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BIRATIONAL RIGIDITY IS NOT AN OPEN PROPERTY

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Abstract. We show that birational rigidity of Mori fibre spaces is not open in moduli.

We assume that all varieties are normal, projective and defined over the field $\mathbb{C}$.

1. Introduction

In this paper we give a negative answer to the question that is closely related to the nature of birationally rigid Mori fibre spaces: whether birational rigidity is open in moduli.

Definition 1.1. A Mori fibre space is a surjective morphism $\pi\colon X \to S$ such that

- the variety X has terminal and $\mathbb{Q}$-factorial singularities,
- the inequality $\dim(S) < \dim(X)$ holds and $\pi_*(\mathcal{O}_X) = \mathcal{O}_S$,
- the divisor $-K_X$ is relatively ample for π,
- the equality $\operatorname{rk}\operatorname{Pic}(X) = \operatorname{rk}\operatorname{Pic}(X) + 1$ holds.

Let $\pi\colon X \to S$ be a Mori fibre space such that $\dim(X) = 3$. Then

- either $\dim(S) = 0$ and X is a Fano 3-fold,
- or $\dim(S) = 1$ and $\pi\colon X \to S$ is a del Pezzo fibration,
- or $\dim(S) = 2$ and $\pi\colon X \to S$ is a conic bundle.

Definition 1.2. The Mori fibre space $\pi\colon X \to S$ is birationally rigid if, given any birational map $\xi\colon X \dashrightarrow X'$ to another Mori fibre space $\pi'\colon X' \to S'$, there exists a commutative diagram

$$\begin{array}{ccccc} X & \overset{\rho}{\dashrightarrow} & X & \overset{\xi}{\dashrightarrow} & X' \\ \big\downarrow{\scriptstyle \pi} & & & & \big\downarrow{\scriptstyle \pi'} \\ S & & \overset{\sigma}{\dashrightarrow} & & S' \end{array}$$

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for some birational maps ρ and σ such that the composition map $\xi \circ \rho$ induces an isomorphism of the generic fibers of the Mori fibre spaces π and π'.

We say that X is birationally rigid if $\dim(S) = 0$ and $\pi\colon X \to S$ is birationally rigid.

Example 1.3. Let X be a general complete intersection in $\mathbb{P}^5$ of a quadric and a cubic. Then

$$-K_X \equiv \mathcal{O}_{\mathbb{P}^5}(1)\Big|_X$$

and $\mathrm{Pic}(X) = \mathbb{Z}[-K_X]$. The threefold X is birationally rigid (see [4] and [7, Chapter 2]).

The following conjecture is Conjecture 1.4 in [2].

Conjecture 1.4. *For any scheme T, and a flat family of Mori fibre spaces parametrised by T*

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathcal{S} \\ & \searrow \quad \swarrow & \\ & T & \end{array}$$

the set of all $t \in T$ such that the corresponding fibre $\mathcal{X}_t \to \mathcal{S}_t$ is birationally rigid is open in T.

In this paper, we show that Conjecture 1.4 fails in general.

Definition 1.5. The Mori fibre space $\pi\colon X \to S$ is square birationally equivalent to a Mori fiber space $\pi'\colon X' \to S'$ if there is a birational map $\xi\colon X \dashrightarrow X'$ that fits a commutative diagram

$$\begin{array}{ccc} X & \overset{\xi}{\dashrightarrow} & X' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ S & \overset{\sigma}{\dashrightarrow} & S' \end{array}$$

for some birational map σ such that ξ induces an isomorphism of the generic fibers of π and π'.

The following definition is due to [3].

Definition 1.6. The *pliability* of a variety V is the set

$$\mathcal{P}(V) = \Big\{\text{Mori fibre space } \tau\colon Y \longrightarrow T \ \Big|\ Y \text{ is birational to } V\Big\}\Big/ \approx,$$

where $\approx :=$ square birational equivalence.

Let V_1 be a complete intersection of a quadric $Q_1 \subset \mathbb{P}^5$ and a cubic $T_1 \subset \mathbb{P}^5$ such that V_1 has singular point P, but Q_1 is non-singular at the point P. Then Q_1 can be given by the equation

$$y_5 h(y_0, y_1, y_2, y_3, y_4) = q_1(y_0, y_1, y_2, y_3, y_4)$$

in $\mathrm{Proj}\Big(\mathbb{C}[y_0, y_1, y_2, y_3, y_4, y_5]\Big) \cong \mathbb{P}^5$, where $h(y_0, y_1, y_2, y_3, y_4)$ and $q_1(y_0, y_1, y_2, y_3, y_4)$ are homogeneous polynomials of degree 1 and 2, respectively, and the point P is given by the equations $y_0 = y_1 = y_2 = y_3 = y_4 = 0$.

Similarly, the cubic hypersurface $T_1 \subset \mathbb{P}^5$ can be given by the equation

$$y_5 q_2\big(y_0, y_1, y_2, y_3, y_4\big) = t\big(y_0, y_1, y_2, y_3, y_4\big),$$

where $t(y_0, y_1, y_2, y_3, y_4)$ is a homogeneous polynomial of degree 3, and $q_2(y_0, y_1, y_2, y_3, y_4)$ is a homogeneous polynomial of degree 2.

Let V_2 be a complete intersection in $\mathbb{P}^5$ of a quadric Q_2 and a cubic T_2 such that Q_2 is given by

$$y_5 h\big(y_0, y_1, y_2, y_3, y_4\big) = q_2\big(y_0, y_1, y_2, y_3, y_4\big),$$

and the cubic hypersurface $T_2 \subset \mathbb{P}^5$ is given by

$$y_5 q_1(y_0, y_1, y_2, y_3, y_4) = t(y_0, y_1, y_2, y_3, y_4).$$

Remark 1.7. The threefold V_2 is singular at the point $P \in V_2$ as well.

Suppose that both V_1 and V_2 satisfy the following generality conditions:

(A) the quadric hypersurface $Q_i \subset \mathbb{P}^5$ is non-singular,
(B) the threefold $V_i = Q_i \cap T_i$ is smooth outside of the point $P \in V_i$,
(C) the point P is an ordinary double point of the threefold $V_i \subset \mathbb{P}^5$,
(D) the threefold V_i contains 12 lines that pass through the point $P \in V_i$.

Remark 1.8. The varieties V_1 and V_2 are $\mathbb{Q}$-factorial, and

$$\mathrm{rk}\,\mathrm{Pic}(V_1) = \mathrm{rk}\,\mathrm{Pic}(V_2) = 1 \text{ (see [8]).}$$

The threefolds V_1 and V_2 are birationally equivalent. Indeed, there is a commutative diagram

$$\begin{array}{ccccccc} & & \tilde{V}_1 & \overset{\gamma}{\dashrightarrow} & \tilde{V}_2 & & \\ & \swarrow^{\beta_1} & & \searrow^{\alpha_1} \quad {}^{\alpha_2}\swarrow & & \searrow^{\beta_2} & \\ V_1 & & \overset{\phi_1}{\dashrightarrow} & Y & \overset{\phi_2}{\dashleftarrow} & & V_2, \end{array}$$

where Y is a singular quartic hypersurface in $\mathbb{P}^4$ that is given by the equation

$$h\big(y_0, \ldots, y_4\big)t\big(y_0, \ldots, y_4\big) = q_1\big(y_0, \ldots, y_4\big)q_2\big(y_0, \ldots, y_4\big)$$

in $\mathrm{Proj}\Big(\mathbb{C}[y_0, \ldots, y_4]\Big) \cong \mathbb{P}^4$, the maps ϕ_1 and ϕ_2 are projections from the point P, the morphisms α_1 and α_2 are flopping contractions, the morphisms β_1 and β_2 are blow ups of P, and γ is a flop in 12 smooth curves.

Remark 1.9. Suppose that h, t, q_1 and q_2 are general. Then it follows from [8, Remark 4.3] that Y does not have automorphism that swaps the quadric surfaces given by

$$h\big(y_0, \ldots, y_4\big) = q_1\big(y_0, \ldots, y_4\big) = 0$$

and $h(y_0, \ldots, y_4) = q_2(y_0, \ldots, y_4) = 0$. This implies that $V_1 \not\cong V_2$.

Consider the following additional generality conditions:

(E) for any line $L \subset V_i$, and for any two-dimensional linear subspace $\Pi \subset \mathbb{P}^5$ such that $L \subset \Pi$, the cycle $V_i|_\Pi$ is reduced along the line L,

(F) for any two-dimensional linear subspace $\Pi \subset \mathbb{P}^5$, the intersection $V_i \cap \Pi$ is not three lines with a common point, and if $P \in \Pi$, the intersection $V_i \cap \Pi$ does not consist of three lines,

(G) for any line $L \subset V_i$ such that $P \in L$, and for any three-dimensional linear subspace
$$\Lambda \subset \mathbb{P}^5$$
such that the intersection $Q_i \cap \Lambda$ consists of two different planes, the three-dimensional linear subspace Λ is not a tangent space to the threefold V_i at any point of $L \setminus P$,

(H) for any line $L \subset V_i$ such that $P \in L$, and for any point $O \in L \setminus P$, the complete intersection $V_i \subset \mathbb{P}^5$ contains at most three lines that pass through O,

(I) for any lines $L \subset V_i \supset L'$ such that $L \ni P \notin L'$ and $L \cap L' \neq \varnothing$, and for any three-dimensional linear subspace $\Lambda \subset \mathbb{P}^5$ such that $L \subset \Lambda \supset L'$, the inequality
$$\mathrm{mult}_{L\cap L'}\Big(V_i\Big|_{\Lambda}\Big) \leqslant 4$$
holds in the case when the scheme $V_i|_\Lambda$ is not reduced along the lines L and L'.

In this paper, we prove the following result.

Theorem 1.10. *Suppose that* V_1 *and* V_2 *satisfy the conditions* A*,* B*,* C*,* D*,* E*,* F*,* G*,* H*,* I*. Then*
$$\mathcal{P}\big(V_1\big) = \mathcal{P}\big(V_2\big) = \Big\{V_1, V_2\Big\}.$$

Let $\mathcal{F}$ be the family of all complete intersections in $\mathbb{P}^5$ that are constructed similar to V_1 or V_2. In Section 8, we will show that general threefolds in $\mathcal{F}$ satisfy A, B, C, D, E, F, G, H, I.

Corollary 1.11. *Let* V *be a general threefold in* $\mathcal{F}$*. Then* $|\mathcal{P}(V)| = 2$ *and* V *is non-rational.*

Now we construct a subfamily $\mathcal{R} \subsetneq \mathcal{F}$. Let $\iota \in \mathrm{Aut}(\mathbb{P}^5)$ be an involution that is given by
$$y_0 \to -y_0,\ y_1 \to y_1,\ y_2 \to y_2,\ y_3 \to y_3,\ y_4 \to y_4\ y_5 \to y_5,$$
let U_1 be a complete intersection in $\mathbb{P}^5$ that is given by the equations
$$\begin{cases} y_5 f\big(y_1,y_2,y_3,y_4\big) = q\big(y_0,y_1,y_2,y_3,y_4\big),\\ y_5\iota^*\Big(q\big(y_0,y_1,y_2,y_3,y_4\big)\Big) = g\big(y_0,y_1,y_2,y_3,y_4\big)\end{cases}$$

in $\mathrm{Proj}\Big(\mathbb{C}[y_0, y_1, y_2, y_3, y_4, y_5]\Big) \cong \mathbb{P}^5$, and let U_2 be a complete intersection in $\mathbb{P}^5$ that is given by the equations

$$\begin{cases} y_5 f\big(y_1, y_2, y_3, y_4\big) = \iota^*\Big(q\big(y_0, \ldots, y_4\big)\Big), \\ y_5 q\big(y_0, y_1, y_2, y_3, y_4\big) = g\big(y_0, y_1, y_2, y_3, y_4\big), \end{cases}$$

where f, g and q are homogeneous forms of degree 1, 3 and 2, respectively. Suppose that

- the equality $g(-y_0, y_1, y_2, y_3, y_4) = g(y_0, y_1, y_2, y_3, y_4)$ holds,
- the threefolds U_1 and U_2 satisfy the conditions A, B, C, D.

Remark 1.12. The threefolds U_1 and U_2 are isomorphic, because $\iota(U_1) = U_2$.

For a fixed biregular involution $\iota \in \mathrm{Aut}(\mathbb{P}^5)$, let $\mathcal{R}$ be a family of complete intersections that are constructed similar to U_1 or U_2. Then $\mathcal{R} \subsetneq \mathcal{F}$. In this paper, we prove the following result.

Theorem 1.13. *A general threefold in $\mathcal{R}$ satisfies the conditions* A, B, C, D, E, F, G, H, I.

Corollary 1.14. *Let U be a general threefold in $\mathcal{R}$. Then*

$$\mathcal{P}\big(U\big) = \big\{U\big\},$$

i.e., the threefold U is birationally rigid, and in particular U is non-rational.

Corollary 1.15. *Birational rigidity is not open in moduli.*

We organize the paper in the following way: we prove Theorem 1.10 in Section 2 omitting the proofs of Lemmas 2.2 and 2.6, we prove Lemma 2.2 in Section 3, we prove Lemma 2.6 in Section 4 omitting the proofs of Lemmas 4.1, 4.4 and 4.7, we prove Lemmas 4.1, 4.4 and 4.7 in Sections 5, 6 and 7, respectively, we prove Theorem 1.13 in Section 8.

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2. Pliability count

Let us use the assumptions and notation of Theorem 1.10.

Remark 2.1. It follows from Proposition 3.1.2 in [4] that the following conditions are equivalent:

- for any two-dimensional linear subspace $\Pi \subset \mathbb{P}^5$ such that $L \subset \Pi$, the scheme-theoretical intersection $V_i \cap \Pi$ is reduced along L,
- the normal sheaf $\mathcal{N}_{L/V_i}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$,

where L is any line in the threefold V_i such that $P \notin L$.

Let us prove Theorem 1.10. Suppose that there is a Mori fibre space $\rho\colon U \to S$, and a birational map $\chi\colon U \dashrightarrow V_1$. To prove Theorem 1.10, we must show that $U \cong V_1$ or $U \cong V_2$.

Take a sufficiently big very ample divisor A on the variety S. Consider a linear system

$$\mathcal{M} = \Big| -mK_U + \rho^*\big(A\big)\Big|$$

for $m \gg 0$. Take any $\sigma \in \mathrm{Bir}(V_1)$. Put $\mathcal{D}_1^\sigma = \sigma \circ \chi(\mathcal{M})$ and $\mathcal{D}_2^\sigma = \phi_2^{-1} \circ \phi_1(\mathcal{D}_1^\sigma)$. Then

$$n_1^\sigma K_{V_1} + \mathcal{D}_1^\sigma \equiv 0 \equiv n_2^\sigma K_{V_2} + \mathcal{D}_2^\sigma$$

for some natural numbers n_1^σ and n_2^σ. Choose $\sigma \in \mathrm{Bir}(V_1)$ that minimizes $\min(n_1^\sigma, n_2^\sigma)$.

Without loss of generality, we may assume that $n_1^\sigma \leqslant n_2^\sigma$. Put $\mathcal{D} = \mathcal{D}_1^\sigma$ and $n = n_1^\sigma$.

It follows from Theorem 2.4 in [2] that either $\sigma \circ \chi$ is an isomorphism, or the singularities of the log pair $(V_1, \frac{1}{n}\mathcal{D})$ are not canonical. Thus, we may assume that $(V_1, \frac{1}{n}\mathcal{D})$ is not canonical.

Lemma 2.2. *Let $C \subset V_1$ be an irreducible curve. Suppose that* $\mathrm{mult}_C(\mathcal{D}) > n$. *Then*

- *either the curve C is a line,*
- *or the curve C is a conic such that $\langle C\rangle \subset Q_1$,*
- *or the curve C is a conic such that $\langle C\rangle \not\subset Q_1$ and $P \in C$.*

Proof. See Section 3. ☐

For a curve $C \subset \mathbb{P}^5$, we denote by $\langle C\rangle$ the smallest linear subspace in $\mathbb{P}^5$ containing C.

Lemma 2.3. *Let $C \subset V_1$ be a conic such that $\langle C\rangle \not\subset Q_1$ and $P \in C$. Then* $\mathrm{mult}_C(\mathcal{D}) \leqslant n$.

Proof. Let $\Lambda \subset \mathbb{P}^5$ be a general three-dimensional linear subspace such that $\langle C\rangle \subset \Lambda$. Then

$$V_1\Big|_{\Lambda} = C + Z,$$

where Z is an elliptic curve such that $P \in C \cap Z$. There is a commutative diagram

$$\begin{array}{ccc} \tilde{V}_1 & \xleftarrow{\ \delta\ } & \bar{V} \\ {\scriptstyle\beta_1}\downarrow & & \downarrow{\scriptstyle\omega} \\ V_1 & \underset{\psi}{\dashrightarrow} & \mathbb{P}^2, \end{array}$$

where ψ is the restriction of the projection $\mathbb{P}^5 \dashrightarrow \mathbb{P}^2$ from $\langle C\rangle$, the morphism β_1 is the blow-up of the singular point P, the morphism δ is the blow up of the proper transform of the curve C on the threefold $\tilde{V}_1$, and ω is a rational map whose general fiber is a smooth elliptic curve.

Let $\bar{E}_1$ be the proper transform of the β_1-exceptional divisor on the threefold $\bar{V}$. The map

$$\omega\Big|_{\bar{E}_1}: \bar{E}_1 \dashrightarrow \mathbb{P}^2$$

is birational, which simply means that $\bar{E}_1$ is a section of a rational fibration ω.

For a general point $O \in \bar{V}_1$, let $\bar{Z}$ be the fiber of ω that passes through O. Then $\bar{Z}$ is a smooth elliptic curve such that $\bar{E}_1 \cap \bar{Z}$ consists of a single point. Let O' be a point on $\bar{Z}$ that is a usual reflection of the point O on the elliptic curve $\bar{Z}$ with respect to the point $\bar{E}_1 \cap \bar{Z}$.

Let us define an involution $\tau \in \mathrm{Bir}(\bar{V})$ by putting $\tau(O) = O'$, which implies that $\tau(\bar{E}_1) = \bar{E}_1$, and τ is an isomorphism in codimension one.

Let F be the δ-exceptional divisor, and let $\bar{\mathcal{D}}$ be the proper transform of $\mathcal{D}$ on $\bar{V}$. Then

$$\bar{\mathcal{D}} \equiv \big(\beta_1 \circ \delta\big)^* \Big(- nK_{V_1}\Big) - \nu_0 \bar{E}_1 - \mathrm{mult}_C\big(\mathcal{D}\big)F,$$

where ν_0 is a natural number. It follows from Proposition 4.5 in [8] that

$$\tau\big(\bar{\mathcal{D}}\big) \equiv \big(\beta_1 \circ \delta\big)^* \Big(-\Big(15n-14\mathrm{mult}_C\big(\mathcal{D}\big)\Big)K_{V_1}\Big) - \nu_0\bar{E}_1 - \Big(16n-15\mathrm{mult}_C\big(\mathcal{D}\big)\Big)F,$$

which immediately implies that the equivalence

$$\beta_1 \circ \delta \circ \tau \circ \delta^{-1} \circ \beta_1^{-1}\big(\mathcal{D}\big) \equiv -\Big(15n - 14\mathrm{mult}_C\big(\mathcal{D}\big)\Big)K_{V_1}$$

holds. But $\beta_1 \circ \delta \circ \tau \circ \delta_1^{-1} \in \mathrm{Bir}(V_1)$. But $15n - 14\mathrm{mult}_C(\mathcal{D}) \geqslant n$ by the minimality in the choice of the number $n \in \mathbb{N}$. Thus, $\mathrm{mult}_C(\mathcal{D}) \leqslant n$. □

Lemma 2.4. *Let $C \subset V_1$ be a conic such that $\langle C\rangle \subset Q$ and $P \in C$. Then* $\mathrm{mult}_C(\mathcal{D}) \leqslant n$.

Proof. Arguing as in [4], we construct an involution $\zeta \in \mathrm{Bir}(V_1)$ such that

$$\zeta\big(\mathcal{D}\big) \equiv -\Big(13n - 12\mathrm{mult}_C\big(\mathcal{D}\big)\Big)K_{V_1},$$

which implies that $\mathrm{mult}_C(\mathcal{D}) \leqslant n$ due to the minimality of the number n. □

Lemma 2.5. *Let $C \subset V_1$ be a line. Then* $\mathrm{mult}_C(\mathcal{D}) \leqslant n$.

Proof. Arguing as in [4], we construct an involution $\zeta \in \mathrm{Bir}(V_1)$ such that

$$\zeta\big(\mathcal{D}\big) \equiv -\Big(4n - 3\mathrm{mult}_C\big(\mathcal{D}\big)\Big)K_{V_1},$$

which implies that $\mathrm{mult}_C(\mathcal{D}) \leqslant n$ due to the minimality of the number n. □

Therefore, we see that the log pair $(V_1, \frac{1}{n}\mathcal{D})$ is canonical outside of finitely many points.

Lemma 2.6. *Let O be a point in $V_1 \setminus P$. Then $(V_1, \frac{1}{n}\mathcal{D})$ is canonical at O.*

Proof. See Section 4. □

Thus, the log pair $(V_1, \frac{1}{n}\mathcal{D})$ is not canonical at the point $P = \mathrm{Sing}(V_1)$.

Let E_1 be the β_1-exceptional divisor, and let $\tilde{\mathcal{D}}$ be the proper transform of $\mathcal{D}$ on $\tilde{V}_1$. Then

$$\tilde{\mathcal{D}} \equiv \beta_1^*\Big(-nK_{V_1}\Big) - \nu_0 E_1,$$

where ν_0 is a natural number. It follows from Theorem 3.10 in [2] that $\nu_0 > n$.

Let E_2 be the β_2-exceptional divisor. Then $\gamma(E_1) \equiv \beta_2^*(-K_{V_2}) - 2E_2$ and

$$\gamma\Big(-K_{V_1}\Big) = \beta_2^*\Big(-2K_{V_2}\Big) - 3E_2$$

because $\gamma(K_{\tilde{V}_1}) \equiv K_{\tilde{V}_2} \equiv \beta_2^*(K_{V_2}) + E_2$ and γ is an isomorphism in codimension one. But

$$\gamma(\tilde{\mathcal{D}}) \equiv \beta_2^*\Big(-\big(2n-\nu_0\big)K_{V_2}\Big) - \big(3n - 2\nu_0\big)E_2,$$

which implies that $\mathcal{D}_2^{\sigma} \equiv -\big(2n-\nu_0\big)K_{V_2}$. Then $n_2^{\sigma} = 2n - \nu_0 < n = n_1^{\sigma}$, which is a contradiction, because $n_1^{\sigma} \leqslant n_2^{\sigma}$. The assertion of Theorem 1.10 is proved.

3. Exclusion of curves

Let us use the assumptions and notation of Lemma 2.2.

Remark 3.1. Let $\Lambda \subset \mathbb{P}^5$ be a three-dimensional linear subspace. Then $V_1|_{\Lambda}$ is reduced along any curve that is not contained in two-dimensional linear subspace, because cubic surface does not intersect an irreducible quadric surface by a double twisted cubic.

Put $\nu = \mathrm{mult}_C(\mathcal{D})$. Let Ω be the smallest linear subspace in $\mathbb{P}^5$ such that $C \subseteq \Omega$.

Suppose that $\nu > n$. To prove Lemma 2.2, we must show that

- either $\Omega \subset Q_1$ and $\deg(C) \leqslant 2$,
- or $P \in C$ and $\deg(C) = 2$.

Arguing as in the proof of [4, Lemma 3.3.6], we see that $\Omega \subset Q_1$ and $\deg(C) \leqslant 2$ in the case when $P \notin \Omega$. Therefore, to complete the proof of Lemma 2.2, we may assume that $P \in \Omega$.

Lemma 3.2. *Suppose that* $\deg(C) = 2$. *Then* $P \in C$.

Proof. Suppose that $\Omega \subset Q_1$. Then $T_1|_{\Omega} = C + L$, where L is a line, which immediately implies that $P \in C \cap L$, because $\mathrm{mult}_P(T_1) = 2$ and $P \in \Omega$. ☐

Thus, to complete the proof of Lemma 2.2, we may assume that $\deg(C) \geqslant 3$. Let us show that this assumption leads to a contradiction with the inequality $\nu > n$.

Lemma 3.3. *The inequality* $\deg(C) \leqslant 5$ *holds.*

Proof. Let D_1 and D_2 be general surfaces in $\mathcal{D}$. Then

$$6n^2 = D_1 \cdot D_2 \cdot H \geqslant \mathrm{mult}_C\Big(D_1 \cdot D_2\Big)\deg\big(C\big) > n^2 \deg\big(C\big),$$

where $D_1 \cdot D_2 \cdot H$ is a degree of the zero-cycle of the corresponding scheme-theoretic intersection, and H is a general hyperplane section of the threefold $V_1 \subset \mathbb{P}^5$. □

Lemma 3.4. *The inequality* $\dim(\Omega) \neq 2$ *holds.*

Proof. Suppose that $\dim(\Omega) = 2$. Then $\Omega \subset Q_1$ and $\deg(C) = 3$, because $C \subseteq \mathrm{Supp}(T_1|_\Omega)$.

Let Λ be a sufficiently general three-dimensional linear subspace in $\mathbb{P}^5$ that contains Ω. Then

$$\Omega \cap V_1 = C \cup \bar{C},$$

where $\bar{C}$ is a plane cubic. But $C \cap \bar{C}$ consists of three distinct points different from P. Then

$$3n = D \cdot \bar{B} \geqslant 3\nu > 3n,$$

where D is a general surface in the linear system $\mathcal{D}$. □

Lemma 3.5. *The curve* C *is singular.*

Proof. Suppose that C is non-singular. Then we have the following cases:

- $\deg(C) = \dim(\Omega) \in \{4, 5\}$ and $g(C) = 0$,
- $\deg(C) = 5$, $\dim(\Omega) = 4$ and $g(C) = 0$,
- $\deg(C) = 5$, $\dim(\Omega) = 4$ and $g(C) = 1$,

where $g(C)$ is the genus of the curve C.

Put $d = \deg(C)$. Let m be a natural number such that the curve C is cut out on V_1 by surfaces in $|-mK_{V_1}|$ that pass through C, the scheme-theoretic intersection of two general surfaces in $|-mK_{V_1}|$ that pass through the curve C is reduced in a general point of the curve C.

We have $m \leqslant 3$, and we can put $m = 2$ unless $\deg(C) = 5$, $\dim(\Omega) = 4$ and $g(C) = 0$.

Let $\delta\colon \bar{V} \to V_1$ be a terminal extraction with the center C and exceptional divisor E. Then

$$\Big(\delta^*\Big(-mK_{V_1}\Big) - E\Big) \cdot \Big(\delta^*\Big(-nK_{V_1}\Big) - \nu E\Big)^2 \geqslant 0,$$

because $\delta^*(-mK_{V_1}) - E$ is nef. Thus, the inequality

$$6mn^2 - dm\nu^2 - 2d\nu n - n^2\left(2 - 2g(C) - d - \frac{\mathrm{mult}_P(C)}{2}\right) \geqslant 0,$$

holds, which easily leads to a contradiction. □

Let $\tilde{\mathcal{D}}$ be the proper transform of $\mathcal{D}$ on $\tilde{V}_1$, and let E_1 be the β_1-exceptional divisor. Then

$$\tilde{\mathcal{D}} \equiv \beta_1^*\Big(-nK_{V_1}\Big) - \nu_0 E_1$$

for some integer $\nu_0 \geqslant 0$. Then $\nu_0 \geqslant \nu/2$ in the case when $P \in C$ (see the proof of Lemma 4.12).

Lemma 3.6. *The equality* $\dim(\Omega) = 3$ *holds.*

Proof. Suppose that $\dim(\Omega) \neq 3$. Then $\dim(\Omega) = 4$ and $\deg(C) = 5$ by Lemma 3.5.

Suppose that $P \in \mathrm{Sing}(C)$. Let $\tilde{C}$ be the proper transform of C on the threefold $\tilde{V}_1$. Then

$$c_1\Big(\mathcal{N}_{\tilde{B}/\tilde{V}_1}\Big) = -2-K_{\tilde{V}_1}\cdot\tilde{C} = -2-\Big(\beta_1^*(K_{V_1})+E_1\Big)\cdot\tilde{C} = -2-K_{V_1}\cdot C-E_1\cdot\tilde{C} = 1,$$

because $\tilde{C} \cong \mathbb{P}^1$ and $\mathrm{mult}_P(C) = 2$. Let $\delta\colon \bar{V} \to \tilde{V}_1$ be a blow up of the curve $\tilde{C}$. Then

$$\Big|(\beta_1 \circ \delta)^*\Big(-nK_{V_1}\Big) - \nu_0\delta^*(E_1) - \nu F\Big|$$

does not have fixed components, where F is the exceptional divisor of the blow up δ. But

$$\Big|(\beta_1 \circ \delta)^*\Big(-3K_{V_1}\Big) - \delta^*(E_1) - F\Big|$$

does not have base curves. Thus, we have

$$\Big((\beta_1\circ\delta)^*\Big(-nK_{V_1}\Big)-\nu_0\delta^*(E_1)-\nu F\Big)^2\Big((\beta_1\circ\delta)^*\Big(-3K_{V_1}\Big)-\delta^*(E_1)-F\Big) \geqslant 0,$$

which leads to a contradiction, because $\nu > n$ and

$$0 \leqslant 18n^2 - 10n\nu - 12\nu^2 + 4\nu\nu_0 - \nu_0^2 = \Big(18n^2 - 10n\nu - 8\nu^2\Big) - \big(2\nu - \nu_0\big)^2 < 0.$$

Therefore, there is a point $O \in V_1$ such that $P \neq O$ and $\mathrm{Sing}(C) = O$.

Let $\upsilon\colon \breve{V} \to \tilde{V}_1$ be the blow up of the point that dominates O, let $\breve{C}$ be the proper transform of the curve C on the threefold $\breve{V}$, and let $\zeta\colon \grave{V} \to \breve{V}$ be the blow up of the curve $\breve{C}$. Then

$$\Big|(\beta_1 \circ \delta \circ \zeta)^*\Big(-nK_{V_1}\Big) - \nu_0(\delta \circ \zeta)^*(E_1) - \mathrm{mult}_O(\mathcal{D})\zeta^*(F) - \nu G\Big|$$

has no fixed components, where F and G are exceptional divisors of υ and ζ, respectively.

Suppose that $P \in C$. Then $\mathrm{mult}_O(\mathcal{D}) \geqslant \nu$ and $\nu_0 > \nu/2$. But the linear system

$$\Big|(\beta_1 \circ \delta \circ \zeta)^*\Big(-3K_{V_1}\Big) - \big(\delta \circ \zeta\big)^*(E_1) - \zeta^*(F) - G\Big|$$

does not have base curves. Arguing as in the proof of Lemma 3.6, we see that

$$18n^2 - 10n\nu - 14\nu^2 + 2\nu\nu_0 - \nu_0^2 + 4\nu\mathrm{mult}_O(\mathcal{D}) - \mathrm{mult}_O^2(\mathcal{D}) \geqslant 0$$

which leads to a contradiction. Thus, we see that $P \notin C$. Then the linear system

$$\Big|(\beta_1 \circ \delta \circ \zeta)^*\Big(-3K_{V_1}\Big) - \zeta^*(F) - G\Big|$$

does not have base curves. Arguing as in the case $P \in C$, we obtain a contradiction. ∎

Thus, we proved that $\Omega \cong \mathbb{P}^3$ and $\deg(C) \geqslant 3$. Then

$$V_1\Big|_{\Omega} = C + \sum_{i=1}^{r} m_i C_i,$$

where C_i is an irreducible curve, and $m_i \in \mathbb{N}$. Then

$$\deg\big(C\big) + \sum_{i=1}^{r} m_i \deg\big(C_i\big) = 6,$$

and $C_i \neq C$ for every $i = 1, \ldots, r$ by Remark 3.1.

Remark 3.7. The quadric $Q_1 \cap \Omega$ is irreducible, because $\dim(\Omega) = 3$.

Let H be a general hyperplane section of V_1 such that $C \subset H$. Then $P \in H$ is a singularity of type $\mathbb{A}_k$. Let L be a fiber of a natural projection $E_1 \cong \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$.

Lemma 3.8. *Let Z_1 and Z_2 be lines on the threefold V_1 such that $Z_1 \cap Z_2 = P$. Then*

$$\tilde{Z}_1 \cap L \neq \varnothing \Rightarrow \tilde{Z}_2 \cap L = \varnothing,$$

where $\tilde{Z}_1$ and $\tilde{Z}_2$ be the proper transform of the lines Z_1 and Z_2 on the threefold $\tilde{V}_1$, respectively.

Proof. The surface $\alpha_1(E_1)$ is a quadric surface in $\mathbb{P}^4$ that is given by the equations

$$h\big(y_0, y_1, y_2, y_3, y_4\big) = q_2\big(y_0, y_1, y_2, y_3, y_4\big) = 0$$

in $\mathrm{Proj}\Big(\mathbb{C}[y_0, y_1, y_2, y_3, y_4]\Big) \cong \mathbb{P}^4$, the curve $\alpha_1(L)$ is a line, and $\alpha_1(\tilde{Z}_1)$ and $\alpha_1(\tilde{Z}_2)$ are singular points of the quartic $Y \subset \mathbb{P}^4$.

Let Π be the two-dimensional linear subspace in $\mathbb{P}^5$ that contains Z_1 and Z_2, and let

$$\phi \colon \mathbb{P}^5 \dashrightarrow \mathbb{P}^4$$

be a projection from the point P. Then $\phi(\Pi)$ is a line in $\mathbb{P}^4$ such that

$$\alpha_1\big(\tilde{Z}_1\big) \in \phi\big(\Pi\big) \ni \alpha_1\big(\tilde{Z}_2\big).$$

Suppose that $\tilde{Z}_1 \cap L \neq \varnothing$ and $\tilde{Z}_2 \cap L \neq \varnothing$. Then $\phi(\Pi) = \alpha_1(L)$, which implies that $\Pi \subset Q_2$.

The linear subspace $\Pi \subset \mathbb{P}^5$ contains two lines Z_1' and Z_2' such that

$$P \in Z_1' \subset V_2 \supset Z_2' \ni P$$

and $\phi_2(Z_1') = \alpha_1(\tilde{Z}_1)$ and $\phi_2(Z_2') = \alpha_1(\tilde{Z}_2)$. Then $Z_1' \subseteq V_2 \cap \Pi \supseteq Z_2'$ and $\Pi \subset Q_2$, which is impossible, because V_2 satisfies the conditions E and F. $\square$

Let $\Upsilon \subset \mathbb{P}^5$ be a hyperplane that is tangent to the quadric Q_1 at the point $P \in V_1$. Then

$$\mathrm{mult}_P\big(C\big) + \sum_{i=1}^{r} m_i \mathrm{mult}_P\big(C_i\big) \geqslant 4$$

in the case when $\Omega \subset \Upsilon$. Let $\tilde{H}$ be a proper transform of H on the threefold $\tilde{V}_1$.

Lemma 3.9. *Suppose that* $\Omega \not\subset \Upsilon$. *Then* $k \leqslant 2$.

Proof. Suppose that $\Omega \not\subset \Upsilon$ and $k \geqslant 3$. Let us show that this assumption leads to a contradiction.

Let $\mathcal{H}$ be a linear subsystem in $|-K_X|$ consisting of surfaces passing through the curve C, and let $\tilde{\mathcal{H}}$ be the proper transform of the linear system $\mathcal{H}$ on the threefold $\tilde{V}_1$. Then

$$\tilde{\mathcal{H}}\Big|_{E_1} = L_1 + \big|L_2\big|,$$

where L_1 and L_2 are fibers of two different projections $\mathbb{P}^1 \times \mathbb{P}^1 \cong E_1 \to \mathbb{P}^1$.

The surface $\tilde{H}$ is a general surface in the linear system $\tilde{\mathcal{H}}$ and

$$E_1 \cap \tilde{H} = L_1 \cup L_{\tilde{H}},$$

where $L_{\tilde{H}} \in |L_2|$. Put $O = L_1 \cap L_{\tilde{H}}$. Then $\tilde{H}$ is singular at the point O.

Let $\tilde{H}'$ be another sufficiently general surface in the linear system $\mathcal{H}$. Then

$$\tilde{H}'\Big|_{\tilde{H}} = mL_1 + L_{\tilde{H}'} + \tilde{C} + \sum_{i=1}^{r} m_i \tilde{C}_i$$

for some curve $L_{\tilde{H}'} \in |L_2|$ and for some natural number m, where $\tilde{C}$ and $\tilde{C}_i$ are proper transforms of the irreducible curves C and C_i on the surface $\tilde{H}$, respectively. Then $m \geqslant \mathrm{mult}_O(\tilde{H}) \geqslant 2$.

Put $\breve{H} = \alpha_1(\tilde{H})$. Then $\breve{H}$ is a general hyperplane section of the quartic $Y \subset \mathbb{P}^4$ given by

$$\begin{aligned} h\big(y_0, \ldots, y_4\big)t\big(y_0, \ldots, y_4\big) &= q_1\big(y_0, \ldots, y_4\big)q_2\big(y_0, \ldots, y_4\big) \\ &\subset \mathrm{Proj}\Big(\mathbb{C}[y_0, \ldots, y_4]\Big) \cong \mathbb{P}^4 \end{aligned}$$

such that $\alpha_1(L_1) \subset \breve{H}$. Similarly, we have $\alpha_1(L_1) \subset \alpha_1(\tilde{H}')$. Then

$$\alpha_1\big(\tilde{H}'\big)\Big|_{\breve{H}} = m\alpha_1\big(L_1\big) + \alpha_1\big(L_{\tilde{H}'}\big) + \alpha_1\big(\tilde{C}\big) + \sum_{i=1}^{r} m_i \alpha_1\big(\tilde{C}_i\big),$$

which implies that the cubic $t(y_0, \ldots, y_4) = 0$ contains the line $\alpha_1(L_1)$, because $m \geqslant 2$.

The quartic Y has 12 different singular points that are given by the equations

$$\begin{aligned} h\big(y_0, y_1, y_2, y_3, y_4\big) &= t\big(y_0, y_1, y_2, y_3, y_4\big) \\ &= q_1\big(y_0, y_1, y_2, y_3, y_4\big) \end{aligned}$$

$$= q_2\big(y_0, y_1, y_2, y_3, y_4\big) = 0,$$

which implies that $|\alpha_1(L_1) \cap \mathrm{Sing}(Y)| = 2$. The latter is impossible by Lemma 3.8. □

For a given point $O \in V_1 \setminus P$, the inequality

$$\mathrm{mult}_O\big(C\big) + \sum_{i=1}^{r} m_i \mathrm{mult}_O\big(C_i\big) \geqslant 4$$

holds in the case when Ω is the tangent linear subspace to V_1 at the point O.

Lemma 3.10. *Suppose that the subspace $\Omega \subset \mathbb{P}^5$ is a tangent linear subspace to the complete intersection V_1 at some point $O \in V_1 \setminus P$. Then O is an ordinary double point of the surface H.*

Proof. An affine part of the complete intersection $V_1 \subset \mathbb{P}^5$ can be given by the equations

$$\mathbb{C}^5 \cong \mathrm{Spec}\Big(\mathbb{C}\big[x, y, z, t, w\big]\Big) \supset \begin{cases} w = xh_1\big(x, y, z, t\big) + y^2 + z^2 + t^2, \\ x = xh_2\big(x, y, z, t\big) + g_2\big(y, z, t\big) + g_3\big(x, y, z, t, w\big) \end{cases}$$

such that O is given by $x = y = z = t = w = 0$, where h_i and g_i are homogeneous polynomials of degree i. Then Ω is given by $w = x = 0$, and the surface H is given by

$$\begin{cases} x = \lambda w, \\ w = xh_1\big(x, y, z, t\big) + y^2 + z^2 + t^2, \\ x = xh_2\big(x, y, z, t\big) + g_2\big(y, z, t\big) + g_3\big(x, y, z, t, w\big) \end{cases}$$

for some general $\lambda \in \mathbb{C}$. We can consider monomials y, z, t as local coordinates on the quadric

$$Q = \begin{cases} x = \lambda w, \\ w = xh_1\big(x, y, z, t\big) + y^2 + z^2 + t^2 \end{cases}$$

in a neighbourhood of the point O. Then the surface $H \subset Q$ is locally given by

$$g_2\big(y, z, t\big) - \lambda\Big(y^2 + z^2 + t^2\Big) + \text{higher degree terms},$$

which implies that O is an ordinary double points of the surface H, because λ is general. □

The subspace $\Omega \subset \mathbb{P}^5$ can be a tangent linear subspace to V_1 at no more than one point, because the quadric $Q_1|_\Omega$ is irreducible and reduced (see Lemma 3.4.1 in [4]).

Lemma 3.11. *The intersection form of $C_1, \ldots, C_r$ on H is not semi-negative definite.*

Proof. Suppose that the intersection form of $C_1, \ldots, C_r$ is semi-negative definite. Then

$$\mathcal{D}\Big|_H = \nu C + \sum_{i=1}^{r} \nu_i C_i + \mathcal{B} \equiv nC + \sum_{i=1}^{r} n m_i C_i,$$

where ν_i is a non-negative integer, and $\mathcal{B}$ is a linear system that does not have fixed curves. Then

$$\left((\nu - n)C + \mathcal{B} + \sum_{\nu_i \geqslant n m_i} (\nu_i - n m_i) C_i\right) \left(\sum_{n m_i \geqslant \nu_i} (n m_i - \nu_i) C_i\right)$$
$$= \left(\sum_{n m_i > \nu_i} (n m_i - \nu_i) C_i\right)^2,$$

which implies that $n m_i \leqslant \nu_i$ for every i, because $\nu > n$ and $C \cap C_i \neq \varnothing$ for every i. Then

$$(\nu - n)C + \sum_{i=1}^{r} (n m_i - \nu_i) C_i + \mathcal{B} \equiv 0,$$

which is a contradiction, because $\nu > n$. □

It follows from [1] that the intersection form of $C_1, \ldots, C_r$ on the surface H is negative definite if and only if they can be contracted on the surface H to an isolated singular point.

Lemma 3.12. *The inequality* $\deg(C) \neq 5$ *holds.*

Proof. Suppose that $\deg(C) = 5$. Then $r = m_1 = \deg(C_1) = 1$, and H is smooth outside of

$$\mathrm{Sing}(C) \cup \Big(C \cap C_1\Big),$$

where H has singularity of type $\mathbb{A}_k$ at the point P.

Suppose that $\Omega \not\subset \Upsilon$. Then $k \leqslant 2$ by Lemma 3.9. Therefore, the set $\mathrm{Sing}(H) \setminus P$ contains at most one point that must be ordinary double point of the surface H by Lemma 3.10. Then

$$C_1 \cdot C_1 \leqslant -2 + \frac{1}{2} + \frac{k}{k+1} < 0,$$

which is impossible by Lemma 3.11. Thus, we see that $\Omega \subset \Upsilon$. Then

$$\mathrm{mult}_P\big(C\big) + 1 = \mathrm{mult}_P\big(C\big) + \mathrm{mult}_P\big(C_1\big) \geqslant 4,$$

which implies that $P = \mathrm{Sing}(C) \in C \cap C_1$ and $\mathrm{mult}_P(C) = 3$. We see that $\mathrm{Sing}(H) = \{P\}$.

The inequality $k \geqslant 3$ holds, because it follows from the subadjunction formula that

$$C_1 \cdot C_1 = -2 + \frac{k}{k+1}$$

if $k \leqslant 2$. But $C_1 \cdot C_1 > 0$ by Lemma 3.11. Then $E_1|_{\tilde{H}} = \tilde{L}_1 + \tilde{L}_k$, where $\tilde{L}_1$ and $\tilde{L}_2$ are irreducible curves. Put $\pi = \beta_1|_{\tilde{H}} \colon \tilde{H} \to H$ and $O = \tilde{L}_1 \cap \tilde{L}_k$. Then $\tilde{H}$ has singularity of type $\mathbb{A}_{k-2}$ at O, and π contracts $\tilde{L}_1$ and $\tilde{L}_k$. Let $\tilde{C}_1$ be the proper transform of C_1 on $\tilde{H}$. Then

$$\tilde{C}_1 \cap \left(\tilde{L}_1 \cup \tilde{L}_k\right) = O,$$

because $C_1 \cdot C_1 = -2 + \frac{k}{k+1}$ in the case when $O \notin \tilde{C}_1$. But $C_1 \cdot C_1 > 0$ by Lemma 3.11.

Let $\eta \colon \bar{H} \to \tilde{H}$ be the minimal resolution of singularities of $\tilde{H}$, let $\bar{L}_1$ and $\bar{L}_k$ be the proper transforms of the curves $\tilde{L}_1$ and $\tilde{L}_k$ on the surface $\bar{H}$, respectivley. Then η contracts a chain of smooth rational curves $\bar{L}_2, \ldots, \bar{L}_{k-1}$ to the point O such that

$$\bar{L}_1 \cdot \bar{L}_1 = \cdots = \bar{L}_k \cdot \bar{L}_k = -2,\ \bar{L}_1 \cdot \bar{L}_2 = \bar{L}_2 \cdot \bar{L}_3 = \cdots = \bar{L}_{k-2} \cdot \bar{L}_{k-1} = \bar{L}_{k-1} \cdot \bar{L}_k = 1.$$

Let $\bar{C}_1$ be the proper transform of the curve $\tilde{C}_1$ on the surface $\bar{H}$. Then $\bar{C}_1 \cdot \bar{C}_1 = -2$ and

$$\bar{C}_1 \cap \bar{L}_1 = \bar{C}_1 \cap \bar{L}_k = \varnothing,$$

where $\bar{C}_1$ intersects only one curve among the curves $\bar{L}_1, \bar{L}_2, \bar{L}_3, \ldots, \bar{L}_{k-1}, \bar{L}_k$.

Arguing as in the proof of Lemma 3.9, we easily see that the morphism $\alpha_1|_{\tilde{H}} \colon \tilde{H} \to \alpha_1(\tilde{H})$ contracts the curve $\tilde{C}_1$ to a singular point of the surface $\alpha_1(\tilde{H})$ of type $\mathbb{A}_s$. Then the curves

$$\bar{C}_1, \bar{L}_2, \bar{L}_3, \ldots, \bar{L}_{k-1}$$

must form a chain and $s = k-1$, where either $\bar{C}_1 \cap \bar{L}_2 \neq \varnothing$, or $\bar{C}_1 \cap \bar{L}_{k-2} \neq \varnothing$.

Therefore, the curves $\bar{C}_1, \bar{L}_1, \bar{L}_2, \bar{L}_3, \ldots, \bar{L}_{k-1}, \bar{L}_k$ form a graph of type $\mathbb{D}_{k+1}$, which implies that their intersection form must be negative definite. Therefore, the inequality $C_1 \cdot C_1 < 0$ holds, which is a impossible by Lemma 3.11. □

Recall that the singular point $P \in V_1$ is a singular point of type $\mathbb{A}_k$ of the surface H.

Lemma 3.13. *The inequality* $\deg(C) \neq 4$ *holds.*

Proof. Suppose that $\deg(C) = 4$. Then either C is smooth, or C has one double point.

Suppose that $r = m_1 = 1$. Then C_1 is a smooth conic. Then H is smooth, which implies that $C_1 \cdot C_1 = -2$. But the inequality $C_1 \cdot C_1 > 0$ holds by Lemma 3.11.

Suppose that $r = 2$ and $m_1 = m_2 = 1$. Then C_1 and C_2 are lines in $\mathbb{P}^5$. The equalities

$$C_1 \cdot C_1 = -2,\ C_2 \cdot C_2 = -2,\ C_1 \cdot C_2 \leqslant 1$$

hold in the case when $C_1 \cap C_2 \in H \setminus \mathrm{Sing}(H)$, which is impossible by Lemma 3.11. Then

$$C_1 \cdot C_1 = -3/2,\ C_2 \cdot C_2 = -3/2,\ C_1 \cdot C_2 = 1/2$$

by Lemma 3.10 in the case when $P \neq C_1 \cap C_2 \in \mathrm{Sing}(H)$. Thus, we see that

$$C_1 \cap C_2 = P = \mathrm{Sing}\big(H\big)$$

by Lemma 3.11. Similarly, we see that P is not an ordinary double point of the surface H.

Thus, the intersection $\tilde{H} \cap E_1$ consists of two smooth irreducible curves $\tilde{L}_1$ and $\tilde{L}_2$.

Let $\tilde{C}_1$ and $\tilde{C}_2$ be the proper transforms of C_1 and C_2 on the surface $\tilde{H}$, respectively. Then

$$\tilde{C}_1 \cap \tilde{L}_i \neq \varnothing \Rightarrow \tilde{C}_2 \cap \tilde{L}_i = \varnothing$$

for $i = 1$ and $i = 2$ by Lemma 3.8. The curves $\tilde{C}_1, \tilde{C}_2, \tilde{L}_1, \tilde{L}_2$ on the surface $\tilde{H}$ can be contracted to an isolated singular point of type $\mathbb{A}_{k+2}$, which is impossible by Lemma 3.11.

Therefore, we proved that $r = 1$ and $m_1 = 2$. Then C_1 is a line, and

$$C_1 \cdot C_1 = \frac{1 - C \cdot C_1}{2}$$

on the surface H. So, the inequality $C \cdot C_1 < 1$ holds, because $C_1 \cdot C_1 > 0$ by Lemma 3.11. Then

$$C \cap C_1 \subset \mathrm{Sing}\big(H\big),$$

because $C \cap C_1 \neq \varnothing$. To complete the proof, we must show that $C \cdot C_1 \geqslant 1$.

Suppose that $P \neq C \cap C_1$. There is a point $O \in C \cap C_1$ such that $O \in \mathrm{Sing}(H)$. Then

$$\mathrm{mult}_O\big(C\big) + 2\mathrm{mult}_O\big(C_1\big) \geqslant 4,$$

because Ω must be the tangent linear subspace to V_1 at the point O. Then $\mathrm{mult}_O(C) = 2$ and

$$1 > C \cdot C_1 \geqslant \frac{\mathrm{mult}_O\big(C\big)}{2} = 1,$$

because O is an ordinary double point of the surface H by Lemma 3.10.

Thus, we see that $C \cap C_1 = P$. Put $\bar{Q} = Q_1|_\Omega$ and $\bar{T} = T_1|_\Omega$.

Suppose that $\bar{Q}$ is a cone. Then C_1 is its rulings. Either the cubic $\bar{T}$ is singular along C_1, or the cubic $\bar{T}$ is tangent to the cone $\bar{Q}$ along the line C_1. Hence, there is a two-dimensional linear subspace $\Pi \subset \Omega$ that is tangent to both $\bar{T}$ and $\bar{Q}$ along the line C_1. The sub-scheme $V_1|_\Pi$ is not reduced along the line C_1, which is impossible, because V_1 satisfies the condition E.

We see that the surface $\bar{Q}$ is smooth. Then $\bar{Q} \cong \mathbb{P}^1 \times \mathbb{P}^1$, where C and C_1 are divisors of bi-degree $(3, 1)$ and $(0, 1)$ on the quadric $\bar{Q}$, respectively.

It follows from $C \cap C_1 = P$ that C_1 is tangent to the curve C at the point P with multiplicity 3, because the equality $C_1 \cdot C = 3$ holds on the quadric surface $\bar{Q}$.

Let $\tilde{H}$ be a proper transform of H on the threefold $\tilde{V}$, let $\tilde{C}$ and $\tilde{C}_1$ be the proper transforms of the curves C and C_1 on the surface $\tilde{H}$, respectively. Then

$\tilde{H}$ is smooth by Lemma 3.9, and

$$C \cdot C_1 = \begin{cases} 3/2 \text{ in the case when } k = 1, \\ 4/3 \text{ in the case when } k = 2, \end{cases}$$

because $\tilde{C} \cdot \tilde{C}_1 = 2$ on the surface $\tilde{H}$. But $C \cdot C_1 < 1$. □

Thus, the curve C is a smooth rational curve of degree 3.

Lemma 3.14. *Suppose that $r = 1$. Then $m_1 \neq 3$.*

Proof. Suppose that $m_1 = 3$. Then C_1 is a line. Put $\bar{Q} = Q_1|_\Omega$ and $\bar{T}_1 = T|_\Omega$. Then

$$\bar{Q} \cdot \bar{T} = C + 3C_1$$

in $\Omega \cong \mathbb{P}^3$. But the quadric surface $\bar{Q}$ is irreducible.

Suppose that $\bar{Q}$ is smooth. Then C is a divisor of type $(3, 0)$ on the surface $\bar{Q} \cong \mathbb{P}^1 \times \mathbb{P}^1$, which is impossible, because the curve C is irreducible and reduced. Thus, the quadric $\bar{Q}$ is a cone.

The line C_1 is a ruling of the cone $\bar{Q}$. Then either $\bar{T}$ is singular along C_1, or $\bar{T}$ is tangent to the quadric $\bar{Q}$ along C_1. Then there is a two-dimensional linear subspace $\Pi \subset \Omega$ that is tangent to $\bar{T}$ and $\bar{Q}$ along C_1. Then $V_1|_\Pi$ is not reduced along C_1, which contradicts the condition E. □

Lemma 3.15. *The inequality $r \neq 1$ holds.*

Proof. Suppose that $r = 1$. Then $m_1 = 1$ by Lemma 3.14, and $V_1|_\Omega = C + C_1$, where C_1 is an irreducible reduced cubic curve. Then the curve C_1 is not contained in any two-dimensional linear subspace in $\Omega \cong \mathbb{P}^3$, because $Q_1|_\Omega$ is irreducible. Then C_1 is a smooth rational cubic curve.

It follows from Lemmas 3.10 and 3.9 that H is smooth outside of P, and either P is an ordinary double point of the surface H, or P is a singular point of the surface H of type $\mathbb{A}_2$. Then

$$C_1 \cdot C_1 = \begin{cases} -3/2 \text{ in the case when } k = 1, \\ -4/3 \text{ in the case when } k = 2, \end{cases}$$

on the surface H. But $C_1 \cdot C_1 > 0$ by Lemma 3.11. □

Let $\pi \colon S \to H$ be a minimal resolution of singularities. Then π contracts a chain or smooth rational curves $L_1, \ldots, L_k$ to the point P such that $L_i^2 = -2$ on the surface S for all i, and

$$L_1 \cdot L_2 = L_2 \cdot L_3 = \cdots = L_{k-2} \cdot L_{k-1} = L_k \cdot L_{k-1} = 1$$

on the surface S. Then $L_1 \cdot L_i = L_k \cdot L_j = 0$ for all $i \neq 2$ and $j \neq k - 1$.

Lemma 3.16. *The inequality $r \neq 3$ holds.*

Proof. Suppose that $r=3$. Then the curves C_1, C_2 and C_3 are distinct lines.

The intersection $C_1 \cap C_2 \cap C_3$ contains no smooth points of the surface H, because otherwise there is a two-dimensional linear subspace $\Pi \subset \mathbb{P}^5$ that contains C_1, C_2, C_2, which is impossible, because $\dim(\Omega)=3$ and the quadric surface $Q_1|_\Omega$ is irreducible and reduced.

To complete the proof, we must consider the following possible cases:

- the intersection $C_1 \cap C_2 \cap C_3$ consists of the point P,
- the intersection $C_1 \cap C_2 \cap C_3$ consists of a point in $\mathrm{Sing}(H) \setminus P$,
- the intersection $C_1 \cap C_2 \cap C_3$ is empty.

Suppose that $C_1 \cap C_2 \cap C_3 = P$. So, the surface H must be smooth outside of the point P, and it follows from Lemma 3.8 that $k=1$. Hence, we can contract the curves C_1, C_2 and C_3 to an isolated singular points of type $\mathbb{D}_4$, which is a contradiction.

Suppose that $C_1 \cap C_2 \cap C_3$ consists of a point $O \in \mathrm{Sing}(H) \setminus P$. Then O is an ordinary double point of the surface H by Lemma 3.10. But $P \in C_1 \cup C_2 \cup C_3$, and $k \leqslant 2$ by Lemma 3.9, which implies that C_1, C_2 and C_3 can be contracted to a points of type $\mathbb{D}_{k+4}$, which is a contradiction.

Thus, we see that $C_1 \cap C_2 \cap C_3 = \varnothing$. Then $Q_1|_\Omega$ is smooth. Thus, we may assume that

$$C_1 \cap C_2 \neq \varnothing,\ C_2 \cap C_3 \neq \varnothing,\ C_1 \cap C_3 = \varnothing.$$

The surface H has singularity of type $\mathbb{A}_k$ at the point $P = \mathrm{Sing}(H)$, and $k \leqslant 2$ by Lemma 3.9. Moreover, the blow up $\beta_1 \colon \tilde{V}_1 \to V_1$ induces a partial resolution of singularities of the surface H. Thus, it follows from Lemma 3.8 that C_1, C_2, C_3 can be contracted to the following points:

- a point of type $\mathbb{D}_{k+3}$ in the case when $P \in C_2$ and $C_1 \not\ni P \notin C_3$,
- a point of type $\mathbb{A}_{k+3}$ in the case when $P \in C_1$ or $P \in C_3$,

which is a contradiction. The obtained contradiction completes the proof. □

Hence, we see that $r=2$ by Lemmas 3.14 and 3.16.

Lemma 3.17. *Either $m_1 = 2$, or $m_2 = 2$.*

Proof. Suppose that $m_1 = m_2 = 1$. We may assume that C_1 is a line, and C_2 is a conic, which implies that $\mathrm{Sing}(H) = P$ and $k \leqslant 2$ by Lemma 3.9. Then C_1 and C_2 can be contracted on the surface H to the following points:

- a singular point of type $\mathbb{A}_{k+2}$ in the case when $P \notin C_2 \cap C_2$,
- a singular point of type $\mathbb{A}_{k+2}$ or $\mathbb{D}_{k+2}$ in the case when $P \in C_1 \cap C_2$,

because $C_1 \cup C_2$ is not contained in a two-dimensional linear subspace of $\mathbb{P}^5$. □

We see that $r=2$, the curves C_1 and C_2 are lines. We may assume that $m_1 = 1$ and $m_2 = 2$.

Lemma 3.18. *Let O be a point in $V_1 \setminus P$. Then Ω is not a tangent subspace to V_1 at the point O.*

Proof. Suppose that Ω is a tangent linear subspace to V_1 at the point O. Then

$$\mathrm{mult}_O\big(C\big)+\mathrm{mult}_O\big(C_1\big)+2\mathrm{mult}_O\big(C_2\big)\geqslant 4,$$

which implies that $O=C\cap C_1\cap C_2$. Put $\bar{Q}=Q_1|_\Omega$ and $\bar{T}_1=T|_\Omega$. Then

$$\bar{Q}\cdot\bar{T}=C+C_1+2C_2$$

in $\Omega\cong\mathbb{P}^3$, where $\bar{Q}$ is an irreducible quadric cone, whose vertex is O.

The line C_2 is a ruling of the cone $\bar{Q}$. Then

- either the cubic $\bar{T}$ is singular along C_2,
- or the cubic $\bar{T}$ is tangent to the quadric $\bar{Q}$ along C_2.

There is a two-dimensional linear subspace $\Pi\subset\Omega$ that is tangent to $\bar{T}$ and $\bar{Q}$ along C_2, which implies that $V_1|_\Pi$ is not reduced along the line C_2. The latter contradicts the condition E. □

Arguing as in the proof of Lemma 3.18, we see that $Q_1|_\Omega$ is smooth and $\Omega\not\subset\Upsilon$.

Remark 3.19. It follows from $Q_1|_\Omega\cong\mathbb{P}^1\times\mathbb{P}^1$ that C, C_1, C_2 form the following configuration:

- the curve C intersects the curve C_1 transversally in one point,
- the curve C_1 intersects the curve C_2 transversally in one point,
- either C intersects C_2 transversally in two points, or C is tangent to C_2 in one point.

The subspace $\Omega\subset\mathbb{P}^3$ is not a tangent linear subspace to V_1 at any smooth point of $V_1\subset\mathbb{P}^5$.

Remark 3.20. The surface H is smooth outside of the set $C_2\cup P$. Moreover, we have

$$\mathrm{Sing}(H)\subsetneq P\cup\Big(C_2\setminus\Big(\big(C_2\cap C_1\big)\cup\big(C_2\cap C_1\big)\Big)\Big),$$

and H has singularity of type $\mathbb{A}_k$ at the point $P\in C_2\cup(C_1\cap C)$, where $k\leqslant 2$ by Lemma 3.9.

The equivalence $K_H\sim 0$ holds. Thus, it follows from the adjunction formula that

$$C\cdot C-\frac{k}{k+1}\mathrm{mult}_P\big(C\big)=C_1\cdot C_1-\frac{k}{k+1}\mathrm{mult}_P\big(C_1\big)=-2,$$

because C and C_1 are smooth rational curves. It follows from $C+C_1+2C_2\equiv\mathcal{O}_{\mathbb{P}^5}(1)|_H$ that

$$\big(C+C_1+2C_2\big)\cdot C_1=\big(C+C_1+2C_2\big)\cdot C_2=1.$$

Lemma 3.21. *The equality $k=2$ holds.*

Proof. Suppose that $k=1$. In the case when $P=C\cap C_1\cap C_2$, we have

$$C\cdot C_1=1/2=C_2\cdot C_1=1/2,$$

which implies that $C_1 \cdot C_1 = -1/2$. But $C_1 \cdot C_1 = -3/2$ by the adjunction formula.

Therefore, we see that $P \neq C \cap C_1 \cap C_2$. Then

- in the case when $P \in C_2$ and $P \notin C \cup C_1$, we have

$$C \cdot C_1 = 1,\ C \cdot C_2 = 2,\ C_1 \cdot C_2 = 1,\ C_1 \cdot C_1 = -2,\ C_2 \cdot C_2 = -1,$$

- in the case when $P \in C \cap C_2$ and $P \neq C \cap C_1$, we have

$$C \cdot C_1 = 1,\ C \cdot C_2 = 3/2,\ C_1 \cdot C_2 = 1,\ C_1 \cdot C_1 = -2,\ C_2 \cdot C_2 = -3/4,$$

- in the case when $P = C \cap C_1$ and $P \notin C_2$, we have

$$C \cdot C_1 = 1/2,\ C \cdot C_2 = 2,\ C_1 \cdot C_2 = 1,\ C_1 \cdot C_1 = -3/2,\ C_2 \cdot C_2 = -1,$$

which is impossible by Lemma 3.11. Thus, we see that $P = C_1 \cap C_2 \neq C \cap C_1$. Then

$$2 + C_1 \cdot C_1 = 1 + C_1 \cdot C_1 + 2C_2 \cdot C_1 = \big(C + C_1 + 2C_2\big) \cdot C_1 = 1,$$

which implies that $C_1 \cdot C_1 = -1$. But $C_1 \cdot C_1 = -3/2$ by the adjunction formula. ☐

Hence, we see that H has singularity of type $\mathbb{A}_2$ at the point $P = \mathrm{Sing}(V_1)$.

Lemma 3.22. *The case $P \notin C \cup C_1$ is impossible.*

Proof. Suppose that $P \notin C \cup C_1$. Then $P \in C_2$ and

$$C \cdot C_1 = 1,\ C \cdot C_2 = 2,\ C_1 \cdot C_2 = 1,\ C_1 \cdot C_1 = -2,\ C_2 \cdot C_2 = -1,$$

which is impossible by Lemma 3.11. ☐

Lemma 3.23. *The case $P \notin C_2$ is impossible.*

Proof. Suppose that $P \notin C_2$. Then $P = C \cap C_1$. Therefore, we have

$$C \cdot C_2 = 2,\ C_1 \cdot C_2 = 1,\ C_1 \cdot C_1 = -4/3,$$

which immediately implies that $C \cdot C_1 = 1/3$ and $C_2 \cdot C_2 = -1$. Thus, we see that the intersection form of the curves C_1 and C_2 is negative definite, which is impossible by Lemma 3.11. ☐

Therefore, we see that $P \in C_2 \cap (C_1 \cup C)$.

Lemma 3.24. *The case $C_1 \not\ni P \in C \cap C_2$ is impossible.*

Proof. Suppose that $C_1 \not\ni P \in C \cap C_2$. Then $C \cdot C = -4/3$ by the adjunction formula. But

$$C \cdot C + C_1 \cdot C + 2C_2 \cdot C = 3,$$

which implies that $C \cdot C_2 = 5/3$, because $C \cdot C_1 = 1$. Thus, we have

$$C_1 \cdot C_2 = 1,\ C_1 \cdot C_1 = -2,\ C_2 \cdot C_2 = -5/6,$$

which is impossible by Lemma 3.11. ☐

Lemma 3.25. *The case $C \not\ni P = C_1 \cap C_2$ is impossible.*

Proof. Suppose that $C \not\ni P = C_1 \cap C_2$. Then it follows from the adjunction formula that

$$C_1 \cdot C_1 = C_2 \cdot C_2 = -4/3,$$

but $C_1 \cdot C_2 = 1/3$ by Lemma 3.8, which is impossible by Lemma 3.11. $\square$

Hence, we see that $P = C_1 \cap C \cap C_2$. Then

$$C \cdot C = C_1 \cdot C_1 = C_2 \cdot C_2 = -4/3$$

by the adjunction formula. But it follows from Lemma 3.8 that $C_1 \cdot C_2 = 1/3$. Then

$$C \cdot C_1 + C_1 \cdot C_1 + 2C_2 \cdot C_1 = \deg\big(C_1\big) = 1,$$

which implies that $C \cdot C_1 = 5/3$. But $C \cdot C_1 \leqslant 2/3$, because the curves C and C_1 intersect transversally in the point P. The obtained contradiction completes the proof of Lemma 2.2.

4. Exclusion of points

Let us use the assumptions and notation of Lemma 2.6.

Lemma 4.1. *The inequality* $\mathrm{mult}_O(\mathcal{D}) \leqslant 2n$ *holds.*

Proof. See Section 5. $\square$

Thus, it follows from Theorem 1.1 in [5] that there is a sequence of blow ups

$$X_N \xrightarrow{\pi_N} X_{N-1} \xrightarrow{\pi_{N-1}} \cdots \xrightarrow{\pi_{L+1}} X_{L+1} \xrightarrow{\pi_L} X_L$$

$$\xrightarrow{\pi_{L-1}} X_{L-1} \xrightarrow{\pi_{L-2}} \cdots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} V_1$$

such that $1 \leqslant L < N$ and the following conditions are satisfied:

- the morphism π_1 is a blow up of the point $O \in V_1$,
- for $i \geqslant 2$, the morphism π_i is a blow up of a smooth subvariety $B_{i-1} \subset X_{i-1}$,
- let $E_i \subset X_i$ be an exceptional divisor of the blow up π_i, then $B_i \subset E_i$,
- for $s > i$, let E_i^s be a proper transform of E_i on the threefold V_s, then $B_s \not\subset E_i^s \subset X_s$,
- for $i \leqslant L-1$, the subvariety $B_i \subset E_i$ is a point,
- for $i \geqslant L$, the subvariety $B_i \subset E_i$ is a smooth curve such that
  - the curve $B_L \subset E_L$ is a line in $E_L \cong \mathbb{P}^2$,
  - for $i > L$, the curve $B_i \subset E_i$ is a section of the $\mathbb{P}^1$-bundle $\pi_i|_{E_i}\colon E_i \to B_{i-1}$,
- let $\mathcal{D}_i$ be the proper transform of the linear system $\mathcal{D}$ on the threefold X_i, then

$$K_{X_s} + \frac{1}{n}\mathcal{D}_s \equiv \sum_{i=1}^{L}\left(\frac{\nu_1 + \cdots + \nu_i}{n} - 2i\right)E_i^s + \sum_{i=L+1}^{s}\left(\frac{\nu_1 + \cdots + \nu_i}{n} - L - s\right)E_i^s$$

for $s > L$, where $\nu_{i+1} = \mathrm{mult}_{B_i}(\mathcal{D}_i)$ and $\nu_1 = \mathrm{mult}_O(\mathcal{D})$,

- the inequality $\sum_{i=1}^{s} \nu_i \leqslant n(L+s)$ holds for every $s > L$ such that $s \neq N$, but

$$\nu_1 + \cdots + \nu_N > n\big(L+N\big). \tag{4.2}$$

Let Λ be the three-dimensional linear subspace in $\mathbb{P}^5$ that is tangent to the threefold V_1 at the point O. Arguing as in the proof of [4, Proposition 3.3.1], we see that $P \in \Lambda$.

Let $L_1, \ldots, L_r$ be lines in V_1 that pass through the point O. Then $P \in \cup_{i=1}^{r} L_i$, and we may assume that $P \in L_1$. Let D_1 and D_2 be general surfaces in $\mathcal{D}$. Put

$$D_1 \cdot D_2 = \alpha_t L_t + C_t,$$

where $D_1 \cdot D_2$ is an effective one-cycle that corresponds to the scheme-theoretic intersection of the divisors D_1 and D_2, and C_t is an effective one-cycle on V_1 such that $L_t \not\subset \mathrm{Supp}(C_t)$. Then

$$6n^2 = -K_{V_1} \cdot \Big(\alpha_t L_t + C_t\Big) = \alpha_t + \deg\big(C_t\big) \geqslant \alpha_t.$$

Let L_t^s be a proper transform of the line L_t on the threefold X_s. Put

$$k_t = \max\Big\{ s \leqslant L \ \Big|\ B_{s-1} \in L_t^{s-1} \Big\}$$

whenever $B_1 \in L_t^1$. In the case when $B_1 \notin L_t^1$, we put $k_t = 1$. Then either $B_{k_t} \notin L_t^{k_t}$ or $k_t = L$.

Let C_t^s be a proper transform of the one-cycle C_t on the threefold X_s. Then

$$k_t \alpha_t + \mathrm{mult}_O\big(C_t\big) + \sum_{i=1}^{L-1} \mathrm{mult}_{B_i}\big(C_t^i\big) \geqslant \sum_{i=1}^{N} \nu_i^2 > \frac{\big(N+L\big)^2}{N} n^2 \tag{4.3}$$

by Theorem 7.5 in [6], because $\sum_{i=1}^{N} \nu_i > n(L+N)$. It should be pointed out that

$$\mathrm{mult}_O\big(C_t\big) \geqslant \mathrm{mult}_{B_1}\big(C_t^1\big) \geqslant \cdots \geqslant \mathrm{mult}_{B_{L-1}}\big(C_t^{L-1}\big).$$

Lemma 4.4. *The inequality $L \neq 1$ holds.*

Proof. See Section 6. □

The inequalities $3 \geqslant r \geqslant 1$ hold, because V_1 satisfies the condition H.

Lemma 4.5. *The inequality $k_1 \cdots k_r > 1$ holds.*

Proof. Suppose that $k_1 = \cdots = k_r = 1$. Then the linear system

$$\Big|\big(\pi_1 \circ \pi_2\big)^* \Big(-K_{V_1}\Big) - \pi_2^*\big(E_1\big) - E_2\Big|$$

does not have base curves. Therefore, we have

$$\begin{aligned} &6n^2 - \alpha_t - \mathrm{mult}_O\big(C_t\big) - \mathrm{mult}_{B_1}\big(C_t^1\big) \\ &= \Big(\big(\pi_1 \circ \pi_2\big)^* \Big(-K_{V_1}\Big) - \pi_2^*\big(E_1\big) - E_2\Big) \cdot C_t^2 \geqslant 0, \end{aligned}$$

but it follows from $\mathrm{mult}_O(C_t) \geqslant \mathrm{mult}_{B_1}(C_t^1) \geqslant \cdots \geqslant \mathrm{mult}_{B_{L-1}}(C_t^{L-1})$ that the inequality

$$\mathrm{mult}_O\big(C_t\big) + \sum_{i=1}^{L-1} \mathrm{mult}_{B_i}\big(C_t^i\big) \leqslant \frac{\Big(\mathrm{mult}_O\big(C_t\big) + \mathrm{mult}_{B_1}\big(C_t^1\big)\Big)L}{2}$$

holds. Thus, we have

$$3n^2 L \geqslant \alpha_t + \Big(3n^2 - \frac{\alpha_t}{2}\Big) L \geqslant \alpha_t + \mathrm{mult}_O\big(C_t\big) + \sum_{i=1}^{L-1} \mathrm{mult}_{B_i}\big(C_t^i\big)$$
$$\geqslant \sum_{i=1}^{N} \nu_i^2 > \frac{\big(N+L\big)^2}{N} n^2 > 4Ln^2,$$

which is a contradiction. □

Let H_t be a proper transform on the threefold V_{k_t} of a sufficiently general hyperplane section of the complete intersection $V_1 \subset \mathbb{P}^5$ that passes through the line L_t. Then

$$H_t \sim \big(\pi_1 \circ \pi_2 \circ \cdots \circ \pi_{k_t}\big)^* \Big(- K_{V_1}\Big) - \big(\pi_2 \circ \cdots \circ \pi_{k_t}\big)^* \big(E_1\big) - \cdots - E_{k_t}$$

and $L_t^{k_t}$ is the only curve on V_{k_t} that has negative intersection with H_t. Then

$$0 \leqslant H_t \cdot C_t^{k_t} \leqslant 6n^2 - \alpha_t - \mathrm{mult}_O\big(C_t\big) - \sum_{i=1}^{k_t-1} \mathrm{mult}_{B_i}\big(C_t^i\big),$$

and it follows from the inequality (4.3) that

$$\text{(4.6)}\quad \big(k_t - 1\big)\alpha_t + 6n^2 \frac{L}{k_t} \geqslant \big(k_t - 1\big)\alpha_t + 6n^2 \frac{L}{k_t} + \alpha_t \left(1 - \frac{L}{k_t}\right) > \frac{\big(N+L\big)^2}{N} n^2,$$

because $\mathrm{mult}_O(C_t) \geqslant \mathrm{mult}_{B_1}(C_t^1) \geqslant \cdots \geqslant \mathrm{mult}_{B_{L-1}}(C_t^{L-1})$ and $L \geqslant k_t$.

Lemma 4.7. *The inequality $k_1 \neq 1$ holds.*

Proof. See Section 7. □

Put $k = k_1$ and $\alpha = \alpha_1$ and $\mu = \mathrm{mult}_{L_1}(\mathcal{D})$.

Remark 4.8. The inequality $\mu \leqslant n$ holds by Lemma 2.5.

Let $\upsilon_k\colon \tilde{X}_k \to X_k$ be the blow up of the point dominating P, let $\omega_k\colon \bar{X}_k \to \tilde{X}_k$ be the blow up of the proper transform of L_1, let F_k and G_k be the exceptional divisor of υ_k and ω_k, respectively.

Lemma 4.9. *The isomorphisms $F_k \cong G_k \cong \mathbb{P}^1 \times \mathbb{P}^1$ hold.*

Proof. The isomorphism $F_k \cong \mathbb{P}^1 \times \mathbb{P}^1$ is obvious. There is a commutative diagram

$$\begin{array}{ccccccccc}
\bar{X}_k & \xrightarrow{\bar{\pi}_k} & \bar{X}_{k-1} & \xrightarrow{\bar{\pi}_{k-1}} & \cdots & \xrightarrow{\bar{\pi}_2} & \bar{X}_1 & \xrightarrow{\bar{\pi}_1} & \bar{V}_1 \\
\downarrow{\scriptstyle \omega_k} & & \downarrow{\scriptstyle \omega_{k-1}} & & & & \downarrow{\scriptstyle \omega_1} & & \downarrow{\scriptstyle \delta_1} \\
\tilde{X}_k & \xrightarrow{\tilde{\pi}_k} & \tilde{X}_{k-1} & \xrightarrow{\tilde{\pi}_{k-1}} & \cdots & \xrightarrow{\tilde{\pi}_2} & \tilde{X}_1 & \xrightarrow{\tilde{\pi}_1} & \tilde{V}_1 \\
\downarrow{\scriptstyle \upsilon_k} & & \downarrow{\scriptstyle \upsilon_{k-1}} & & & & \downarrow{\scriptstyle \upsilon_1} & & \downarrow{\scriptstyle \beta_1} \\
X_k & \xrightarrow{\pi_k} & X_{k-1} & \xrightarrow{\pi_{k-1}} & \cdots & \xrightarrow{\pi_2} & X_1 & \xrightarrow{\pi_1} & V_1,
\end{array}$$

where $\tilde{\pi}_i$ and $\bar{\pi}_i$ are birational morphisms, υ_i is the blow up of the point that dominates $P \in V_1$, the morphism ω_i is the blow up of the proper transform of the curve L_1, and δ_1 is the blow up of the proper transform of the line L_1 on the threefold $\tilde{V}_1$,

Let $\tilde{O}$ be the point in $\tilde{V}_1$ that dominates $O \in V_1$. Then $\tilde{\pi}_1$ is the blow up of the point $\tilde{O}$.

Let G be the exceptional divisor of δ_1. Then $G \cong \mathbb{P}^1 \times \mathbb{P}^1$, because V_1 satisfies the generality condition D. But $G_1 \cong G \cong \mathbb{P}^1 \times \mathbb{P}^1$, because $\bar{\pi}_1$ is a blow up of a smooth curve in G.

Let G_i be the exceptional divisor of ω_i. Arguing as above, we see that

$$G_k \cong G_{k-1} \cong \cdots \cong G_1 \cong G \cong \mathbb{P}^1 \times \mathbb{P}^1,$$

which completes the proof. $\square$

Let Z_1 and Z_1 be curves on G_k such that $Z_1 \cdot Z_1 = Z_2 \cdot Z_2 = 0$, $Z_1 \cdot Z_2 = 1$ and Z_2 is contracted by the morphism ω_k to a point in $\tilde{X}_k$.

Lemma 4.10. *The equivalence $-G_k|_{G_k} \sim Z_1 + (k+1)Z_2$ holds.*

Proof. There is an integer ϵ such that $-G_k|_{G_k} \sim Z_1 + \epsilon Z_2$. Then

$$2\epsilon = \Big(Z_1 + \epsilon Z_2\Big) \cdot \Big(Z_1 + \epsilon Z_2\Big) = G_k^3 = -c_1\Big(\mathcal{N}_{\tilde{L}_1^k/\tilde{X}_k}\Big) = 2 + K_{\tilde{X}_k} \cdot \tilde{L}_1^k = 2 + 2k,$$

where $\tilde{L}_1^k$ is the proper transform of the line L_1 on the threefold $\tilde{X}_k$. $\square$

Let $\bar{\mathcal{D}}_k$ be the proper transform of the linear system $\mathcal{D}$ on the threefold $\bar{X}_k$. Then

$$\begin{aligned}\bar{\mathcal{D}}_k \sim \Big(\upsilon_k \circ \omega_k\Big)^* \Big(\Big(\pi_1 \circ \cdots \circ \pi_k\Big)^* \Big(-nK_{V_1}\Big) - \Big(\pi_2 \circ \cdots \circ \pi_k\Big)^* \Big(\nu_1 E_1\Big) \\ - \cdots - \nu_k E_k\Big) - \omega_k^*\Big(\nu_0 F_k\Big) - \mu G_k,\end{aligned}$$

where ν_0 is an integer number. Therefore, it follows from Lemma 4.10 that

$$\bar{\mathcal{D}}_k\Big|_{G_k} \sim \mu Z_1 + \Big(n + \big(k+1\big)\mu - \nu_1 - \cdots - \nu_k - \nu_0\Big) Z_2,$$

which implies that $\sum_{i=1}^k \nu_i \leqslant n + \mu(k+1) - \nu_0$, because $\bar{\mathcal{D}}_k|_{G_k}$ is effective.

Lemma 4.11. *The inequality*

$$\sum_{i=1}^{s}\nu_i N/s > (N+L)n$$

holds for every $1 \leqslant s \leqslant N$.

Proof. The inequality $\sum_{i=1}^{N}\nu_i > n(N+L)$ implies that

$$\left(\sum_{i=1}^{s}\nu_i\right)\frac{N}{s} > n\big(N+L\big)$$

for every $1 \leqslant s \leqslant N$, because $\nu_1 \geqslant \nu_2 \geqslant \cdots \geqslant \nu_N$. □

Put $\theta = \sum_{i=1}^{k}\nu_i/k$. Then $n+\mu(k+1)-\nu_0 \geqslant k\theta$.

Lemma 4.12. *The inequality* $\nu_0 \geqslant \mu/2$ *holds.*

Proof. Let $\tilde{D}_k$ be the proper transform on the threefold $\tilde{X}_k$ of a general surface in $\mathcal{D}$. Then

$$\tilde{D}_k\Big|_{F_k} \equiv -\nu_0 F_k\Big|_{F_k},$$

which implies that $\tilde{D}_k|_{F_k}$ is an effective divisor of bi-degree (ν_0,ν_0) on $F_k \cong \mathbb{P}^1\times\mathbb{P}^1$.

Let $\tilde{L}_1^k$ be the proper transform of the line L_1 on the threefold $\tilde{X}_k$. Then

$$\mathrm{mult}_Q\left(\tilde{D}_k\Big|_{F_k}\right) \geqslant \mathrm{mult}_Q\big(\tilde{D}_k\big) \geqslant \mathrm{mult}_{\tilde{L}_1^k}\big(\tilde{D}_k\big) = \mu,$$

where $Q = \tilde{L}_1^k \cap F_k$. Thus, we see that $\nu_0 \geqslant \mu/2$. □

Corollary 4.13. *The inequality* $(k+1/2)\mu + n \geqslant k\theta$ *holds.*

Let $\bar{\mathcal{H}}$ be the proper transform on $\bar{X}_k$ of the linear system that is cut out on threefold $V_1 \subset \mathbb{P}^5$ by hyperplanes that pass through the line L_1. Then $\bar{\mathcal{H}}$ has no base curves and

$$\bar{\mathcal{H}} = \Big|\big(\upsilon_k\circ\omega_k\big)^*\Big(\big(\pi_1\circ\cdots\circ\pi_k\big)^*\Big(-K_{V_1}\Big) - \big(\pi_2\circ\cdots\circ\pi_k\big)^*\big(E_1\big) - \cdots - E_k\Big) - \omega_k^*\big(F_k\big) - G_k\Big|,$$

which implies that the equivalence $\bar{\mathcal{H}}|_{G_k} \sim Z_1 + Z_2$ holds by Lemma 4.12.

Lemma 4.14. *The inequality* $\alpha \leqslant 6n^2 - 2n\theta + n^2/k$ *holds.*

Proof. Let $\bar{D}_1$ and $\bar{D}_2$ be general surfaces in $\bar{\mathcal{D}}_k$, and let $\bar{H}$ be general surface in $\bar{\mathcal{H}}$. Then

$$\bar{D}_1\cdot\bar{D}_2\cdot\bar{H} = 6n^2 - \sum_{i=1}^{k}\nu_i^2 - 2\mu\Big(n - \sum_{i=1}^{k}\nu_i\Big) - \nu_0^2 - \big(\mu-\nu_0\big)^2 - \big(k+1\big)\mu^2,$$

but $\alpha \leqslant \mu^2 + \bar{D}_1\cdot\bar{D}_2\cdot\bar{H}$, because $\bar{H}|_{G_k}$ is ample. Thus, we have

$$\alpha \leqslant 6n^2 - k\theta^2 - 2\mu(n-k\theta) - k\mu^2,$$

because the inequality $\sum_{i=1}^{k}\nu_i^2 \geqslant k\theta^2$ holds. Put $\phi(\mu) = 6n^2 - k\theta^2 - 2\mu(n - k\theta) - k\mu^2$. Then

$$\phi(\mu) \leqslant \phi\left(\frac{k\theta - n}{k}\right) = 6n^2 - 2n\theta + \frac{n^2}{k},$$

which completes the proof. □

The inequality $\theta N > n(N + L)$ holds by Lemma 4.11.

Lemma 4.15. *The inequality $\theta > 5n/4$ holds.*

Proof. Suppose that $\theta \leqslant 5n/4$. Then $N > 4L$ by Lemma 4.11. It follows from the inequalities (4.3) that $\alpha > 6n^2$. On the other hand, we know that $\alpha \leqslant 6n^2$. □

Therefore, the inequalities $\alpha \leqslant 6n^2 - 2n\theta + n^2/k \leqslant (7/2 + 1/k)n^2 \leqslant 4n^2$ hold.

Lemma 4.16. *The equality $L = k$ holds.*

Proof. Suppose that $L > k \geqslant 3$. Then it follows from Lemma 4.14 that the inequality

$$\alpha \leqslant 6n^2 - 2n\theta + \frac{n^2}{k} < \frac{7n^2}{2} + \frac{n^2}{3} \leqslant \frac{23n^2}{6}$$

holds, because $\theta > 5n/4$. But $\mu \leqslant n$. Therefore, it follows from Corollary 4.13 that

$$\theta \leqslant \frac{(k + 1/2)\mu + n}{k} \leqslant 3n/2,$$

which implies that $N > 2L$ by Lemma 4.11. Then it follows from the inequalities (4.6) that

$$\left(\frac{23(k-1)}{6} + \frac{6L}{k}\right)n^2 \geqslant (k-1)\alpha + 6n^2\frac{L}{k} > \frac{(N+L)^2}{N}n^2 > \frac{9}{2}Ln^2,$$

which implies that $k \leqslant 1$. Therefore, the inequalities $L > k \geqslant 3$ are inconsistent.

To complete the proof, we may assume that $L > k = 2$. Then

$$\alpha \leqslant 6n^2 - 2n\theta + \frac{n^2}{k} < 4n^2$$

by Lemma 4.14, because $\theta > 5n/4$. On the other hand, it follows from Corollary 4.13 that

$$\theta \leqslant \frac{(k + 1/2)\mu + n}{k} \leqslant 7n/4,$$

which implies that $3N > 4L$ by Lemma 4.11. Then it follows from the inequalities (4.6) that

$$3Ln^2 + \alpha(2 - L/2) = (k-1)\alpha + 6n^2\frac{L}{k} + \alpha\left(1 - \frac{L}{k}\right) > \frac{(N+L)^2}{N}n^2 > \frac{49}{12}Ln^2,$$

which implies that $L \leqslant 2$, because $\alpha \leqslant 4n^2$. But $L > k = 2$. $\square$

Therefore, we have $L = k \geqslant 2$.

Lemma 4.17. *The inequality $\theta > 4n/3$ holds.*

Proof. Suppose that $\theta \leqslant 4n/3$. Then $N > 3L$. Now it follows from the inequalities (4.6) that

$$\big(k-1\big)\alpha + 6n^2 > \frac{\big(N+L\big)^2}{N}n^2 > \frac{16}{3}Ln^2,$$

which implies that $L < 3/2$, because $\alpha \leqslant 4n^2$ by Lemma 4.14. But $L \neq 1$ by Lemma 4.4. $\square$

It follows from Lemma 4.14 that $\alpha \leqslant 6n^2 - 2n\theta + n^2/k < (10/3 + 1/k)n^2 \leqslant 23n^2/6$. But

$$\big(k+3/2\big)n \leqslant \big(k+1/2\big)\mu + n \geqslant k\theta > 4k/3$$

by Corollary 4.13. In particular, we have $L = k \leqslant 4$.

Lemma 4.18. *The inequality $L \neq 4$ holds.*

Proof. Suppose that $L = 4$. Then $\theta \leqslant 11n/8$ by Corollary 4.13, which implies that

$$\frac{35}{2}n^2 > 3\alpha + 6n^2 > \frac{\big(N+L\big)^2}{N}n^2 > \frac{98}{5}n^2,$$

because $N \geqslant 10$ by Lemma 4.11. The obtained contradiction completes the proof. $\square$

Therefore, we proved that either $L = k = 2$, or $L = k = 3$.

Lemma 4.19. *The equality $L = 2$ holds.*

Proof. Suppose that $L = 3$. Then $\theta \leqslant n(L+3/2)/L = 3n/2$ by Corollary 4.13. But

$$\alpha \leqslant 6n^2 - 2n\theta + \frac{n^2}{3} < \frac{11}{3}n^2$$

by Lemma 4.14, because $\theta > 4n/3$. Thus, it follows from the inequalities (4.6) that

$$\frac{40}{3}n^2 > 2\alpha + 6n^2 > \frac{\big(N+L\big)^2}{N}n^2 > \frac{27}{2}n^2,$$

because $N > 2L = 6$ by Lemma 4.11. The obtained contradiction completes the proof. $\square$

Therefore, it follows from the inequalities (4.6) that

$$\frac{59}{6}n^2 > \alpha + 6n^2 > \frac{\big(N+2\big)^2}{N}n^2,$$

because $\alpha < 23n^2/6$ by Lemmas 4.14 and 4.17. Thus, the inequality $N \leqslant 5$ holds.

Lemma 4.20. *The inequality $N \neq 5$ holds.*

Proof. Suppose that $N = 5$. Then $5\theta > 7n$ by Lemma 4.11. Then

$$\alpha \leqslant 6n^2 - 2n\theta + \frac{n^2}{2} < \frac{37}{10}n^2$$

by Lemma 4.14. Thus, it follows from the inequalities (4.6) that

$$\frac{97}{10}n^2 > \alpha + 6n^2 > \frac{\big(N+L\big)^2}{N}n^2 = \frac{49}{5}n^2,$$

which is a contradiction. □

Let $\bar{B}_2$ be a proper transform of the curve B_2 on the threefold $\bar{X}_2$.

Lemma 4.21. *The inequality $G_2 \cap \bar{B}_2 \neq \varnothing$ holds.*

Proof. Suppose that $G_2 \cap \bar{B}_2 = \varnothing$. Then $L_1^2 \cap B_2$ consists of a point $Q \notin B_2$.

Recall that B_2 is a line in $E_2 \cong \mathbb{P}^2$. Let Γ be a general line in $E_2 \cong \mathbb{P}^2$ that passes through the point Q. Then $\nu_1 \geqslant \nu_2 = \mathcal{D}_2 \cdot \Gamma \geqslant \mu + \nu_3 > \mu + n$. Then $5\mu/2 + n > 2(\mu + n)$ by Corollary 4.13, which implies that $\mu > 2n$. But $\mu \leqslant n$. □

Let $\Omega \subset G_2 \cong \mathbb{P}^1 \times \mathbb{P}^1$ be a general curve in $|Z_1 + Z_2|$ that contains the point $G_2 \cap \bar{B}_2$. Then

$$4\mu + n - \nu_1 - \nu_2 - \nu_0 = \bar{\mathcal{D}}_2\Big|_{G_2} \cdot \Omega \geqslant \mathrm{mult}_{\bar{B}_2}\big(\bar{\mathcal{D}}_2\big) = \nu_3,$$

which gives $\nu_1 + \nu_2 + \nu_3 \leqslant 9n/2$, because $\mu \leqslant n$ by Lemma 2.5 and $\nu_0 \geqslant \mu/2$ by Lemma 4.12. But

$$\nu_1 + \nu_2 + \nu_3 > \frac{3\big(N+2\big)}{N}n \geqslant \frac{9}{2}n$$

by Lemma 4.11, because $N \leqslant 4$ by Lemma 4.20. The assertion of Lemma 2.6 is proved.

5. Exclusion of non-infinitely close points

We use the assumptions and notation of Lemma 4.1. Suppose that $\mathrm{mult}_O(\mathcal{D}) > 2n$, and let us show that this assumption leads to a contradiction. Let Λ be the three-dimensional linear subspace in $\mathbb{P}^5$ that is tangent to the threefold V_1 at the point O. Put $\nu = \mathrm{mult}_O(\mathcal{D})$.

Remark 5.1. The quadric $Q_1|_\Lambda$ and the cubic $T_1|_\Lambda$ are both singular at the point O.

Arguing as in the proof of [4, Proposition 3.3.1], we see that $P \in \Lambda$. Let L_1 be a line in $\mathbb{P}^5$ such that $P \in L_1 \ni O$. Then $L_1 \subset V_1 \cap \Lambda$, and Λ is not contained in the hyperplane in $\mathbb{P}^5$ that is tangent to Q_1 at the point P.

Remark 5.2. The quadric $Q_1|_\Lambda$ is irreducible and reduced, because V_1 satisfies the condition G.

Let H be a general hyperplane section of the threefold V_1 such that $\Lambda \cap V_1 \subset H$.

Lemma 5.3. *The point O is an isolated ordinary double point of the surface H.*

Proof. Arguing as in the proof of Lemma 3.10, we see that the point O is an isolated ordinary double point of the surface H, because the quadric $Q_1|_\Lambda$ is irreducible and reduced. □

Arguing as in the proof of the Lemma 3.14, we see that $V_1|_\Lambda$ is reduced.

Corollary 5.4. *The surface H is smooth outside of the points P and O.*

The surface H has singularity of type $\mathbb{A}_k$ at the point P, where $k \leqslant 2$ by Lemma 3.9. Put

$$V_1\Big|_\Lambda = L_1 + \sum_{i=1}^{r} C_i,$$

where C_i is an irreducible reduced curve such that $C_i \neq C_j \iff i \neq j$ and $C_i \neq L_1$ for all i. Then

$$\Big(\deg\big(C_i\big),\ \mathrm{mult}_O\big(C_i\big)\Big) \notin \Big\{\big(3,2\big),\ \big(2,1\big)\Big\}$$

and $\deg(C_i) = 1 \Rightarrow O \in C_i$ for all i, because $Q_1|_\Lambda$ is a quadric cone whose vertex is the point O.

Remark 5.5. The inequality $r \leqslant 3$ holds, because V_1 satisfies the condition H. Then

$$O = L_1 \cap C_1 \cap \cdots \cap C_r.$$

Let $\pi\colon \bar{V}_1 \to V_1$ be a blow up of O, let E be the exceptional of π, and let $\bar{H}$ be the proper transforms of the surface H on the threefold $\bar{V}_1$. Then $\bar{H} \cap E$ is an irreducible conic in $E \cong \mathbb{P}^2$.

Lemma 5.6. *The equality $\sum_{i=1}^{r} \mathrm{mult}_O(C_i) = 3$ holds.*

Proof. Let H' be a general hyperplane section of V_1 such that $\Lambda \cap V_1 \subset H'$. Then

$$H \cdot H' = L_1 + \sum_{i=1}^{r} C_i,$$

and the inequality $\sum_{i=1}^{r} \mathrm{mult}_O(C_i) \geqslant 3$ holds by construction.

Let $\bar{H}'$ be the proper transforms of the surface H' on the threefold $\bar{V}_1$. Then either

$$1 + \sum_{i=1}^{r} \mathrm{mult}_O\big(C_i\big) = \mathrm{mult}_O\big(L_1\big) + \sum_{i=1}^{r} \mathrm{mult}_O\big(C_i\big) = \mathrm{mult}_O\big(H\big)\mathrm{mult}_O\big(H'\big) = 4,$$

or we have $H \cap E = H' \cap E$. But in the latter case, we have

$$\mathrm{mult}_O\big(L_1\big) + \sum_{i=1}^{r} \mathrm{mult}_O\big(C_i\big) \geqslant \mathrm{mult}_O\big(H\big)\mathrm{mult}_O\big(H'\big) + 2 = 6,$$

which implies that $r = 5$. But $r \leqslant 3$. □

Let $\bar{L}_1$ and $\bar{C}_i$ be the proper transforms of L_1 and C_i on the threefold V_1, respectively.

Lemma 5.7. *The intersection form of the curves*

$$\bar{L}_1, \bar{C}_1, \ldots, \bar{C}_r$$

on the normal surface $\bar{H}$ is not semi-negative definite.

Proof. Let $\mathcal{B}$ be the proper transform of the linear system $\mathcal{D}$ on the threefold $\bar{V}_1$. Then

$$\mathcal{B}\Big|_{\bar{H}} = \epsilon L_1 + \sum_{i=1}^{r} \nu_i \bar{C}_i + \mathcal{R} \equiv n\bar{L}_1 + \sum_{i=1}^{r} n\bar{C}_i + \big(2n - \nu\big)E\Big|_{\bar{H}},$$

where ϵ and ν_i are non-negative integers, and $\mathcal{R}$ is a linear system that has no fixed curves.

The inequalities $\epsilon \leqslant n$ and $\nu_i \leqslant n$ hold for every $i \in \{1, \ldots, r\}$ by Lemmas 2.2, 2.3, 2.4, 2.5.

Suppose that the intersection form of $\bar{L}_1, \bar{C}_1, \ldots, \bar{C}_r$ is semi-negative definite. Then

$$\begin{aligned} 0 &\leqslant \Big(\big(\nu - 2n\big)E\Big|_{\bar{H}} + \mathcal{R}\Big) \cdot \Bigg(\big(n - \epsilon\big)L_1 + \sum_{i=1}^{r} \big(n - \nu_i\big)\bar{C}_i\Bigg) \\ &= \Bigg(\big(n - \epsilon\big)L_1 + \sum_{i=1}^{r} \big(n - \nu_i\big)\bar{C}_i\Bigg)^2 \leqslant 0, \end{aligned}$$

which gives $\epsilon = \nu_1 = \cdots = \nu_r = n$, because $O = L_1 \cap C_1 \cap \cdots \cap C_r$. Then $\nu = 2n$. But $\nu > 2n$. □

The equality $(\bar{L}_1 + \sum_{i=1}^{r} \bar{C}_i) \cdot \bar{L}_1 = -1$ holds on the surface $\bar{H}$, because the equivalences

$$\bar{L}_1 + \sum_{i=1}^{r} \bar{C}_i \equiv -K_{\bar{V}_1}\Big|_{\bar{H}} \equiv \Big(\pi^*\big(-K_{V_1}\big) - 2E\Big)\Big|_{\bar{H}}$$

hold on the surface $\bar{H}$. Similarly, the equality

$$\Big(\bar{L}_1 + \sum_{i=1}^{r} \bar{C}_i\Big) \cdot \bar{C}_t = \deg\big(C_t\big) - 2\mathrm{mult}_O\big(C_t\big)$$

holds for every $t \in \{1, \dots, r\}$. Therefore, it follows from Lemma 5.7 and [1] that

$$\Big(\bar{L}_1 + \sum_{i=1}^{r} \bar{C}_i\Big) \cdot \bar{C}_s = \deg\big(C_k\big) - 2\mathrm{mult}_O\big(C_s\big) > 0$$

for some $s \in \{1, \dots, r\}$. We may assume that $s = r$.

The equalities $\sum_{i=1}^{r} \deg(C_i) = 5$ and $\sum_{i=1}^{r} \mathrm{mult}_O(C_i) = 3$ hold. Then $r = \deg(C_3) = 3$ and

$$\deg\big(C_1\big) = \deg\big(C_2\big) = \mathrm{mult}_O\big(C_1\big) = \mathrm{mult}_O\big(C_2\big) = \mathrm{mult}_O\big(C_3\big) = 1,$$

which implies, in particular, that $\mathrm{mult}_P(L_1) + \sum_{i=1}^{r} \mathrm{mult}_P(C_i) = 2$.

Corollary 5.8. *The point P is an ordinary double point of the surface H.*

On the surface H, the curves $\bar{L}_1$, $\bar{C}_1$, $\bar{C}_2$, $\bar{C}_3$ can be contracted to an isolated singular point of type $\mathbb{D}_5$. So, their intersection form is negative definite by [1], which is impossible by Lemma 5.7.

The obtained contradiction completes the proof of Lemma 4.1.

6. Infinitely close points

Let us use the assumptions and notation of Lemma 4.4. Suppose that $L = 1$.

Lemma 6.1. *The inequality $N \leqslant 3$ holds.*

Proof. The linear system $|\pi_1^*(-K_{V_1}) - E_1|$ has no base points. Then

$$6n^2 - \alpha_t - \mathrm{mult}_O\big(C_t\big) = \Big(\pi_1^*\Big(-K_{V_1}\Big) - E_1\Big) \cdot C_t^1 \geqslant 0,$$

which implies that $\mathrm{mult}_O(C_t) \leqslant 6n^2 - \alpha_t$. Thus, we have

$$6n^2 \geqslant \alpha_t + \mathrm{mult}_O\big(C_t\big) \geqslant \sum_{i=1}^{N} \nu_i^2 > n^2 \frac{\big(N+1\big)^2}{N},$$

which implies that $N \leqslant 3$. □

Let $\mathcal{R}$ be a proper transform of the linear system

$$\Big|\big(\pi_1 \circ \pi_2\big)^*\Big(-K_{V_1}\Big) - \pi_2^*\big(E_1\big) - E_2\Big|$$

on V_1. There is a two-dimensional linear subspace $\Pi \subset \Lambda$ such that $\mathcal{R}$ is cut out on V_1 by hyperplanes that pass through Π, and $\mathrm{Bs}(\mathcal{R}) = \Pi \cap V_1$.

Arguing as in the proof of [4, Proposition 3.3.1], we see that $P \in \Pi$. Let Υ be a hyperplane in $\mathbb{P}^5$ that is tangent to the quadric Q_1 at the point P. Then

$$L_1 \subset \Pi \subset \Lambda \not\subset \Upsilon \subset \mathbb{P}^5,$$

because V_1 satisfies the condition G. Let H be a sufficiently general surface in $\mathcal{R}$.

Arguing as in the proof of [4, Lemma 3.5.3], we see that H is smooth outside of P.

The surface H has singularity of type $\mathbb{A}_k$ at the point P. Then $k \leqslant 2$ by Lemma 3.9, and the point P is an ordinary double point of the surface H in the case when $\Pi \not\subset \Upsilon$. But

$$\Pi \subset \Upsilon \iff Q_1 \cap \Pi = L_1.$$

Let S be a proper transform of H on the threefold X_2, and let $\pi\colon S \to H$ be a birational morphism induced by the composition $\pi_1 \circ \pi_2$. Then π is a blow up of the point O that contracts an irreducible smooth curve $E \subset S$ such that $E = E_2 \cap S$, where $E_2 \cong \mathbb{F}_2$ and $B_2 \subset E_2$.

Let C be a section of the natural projection $E_2 \to \mathbb{P}^1$ such that $C^2 = -2$, and let F be a fiber of the natural projection $E_2 \to \mathbb{P}^1$. Then $E \sim C + 2F$. It follows from [5] that $B_2 \sim C + 2F$ in the case when $N = 3$, where $E \neq B_2$ due to generality in the choice of the surface $H \in \mathcal{R}$.

Let $\bar{L}_1$ be a proper transform of L_1 on the surface S. Then

$$\bar{L}_1 \cdot \bar{L}_1 = -3 + k/(k+1).$$

Lemma 6.2. *We inequality* $\Pi \cap V_1 \neq L_1$ *holds.*

Proof. Suppose that $\Pi \cap V_1 = L_1$. Then $V_1|_\Pi = L_1$ and

$$\mathcal{D}_2\Big|_S = \mathrm{mult}_{L_1}(\mathcal{D})\bar{L}_1 + \mathcal{M} \equiv \pi^*\Big(\mathcal{O}_{\mathbb{P}^5}(n)\Big|_H\Big) - (\nu_1 + \nu_2)E,$$

where $\mathcal{M}$ is a linear system on S that has no fixed curves. Then

$$\begin{aligned} M_1 \cdot M_2 = 6n^2 &- 2\mathrm{mult}_{L_1}(\mathcal{D}) - \mathrm{mult}^2_{L_1}(\mathcal{D})\bar{L}_1 \cdot \bar{L}_1 \\ &+ 2\mathrm{mult}_{L_1}(\mathcal{D})(\nu_1 + \nu_2) - (\nu_1 + \nu_2)^2, \end{aligned}$$

where M_1 and M_2 are general curves in $\mathcal{M}$. But $M_1 \cdot M_2 \geqslant 0$ and

$$\nu_1 + \nu_2 + \cdots + \nu_N > (N+1)n,$$

which implies that $N = 3$, because $\bar{L}_1 \cdot \bar{L}_1 \leqslant -7/3$. Let Q be a point in $E \cap B_2 \neq \varnothing$. Then

$$\mathrm{mult}_{L_1}(\mathcal{D})\mathrm{mult}_Q(\bar{L}_1) + \mathrm{mult}_Q(\mathcal{M}) \geqslant \nu_3,$$

which implies that the inequality

$$M_1 \cdot M_2 \geqslant \Big(\nu_3 - \mathrm{mult}_{L_1}(\mathcal{D})\Big)^2$$

holds. Therefore, we see that

$$\begin{aligned} &6n^2 - 2\mathrm{mult}_{L_1}(\mathcal{D}) - \mathrm{mult}^2_{L_1}(\mathcal{D})\bar{L}_1 \cdot \bar{L}_1 + 2\mathrm{mult}_{L_1}(\mathcal{D})(\nu_1 + \nu_2) - (\nu_1 + \nu_2)^2 \\ &\geqslant \Big(\nu_3 - \mathrm{mult}_{L_1}(\mathcal{D})\Big)^2, \end{aligned}$$

where $\nu_1 + \nu_2 + \nu_3 > 4n$. The obtained inequalities are inconsistent. We see that $\Pi \cap V_1 \neq L_1$. □

The quadric $Q_1|_\Lambda$ is irreducible. Then $\Pi\not\subset Q_1$, because $\Pi\subset\Lambda$. So, we may assume that

$$\Pi\cap Q_1=L_1+L_2,$$

where L_2 is a line on V_1 such that $O\in L_2\ne L_1$. Then $k=1$ and $\bar{L}_1\cdot\bar{L}_1=-5/2$.

Let $\bar{L}_2$ be a proper transform of the line L_2 on the surface S. Then

$$\mathcal{D}_2\Big|_S=\mathrm{mult}_{L_1}\big(\mathcal{D}\big)\bar{L}_1+\mathrm{mult}_{L_2}\big(\mathcal{D}\big)\bar{L}_2+\mathcal{T}\equiv\pi^*\Big(\mathcal{O}_{\mathbb{P}^5}\big(n\big)\Big|_H\Big)-\big(\nu_1+\nu_2\big)E,$$

where $\mathcal{T}$ is a linear system on S that has no fixed curves. Then $\nu_1+\cdots+\nu_N>(N+1)n$ and

$$\begin{aligned}T_1\cdot T_2=6n^2-\sum_{i=1}^{2}2\mathrm{mult}_{L_i}\big(\mathcal{D}\big)-\frac{5\mathrm{mult}^2_{L_1}\big(\mathcal{D}\big)}{2}-3\mathrm{mult}^2_{L_1}\big(\mathcal{D}\big)\\+\sum_{i=1}^{2}2\big(\nu_1+\nu_2\big)\mathrm{mult}_{L_i}\big(\mathcal{D}\big)-\big(\nu_1+\nu_2\big)^2,\end{aligned}$$

where T_1 and T_2 are general curves in $\mathcal{L}$. But $T_1\cdot T_2\geqslant 0$, which implies that $N=3$.

Lemma 6.3. *The equality* $|E\cap B_2|=2$ *holds.*

Proof. The equality $|E\cap B_2|=2$ holds, because the restriction of the linear system

$$\Big|\big(\pi_1\circ\pi_2\big)^*\Big(-K_{V_1}\Big)-\pi_2^*\big(E_1\big)-E_2\Big|$$

to the surface E_2 is a pencil in $|C+2F|$, whose base locus consists of $\bar{L}_1\cap E_2$ and $\bar{L}_2\cap E_2$. □

Let Q_1 and Q_2 be two points in $E\cap B_2$ such that $Q_1\ne Q_2$. Then

$$\mathrm{mult}_{L_1}\big(\mathcal{D}\big)\mathrm{mult}_{Q_i}\big(\bar{L}_1\big)+\mathrm{mult}_{L_2}\big(\mathcal{D}\big)\mathrm{mult}_{Q_i}\big(\bar{L}_2\big)+\mathrm{mult}_{Q_i}\big(\mathcal{M}\big)\geqslant\nu_3.$$

Lemma 6.4. *Either* $Q_1\in\bar{L}_1\cup\bar{L}_2$, *or* $Q_2\in\bar{L}_1\cup\bar{L}_2$.

Proof. Suppose that $Q_1\notin\bar{L}_1\cup\bar{L}_2\not\ni Q_2$. Then $T_1\cdot T_2\geqslant 2\nu_3^2$. Therefore, we have

$$\begin{aligned}6n^2-\sum_{i=1}^{2}2\mathrm{mult}_{L_i}\big(\mathcal{D}\big)-\frac{5\mathrm{mult}^2_{L_1}\big(\mathcal{D}\big)}{2}-3\mathrm{mult}^2_{L_1}\big(\mathcal{D}\big)\\+\sum_{i=1}^{2}2\big(\nu_1+\nu_2\big)\mathrm{mult}_{L_i}\big(\mathcal{D}\big)-\big(\nu_1+\nu_2\big)^2\\\geqslant 2\nu_3^2,\end{aligned}$$

which is impossible, because $\nu_1+\nu_2+\nu_3>4n$. □

We may assume that $Q_1\in\bar{L}_1\cup\bar{L}_2$.

Lemma 6.5. *The set* $\bar{L}_1\cup\bar{L}_2$ *contains* Q_2.

Proof. Suppose that $Q_1 \not\in \bar{L}_1 \cup \bar{L}_2$. Then

$$T_1 \cdot T_2 \geqslant \nu_3^2 + \Big(\nu_3 - \mathrm{mult}_{L_1}\big(\mathcal{D}\big)\mathrm{mult}_{Q_1}\big(\bar{L}_1\big) - \mathrm{mult}_{L_2}\big(\mathcal{D}\big)\mathrm{mult}_{Q_1}\big(\bar{L}_2\big)\Big)^2,$$

which leads to a contradiction, because $\nu_1 + \nu_2 + \nu_3 > 4n$. □

We may assume that $Q_1 \in \bar{L}_1$ and $Q_2 \in \bar{L}_1$. Put

$$\mathcal{B} = \Big|\big(\pi_1 \circ \pi_2 \circ \pi_3\big)^*\Big(-K_{V_1}\Big) - \big(\pi_2 \circ \pi_3\big)^*\big(E_1\big) - \pi_3^*\big(E_2\big) - E_3\Big|$$

and $\mathcal{P} = \pi_1 \circ \pi_2 \circ \pi_3(\mathcal{B})$. Then there is a three-dimensional linear subspace $\Sigma \subset \mathbb{P}^5$ such that the system $\mathcal{P}$ is cut out on $V_1 \subset \mathbb{P}^5$ by hyperplanes in $\mathbb{P}^5$ that pass through Σ. Then $\Pi \subset \Sigma$.

Lemma 6.6. *The inequality $\Sigma \neq \Lambda$ holds.*

Proof. Suppose that $\Sigma = \Lambda$. Then

$$\pi_2 \circ \pi_3\Big(\mathcal{B}\Big) = \Big|\pi_1^*\Big(-K_{V_1}\Big) - 2E_1\Big| + E_1,$$

but $|\pi_1^*(-K_{V_1}) - 2E_1|$ does not have base curves in E_1 (see the proof of Lemma 3.10). Then

$$\mathcal{B} \not\sim \big(\pi_1 \circ \pi_2 \circ \pi_3\big)^*\Big(-K_{V_1}\Big) - \big(\pi_2 \circ \pi_3\big)^*\big(E_1\big) - \pi_3^*\big(E_2\big) - E_3,$$

which is a contradiction. □

Let B and D_3 be general surfaces in $\mathcal{B}$ and $\mathcal{D}_3$, respectively. Then

$$D_3\Big|_B = m_1\breve{L}_1 + m_2\breve{L}_2 + \Delta \equiv \Big(\big(\pi_1 \circ \pi_2 \circ \pi_3\big)^*\Big(-nK_{V_1}\Big) - \big(\nu_1 + \nu_2 + \nu_3\big)E_3\Big)\Big|_B$$

for some non-negative integers m_1 and m_2, where Δ is an effective divisor such that

$$\breve{L}_2 \not\subset \mathrm{Supp}\big(\Delta\big) \not\supset \breve{L}_2,$$

and $\breve{L}_1$ and $\breve{L}_2$ are proper transforms of the curves L_1 and L_2 on the threefold V_3, respectively.

Lemma 6.7. *The scheme $V_1|_\Sigma$ is not reduced along L_1 and is not reduced along L_2.*

Proof. Suppose that $V_1|_\Sigma$ is reduced along L_i, where $i \in \{1, 2\}$. Then $m_i = \mathrm{mult}_{L_i}(\mathcal{D})$. But

$$-3m_i \leqslant m_i\breve{L}_i \cdot \breve{L}_i \leqslant \Big(m_1\breve{L}_1 + m_2\breve{L}_2 + \Delta\Big) \cdot \breve{L}_i = n - \nu_1 - \nu_2 - \nu_3 < -3n,$$

because $\breve{L}_i \cdot \breve{L}_i \geqslant -3$. Then $\mathrm{mult}_{L_i}\big(\mathcal{D}\big) > n$, which is impossible by Lemma 2.2. □

The quadric hypersurface $Q_1|_{\Sigma} \subset \Sigma \cong \mathbb{P}^3$ must be irreducible, because V_1 satisfies the generality conditions E and F. Arguing as in the proof of Lemma 3.14, we see that $Q_1|_{\Sigma}$ is smooth. Then

$$V_1\Big|_{\Sigma} = 2L_1 + 2L_2 + Z,$$

where Z is a conic such that $O \notin \mathrm{Supp}(Z)$, because V_1 satisfies the generality condition I.

Lemma 6.8. *The curve Z is reduced.*

Proof. The curve Z is a divisor of bi-degree $(1,1)$ on $Q_1|_{\Sigma} \cong \mathbb{P}^1 \times \mathbb{P}^1$. Then Z is reduced. □

Lemma 6.9. *The surface B is smooth outside of the set $\breve{L}_1 \cup \breve{L}_1$.*

Proof. For every point $Q \in V_1 \backslash \{P, O\}$, we have $\mathrm{mult}_Q(V_1|_{\Sigma}) \leqslant 3$, which implies that Σ is not a tangent linear subspace to V_1 at the point Q. Then

$$\mathrm{Sing}\big(B\big) \subset \breve{L}_1 \cup \breve{L}_2 \cup \Big(E_3 \cap B\Big),$$

because Z is reduced. But B is smooth along $E_3 \cap B$. □

Let $\breve{P} \in X_3$ be a point such that $\pi_1 \circ \pi_2 \circ \pi_3(\breve{P}) = P$. Put $\breve{E}_3 = E_3|_B$. Then

$$\breve{L}_1 \cap \breve{E}_3 \not\in \mathrm{Sing}\big(B\big) \not\ni \breve{L}_2 \cap \breve{E}_3$$

and B has singularity at $\breve{P}$ of type $\mathbb{A}_q$. Then $q \leqslant 2$ by Lemma 3.9, because $\mathrm{mult}_P(V_1|_{\Sigma}) \leqslant 4$.

Let $\breve{Z}$ be a proper transform of the curve Z on the threefold X_3. Then

$$\begin{aligned} B\Big|_B \equiv 2\breve{L}_1 + 2\breve{L}_2 + \breve{Z} + \breve{E}_3 &\equiv \Big(\big(\pi_1 \circ \pi_2 \circ \pi_3\big)^*\Big(-K_{V_1}\Big) - 3E_3\Big)\Big|_B \\ &\equiv \big(\pi_1 \circ \pi_2 \circ \pi_3\big)^*\Big(-K_{V_1}\Big)\Big|_B + 3\breve{E}_3 \end{aligned}$$

and $\breve{L}_1 \cap \breve{Z} \not\subseteq \mathrm{Sing}(B) \not\supseteq (\breve{L}_1 \setminus \breve{P}) \cap \breve{Z}$.

Lemma 6.10. *The conic Z is reducible.*

Proof. Suppose that Z is irreducible. Put

$$\begin{aligned} \mathcal{D}_3\Big|_B &= m_1\breve{L}_1 + m_2\breve{L}_2 + \mathrm{mult}_Z\big(\mathcal{D}\big)\breve{Z} + \epsilon\breve{E}_3 + \mathcal{F} \\ &\equiv \big(\pi_1 \circ \pi_2 \circ \pi_3\big)^*\Big(-nK_{V_1}\Big)\Big|_B - \big(\nu_1 + \nu_2 + \nu_3\big)\breve{E}_3, \end{aligned}$$

where $\epsilon \geqslant 0$ is an integer, and $\mathcal{F}$ is a linear system on B that has no fixed components. Then

$$\big(2n - m_1\big)\breve{L}_1 + \big(2n - m_2\big)\breve{L}_2 + \Big(n - \mathrm{mult}_Z\big(\mathcal{D}\big)\Big)\breve{Z} \equiv \mathcal{F} + \big(\nu_1 + \nu_2 + \nu_3 + \epsilon - 4\big)\breve{E}_3,$$

on the surface B. Arguing as in the proof of Lemma 3.11, we easily see that intersection form of the curves $\breve{L}_1, \breve{L}_2, \breve{Z}$ on the surface B is not negative definite. But

$$\breve{L}_1 \cdot \breve{Z} = \breve{L}_2 \cdot \breve{Z} = 1,\ \breve{Z} \cdot \breve{Z} = \breve{L}_1 \cdot \breve{L}_1 = \breve{L}_2 \cdot \breve{L}_2 = -2,\ \breve{L}_2 \cdot \breve{L}_2 = 0$$

in the case when $P \notin Z$. Therefore, we have $P \in Z$.

The point $\breve{P}$ must be a singular point of the surface B of type $\mathbb{A}_2$, because

$$\frac{3}{2} = \breve{Z} \cdot \breve{Z} + 3 = \breve{Z} \cdot \Big(2\breve{L}_1 + 2\breve{L}_1 + \breve{Z}\Big) = 2$$

in the case when $\breve{P}$ is an ordinary double point of the surface B.

We have $\breve{Z} \cdot \breve{L}_2 = 1$ and $\breve{L}_1 \cdot \breve{L}_2 = 0$. But $\breve{Z} \cdot \breve{Z} = -4/3$ by the subadjunction formula. Then

$$2\breve{Z} \cdot \breve{L}_1 + 2/3\breve{Z} \cdot \Big(2\breve{L}_1 + 2\breve{L}_1 + \breve{Z}\Big) = 2,$$

which implies that $\breve{Z} \cdot \breve{L}_1 = 2/3$ on the surface B. Then $\breve{L}_1 \cdot \breve{L}_1 = -11/6$, because the equalities

$$2\breve{L}_1 \cdot \breve{L}_1 + 2/3 = \breve{L}_1 \cdot \Big(2\breve{L}_1 + 2\breve{L}_1 + \breve{Z}\Big) = -3$$

hold. Similarly, we easily see that $\breve{L}_2 \cdot \breve{L}_2 = -2$. Therefore, we proved that the intersection form of the curves $\breve{L}_1, \breve{L}_2, \breve{Z}$ is negative definite, which is a contradiction. □

We have $Z = Z_1 + Z_2$, where Z_1 and Z_2 are lines such that $Z_1 \cap Z_2 \neq \varnothing$. We may assume that

$$Z_1 \cap L_2 = Z_2 \cap L_2 = \varnothing$$

and $Z_1 \cap L_1 \neq \varnothing \neq Z_2 \cap L_2$. Then it follows from Lemma 3.8 that $P \notin Z_1$.

Let $\breve{Z}_1$ and $\breve{Z}_2$ be the proper transform of Z_1 and Z_2 on X_3, respectively. Then

$$\breve{Z}_1 \cdot \breve{Z}_1 = \breve{Z}_2 \cdot \breve{Z}_2 = \breve{L}_1 \cdot \breve{L}_1 = \breve{L}_2 \cdot \breve{L}_2 = -2,$$

and $\breve{Z}_1 \cdot \breve{Z}_2 = \breve{Z}_1 \cdot \breve{L}_1 = \breve{Z}_2 \cdot \breve{L}_2 = 1$ on the surface B. Therefore, we see that the intersection form of the curves $\breve{L}_1, \breve{L}_2, \breve{Z}_1, \breve{Z}_2$ on the surface B is negative definite. Arguing as in the proof of Lemma 6.10, we get a contradiction that proves Lemma 4.4.

7. Lines in smooth locus

Let us use the assumptions and notation of Lemma 4.7. Suppose that $k_1 = 1$. Then $k_1 k_2 \cdots k_r \neq 1$ by Lemma 4.5. Thus, we may assume that $k_2 \geqslant 2$. Put $k = k_2$ and $\alpha = \alpha_2$.

Remark 7.1. The isomorphism $\mathcal{N}_{L_2/V_1} \cong \mathcal{O}_{\mathbb{P}_1} \oplus \mathcal{O}_{\mathbb{P}_1}(-1)$ holds by Remark 2.1.

Let $\omega_k\colon \bar{X}_k \to X_k$ be the blow up of L_2^k, and let G_k be the exceptional divisor of ω_k.

Lemma 7.2. *The isomorphism $G_k \cong \mathbb{F}_1$ hold.*

Proof. Arguing as in the proof of Lemma 4.9, we see that $G_k \cong \mathbb{F}_1$. □

Let Z_1 and Z_1 be curves on $G_k \cong \mathbb{F}_1$ such that $Z_1 \cdot Z_1 = -1$, $Z_2 \cdot Z_2 = 0$, $Z_1 \cdot Z_2 = 1$.

Lemma 7.3. *The equivalence* $-G_k|_{G_k} \sim Z_1 + (k+1)Z_2$ *holds.*

Proof. There is an integer ϵ such that $-G_k|_{G_k} \sim Z_1 + \epsilon Z_2$. Then

$$-1+2\epsilon = \Big(Z_1+\epsilon Z_2\Big)\cdot\Big(Z_1+\epsilon Z_2\Big) = G_k^3 = -c_1\Big(\mathcal{N}_{L_2^k/X_k}\Big) = 2+K_{X_k}\cdot L_2^k = 2k+1,$$

which implies that $\epsilon = k+1$. □

Let $\bar{\mathcal{D}}_k$ be the proper transform of $\mathcal{D}$ on $\bar{X}_k$. Then

$$\bar{\mathcal{D}}_k\Big|_{G_k} \sim \mathrm{mult}_{L_2}(\mathcal{D})Z_1 + \Big(n + \big(k+1\big)\mathrm{mult}_{L_2}(\mathcal{D}) - \sum_{i=1}^{k}\nu_i\Big)Z_2$$

by Lemma 7.3. Put $\mu = \mathrm{mult}_{L_2}(\mathcal{D})$ and $\theta = \sum_{i=1}^{k}\nu_i/k$. Then $\mu \leqslant n$ by Lemma 2.5.

Corollary 7.4. *The inequality* $(k+1)\mu + n \geqslant k\theta$ *holds.*

Let $\bar{\mathcal{H}}$ be the proper transform on $\bar{X}_k$ of the linear system that is cut out on V_1 by hyperplanes that pass through L_2. Then $\bar{\mathcal{H}}|_{G_k} \sim Z_1 + 2Z_2$ by Lemma 7.3. Then $\bar{\mathcal{H}}$ has no base curves.

Lemma 7.5. *The inequality* $\alpha \leqslant 6n^2 - \mu^2(k+1) + 2\mu(k\theta - n) - k\theta^2$ *holds.*

Proof. Let $\bar{D}_1$ and $\bar{D}_2$ be general surfaces in $\bar{\mathcal{D}}_k$, and let $\bar{H}$ be general surface in $\bar{\mathcal{H}}$. Then

$$\alpha \leqslant \mu^2 + \bar{D}_1\cdot\bar{D}_2\cdot\bar{H} = 6n^2 - \sum_{i=1}^{k}\nu_i^2 - \mu^2\big(k+1\big) + 2\mu\big(k\theta - n\big),$$

because the divisor $\bar{H}|_{G_k}$ is ample, and the inequality $\sum_{i=1}^{k}\nu_i^2 \geqslant k\theta^2$ holds. □

Arguing as in the proof of Lemma 4.15, we see that $\theta > 5n/4$. Then it follows from by Lemma 7.5 that $\alpha < 29n^2/8$.

Lemma 7.6. *The equality* $k = 2$ *holds.*

Proof. Suppose that $k \geqslant 3$. Then $\theta \leqslant 5n/3$ by Lemma 7.4. Hence, we have

$$\frac{\big(N+L\big)^2}{N}n^2 > \frac{25}{6}Ln^2,$$

because $2N > 3L$ by Lemma 4.11. But it follows from Lemma 7.5 that

$$\alpha \leqslant 6n^2 - \mu^2(k+1) + 2\mu(k\theta - n) - k\theta^2 < \frac{205}{64}n^2,$$

because $k \geqslant 3$ and $\theta > 5n/4$. Then it follows from the inequalities (4.6) that

$$\left(\frac{205\big(k-1\big)}{64}+\frac{6L}{k}\right)n^2 \geqslant \big(k-1\big)\alpha + 6n^2\frac{L}{k} > \frac{\big(N+L\big)^2}{N}n^2 > \frac{25}{6}Ln^2,$$

which implies that $k \leqslant 2$, because $L \geqslant k$. The obtained contradiction completes the proof. □

We have $N > L \geqslant k = 2$. Then $L \leqslant 3$, because it follows from the inequalities (4.6) that

$$\left(\frac{29}{8}+3L\right)n^2 \geqslant \alpha + 3Ln^2 > \frac{\big(N+L\big)^2}{N}n^2 > 4Ln^2.$$

Lemma 7.7. *The equality $L = 2$ holds.*

Proof. Suppose that $L = 3$. Then it follows from the inequalities (4.6) that

$$\frac{101}{8}n^2 \geqslant \alpha + 9n^2 > \frac{\big(N+3\big)^2}{N}n^2,$$

which implies that $N = 4$, because $N > L = 3$. Then $\theta > 7n/4$ by Lemma 4.11, and

$$\alpha \leqslant 6n^2 - \mu^2(k+1) + 2\mu(k\theta - n) - k\theta^2 < \frac{49}{25}n^2,$$

by Lemma 7.5. Then it follows from the inequalities (4.6) that

$$\frac{274}{25}n^2 \geqslant \alpha + 9n^2 > \frac{\big(N+3\big)^2}{N}n^2 = \frac{49}{4}n^2,$$

which is a contradiction. □

It follows from the inequalities (4.6) that $77/8 > (N+2)^2/N$. Then $N \leqslant 4$.

Lemma 7.8. *The equality $N = 3$ holds.*

Proof. Suppose that $N = 4$. Then $\theta > 3n/2$ by Lemma 4.11, which implies that

$$\alpha \leqslant 6n^2 - \mu^2(k+1) + 2\mu(k\theta - n) - k\theta^2 < \frac{71}{25}n^2,$$

by Lemma 7.5. Then it follows from the inequalities (4.6) that

$$\frac{221}{25}n^2 \geqslant \alpha + 6n^2 > \frac{\big(N+2\big)^2}{N}n^2 = 9n^2,$$

which is a contradiction. □

Therefore, the inequality $\theta > 5n/3$ holds by Lemma 4.11. Then

$$\alpha \leqslant 6n^2 - \mu^2(k+1) + 2\mu(k\theta - n) - k\theta^2 < \frac{113}{50}n^2,$$

by Lemma 7.5. Then it follows from the inequalities (4.6) that

$$\frac{413}{50}n^2 \geqslant \alpha + 6n^2 > \frac{(N+2)^2}{N}n^2 = \frac{25}{3}n^2,$$

which is a contradiction. The assertion of Lemma 4.7 is proved.

8. Generality conditions

Let us use the assumption and notation of Section 1, and let us prove Theorem 1.13. Put

- $h(y_1, y_2, y_3, y_4) = \alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3 + \alpha_4 y_4$,
- $q(y_0, y_1, y_2, y_3, y_4) = Ay_0^2 + y_0 \sum_{i=1}^4 \beta_i y_i + \sum_{1\leqslant i\leqslant j\leqslant 4} \gamma_{ij} y_i y_j$,
- $t(y_0, y_1, y_2, y_3, y_4) = \sum_{1\leqslant i\leqslant j\leqslant k\leqslant 4} \delta_{ijk} y_i y_j y_k + y_0^2 \sum_{i=1}^4 \epsilon_i y_i$,

where α_i, β_i, γ_{ij}, A, δ_{ijk}, ϵ_i are complex numbers. Let $Q \subset \mathbb{P}^5$ be a quadric in that is given by

$$y_5 \sum \alpha_i y_i = Ay_0^2 + y_0 \sum \beta_i y_i + \sum \gamma_{ij} y_i y_j$$

in $\mathrm{Proj}\Big(\mathbb{C}[y_0, y_1, y_2, y_3, y_4, y_5]\Big) \cong \mathbb{P}^5$, let $T \subset \mathbb{P}^5$ be a cubic hypersurface that is given by

$$y_5\left(Ay_0^2 - y_0 \sum \beta_i y_i + \sum \gamma_{ij} y_i y_j\right) = \sum \delta_{ijk} y_i y_j y_k + y_0^2 \sum \epsilon_i y_i,$$

and let $P \in \mathbb{P}^5$ be the point $\{y_0 = \cdots = y_4 = 0\}$. Put $V = Q \cap T$. Suppose that

- the threefold V satisfy the conditions A, B, C, D,
- the inequality $A \neq 0$ holds.

Remark 8.1. To prove Theorem 1.13, we must show that the threefold V satisfies the generality conditions E, F, G, H, I in the case when the polynomials h, q and t are sufficiently general.

Let $F \subset \mathbb{P}^5$ be a hyperplane $\{y_0 = 0\}$, let $\iota \in \mathrm{Aut}(\mathbb{P}^5)$ be an involution that is given by

$$y_0 \to -y_0,\ y_1 \to y_1,\ y_2 \to y_2,\ y_3 \to y_3,\ y_4 \to y_4\ y_5 \to y_5,$$

and let $\zeta \in \mathbb{P}^5$ be the point $\{y_1 = \cdots = y_5 = 0\}$. Then ι fixes F and ζ.

Remark 8.2. It follows from $A \neq 0$ that $\zeta \notin V$.

Let L be a line that pass through P and ζ. Then $L \not\subset V$, and L is given by the equations

$$y_1 = y_2 = y_3 = y_4 = 0.$$

Lemma 8.3. *Let L_1 be a line in V such that $P \notin L_1$. Then $L \cap L_1 = \varnothing$.*

Proof. Suppose that $L \cap L_1 \neq \varnothing$. Let $\Pi \subset \mathbb{P}^5$ be a two-dimensional linear subspace that contains both lines L_1 and L. We may assume that $L_1 \cap F = \{y_0 = y_2 = \cdots = y_5 = 0\}$. Then Π is given by $y_2 = y_3 = y_4 = 0$, which implies that $\Pi \not\subset Q$, because $A \neq 0$. Then the conic $Q|_{\Pi}$ is given by

$$Ay_0^2 + \beta_1 y_0 y_1 + \gamma_{11} y_1^2 - \alpha_1 y_1 y_5 = y_2 = y_3 = y_4 = 0,$$

but $L_1 \subset \mathrm{Supp}(Q|_{\Pi})$. Therefore, we have $\alpha_1 = 0$, because the homogeneous polynomial

$$Ay_0^2 + \beta_1 y_0 y_1 + \gamma_{11} y_1^2 - \alpha_1 y_1 y_5$$

must be a product of two linear forms. Then $P \in L_1$, which is a contradiction. □

Now we suppose that the polynomials h, q and t are sufficiently general.

Lemma 8.4. *Let L_1 be a line in V such that $P \notin L_1$. Then $L_1 \not\subset F$.*

Proof. Let $\phi\colon \mathbb{P}^5 \dashrightarrow \mathbb{P}^4$ be a projection from the point P. Put $X = \phi(V)$. Then X is given by

$$h(y_1 \ldots, y_4) t(y_0, \ldots, y_4) = q(y_0, \ldots, y_4) q(-y_0, \ldots, y_4)$$

in $\mathrm{Proj}\Big(\mathbb{C}[y_0, \ldots, y_4]\Big) \cong \mathbb{P}^4$.

Put $\bar{L}_1 = \phi(L_1)$ and $\bar{F} = \phi(F)$. Suppose that $\bar{L}_1 \subset \bar{F}$. Then $X|_{\bar{F}}$ is given by

$$\Big(\sum \alpha_i y_i\Big)\Big(\sum \delta_{ijk} y_i y_j y_k\Big) = \Big(\sum \gamma_{ij} y_i y_j\Big)^2 \tag{8.5}$$

in $\mathrm{Proj}\Big(\mathbb{C}[y_1, y_2, y_3, y_4]\Big) \cong \mathbb{P}^3$.

We may assume $\bar{L}_1$ is given by $y_0 = y_3 = y_4 = 0$. Then it follows from $\bar{L}_1 \subset X$ that

$$\begin{cases} \alpha_1\delta_{111} + \gamma_{11}^2 = 0, \\ \alpha_1\delta_{112} + \alpha_2\delta_{111} + 2\gamma_{11}\gamma_{12} = 0, \\ \alpha_1\delta_{122} + \alpha_2\delta_{112} + 2\gamma_{11}\gamma_{22} + \gamma_{12}^2 = 0, \\ \alpha_1\delta_{222} + \alpha_2\delta_{122} + 2\gamma_{12}\gamma_{22} = 0, \\ \alpha_2\delta_{222} + \gamma_{22}^2 = 0. \end{cases} \tag{8.6}$$

Let $\mathcal{X}$ be the set of all quartic hypersurfaces in $\mathbb{P}^3$ that are given by the equations (8.5). Put

$$\mathcal{I} = \Big\{(\Gamma,\ \bar{X})\ \Big|\ \Gamma \subset \bar{X}\Big\} \subset Gr(2,4) \times \mathcal{X},$$

and let $\omega\colon \mathcal{I} \to \mathcal{X}$ be the natural projections. Then it follows from the equations (8.6) that

$$\dim(\mathcal{I}) = \dim(\mathcal{X}) - 5 + \dim\Big(Gr(2,4)\Big) = \dim(\mathcal{X}) - 1 < \dim(\mathcal{X}),$$

which implies that ω is not surjective. But the polynomials h, q and t are chosen to be sufficiently general by assumption, which implies that $\bar{L}_1 \subset \bar{F}$. Therefore, we see that $L_1 \not\subset F$. $\square$

Let L_1 be a line on V such that $P \notin L_1$, and let $[x_0 : \cdots : x_5]$ be coordinates on $\mathbb{P}^5$ such that

- the hyperplane $F \subset \mathbb{P}^5$ is given by $x_0 = 0$,
- the line $L_1 \subset V$ is given by $x_2 = x_3 = x_4 = x_5 = 0$,
- the point P is given by $x_0 = x_1 = x_2 = x_3 = x_4 = 0$,

and $\zeta = (1 : -a_1 : -a_2 : -a_3 : -a_4 : -a_5)$. Then

$$\begin{cases} y_0 = x_0, \\ y_1 = x_1 + a_1 x_0, \\ y_2 = x_2 + a_2 x_0, \\ y_3 = x_3 + a_3 x_0, \\ y_4 = x_4 + a_4 x_0, \\ y_5 = x_5 + b x_1 + (a_5 + b a_1) x_0, \end{cases} \tag{8.7}$$

where b and a_i are complex numbers. Then the reflection ι acts as

$$x_0 \to -x_0,\ x_1 \to x_1 + 2a_1 x_0,\ \ldots,\ x_5 \to x_5 + 2a_5 x_0.$$

Let $\Pi \subset \mathbb{P}^5$ be a two-dimensional linear subspace such that $L_1 \subset \Pi$.

Lemma 8.8. *The scheme $V|_\Pi$ is reduced along L_1.*

Proof. It follows from Lemmas 8.3 and 8.4 that $L \cap L_1 = \varnothing$ and $L_1 \not\subset F$. Let us consider an open subset of a variety of $(1, 2)$–flags

$$\mathcal{T} = \Big\{(\Gamma,\ \Sigma)\ \Big|\ \Gamma \subset \Sigma,\ \Gamma \not\subset F,\ \Gamma \cap L = \varnothing\Big\},$$

and a closed subset $\mathcal{S} = \{(\Gamma,\ \Sigma) \mid \Sigma \subset \langle \Gamma, L\rangle\} \subset \mathcal{T}$, where $\langle \Gamma, L\rangle$ is a three-dimensional linear subspace in $\mathbb{P}^5$ that contains the lines Γ and L. Then $\dim(\mathcal{T}) = 11$ and $\dim(\mathcal{S}) = 6$.

Choose the coordinates $[x_0 : \cdots : x_5]$ such that Σ is given by $x_3 = x_4 = x_5 = 0$.

Suppose that $V|_\Sigma$ is not reduced along Γ. Then the scheme

$$x_3 = x_4 = x_5 = Ay_0^2 + y_0 \sum \beta_i y_i + \sum \gamma_{ij} y_i y_j - y_5 \sum \alpha_i y_i = 0$$

is not reduced along the line L_1, and the scheme

$$\begin{aligned} x_3 &= x_4 = x_5 \\ &= y_5 \Big(Ay_0^2 - y_0 \sum \beta_i y_i + \sum \gamma_{ij} y_i y_j\Big) - \sum \delta_{ijk} y_i y_j y_k - y_0^2 \sum \epsilon_i y_i = 0 \end{aligned}$$

is not reduced along L_1, where $y_0, y_1, \ldots, y_5$ are given by the equations 8.7.

Suppose that $(L_1, \Pi) \notin \mathcal{S}$. Then $a_3 \neq 0$ or $a_4 \neq 0$, because $\Pi \not\subset \langle L_1, L\rangle$, which implies that the non-reducedness of the scheme $V|_\Pi$ along the line L_1 imposes 12 independent linear conditions on the coefficients α_i, β_i, γ_{ij}, A, δ_{ijk}, ϵ_i.

Let $\mathcal{R}$ be a family of threefolds that are constructed like the threefold V. Put

$$\mathcal{I} = \Big\{ \Big((\Gamma,\ \Sigma), Y \Big) \ \Big|\ \Gamma \subset Y,\ P \not\in \Gamma,\ Y\Big|_{\Sigma} \text{ is not reduced along } \Gamma \Big\} \subset \mathcal{T} \setminus \mathcal{S} \times \mathcal{R},$$

and let $\alpha\colon \mathcal{I} \to \mathcal{R}$ be the natural projection. Then

$$\dim\big(\mathcal{I}\big) = \dim\Big(\mathcal{T} \setminus \mathcal{S}\Big) + \dim\big(\mathcal{R}\big) - 12 = \dim\big(\mathcal{R}\big) - 1,$$

which implies that α is not surjective. Thus, the scheme $V|_{\Pi}$ is reduced along L_1 if $\Pi \not\subset \langle L_1, L_2 \rangle$.

We see that $(L_1, \Pi) \in \mathcal{S}$. Then $a_3 = a_4 = 0$, but $a_2 \neq 0$, because $L_1 \cap L = \varnothing$, which implies that the non-reducedness of the scheme $V|_{\Pi}$ along L_1 imposes at least 9 independent linear conditions on α_i, β_i, γ_{ij}, A, δ_{ijk}, ϵ_i. But $\dim(\mathcal{S}) = 6$, which is a contradiction. ☐

Arguing as in the proof of Lemma 8.8, we see that V satisfies the conditions E, F, G, H, I.

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