x}(D),$$

$$\frac{4}{33} > \frac{2}{29} = 11R_z \cdot D \geqslant \mathrm{mult}_{O_x}(D)$$

must hold, and hence the point P cannot be the point O_x. Since R_t is singular at the point O_z with multiplicity 4, we can apply the same method to C_t, i.e., one of the following inequalities must be satisfied:

$$\frac{4}{33} > \frac{1}{29} = 39L_{xt} \cdot D \geqslant \mathrm{mult}_{O_z}(D);$$
$$\frac{4}{33} > \frac{1}{11} = \frac{39}{4}R_t \cdot D \geqslant \frac{1}{4}\mathrm{mult}_{O_z}(D)\mathrm{mult}_{O_z}(R_t) = \mathrm{mult}_{O_z}(D).$$

Thus, the point P cannot be O_z.

Write $D = \mu R_x + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $R_x \not\subset \mathrm{Supp}\,(\Omega)$. If $\mu > 0$, then L_{xt} is not contained in the support of D. Thus,

$$\frac{2}{29}\mu = \mu R_x \cdot L_{xt} \leqslant D \cdot L_{xt} = \frac{1}{29 \cdot 39},$$

and hence $\mu \leqslant \frac{1}{78}$. We have

$$49\Omega \cdot R_x = 49(D \cdot R_x - \mu R_x^2) = \frac{2 + 76\mu}{29} < \frac{4}{33}.$$

Then Lemma 2.5 shows that the point P cannot belong to R_x. In particular, the point P cannot be O_t.

Put $D = \epsilon L_{xt} + \Delta$, where Δ is an effective $\mathbb{Q}$-divisor such that $L_{xt} \not\subset \mathrm{Supp}\,(\Delta)$. Since $(X, \frac{33}{4}D)$ is log canonical at the point O_y, $\epsilon \leqslant \frac{4}{33}$ and hence

$$\Delta \cdot L_{xt} = D \cdot L_{xt} - \epsilon L_{xt}^2 = \frac{1 + 67\epsilon}{29 \cdot 39} < \frac{4}{33}.$$

Then Lemma 2.5 implies that the point P cannot belong to L_{xt}.

Consequently, the point P must be a smooth point in the outside of C_x. Then an absurd inequality

$$\frac{4}{33} < \mathrm{mult}_P(D) \leqslant \frac{539 \cdot 127}{11 \cdot 29 \cdot 39 \cdot 49} < \frac{4}{33}$$

follows from Lemma 2.7 since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(539))$ contains $x^{20}y^{11}$, x^{49}, $x^{10}z^{11}$, and t^{11}. The obtained contradiction completes the proof. □

Lemma 5.3 *Let X be a quasismooth hypersurface of degree* 57 *in* $\mathbb{P}(5, 13, 19, 22)$. *Then* $\mathrm{lct}(X) = \frac{25}{12}$.

Proof The surface X can be defined by the quasihomogeneous equation

$$z^3 + yt^2 + xy^4 + x^7t + \epsilon x^5yz = 0,$$

where $\epsilon \in \mathbb{C}$. The surface X is singular only at the points O_x, O_y, and O_t.

The curves C_x and C_y are irreducible. Moreover, we have

$$\frac{25}{12} = \mathrm{lct}\left(X, \frac{2}{5}C_x\right) < \mathrm{lct}\left(X, \frac{2}{13}C_y\right) = \frac{65}{21}.$$

Suppose that $\mathrm{lct}(X) < \frac{25}{12}$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the pair $(X, \frac{25}{12}D)$ is not log canonical at some point P. By Lemma 2.6, we may assume that the support of the divisor D contains neither C_x nor C_y.

Since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(110))$ contains the monomials x^9y^5, x^{22}, and t^5, it follows from Lemma 2.7 that the point P is either a singular point of X or a smooth point on C_x. However, this is impossible since $22D \cdot C_x = \frac{6}{13} < \frac{12}{25}$ and $5D \cdot C_y = \frac{3}{11} < \frac{12}{25}$. □

Lemma 5.4 *Let X be a quasismooth hypersurface of degree* 48 *in* $\mathbb{P}(9, 12, 13, 16)$. *Then* $\mathrm{lct}(X) = \frac{63}{24}$.

Proof The surface X can be defined by the quasihomogeneous equation

$$t^3 - y^4 + xz^3 + x^4y = 0.$$

The surface X is singular at the points O_x, O_z, $Q_4 = [0 : 1 : 0 : 1]$ and $Q_3 = [1 : 1 : 0 : 0]$.

The curves C_x, C_y, C_z, and C_t are irreducible and reduced. We have

$$\begin{aligned}\frac{63}{24} &= \mathrm{lct}\left(X, \frac{2}{9}C_x\right) < \mathrm{lct}\left(X, \frac{2}{12}C_y\right) = 4 < \mathrm{lct}\left(X, \frac{2}{13}C_z\right) \\ &= \frac{13}{2} < \mathrm{lct}\left(X, \frac{2}{16}C_t\right) = \frac{16}{2}.\end{aligned}$$

Therefore, $\mathrm{lct}(X) \leqslant \frac{63}{24}$.

Suppose that $\mathrm{lct}(X) < \frac{63}{24}$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the pair $(X, \frac{63}{24}D)$ is not log canonical at some point P. By Lemma 2.6, we may assume that the support of the divisor D contains none of the curves C_x, C_y, C_z, and C_t.

Note that the curve C_x is singular at O_z with multiplicity 3 and the curve C_y is singular at O_x with multiplicity 3. Then the inequalities

$$\frac{13}{3}D \cdot C_x = \frac{1}{6} < \frac{24}{63}, \qquad \frac{9}{3}D \cdot C_y = \frac{2}{13} < \frac{24}{63},$$
$$3D \cdot C_z = \frac{1}{6} < \frac{24}{63}, \qquad D \cdot C_t = \frac{8}{9 \cdot 13} < \frac{24}{63}$$

show that the point P must be located in the outside of $C_x \cup C_y \cup C_z \cup C_t$.

Consider the pencil $\mathcal{L}$ on X defined by the equations $\lambda xt + \mu yz = 0$, $[\mu, \lambda] \in \mathbb{P}^1$. Then there is a unique curve Z in the pencil $\mathcal{L}$ passing through the point P. Then the curve Z is defined by an equation of the form $xt - \alpha yz = 0$, where α is a non-zero constant. We see that $C_x \not\subset \mathrm{Supp}(Z)$. But the open subset $Z \setminus C_x$ of the curve Z is a $\mathbb{Z}_9$-quotient of the affine curve

$$t - \alpha yz = t^3 + y^4 + z^3 + y = 0 \subset \mathbb{C}^3 \cong \mathrm{Spec}\Big(\mathbb{C}\big[y, z, t\big]\Big),$$

which is isomorphic to the plane affine curve given by the equation

$$\alpha^3 y^3 z^3 + y^4 + z^3 + y = 0 \subset \mathbb{C}^2 \cong \mathrm{Spec}\Big(\mathbb{C}\big[y, z\big]\Big).$$

Then, it is easy to see that the curve Z is irreducible and $\mathrm{mult}_P(Z) \leqslant 4$. Thus, we may assume that $\mathrm{Supp}(D)$ does not contain the curve Z by Lemma 2.6. However,

$$\frac{25}{18 \cdot 13} = D \cdot Z \geqslant \mathrm{mult}_P\big(D\big) > \frac{24}{63}.$$

Consequently, $\mathrm{lct}(X) = \frac{63}{24}$. □

Lemma 5.5 *Let X be a quasismooth hypersurface of degree* 79 *in* $\mathbb{P}(13, 14, 23, 33)$. *Then* $\mathrm{lct}(X) = \frac{65}{32}$.

Proof The surface X can be defined by the quasihomogeneous equation

$$z^2 t + y^4 z + xt^2 + x^5 y = 0.$$

The surface X is singular at O_x, O_y, O_z, and O_t. We have

$$\mathrm{lct}\left(X, \frac{4}{13}C_x\right) = \frac{65}{32} < \mathrm{lct}\left(X, \frac{4}{13}C_x\right) = \frac{21}{8} < \mathrm{lct}\left(X, \frac{5}{25}C_t\right)$$
$$= \frac{33}{10} < \mathrm{lct}\left(X, \frac{4}{23}C_z\right) = \frac{69}{20}.$$

In particular, $\mathrm{lct}(X) \leqslant \frac{65}{32}$.

Each of the divisors C_x, C_y, C_z, and C_t consists of two irreducible and reduced components. The divisor C_x (resp. C_y, C_z, C_t) consists of L_{xz} (resp. L_{yt}, L_{xz}, L_{yt}) and $R_x = \{x = y^4 + zt = 0\}$ (resp. $R_y = \{y = z^2 + xt = 0\}$, $R_z = \{z = x^4y + t^2 = 0\}$, $R_t = \{t = x^5 + y^3z = 0\}$). The curve L_{xz} intersects R_x (resp. R_z) only at the point O_t (resp. O_y). The curve L_{yt} intersects R_y (resp. R_t) only at the point O_x (resp. O_z).

We suppose that $\mathrm{lct}(X) < \frac{65}{32}$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \frac{65}{32}D)$ is not log canonical at some point $P \in X$.

The intersection numbers among the divisors D, L_{xz}, L_{yt}, R_x, R_y, R_z, R_t are as follows:

$$L_{xz}^2 = -\frac{43}{14 \cdot 33}, \qquad R_x^2 = -\frac{40}{23 \cdot 33}, \qquad L_{xz} \cdot R_x = \frac{4}{33},$$

$$D \cdot L_{xz} = \frac{4}{14 \cdot 33}, \qquad D \cdot R_x = \frac{16}{23 \cdot 33}, \qquad L_{yt}^2 = -\frac{32}{13 \cdot 23},$$

$$R_y^2 = -\frac{38}{13 \cdot 33}, \qquad L_{yt} \cdot R_y = \frac{2}{13}, \qquad D \cdot L_{yt} = \frac{4}{13 \cdot 23}, \qquad D \cdot R_y = \frac{8}{13 \cdot 33},$$

$$R_z^2 = \frac{20}{13 \cdot 14}, \qquad L_{xz} \cdot R_z = \frac{2}{14}, \qquad D \cdot R_z = \frac{8}{13 \cdot 14},$$

$$R_t^2 = \frac{95}{14 \cdot 13}, \qquad L_{yt} \cdot R_t = \frac{5}{23}, \qquad D \cdot R_t = \frac{20}{14 \cdot 23}.$$

By Lemma 2.6 we may assume that the support of D does not contain at least one component of each divisor C_x, C_y, C_z, C_t. Since the curve R_t is singular at the point O_z with multiplicity 3 and the curve R_z is singular at the point O_y, in each of the following pairs of inequalities, at least one of two must hold:

$$\mathrm{mult}_{O_x}(D) \leqslant 13 D \cdot L_{yt} = \frac{4}{23} < \frac{32}{65}, \qquad \mathrm{mult}_{O_x}(D) \leqslant 13 D \cdot R_y = \frac{8}{33} < \frac{32}{65};$$

$$\mathrm{mult}_{O_y}(D) \leqslant 14 D \cdot L_{xz} = \frac{4}{33} < \frac{32}{65}, \qquad \mathrm{mult}_{O_y}(D) \leqslant \frac{14}{2} D \cdot R_z = \frac{4}{13} < \frac{32}{65};$$

$$\mathrm{mult}_{O_z}(D) \leqslant 23 D \cdot L_{yt} = \frac{4}{13} < \frac{32}{65}, \qquad \mathrm{mult}_{O_z}(D) \leqslant \frac{23}{3} D \cdot R_t = \frac{10}{21} < \frac{32}{65}.$$

Therefore, the point P can be none of O_x, O_y, O_z.

Put $D = m_0 L_{xz} + m_1 L_{yt} + m_2 R_x + m_3 R_y + m_4 R_z + m_5 R_t + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor whose support contains none of L_{xz}, L_{yt}, R_x, R_y, R_z, R_t. Since the pair $(X, \frac{65}{32}D)$ is log canonical at the points O_x, O_y, O_z, we have $m_i \leqslant \frac{32}{65}$ for each i. Since

$$(D - m_0 L_{xz}) \cdot L_{xz} = \frac{4 + 43 m_0}{14 \cdot 33} \leqslant \frac{32}{65}, \qquad (D - m_1 L_{yt}) \cdot L_{yt} = \frac{4 + 32 m_1}{13 \cdot 23} \leqslant \frac{32}{65},$$

$$(D - m_2 R_x) \cdot R_x = \frac{16 + 40 m_2}{23 \cdot 33} \leqslant \frac{32}{65}, \qquad (D - m_3 R_y) \cdot R_y = \frac{8 + 38 m_3}{13 \cdot 33} \leqslant \frac{32}{65},$$

$$(D - m_4 R_z) \cdot R_z = \frac{8 - 20m_4}{13 \cdot 14} \leqslant \frac{32}{65}, \qquad (D - m_5 R_t) \cdot R_t = \frac{20 - 95m_5}{14 \cdot 23} \leqslant \frac{32}{65}$$

Lemma 2.5 implies that the point P cannot be a smooth point of X on $C_x \cup C_y \cup C_z \cup C_t$. Therefore, the point P is either a point in the outside of $C_x \cup C_y \cup C_z \cup C_t$ or the point O_t.

Suppose that $P \notin C_x \cup C_y \cup C_z \cup C_t$. Then we consider the pencil $\mathcal{L}$ on X defined by the equations $\lambda xt + \mu z^2 = 0$, $[\lambda : \mu] \in \mathbb{P}^1$. There is a unique curve Z_α in the pencil passing through the point P. This curve is cut out by $xt + \alpha z^2 = 0$, where α is a non-zero constant.

The curve Z_α is reduced. But it is always reducible. Indeed, one can check that $Z_\alpha = C_\alpha + L_{xz}$, where C_α is a reduced curve whose support contains no L_{xy}. Let us prove that C_α is irreducible if $\alpha \neq 1$.

Any component of the curve C_t is not contained in the curve Z_α. The open subset $Z_\alpha \setminus C_t$ of the curve Z_α is a $\mathbb{Z}_{33}$-quotient of the affine curve

$$x + \alpha z^2 = z^2 + y^4 z + x + x^5 y = 0 \subset \mathbb{C}^3 \cong \operatorname{Spec}\Big(\mathbb{C}\big[x, y, z\big]\Big),$$

which is isomorphic to a plane affine curve defined by the equation

$$z\left((\alpha - 1)z + y^4 - \alpha^5 y z^9\right) = 0 \subset \mathbb{C}^2 \cong \operatorname{Spec}\Big(\mathbb{C}\big[y, z\big]\Big).$$

Thus, if $\alpha \neq 1$, then the curve Z_α consists of two irreducible and reduced curves L_{xz} and C_α. If $\alpha = 1$, then the curve Z_α consists of three irreducible and reduced curves L_{xz}, R_y, and C_1. In both the cases, the curve C_α (including $\alpha = 1$) is smooth at the point P. By Lemma 2.6, we may assume that $\operatorname{Supp}(D)$ does not contain at least one irreducible component of the curve Z_α.

If $\alpha \neq 1$, then

$$D \cdot C_\alpha = \frac{8}{13 \cdot 14},$$

$$C_\alpha^2 = Z_\alpha \cdot C_\alpha - L_{xz} \cdot C_\alpha \geqslant Z_\alpha \cdot C_\alpha - (R_x + L_{xz}) \cdot C_\alpha = \frac{33}{4} D \cdot C_\alpha > 0.$$

If $\alpha = 1$, then

$$D \cdot C_1 = \frac{152}{13 \cdot 14 \cdot 33},$$

$$C_1^2 = Z_1 \cdot C_1 - (L_{xz} + R_y) \cdot C_1 \geqslant Z_1 \cdot C_1 - (R_x + L_{xz}) \cdot C_1 - (L_{yt} + R_y) \cdot C_1 = \frac{19}{4} D \cdot C_1 > 0.$$

We put $D = mC_\alpha + \Delta_\alpha$, where Δ_α is an effective $\mathbb{Q}$-divisor such that $C_\alpha \not\subset \operatorname{Supp}(\Delta_\alpha)$. Since $C\alpha$ intersects the curve C_t and the pair $(X, \frac{65}{32}D)$ is log canonical along the curve C_t, we obtain $m \leqslant \frac{32}{65}$. Then, the inequality

$$(D - mC_\alpha) \cdot C_\alpha \leqslant D \cdot C_\alpha < \frac{32}{65}$$

implies that the pair $(X, \frac{65}{32}D)$ is log canonical at the point P by Lemma 2.5. The obtained contradiction concludes that the point P must be the point O_t.

If the irreducible component L_{xz} is not contained in the support of D, then the inequality

$$\mathrm{mult}_{O_t}(D) \leqslant 33D \cdot L_{xz} = \frac{2}{7} < \frac{32}{65}$$

is a contradiction. Therefore, the irreducible component L_{xz} must be contained in the support of D, and hence the curve R_x is not contained in the support of D. Put $D = aL_{xz} + bR_y + \Delta$, where Δ is an effective $\mathbb{Q}$-divisor whose support contains neither L_{xz} nor R_y. Then

$$\frac{16}{23 \cdot 33} = D \cdot R_x \geqslant aL_{xz} \cdot R_x + \frac{\mathrm{mult}_{O_t}(D) - a}{33} > \frac{3a}{33} + \frac{32}{33 \cdot 65}$$

and hence $a < \frac{304}{3 \cdot 23 \cdot 65}$. If $b \neq 0$, then L_{yt} is not contained in the support of D. Therefore,

$$\frac{4}{13 \cdot 23} = D \cdot L_{yt} \geqslant bR_y \cdot L_{yt} = \frac{2b}{13},$$

and hence $b \leqslant \frac{2}{23}$.

Let $\pi : \bar{X} \to X$ be the weighted blow up at the point O_t with weights $(13, 19)$ and let F be the exceptional curve of the morphism π. Then F contains two singular points Q_{13} and Q_{19} of $\bar{X}$ such that Q_{13} is a singular point of type $\frac{1}{13}(1, 1)$, and Q_{19} is a singular point of type $\frac{1}{19}(3, 7)$. Then

$$K_{\bar{X}} \sim_{\mathbb{Q}} \pi^*(K_X) - \frac{1}{33}F, \qquad \bar{L}_{xz} \sim_{\mathbb{Q}} \pi^*(L_{xz}) - \frac{19}{33}F,$$
$$\bar{R}_y \sim_{\mathbb{Q}} \pi^*(R_y) - \frac{13}{33}F, \qquad \bar{\Delta} \sim_{\mathbb{Q}} \pi^*(\Delta) - \frac{c}{33}F,$$

where $\bar{L}_{xz}$, $\bar{R}_y$, and $\bar{\Delta}$ are the proper transforms of L_{xz}, R_y, and Δ by π, respectively, and c is a non-negative rational number. Note that $F \cap \bar{R}_y = \{Q_{19}\}$ and $F \cap \bar{L}_{xz} = \{Q_{13}\}$.

The log pull-back of the log pair $(X, \frac{65}{32}D)$ by π is the log pair

$$\left(\bar{X},\ \frac{65a}{32}\bar{L}_{xz} + \frac{65b}{32}\bar{R}_y + \frac{65}{32}\bar{\Delta} + \theta_1 F\right),$$

where

$$\theta_1 = \frac{32 + 65(19a + 13b + c)}{32 \cdot 33}.$$

This is not log canonical at some point $Q \in F$. We have

$$0 \leqslant \bar{\Delta} \cdot \bar{L}_{xz} = \frac{4 + 43a}{14 \cdot 33} - \frac{b}{33} - \frac{c}{13 \cdot 33}.$$

This inequality shows $13b+c \leqslant \frac{13}{14}(4+43a)$. Since $a \leqslant \frac{304}{3\cdot 23\cdot 65}$, we obtain

$$\theta_1 = \frac{32+1235a}{32\cdot 33} + \frac{65(13b+c)}{32\cdot 33} \leqslant \frac{32+1235a}{32\cdot 33} + \frac{13\cdot 65(4+43a)}{14\cdot 32\cdot 33} < 1.$$

Suppose that the point Q is neither Q_{13} nor Q_{19}. Then, the point Q is not in $\bar{L}_{xz} \cup \bar{R}_y$. Therefore, the pair $(\bar{X}, \frac{65}{32}\bar{\Delta} + F)$ is not log canonical at the point Q, and hence

$$1 < \frac{65}{32}\bar{\Delta}\cdot F = \frac{65c}{13\cdot 19\cdot 32}.$$

But $c \leqslant 13b+c \leqslant \frac{13}{14}(4+43a) < \frac{13\cdot 19\cdot 32}{65}$ since $a \leqslant \frac{304}{3\cdot 23\cdot 65}$. Therefore, the point Q is either Q_{13} or Q_{19}.

Suppose that the point Q is Q_{13}. Then the point Q is in $\bar{L}_{xz}$ but not in $\bar{R}_y$. Therefore, the pair $(\bar{X}, \bar{L}_{xz} + \frac{65}{32}\bar{\Delta} + \theta_1 F)$ is not log canonical at the point Q. However, this is impossible since

$$\begin{aligned} 13\left(\frac{65}{32}\bar{\Delta} + \theta_1 F\right)\cdot \bar{L}_{xz} &= \frac{13\cdot 65}{32}\left(\frac{4+43a}{14\cdot 33} - \frac{b}{33} - \frac{c}{13\cdot 33}\right) + \theta_1 \\ &= \frac{32+1235a}{32\cdot 33} + \frac{13\cdot 65(4+43a)}{14\cdot 32\cdot 33} < 1. \end{aligned}$$

Therefore, the point Q must be the point Q_{19}.

Let $\psi\colon \tilde{X} \to \bar{X}$ be the weighted blow up at the point Q_{19} with weights $(3,7)$ and let E be the exceptional curve of the morphism ψ. The exceptional curve E contains two singular points O_3 and O_7 of $\tilde{X}$. The point O_3 is of type $\frac{1}{3}(1,2)$ and the point O_7 is of type $\frac{1}{7}(4,5)$. Then

$$K_{\tilde{X}} \sim_{\mathbb{Q}} \psi^*(K_{\bar{X}}) - \frac{9}{19}E, \qquad \tilde{R}_y \sim_{\mathbb{Q}} \psi^*(\bar{R}_y) - \frac{3}{19}E,$$
$$\tilde{F} \sim_{\mathbb{Q}} \psi^*(F) - \frac{7}{19}E, \qquad \tilde{\Delta} \sim_{\mathbb{Q}} \psi^*(\bar{\Delta}) - \frac{d}{19}E,$$

where $\tilde{R}_y$, $\tilde{F}$, and $\tilde{\Delta}$ are the proper transforms of $\bar{R}_y$, F, and $\bar{\Delta}$ by ψ, respectively, and d is a non-negative rational number.

The log pull-back of the log pair $(X, \frac{65}{32}D)$ by $\pi\circ\psi$ is the log pair

$$\left(\tilde{X},\ \frac{65a}{32}\tilde{L}_{xz} + \frac{65b}{32}\tilde{R}_y + \frac{65}{32}\tilde{\Delta} + \theta_1\tilde{F} + \theta_2 E\right),$$

where $\tilde{L}_{xz}$ is the proper transform of $\bar{L}_{xz}$ by ψ and

$$\theta_2 = \frac{65(3b+d)}{19\cdot 32} + \frac{7}{19}\theta_1 + \frac{9}{19} = \frac{9728 + 65(133a+190b+7c+33d)}{19\cdot 32\cdot 33}.$$

This is not log canonical at some point $O \in E$.

We have

$$0 \leqslant \tilde{\Delta}\cdot\tilde{R}_y = \bar{\Delta}\cdot\bar{R}_y - \frac{d}{7\cdot 19} = \frac{8+38b}{13\cdot 33} - \frac{19a+c}{19\cdot 33} - \frac{d}{7\cdot 19},$$

and hence $133a+7c+33d \leqslant \frac{133}{13}(8+38b)$. Therefore, this inequality together with $b<\frac{2}{23}$ gives us

$$\begin{aligned}\theta_2 &= \frac{9728+65\cdot 190b}{19\cdot 32\cdot 33}+\frac{65(133a+7c+33d)}{19\cdot 32\cdot 33}\\ &\leqslant \frac{9728+65\cdot 190b}{19\cdot 32\cdot 33}+\frac{65\cdot 7(8+38b)}{13\cdot 32\cdot 33}<1.\end{aligned}$$

Suppose that the point O is in the outside of $\tilde{R}_y$ and $\tilde{F}$. Then the log pair $(E, \frac{65}{32}\tilde{\Delta}|_E)$ is not log canonical at the point O and hence

$$1<\frac{65}{32}\tilde{\Delta}\cdot E=\frac{65d}{3\cdot 7\cdot 32}.$$

However,

$$d\leqslant \frac{1}{33}(133a+7c+33d)\leqslant \frac{133}{13\cdot 33}(8+38b)<\frac{3\cdot 7\cdot 32}{65}$$

since $b\leqslant \frac{2}{23}$. This is a contradiction.

Suppose that the point O belongs to $\tilde{R}_y$. Then the log pair $(\tilde{X}, \frac{65b}{32}\tilde{R}_y+\frac{65}{32}\tilde{\Delta}+\theta_2 E)$ is not log canonical at the point O and hence

$$1<7\left(\frac{65}{32}\tilde{\Delta}+\theta_2 E\right)\cdot \tilde{R}_x=\frac{7\cdot 65}{32}\left(\frac{8+38b}{13\cdot 33}-\frac{19a+c}{19\cdot 33}-\frac{d}{7\cdot 19}\right)+\theta_2.$$

However,

$$\begin{aligned}\frac{7\cdot 65}{32}\left(\frac{8+38b}{13\cdot 33}-\frac{19a+c}{19\cdot 33}-\frac{d}{7\cdot 19}\right)+\theta_2 &= \frac{9728+65\cdot 190b}{19\cdot 32\cdot 33}+\frac{65\cdot 7(8+38b)}{13\cdot 32\cdot 33}\\ &<1.\end{aligned}$$

This is a contradiction. Therefore, the point O is the point O_3.

Suppose that the point O belongs to $\tilde{F}$. Then the log pair $(\tilde{X}, \frac{65}{32}\tilde{\Delta}+\theta_1\tilde{F}+\theta_2 E)$ is not log canonical at the point O and hence

$$1<3\left(\frac{65}{32}\tilde{\Delta}+\theta_2 E\right)\cdot \tilde{F}=\frac{3\cdot 65}{32}\left(\frac{c}{13\cdot 19}-\frac{d}{3\cdot 19}\right)+\theta_2.$$

However,

$$\begin{aligned}\frac{3\cdot 65}{32}\left(\frac{c}{13\cdot 19}-\frac{d}{3\cdot 19}\right)+\theta_2 &= \frac{3\cdot 65c}{13\cdot 19\cdot 32}+\frac{9728+65(133a+190b+7c)}{19\cdot 32\cdot 33}\\ &= \frac{512+455a}{32\cdot 33}+\frac{65\cdot 190(13b+c)}{13\cdot 19\cdot 32\cdot 33}\\ &\leqslant \frac{512+455a}{32\cdot 33}+\frac{65\cdot 190(4+43a)}{14\cdot 19\cdot 32\cdot 33}<1\end{aligned}$$

since $13b+c\leqslant \frac{13}{14}(4+43a)$ and $a\leqslant \frac{304}{3\cdot 23\cdot 65}$. This is a contradiction. $\square$

6 Tables

Log del Pezzo surfaces with $I = 1$

Weights	Degree	lct
$(2, 2n+1, 2n+1, 4n+1)$	$8n+4$	1
$(1, 2, 3, 5)$	10	1^{a} $\frac{7}{10}^{b}$
$(1, 3, 5, 7)$	15	1^{c} $\frac{8}{15}^{d}$
$(1, 3, 5, 8)$	16	1
$(2, 3, 5, 9)$	18	2^{e} $\frac{11}{6}^{f}$
$(3, 3, 5, 5)$	15	2
$(3, 5, 7, 11)$	25	$\frac{21}{10}$
$(3, 5, 7, 14)$	28	$\frac{9}{4}$
$(3, 5, 11, 18)$	36	$\frac{21}{10}$
$(5, 14, 17, 21)$	56	$\frac{25}{8}$
$(5, 19, 27, 31)$	81	$\frac{25}{6}$
$(5, 19, 27, 50)$	100	$\frac{25}{6}$
$(7, 11, 27, 37)$	81	$\frac{49}{12}$
$(7, 11, 27, 44)$	88	$\frac{35}{8}$
$(9, 15, 17, 20)$	60	$\frac{21}{4}$
$(9, 15, 23, 23)$	69	6
$(11, 29, 39, 49)$	127	$\frac{33}{4}$
$(11, 49, 69, 128)$	256	$\frac{55}{6}$
$(13, 23, 35, 57)$	127	$\frac{65}{8}$
$(13, 35, 81, 128)$	256	$\frac{91}{10}$

[a]if C_x has an ordinary double point
[b]if C_x has a non-ordinary double point
[c]if the defining equation of X contains yzt
[d]if the defining equation of X contains no yzt
[e]if C_y has a tacnodal point
[f]if C_y has no tacnodal points

Log del Pezzo surfaces with $I = 3$

Weights	Degree	lct
$(5, 7, 11, 13)$	33	$\frac{49}{36}$
$(5, 7, 11, 20)$	40	$\frac{25}{18}$
$(11, 21, 29, 37)$	95	$\frac{11}{4}$
$(11, 37, 53, 98)$	196	$\frac{55}{18}$
$(13, 17, 27, 41)$	95	$\frac{65}{24}$
$(13, 27, 61, 98)$	196	$\frac{91}{30}$
$(15, 19, 43, 74)$	148	$\frac{57}{14}$

Log del Pezzo surfaces with $I = 2$

Weights	Degree	lct
$(3, 3n, 3n+1, 3n+1)$	$9n+3$	1
$(3, 3n+1, 3n+2, 3n+2)$	$9n+6$	1
$(3, 3n+1, 3n+2, 6n+1)$	$12n+5$	1
$(3, 3n+1, 6n+1, 9n)$	$18n+3$	1
$(3, 3n+1, 6n+1, 9n+3)$	$18n+6$	1
$(4, 2n+1, 4n+2, 6n+1)$	$12n+6$	1
$(4, 2n+3, 2n+3, 4n+4)$	$8n+12$	1
$(2, 3, 4, 5)$	12	1^{a} $\frac{7}{12}^{b}$
$(2, 3, 4, 7)$	14	1
$(3, 4, 5, 10)$	20	$\frac{3}{2}$
$(3, 4, 10, 15)$	30	$\frac{3}{2}$
$(5, 13, 19, 22)$	57	$\frac{25}{12}$
$(5, 13, 19, 35)$	70	$\frac{25}{12}$
$(6, 9, 10, 13)$	36	$\frac{25}{12}$
$(7, 8, 19, 25)$	57	$\frac{49}{24}$
$(7, 8, 19, 32)$	64	$\frac{35}{16}$
$(9, 12, 13, 16)$	48	$\frac{63}{24}$
$(9, 12, 19, 19)$	57	3
$(9, 19, 24, 31)$	81	3
$(10, 19, 35, 43)$	105	$\frac{57}{14}$
$(11, 21, 28, 47)$	105	$\frac{77}{30}$
$(11, 25, 32, 41)$	107	$\frac{11}{3}$
$(11, 25, 34, 43)$	111	$\frac{33}{8}$
$(11, 43, 61, 113)$	226	$\frac{55}{12}$
$(13, 18, 45, 61)$	135	$\frac{91}{30}$
$(13, 20, 29, 47)$	107	$\frac{65}{18}$
$(13, 20, 31, 49)$	111	$\frac{65}{16}$
$(13, 31, 71, 113)$	226	$\frac{91}{20}$
$(14, 17, 29, 41)$	99	$\frac{51}{10}$

[a]if the defining equation of X contains yzt
[b]if the defining equation of X contains no yzt

Log del Pezzo surfaces with $I = 4$

Weights	Degree	lct
$(6, 6n+3, 6n+5, 6n+5)$	$18n+15$	1
$(6, 6n+5, 12n+8, 18n+9)$	$36n+24$	1
$(6, 6n+5, 12n+8, 18n+15)$	$36n+30$	1
$(5, 6, 8, 9)$	24	1
$(5, 6, 8, 15)$	30	1

Log del Pezzo surfaces with $I=4$

Weights	Degree	lct
$(9,11,12,17)$	45	$\frac{77}{60}$
$(10,13,25,31)$	75	$\frac{91}{60}$
$(11,17,20,27)$	71	$\frac{11}{6}$
$(11,17,24,31)$	79	$\frac{33}{16}$
$(11,31,45,83)$	166	$\frac{55}{24}$
$(13,14,19,29)$	71	$\frac{65}{36}$
$(13,14,23,33)$	79	$\frac{65}{32}$
$(13,23,51,83)$	166	$\frac{91}{40}$

Log del Pezzo surfaces with $I=5$

Weights	Degree	lct
$(11,13,19,25)$	63	$\frac{13}{8}$
$(11,25,37,68)$	136	$\frac{11}{6}$
$(13,19,41,68)$	136	$\frac{91}{50}$

Log del Pezzo surfaces with $I=6$

Weights	Degree	lct
$(8,4n+5,4n+7,4n+9)$	$12n+23$	1
$(9,3n+8,3n+11,6n+13)$	$12n+35$	1
$(7,10,15,19)$	45	$\frac{35}{54}$
$(11,19,29,53)$	106	$\frac{55}{36}$
$(13,15,31,53)$	106	$\frac{91}{60}$

Log del Pezzo surfaces with $I=7$

Weights	Degree	lct
$(11,13,21,38)$	76	$\frac{13}{10}$

Log del Pezzo surfaces with $I=8$

Weights	Degree	lct
$(7,11,13,23)$	46	$\frac{35}{48}$
$(7,18,27,37)$	81	$\frac{35}{72}$

Log del Pezzo surfaces with $I=9$

Weights	Degree	lct
$(7,15,19,32)$	64	$\frac{35}{54}$

Log del Pezzo surfaces with $I=10$

Weights	Degree	lct
$(7,19,25,41)$	82	$\frac{7}{12}$
$(7,26,39,55)$	117	$\frac{7}{18}$

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