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Double cubics and double quartics

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Abstract We study a double cover $\psi : X \to V \subset \mathbb{P}^n$ branched over a smooth divisor $R \subset V$ such that R is cut on V by a hypersurface of degree $2(n - \deg(V))$, where $n \geqslant 8$ and V is a smooth hypersurface of degree 3 or 4. We prove that X is nonrational and birationally superrigid.

1 Introduction

Let $\psi : X \to V \subset \mathbb{P}^n$ be a double cover branched over a smooth divisor $R \subset V$, where $n \geqslant 4$ and V is a smooth hypersurface[1]. Then $\mathrm{rk}\,\mathrm{Pic}(X) = 1$ (see [4]) and

$$-K_X \sim \psi^*(\mathcal{O}_{\mathbb{P}^n}(d + r - 1 - n)|_V),$$

where $d = \deg V$ and r is a natural number such that $R \sim \mathcal{O}_{\mathbb{P}^n}(2r)|_V$. Therefore X is nonrational in the case when $d + r \geqslant n + 1$. The variety X is rationally connected if $d + r \leqslant n$, because it is a smooth Fano variety (see [8]). Moreover, the following result is due to [11].

Theorem 1 *The variety X is birationally superrigid*[2] *if it is general and $d + r = n \geqslant 5$.*

In this paper we prove the following result.

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[1] All varieties are assumed to be projective, normal, and defined over $\mathbb{C}$.

[2] Namely, we have $\mathrm{Bir}(X) = \mathrm{Aut}(X)$, and X is not birational to the following varieties: a variety Y such that there is a morphism $\tau : Y \to Z$ whose general fiber has negative Kodaira dimension and $\dim(Y) \neq \dim(Z) \neq 0$; a Fano variety of Picard rank 1 having terminal $\mathbb{Q}$-factorial singularities that is not biregular to X.

Theorem 2 *The variety X is birationally superrigid if $d + r = n \geqslant 8$ and $d = 3$ or 4.*

One can use Theorem 2 to construct explicit examples of nonrational Fano varieties.

Example 3 The complete intersection

$$\sum_{i=0}^{8} x_i^4 = z^2 - x_0^4 x_1^4 + x_2^4 x_3^4 + x_4^4 x_5^4 + x_6^4 x_7^4$$
$$= 0 \subset \mathbb{P}(1^9, 3) \cong \mathrm{Proj}(\mathbb{C}[x_0, \dots, x_8, z])$$

is smooth. Hence, it is birationally superrigid and nonrational by Theorem 2.

In the case when $d + r = n \geqslant 4$ and $d = 1$ or 2 the birational superrigidity of X is proved in [5] and [10]. In the case when $d + r = n = 4$ and $d = 3$ the variety X is not birationally superrigid, but it is nonrational (see [6], [3]). In the case when $d + r < n$ the only known way to prove the nonrationality of X is the method of §V in [8], which implies the following result.

Proposition 4 *The variety X is nonrational if it is very general, $n \geqslant 4$ and $r \geqslant \frac{d+n+2}{2}$.*

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2 Preliminaries

Let X be a variety and $B_X = \sum_{i=1}^{\epsilon} a_i B_i$ be a boundary on X, where $a_i \in \mathbb{Q}$ and B_i is either a prime divisor on X or a linear system on X having no base components. We say that B_X is effective if every $a_i \geqslant 0$, we say that B_X is movable if every B_i is a linear system having no fixed components[3]. In the rest of the section we assume that all varieties are $\mathbb{Q}$-factorial.

Remark 5 We can consider B_X^2 as an effective codimension-two cycle if B_X is movable.

The notions such as discrepancies, terminality, canonicity, log terminality and log canonicity can be defined for the log pair (X, B_X) as for usual log pairs (see [7]).

Definition 6 The log pair (X, B_X) has canonical (terminal, respectively) singularities if for every birational morphism $f : W \to X$ there is an equivalence

$$K_W + B_W \sim_{\mathbb{Q}} f^*(K_X + B_X) + \sum_{i=1}^{n} a(X, B_X, E_i) E_i$$

such that every number $a(X, B_X, E_i)$ is non-negative (positive, respectively), where B_W is a proper transform of B_X on W, and E_i is an f-exceptional divisor. The number $a(X, B_X, E_i)$ is called the discrepancy of the log pair (X, B_X) in the divisor E_i.

[3] Every effective movable log pair can be considered as a usual log pair (see [7]).

The application of Log Minimal Model Program (see [7]) to an effective movable log pair having canonical or terminal singularities preserves its canonicity or terminality respectively.

Definition 7 An irreducible subvariety $Y \subset X$ is a center of canonical singularities of the log pair (X, B_X) if there is a birational morphism $f : W \to X$ and an f-exceptional divisor E such that $f(E) = Y$ and the inequality $a(X, B_X, E) \leqslant 0$ holds. The set of all centers of canonical singularities of the log pair (X, B_X) is denoted as $\mathbb{CS}(X, B_X)$.

In particular, the log pair (X, B_X) has terminal singularities if and only if $\mathbb{CS}(X, B_X) = \varnothing$.

Remark 8 Let H be a general hyperplane section of X. Then every component of $Z \cap H$ is contained in the set $\mathbb{CS}(H, B_X|_H)$ for every subvariety $Z \subset X$ contained in $\mathbb{CS}(X, B_X)$.

Remark 9 Let $Z \subset X$ be a proper irreducible subvariety such that X is smooth at the generic point of Z. Suppose that B_X is effective. Then $Z \in \mathbb{CS}(X, B_X)$ implies $\mathrm{mult}_Z(B_X) \geqslant 1$, but in the case $\mathrm{codim}(Z \subset X) = 2$ the inequality $\mathrm{mult}_Z(B_X) \geqslant 1$ implies $Z \in \mathbb{CS}(X, B_X)$.

The following result is Lemma 3.18 in [1].

Lemma 10 *Suppose that X is a smooth complete intersection $\cap_{i=1}^k G_i \subset \mathbb{P}^n$, and B_X is effective such that $B_X \sim_{\mathbb{Q}} rH$ for some $r \in \mathbb{Q}$, where G_i is a hypersurface in $\mathbb{P}^n$, and H is a hyperplane section of X. Then* $\mathrm{mult}_Z(B_X) \leqslant r$ *for every irreducible subvariety $Z \subset X$ such that* $\dim(Z) \geqslant k$.

The following result is well known (see [2], [3]).

Theorem 11 *Let X be a Fano variety of Picard rank 1 having terminal $\mathbb{Q}$-factorial singularities that is not birationally superrigid. Then there is a linear system $\mathcal{M}$ on the variety X whose base locus has codimension at least 2 such that the singularities of the log pair $(X, \mu\mathcal{M})$ are not canonical, where μ is a positive rational number such that $K_X + \mu\mathcal{M} \sim_{\mathbb{Q}} 0$.*

Let $f : V \to X$ be a birational morphism such that the union of $\cup_{i=1}^{\epsilon} f^{-1}(B_i)$ and all f-exceptional divisors forms a divisor with simple normal crossing. Then f is called a log resolution of the log pair (X, B_X), and the log pair (V, B^V) is called the log pull back of (X, B_X) if

$$B^V = f^{-1}(B_X) - \sum_{i=1}^{n} a(X, B_X, E_i)E_i$$

such that $K_V + B^V \sim_{\mathbb{Q}} f^*(K_X + B_X)$, where E_i is an f-exceptional divisor and $a(X, B_X, E_i) \in \mathbb{Q}$.

Definition 12 The log canonical singularity subscheme $\mathcal{L}(X, B_X)$ is the subscheme associated to the ideal sheaf $\mathcal{I}(X, B_X) = f_*(\mathcal{O}_V(\lceil -B^V \rceil))$. A proper irreducible subvariety $Y \subset X$ is called a center of log canonical singularities of the log pair (X, B_X) if there is a divisor $E \subset V$ that is contained in the effective part of the support of $\lfloor B^V \rfloor$ and $f(E) = Y$. The set of all centers of log canonical singularities of (X, B_X) is denoted as $\mathbb{LCS}(X, B_X)$, the set-theoretic union of the elements of $\mathbb{LCS}(X, B_X)$ is denoted as $LCS(X, B_X)$.

In particular, we have $\mathrm{Supp}(\mathcal{L}(X, B_X)) = LCS(X, B_X)$.

Remark 13 Let H be a general hyperplane section of X and $Z \in \mathbb{LCS}(X, B_X)$. Then every component of the intersection $Z \cap H$ is contained in the set $\mathbb{LCS}(H, B_X|_H)$.

The following result is Theorem 17.4 in [9].

Theorem 14 *Let $g : X \to Z$ be a morphism. Then $LCS(X, B_X)$ is connected in a neighborhood of every fiber of the morphism $g \circ f$ if the following conditions hold:*

- *the morphism g has connected fibers;*
- *the divisor $-(K_X + B_X)$ is g-nef and g-big;*
- *the inequality $\mathrm{codim}(g(B_i) \subset Z) \geqslant 2$ holds if $a_i < 0$;*

The following corollary of Theorem 14 is Theorem 17.6 in [9].

Theorem 15 *Let Z be an element of the set $\mathbb{CS}(X, B_X)$, and H be an effective Cartier divisor on the variety X. Suppose that the boundary B_X is effective, the varieties X and H are smooth in the generic point of Z and $Z \subset H \not\subset \mathrm{Supp}(B_X)$. Then $\mathbb{LCS}(H, B_X|_H) \neq \varnothing$.*

The following result is Theorem 3.1 in [3].

Theorem 16 *Suppose that $\dim(X) = 2$, the boundary B_X is effective and movable, and there is a smooth point $O \in X$ such that $O \in \mathbb{LCS}(X, (1 - a_1)\Delta_1 + (1 - a_2)\Delta_2 + M_X)$, where Δ_1 and Δ_2 are smooth curves on X intersecting normally at O, and a_1 and a_2 are arbitrary non-negative rational numbers. Then we have*

$$\mathrm{mult}_O(B_X^2) \geqslant \begin{cases} 4a_1a_2 \text{ if } a_1 \leqslant 1 \text{ or } a_2 \leqslant 1 \\ 4(a_1 + a_2 - 1) \text{ if } a_1 > 1 \text{ and } a_2 > 1. \end{cases}$$

3 Main local inequality

Let X be a variety, O be a smooth point on X, $f : V \to X$ be a blow up of the point O, E be an exceptional divisor of f, $B_X = \sum_{i=1}^{\epsilon} a_i \mathcal{B}_i$ be a movable boundary on X, and $B_V = f^{-1}(B_X)$, where a_i is a non-negative rational number and $\mathcal{B}_i$ is a linear system on X having no base components. Suppose that $O \in \mathbb{CS}(X, B_X)$, but the singularities of (X, B_X) are log terminal in some punctured neighborhood of the point O. The following result is Corollary 3.5 in [3].

Lemma 17 *Suppose that* $\dim(X) = 3$ *and* $\mathrm{mult}_O(B_X) < 2$. *Then there is a line* $L \subset E \cong \mathbb{P}^2$ *such that* $L \in \mathbb{LCS}(V, B_V + (\mathrm{mult}_O(B_X) - 1)E)$.

Suppose that $\dim(X) = 4$ and $\mathrm{mult}_O(B_X) < 3$. Then the proof of Lemma 17 and Theorem 14 implies the following result.

Proposition 18 *One of the following possibilities holds:*

- *there is a surface* $S \subset E$ *such that* $S \in \mathbb{LCS}(V, B_V + (\mathrm{mult}_O(B_X) - 2)E)$;
- *there is a line* $L \subset E \cong \mathbb{P}^3$ *such that* $L \in \mathbb{LCS}(V, B_V + (\mathrm{mult}_O(B_X) - 2)E)$.

Now suppose that the set $\mathbb{LCS}(V, B_V + (\mathrm{mult}_O(B_X) - 2)E)$ does not contain surfaces that are contained in the divisor E and contains a line $L \subset E \cong \mathbb{P}^3$. Let $g : W \to V$ be a blow up of in L, $F = g^{-1}(L)$, $\bar{E} = g^{-1}(E)$, and $B_W = g^{-1}(B_V)$. Then

$$B^W = B_W + (\mathrm{mult}_O(B_X) - 3)\bar{E} + (\mathrm{mult}_O(B_X) + \mathrm{mult}_L(B_V) - 5)F.$$

Proposition 19 *One of the following possibilities holds:*

- *the divisor* F *is contained in* $\mathbb{LCS}(W, B^W + \bar{E} + 2F)$;
- *there is a surface* $Z \subset F$ *such that* $Z \in \mathbb{LCS}(W, B^W + \bar{E} + 2F)$ *and* $g(Z) = L$.

The following result is implied by Proposition 19.

Theorem 20 *Let* Y *be a variety,* $\dim(Y) = 4$, $\mathcal{M}$ *be a linear system on the variety* Y *having no base components,* S_1 *and* S_2 *be sufficiently general divisors in* $\mathcal{M}$, P *be a smooth point on the variety* Y *such that* $P \in \mathbb{CS}(Y, \frac{1}{n}\mathcal{M})$ *for* $n \in \mathbb{N}$, *but the singularities of* $(Y, \frac{1}{n}\mathcal{M})$ *are canonical in some punctured neighborhood of the point* P, $\pi : \hat{Y} \to Y$ *be a blow up of* P, *and* Π *be an exceptional divisor of* π. *Then there is a line* $C \subset \Pi \cong \mathbb{P}^3$ *such that the inequality*

$$\mathrm{mult}_P(S_1 \cdot S_2 \cdot \Delta) \geqslant 8n^2$$

holds for any divisor Δ *on* Y *such that the following conditions hold:*

- *the divisor* Δ *contains the point* P *and* Δ *is smooth at* P;
- *the line* $C \subset \Pi \cong \mathbb{P}^3$ *is contained in the divisor* $\pi^{-1}(\Delta)$;
- *the divisor* Δ *does not contain subvarieties of dimension* 2 *contained in* $\mathrm{Bs}(\mathcal{M})$.

Proof Let Δ be a divisor on Y such that $P \in \Delta$, the divisor Δ is smooth at P, and Δ does not contain any surface that is contained in the base locus of $\mathcal{M}$. Then the base locus of the linear system $\mathcal{M}|_\Delta$ has codimension 2 in Δ. In particular, the intersection $S_1 \cdot S_2 \cdot \Delta$ is an effective one-cycle. Let $\bar{S}_1 = S_1|_\Delta$ and $\bar{S}_2 = S_2|_\Delta$. Then we must prove that the inequality

$$\mathrm{mult}_P(\bar{S}_1 \cdot \bar{S}_2) \geqslant 8n^2 \tag{21}$$

holds, perhaps, under certain additional conditions on Δ. Put $\bar{\mathcal{M}} = \mathcal{M}|_\Delta$. Then

$$P \in \mathbb{LCS}\left(\Delta, \frac{1}{n}\bar{\mathcal{M}}\right)$$

by Theorem 15. Let $\bar{\pi} : \hat{\Delta} \to \Delta$ be a blow up of P and $\bar{\Pi} = \bar{\pi}^{-1}(P)$. Then the diagram

$$\begin{array}{ccc} \hat{\Delta} & \hookrightarrow & \hat{Y} \\ {\scriptstyle \bar{\pi}}\downarrow & & \downarrow{\scriptstyle \pi} \\ \Delta & \hookrightarrow & Y \end{array} \tag{22}$$

is commutative, where $\hat{\Delta}$ is identified with $\pi^{-1}(\Delta) \subset \hat{Y}$. We have $\bar{\Pi} = \Pi \cap \hat{\Delta}$.

Let $\hat{\mathcal{M}} = \bar{\pi}^{-1}(\bar{\mathcal{M}})$. The inequality 21 is obvious if $\mathrm{mult}_P(\bar{\mathcal{M}}) \geqslant 3n$. Hence we may assume that $\mathrm{mult}_P(\bar{\mathcal{M}}) < 3n$. Then

$$\bar{\Pi} \notin \mathbb{LCS}(\hat{\Delta}, \frac{1}{n}\hat{\mathcal{M}} + \left(\frac{1}{n}\mathrm{mult}_P(\bar{\mathcal{M}}) - 2\right)\bar{\Pi}),$$

which implies the existence of a subvariety $\Xi \subset \bar{\Pi} \cong \mathbb{P}^2$ such that Ξ is a center of log canonical singularities of $(\hat{\Delta}, \frac{1}{n}\hat{\mathcal{M}} + (\frac{1}{n}\mathrm{mult}_P(\bar{\mathcal{M}}) - 2)\bar{\Pi})$.

Suppose that Ξ is a curve. Put $\hat{S}_i = \bar{\pi}^{-1}(S_i)$. Then

$$\mathrm{mult}_P(\bar{S}_1 \cdot \bar{S}_2) \geqslant \mathrm{mult}_P(\bar{\mathcal{M}})^2 + \mathrm{mult}_\Xi(\hat{S}_1 \cdot \hat{S}_2),$$

but we can apply Theorem 16 to the log pair $(\hat{\Delta}, \frac{1}{n}\hat{\mathcal{M}} + (\frac{1}{n}\mathrm{mult}_P(\bar{\mathcal{M}}) - 2)\bar{\Pi})$ in the generic point of the curve Ξ. The latter implies that the inequality

$$\mathrm{mult}_\Xi(\hat{S}_1 \cdot \hat{S}_2) \geqslant 4(3n^2 - n\,\mathrm{mult}_P(\bar{\mathcal{M}}))$$

holds. Therefore we have

$$\mathrm{mult}_P(\bar{S}_1 \cdot \bar{S}_2) \geqslant \mathrm{mult}_P(\bar{\mathcal{M}})^2 + 4(3n^2 - n\,\mathrm{mult}_P(\bar{\mathcal{M}})) \geqslant 8n^2,$$

which implies the inequality 21.

Suppose now that the subvariety $\Xi \subset \bar{\Pi}$ is a point. In this case Proposition 18 implies the existence of a line $C \subset \Pi \cong \mathbb{P}^3$ such that

$$C \in \mathbb{LCS}\left(\hat{Y}, \frac{1}{n}\pi^{-1}(\mathcal{M}) + (\mathrm{mult}_P(\mathcal{M})/n - 2)\Pi\right)$$

and $\Xi = C \cap \hat{\Delta}$. The line $C \subset \Pi$ depends only on the properties of the log pair $(Y, \frac{1}{n}\mathcal{M})$.

Suppose that initially we take Δ such that $C \subset \pi^{-1}(\Delta)$. Then we can repeat all the previous steps of our proof. Moreover, the geometrical meaning of Proposition 19 is the following: the condition $C \subset \hat{\Delta} = \pi^{-1}(\Delta)$ implies that

$$C \in \mathbb{LCS}\left(\hat{\Delta}, \frac{1}{n}\hat{\mathcal{M}} + (\mathrm{mult}_P(\bar{\mathcal{M}})/n - 2)\bar{\Pi}\right)$$

in the case when the set $\mathbb{LCS}(\hat{\Delta}, \frac{1}{n}\hat{\mathcal{M}} + (\frac{1}{n}\mathrm{mult}_P(\bar{\mathcal{M}}) - 2)\bar{\Pi})$ does not contain any other curve in $\bar{\Pi}$. Thus we can apply the previous arguments to the divisor Δ such that $C \subset \hat{\Delta}$ and obtain the proof of the inequality 21. □

In the rest of the section we prove Proposition 19. We may assume that $X \cong \mathbb{C}^4$. Let H be a general hyperplane section of X such that $L \subset f^{-1}(H)$, $T = f^{-1}(H)$ and $S = g^{-1}(T)$. Then

$$K_W + B^W + \bar{E} + 2F + S \sim_{\mathbb{Q}} (f \circ g)^*(K_X + B_X + H)$$

and

$$B^W + \bar{E} + 2F = B_W + (\mathrm{mult}_O(B_X) - 2)\bar{E} + (\mathrm{mult}_O(B_X) + \mathrm{mult}_L(B_V) - 3)F,$$

which implies that

$$F \in \mathbb{LCS}(W, B^W + \bar{E} + 2F) \iff \mathrm{mult}_O(B_X) + \mathrm{mult}_L(B_V) \geqslant 4$$

by Definition 12. Thus we may assume that $\mathrm{mult}_O(B_X) + \mathrm{mult}_L(B_V) < 4$. We must prove that there is a surface $Z \subset F$ such that $Z \in \mathbb{LCS}(W, B^W + \bar{E} + 2F)$ and $g(Z) = L$.

Now let $\bar{H}$ be a sufficiently general hyperplane section of the variety X passing through the point O, $\bar{T} = f^{-1}(\bar{H})$ and $\bar{S} = g^{-1}(\bar{T})$. Then $O \in \mathbb{LCS}(\bar{H}, B_X|_{\bar{H}})$ by Theorem 15 and

$$K_W + B^W + \bar{E} + F + \bar{S} \sim_{\mathbb{Q}} (f \circ g)^*(K_X + B_X + H),$$

which implies that the log pair $(\bar{S}, (B^W + \bar{E} + F)|_{\bar{S}})$ is not log terminal. We can apply Theorem 14 to the morphism $f \circ g : \bar{S} \to \bar{H}$. Therefore either the locus $LCS(\bar{S}, (B^W + \bar{E} + F)|_{\bar{S}})$ consists of a single isolated point in the fiber of the morphism $g|_F : F \to L$ over the point $\bar{T} \cap L$ or it contains a curve in the fiber of the morphism $g|_F : F \to L$ over the point $\bar{T} \cap L$.

Remark 23 Every element of the set $\mathbb{LCS}(\bar{S}, (B^W + \bar{E} + F)|_{\bar{S}})$ that is contained in the fiber of the $\mathbb{P}^2$-bundle $g|_F : F \to L$ over the point $\bar{T} \cap L$ is an intersection of $\bar{S}$ with some element of the set $\mathbb{LCS}(W, B^W + \bar{E} + F)$ due to the generality in the choice of $\bar{H}$.

Therefore the generality of $\bar{H}$ implies that either $\mathbb{LCS}(W, B^W + \bar{E} + F)$ contains a surface in the divisor F dominating the curve L or the only center of log canonical singularities of the log pair $(W, B^W + \bar{E} + F)$ that is contained in the divisor F and dominates the curve L is a section of the $\mathbb{P}^2$-bundle $g|_F : F \to L$. On the other hand, we have

$$\mathbb{LCS}(W, B^W + \bar{E} + F) \subseteq \mathbb{LCS}(W, B^W + \bar{E} + 2F),$$

which implies that in order to prove Proposition 19 we may assume that the divisor F contains a curve C such that the following conditions hold:

- the curve C is a section of the $\mathbb{P}^2$-bundle $g|_F : F \to L$;
- the curve C is the unique element of the set $\mathbb{LCS}(W, B^W + \bar{E} + 2F)$ that is contained in the g-exceptional divisor F and dominates the curve L;
- the curve C is the unique element of the set $\mathbb{LCS}(W, B^W + \bar{E} + F)$ that is contained in the g-exceptional divisor F and dominates the curve L.

We have $O \in \mathbb{LCS}(H, M_X|_H)$ by Theorem 15, but $\mathbb{LCS}(S, (B^W + \bar{E} + 2F)|_S) \neq \varnothing$, where S is the proper transform of H on W. We can apply Theorem 14 to the log pair $(S, (B^W + \bar{E} + 2F)|_S)$ and the birational morphism $f \circ g|_S : S \to H$, which implies that one of the following holds:

- the locus $LCS(S, (B^W + \bar{E} + 2F)|_S)$ consists of a single point;
- the locus $LCS(S, (B^W + \bar{E} + 2F)|_S)$ contains the curve C.

Corollary 24 *Either $C \subset S$ or $S \cap C$ consists of a single point.*

By construction we have $L \cong C \cong \mathbb{P}^1$ and

$$F \cong \mathrm{Proj}(\mathcal{O}_L(-1) \oplus \mathcal{O}_L(1) \oplus \mathcal{O}_L(1))$$

and $S|_F \sim B + D$, where B is the tautological line bundle on F and D is a fiber of the natural projection $g|_F : F \to L \cong \mathbb{P}^1$.

Lemma 25 *The group $H^1(\mathcal{O}_W(S - F))$ vanishes.*

Proof The intersection of the divisor $-g^*(E) - F$ with every curve that is contained in the divisor $\bar{E}$ is non-negative and $(-g^*(E) - F)|_F \sim B + D$. Hence $-4g^*(E) - 4F$ is h-big and h-nef, where $h = f \circ g$. However, we have $X \cong \mathbb{C}^4$ and

$$K_W - 4g^*(E) - 4F = S - F,$$

which implies $H^1(\mathcal{O}_W(S - F)) = 0$ by the Kawamata–Viehweg vanishing (see [7]). □

Thus the restriction map

$$H^0(\mathcal{O}_W(S)) \to H^0(\mathcal{O}_F(S|_F))$$

is surjective, but $|S|_F|$ has no base points (see §2.8 in [12]).

Corollary 26 *The curve C is not contained in S.*

Let $\tau = g|_F$ and $\mathcal{I}_C$ be an ideal sheaf of C on F. Then $R^1\,\tau_*(B \otimes \mathcal{I}_C) = 0$ and the map

$$\pi : \mathcal{O}_L(-1) \oplus \mathcal{O}_L(1) \oplus \mathcal{O}_L(1) \to \mathcal{O}_L(k)$$

is surjective, where $k = B \cdot C$. The map π is given by a an element of the group

$$H^0(\mathcal{O}_L(k+1)) \oplus H^0(\mathcal{O}_L(k-1)) \oplus H^0(\mathcal{O}_L(k-1)),$$

which implies $k \geqslant -1$.

Lemma 27 *The equality $k = 0$ is impossible.*

Proof Suppose $k = 0$. Then the map π is given by matrix $(ax + by, 0, 0)$, where a and b are complex numbers and $(x : y)$ are homogeneous coordinates on $L \cong \mathbb{P}^1$. Thus the map π is not surjective over the point of L at which $ax + by$ vanishes. □

Therefore the divisor B can not have trivial intersection with C. Hence the intersection of the divisor S with the curve C is either trivial or consists of more than one point, but we already proved that $S \cap C$ consists of one point. The obtained contradiction proves Proposition 19.

The following result is a generalization of Theorem 20.

Theorem 28 *Let Y be a variety of dimension $r \geqslant 5$, $\mathcal{M}$ be a linear system on Y having no base components, S_1 and S_2 be general divisors in the linear system $\mathcal{M}$, P be a smooth point of the variety Y such that $P \in \mathbb{CS}(Y, \frac{1}{n}\mathcal{M})$ for some natural number n, but the singularities of the log pair $(Y, \frac{1}{n}\mathcal{M})$ are canonical in some punctured neighborhood of P, $\pi : \hat{Y} \to Y$ be a blow up of the point P, and Π be a π-exceptional divisor. Then there is a linear subspace $C \subset \Pi \cong \mathbb{P}^{r-1}$ having codimension 2 such that $\mathrm{mult}_P(S_1 \cdot S_2 \cdot \Delta) > 8n^2$, where Δ is a divisor on Y passing through P such that Δ is smooth at P, the divisor $\pi^{-1}(\Delta)$ contains C, the divisor Δ does not contain any subvarieties of Y of codimension 2 that are contained in the base locus of $\mathcal{M}$.*

Proof We consider only the case $r = 5$. Let H_1, H_2, H_3 be general hyperplane sections of the variety Y passing through P. Put $\bar{Y} = \cap_{i=1}^{3} H_i$ and $\bar{\mathcal{M}} = \mathcal{M}|_{\bar{Y}}$. Then $\bar{Y}$ is a surface, which is smooth at P, and $P \in \mathbb{LCS}(\bar{Y}, \frac{1}{n}\bar{\mathcal{M}})$ by Theorem 15. Let $\pi : \hat{Y} \to Y$ be a blow up of P, Π be an exceptional divisor of π, and $\hat{\mathcal{M}} = \pi^{-1}(\mathcal{M})$. Then the set

$$\mathbb{LCS}\left(\hat{Y}, \frac{1}{n}\hat{\mathcal{M}} + (\mathrm{mult}_P(\mathcal{M})/n - 2)\Pi\right)$$

contains a subvariety $Z \subset \Pi$ such that $\dim(Z) \geqslant 2$.

In the case $\dim(Z) = 4$ the claim is obvious. In the case $\dim(Z) = 3$ we can proceed as in the proof of Theorem 20 to prove that

$$\mathrm{mult}_P(S_1 \cdot S_2 \cdot \Delta) > 8n^2$$

for any divisor Δ on Y such that the divisor Δ contains the point P, the divisor Δ is smooth at the point P, the divisor Δ does not contain any subvariety $\Gamma \subset Y$ of codimension 2 that is contained in the base locus of the linear system $\mathcal{M}$.

It should be pointed out that in the cases when $\dim(Z) \geqslant 3$ we do not need to fix any linear subspace $C \subset \Pi$ of codimension 2 such that $\pi^{-1}(\Delta)$ contains C. The latter condition is vacuous posteriori when $\dim(Z) \geqslant 3$.

Suppose that $\dim(Z) = 2$. Then the surface Z is a linear subspace of $\Pi \cong \mathbb{P}^4$ having codimension 2 by Theorem 14. Moreover, the surface Z does not depend on the choice of our divisors H_1, H_2, H_3, because it depends only on the properties of the log pair $(Y, \frac{1}{n}\mathcal{M})$.

Put $C = Z$. Let H be a sufficiently general hyperplane section of Y passing through the point P, and Δ be a divisor on Y such that Δ contains point P, the divisor Δ is smooth at the point P, the divisor $\pi^{-1}(\Delta)$ contains C, the divisor Δ does not contain any subvariety of Y of codimension 2 contained in the base locus of the linear system $\mathcal{M}$. Then

$$\mathrm{mult}_P(S_1 \cdot S_2 \cdot \Delta) > 8n^2 \iff \mathrm{mult}_P(S_1|_H \cdot S_2|_H \cdot \Delta|_H) > 8n^2$$

due to the generality of H. However, we have $\mathrm{mult}_P(S_1|_H \cdot S_2|_H \cdot \Delta|_H) > 8n^2$ by Theorem 20, because $P \in \mathbb{CS}(H, \mu\mathcal{M}|_H)$ for some positive rational number $\mu < 1/n$ by Theorem 15. □

4 Birational superrigidity

In this section we prove Theorem 2. Let $\psi : X \to V \subset \mathbb{P}^n$ be a double cover branched over a smooth divisor $R \subset V$ such that $n \geqslant 7$. Then $R \sim \mathcal{O}_{\mathbb{P}^n}(2r)|_V$ for some $r \in \mathbb{N}$, and

$$-K_X \sim \psi^*(\mathcal{O}_{\mathbb{P}^n}(d+r-1-n)|_V),$$

where $d = \deg V$. Suppose that $d + r = n$ and $d = 3$ or 4. Then the group $\mathrm{Pic}(X)$ is generated by the divisor $-K_X$, and $(-K_X)^2 = 2d \leqslant 8$. Suppose that X is not birationally superrigid. Then Theorem 11 implies the existence of a linear system $\mathcal{M}$ whose base locus has codimension at least 2 and the singularities of the log pair $(X, \frac{1}{m}\mathcal{M})$ are not canonical, where m is a natural number such that the equivalence $\mathcal{M} \sim -mK_X$ holds. Hence the set $\mathbb{CS}(X, \frac{1}{m}\mathcal{M})$ contains a proper irreducible subvariety $Z \subset X$ such that $Z \in \mathbb{CS}(X, \mu\mathcal{M})$ for some rational $\mu < 1/m$.

Corollary 29 *For a general $S \in \mathcal{M}$ the inequality* $\mathrm{mult}_Z(S) > m$ *holds.*

A priori we have $\dim(Z) \leqslant \dim(X) - 2 = n - 3$. We may assume that Z has maximal dimension among subvarieties of X such that the singularities of the log pair $(X, \frac{1}{m}\mathcal{M})$ are not canonical in their generic points.

Lemma 30 *The inequality* $\dim(Z) \neq 0$ *holds.*

Proof Suppose that Z is a point. Let S_1 and S_2 be sufficiently general divisors in the linear system $\mathcal{M}$, $f : U \to X$ be a blow up of Z, and E be an f-exceptional divisor. Then Theorem 28 implies the existence of a linear subspace $\Pi \subset E \cong \mathbb{P}^{n-2}$ of codimension 2 such that

$$\mathrm{mult}_Z(S_1 \cdot S_2 \cdot D) > 8m^2$$

holds for any $D \in |-K_X|$ such that $\Pi \subset f^{-1}(D)$, the divisor D is smooth at Z, and D does not contain any subvariety of X of codimension 2 that is contained in the base locus of $\mathcal{M}$.

Let $\mathcal{H}$ be a linear system of hyperplane sections of the hypersurface V such that $H \in \mathcal{H}$ if and only if $\Pi \subset (\psi \circ f)^{-1}(H)$. Then there is a linear subspace $\Sigma \subset \mathbb{P}^n$ of dimension $n - 3$ such that the divisors in the linear system $\mathcal{H}$ is cut on V by the hyperplanes in $\mathbb{P}^n$ that contains the linear subspace Σ. Hence the base locus of the linear system $\mathcal{H}$ consists of the intersection $\Sigma \cap V$, but we have $\Sigma \not\subset V$ by the Lefschetz theorem. In particular, $\dim(\Sigma \cap V) = n - 4$.

Let H be a general divisor in $\mathcal{H}$ and $D = \psi^{-1}(H)$. Then $\Pi \subset f^{-1}(D)$, and D is smooth at the point Z. Moreover, the divisor D does not contain any subvariety $\Gamma \subset X$ of codimension 2 that is contained in the base locus of $\mathcal{M}$, because otherwise $\psi(\Gamma) \subset \Sigma \cap V$, but $\dim(\psi(\Gamma)) = n - 3$ and $\dim(\Sigma \cap V) = n - 4$. Let

$H_1, H_2, \dots, H_k$ be general divisors in $|-K_X|$ passing through the point Z, where $k = \dim(Z) - 3$. Then we have

$$2dm^2 = H_1 \cdot \dots \cdot H_k \cdot S_1 \cdot S_2 \cdot D \geqslant \mathrm{mult}_Z(S_1 \cdot S_2 \cdot D) > 8m^2,$$

which is a contradiction. □

Lemma 31 *The inequality* $\dim(Z) \geqslant \dim(X) - 4$ *holds.*

Proof Suppose that $\dim(Z) \leqslant \dim(X) - 5$. Let $H_1, H_2, \dots, H_k$ be sufficiently general hyperplane sections of the hypersurface $V \subset \mathbb{P}^n$, where $k = \dim(Z) > 0$. Put

$$\bar{V} = \cap_{i=1}^k H_i,\ \bar{X} = \psi^{-1}(\bar{V}),\ \bar{\psi} = \psi|_{\bar{X}} : \bar{X} \to \bar{V},$$

and $\bar{\mathcal{M}} = \mathcal{M}|_{\bar{X}}$. Then $\bar{V}$ is a smooth hypersurface of degree d in $\mathbb{P}^{n-k}$, $\bar{\psi}$ is a double cover branched over a smooth divisor $R \cap \bar{V}$, $\bar{\mathcal{M}}$ has no base components, and $\bar{V}$ does not contains linear subspaces of $\mathbb{P}^{n-k}$ of dimension $n-k-3$ by the Lefschetz theorem. Let P be any point of the intersection $Z \cap \bar{X}$. Then $P \in \mathbb{CS}(\bar{X}, \frac{1}{m}\bar{\mathcal{M}})$ and we can repeat the proof of Lemma 30 to get a contradiction. □

Lemma 32 *The inequality* $\dim(Z) \neq \dim(X) - 2$ *holds.*

Proof Suppose that $\dim(Z) = \dim(X) - 2$. Let S_1 and S_2 be sufficiently general divisors in the linear system $\mathcal{M}$, and $H_1, H_2, \dots, H_{n-3}$ be general divisors in $|-K_X|$. Then

$$2dm^2 = H_1 \cdot \dots \cdot H_{n-3} \cdot S_1 \cdot S_2 \geqslant \mathrm{mult}_Z(S_1)\mathrm{mult}_Z(S_2)(-K_X)^{n-3} \cdot Z > m^2(-K_X)^{n-3} \cdot Z,$$

because $\mathrm{mult}_Z(\mathcal{M}) > m$. Therefore $(-K_X)^{n-3} \cdot Z < 2d$. On the other hand, we have

$$(-K_X)^{n-3} \cdot Z = \begin{cases} \deg(\psi(Z) \subset \mathbb{P}^n) \text{ when } \psi|_Z \text{ is birational,} \\ 2\deg(\psi(Z) \subset \mathbb{P}^n) \text{ when } \psi|_Z \text{ is not birational.} \end{cases}$$

The Lefschetz theorem implies that $\deg(\psi(Z))$ is a multiple of d. Therefore $\psi|_Z$ is a birational morphism and $\deg(\psi(Z)) = d$. Hence either $\psi(Z)$ is contained in R, or the scheme-theoretic intersection $\psi(Z) \cap R$ is singular in every point. However, we can apply the Lefschetz theorem to the smooth complete intersection $R \subset \mathbb{P}^n$, which gives a contradiction. □

Lemma 33 *The inequality* $\dim(Z) \leqslant \dim(X) - 5$ *holds.*

Proof Suppose that $\dim(Z) \geqslant \dim(X) - 4 \geqslant 3$. Let S be a sufficiently general divisor in the linear system $\mathcal{M}$, $\hat{S} = \psi(S \cap R)$ and $\hat{Z} = \psi(Z \cap R)$. Then $\hat{S}$ is a divisor on the complete intersection $R \subset \mathbb{P}^n$ such that $\mathrm{mult}_{\hat{Z}}(\hat{S}) > m$ and $\hat{S} \sim \mathcal{O}_{\mathbb{P}^n}(m)|_R$, because R is a ramification divisor of ψ. Hence, the inequality $\dim(\hat{Z}) \geqslant 2$ is impossible by Lemma 10. □

Therefore Theorem 2 is proved.

References

1. Cheltsov, I.: Nonrationality of a four-dimensional smooth complete intersection of a quadric and a quadric not containing a plane. Sb. Math. **194**, 1679–1699 (2003)
2. Corti, A.: Factorizing birational maps of threefolds after Sarkisov. J. Alg. Geometry **4**, 223–254 (1995)
3. Corti, A.: Singularities of linear systems and 3-fold birational geometry. L.M.S. Lecture Note Series **281**, 259–312 (2000)
4. Dolgachev, I.: Weighted projective varieties. Lecture Notes in Mathematics **956**, 34–71 (1982)
5. Iskovskikh, V.: Birational automorphisms of three-dimensional algebraic varieties. J. Soviet Math. **13**, 815–868 (1980)
6. Iskovskikh, V., Pukhlikov, A.: Birational automorphisms of multidimensional algebraic manifolds. J. Math. Sci. **82**, 3528–3613 (1996)
7. Kawamata, Y., Matsuda, K., Matsuki, K.: Introduction to the minimal model problem. Adv. Stud. Pure Math. **10**, 283–360 (1987)
8. Kollár, J.: Rational curves on algebraic varieties. Springer-Verlag, Berlin (1996)
9. Kollár, J. et al.: Flips and abundance for algebraic threefolds, in "A summer seminar at the University of Utah, Salt Lake City, 1991". Astérisque. **211** (1992)
10. Pukhlikov, A.: Birational automorphisms of a double space and a double quadric. Math. USSR Izv. **32**, 233–243 (1989)
11. Pukhlikov, A.: Birationally rigid double Fano hypersurfaces. Sb. Math. **191**, 883–908 (2000)
12. Reid, M.: Chapters on algebraic surfaces, Complex algebraic geometry (Park City, UT, 1993). IAS/Park City Math. Ser. **3** AMS, Providence, RI, 3–159 (1997)