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# Total log canonical thresholds and generalized Eckardt points

I.A. Cheltsov and Jihun Park

**Abstract.** Let X be a smooth hypersurface of degree  $n \ge 3$  in  $\mathbb{P}^n$ . It is proved that the log canonical threshold of an arbitrary hyperplane section H of it is at least (n-1)/n. Under the assumption of the log minimal model program it is also proved that the log canonical threshold of  $H \subset X$  is (n-1)/n if and only if H is a cone in  $\mathbb{P}^{n-1}$  over a smooth hypersurface of degree n in  $\mathbb{P}^{n-2}$ .

Bibliography: 16 titles.

### §1. Introduction

Shokurov introduced log canonical thresholds in [1] to measure how far log pairs are from log canonicity. It has been subsequently shown that log canonical thresholds have many amazing and useful properties.

**Definition 1.1.** Let (X, B) be a log canonical pair, and Z a closed subvariety of X. Let D be a Q-Cartier divisor on X.<sup>1</sup> The log canonical threshold of D along Z with respect to the log canonical divisor  $K_X + B$  is the quantity

 $lct_Z(D; X, B) = \sup\{c : \text{ the divisor } K_X + B + cD \text{ is log canonical along } Z\}.$ 

It is easy to verify using the definition that  $lct_Z(D; X, B) \in [0, 1]$ .

Remark 1.2. In the case of a trivial boundary B = 0 we write  $lct_Z(D; X)$  instead of  $lct_Z(D; X, 0)$ , and in the case Z = X we use the notation lct(D; X, B) in place of  $lct_X(D; X, B)$ .

One comes across log canonical thresholds in various disguises in other branches of mathematics.

**Example 1.3.** (1) Consider a hypersurface  $D \subset \mathbb{C}^n$  defined as the zero set of a non-constant holomorphic function f in the neighbourhood of the origin O. Then we can see ([3]) that the log canonical threshold  $\operatorname{lct}_O(D; \mathbb{C}^n)$  is equal to the quantity

 $\sup\{c: \text{ the function } |f|^{-c} \text{ belongs to the class } L^2 \text{ in the neighbourhood of } O\}.$ 

<sup>&</sup>lt;sup>1</sup> Unless otherwise stated all varieties are assumed to be projective and defined over  $\mathbb{C}$ . The main definitions and notation can be found in [1] and [2].

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(2) Bernstein–Sato polynomials appearing in the theory of differential operators and, in particular, in D-module theory (see [4]) provide another example. We briefly explain what we understand by Bernstein–Sato polynomials. It is well known that for each convergent power series

$$f \in \mathbb{C}\{z_1, \ldots, z_n\}$$

there exists a non-trivial polynomial  $b(s) \in \mathbb{C}[s]$  and a linear differential operator

$$Q = \sum_{I,j} f_{I,j} s^j \frac{\partial^I}{\partial z_I}$$

such that  $b(s)f^s = Qf^{s+1}$ , where each  $f_{I,j}$  is a convergent power series. For a fixed series the polynomials b(s) form an ideal of  $\mathbb{C}[s]$ . The monic generator of this ideal is called the *Bernstein–Sato polynomial* of f. We can show that  $lct_0(D; \mathbb{C}^n)$  is the absolute value of the largest zero of the Bernstein–Sato polynomial of f (see [3]).

Mustață [5] has recently investigated log canonical thresholds via jet schemes. In particular, he obtained the following result.

**Theorem 1.4.** Let X be a smooth variety, and D an integral effective divisor on X. Then

$$\operatorname{lct}(D; X) = \dim X - \sup_{m \ge 0} \frac{\dim D_m}{m+1},$$

where  $D_m$  is the mth jet scheme of D.

Recall that the *mth jet scheme*  $X_m$  of a variety X is a scheme with closed points over  $x \in X$  that are morphisms

$$\mathcal{O}_{X,x} \to k[t]/(t^{m+1}).$$

If X is smooth, then  $X_m$  is an affine bundle over X of dimension  $(m+1) \dim X$ .

To understand the geometry of a fixed variety it is often important to investigate linear systems related to the canonical divisor. One way of investigating is to find 'extreme' elements of such linear systems.

There can be two kinds of extreme elements in a linear system: a 'good' element and a 'bad' one. We now explain what we mean by 'good' and 'bad' elements.

For a 'good' element Reid considered a general elephant. Following in his footsteps Shokurov introduced a more general concept.

**Definition 1.5.** Let X be a normal variety, and D = S + B a subboundary on X such that the divisors S and B have no common components, S is a reduced divisor, and  $\lfloor B \rfloor \leq 0$ . Then we say that the divisor  $K_X + D$  is *n*-complementary if there exists a divisor  $D^+$  on X satisfying the following conditions:

(1)  $nD^+$  is integral and  $n(K_X + D^+)$  is linearly trivial;

- (2)  $nD^+ \ge nS + \lfloor (n+1)B \rfloor;$
- (3)  $K_X + D^+$  is log canonical.

If these conditions hold, then the divisor  $K_X + D^+$  is called an *n*-complement of  $K_X + D$ .

We now introduce a property of elements of linear systems that is converse to being 'good'. It was originally introduced by Keel and McKernan [6]. Strictly speaking, it is antithetical to Reid's 'general elephant'.

**Definition 1.6.** Let X be a normal variety, and B an effective  $\mathbb{Q}$ -divisor on X. A special tiger for the log canonical divisor  $K_X + B$  is an effective  $\mathbb{Q}$ -divisor D such that  $K_X + B + D$  is numerically trivial but not Kawamata log terminal.

Assume that a log pair (X, B) is log canonical. Then using log canonical thresholds we can describe special tigers for the log canonical divisor  $K_X + B$ .

**Definition 1.7.** Let (X, B) be a log canonical pair with non-empty  $|-(K_X + B)|$ . The *total log canonical threshold* of (X, B) is the real quantity

 $glct(X, B) = \sup\{r : K_X + B + rD \text{ is log canonical for all } D \in |-(K_X + B)|\}.$ 

*Remark* 1.8. It follows from this definition that  $glct(X, B) \in [0, 1]$ .

If B = 0, then the total log canonical threshold of the log pair (X, B) will be denoted by glct(X).

The total log canonical threshold of a log pair (X, B) measures how 'bad' elements of the linear system  $|-(K_X + B)|$  can be. It is worthwhile paying attention also to the special tigers realizing the total log canonical threshold.

**Definition 1.9.** Let X be a normal variety and B an effective  $\mathbb{Q}$ -divisor on X. An effective  $\mathbb{Q}$ -divisor D on X is called a *wild tiger* for the log canonical divisor  $K_X + B$  if the following conditions are satisfied: the divisor  $K_X + B + D$  is linearly trivial and lct(D; X, B) = glct(X, B). By a wild tiger for X we mean a wild tiger for the canonical divisor  $K_X$ .

We note that in the definition of a wild tiger linear triviality replaces numerical triviality participating in the definition of a special tiger.

*Remark* 1.10. The concept of wild tiger is a sort of opposite to 1-complement of the log canonical divisor  $K_X + B$ . One can find an interesting detailed description of interactions between 1-complements and wild tigers in [7].

In the present paper we are mainly interested in finding wild tigers for smooth hypersurfaces of degree  $n \ge 3$  in  $\mathbb{P}^n$ .

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#### §2. Eckardt points

An *Eckardt point* is a point on a smooth cubic surface  $\Sigma$  at which three lines on  $\Sigma$  intersect each other. In other words a point p on  $\Sigma$  is an Eckardt point if there exists an element in the linear system  $|-K_{\Sigma}|$  that is a cone with vertex p and base consisting of three distinct points.

We shall now investigate smooth del Pezzo surfaces and, in particular, smooth cubic surfaces to find their wild tigers. This will allow us to discover special features of the Eckardt points.

In hunting for wild tigers on a variety the first step is to calculate the total log canonical threshold. However, in view of the explicit geometric description of smooth del Pezzo surfaces we can find wild tigers for them by investigating cubic curves and points in general position on  $\mathbb{P}^2$ .

**Proposition 2.1.** Let S be a smooth del Pezzo surface of degree d. Then one of the following cases occurs:

- (1)  $d = 9, S \cong \mathbb{P}^2$ ,  $glct(S) = \frac{1}{3}$ , a wild tiger is a triple line;
- (2) d = 8,  $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ ,  $\operatorname{glct}(S) = \frac{1}{2}$ , a wild tiger consists of a double fibre of the projection onto one factor and two, possibly coincident, fibres of the projection onto the other factor;
- (3) d = 8,  $S \cong \mathbb{F}_1$ ,  $glct(S) = \frac{1}{3}$ , a wild tiger consists of a triple fibre of the projection onto  $\mathbb{P}^1$  and the double exceptional section;
- (4) d = 7,  $glct(S) = \frac{1}{3}$ , a wild tiger consists of a chain of three exceptional curves considered with multiplicities 2, 3, and 2, respectively;
- (5) d = 6, glct(S) = 1/2, a wild tiger consists either of a chain of three smooth rational curves with self-intersections -1, 0, -1 considered with multiplicities 1, 2, and 1, respectively, or of a chain of four exceptional curves taken with multiplicities 1, 2, 2, and 1, respectively, or of four intersecting smooth rational curves making up the configuration D<sub>4</sub>, where the central curve is exceptional and of multiplicity 2, two curves at the vertices of the graph D<sub>4</sub> are also exceptional, of multiplicity one, and the remaining curve has multiplicity one and self-intersection index zero;
- (6) d = 5, glct(S) = <sup>1</sup>/<sub>2</sub>, a wild tiger consists of four exceptional curves making up the configuration D₄, where the central curve has multiplicity 2 and all other curves have multiplicity 1;
- (7) d = 4, glct(S) = <sup>2</sup>/<sub>3</sub>, a wild tiger consists of three smooth rational curves intersecting transversally at one point, where all the three curves are taken with multiplicity 1, two of them are exceptional, and one curve has selfintersection index zero;
- (8) d = 3, S is a smooth cubic with Eckardt point  $glct(S) = \frac{2}{3}$ , a wild tiger consists of three exceptional points intersecting at one point, of multiplicity 1 each;
- (9) d = 3, S is a generic smooth cubic in  $\mathbb{P}^3$ ,  $glct(S) = \frac{3}{4}$ , a wild tiger consists of a line and a conic, both of multiplicity 1 and intersecting tangentially;
- (10) d = 2, the surface S contains an anticanonical divisor with one tacnode, glct(S) =  $\frac{3}{4}$ , a wild tiger consists of two exceptional curves, both of multiplicity 1, intersecting tangentially with intersection number 2;
- (11) d = 2, the surface S contains no anticanonical divisors with tacnodes,  $glct(S) = \frac{5}{6}$ , a wild tiger consists of a cuspidal rational curve of multiplicity 1;
- (12) d = 1, the anticanonical linear system  $|-K_S|$  contains a cuspidal rational curve,  $glct(S) = \frac{5}{6}$ , a wild tiger consists of a cuspidal rational curve of multiplicity 1;
- (13) d = 1, the anticanonical linear system  $|-K_S|$  contains no cuspidal rational curves, glct(S) = 1, a wild tiger consists of an arbitrary curve in the anticanonical linear system  $|-K_S|$  considered with multiplicity 1.

We leave Proposition 2.1 without proof (see [7]) and proceed to smooth cubic surfaces. We have already shown that there exist in this case only two opportunities for wild tigers. If a cubic surface has an Eckardt point, then a wild tiger consists of three lines intersecting at the Eckardt point. Otherwise a wild tiger is a hyperplane section consisting of a line and a conic intersecting tangentially.

*Remark* 2.2. The existence of an Eckardt point is a codimension-one condition on a cubic.

It is expected that Eckardt points are an indication of certain birational properties of del Pezzo fibrations (see [8] and [7]). Hence an attempt at a reasonable generalization of the concept of Eckardt point to hypersurfaces of degree  $n \ge 4$ in  $\mathbb{P}^n$  seems logical.

**Definition 2.3.** Let X be a smooth hypersurface of degree  $n \ge 3$  in  $\mathbb{P}^n$ . A point p on X is called a *generalized Eckardt point* if there exists an element S of  $|-K_X|$  that is a cone in  $\mathbb{P}^{n-1}$  over a smooth hypersurface of degree n in  $\mathbb{P}^{n-2}$  with vertex at p.

It is clear that the concept of a generalized Eckardt point coincides with the classical concept for n = 3. Hence in what follows we shall call generalized Eckardt points simply Eckardt points.

As noted before, if a smooth cubic surface has an Eckardt point, then a wild tiger is a cone over three distinct points. This result can be generalized as follows.

**Theorem 2.4.** Let X be a smooth hypersurface of degree  $n \ge 3$  in  $\mathbb{P}^n$ . If X has an Eckardt point p, then the cone S in the linear system  $|-K_X|$  involved in the definition of a generalized Eckardt point is a wild tiger for X. In particular,

$$\operatorname{LCS}\left(X, \frac{n-1}{n}S\right) = \{p\} \quad and \quad \operatorname{lct}(S; X) = \operatorname{glct}(X) = \frac{n-1}{n}$$

We shall prove Theorem 2.4 in the next section.

If a smooth cubic surface has total log canonical threshold  $\frac{2}{3}$ , then it has an Eckardt point. It is natural to conjecture a similar result also in the multidimensional case.

**Conjecture 2.5.** Let X be a smooth hypersurface of degree  $n \ge 3$  in  $\mathbb{P}^n$ . If the total log canonical threshold of X is (n-1)/n, then a wild tiger S of the hypersurface X is a cone in  $\mathbb{P}^{n-1}$  with vertex p over a smooth hypersurface of degree n in  $\mathbb{P}^{n-2}$ . In particular, p is an Eckardt point and

$$\operatorname{LCS}\left(X, \frac{n-1}{n}S\right) = \{p\}.$$

Of course, the conjecture holds for n = 3. In §4 we shall prove Conjecture 2.5 under the additional assumption of the log minimal model program. Since the log minimal model program holds up to dimension 3, this will prove Conjecture 2.5 for n = 4.

We mention now another problem related to total log canonical thresholds and wild tigers for smooth Fano varieties. We have already seen that a generic cubic surface has total log canonical threshold  $\frac{3}{4}$ , and its wild tiger consists of a line and a conic intersecting tangentially with intersection number 2. A generic del Pezzo surface of degree 2 has total log canonical threshold  $\frac{5}{6}$ , and its wild tiger is a cuspidal cubic.

**Question 2.6.** What is the total log canonical threshold of a generic hypersurface of degree n in  $\mathbb{P}^n$  and what is a wild tiger for this surface?

### $\S$ 3. A lower bound for total log canonical thresholds

Let W be a smooth hypersurface of degree m in  $\mathbb{P}^n$ , where  $n \ge 4$ , and let H be a hyperplane section of W. It follows from Lefschetz's theorem that  $\operatorname{Pic} W = \mathbb{Z}H$ . In particular, a hyperplane section H of W is irreducible and reduced.

**Lemma 3.1.** For an arbitrary curve C on H,  $\operatorname{mult}_C H = 1$ .

*Proof.* Let p be a sufficiently generic point in  $\mathbb{P}^n$ . We consider a cone  $P_p$  with base C and vertex p. Then we have  $P_p \cap W = C \cup R_p$ , where  $R_p$  is a curve on W of degree  $(m-1) \deg C$ .

Because p is a generic point, the curves C and  $R_p$  intersect transversally at precisely  $(m-1) \deg C$  distinct points (see [9]). On the other hand, we have  $H \cdot R_p = \deg R_p = (m-1) \deg C$ . That is, we have the inequality

$$H \cdot R_p \ge \deg R_p \operatorname{mult}_C H = (m-1) \deg C \operatorname{mult}_C H$$

**Corollary 3.2.** A hyperplane section H has only isolated singularities. In particular, it is normal.

**Theorem 3.3.** The log canonical threshold of the divisor H in the hypersurface W is at least  $\lambda = \min\{(n-1)/m, 1\}$ .

*Proof.* Let  $\alpha \in (0, \lambda)$ . We shall consider the log pair  $(\mathbb{P}^{n-1}, \alpha H)$  instead of  $(W, \alpha H)$ . We assume that the singularities of  $(\mathbb{P}^{n-1}, \alpha H)$  are not Kawamata log terminal. Then the log canonical singularity subscheme

$$\mathcal{L} = \mathrm{LCS}(\mathbb{P}^{n-1}, \alpha H)$$

is zero-dimensional.

We consider now a Cartier divisor D such that

$$D \equiv K_{\mathbb{P}^{n-1}} + \alpha H + (\lambda - \alpha)H'$$

for a generic element H' in |H|. Then

$$\mathcal{O}_{\mathbb{P}^{n-1}}(D) = \begin{cases} \mathcal{O}_{\mathbb{P}^{n-1}}(-1) & \text{if } m-n \ge -1, \\ \mathcal{O}_{\mathbb{P}^{n-1}}(m-n) & \text{if } m-n < -1. \end{cases}$$

By Shokurov's vanishing theorem (see [10]) we obtain an exact sequence

$$H^0(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(D)) \to H^0(\mathcal{L}, \mathcal{O}_{\mathcal{L}}(D)) \to 0$$

On the other hand,  $H^0(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(D)) = 0$  whereas  $H^0(\mathcal{L}, \mathcal{O}_{\mathcal{L}}(D)) \neq 0$ .

**Corollary 3.4.** Suppose that n = m. Then the total log canonical threshold of W is at least (n-1)/n.

Theorem 2.4 immediately follows from Corollary 3.4.

Remark 3.5. It follows from the proof of Theorem 3.3 that if the log canonical threshold  $\alpha$  of a hyperplane section H in a hypersurface W is not 1, then the locus of log canonical singularities LCS $(W, \alpha H)$  consists of a single point.

**Example 3.6.** Assume that a hypersurface W in  $\mathbb{P}^n$  is defined by the equation

$$x_0^m - x_1^m + \sum_{i=2}^n x_i^m = 0$$

and its hyperplane section H is described by the equation  $x_0 - x_1 = 0$ . Then the log canonical threshold of H is  $\lambda$ . Hence  $\lambda$  is the greatest lower bound for the log canonical thresholds of hyperplane sections of smooth hypersurfaces of degree m in  $\mathbb{P}^n$ .

### §4. Proof of Conjecture 2.5 in the framework of the log minimal model program

Let X be a smooth hypersurface of degree  $n \ge 4$  in  $\mathbb{P}^n$ . Let S be a hyperplane section of X.

The aim of this section is to prove Conjecture 2.5 for n = 4. More specifically, under the assumption of the log minimal model program in dimension n - 1 we shall show that S is a cone in  $\mathbb{P}^{n-1}$  over a smooth hypersurface of degree n in  $\mathbb{P}^{n-2}$ 

if and only if the log pair  $\left(X, \frac{n-1}{n}S\right)$  is not Kawamata log terminal.

Assume that  $\left(X, \frac{n-1}{n}S\right)$  is not Kawamata log terminal. Then the set of

centres of log canonical singularities  $LCS\left(X, \frac{n-1}{n}S\right)$  is non-empty and contains only finitely many points. Moreover, it follows by Remark 3.5 that

$$\operatorname{LCS}\left(X, \frac{n-1}{n}S\right) = \{p\}$$

for some point  $p \in X$ .

In place of 
$$\left(X, \frac{n-1}{n}S\right)$$
 we can consider the log pair  $\left(\mathbb{P}^{n-1}, \frac{n-1}{n}S\right)$ .

Throughout what follows we assume that the log minimal model program holds for dimension n-1.

Lemma 4.1. There exists a birational morphism

$$f: V \to \mathbb{P}^{n-1}$$

with the following properties:

- (1) f is an isomorphism outside  $LCS\left(\mathbb{P}^{n-1}, \frac{n-1}{n}S\right) = \{p\};$
- (2) the variety V has  $\mathbb{Q}$ -factorial terminal singularities;
- (3) there exists an effective f-exceptional  $\mathbb{Q}$ -divisor E on V such that the support of E is the entire f-exceptional locus on V,  $\lfloor E \rfloor \neq 0$ , and

$$K_V + \frac{n-1}{n} f_*^{-1}(S) + E = f^* \left( K_{\mathbb{P}^{n-1}} + \frac{n-1}{n} S \right).$$

Proof. Let

$$g: V' \to \mathbb{P}^{n-1}$$

be a log terminal blow-up of  $\left(\mathbb{P}^{n-1}, \frac{n-1}{n}S\right)$ . Since V' has  $\mathbb{Q}$ -factorial Kawamata log terminal singularities, we can consider a terminal blow-up  $h: V \to V'$  of V'. Then the birational morphism

$$f = q \circ h \colon V \to \mathbb{P}^{n-1}$$

has all the required properties.

We fix a birational morphism  $f: V \to \mathbb{P}^{n-1}$ . Let  $\widetilde{S} = f_*^{-1}(S)$ . Since the log pair  $\left(V, \frac{n-1}{n}\widetilde{S} + E\right)$  is log canonical, the log pair  $\left(\mathbb{P}^{n-1}, \frac{n-1}{n}S\right)$ is also log canonical. On the other hand,  $K_V + \frac{n-1}{n}\widetilde{S} + E$  is not nef. Hence there

is also log canonical. On the other hand,  $K_V + \frac{-S + E}{n}$  is not net. Hence there exists an extremal contraction  $g \colon V \to W$ 

such that  $-\left(K_V + \frac{n-1}{n}\widetilde{S} + E\right)$  is *g*-ample.

Remark 4.2. Curves contracted by g cannot lie in fibres of the birational morphism f because  $-\left(K_V + \frac{n-1}{n}\tilde{S} + E\right)$  is g-ample and f-numerically trivial. In particular, W is not a point.

**Lemma 4.3.** Suppose that the extremal contraction g contracts a subvariety F of V to a subvariety Z of W. Then dim F - dim Z = 1.

*Proof.* Assume that dim  $F - \dim Z > 1$ . Let G be a fibre of g over a point in Z. Then  $G \cap E \neq \emptyset$ , and since V is  $\mathbb{Q}$ -factorial, there exists a curve on  $G \cap E$  contracted by both f and g.

**Corollary 4.4.** If  $g: V \to W$  is a Mori fibre space, then g is a conic bundle.

The following lemma is due to Shokurov (see [11]).

**Lemma 4.5.** Let Y be a variety with at worst terminal singularities. Let  $h: Y \to Z$  be a birational contraction. If a curve C on Y is an irreducible component of the exceptional locus of h, then  $K_Y \cdot C > -1$ .

*Proof.* This is a local result as regards the base, and it is sufficient to consider an analytic neighbourhood of the point h(C) = q on Z. In addition, we can assume that the exceptional locus of h is the curve C.

Assume that  $K_Y \cdot C \leq -1$ . We choose a divisor H on Y such that  $H \cdot C = 1$ . Then  $(K_Y + H) \cdot C = 0$  (see [11]).

We consider now an  $K_Y$ -flip of h:

$$\begin{array}{cccc} Y & & \stackrel{\varphi}{\dashrightarrow} & & Y^+ \\ & h \searrow & \swarrow h^+ & \\ & & Z \end{array}$$

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Let  $H^+ = \varphi(H)$ . Then the dimension of the exceptional locus  $Ex(h^+)$  of  $h^+$  is  $\dim Y - 2$  because the sum of the dimensions of the exceptional loci of the birational morphisms h and  $h^+$  cannot be less than  $\dim Y - 1$  (see [2] and [11]).

Note that the numerical h-triviality of  $K_Y + H$  yields the inequality

$$H^+ \cdot C' < 0$$

for each curve C' on the exceptional locus of  $h^+$ . Therefore,  $Ex(h^+) \subset H^+$ .

Let E be the exceptional divisor of the blow up with centre at a component of  $Ex(h^+)$ . We can assume that the centre of the divisor E on Y does not lie in H. Then we have the following inequality:

$$a(E; Y, H) \leqslant a(E; Y^+, H^+) \leqslant 0,$$

where a(E; Y, H) and  $a(E; Y^+, H^+)$  are the discrepancies of E with respect to the log canonical divisors  $K_Y + H$  and  $K_{Y^+} + H^+$ . This is a contradiction because Y is terminal.

**Lemma 4.6.** The morphism g cannot be a small contraction with exceptional locus containing a curve as an irreducible component.

*Proof.* Assume that g is a small contraction and a curve C is an irreducible component of the exceptional locus of the birational morphism g. By Lemma 4.5 we obtain  $K_V \cdot C > -1$ . On the other hand,

$$\left(K_V + \frac{n-1}{n}\widetilde{S}\right) \cdot C = \left(f^*(\mathcal{O}_{\mathbb{P}^{n-1}}(-1)) - E\right) \cdot C = -\deg f_*(C) - E \cdot C \leqslant -1$$

Thus,  $\widetilde{S} \cdot C < 0$  and  $C \subset \widetilde{S}$ , therefore

$$(K_V + \widetilde{S}) \cdot C < \left(K_V + \frac{n-1}{n}\widetilde{S}\right) \cdot C \leq -1.$$

Let  $\nu: \widehat{S} \to \widetilde{S}$  be a normalization of  $\widetilde{S}$ . By the adjunction formula we obtain

$$K_{\widehat{S}} + \operatorname{Diff}_{\widetilde{S}}(0) = \nu^* \left( (K_V + \widetilde{S})|_{\widetilde{S}} \right)$$

The curve  $\nu_*^{-1}(C)$  cannot lie in  $\operatorname{Diff}_{\widetilde{S}}(0)$  since  $\widetilde{S}$  is smooth at a generic point on C. Hence  $K_{\widehat{S}} \cdot \nu_*^{-1}(C) < -1$ . On the other hand,

$$K_{\widehat{S}} \cdot \nu_*^{-1}(C) \ge -1$$

since the curve  $\nu_*^{-1}(C)$  is contractible.

We have proved that the extremal contraction g is a contraction of a subvariety F of V to a subvariety Z of W such that  $\dim F - \dim Z = 1$  and  $\dim Z \ge 1$ . Let C be a generic fibre of g over Z. Then

Let C be a generic fibre of g over Z. Then

$$\left(K_V + \frac{n-1}{n}\widetilde{S}\right) \cdot C = \left(f^*(\mathcal{O}_{\mathbb{P}^{n-1}}(-1)) - E\right) \cdot C = -\deg f_*(C) - E \cdot C < -\deg f_*(C)$$

because the curve C meets the exceptional locus of the birational morphism f.

**Lemma 4.7.** If g is not a conic bundle, then  $F = \tilde{S}$ .

*Proof.* Let C be a sufficiently generic fibre of the morphism g. The inequality

$$-1 \leqslant K_V \cdot C = \left(K_V + \frac{n-1}{n}\widetilde{S}\right) \cdot C - \frac{n-1}{n}\widetilde{S} \cdot C < -\deg f_*(C) - \frac{n-1}{n}\widetilde{S} \cdot C$$

shows that  $F \subset \widetilde{S}$ . If the codimension of F in V is greater than 1, then

$$-1 \ge -\deg f_*(C) > \left(K_V + \frac{n-1}{n}\widetilde{S}\right) \cdot C = (K_V + \widetilde{S}) \cdot C - \frac{1}{n}\widetilde{S} \cdot C > -1 - \frac{1}{n}\widetilde{S} \cdot C > -1,$$

where the next to last inequality is a consequence of Lemma 4.5. Hence the subvariety  $F \subset V$  has codimension 1 and  $F = \tilde{S}$ .

We now know that the extremal contraction g is either a conic bundle or a contraction of the divisor  $\widetilde{S}$  to a subvariety Z of W of dimension n-3.

**Theorem 4.8.** If the log minimal model program holds in dimension n - 1, then Conjecture 2.5 holds for dimension n.

*Proof.* Let g be a conic bundle. Then  $\lfloor E \rfloor \cap C \neq \emptyset$  since no components of the divisor E lie in fibres of g. Hence

$$\deg f_*(C) = -\left(K_V + \frac{n-1}{n}\widetilde{S} + E\right) \cdot C = 2 - \frac{n-1}{n}\widetilde{S} \cdot C - E \cdot C < 2.$$

Consequently,  $f_*(C)$  is a line on S and S is a cone in  $\mathbb{P}^{n-1}$ .

Assume now that g contracts the divisor  $\widetilde{S}$  to an (n-3)-dimensional subvariety of Z. Then

$$-\deg f_*(C) > \left(K_V + \frac{n-1}{n}\widetilde{S}\right) \cdot C = (K_V + \widetilde{S}) \cdot C - \frac{1}{n}\widetilde{S} \cdot C = -2 - \frac{1}{n}\widetilde{S} \cdot C > -2.$$

Thus,  $f_*(C)$  is a line on S. Consequently, S is a cone in  $\mathbb{P}^{n-1}$ .

**Corollary 4.9.** Conjecture 2.5 holds for n = 4.

**Corollary 4.10.** The total log canonical threshold of a smooth quartic X in  $\mathbb{P}^4$  is equal to  $\frac{3}{4}$  if and only if the quartic X has an Eckardt point.

#### § 5. An application

Let  $\mathcal{O}$  be a discrete valuation ring with quotient field K. We assume that the residue field has characteristic zero.

For a scheme  $\pi: X \to \text{Spec } \mathcal{O}$  we denote its scheme-theoretic fibre  $\pi^*(O)$  by  $S_X$ , where O is the closed point of Spec  $\mathcal{O}$ .

**Theorem 5.1.** Let X and Y be two smooth Fano fibrations over Spec O with generic fibres isomorphic to a smooth hypersurface of degree n in  $\mathbb{P}^n_K$ , where  $n \ge 3$ . Then each birational map of X into Y over Spec O that is biregular on a generic fibre is biregular.

*Proof.* Note that our birational map cannot be an isomorphism in codimension one (see [12]). The anticanonical divisors of  $S_X$  and  $S_Y$  are very ample. Moreover, their total log canonical thresholds are strictly larger than  $\frac{1}{2}$ . The required result can therefore be obtained with the help of the same method as in [7].

We point out that Theorem 5.1 follows from the birational rigidity of smooth hypersurfaces of degree n in  $\mathbb{P}^n$ . However, birational rigidity has been established so far only for generic smooth hypersurfaces of degree n > 3 in  $\mathbb{P}^n$  and for all smooth hypersurfaces of degree  $n \in [4, 8]$  in  $\mathbb{P}^n$  (see [13]–[16]).

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