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Anticanonical models of three-dimensional Fano varieties of degree 4

I. A. Cheltsov

Abstract. All birational transformations of three Fano 3-folds of degree 4 into canonical Fano 3-folds, elliptic fibrations, and fibrations of K3 surfaces are described.

Bibliography: 9 titles.

All varieties under consideration are assumed to be complex projective. The main definitions, concepts, and notation can be found in [1], [2]. The author wishes to thank M. M. Grinenko, V. A. Iskovskikh, Yu. G. Prokhorov, A. V. Pukhlikov, and V. V. Shokurov for useful conversations.

§ 1. Introduction

Let X be one of the following 3-folds: a smooth quartic; a general¹ quartic with simple double point O ; a smooth double cover of a quadric Q ramified in an octic surface S . Then X is a Fano 3-fold with terminal \mathbb{Q} -factorial singularities, $\text{Pic } X = \mathbb{Z}K_X$, $-K_X^3 = 4$, and $\varphi_{|-K_X|}$ is an embedding for a quartic and a double cover otherwise.

Definition 1.1. A Fano variety V with terminal \mathbb{Q} -factorial singularities and Picard group \mathbb{Z} is said to be *birationally rigid* if V is not birationally equivalent to another (non-isomorphic to V) Fano variety with terminal \mathbb{Q} -factorial singularities and Picard group \mathbb{Z} , or to a fibration with fibres of Kodaira dimension $-\infty$.

A Fano 3-fold X is birationally rigid (see [3]–[5]), but a priori X can be birationally transformed into a canonical or even terminal Fano 3-fold distinct from X .

Example 1.2. Let X be a general quartic 3-fold with simple double point O and let $\rho: V \rightarrow X$ be a blow up of O . Then $\psi_O = \varphi_{|-2K_V|} \circ \rho^{-1}$ is birational and $X_O = \psi_O(X)$ is a terminal Fano 3-fold with $-K_{X_O}^3 = 2$.

Proposition 1.3. *Let X be a general quartic 3-fold with simple double point O . Then for each line C on X passing through O there exists birational transformation $\psi_C: X \dashrightarrow X_C$ such that X_C is a Fano 3-fold with terminal singularities and $-K_{X_C}^3 = \frac{1}{2}$.*

¹We shall assume that X contains precisely 24 lines passing through the singular point.

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Proposition 1.4. *Let X be a smooth double cover of a quadric Q ramified in an octic surface S . Then for each curve C on X with $-K_X \cdot C = 1$ there exists a birational map $\psi_C: X \dashrightarrow X_C$ such that X_C is a canonical Fano 3-fold with $-K_{X_C}^3 = \frac{1}{2}$ and X_C has terminal singularities if and only if the image of C on Q does not lie in S .*

We prove Propositions 1.3 and 1.4 in § 2. The main aim of this paper is the proof of the following result.

Main theorem. *Let $Y \not\cong X$ be a canonical Fano 3-fold that is birationally equivalent to X . Then X is not a smooth quartic 3-fold; if X is a general quartic 3-fold with simple double point O , then $Y \cong X_O$ or $Y \cong X_C$ for a line C on X passing through O ; if X is a double cover of a smooth quadric ramified in an octic, then $Y \cong X_C$ for an irreducible curve C on X with $-K_X \cdot C = 1$.*

The group of birational automorphisms of the 3-fold X was described in [3]–[5]; in particular, there a set \mathcal{S}_X of birational involutions of X was described such that the sequence of groups $1 \rightarrow \prod_{\tau \in \mathcal{S}_X} \tau \rightarrow \text{Bir } X \rightarrow \text{Aut } X \rightarrow 1$ is exact; $\mathcal{S}_X = \emptyset$ for a smooth quartic 3-fold, \mathcal{S}_X consists of 25 involutions in the case of a quartic 3-fold with a simple double point – one of them is induced by the double point and 24 involutions are induced by the lines on the quartic passing through the singular point; in the case of a smooth double cover of a quadric Q ramified in an octic S all involutions in the set \mathcal{S}_X are induced by curves C in X with $-K_X \cdot C = 1$ such that their images on Q do not lie in S . Thus, \mathcal{S}_X is in a one-to-one correspondence with the terminal Fano manifolds birationally equivalent but non-isomorphic to X .

Proposition 1.5. *Let V be a birationally rigid Fano 3-fold, and G a finite subgroup of $\text{Bir } V$. Then there exists a birational map $\gamma: V \dashrightarrow Y$ such that Y is a Fano 3-fold with terminal singularities and $\gamma \circ G \circ \gamma^{-1} \subset \text{Aut } Y$.*

Proof. Let $K(V)$ be the field of rational functions on V , W a normal projective model of the field $K(V)^G$, and Y a normalization of the 3-fold W in $K(V)$. Then Y is birationally equivalent to V and the group G acts biregularly on Y . The required result now follows from the G -equivariant resolution of singularities of Y and the G -equivariant Minimal Model Program.

Thus, for each $\tau \in \mathcal{S}_X$ there exists a birational map $\gamma: X \dashrightarrow Y$ onto a terminal Fano 3-fold Y such that $\gamma \circ \tau \circ \gamma^{-1}$ is biregular on Y . Conversely, on each terminal Fano variety Y that is birationally isomorphic but non-equivalent to X there exists a biregular involution τ_Y inducing a birational involution in the set \mathcal{S}_X .

The variety X is not birationally equivalent to a fibration of varieties of Kodaira dimension $-\infty$; however, X is always birationally equivalent to a fibration of varieties of Kodaira dimension 0.

Example 1.6. Let \mathcal{H} be a pencil in $|-K_X|$, $L \subset \mathbb{P}^4$ a line in the anticanonical image of X , and let γ be the composite of $\varphi_{|-K_X|}$ and the projection from L . Then the resolution of indeterminacy of γ transforms X birationally into an elliptic fibration and the resolution of indeterminacy of $\varphi_{\mathcal{H}}$ produces a birational transformation of it into a fibration of K3 surfaces.

The fibrations described in Example 1.6 can be considered from a common point of view: they are defined by projections from linear subspaces of \mathbb{P}^4 restricted to

the anticanonical image of X . For this reason we shall call them simply “fibrations defined by projections”.

The results of [2] and [6], [7] lead one to the natural suggestion that up to an action of $\text{Bir } X$ and birational equivalence of fibrations, fibrations defined by projections are the only fibrations of varieties of Kodaira dimension 0 that are birationally equivalent to X . Moreover, this has been *proved* in [2] for a smooth quartic 3-fold and a smooth double cover of a quadric 3-fold, but the proof there contained a gap, and one type of fibration was left out, as pointed out to this author by Iskovskikh (see [8]).

Definition 1.7. A smooth point P in X is said to be *K-special* if for some positive integer m the linear system $|-mK_V|$ has no fixed components, where V is a blow up of P .

Proposition 1.8. Let $\rho: V \rightarrow X$ be a blow up of a smooth point P in X and let $m \in \mathbb{N}$ be an integer such that $|-mK_V|$ has no fixed components. Then $|-mK_V|$ is a pencil and its general element is a union of surfaces of Kodaira dimension zero.

Proof. Consider a general element $H \in |-K_X|$ passing through P and its proper transform \tilde{H} on V . Then $|-nK_V - \tilde{H}| = \emptyset$ for $n \in \mathbb{N}$, because otherwise $|-mK_V|$ has a fixed component. Now, $K_V^2 \cdot \tilde{H} = 0$ and $H^0(\mathcal{O}_V(-nK_V)) \subset H^0(\mathcal{O}_{\tilde{H}}(-nK_V))$, therefore the linear system $|-mK_V|$ is a pencil.

Assume now that m is the smallest positive integer such that $|-mK_V|$ has no fixed components. Let S_1 and S_2 be two sufficiently general surfaces in $|-mK_V|$ and $\overline{S}_1, \overline{S}_2$ their respective images on X . The surfaces \overline{S}_1 and \overline{S}_2 are irreducible, reduced and have multiplicity $2m$ at P . To complete the proof we must show that S_1 and S_2 have Kodaira dimension 0.

Let $\mathcal{H} \subset |-K_X|$ be a linear system of surfaces passing through P , and H a general element of \mathcal{H} . Then it follows from the equality $\text{mult}_P(\overline{S}_1 \cdot \overline{S}_2) = 4m^2$ that $S_1 \cap S_2$ has no curves from the exceptional divisor of ρ and all curves in $\overline{S}_1 \cap \overline{S}_2$ are contracted by $\varphi_{\mathcal{H}}$. Hence if X is a quartic, then $\overline{S}_1 \cap \overline{S}_2$ is the union of all lines passing through P . If X is a double cover of a smooth quadric $Q \subset \mathbb{P}^4$, then it follows from the equality $H \cdot \overline{S}_1 \cdot \overline{S}_2 = \text{mult}_P(\overline{S}_1 \cdot \overline{S}_2)$ that P is the unique base point of the linear system \mathcal{H} contained in $\overline{S}_1 \cap \overline{S}_2$. Thus, the strict transforms of all curves in $\overline{S}_1 \cap \overline{S}_2$ on the surfaces S_1 and S_2 are simultaneously regularly contractible. The subadjunction formula shows that the canonical divisors of the resolutions of singularities of S_1 and S_2 are regularly contractible, therefore the Kodaira dimension of S_1 and S_2 is either 0 or $-\infty$; however, X is birationally rigid and can contain no pencils of surfaces of Kodaira dimension $-\infty$.

Corollary 1.9. Each *K-special* point on X defines a birational map of X into a fibration of surfaces of Kodaira dimension zero.

Example 1.10. Let $X \subset \mathbb{P}^4$ be a quartic and P a smooth point in X such that in a neighbourhood of P the local equation of X is $q_1 + q_2 + q_1q_2 + q_4 = 0$, where q_i is a homogeneous polynomial of degree i and the system $q_1 = q_2 = q_4 = 0$ has only finitely many homogeneous solutions. Then P is *K-special* and defines a birational transformation of X into a fibration of K3-surfaces with one multiple fibre of multiplicity 2, the strict transform of the hyperplane section of X tangent at P .

In this paper we prove the following result.

Auxiliary theorem. *Up to birational equivalence of fibrations and up to an action of the group $\text{Bir}(X)$ fibrations defined by projections and K -special points are the only fibrations of varieties of Kodaira dimension zero birationally equivalent to X .*

To complete the section we shall show that in the case of a double cover of a quadric 3-fold all fibrations given by K -special points are fibrations given by projections.

Lemma 1.11. *Let X be a smooth double cover of a quadric 3-fold Q ramified in an octic surface $S \subset Q$. Then a point P in X is K -special if and only if the image of P on Q lies in the ramification surface S .*

Proof. Assume that the image of P on Q lies in S . Let $\mathcal{H}_P \subset |-K_X|$ be a linear system of surfaces whose images on Q are tangent to S at the image of P . By construction each surface in \mathcal{H}_P is singular in P and \mathcal{H}_P has no fixed components, therefore P is K -special.

Assume that P is K -special, but its image on Q does not belong to S . Let $\rho: V \rightarrow X$ be a blow up of P , let S_1 and S_2 be general elements of $|-mK_V|$, and \overline{S}_1 and \overline{S}_2 their images on X , respectively, where m is the smallest integer such that $|-mK_V|$ has no fixed components. Then the \overline{S}_i are reduced, irreducible and have multiplicity $2m$ at P . Let $\mathcal{H} \subset |-K_X|$ be a linear system of divisors in $|-K_X|$ passing through P , and H a general element of \mathcal{H} . Then the base locus of \mathcal{H} contains a point $\hat{P} \neq P$. Now, the equality $\text{mult}_P(\overline{S}_1 \cdot \overline{S}_2) = 4m^2$ shows that $\hat{P} \notin \overline{S}_1 \cap \overline{S}_2$ and all curves in $\overline{S}_1 \cap \overline{S}_2$ are contracted by $\varphi_{\mathcal{H}}$. Hence the intersection $\overline{S}_1 \cap \overline{S}_2$ consists of finitely many curves C_λ such that $P \in C_\lambda$ and $-K_X \cdot C_\lambda = 1$. Let C be an arbitrary curve in $\overline{S}_1 \cap \overline{S}_2$, and \hat{C} a curve on X passing through \hat{P} whose image on Q coincides with the image of C . Consider the restriction of S_1 to a general surface D in $|-K_X|$ containing C and \hat{C} . By construction $S_1|_D = \text{mult}_C(S_1)C + B$ and the support of B does not contain C and \hat{C} , since otherwise S_1 contains \hat{C} and S_2 also contains \hat{C} in view of the generality of the S_i , whereas $\hat{P} \notin \overline{S}_1 \cap \overline{S}_2$. On the other hand, $C \cdot \hat{C} = 4$ and $C^2 = -2$ on D and $\text{mult}_P(B) \geq 2m - \text{mult}_C(S_1)$. Intersecting B with C and \hat{C} we obtain a contradiction: $m \geq 4 \text{mult}_C(S_1)$ and $3 \text{mult}_C(S_1) \geq m$.

Corollary 1.12. *Let X be a smooth double cover of a quartic 3-fold $Q \subset \mathbb{P}^4$ ramified in an octic surface $S \subset Q$, P a K -special point in X , and $\hat{P} \in S$ its image in Q . Then the fibration defined by P is a fibration of K3 surfaces defined by the projection from the two-dimensional linear subspace T of \mathbb{P}^4 tangent to S at the point \hat{P} .*

§ 2. Construction of two birational maps

Let X be a quartic 3-fold with simple double point O containing precisely 24 lines passing through O . Then $\text{Pic } X = \mathbb{Z}K_X$ and $-K_X$ is a hyperplane section of X . We fix a line C on X through O , consider the blow up $f: W \rightarrow X$ of O , set $Q = f^{-1}(O)$, blow up $g: V \rightarrow W$ the proper transform of C , and set $E = g^{-1}(f^{-1}(C))$ and $Q_V = g^{-1}(Q)$. Then $|-K_V|$ is free and $\psi_{|-K_V|}$ is an elliptic fibration with section Q_V . Its restriction to Q_V is birational and contracts two smooth rational

curves C_1 and C_2 . Note that the nef and big divisor $(f \circ g)^*(-K_X) - K_V$ has intersection zero only with C_1 and C_2 and there exists a flop $\rho: V \dashrightarrow \widehat{V}$ in C_1 and C_2 . The restriction of ρ to a general element of $|-K_V|$ is an isomorphism, $\psi|_{-K_{\widehat{V}}} = \psi|_{-K_V} \circ \rho$, and the linear system $|-K_{\widehat{V}}|$ is free. The surface $\rho(Q_V)$ is isomorphic to \mathbb{P}^2 and its normal bundle in \widehat{V} is $\mathcal{O}_{\mathbb{P}^2}(-2)$. We now contract by $\widehat{g}: \widehat{V} \rightarrow \widehat{W}$ the divisor $\rho(Q_V)$ into a point $P \cong \frac{1}{2}(1, 1, 1)$; then $-K_{\widehat{W}}^3 = \frac{1}{2}$ and the base locus of $|-K_{\widehat{W}}|$ consists of P . Hence $-K_{\widehat{W}}$ is nef and big and for $n \gg 0$ the linear system $|-nK_{\widehat{W}}|$ defines a birational morphism onto a normal variety X_C , which is terminal Fano with $-K_{X_C}^3 = \frac{1}{2}$. This proves Proposition 1.3.

We now prove Proposition 1.4.

Let $\theta: X \rightarrow Q \subset \mathbb{P}^4$ be a smooth double cover of a quartic 3-fold Q ramified in an octic S . Then $\text{Pic } X = \mathbb{Z}K_X$ and $-K_X \sim \theta^*(\mathcal{O}_{\mathbb{P}^4}(1)|_Q)$. We consider a smooth rational curve $C \subset X$ with $-K_X \cdot C = 1$ and its blow up $f: W \rightarrow X$, set $E = f^{-1}(C)$, and denote by C_1 the unique base curve of $|-K_W|$. Let $g: V \rightarrow W$ be a blow up of C_1 ; we set $G = g^{-1}(C_1)$. Then $|-K_V|$ is free and $\varphi|_{-K_V}$ is an elliptic fibration with section $G \cong \mathbb{F}_1$. Let $C_2 \subset G$ be an exceptional section. If $N_{C_2/V} \cong \mathcal{O}_{C_2}(m) \oplus \mathcal{O}_{C_2}(n)$, then $m + n = -2$ and the embedding of normal bundles $N_{C_2/G} \subset N_{C_2/V}$ shows that $N_{C_2/V} \cong \mathcal{O}_{C_2}(-1) \oplus \mathcal{O}_{C_2}(-1)$. Let $r: Y \rightarrow V$ be the blow up of C_2 and let $R = r^{-1}(C_2) \cong \mathbb{P}^1 \times \mathbb{P}^1$. We contract by $\widehat{r}: Y \rightarrow \widehat{V}$ the surface R to a curve $\widehat{C}_2 \subset \widehat{V}$ so that $\widehat{r} \circ r^{-1}$ is a flop in C_2 ; \widehat{r} contracts an exceptional section of the surface $r^{-1}(G) \cong \mathbb{F}_1$. We set $\widehat{G} = \widehat{r} \circ r^{-1}(G) \cong \mathbb{P}^2$; then $N_{\widehat{G}/\widehat{V}} \cong \mathcal{O}_{\widehat{G}}(-2)$. There exists a contraction $\widehat{g}: \widehat{V} \rightarrow \widehat{W}$ of the surface \widehat{G} into a singular point $\widehat{P} \cong \frac{1}{2}(1, 1, 1)$, and the map $\rho = \widehat{g} \circ \widehat{r} \circ r^{-1} \circ g^{-1}$ is a log flip for the log pair $(W, \varepsilon|-K_W|)$ for $\varepsilon > 1$, therefore \widehat{W} is projective and \mathbb{Q} -factorial, $-K_{\widehat{W}}^3 = \frac{1}{2}$, the base locus of $|-K_{\widehat{W}}|$ consists of the point \widehat{P} , $-K_{\widehat{W}}$ is nef and big, and for $n \gg 0$ the linear system $|-nK_{\widehat{W}}|$ defines a birational morphism onto a canonical Fano 3-fold X_C with $-K_{X_C}^3 = \frac{1}{2}$. If $\theta(C) \not\subset S$, then the map so constructed contracts no divisors on X , therefore the singularities of X_C are terminal. If $\theta(C) \subset S$ then our map contracts the surface spanned by the inverse images on X of the lines on Q tangent to S at $\theta(C)$, therefore X_C has a singularity of type \mathbb{A}_1 at the general point of one curve. The proof of Proposition 1.4 is now complete.

§ 3. Smooth quartic 3-fold

Let X be a smooth quartic 3-fold in \mathbb{P}^4 . Then $\text{Pic } X = -\mathbb{Z}K_X$ and $-K_X$ is rationally equivalent to a hyperplane section of X . Consider a movable log pair (X, M_X) such that $K_X + M_X \sim_{\mathbb{Q}} 0$.

Theorem 3.1. *The log pair (X, M_X) is canonical and $\text{CS}(X, M_X)$ is one of the following sets: \emptyset ; $\{P\}$ for a K -special point P ; $\{L\}$ for a line L in X ; $\{X \cap H\}$ for a two-dimensional linear subspace $H \subset \mathbb{P}^4$. Moreover, if $\text{CS}(X, M_X) = \{P\}$, then the boundary M_X lies in the fibres of the rational map defined by the K -special point P and if $\text{CS}(X, M_X)$ contains a curve, then M_X lies in the fibres of the projection from $\text{CS}(X, M_X)$.*

Note that Theorem 3.1 yields the Main Theorem for X (see [7]).

Proposition 3.2. *For a smooth quartic X Theorem 3.1 yields the Auxiliary Theorem.*

Proof. Assume that there exists a birational map $\theta: X \dashrightarrow Y$ of X into a fibration $\tau: Y \rightarrow Z$ with an elliptic curve or a surface of Kodaira dimension zero as a general fibre. Let $M_X = \lambda\theta^{-1}(|\tau^*(D)|)$ for a sufficiently big divisor D on Z and positive rational λ such that $K_X + M_X \sim_{\mathbb{Q}} 0$. Assume that (X, M_X) is terminal. Then $(X, \gamma M_X)$ is a canonical model for some $\gamma > 1$, but $\kappa(X, \gamma M_X) < 3$ by construction. Hence $\text{CS}(X, M_X) \neq \emptyset$ and by Theorem 3.1 either $\text{CS}(X, M_X) = \{P\}$ for a K -special point P in X , or $\text{CS}(X, M_X) = \{L\}$ for a line L , or $\text{CS}(X, M_X) = \{X \cap H\}$ for a two-dimensional linear subspace $H \subset \mathbb{P}^4$.

If $\text{CS}(X, M_X)$ is not a line or the dimension of Y is 2, then Theorem 3.1 yields the required result, therefore we shall assume that τ is a fibration of surfaces and $\text{CS}(X, M_X) = \{L\}$. Let $f: V \rightarrow X$ be a blow up of the line L ; we set $E = f^{-1}(L)$ and $M_V = f^{-1}(M_X)$. By Theorem 3.1 the boundary M_V lies in the fibres of the elliptic fibration $\varphi|_{-K_V|}$ and $K_V + M_V \sim_{\mathbb{Q}} 0$. The singularities of (V, M_V) are not terminal because otherwise $\kappa(V, \delta M_V) = 2$ for $\delta > 1$, but $\kappa(V, \delta M_V) \leq 1$ by construction. Hence $\text{CS}(V, M_V) \neq \emptyset$. Assume that $\text{CS}(V, M_V)$ contains a point P_V . Then $f(P_V) \in \text{CS}(X, M_X)$, which contradicts our assumption. Hence the set $\text{CS}(V, M_V)$ contains a curve $\widehat{L} \subset V$ such that $\widehat{L} \subset E$, but M_V lies in the fibres of $\varphi|_{-K_V|}$ and \widehat{L} is contracted by $\varphi|_{-K_V|}$.

Let \mathcal{H} be a pencil of surfaces in $|-K_V|$ passing through \widehat{L} . The rational map $\varphi_{\mathcal{H}} \circ f^{-1}$ is a projection from some 2-dimensional subspace T of \mathbb{P}^4 containing L . Let $g: W \rightarrow V$ be the resolution of indeterminacy of $\varphi_{\mathcal{H}}$, let $M_W = g^{-1}(M_V)$ and $h = \varphi_{\mathcal{H}} \circ g^{-1}$, and consider a general fibre D_W of h . Then

$$M_W|_{D_W} \sim_{\mathbb{Q}} \sum_{i \in I} c_i F_i|_{D_W}$$

for some rational c_i and g -exceptional divisors F_i , where at least one g -exceptional divisor dominating \widehat{L} does not lie in the support of $\sum_{i \in I} c_i F_i$. By examining all possible cases one can show that the intersection form of the curves $F_i|_{D_W}$ on the surface D_W is negative-definite. In view of the movability of $M_W|_{D_W}$, this means that $M_W|_{D_W} = \emptyset$. Hence M_W lies in the fibres of h , which completes the proof and also shows that L is cut on X by T with multiplicity 4.

In the rest of this section we prove Theorem 3.1. The canonicity of each movable log pair on X with \mathbb{Q} -rationally trivial log canonical divisor is equivalent to the birational rigidity of X , therefore we shall assume that (X, M_X) has canonical singularities.

Lemma 3.3. *Suppose that $\text{CS}(X, M_X)$ contains a point P . Then P is K -special, $\text{CS}(X, M_X) = \{P\}$, and the boundary M_X lies in the fibres of the map defined by the K -special point P .*

Proof. Let H_P be a general hyperplane section of X through P . The inequality $H_P \cdot M_X^2 \geq \text{mult}_P(M_X^2)$ and Pukhlikov's local inequality in the form of Theorem 4.1 of [2] show that $\text{mult}_P(M_X) = 2$. Let $f: W \rightarrow X$ be a blow up of the point P , and

$M_W = f^{-1}(M_X)$. Then $M_W \sim_{\mathbb{Q}} -K_W$ and $|-nK_W|$ has no fixed components for some $n \gg 0$, the point P is K -special, and $|-nK_W|$ is a pencil by Proposition 1.8. It follows from the equality $\text{mult}_P(M_X^2) = 4$ that the support of M_X^2 is the union of all lines passing through P . In particular, there are only finitely many lines on X passing through P . The base locus of $f(|-nK_W|)$ also consists of all lines passing through P , therefore M_X lies in the fibres of $\varphi_{f(|-nK_W|)}$.

Assume that $\text{CS}(X, M_X)$ contains elements distinct from P . We have already proved that the support of the effective cycle M_X^2 is the union of all lines passing through P . Hence there exists a line C on X passing through P such that either $C \in \text{CS}(X, M_X)$ or $\text{CS}(X, M_X)$ contains another K -special point on C . Consider a general hyperplane section H_C of X containing C . The surface H_C is smooth and we have $M_X|_{H_C} = \text{mult}_C(M_X)C + B_C$ for some movable boundary B_C . The inequality $B_C^2 \geq (2 - \text{mult}_C(M_X))^2$ shows that $\text{mult}_C(M_X) \leq \frac{2}{3}$. If the set $\text{CS}(X, M_X)$ contains another point in C , then we arrive at a contradiction: $B_C^2 = 4 - 2\text{mult}_C(M_X) - 2\text{mult}_C^2(M_X) \geq 2(2 - \text{mult}_C(M_X))^2$.

We shall assume that $\text{CS}(X, M_X)$ contains no points, but it contains an irreducible reduced curve C . The canonicity of (X, M_X) means that $\text{mult}_C(M_X) = 1$. It follows from the inequality $4 = -K_X \cdot M_X^2 \geq -K_X \cdot C$ that the degree of $C \subset \mathbb{P}^4$ is at most 4.

Lemma 3.4. *The curve C lies in a 2-dimensional linear subspace of \mathbb{P}^4 .*

Proof. Assume that the assertion does not hold. Then C is either a smooth curve of degree 3 or 4, or it is a rational curve of degree 4 with one double point.

Let C be a smooth curve, $f: W \rightarrow X$ a blow up of it, $E = f^{-1}(C)$, $M_W = f^{-1}(M_X)$, and $d = -K_X \cdot C$. The base locus of the linear system $|f^*(-dK_X) - qE|$ contains no curves; however, $(f^*(-dK_X) - E) \cdot M_W^2 < 0$.

Thus, we can assume that C is a rational quartic curve with double point P . Let $f: W \rightarrow X$ be the composite of the blow up of P and the strict transform of C and let G and E be f -exceptional divisors such that $f(E) = C$ and $f(G) = P$. Then $|f^*(-4K_X) - E - 2G|$ has no base curves, but $(f^*(-4K_X) - E - 2G) \cdot M_W^2 < 0$ for $M_W = f^{-1}(M_X)$ because $2 > \text{mult}_P(M_X) \geq \text{mult}_C(M_X) = 1$.

Lemma 3.5. *If C is a line, then M_X lies in the fibres of the projection from C .*

Proof. Let $f: W \rightarrow X$ be the blow up of C , and $M_W = f^{-1}(M_X)$. Then the linear system $|-K_W|$ is free, the morphism $\varphi_{|-K_W|}$ is an elliptic fibration, and $M_W \sim_{\mathbb{Q}} -K_W$.

Lemma 3.6. *Let C be a line such that $\text{CS}(X, M_X) \neq \{C\}$. Then there exists a 2-dimensional linear subspace T of \mathbb{P}^4 containing C such that $\text{CS}(X, M_X) = \{X \cap T\}$ and the boundary M_X lies in the fibres of the projection from T .*

Proof. Let $f: V \rightarrow X$ be a blow up of the line C , let E be the exceptional divisor of f , and $M_V = f^{-1}(M_X)$. By Lemma 3.5, M_V lies in the fibres of $\varphi_{|-K_V|}: V \rightarrow \mathbb{P}^2$. By assumption $\text{CS}(V, M_V)$ contains a curve \widehat{L} not lying in E . The movability of M_V means that \widehat{L} is contracted by $\varphi_{|-K_V|}$. Let \mathcal{H} be a pencil of surfaces in $|-K_V|$ passing through \widehat{L} , and D a general element of \mathcal{H} . Then D is a K3 surface with

at most canonical singularities, the map $\varphi_{\mathcal{H}} \circ f^{-1}$ is a projection from some 2-dimensional linear subspace T of \mathbb{P}^4 containing both C and $L = f(\widehat{L})$. Moreover, C is cut on X by T with multiplicity at most 3. The required result follows now by the arguments used at the end of the proof of Proposition 3.2.

Lemma 3.7. *If the degree of C is 4, then M_X lies in the fibres of the projection from C .*

Proof. By Lemma 3.4 the quartic curve C is cut on X by a 2-dimensional linear subspace $T \subset \mathbb{P}^4$. Let \mathcal{H}_T be a pencil of hyperplane sections of X passing through T , let $f: W \rightarrow X$ be a resolution of indeterminacy of $\varphi_{\mathcal{H}_T}$, and $g = \varphi_{\mathcal{H}_T} \circ f^{-1}$, where f is biregular outside C and W is smooth and contains one exceptional divisor E dominating C . Let $M_W = f^{-1}(M_X)$ and let D be a general fibre of g . Then $M_W|_D$ is equivalent to $\sum_{i \in I} c_i F_i|_D$ for some $c_i \in \mathbb{Q}$, where all the $f(F_i)$ are points in C . The movability of $M_W|_D$ means that $M_W|_D = \emptyset$. Hence M_W lies in the fibres of g , which yields the required result.

Lemma 3.8. *Let C be a conic or a cubic lying on a 2-dimensional linear subspace T . Then the set $\text{CS}(X, M_X)$ consists of all irreducible components of $X \cap T$ and M_X lies in the fibres of the projection from T .*

Proof. Let \mathcal{H}_T be a pencil of hyperplane sections of the quartic X containing T , and D a general surface in \mathcal{H}_T . Then D is a K3 surface with canonical singularities. Let C be a plane cubic. Then $X \cap T = C \cup L$ and $M_X|_D = C + \text{mult}_L(M_X)L + B_D$ for some movable boundary B_D equivalent to $(1 - \text{mult}_L(M_X))L$. Since $L^2 < 0$, it follows that $L \in \text{CS}(X, M_X)$, and the required result follows from Lemma 3.6.

Let C be a conic and let $X \cap T = C \cup L_1 \cup L_2$ for lines L_1 and L_2 , $L_1 \neq L_2$. Then $M_X|_D = C + \text{mult}_{L_1}(M_X)L_1 + \text{mult}_{L_2}(M_X)L_2 + B_D$ for a movable boundary B_D equivalent to $(1 - \text{mult}_{L_1}(M_X))L_1 + (1 - \text{mult}_{L_2}(M_X))L_2$, and the intersection form of the lines L_1 and L_2 is negative-definite. Thus, $L_i \in \text{CS}(X, M_X)$ and the required result follows from Lemma 3.6.

Let C be a conic and let $X \cap T = C \cup R$ for a line R on X . Then we have the equality $M_X|_D = C + \alpha R + B_D$ for some rational $\alpha \geq \text{mult}_R(M_X)$ and a movable boundary B_D . However, B_D is equivalent to $(2 - \alpha)R$ and $R^2 < 0$, which means that the boundary B_D is empty. On the other hand, the base locus of the pencil \mathcal{H}_T consists of C and R , which shows that M_X lies in the fibres of $\varphi_{\mathcal{H}_T}$ and $R \in \text{CS}(X, M_X)$.

Let C be a conic and $X \cap T = C$. Consider a blow up $f: V \rightarrow X$ of C and let $E = f^{-1}(C)$ and $M_V = f^{-1}(M_X)$. Let S be a general surface in $|-K_V| = f^{-1}(\mathcal{H}_T)$. Then for some movable boundary B_S on S we have $M_V|_S = \text{mult}_{\widehat{C}}(M_V)\widehat{C} + B_S$, where \widehat{C} is the unique base curve of $|-K_V|$. On the other hand, the boundary B_S is equivalent to $\text{mult}_{\widehat{C}}(M_V)\widehat{C}$ and $\widehat{C}^2 < 0$. Hence $B_S = \emptyset$ and M_V lies in the fibres of $\varphi_{|-K_V|}$ because the base locus of $|-K_V|$ is \widehat{C} .

The proof of Theorem 3.1 is now complete.

§ 4. Singular quartic 3-fold

Let X be a quartic 3-fold with one simple double point O such that there exist precisely 24 lines on X passing through O . In Example 1.2 we constructed a birational map $\psi_O: X \dashrightarrow X_O$ onto a terminal Fano 3-fold with $-K_{X_O}^3 = 2$, and in § 2 for each line $C \subset X$ containing O we constructed a birational map $\psi_C: X \dashrightarrow X_C$ onto a terminal Fano 3-fold with $-K_{X_C}^3 = \frac{1}{2}$.

Proposition 4.1. *Suppose that there exists a birational map $\sigma: X \dashrightarrow Y$ onto a canonical Fano 3-fold Y . Let $M_Y = \frac{1}{n}|-nK_Y|$ for some $n \gg 0$ and assume that the log pairs (X, M_X) , (X_O, M_{X_O}) , and (X_C, M_{X_C}) are birationally equivalent to the log pair (Y, M_Y) . Then up to an action of $\text{Bir } X$ one of the log pairs (X, M_X) , (X_O, M_{X_O}) , or (X_C, M_{X_C}) has semiterminal singularities² and a \mathbb{Q} -rationally trivial log canonical divisor.*

It follows from the uniqueness of the canonical model that Proposition 4.1 yields the Main Theorem for a quartic X (see [7]). The next result is equivalent to the birational rigidity of X , which is proved in [5].

Proposition 4.2. *For a movable log pair (X, M_X) of Kodaira dimension 0 there exists an automorphism $\tau \in \text{Bir } X$ such that the singularities of $(X, \tau(M_X))$ are canonical and $K_X + \tau(M_X) \sim_{\mathbb{Q}} 0$.*

For a proof of Proposition 4.1 we require a description of the set of centres of canonical singularities of log pairs with canonical singularities and Kodaira dimension 0 on a quartic X and one additional auxiliary result.

Theorem 4.3. *Let (X, M_X) be a canonical movable log pair with $K_X + M_X \sim_{\mathbb{Q}} 0$. Then $\text{CS}(X, M_X)$ is one of the following sets: \emptyset ; $\{O\}$; $\{P\}$ for a K -special point P ; $\{C\}$ for a line C passing through the point O ; $\{O, C\}$ for a line C passing through O ; $\{C\}$ for a line C not passing through O ; $\{O, X \cap H\}$ for a two-dimensional linear subspace H passing through O ; $\{X \cap H\}$ for a two-dimensional linear subspace H not passing through O . Moreover, assume in addition that $\text{CS}(X, M_X)$ is non-empty and does not consist of a point O or a line C on X passing through O . If $\text{CS}(X, M_X)$ contains a curve, then M_X lies in the fibres of the projection from $\text{CS}(X, M_X)$, and if $\text{CS}(X, M_X) = \{P\}$ for a K -special point P , then the movable boundary M_X lies in the fibres of the map defined by the K -special point P .*

Lemma 4.4. *Let (V, B_V) be a log pair such that $V \cong \mathbb{P}^1 \times \mathbb{P}^1$ and B_V is an effective divisor of type (α, β) in $\text{Pic } V \otimes \mathbb{Q}$ for some α and β in $\mathbb{Q} \cap [0, 1)$. Then the log pair (V, B_V) has at most log terminal singularities.*

Proof. Intersecting B_V with the rulings on V we conclude that $\text{LCS}(V, B_V)$ contains no curves. Assume that $\text{LCS}(V, B_V)$ contains a point $O \in V$. Let H be a divisor on V of type $(1 - \alpha, 1 - \beta)$ in the group $\text{Pic } V \otimes \mathbb{Q}$. Then H is ample and $H^0(\mathcal{O}_V(D)) = 0$ for $D = K_V + B_V + H$; however, by Shokurov's vanishing theorem (see [2]) the map $H^0(\mathcal{O}_V(D)) \rightarrow H^0(\mathcal{O}_{\mathcal{L}(V, B_V)}(D))$ is surjective.

²A log pair (X, M_X) is semiterminal if there exists $\lambda > 1$ such that $(X, \lambda M_X)$ is canonical.

Proposition 4.5. *Theorem 4.3 yields the Main Theorem for the quartic X .*

Proof. Assume that there exists a non-trivial birational map $\sigma: X \dashrightarrow Y$ onto a canonical Fano 3-fold Y . Let $M_Y = \frac{1}{n}|-nK_Y|$ for a sufficiently large integer n and let $M_X = \sigma^{-1}(M_Y)$. Then $\kappa(X, M_X) = 0$ and by Proposition 4.2 we can replace σ by its composite with a birational automorphism of X so that the singularities of the movable log pair (X, M_X) are canonical and $K_X + M_X \sim_{\mathbb{Q}} 0$.

Let $M_{X_O} = \psi_O(M_X)$ and let $M_{X_C} = \psi_C(M_X)$ for an arbitrary line C on X passing through the point O , where ψ_O and ψ_C are the birational maps from Example 1.2 and Proposition 1.3, respectively. By Proposition 4.1, for the proof of the Main Theorem it is sufficient to show that one of the log pairs (X, M_X) , (X_O, M_{X_O}) , and (X_C, M_{X_C}) has semiterminal singularities and a \mathbb{Q} -rationally trivial log canonical divisor.

We may assume that the singularities of the log pair (X, M_X) are not terminal. Then Theorem 4.3 and the construction of the movable boundary M_Y imply that either $\text{CS}(X, M_X) = \{O\}$ or $\text{CS}(X, M_X) = \{C\}$ for some line C on X containing the singular point O .

Let $\text{CS}(X, M_X) = \{C\}$ for some line C on the quartic X passing through O . We consider the blow ups $f: W \rightarrow X$ of O and $g: V \rightarrow W$ of the curve $f^{-1}(C)$. Let E be the g -exceptional divisor, $Q_V = g^{-1}(f^{-1}(O))$, and $M_V = (f \circ g)^{-1}(M_X)$. Then $K_V + M_V \sim_{\mathbb{Q}} a(V, M_V, Q_V)Q_V$ and $E \cong \mathbb{P}^1 \times \mathbb{P}^1$.

Assume that (V, M_V) is not terminal. Then the set $\text{CS}(V, M_V)$ contains a curve $C_V \subset E$ dominating the line C . This, however, contradicts the relation $M_V \cdot D = a(V, M_V, Q_V) \in \mathbb{Q} \cap (0, 1)$, where D is a fibre of the morphism $\varphi_{|-K_V|}$ passing through some point of the curve C_V . Hence the singularities of (V, M_V) are terminal and it follows from the construction of ψ_C that (X_C, M_{X_C}) has semiterminal singularities and $K_{X_C} + M_{X_C} \sim_{\mathbb{Q}} 0$.

To complete the proof we may assume that the singular point O is the unique centre of canonical singularities of the log pair (X, M_X) . Let $f: W \rightarrow X$ be a blow up of O , let $Q = f^{-1}(O)$, and $M_W = f^{-1}(M_X)$. Then Theorem 3.11 of [9] shows that $K_W + M_W \sim_{\mathbb{Q}} f^*(K_X + M_X)$. Moreover, the morphism from W into X_O is crepant for (W, M_W) . Thus, it remains to show that the log pair (W, M_W) has terminal singularities. All elements of the set $\text{CS}(W, M_W)$ lie in the exceptional divisor Q .

Assume that $\text{CS}(W, M_W)$ contains a point $O_W \in Q$ that belongs to no strict transform of a line on X passing through O . Then Pukhlikov's local inequality in the form of Theorem 4.1 in [2] shows that $\text{mult}_{O_W}(M_W^2) \geq 4$. Intersecting M_W^2 with a general surface in $|-K_W|$ passing through O_W we obtain a contradiction.

Assume that $\text{CS}(W, M_W)$ contains a point $O_W \in Q$ such that $O_W \in f^{-1}(C)$ for some line C on X . Let $g: V \rightarrow W$ be a blow up of $f^{-1}(C)$, let E be the exceptional divisor, and $M_V = g^{-1}(M_W)$. By Shokurov's connectedness theorem $\text{LCS}(E, M_V|_E) \neq \emptyset$, which contradicts Lemma 4.4. Hence the set $\text{CS}(W, M_W)$ does not contain points.

Assume that $\text{CS}(W, M_W)$ contains a curve Z in Q . Intersecting M_W with the rulings on $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ we conclude that Z is itself one such fibre. Let $g: V \rightarrow W$ be a blow up of Z , let H be a general surface in the pencil $|-K_V|$,

and $M_V = g^{-1}(M_W)$. Then H is a K3 surface with at most canonical singularities and $M_V|_H = \text{mult}_{\widehat{Z}}(M_V)\widehat{Z} + \sum_{i \in I} \alpha_i C_i + B_H$ for some rational numbers $\alpha_i \geq \text{mult}_{C_i}(M_V)$ and a movable boundary B_H , where \widehat{Z} and C_i are base curves of the pencil $| -K_V |$ such that $\varphi_{|-K_W|}(g(\widehat{Z})) = \varphi_{|-K_W|}(Z)$, and the curves C_i are mapped into lines on X passing through O . On the other hand, the intersection form of the curves \widehat{Z} and C_i on H is negative-definite and $M_V|_H$ is equivalent to $H|_H$, which is equivalent to $\widehat{Z} + \sum_{i \in I} a_i C_i$ for some positive integers a_i . Hence $B_H = \emptyset$ and M_V lies in the fibres of $\varphi_{|-K_V|}$, which in turn means that $f \circ g(C_i) \in \text{CS}(X, M_X)$. However, M_V cannot lie in the fibres of $\varphi_{|-K_V|}$ by construction and by our assumption that $\text{CS}(X, M_X) = \{O\}$.

Proposition 4.6. *Theorem 4.3 yields the auxiliary theorem for X .*

Proof. Assume that the quartic X can be birationally transformed by a map $\theta: X \dashrightarrow Y$ into a fibration $\tau: Y \rightarrow Z$ whose general fibre is either an elliptic curve or a surface of Kodaira dimension zero. Let $M_X = \lambda \theta^{-1}(|\tau^*(D_Z)|)$ for a sufficiently big divisor D_Z on Z and λ such that $K_X + M_X \sim_{\mathbb{Q}} 0$. By Proposition 4.2 we can assume that the movable log pair (X, M_X) has canonical singularities.

Assume that the log pair (X, M_X) has terminal singularities. Then for some rational γ the log pair $(X, \gamma M_X)$ is a canonical model, whereas $\kappa(X, \gamma M_X) < 3$ by construction. Hence $\text{CS}(X, M_X) \neq \emptyset$. Moreover, the same arguments show that the singularities of the log pairs (X_O, M_{X_O}) and (X_C, M_{X_C}) are not semi-terminal, where $M_{X_O} = \psi_O(M_X)$ and $M_{X_C} = \psi_C(M_X)$ for a line C on X passing through O and ψ_O and ψ_C are the birational maps from Example 1.2 and Proposition 1.3, respectively. Thus, it follows from the proof of Proposition 4.5 that $\text{CS}(X, M_X) \neq \{O\}$ and $\text{CS}(X, M_X) \neq \{C\}$ for all lines C on X containing O . By Theorem 4.3, $\text{CS}(X, M_X)$ is one of the following sets: $\{P\}$ for some K -special point P ; $\{O, C\}$ for a line C passing through the point O ; $\{C\}$ for a line C not passing through O ; $\{O, X \cap H\}$ for a 2-dimensional linear subspace H passing through O ; $\{X \cap H\}$ for a two-dimensional linear subspace H not passing through O . If $\text{CS}(X, M_X)$ contains a curve, then M_X lies in the fibres of the projection from $\text{CS}(X, M_X)$, while if $\text{CS}(X, M_X) = \{P\}$ for a K -special point P , then M_X lies in the fibres of the map defined by P . In particular, if τ is an elliptic fibration, or $\text{CS}(X, M_X) \neq \{O, C\}$ for any line C on X passing through O or $\text{CS}(X, M_X) \neq \{C\}$ for any line C on X not passing through O , then Theorem 4.3 yields all the required results. Thus, we can assume for the completion of the proof that τ is a fibration of surfaces of Kodaira dimension zero, and the set $\text{CS}(X, M_X)$ is either $\{O, C\}$ for a line C passing through C or $\{C\}$ for a line C not passing through C , and M_X lies in the fibres of the projection from C . The case when $\text{CS}(X, M_X) = \{C\}$ for some line C on X not passing through the point O is covered by the proof of Proposition 3.2. Hence we shall assume that the set $\text{CS}(X, M_X)$ consists of the point O and a line C on X passing through O .

Let $f: W \rightarrow X$ be a blow up of O , $Q = f^{-1}(O)$, and $M_W = f^{-1}(M_X)$. Then it follows from Theorem 3.1 in [9] that $\text{mult}_O(M_X) = 1$ and $K_W + M_W \sim_{\mathbb{Q}} 0$.

Assume that $\text{CS}(W, M_W)$ contains a point O_W . Then we have $O_W \in Q$ and $\text{mult}_{O_W}(M_W^2) \geq 4$ by Pukhlikov's local inequality in the form of Theorem 4.1 of [2]. If O_W does not lie in the strict transform of a line on X , then intersecting M_W^2

with the general surface in $|-K_W|$ passing through O_W we obtain a contradiction. Hence $O_W \in f^{-1}(L)$ for some line L on X . Let $g: V \rightarrow W$ be a blow up of $f^{-1}(L)$, E the exceptional divisor of g , and $M_V = g^{-1}(M_W)$. Then $E \cong \mathbb{P}^1 \times \mathbb{P}^1$ and by Shokurov's connectedness theorem $\text{LCS}(E, M_V|_E) \neq \emptyset$, in contradiction with Lemma 4.4.

Assume that $\text{CS}(W, M_W)$ contains a curve $Z \subset Q$. Then $f^{-1}(C) \cap Z \neq \emptyset$ and Z is a ruling on $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$. Let $g: V \rightarrow W$ be a blow up of the curve Z , let H be a sufficiently general divisor in $|-K_V|$, and let $M_V = g^{-1}(M_W)$. Then $M_V|_H = \text{mult}_{\widehat{Z}}(M_V)\widehat{Z} + \sum_{i \in I} \alpha_i C_i + B_H$ for some rational $\alpha_i \geq \text{mult}_{C_i}(M_V)$ and a movable boundary B_H , where \widehat{Z} and the C_i are base curves of the pencil $|-K_V|$ such that $\varphi|_{-K_W|}(g(\widehat{Z})) = \varphi|_{-K_W|}(Z)$, the images of the curves C_i on X are lines through the point O , and $f \circ g(C_j) = C$ for some $j \in I$. On the other hand, the intersection form of the curves \widehat{Z} and C_i on H is negative-definite and the divisor $M_V|_H$ is equivalent to $H|_H$, which in its turn is equivalent to $\widehat{Z} + \sum_{i \in I} a_i C_i$ for some $a_i \in \mathbb{N}$. Hence $B_H = \emptyset$ and M_V lies in the fibres of the rational map $\varphi|_{-K_V|}$, which yields the required result.

We can therefore assume that $\text{CS}(W, M_W) = \widehat{C} = f^{-1}(C)$. Let $g: V \rightarrow W$ be a blow up of \widehat{C} , E the g -exceptional divisor, $Q_V = g^{-1}(Q)$, and $M_V = g^{-1}(M_W)$. Then $K_V + M_V \sim_{\mathbb{Q}} 0$, $|-K_V|$ is free, $\varphi|_{-K_V|}$ is an elliptic fibration with section Q_V , M_V lies in the fibres of $\varphi|_{-K_V|}$, and $E \cong \mathbb{P}^1 \times \mathbb{P}^1$.

The set $\text{CS}(V, M_V)$ is non-empty, but $\text{CS}(V, M_V)$ does not contain points or curves contacted by g , for otherwise their images in W would belong to $\text{CS}(W, M_W) = \{\widehat{C}\}$. Hence $\text{CS}(V, M_V)$ contains a curve $\widetilde{C} \subset E$ dominating \widehat{C} . However, M_V lies in the fibres of $\varphi|_{-K_V|}$, therefore $\varphi|_{-K_V|}$ contracts \widetilde{C} ; on the other hand $\varphi|_{-K_V|}|_E$ does not contract curves.

In the rest of this section we prove Theorem 4.3. We fix a movable log pair (X, M_X) such that the singularities of (X, M_X) are canonical, $K_X + M_X \sim_{\mathbb{Q}} 0$, and $\text{CS}(X, M_X) \neq \emptyset$.

Lemma 4.7. *Assume that $\text{CS}(X, M_X)$ contains a smooth point P . Then P is K -special, $\text{CS}(X, M_X) = \{P\}$, and M_X lies in the fibres of a rational map defined by the K -special point P .*

Proof. It follows from the proof of Lemma 3.3 that P is K -special, $\text{mult}_P(M_X) = 2$, the boundary M_X lies in the fibres of the map defined by the point P , and either $\text{CS}(X, M_X) = \{P\}$ or there exists a centre of canonical singularities of (X, M_X) lying on some line C on X passing through P and O . Assume that $\text{CS}(X, M_X) \neq \{P\}$ and consider a general hyperplane section H_C of X passing through C . Then $M_X|_{H_C} = \text{mult}_C(M_X)C + B_C$ for some movable boundary B_C . Moreover, $\text{mult}_P(B_C) \geq 2 - \text{mult}_C(M_X)$ and $B_C^2 = 4 - 2\text{mult}_C(M_X) - \frac{3}{2}\text{mult}_C^2(M_X)$, therefore $\text{mult}_C(M_X) \leq \frac{4}{5}$. In particular, $C \notin \text{CS}(X, M_X)$ and either $\text{CS}(X, M_X)$ contains another K -special point on C or $O \in \text{CS}(X, M_X)$.

Assume that the line C contains two K -special points in $\text{CS}(X, M_X)$. Then $B_C^2 = 4 - 2\text{mult}_C(M_X) - \frac{3}{2}\text{mult}_C^2(M_X) \geq 2(2 - \text{mult}_C(M_X))^2$, which is impossible. Hence $O \in \text{CS}(X, M_X)$ and $\text{mult}_O(M_X) = 1$ by Theorem 3.11 of [9], which yields

the inequality $B_C^2 \geq (2 - \text{mult}_C(M_X))^2 + 2(1 - \frac{1}{2} \text{mult}_C(M_X))^2$, contradicting the equality $B_C^2 = 4 - 2 \text{mult}_C(M_X) - \frac{3}{2} \text{mult}_C^2(M_X)$.

We can assume that $\text{CS}(X, M_X)$ contains no smooth points, but contains a curve C such that the degree of $C \subset \mathbb{P}^4$ is at most 4 and $\text{mult}_C(M_X) = 1$.

Lemma 4.8. *The curve C lies in a two-dimensional linear subspace of \mathbb{P}^4 .*

Proof. Assume that the assertion is false. Then C is either a smooth curve or a rational curve of degree 4 with one double point. If $O \notin C$, then the proof of Lemma 3.4 leads to a contradiction, therefore we shall assume that $O \in C$.

Let C be a smooth curve of degree 3 or 4. Let $f: W \rightarrow X$ be a blow up of O , $g: V \rightarrow W$ a blow up of $f^{-1}(C)$, let E be the g -exceptional divisor, $Q_V = (f \circ g)^{-1}(O)$, $M_V = (f \circ g)^{-1}(M_X)$, and $d = -K_X \cdot C$. Then $(f \circ g)^*(-dK_X) - Q_V - E$ is a nef divisor and the boundary M_V is equivalent to $(f \circ g)^*(-K_X) - \text{mult}_O(M_X)Q_V - E$; however, $((f \circ g)^*(-dK_X) - Q_V - E) \cdot M_V^2 < 0$.

Let C be a rational curve of degree 4 with double point $P \neq O$. Let $h: U \rightarrow X$ be the composite of blow ups of the points P and O with a blow up of the strict transform of C , $M_U = h^{-1}(M_X)$, $Q_U = h^{-1}(O)$, $F = h^{-1}(P)$, and G an h -exceptional divisor such that $h(G) = C$. Then $M_U \sim_{\mathbb{Q}} h^*(-K_X) - \text{mult}_O(M_X)Q_U - \text{mult}_P(M_X)F - G$ and the linear system $|h^*(-4K_X) - Q_U - 2F - G|$ is free, whereas direct calculations show that $(h^*(-4K_X) - Q_U - 2F - G) \cdot M_U^2 < 0$.

Now let C be a rational curve of degree 4 with double point O . Let $f: W \rightarrow X$ be a blow up of O , and $g: V \rightarrow W$ a blow up of $f^{-1}(C)$. Let $M_V = (f \circ g)^{-1}(M_X)$. Then $M_V \sim_{\mathbb{Q}} (f \circ g)^*(-K_X) - \text{mult}_O(M_X)Q_V - E$, where $Q_V = (f \circ g)^{-1}(O)$ and $E = g^{-1}(f^{-1}(C))$. Direct calculations show that $((f \circ g)^*(-4K_X) - Q_V - E) \cdot M_V^2 < 0$, but it is easy to see that the linear system $|(f \circ g)^*(-4K_X) - Q_V - E|$ has no base curves.

Lemma 4.9. *Let C be a line and let $O \in \text{CS}(X, M_X)$ if $O \in C$. Then M_X lies in the fibres of the projection from C .*

Proof. In the case $O \notin C$ let $f: W \rightarrow X$ be a blow up of C , otherwise let $f: W \rightarrow X$ be the composite of a blow up of O with a blow up of the strict transform of C . Then $M_W \sim_{\mathbb{Q}} -K_W$, $M_W \cdot K_W^2 = 0$, $|-K_W|$ is free, and $\varphi_{|-K_W|}$ is an elliptic fibration. Hence M_W lies in the fibres of $\varphi_{|-K_W|}$.

Lemma 4.10. *Let C be a line, $O \notin C$, and assume that $\text{CS}(X, M_X)$ contains a curve $L \neq C$. Then there exists a two-dimensional linear subspace T of \mathbb{P}^4 containing both C and L such that $\text{CS}(X, M_X) = \{O, X \cap T\}$ if $O \in T$ and $\text{CS}(X, M_X) = \{X \cap T\}$ if $O \notin T$, and the boundary M_X lies in the fibres of the projection from T .*

Proof. Let $f: V \rightarrow X$ be a blow up of C , let E be the exceptional divisor of f , and let $M_V = f^{-1}(M_X)$. Then $|-K_V|$ is free, the morphism $\varphi_{|-K_V|}$ is an elliptic fibration, and M_V lies in the fibres of $\varphi_{|-K_V|}$ by Lemma 4.9, while $\text{CS}(V, M_V)$ contains $\widehat{L} = f^{-1}(L)$. Hence \widehat{L} is contracted to a point by $\varphi_{|-K_V|}$.

Let \mathcal{H} be a pencil of surfaces in $|-K_V|$ passing through \widehat{L} and let D be a general surface in \mathcal{H} . Then D is a K3 surface with canonical singularities and the map $\varphi_{\mathcal{H}} \circ f^{-1}$ is the projection from a two-dimensional linear subspace $T \subset \mathbb{P}^4$

containing both C and L . In the case $O \notin T$ the required result follows from the proof of Lemma 3.6. We shall therefore assume that $O \in T$.

Assume that M_V lies in the fibres of $\varphi_{\mathcal{H}}$ and $\text{CS}(X, M_X)$ contains all curves in $X \cap T$. For a resolution $h: U \rightarrow V$ of indeterminacy of $\varphi_{\mathcal{H}}$ we take a general fibre S of $\varphi_{\mathcal{H}} \circ h^{-1}$ and set $M_U = h^{-1}(M_V)$. Then $M_U|_S \sim_{\mathbb{Q}} (1 - \text{mult}_O(M_X))Q_U|_S$ for a divisor Q_U whose image on X is the point O . The divisor Q_U does not lie in the fibres of $\varphi_{\mathcal{H}} \circ h^{-1}$, therefore $O \in \text{CS}(X, M_X)$. To complete the proof we must show that all curves in $X \cap T$ belong to $\text{CS}(X, M_X)$ and M_V lies in the fibres of $\varphi_{\mathcal{H}}$. However, this follows from the end of the proof of Proposition 3.2.

Lemma 4.11. *Let C be a line passing through O and assume that $\text{CS}(X, M_X)$ contains a curve $L \neq C$. Then there exists a two-dimensional linear subspace $T \subset \mathbb{P}^4$ such that $\text{CS}(X, M_X) = \{O, X \cap T\}$ and M_X lies in the fibres of the projection from T .*

Proof. Let S be a general hyperplane section of X passing through C ; S is a K3 surface with one simple double point O . Consider the restriction $M_X|_S$, which is $C + B_S$ for a movable boundary B_S on S . Assume that C and L do not lie in a two-dimensional subspace of \mathbb{P}^4 . Then the intersection $S \cap L$ contains a point P such that $P \notin C$. Hence $1 > B_S^2 \geq \text{mult}_P^2(M_X) = 1$. This means that there exists a two-dimensional linear subspace $T \subset \mathbb{P}^4$ containing the curves C and L .

Assume that $\text{CS}(X, M_X)$ contains all curves in $X \cap T$ and M_X lies in the fibres of the projection ψ from T . Let $h: U \rightarrow V$ be the resolution of indeterminacy of ψ , S_U a general fibre of $\psi \circ h^{-1}$, and $M_U = h^{-1}(M_X)$. Then we have

$$M_U|_{S_U} \sim_{\mathbb{Q}} (1 - \text{mult}_O(M_X))Q_U|_{S_U},$$

where Q_U is a divisor such that $h(Q_U) = O$. The surface Q_U does not lie in the fibres of $\psi \circ h^{-1}$. Hence $O \in \text{CS}(X, M_X)$ and to complete the proof we must show that $\text{CS}(X, M_X)$ contains all components of $X \cap T$ and M_X lies in the fibres of the projection from T . But this follows from the end of the proof of Proposition 3.2.

Lemma 4.12. *If C is a quartic curve lying in a two-dimensional linear subspace $T \subset \mathbb{P}^4$, then M_X lies in the fibres of the projection from T and $O \in \text{CS}(X, M_X)$ if and only if $O \in T$.*

Proof. We consider a pencil \mathcal{H}_T of hyperplane sections of X passing through $X \cap T$ and resolve indeterminacy of $\varphi_{\mathcal{H}_T}$ by means of a birational morphism $f: W \rightarrow X$ such that W contains a unique f -exceptional divisor E dominating C and f is an isomorphism outside C . Let D be a general fibre of the fibration $g = \varphi_{\mathcal{H}_T} \circ f^{-1}$ of K3 surfaces and let $M_W = f^{-1}(M_X)$. Then

$$D \sim f^*(-K_X) - E - \sum_{i \in I} a_i F_i,$$

where the images of all the F_i on X are points in C and $a_i \in \mathbb{N}$. On the other hand,

$$M_W|_D \sim_{\mathbb{Q}} \sum_{i \in I} c_i F_i|_D \text{ for some } c_i \in \mathbb{Q}.$$

Hence the movability of $M_W|_D$ means that M_W lies in the fibres of the morphism g .

Assume now that the curve C contains the point O . In this case we can choose a birational morphism f such that $f = g \circ h$, where $g: V \rightarrow X$ is a blow up of O and h is birational. Let Q be an exceptional divisor of g and $Q_W = h^{-1}(Q)$. Then $F_j = Q_W$ for some $j \in I$ and $F_j|_D$ is non-trivial. However, $c_j = 1 - \text{mult}_O(M_X)$, therefore $O \in \text{CS}(X, M_X)$.

Lemma 4.13. *Let C be either a conic or a plane cubic lying in a two-dimensional linear subspace $T \subset \mathbb{P}^4$. Then the boundary M_X lies in the fibres of the projection from T and either $O \in T$ and $\text{CS}(X, M_X) = \{O, X \cap T\}$, or $O \notin T$ and $\text{CS}(X, M_X) = \{X \cap T\}$.*

Proof. Let \mathcal{H}_T be a pencil of hyperplane sections of the quartic X passing through C , and D a sufficiently general divisor in \mathcal{H}_T . Then D is a K3 surface with at most canonical singularities.

Let C be a plane cubic. Then $M_X|_D = C + \text{mult}_L(M_X)L + B_D$ for some line $L \subset D$ and a movable boundary B_D on D . Moreover, B_D is equivalent to $(1 - \text{mult}_L(M_X))L$ and $L^2 < 0$. Hence $L \in \text{CS}(X, M_X)$ and the required result follows from Lemmas 4.10 and 4.11.

Assume that C is a conic and $X \cap T = C \cup L_1 \cup L_2$ for lines L_1 and L_2 on X . From the numerical properties of $M_X|_D$ we see that $L_i \in \text{CS}(X, M_X)$ and the required result follows by Lemmas 4.10 and 4.11.

Let C be a conic and let $X \cap T = C$. If $O \notin C$, then the required result follows from the proof of Lemma 3.9, therefore we assume that $O \in C$. Let $f: V \rightarrow X$ be the composite of a blow up of O and a blow up of the strict transform of C , let $Q_V = f^{-1}(O)$, let E be an exceptional divisor dominating C , $M_V = f^{-1}(M_X)$, and S a general surface in $|-K_V|$. Then $M_V|_S = \text{mult}_{\widehat{C}}(M_V)\widehat{C} + \sum_{i=1}^k \alpha_i Z_i + B_S$ on S , where $\widehat{C} \subset E$, the $Z_i \subset Q_V$ are base curves of $|-K_V|$, and B_S is a movable boundary. However, on S the boundary B_S is \mathbb{Q} -rationally equivalent to $\text{mult}_{\widehat{C}}(M_V)\widehat{C} + \sum_{i=1}^k \beta_i Z_i$ for some $\beta_i \in \mathbb{Q}$ and the intersection form of the curves \widehat{C} and Z_i is negative-definite. Hence $B_S = \emptyset$, M_V lies in the fibres of $\varphi_{|-K_V|}$, and we see from the explicit description of M_X that $O \in \text{CS}(X, M_X)$.

The proof of Theorem 4.3 is thus complete.

§ 5. Double cover of a quadric

Let $\theta: X \rightarrow Q$ be a smooth double cover of a quadric 3-fold Q ramified in a smooth octic surface $S \subset Q$. Then $\text{Pic } X = -\mathbb{Z}K_X$ and $-K_X$ is the pullback of a hyperplane section of Q . In § 4 for each curve C on X with $-K_X \cdot C = 1$ we constructed a birational map $\psi_C: X \dashrightarrow X_C$ onto a terminal Fano 3-fold X_C with $-K_{X_C}^3 = \frac{1}{2}$.

Proposition 5.1. *Let $\sigma: X \dashrightarrow Y$ be a birational map onto a Fano 3-fold Y with canonical singularities. For $n \gg 0$ let $M_Y = \frac{1}{n}|-nK_Y|$ and let (X, M_X) and (X_C, M_{X_C}) be log pairs birationally equivalent to (Y, M_Y) . Then up to an action of the group $\text{Bir } X$ either (X, M_X) or some (X_C, M_{X_C}) has semiterminal singularities and \mathbb{Q} -rationally trivial log canonical divisor.*

Similarly to the quartic case, the main theorem for X follows from Proposition 5.1, and the following result is equivalent to the birational rigidity of X proved in [4].

Proposition 5.2. *Let (X, M_X) be a movable log pair on X with $\kappa(X, M_X) = 0$. Then for some automorphism $\tau \in \text{Bir } X$ the singularities of $(X, \tau(M_X))$ are canonical and $K_X + \tau(M_X) \sim_{\mathbb{Q}} 0$.*

Theorem 5.3. *Let (X, M_X) be a canonical movable log pair with $K_X + (M_X) \sim_{\mathbb{Q}} 0$. Then*

$$\text{CS}(X, M_X) = \begin{cases} \emptyset, \\ \{C\} \text{ for a curve } C \subset X \text{ with } -K_X \cdot C = 1, \\ \{C\} \text{ for a curve } C \subset X \text{ such that } \theta(C) \text{ is a line} \\ \quad \text{and } -K_X \cdot C = 2, \\ \{C_1 \cup C_2\}, \text{ where } \theta(C_1) = \theta(C_2) \text{ is a line} \\ \quad \text{and } -K_X \cdot (C_1 + C_2) = 2, \\ \left\{ \bigcup_{i \in I} C_i, \bigcup_{j \in J} P_j \right\}, \text{ where the } C_i \text{ are all base curves of a pencil} \\ \quad \mathcal{H} \subset |-K_X|, \text{ and the } P_j \text{ are singular points} \\ \quad \text{on a general surface in } \mathcal{H}. \end{cases}$$

Moreover, one of the following results hold: (X, M_X) is terminal; $\text{CS}(X, M_X)$ consists of a curve C with $-K_X \cdot C = 1$, the log pair $(\psi_C(X), \psi_C(M_X))$ has semi-terminal singularities and \mathbb{Q} -rationally trivial log canonical divisor, where ψ_C is the birational map from Proposition 1.4; the set $\text{CS}(X, M_X)$ contains a point P such that $\theta(P) \in S$ and M_X lies in the fibres of the composite of θ with the projection from the tangent space to S at the point $\theta(P)$; M_X lies in the fibres of the composite of θ with the projection from $\theta(\text{CS}(X, M_X))$.

Similarly to the proof of Proposition 4.5 one can show that the main theorem for X follows from Theorem 5.3.

Proposition 5.4. *Theorem 5.3 yields the auxiliary theorem for X .*

Proof. Assume that X can be birationally transformed by a map $\sigma: X \dashrightarrow Y$ into a fibration $\tau: Y \rightarrow Z$ whose general fibre is an elliptic curve or a surface of Kodaira dimension zero. Let $M_X = \lambda \sigma^{-1}(|\tau^*(D_Z)|)$ for a sufficiently big divisor D_Z and $\lambda \in \mathbb{Q}_{>0}$. Then we can assume by Proposition 5.2 that (X, M_X) has canonical singularities and $K_X + M_X \sim_{\mathbb{Q}} 0$.

Assume that the singularities of the log pair (X, M_X) are terminal. Then for some $\gamma > 0$ the log pair $(X, \gamma M_X)$ is a canonical model. On the other hand it follows by the construction of M_Y that $\kappa(Y, \gamma M_Y) < 3$, therefore $\text{CS}(X, M_X) \neq \emptyset$.

Let L be a curve on X with $-K_X \cdot L = 1$ and let (X_L, M_{X_L}) be a movable log pair with $M_{X_L} = \psi_L(M_X)$ for the birational map $\psi_L: X \dashrightarrow X_L$ from Proposition 1.4. Assume that the singularities of the log pair (X_L, M_{X_L}) are semiterminal and $K_{X_L} + M_{X_L} \sim_{\mathbb{Q}} 0$. Then the movable log pair $(X_L, \gamma M_{X_L})$ is a canonical model for some positive rational γ , which contradicts the inequality $\kappa(Y, \gamma M_Y) < 3$.

By Theorem 5.3 either $\text{CS}(X, M_X)$ contains a point P and $\theta(P) \in S$ or

$$\text{CS}(X, M_X) = \begin{cases} \{L\} \text{ for a curve } L \subset X \text{ with } -K_X \cdot L = 1, \\ \{C\} \text{ for a curve } C \subset X \text{ such that } \theta(C) \text{ is a line} \\ \quad \text{and } -K_X \cdot C = 2, \\ \{L_1 \cup L_2\}, \text{ where } \theta(L_1) = \theta(L_2) \text{ is a line} \\ \quad \text{and } -K_X \cdot (L_1 + L_2) = 2, \\ \left\{ \bigcup_{i \in I} C_i \right\}, \text{ where the } C_i \text{ are all base curves} \\ \quad \text{of a pencil } \mathcal{H} \subset |-K_X|. \end{cases}$$

In both cases M_X lies in the fibres of ψ , where ψ is either the composite of θ and the projection from $\theta(\text{CS}(X, M_X))$ or the composite of θ and the projection from the tangent space to S at $\theta(P)$ if $P \in \text{CS}(X, M_X)$.

In particular, if $\text{CS}(X, M_X)$ contains a point P on the 3-fold X , then it follows from Theorem 5.3 that τ is a fibration of K3 surfaces equivalent to the fibration defined by the projection from the tangent space to the surface S at $\theta(P)$. In the case when the set $\text{CS}(X, M_X)$ consists of the base curves of the pencil $\mathcal{H} \subset |-K_X|$ and the image of the locus of $\text{CS}(X, M_X)$ on Q is not a line, Theorem 5.3 shows that τ is a fibration of K3 surfaces equivalent to the one defined by the rational map $\varphi_{\mathcal{H}}$. In the case when τ is an elliptic fibration Theorem 5.3 shows that τ is birationally equivalent as a fibration to the fibration defined by the composite of the double cover θ and a projection from some line on Q .

Thus, we can assume that τ is a fibration of surfaces of Kodaira dimension 0,

$$\text{CS}(X, M_X) = \begin{cases} \{L\} \text{ for a curve } C \subset X \text{ with } -K_X \cdot L = 1, \\ \{C\} \text{ for a curve } C \subset X \text{ such that } \theta(C) \text{ is a line} \\ \quad \text{and } -K_X \cdot C = 2, \\ \{L_1 \cup L_2\}, \text{ where } \theta(L_1) = \theta(L_2) \text{ is a line} \\ \quad \text{and } -K_X \cdot (L_1 + L_2) = 2, \end{cases}$$

and M_X lies in the fibres of the composite of θ with the projection from the line $\theta(\text{CS}(X, M_X))$. We must show that M_X lies in the fibres of $\varphi_{\mathcal{H}}$ for some pencil $\mathcal{H} \subset |-K_X|$.

Assume that the unique centre of canonical singularities of the movable log pair (X, M_X) is a smooth rational curve L such that $-K_X \cdot L = 1$. Let $f: W \rightarrow X$ be a blow up of L , E an exceptional divisor of f , $M_W = f^{-1}(M_X)$, and \widehat{L} a base curve of the linear system $|-K_W|$. Then \widehat{L} is a smooth rational curve and it follows by the construction of the birational map ψ_L in Proposition 1.4 that for a sufficiently large integer n we have $\psi_L = \varphi_{|-nK_{\widehat{W}}|} \circ \rho$, where $\rho: W \dashrightarrow \widehat{W}$ is an antiflip in \widehat{L} .

Assume that the log pair (W, M_W) has terminal singularities and let $\zeta > 1$ be a rational number such that the singularities of the log pair $(W, \zeta M_W)$ are still terminal. Let $M_{\widehat{W}} = \rho(M_W)$. Then the singularities of the movable log pair $(\widehat{W}, \zeta M_{\widehat{W}})$ are also terminal because ρ is a log flip for the movable log pair $(W, \zeta M_W)$. On the other hand, the log canonical divisor $K_{\widehat{W}} + M_{\widehat{W}}$ is \mathbb{Q} -rationally trivial. In particular, $\varphi_{|-nK_{\widehat{W}}|}$ is crepant for the terminal log pair $(\widehat{W}, \zeta M_{\widehat{W}})$. Hence $(X_L, \psi_L(M_X))$

has semiterminal singularities and a \mathbb{Q} -rationally trivial canonical divisor; but we have already proved that this is impossible. Thus, the set $\text{CS}(W, M_W)$ contains an irreducible curve $T \subset E$ such that $f(T) = C$.

Assume that $T \neq \widehat{L}$ and $\widehat{L} \subset E$. Intersecting M_W with a sufficiently general fibre of $f|_E : E \rightarrow C$ we obtain a contradiction.

Assume that $T \neq \widehat{L}$ and $\widehat{L} \not\subset E$, and let D_W be a general surface in $|-K_W|$. Then D_W is a smooth K3 surface, $T \cap D_W \neq \emptyset$, and we have the equivalence $M_W|_{D_W} = \text{mult}_{\widehat{L}}(M_W)\widehat{L} + R_{D_W} \sim_{\mathbb{Q}} \widehat{L} + F$ for some movable boundary R_{D_W} and elliptic curve F such that $F \cdot \widehat{L} = 1$; however, $M_W \cdot \widehat{L} = -1$ and therefore $\text{mult}_{\widehat{L}}(M_W) > 0$. Intersecting $M_W|_{D_W}$ with a curve in the pencil $|F|$ passing through a point in $T \cap D_W$ we obtain a contradiction.

Hence $T = \widehat{L}$, $\widehat{L} \subset E$, the image of L on Q lies in S , and \widehat{L} is an exceptional section of the ruled surface E . Let $g: V \rightarrow W$ be a blow up of \widehat{L} , $G_V = f^{-1}(\widehat{L})$, and let $M_V = g^{-1}(M_W)$. Then $\varphi|_{-K_V|}$ is an elliptic fibration with section $G_V \cong \mathbb{F}_1$ and $M_V \cdot K_V^2 = 0$, therefore M_V lies in the fibres of the morphism $\varphi|_{-K_V|}$.

The singularities of (V, M_V) are not terminal and $\text{CS}(V, M_V)$ contains a curve $\tilde{L} \subset G_V$. However, we have already proved that M_V is contained in the fibres of $\varphi|_{-K_V|}$. Hence \tilde{L} is contracted to a point by $\varphi|_{-K_V|}$ and \tilde{L} is an exceptional section of the surface $G_V \cong \mathbb{F}_1$.

Let $h: U \rightarrow V$ be a blow up of the curve \tilde{L} , $R = h^{-1}(\tilde{L})$, and $M_U = h^{-1}(M_V)$. Let D be a sufficiently general surface in $|-K_U|$. Then D is a K3 surface with at most canonical singularities and $M_U|_D = \sum_{i=1}^k \alpha_i \tilde{L}_i + B_D$, where the \tilde{L}_i are base curves of $|-K_U|$, B_D is a movable boundary on D , and $\alpha_i \in \mathbb{Q}_{\geq 0}$. On the other hand, $M_U|_D$ is equivalent to $D|_D$, which is in its turn equivalent to $\sum_{i=1}^k \beta_i \tilde{L}_i$ for $\beta_i \in \mathbb{N}$. The intersection form of the curves \tilde{L}_i on D is negative-definite. Hence $B_D = \emptyset$, which means that M_U lies in the fibres of $\varphi|_{-K_U|}$.

Let $(X, M_X) = \{C\}$, where $-K_X \cdot C = 2$ and $\theta(C)$ is a line. Then C is a smooth elliptic curve or a rational curve with one double point. Let \mathcal{H}_C be a linear subsystem of $|-K_X|$ consisting of surfaces passing through the curve C . The map $\varphi_{\mathcal{H}_C}$ is the composite of the double cover θ and the projection from the line $\theta(C)$. Let $r: V_C \rightarrow X$ be a birational map such that r has a unique exceptional divisor G , which dominates C , V_C has terminal singularities, $\mathcal{H}_{V_C} = r^{-1}(\mathcal{H}_C)$ is free, and $\varphi_{\mathcal{H}_{V_C}}$ is an elliptic fibration. Let $M_{V_C} = r^{-1}(M_X)$. Then the equality $\text{mult}_C(M_X) = 1$ yields $\mathcal{H}_{V_C}^2 \cdot M_{V_C} = 0$, therefore M_{V_C} lies in the fibres of $\varphi_{\mathcal{H}_{V_C}}$ and M_{V_C} is \mathbb{Q} -rationally equivalent to \mathcal{H}_{V_C} . Hence the singularities of (V_C, M_{V_C}) are not terminal because otherwise $\kappa(V_C, \gamma M_{V_C}) = 2$ for each rational $\gamma > 1$, although it follows from the construction of the movable boundary M_X that $\kappa(X, \gamma M_X) \leq 1$. Thus, the set $\text{CS}(V_C, M_{V_C})$ contains a curve \widehat{C} such that $r(\widehat{C}) = C$. However, we have already proved that the boundary M_{V_C} lies in the fibres of $\varphi|_{-K_V|}$. Hence \widehat{C} is contracted by $\varphi|_{-K_V|}$ and is an exceptional section of the ruled surface G . Let $\mathcal{H}_{\widehat{C}}$ be a pencil in the linear system \mathcal{H}_{V_C} containing surfaces passing through the curve \widehat{C} , and let $D_{\widehat{C}}$ be a general surface in \mathcal{H}_{V_C} . Then $M_{V_C}|_{D_{\widehat{C}}} \sim_{\mathbb{Q}} \widehat{C}$ on $D_{\widehat{C}}$, therefore $M_{V_C}|_{D_{\widehat{C}}} = \widehat{C}$. On the other hand, the curve \widehat{C} is the base locus of the pencil $\mathcal{H}_{\widehat{C}}$. Hence M_{V_C} lies in the fibres of $\varphi_{\mathcal{H}}$.

Finally, we can assume that $\text{CS}(X, M_X)$ consists of two smooth rational curves L_1 and L_2 , $L_1 \neq L_2$, mapped by θ into the same line on Q . Let $s: \overline{X} \rightarrow X$ be the composite of a blow up of L_1 and a blow up of the strict transform of L_2 . Let $M_{\overline{X}} = s^{-1}(M_X)$ and let E_1 and E_2 be exceptional divisors of the morphism s dominating the curves L_1 and L_2 , respectively. Then $|-K_{\overline{X}}|$ is free, $\varphi_{|-K_{\overline{X}}|}$ is an elliptic fibration with two sections E_1 and E_2 , $M_{\overline{X}}$ lies in the fibres of $\varphi_{|-K_{\overline{X}}|}$, $E_2 \cong \mathbb{F}_1$, and the singularities of the log pair $(\overline{X}, M_{\overline{X}})$ are not terminal. Hence the set $\text{CS}(V_C, M_{V_C})$ contains a curve \overline{L}_2 such that the image of \overline{L}_2 on X is either L_1 or L_2 . Moreover, we may assume that the curve \overline{L}_2 dominates L_2 because otherwise we can take s to be the composite of a blow up of L_2 with the blow up of the strict transform of L_1 . The boundary $M_{\overline{X}}$ lies in the fibres of the elliptic fibration $\varphi_{|-K_{\overline{X}}|}$. Hence \overline{L}_2 is an exceptional section of the surface E_2 . Let $\mathcal{H}_{\overline{X}} \subset |-K_{\overline{X}}|$ be a pencil of surfaces passing through \overline{L}_2 and let $D_{\overline{X}}$ be a sufficiently general surface in $\mathcal{H}_{\overline{X}}$. Then $M_{\overline{X}}|_{D_{\overline{X}}} = \overline{L}_2 + \sum_{i=1}^k a_i \overline{Z}_i + B_{D_{\overline{X}}}$, where $B_{D_{\overline{X}}}$ is a movable boundary, the \overline{Z}_i are base curves of the pencil $\mathcal{H}_{\overline{X}}$ distinct from \overline{L}_2 , and $a_i \in \mathbb{Q}_{\geq 0}$. On the surface $D_{\overline{X}}$ the restriction $M_{\overline{X}}|_{D_{\overline{X}}}$ is equivalent to $D_{\overline{X}}|_{D_{\overline{X}}}$, which is equivalent to $\overline{L}_2 + \sum_{i=1}^k b_i \overline{Z}_i$ for some integers b_i . However, the intersection form of the curves \overline{Z}_i on the surface $D_{\overline{X}}$ is negative-definite. Hence $B_{D_{\overline{X}}} = \emptyset$ and the support of $M_{\overline{X}}|_{D_{\overline{X}}}$ consists of the base locus of the pencil $\mathcal{H}_{\overline{X}}$. Thus, $M_{\overline{X}}$ lies in the fibres of $\varphi_{\mathcal{H}_{\overline{X}}}$.

In the rest of this section we prove Theorem 5.3. Consider a movable log pair (X, M_X) with canonical singularities such that $K_X + M_X \sim_{\mathbb{Q}} 0$ and $\text{CS}(X, M_X) \neq \emptyset$.

Lemma 5.5. *Suppose that $\text{CS}(X, M_X)$ contains a point P . Then the image of P on Q lies in S , $\text{CS}(X, M_X) = \{P, \bigcup_{i \in I} C_i, \bigcup_{j \in I} P_j\}$, and M_X lies in the fibres of $\varphi_{\mathcal{H}}$, where \mathcal{H} is a pencil in $|-K_X|$ consisting of surfaces singular at P , the C_i are all base curves of \mathcal{H} , and the P_j are base points of the pencil \mathcal{H} such that $\theta(P_j) \in S$ and the tangent spaces to S at the points $\theta(P_j)$ coincide with the tangent space to S at $\theta(P)$.*

Proof. Let H_P be a general surface in $|-K_X|$ containing P . It follows from the relation $H_P \cdot M_X^2 \geq \text{mult}_P(M_X^2)$ and Pukhlikov's local inequality in the form of Theorem 4.1 of [2] that $\text{mult}_P(M_X) = 2$.

Let $f: W \rightarrow X$ be a blow up of P , $E = f^{-1}(P)$, and $M_W = f^{-1}(M_X)$. Then $M_W \sim_{\mathbb{Q}} -K_W$, $|-nK_W|$ has no fixed components for $n \gg 0$, P is a K -special point, and $\theta(P) \in S$ by Lemma 1.11. In particular, the pencil $|-K_W|$ has no fixed components.

Let $C \subset X$ be a curve such that either $-K_X \cdot C = 1$ or C has a singularity at P and $-K_X \cdot C = 2$. Then C lies in the base locus of $f(|-K_W|)$ and in the support of M_X^2 . On the other hand, $M_W^2 \cdot (f^*(-K_X) - E) = 0$, the divisor $f^*(-K_X) - E$ is nef and big, and $|k(f^*(-K_X) - E)|$ for some $k \gg 0$. Thus, the base locus of the pencil $f(|-K_W|)$ and the support of M_X^2 consist of finitely many curves C_λ such that each C_λ is either smooth and $-K_X \cdot C_\lambda = 1$, or C_λ is singular at P and $-K_X \cdot C_\lambda = 2$. This yields the required result.

Thus, we shall assume that $\text{CS}(X, M_X)$ contains no points, but contains a curve C such that $-K_X \cdot C \leq 4$.

Lemma 5.6. *Let C be the unique centre in $\text{CS}(X, M_X)$ and let $-K_X \cdot C = 1$. Then either M_X lies in the fibres of the composite of θ with a projection from $\theta(C)$ or $(X_C, \psi_C(M_X))$ has terminal singularities and a \mathbb{Q} -rationally trivial log canonical divisor, where $\psi_C: X \dashrightarrow X_C$ is the birational map from Proposition 1.4.*

Proof. Note that C is a smooth rational curve and its image on Q is a line. Let $f: W \rightarrow X$ be a blow up of C . Let E be the exceptional divisor of f , $M_W = f^{-1}(M_X)$, and C_1 the unique base curve of the linear system $|-K_W|$. Then C_1 is a smooth rational curve and it follows from the construction of the birational map ψ_C that $\psi_C = \varphi_{|-nK_{\widehat{W}}|} \circ \rho$ for $n \gg 0$, where the birational map $\rho: W \dashrightarrow \widehat{W}$ is an antiflip in the curve \widehat{C}_1 .

Assume that the log pair (W, M_W) has terminal singularities. We take $\zeta \in \mathbb{Q}_{>1}$ such that the singularities of $(W, \zeta M_W)$ are still terminal. Let $M_{\widehat{W}} = \rho(M_W)$. Then the singularities of $(\widehat{W}, \zeta M_{\widehat{W}})$ are terminal because ρ is a log flip for $(W, \zeta M_W)$. The birational morphism $\varphi_{|-nK_{\widehat{W}}|}$ is crepant for $(\widehat{W}, \zeta M_{\widehat{W}})$, therefore (X_C, M_{X_C}) has semiterminal singularities and $K_{X_C} + M_{X_C} \sim_{\mathbb{Q}} 0$, where $M_{X_C} = \psi_C(M_X)$.

Assume that (X_C, M_{X_C}) has semiterminal singularities and $K_{X_C} + M_{X_C} \sim_{\mathbb{Q}} 0$. Then the singularities of (W, M_W) are also semiterminal because $(\widehat{W}, M_{\widehat{W}})$ is a crepant pull back of the log pair (X_C, M_{X_C}) and ρ^{-1} is a log flop for $(\widehat{W}, M_{\widehat{W}})$.

To complete the proof we can assume that the singularities of the log pair (W, M_W) are not terminal. Then the equivalence $K_W + M_W \sim_{\mathbb{Q}} 0$ shows that $\text{CS}(W, M_W)$ contains a curve $T \subset E$ such that $f(T) = C$.

Let $T = C_1 \subset E$, let $g: V \rightarrow W$ be a blow up of the curve C_1 , and let $M_V = g^{-1}(M_W)$. Then $|-K_V|$ is free, $\varphi_{|-K_V|}$ is an elliptic fibration, and $M_V \cdot K_V^2 = 0$. Hence M_V lies in the fibres of $\varphi_{|-K_V|}$ and M_X lies in the fibres of $\varphi_{|-K_V|} \circ g^{-1} \circ f^{-1}$, but $\varphi_{|-K_V|} \circ g^{-1} \circ f^{-1}$ is the composite of θ and the projection from $\theta(C)$. Thus, the inequality $\kappa(V, \gamma M_V) < 3$ holds for each rational number $\gamma > 1$. In particular, (X_C, M_{X_C}) has no semiterminal singularities and a \mathbb{Q} -rationally trivial canonical divisor.

Assume that $T \neq C_1$ and $C_1 \subset E$. Then intersecting M_W with a sufficiently general fibre of $f|_E: E \rightarrow C$ we see that $E \subset M_W$, which contradicts the movability of the boundary M_W .

Assume now that $T \neq C_1$ and $C_1 \not\subset E$. Let D_W be a general divisor in $|-K_W|$. Then D_W is a smooth K3 surface, $T \cap D_W \neq \emptyset$, and we have the equality $M_W|_{D_W} = \text{mult}_{C_1}(M_W)C_1 + R_{D_W} \sim_{\mathbb{Q}} C_1 + F$ for some movable boundary R_{D_W} and an elliptic curve F such that $F \cdot C_2 = 1$. On the other hand $M_W \cdot C_1 = -1$, therefore $\text{mult}_{C_1}(M_W) > 0$. Intersecting $M_W|_{D_W}$ with a curve in $|F|$ passing through a point in $T \cap D_W$ we obtain a contradiction.

Lemma 5.7. *The image of C on the quadric Q is a line or a conic.*

Proof. Assume that the degree of $\theta(C)$ is greater than 2. The restriction $\theta|_C$ is an isomorphism and the image of C on Q is either a singular rational curve of degree 4 with one double point or a smooth curve of degree 3 or 4.

Let C be a smooth curve. Then either C is a rational curve or C is an elliptic curve and $-K_X \cdot C = 4$. Let $f: W \rightarrow X$ be a blow up of the curve C , E an exceptional divisor of the birational morphism f , $M_W = f^{-1}(M_X)$, and $d = -K_X \cdot C$.

We shall show that the divisor $f^*(-dK_X) - E$ is nef. In the case when the image of the curve C on the quadric Q does not lie in the ramification surface S let $\tilde{C} \neq C$ be a curve on X whose image on Q coincides with the image of C . Then the base locus of the linear system $|f^*(-dK_X) - E|$ lies in the curve $\hat{C} = f^{-1}(\tilde{C})$, therefore the divisor $f^*(-dK_X) - E$ has a non-negative intersection with all curves on the 3-fold W with the possible exception of \hat{C} ; however, direct calculations show that $(f^*(-dK_X) - E) \cdot \hat{C} > 0$. In the case when the image of C on the quadric Q lies in the surface S the base locus of the linear system $|f^*(-dK_X) - E|$ lies in the exceptional divisor E . We can assume that the normal bundle $N_{C/X}$ is decomposable. Explicit calculations yield $(f^*(-dK_X) - E)|_E \cdot s_\infty \geq 0$ for an exceptional section s_∞ of the ruled surface E . We have thus proved that $f^*(-dK_X) - E$ is nef, while on the other hand $(f^*(-dK_X) - E) \cdot M_W^2 < 0$.

We can now assume that $-K_X \cdot C = 4$ and C is a rational curve with one double point P . Let $g: V \rightarrow X$ be the composite of a blow up of P and a blow up of the strict transform of C , let $M_V = g^{-1}(M_X)$, and let G and H be exceptional divisors of the birational morphism g , where H is an exceptional divisor of the blow up of the strict transform of C . Using the same arguments as in the previous case one can show that the divisor $f^*(-4K_X) - H - 2G$ is nef, while direct calculations show that $(f^*(-4K_X) - H - 2G) \cdot M_W^2 < 0$.

Lemma 5.8. *Let $\theta(C)$ be a conic. Then $\text{CS}(X, M_X)$ consists of all base curves of the pencil $\mathcal{H} \subset |-K_X|$ of surfaces passing through C and M_X lies in the fibres of $\varphi_{\mathcal{H}}$.*

Proof. We have either $-K_X \cdot C = 2$ and $\theta|_C$ is an isomorphism, or $-K_X \cdot C = 4$ and $\theta|_C$ is a double cover. Let $-K_X \cdot C = 4$ and let $f: W \rightarrow X$ be a resolution of indeterminacy of the rational map $\varphi_{\mathcal{H}}$ such that f is an isomorphism outside the curve C and W is smooth and contains precisely one f -exceptional divisor E dominating the curve C . Let $\tau = \varphi_{\mathcal{H}} \circ f^{-1}$ and $M_W = f^{-1}(M_X)$. Then a sufficiently general fibre D of τ is rationally equivalent to a divisor $f^*(-K_X) - E - \sum_{i \in I} a_i F_i$, where all the $f(F_i)$ are points in C and all the a_i are integers. On the other hand, on the surface D the restriction $M_W|_D$ is \mathbb{Q} -rationally equivalent to $\sum_{i \in I} c_i F_i|_D$ for some rational numbers c_i . Hence $M_W|_D = \emptyset$ and M_W lies in the fibres of τ , which yields the required result.

Now let $-K_X \cdot C = 2$ and assume that the image of C on Q does not lie in the ramification surface S . Consider a smooth rational curve \tilde{C} on X such that $\tilde{C} \neq C$ and the images of the curves \tilde{C} and C on Q are the same. Let D_C be a general divisor in \mathcal{H} . Then $\tilde{C} \subset D_C$ and $M_X|_{D_C} = C + \text{mult}_{\tilde{C}}(M_X)\tilde{C} + R_{D_C} \sim_{\mathbb{Q}} C + \tilde{C}$ for some movable boundary R_{D_C} . On the other hand, $\tilde{C}^2 < 0$ on D . Hence the boundary R_{D_C} is empty, $\text{mult}_{\tilde{C}}(M_X) = 1$, and $\text{CS}(X, M_X)$ contains the curve \tilde{C} . However, the base locus of \mathcal{H} consists of C and \tilde{C} , therefore M_X lies in the fibres of $\varphi_{\mathcal{H}}$.

Let $-K_X \cdot C = 2$ and $\theta(C) \subset S$. Let $g: V \rightarrow X$ be a blow up of C , F an exceptional divisor of g , and let $M_V = g^{-1}(M_X)$ and $\mathcal{H}_V = f^{-1}(\mathcal{H})$. Let D_V be a general surface in \mathcal{H}_V . Then the base locus of \mathcal{H}_V consists of a section \hat{C} of the ruled surface F and we have $M_V|_{D_V} = \text{mult}_{\hat{C}}(M_V)\hat{C} + R_{D_V}$ for a movable boundary R_{D_V} .

On the surface D_V the restriction $M_V|_{D_V}$ is equivalent to $-K_V|_{D_V} \sim \widehat{C}$. On the other hand, $\widehat{C}^2 < 0$ on D_V . Thus, the boundary R_{D_V} is empty and the support of the restriction $M_V|_{D_V}$ lies in the base locus of the pencil \mathcal{H}_V . Hence M_V lies in the fibres of $\varphi_{\mathcal{H}_V}$.

We can assume that the image of the curve C on the quadric Q is a line.

Lemma 5.9. *Assume that $\text{CS}(X, M_X)$ contains a curve L and $\theta(L) \neq \theta(C)$. Then $\theta(L)$ is a line, $\theta(C) \cap \theta(L) \neq \emptyset$, the set $\text{CS}(X, M_X)$ consists of all base curves of the pencil \mathcal{H} of surfaces in $|-K_X|$ passing through both C and L , and M_X lies in the fibres of $\varphi_{\mathcal{H}}$.*

Proof. It follows from Lemmas 5.7 and 5.8 that $\theta(L)$ is a line. Let $-K_X \cdot C = 1$ and assume that $\theta(L) \cap \theta(C) = \emptyset$. Let $f: W \rightarrow X$ be a blow up of the curve C , let E be an exceptional divisor of the morphism f , $\widehat{L} = f^{-1}(L)$, $M_W = f^{-1}(M_X)$, and let D_W be a general divisor in $|-K_W|$. Then the base locus of $|-K_W|$ consists of one smooth rational curve C_1 , D_W is a smooth K3 surface, $\widehat{L} \cap D_W \neq \emptyset$, and for some movable boundary R_{D_W} we have $M_W|_{D_W} = \text{mult}_{C_1}(M_W)C_1 + R_{D_W} \sim_{\mathbb{Q}} C_1 + F$ on D_W , where the curve F is elliptic and $F \cdot C_1 = 1$. Moreover, the strict inequality $M_W \cdot C_1 < 0$ yields $\text{mult}_{C_1}(M_W) > 0$. Intersecting $M_W|_{D_W}$ with a curve in the pencil $|F|$ passing through a point in $\widehat{L} \cap D_W$ we obtain a contradiction.

Let $-K_X \cdot C = 2$ and assume that $\theta(L) \cap \theta(C) = \emptyset$. Then C is either a smooth elliptic curve or a rational curve with one singular double point. Let \mathcal{H}_C be a linear system consisting of surfaces in the anticanonical linear system $|-K_X|$ passing through the curve C . Then the map $\varphi_{\mathcal{H}_C}: X \dashrightarrow \mathbb{P}^2$ is the composite of the double cover θ and the projection from $\theta(C)$. Let $r: V_C \rightarrow X$ be a terminal blow up for the canonical log pair (X, \mathcal{H}_C) such that the linear system $\mathcal{H}_{V_C} = r^{-1}(\mathcal{H}_C)$ is free and the morphism $\varphi_{\mathcal{H}_{V_C}}$ is an elliptic fibration. Let $M_{V_C} = r^{-1}(M_X)$. Then $M_{V_C} \sim_{\mathbb{Q}} \mathcal{H}_{V_C}$, therefore M_{V_C} lies in the fibres of the elliptic fibration $\varphi_{\mathcal{H}_{V_C}}$. On the other hand, the strict transform of L on the 3-fold V_C is a centre of canonical singularities of the log pair (V_C, M_{V_C}) and does not lie in the fibres of $\varphi_{\mathcal{H}_{V_C}}$, which contradicts the movability of the boundary M_{V_C} .

We have thus proved that $\theta(L) \cap \theta(C) \neq \emptyset$. Now let \mathcal{H} be a pencil in the linear system $|-K_X|$ consisting of surfaces passing through both C and L . We must prove that the set $\text{CS}(X, M_X)$ contains all base curves of the pencil \mathcal{H} and M_X lies in the fibres of $\varphi_{\mathcal{H}}$.

Let $-K_X \cdot C = -K_X \cdot L = 1$. Let $g: V \rightarrow X$ be the composite of a blow up of C with a blow up of L , D_V a general surface in $|-K_V|$, and $M_V = g^{-1}(M_X)$. Then $M_V|_{D_V} = \sum_{i=1}^k \alpha_i Z_i + R_{D_V}$ for a movable boundary R_{D_V} , base curves Z_i of the pencil $|-K_V|$, and non-negative rational α_i . On the surface D_V the restriction $M_V|_{D_V}$ is equivalent to $-K_V|_{D_V} \sim \sum_{i=1}^k \beta_i Z_i$ for $\beta_i \in \mathbb{N}$. On the other hand, the intersection form of the curves Z_i on the surface D_V is negative-definite, therefore M_V lies in the fibres of $\varphi_{|-K_V|}$ and by implication $\text{CS}(X, M_X)$ contains all base curves of the pencil \mathcal{H} .

Let $-K_X \cdot C = -K_X \cdot L = 2$. Let D be a sufficiently general surface in the pencil \mathcal{H} . Then $M_X|_D = C + L + R_D \sim_{\mathbb{Q}} -K_X|_D \sim C + L$ for some movable

boundary R_D . Hence $R_D = \emptyset$ and $M_X|_D$ consists of two base curves of the pencil \mathcal{H} , which means that M_V lies in the fibres of $\varphi_{|-K_V|}$.

Two cases are now left to consider: either $-K_X \cdot C = 2$ and $-K_X \cdot L = 1$, or $-K_X \cdot C = 1$ and $-K_X \cdot L = 2$. We may assume without loss of generality that $-K_X \cdot C = 2$ and $-K_X \cdot L = 1$. Let $h: U \rightarrow X$ be a blow up of L , \widehat{L} a base curve of $|-K_U|$, $\widehat{C} = h^{-1}(C)$, $M_U = h^{-1}(M_X)$, and $\mathcal{H}_U = h^{-1}(\mathcal{H})$. Let D_U be a general surface in \mathcal{H}_U . The curves \widehat{C} and \widehat{L} are contained in the base locus of \mathcal{H}_U . If C is singular and its singular point belongs to L , then the base locus of \mathcal{H}_U contains a curve Z that is mapped into the singular point of C . In the case when C is smooth or the singular point of C does not belong to L the base locus of \mathcal{H}_U consists of the curves \widehat{L} and \widehat{C} and we set $Z = \emptyset$. Then $M_U|_{D_U} = \widehat{C} + \text{mult}_{\widehat{L}}(M_U)\widehat{L} + \alpha Z + R_{D_U}$ for some movable boundary R_{D_U} and $\alpha = \text{mult}_Z(M_U)$ if $Z \neq \emptyset$. On D_U the restriction $M_U|_{D_U}$ is equivalent to $\widehat{C} + \widehat{L} + Z$ and the intersection form of the curves Z and \widehat{L} is negative-definite. Hence the boundary R_{D_U} is empty, $\text{mult}_{\widehat{L}}(M_U) = 1$, and if $Z \neq \emptyset$, then $\text{mult}_Z(M_U) = 1$. Thus, M_U lies in the fibres of $\varphi_{\mathcal{H}_U}$ and all base curves of the pencil \mathcal{H} belong to $\text{CS}(X, M_X)$. Moreover, in the case when C is singular and its singular point belongs to L it follows from the equality $\text{mult}_Z(M_U) = 1$ that $\text{CS}(X, M_X)$ contains the singular point of C and all surfaces in the pencil \mathcal{H} are singular at this point. However, we have assumed that the set $\text{CS}(X, M_X)$ does not contain points in the 3-fold X .

We can assume therefore that $\text{CS}(X, M_X)$ is taken to a line on the quadric Q .

Lemma 5.10. *Let $-K_X \cdot C = 2$. Then $\text{CS}(X, M_X) = \{C\}$ and M_X lies in the fibres of the composite of the double cover θ with the projection from $\theta(C)$.*

Proof. Consider a linear system \mathcal{H}_C of surfaces in $|-K_X|$ containing the curve C . Let $f: W \rightarrow X$ be a resolution of indeterminacy of the rational map $\varphi_{\mathcal{H}_C}$ such that W is a smooth variety containing a unique divisor $E \subset W$ dominating C . Let $\mathcal{H}_W = f^{-1}(\mathcal{H}_C)$ and $M_W = f^{-1}(M_X)$. Then $M_W \cdot \mathcal{H}_W^2 = 0$, which proves the required result.

We can assume that $-K_X \cdot C = 1$ and $\text{CS}(X, M_X) = \{C, L\}$, where $L \neq C$ and $\theta(L) = \theta(C)$. Using arguments from the proof of Lemma 5.10 we see that M_X lies in the fibres of the composite of θ with the projection from $\theta(C)$. This completes the proof of Theorem 5.3.

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