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Non-rationality of the 4-dimensional smooth complete intersection of a quadric and a quartic not containing planes

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Abstract. The non-rationality and the birational superrigidity is proved for the 4-dimensional smooth complete intersection of a quadric and a quartic in $\mathbb{P}^6$ that contains no 2-dimensional linear subspace of $\mathbb{P}^6$. It is also proved that such an intersection is not birationally isomorphic to an elliptic fibration.

Bibliography: 24 titles.

All varieties are assumed to be projective, normal, and defined over $\mathbb{C}$, all divisors are assumed to be $\mathbb{Q}$-divisors; for all fibrations $\tau\colon V \to Z$ we shall assume that the dimension of Z is less than the dimension of V, τ has connected fibres, and Z is not a point.

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§ 1. Introduction

Let $V_8 = F_2 \cap F_4 \subset \mathbb{P}^6$ be a smooth complete intersection, where F_2 and F_4 are a quadric and a quartic in $\mathbb{P}^6$, respectively. Then V_8 is easily seen to be a Fano variety of dimension 4, the divisor $-K_{V_8}$ is rationally equivalent to the hyperplane section of V_8, the Picard group of V_8 is generated by $-K_{V_8}$, and $K_{V_8}^4 = 8$.

Definition 1.1. A terminal $\mathbb{Q}$-factorial Fano manifold V with Picard group $\mathbb{Z}$ is said to be *birationally superrigid* if the following three conditions hold:

(1) V cannot be birationally transformed into a fibration of varieties whose generic fibre has Kodaira dimension $-\infty$;
(2) the variety V cannot be birationally transformed into a $\mathbb{Q}$-factorial terminal Fano variety with Picard group $\mathbb{Z}$, not biregular to V;
(3) the variety V admits no non-biregular birational automorphisms.

The main aim of this paper is the proof of the following result.

Theorem 1.2. *The complete intersection* V_8 *is birationally superrigid if it contains no* 2-*dimensional linear subspaces of* $\mathbb{P}^6$.

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Corollary 1.3. *Every smooth complete intersection of a quadric and a quartic in $\mathbb{P}^6$ that does not contain 2-dimensional linear subspaces of $\mathbb{P}^6$ is not rational and its group of birational automorphisms is a finite group generated by biregular projective automorphisms.*

Remark 1.4. A dimension count shows that a sufficiently general complete intersection of a quadric and a quartic in $\mathbb{P}^6$ contains no 2-dimensional subspaces of $\mathbb{P}^6$.

Theorem 1.2 is actually a special case of the following general conjecture [1].

Conjecture 1.5. *A smooth complete intersection $\bigcap_{i=1}^{k} F_i \subset \mathbb{P}^M$ with $\sum_{i=1}^{k} d_i = M$ is birationally superrigid if $M-k \geqslant 4$, where F_i is a hypersurface of degree $d_i \geqslant 2$.*

The birational superrigidity of a smooth quartic 2-fold in $\mathbb{P}^4$ was proved in [2], the birational superrigidity of a general hypersurface of degree M in $\mathbb{P}^M$ for $M>5$ was proved in [3], the birational superrigidity of an arbitrary smooth quintic 4-fold in $\mathbb{P}^5$ was proved in [4], the birational superrigidity of a smooth hypersurface of degree N in $\mathbb{P}^N$ for $6 \leqslant N \leqslant 8$ was proved in [5], the birational superrigidity of a general complete intersection $\bigcap_{i=1}^{k} F_i \subset \mathbb{P}^M$, where F_i is a hypersurface of degree $d_i \geqslant 2$, $\sum_{i=1}^{k} d_i = M$, and $M > 3k \geqslant 6$ was proved in [1]. Recently, the birational superrigidity of an arbitrary smooth hypersurface of degree N in $\mathbb{P}^N$ for $N \geqslant 6$ has been established in [6], but the proof contains a small gap that was pointed out in [7]. The birational rigidity of a smooth hypersurface of degree N in $\mathbb{P}^N$ for $4 \leqslant N \leqslant 12$ and the birational superrigidity of a smooth complete intersection of a quadric and a sextic in $\mathbb{P}^8$ were proved in [7].

We believe that Theorem 1.2 holds for every smooth complete intersection V_8. However, in the case when V_8 contains a 2-dimensional linear subspace, its birational geometry has a more complicated structure.

Example 1.6. Assume that V_8 contains a 2-dimensional linear subspace Π of $\mathbb{P}^6$; then the generic fibre of the projection $\tau\colon V_8 \dashrightarrow \mathbb{P}^3$ from Π is an elliptic curve.

Nevertheless, in the last section we prove the following result.

Theorem 1.7. *A complete intersection V_8 is not birationally equivalent to an elliptic fibration if and only if it contains no 2-dimensional linear subspaces of $\mathbb{P}^6$.*

Birational transformations into elliptic fibrations were effectively used in [8] and [9] in the proof of the potential density[1] of the rational points on Fano 3-folds, where the following result was obtained.

Theorem 1.8. *The rational points are potentially dense on all smooth Fano 3-folds with the possible exception of double covers of $\mathbb{P}^3$ ramified in a smooth sextic surface.*

The possible exception appears in Theorem 1.8 for the sole reason that the double cover of $\mathbb{P}^3$ ramified in a smooth sextic is the only smooth Fano 3-fold that cannot be birationally transformed into an elliptic fibration (see [10]). Birational transformations of higher-dimensional varieties into fibrations of varieties of Kodaira dimension 0 have been studied in [5] and [10]–[14].

[1]The rational points of a variety X defined over a number field $\mathbb{F}$ are potentially dense if there exists a finite extension $\mathbb{K}/\mathbb{F}$ such that the set $X(\mathbb{K})$ of $\mathbb{K}$-rational points is Zariski dense in X.

§ 2. Basics of movable log pairs

In this section we consider properties of movable log pairs introduced in [15].

Definition 2.1. A *movable log pair* (X, M_X) is a variety X together with a movable boundary M_X, where $M_X = \sum_{i=1}^{n} a_i \mathcal{M}_i$ is a formal finite linear combination of linear systems $\mathcal{M}_i$ on X without fixed components such that $a_i \in \mathbb{Q}_{\geqslant 0}$.

A movable log pair can be regarded as an ordinary log pair by replacing each linear system by its general element or an appropriate weighted sum of its general elements. In particular, for a fixed movable log pair (X, M_X) we can treat M_X as an effective divisor and we shall call $K_X + M_X$ the *log canonical divisor of the movable log pair* (X, M_X). In the rest of this section we shall assume that the log canonical divisors of the log pairs under consideration are $\mathbb{Q}$-Cartier divisors.

Remark 2.2. For a movable log pair (X, M_X) we can regard M_X^2 as a well-defined effective cycle of codimension two on X, provided that X is $\mathbb{Q}$-factorial.

By contrast to ordinary log pairs, the strict transform of a movable boundary is well and naturally defined for each birational map.

Definition 2.3. Movable log pairs (X, M_X) and (Y, M_Y) are *birationally equivalent* if there exists a birational map $\rho\colon X \dashrightarrow Y$ such that $M_Y = \rho(M_X)$.

Discrepancies, terminality, canonicity, log terminality, and log canonicity can be defined for movable log pairs in the same way as for ordinary log pairs (see [16]).

Remark 2.4. The application of Log Minimal Model Program to canonical and terminal log pairs preserves their canonicity and terminality, respectively.

Each movable log pair is birationally equivalent to a log pair with canonical singularities, and singularities of a movable log pair coincide with those of the variety outside the base loci of components of the boundary.

Definition 2.5. A proper irreducible subvariety Y of X is called a *centre of canonical singularities of a movable log pair* (X, M_X) if there exist a birational morphism $f\colon W \to X$ and an f-exceptional divisor $E_1 \subset W$ such that

$$K_W + f^{-1}(M_X) \sim_{\mathbb{Q}} f^*(K_X + M_X) + \sum_{i=1}^{k} a(X, M_X, E_i) E_i,$$

the inequality $a(X, M_X, E_1) \leqslant 0$ holds, and $f(E_1) = Y$, where $a(X, M_X, E_i) \in \mathbb{Q}$ and E_i is an f-exceptional divisor.

Definition 2.6. The notation $\mathbb{CS}(X, M_X)$ will denote the set of centres of canonical singularities of a movable log pair (X, M_X), and $CS(X, M_X)$ will be the locus of all centres of canonical singularities of a movable log pair (X, M_X) (regarded as a subset of X).

In particular, a movable log pair (X, M_X) is terminal $\iff$ $\mathbb{CS}(X, M_X) = \varnothing$.

Definition 2.7. The quantity

$$\varkappa(X, M_X) = \begin{cases} \dim(\varphi_{|nm(K_W + M_W)|}(W)) \\ \qquad \text{for } n \gg 0 \text{ such that } |n(K_W + M_W)| \neq \varnothing; \\ -\infty \ \text{ if } |nm(K_W + M_W)| = \varnothing \text{ for all positive integers } n \end{cases}$$

is called the *Kodaira dimension of a movable log pair* (X, M_X); here the movable log pair (W, M_W) is birationally equivalent to (X, M_X) and has canonical singularities, and $m \in \mathbb{N}$ is an integer such that $m(K_W + M_W)$ is a Cartier divisor.

One can show that the Kodaira dimension of a movable log pair is well defined and independent of the choice of the birationally equivalent movable log pair with canonical singularities. By definition, the Kodaira dimension of a movable log pair is a birational invariant and a non-decreasing function of the coefficients of the movable boundary.

Definition 2.8. A movable log pair (V, M_V) is called a *canonical model of a movable log pair* (X, M_X) if there exists a birational map $\psi\colon X \dashrightarrow V$ such that $M_V = \psi(M_X)$, the log canonical divisor $K_V + M_V$ is ample, and (V, M_V) has canonical singularities.

Theorem 2.9. *A canonical model is unique if its exists.*

Proof. Suppose that two movable log pairs (X, M_X) and (V, M_V) are canonical models and that $M_X = \rho(M_V)$ for some birational map $\rho\colon V \dashrightarrow X$. Let $g\colon W \to X$ and $f\colon W \to V$ be birational morphisms such that $\rho = g \circ f^{-1}$. Then

$$K_W + M_W \sim_{\mathbb{Q}} g^*(K_X + M_X) + \Sigma_X \sim_{\mathbb{Q}} f^*(K_V + M_V) + \Sigma_V,$$

where $M_W = g^{-1}(M_X) = f^{-1}(M_V)$ and Σ_X and Σ_V are exceptional divisors of the birational morphisms g and f, respectively. The canonicity of the log pairs (X, M_X) and (V, M_V) shows that the divisors Σ_X and Σ_V are effective. Let n be a sufficiently large integer such that the divisors $n(K_W + M_W)$, $n(K_X + M_X)$, and $n(K_V + M_V)$ are Cartier. Then it follows from the effectivity of Σ_X and Σ_V that

$$\varphi_{|n(K_W+M_W)|} = \varphi_{|g^*(n(K_X+M_X))|} = \varphi_{|f^*(n(K_V+M_V))|},$$

and ρ is an isomorphism because $K_X + M_X$ and $K_V + M_V$ are ample.

The existence of a canonical model of a movable log pair shows that its Kodaira dimension is equal to the dimension of the variety.

§ 3. Preliminary results

We already mentioned in the previous section that movable boundaries can be regarded as effective divisors and movable log pairs can be regarded as ordinary log pairs. Hence we can consider log pairs containing movable components.

Warning 3.1. We impose no restrictions on the coefficients of the boundaries; in particular, boundaries are not necessarily effective.

We shall assume that the log canonical divisors of all log pairs are $\mathbb{Q}$-Cartier divisors.

Definition 3.2. Let (X, B_X) be a log pair and $f\colon V \to X$ a birational morphism. Then a log pair (V, B^V) is called a *log pullback of* (X, B_X) if we have $B^V = f^{-1}(B_X) - \sum_{i=1}^{n} a(X, B_X, E_i)E_i$ and $K_V + B^V \sim_{\mathbb{Q}} f^*(K_X + B_X)$, where $a(X, B_X, E_i) \in \mathbb{Q}$ and E_i is an f-exceptional divisor.

Definition 3.3. A proper irreducible subvariety Y of X is called a *centre of log canonical singularities of* (X, B_X) if there exist a birational morphism $f\colon W \to X$ and a divisor $E \subset W$ such that E lies in the support of the effective part of the divisor $\lfloor B^Y \rfloor$.

Definition 3.4. The notation $\mathbb{LCS}(X, B_X)$ will denote the set of centres of log canonical singularities of a log pair (X, B_X), and $LCS(X, B_X)$ will denote the locus of all centres of log canonical singularities of the log pair (X, B_X) (regarded as a proper subset of X).

Consider now the log pair (X, B_X), where $B_X = \sum_{i=1}^{k} a_i B_i$, B_i is an effective prime divisor, and $a_i \in \mathbb{Q}$. We choose a birational morphism $f\colon Y \to X$ such that Y is smooth and the union of all divisors $f^{-1}(B_i)$ and all f-exceptional divisors is a divisor with simple normal crossings. Then the morphism f is called a *log resolution of the log pair* (X, B_X) and we have $K_Y + B^Y \sim_{\mathbb{Q}} f^*(K_X + B_X)$ for the log pullback (Y, B^Y) of (X, B_X).

Definition 3.5. The subscheme associated with the ideal sheaf $\mathcal{I}(X, B_X) = f_*(\lceil -B^Y \rceil)$ is called the *log canonical subscheme of the log pair* (X, B_X); it is denoted by $\mathcal{L}(X, B_X)$.

The support of the subscheme $\mathcal{L}(X, B_X)$ is precisely the locus $LCS(X, B_X) \subset X$. The following result is Shokurov's famous vanishing theorem.

Theorem 3.6. *Let (X, B_X) be a log pair with effective B_X, and let H be a nef and big divisor on X such that $D = K_X + B_X + H$ is a Cartier divisor. Then $H^i(X, \mathcal{I}(X, B_X) \otimes D) = 0$ for all $i > 0$.*

Proof. We have $R^i f_*(f^*(K_X + B_X + H) + \lceil -B^W \rceil) = 0$ for $i > 0$ by the relative Kawamata–Viehweg vanishing theorem. Degeneration of the local-to-global spectral sequence and the equality

$$R^0 f_*(f^*(K_X + B_X + H) + \lceil -B^W \rceil) = \mathcal{I}(X, B_X) \otimes D$$

show that for all $i \geqslant 0$,

$$H^i(X, \mathcal{I}(X, B_X) \otimes D) = H^i(W, f^*(K_X + B_X + H) + \lceil -B^W \rceil),$$

while $H^i(W, f^*(K_X + B_X + H) + \lceil -B^W \rceil) = 0$ for $i > 0$ by the Kawamata–Viehweg vanishing theorem.

For a Cartier divisor D on X we have the exact sequence

$$0 \to \mathcal{I}(X, B_X) \otimes D \to \mathcal{O}_X(D) \to \mathcal{O}_{\mathcal{L}(X, B_X)}(D) \to 0,$$

and Theorem 3.6 yield the following three Shokurov connectedness theorems.

Theorem 3.7. *Let (X, B_X) be a log pair with effective B_X, and suppose that $-(K_X + B_X)$ is a nef and big divisor. Then $LCS(X, B_X)$ is connected.*

Theorem 3.8. *Let (X, B_X) be a log pair with effective B_X, and suppose that $-(K_X + B_X)$ is a g-nef and g-big divisor for some morphism $g\colon X \to Z$ with connected fibres. Then $LCS(X, B_X)$ is connected in the neighbourhood of each fibre of g.*

Theorem 3.9. *Let $g\colon X \to Z$ be a morphism, $D_X = \sum_{i\in I} d_i D_i$ a divisor on X, and $h\colon V \to X$ a resolution of singularities of X such that $g_*(\mathcal{O}_X) = \mathcal{O}_Z$, the divisor $(K_X + D_X)$ is g-nef and g-big, the codimension of every subvariety $g(D_i) \subset Z$ is at least 2 for $d_i < 0$, and the union of all divisors $h^{-1}(D_i)$ and all h-exceptional divisors is a divisor with simple normal crossings. Then the locus $\bigcup_{a_E \leqslant -1} E$ is connected in the neighbourhood of every fibre of $g \circ h$, where the rational numbers a_E are defined by the $\mathbb{Q}$-rational equivalence $K_V \sim_{\mathbb{Q}} f^*(K_X + D_X) + \sum_{E\subset V} a_E E$.*

Note that Theorem 3.9 is Theorem 17.4 of [17].

In the previous section we defined a centre of canonical singularities of a movable log pair and several related concepts; the movability of the boundary had actually nothing to do with these definitions. Hence these concepts can be introduced also for ordinary log pairs.

Theorem 3.10. *Let (X, B_X) be a log pair with effective B_X, suppose that $Z \in \mathbb{CS}(X, B_X)$, and let H be an effective irreducible Cartier divisor on X such that $Z \subset H$, H is not a component of B_X, and H is smooth at the generic point of Z. Then $Z \in \mathbb{LCS}(H, B_X|_H)$.*

Proof. Let $f\colon W \to X$ be a log resolution of $(X, B_X + H)$ and set $\widehat{H} = f^{-1}(H)$. Then

$$K_W + \widehat{H} \sim_{\mathbb{Q}} f^*(K_X + B_X + H) + \sum_{E \neq \widehat{H}} a(X, B_X + H, E)E,$$

and by assumption $\{Z, H\} \subset LCS(X, B_X + H)$. Application of Theorem 3.9 to the log pullback of $(X, B_X + H)$ on W yields $\widehat{H} \cap E \neq \varnothing$ for some f-exceptional divisor E on W such that $f(E) = Z$ and $a(X, B_X, E) \leqslant -1$. Now, the equivalence

$$K_{\widehat{H}} \sim (K_W + \widehat{H})|_{\widehat{H}} \sim_{\mathbb{Q}} f|_{\widehat{H}}^*(K_H + B_X|_H) + \sum_{E \neq \widehat{H}} a(X, B_X + H, E)E|_{\widehat{H}}$$

yields the assertion of the theorem.

The next result is Theorem 3.1 of [18]; it is so useful in what follows that we present here the proof from [18].

Theorem 3.11. *Let H be a surface, O a smooth point in H, M_H an effective movable boundary on H, a_1 and a_2 non-negative rational numbers, Δ_1 and Δ_2 irreducible and reduced curves on H intersecting normally at the point O. If $O \in \mathbb{LCS}(H, (1 - a_1)\Delta_1 + (1 - a_2)\Delta_2 + M_H)$, then*

$$\operatorname{mult}_O(M_H^2) \geqslant \begin{cases} 4a_1a_2 & \text{if } a_1 \leqslant 1 \text{ or } a_2 \leqslant 1; \\ 4(a_1 + a_2 - 1) & \text{if } a_1 > 1 \text{ or } a_2 > 1 \end{cases}$$

and the inequality is strict if the log pair $(H,(1-a_1)\Delta_1+(1-a_2)\Delta_2+M_H)$ *is not log canonical in the neighbourhood of* O.

Proof. Set $D=(1-a_1)\Delta_1+(1-a_2)\Delta_2+M_H$ and let $f\colon S\to H$ be a birational morphism such that the surface S is smooth and

$$K_S+f^{-1}(D)\sim_{\mathbb{Q}} f^*(K_H+D)+\sum_{i=1}^{k}a(H,D,E_i)E_i,$$

where E_i is an f-exceptional curve, $a(H,D,E_i)\in\mathbb{Q}$, and $a(H,D,E_1)\leqslant -1$. Then the birational morphism f is a composite of k blowups of smooth points.

Assume that we have proved the required result for $a_1\leqslant 1$ or $a_2\leqslant 1$. Thus, we can assume that $a_1>1$ and $a_2>1$. We define rational numbers $a(H,E_i)$, $m(H,M_H,E_i)$, and $m(H,\Delta_j,E_i)$ by means of the relations

$$\sum_{i=1}^{k}a(H,E_i)E_i\sim_{\mathbb{Q}} K_S-f^*(K_H),$$
$$\sum_{i=1}^{k}m(H,M_H,E_i)E_i\sim_{\mathbb{Q}} f^{-1}(M_H)-f^*(M_H),$$
$$\sum_{i=1}^{k}m(H,\Delta_j,E_i)E_i\sim_{\mathbb{Q}} f^{-1}(\Delta_j)-f^*(\Delta_j).$$

Then

$$\begin{aligned}a(H,D,E_i)=a(H,E_i)&-m(H,M_H,E_i)\\&+m(H,\Delta_1,E_i)(a_1-1)+m(H,\Delta_2,E_i)(a_2-1),\end{aligned}$$

and we may assume that $m(H,\Delta_1,E_1)\geqslant m(H,\Delta_2,E_1)$. Thus,

$$-1\geqslant a(H,D,E_1)\geqslant a(H,E_i)-m(H,M_H,E_i)+m(H,\Delta_2,E_i)(a_1+a_2-2)$$

and $O\in\mathbb{LCD}(H,(2-a_1-a_2)\Delta_2+M_H)$. Hence $\operatorname{mult}_O(M_H^2)\geqslant 4(a_1+a_2-1)$ because we have assumed that the theorem holds for the log pair $(H,(2-a_1-a_2)\Delta_2+M_H)$.

We can assume that $a_1\leqslant 1$. Let $h\colon T\to H$ be a blowup of O, and let E be an h-exceptional curve. Then $f=g\circ h$ for some birational morphism $g\colon S\to T$ that is the composite of $k-1$ blowups of smooth points, and we have

$$K_T+(1-a_1)\overline{\Delta}_1+(1-a_2)\overline{\Delta}_2+(1-a_1-a_2+m)E+M_T\sim_{\mathbb{Q}} h^*(K_H+D),$$

where $\overline{\Delta}_j=h^{-1}(\Delta_j)$, $m=\operatorname{mult}_O(M_H)$, and $M_T=h^{-1}(M_H)$.

If $k=1$, then $S=T$, $E_1=E$, and $a(H,D,E_1)=a_1+a_2-m-1\leqslant -1$. Thus,

$$\operatorname{mult}_O(M_H^2)\geqslant m^2\geqslant (a_1+a_2)^2\geqslant 4a_1a_2$$

and the proof is complete. Hence we can assume that $k>1$ and $P=g(E_1)$ is a point in E.

By construction $P \in \mathbb{LCD}(T, (1-a_1)\overline{\Delta}_1 + (1-a_2)\overline{\Delta}_2 + (1-a_1-a_2+m)E + M_T)$ and there exist three possible cases: $P \in E \cap \overline{\Delta}_1$, $P \in E \cap \overline{\Delta}_2$, and $P \notin \overline{\Delta}_1 \cup \overline{\Delta}_2$. Moreover, we can assume that the theorem holds for the log pair $(T, (1-a_1)\overline{\Delta}_1 + (1-a_1-a_2+m)E + M_T)$ in the case $P \in E \cap \overline{\Delta}_1$, for the log pair $(T, (1-a_2)\overline{\Delta}_2 + (1-a_1-a_2+m)E + M_T)$ in the case $P \in E \cap \overline{\Delta}_2$, and for the log pair $(T, (1-a_1-a_2+m)E + M_T)$ in the case $P \notin \overline{\Delta}_1 \cup \overline{\Delta}_2$, because all assumptions of the theorem hold in each of these cases and g consists of $k-1$ blowups of smooth points. Moreover, $\mathrm{mult}_O(M_H^2) \geqslant m^2 + \mathrm{mult}_P(M_T^2)$.

Consider the case $P \in E \cap \overline{\Delta}_1$. Then by induction we obtain

$$\mathrm{mult}_O(M_H^2) \geqslant m^2 + 4a_1(a_1 + a_2 - m) = (2a_1 - m)^2 + 4a_1a_2 \geqslant 4a_1 4a_2.$$

Suppose that $P \in E \cap \overline{\Delta}_2$. If $a_2 \leqslant 1$ or $a_1 + a_2 - m \leqslant 1$, then we can proceed as in the previous case. Thus, we can assume that $a_2 < 1$ and $a_1 + a_2 - m < 1$. Then by induction we obtain

$$\mathrm{mult}_O(M_H^2) \geqslant m^2 + 4(a_1 + 2a_2 - m - 1) > 4a_2 \geqslant 4a_1 4a_2.$$

Consider now the case $P \notin \overline{\Delta}_1 \cup \overline{\Delta}_2$. By induction

$$\mathrm{mult}_O(M_H^2) \geqslant m^2 + 4(a_1 + a_2 - m) > m^2 + 4a_1(a_1 + a_2 - m) \geqslant 4a_1 4a_2.$$

Most applications use the following simplified version of Theorem 3.11.

Lemma 3.12. *Let H be a surface, O a smooth point in H, M_H an effective movable boundary on H, and suppose that $O \in \mathbb{LCS}(H, M_H)$. Then* $\mathrm{mult}_O(M_H^2) \geqslant 4$, *and equality holds if and only if* $\mathrm{mult}_O(M_H) = 2$.

The following result is Corollary 7.3 in [19].

Theorem 3.13. *Let X be a 3-fold, O a smooth point in X, M_X an effective movable boundary on X, and suppose that $O \in \mathbb{CS}(X, M_X)$. Then* $\mathrm{mult}_O(M_X^2) \geqslant 4$ *and equality holds only if* $\mathrm{mult}_O(M_X) = 2$.

Proof. Let H be a general hyperplane section of X passing through O. Then O is a centre of log canonical singularities of the log pair $(H, M_X|_H)$, by Theorem 3.10. On the other hand

$$\mathrm{mult}_O(M_X) = \mathrm{mult}_O(M_X|_H), \qquad \mathrm{mult}_O(M_X^2) = \mathrm{mult}_O((M_X|_H)^2),$$

and the claim follows from Theorem 3.11.

An iterative application of Theorem 3.10 leads to the following result.

Theorem 3.14. *Let X be a variety of dimension at least 3, Z a subvariety of X of codimension at least 3, and suppose that $Z \in \mathbb{CS}(X, M_X)$, where M_X is an effective movable boundary on X. Then* $\mathrm{mult}_Z(M_X^2) \geqslant 4$, *and equality holds only if* $\mathrm{mult}_O(M_X) = 2$ *and the codimension of Z is 3.*

The following result is Corollary 3.5 of [18].

Lemma 3.15. *Let X be a 3-fold, O a smooth point in X, M_X an effective movable boundary on X, $f\colon V \to X$ a blowup of the point O, E an exceptional divisor of f, and suppose that $O \in \mathbb{CS}(X, M_X)$. Then either $\mathrm{mult}_O(M_X) \geqslant 2$ or there exists a line $L \subset E \cong \mathbb{P}^2$ such that $L \in \mathbb{LCS}(V, M^V + E)$.*

Proof. The result is local on the variety X. Hence we can assume that $O \in X$ is a smooth 3-fold germ. Consider a general hyperplane section H of X passing through O and let $T = f^{-1}(H)$. Then

$$K_V + M^V + E + T = K_V + f^{-1}(M_X) + (\mathrm{mult}_O(M_X) - 1)E + T \sim_{\mathbb{Q}} f^*(K_X + M_X + H),$$

where we can assume that $\mathrm{mult}_O(M_X) < 2$. Thus, $O \in \mathbb{LCS}(H, M_X|_H)$ by Theorem 3.10 and therefore the log pair $(T, M^V|_T + E|_T)$ is not log terminal. Moreover, applying Theorem 3.9 to the morphism $f\colon T \to H$ we conclude that the locus $LCS(T, M^V|_T + E|_T)$ consists of a simple point, which is the intersection of T with a centre in $\mathbb{LCS}(V, M^V + E)$. Hence the unique 1-dimensional centre of log canonical singularities of the log pair $(V, M^V + E)$ contained in E is a line in $E \cong \mathbb{P}^2$.

The following result is a natural direct generalization of Lemma 3.15.

Lemma 3.16. *Let X be a 4-fold, O a smooth point in X, M_X an effective movable boundary on X, $f\colon V \to X$ a blowup of the point O, E an exceptional divisor of f, suppose that $O \in \mathbb{CS}(X, M_X)$, and assume that $\mathrm{mult}_O(M_X) < 3$. Then either there exists a surface $S \subset E$ such that $S \in \mathbb{LCS}(V, M^V + E)$ or there exists a line $L \subset E \cong \mathbb{P}^3$ such that $L \in \mathbb{LCS}(V, M^V + E)$.*

We shall now use the idea of the proof of Lemma 3.15 to obtain the following result.

Lemma 3.17. *Let X be a 4-fold, O a smooth point in X, M_X an effective movable boundary on X, let $f\colon V \to X$ be a blowup of the point O, E an exceptional divisor of f, suppose that $O \in \mathbb{CS}(X, M_X)$, assume that $\mathrm{mult}_O(M_X) < 3$, assume that $\mathbb{LCS}(V, M^V + E)$ contains no surfaces in E and contains a line $L \subset E \cong \mathbb{P}^3$, let $g\colon W \to V$ be a blowup of L, F an exceptional divisor of g, and set $E_W = g^{-1}(E)$. Then either $F \in \mathbb{LCS}(W, M^W + E_W + 2F)$ or there exists a surface $Z \subset F$ dominating L such that $Z \in \mathbb{LCS}(W, M^W + E_W + 2F)$.*

Proof. The required result is local on X, therefore we can assume that X is a smooth 4-fold germ containing O. Consider a general hyperplane section H of X passing through O such that $L \subset T = f^{-1}(H)$. Set $M_V = f^{-1}(M_X)$, $M_W = g^{-1}(M_V)$, and $S = g^{-1}(T)$. Then

$$K_W + M^W + E_W + 2F + S \sim_{\mathbb{Q}} (f \circ g)^*(K_X + M_X + H)$$

and the divisor $M^W + E_W + 2F + S$ has the representation

$$M_W + (\mathrm{mult}_O(M_X) - 2)E_W + (\mathrm{mult}_O(M_X) + \mathrm{mult}_L(M_V) - 3)F + S,$$

where $\mathrm{mult}_O(M_X) < 3$. We can assume that $\mathrm{mult}_O(M_X) + \mathrm{mult}_L(M_V) < 4$. We must show that there exists a surface $Z \subset F$ such that Z dominates the curve L and $Z \in \mathbb{LCS}(W, M^W + E_W + 2F)$.

Let $\overline{H}$ be a general hyperplane section of X passing through O such that $L \not\subset \overline{T} = f^{-1}(\overline{H})$, and let $\overline{S} = g^{-1}(\overline{T})$. Then $O \in \mathbb{LCS}(\overline{H}, M_X|_{\overline{H}})$, by Theorem 3.10; moreover, the relation

$$K_W + M^W + E_W + F + \overline{S} \sim_{\mathbb{Q}} (f \circ g)^*(K_X + M_X + H)$$

shows that $(\overline{S}, (M^W + E_W + F)|_{\overline{S}})$ is not log terminal. Applying Theorem 3.9 to the birational morphism $f \circ g\colon \overline{S} \to \overline{H}$ we conclude that the part of the locus $LCS(\overline{S}, (M^W + E_W + F)|_{\overline{S}})$ lying in the fibre of g over the point $\overline{T} \cap L$ either consists of a single point or contains a curve in this fibre. Moreover, the elements of $\mathbb{LCS}(\overline{S}, (M^W + E_W + F)|_{\overline{S}})$ lying in the fibre of g over $\overline{T} \cap L$ are the intersections of $\overline{S}$ with elements of $\mathbb{LCS}(W, M^W + E_W + F)$. Hence either $\mathbb{LCS}(W, M^W + E_W + F)$ contains a surface in F or the unique centre of log canonical singularities of the log pair $(W, M^W + E_W + F)$ lying in F is a curve $C \subset F$ that is a section of the $\mathbb{P}^2$-bundle $g\colon F \to L$. However, every element of the set $\mathbb{LCS}(W, M^W + E_W + F)$ is an element of $\mathbb{LCS}(W, M^W + E_W + 2F)$, therefore for the proof of the required result we can assume that C is the unique centre of log canonical singularities of the log pair $(W, M^W + E_W + F)$ and the log pair $(W, M^W + E_W + 2F)$ lying in the exceptional divisor F.

The point O is an element of $\mathbb{LCS}(H, M_X|_H)$ by Theorem 3.10. By assumption the log pair $(S, M^W|_S + E_W|_S + 2F|_S)$ is not log terminal over O. Applying Theorem 3.9 to the morphism $f \circ g\colon S \to H$ we conclude that over O the locus $LCS(S, (M^W + E_W + 2F)|_S)$ consists of the unique point $S \cap C$. Applying the Kawamata–Viehweg vanishing theorem (see [16]) to the divisor $S - F$ we establish the surjectivity of the map $H^0(S) \to H^0(S|_F)$. On the other hand the linear system $|S|_F|$ is free from base points (see [20], § 2.8), therefore $C \not\subset S$ in view of the generality of H. Moreover,

$$L \cong C \cong \mathbb{P}^1, \qquad F \cong \mathbb{P}(\mathcal{O}_L(-1) \oplus \mathcal{O}_L(1) \oplus \mathcal{O}_L(1))$$

and $S|_F \sim B + D$, where $B = \mathcal{O}_F(1)$ and D is a fibre of $\tau = g|_F$. Let $\mathcal{I}_C$ be the ideal sheaf of the curve C on F. Then $R^1\tau_*(B \otimes \mathcal{I}_C) = 0$ and the map $\pi\colon \mathcal{O}_L(-1) \oplus \mathcal{O}_L(1) \oplus \mathcal{O}_L(1) \to \mathcal{O}_L(k)$ is surjective, where $k = B \cdot C$ and the map π lies in $H^0(\mathcal{O}_L(k+1)) \oplus H^0(\mathcal{O}_L(k-1)) \oplus H^0(\mathcal{O}_L(k-1))$. In particular, $k \geqslant -1$. The equality $k = 0$ is impossible because in that case π can be described by a matrix $(ax + by, 0, 0)$, in which a and b are complex numbers and $(x : y)$ are homogeneous coordinates on $L \cong \mathbb{P}^1$, which contradicts the surjectivity of π at the point $ax + by = 0$. Thus, the divisor B on F cannot have trivial intersection with the section C and $S \cap C$ is either trivial or contains more than one point. However, we have already shown that $S \cap C$ consists of a single point.

The following result is a generalization of Theorem 2 in [21].

Lemma 3.18. *Let V be a smooth complete intersection $\bigcap_{i=1}^{k} G_i \subset \mathbb{P}^M$ of dimension at least* 3, *D an effective divisor on V such that $D \equiv \mathcal{O}_{\mathbb{P}^M}(n)|_V$, let $S \subset V$ be an irreducible subvariety of dimension at least k and codimension at least* 2, *where G_i is a hypersurface in $\mathbb{P}^M$. Then $\mathrm{mult}_S(D) \leqslant n$.*

Proof. We can assume that S has dimension $k < (M-1)/2$. Consider a sufficiently general cone $C_S \in \mathbb{P}^M$ over S with vertex at a sufficiently general point $P \in \mathbb{P}^M$.

Then $C_S \cap V = S \cup R_S$ for some curve R_S on the variety V, and this equality is also valid in the scheme-theoretic sense in view of the generality of the cone C_S.

Let $\pi\colon V \to \mathbb{P}^{M-1}$ be the projection from P and $D_\pi \subset V$ the ramification variety of π. We claim that

$$R_S \cap S = D_\pi \cap S$$

in the set-theoretic sense. Suppose that $C_S \cap G_i = S \cup R_S^i$. Then $R_S^i \cap S = D_\pi^i \cap S$ in the set theoretic sense for the smooth ramification divisor $D_\pi^i \subset G_i$ of the projection $\pi^i\colon G_i \to \mathbb{P}^{M-1}$ from the point P, by Lemma 3 of [6]. On the other hand we have $R_S = \bigcap_{i=1}^k R_S^i$ and $D_\pi = \bigcap_{i=1}^k D_\pi^i$, and therefore $R_S \cap S = D_\pi \cap S$.

Consider homogeneous coordinates $(z_0 : \ldots : z_M)$ on $\mathbb{P}^M$ in which G_j is given by an equation $F_j = 0$ and P has coordinates $(p_0 : \ldots : p_M)$. Then D_π is described by k equations

$$\sum_{i=0}^{M} \frac{\partial F_j}{\partial z_i} p_i = 0,$$

and the linear systems

$$\left| \sum_{i=0}^{M} \lambda_i \frac{\partial F_j}{\partial z_i} = 0 \right|$$

are free on V since V is smooth. Hence $D_\pi \cap S$ consists of $d_S \prod_{i=1}^k (d_i - 1)$ distinct sufficiently general points in S, where d_S is the degree of S in $\mathbb{P}^M$; however, the degree of R_S is precisely equal to $d_S \prod_{i=1}^k (d_i - 1)$, and the generality of C_S means that $R_S \not\subset D$, so that $\operatorname{mult}_S(D) \leqslant n$.

§ 4. Movable log pairs on the variety V_8

Let V_8 be a smooth complete intersection $F_2 \cap F_4 \subset \mathbb{P}^6$ such that V_8 contains no planes in $\mathbb{P}^6$, where F_2 and F_4 are a quadric and a quartic in $\mathbb{P}^6$, respectively.

Theorem 4.1. *Let M_{V_8} be an effective movable boundary on the variety V_8 such that $K_{V_8} + M_{V_8} \sim_{\mathbb{Q}} 0$. Then the log pair (V_8, M_{V_8}) is canonical.*

The following implication is well known (see [22]).

Proposition 4.2. *The birational superrigidity of the variety V_8 is a consequence of Theorem* 4.1.

Proof. Let ρ be a birational transformation of V_8 into a variety Y such that either there exists a fibration $\tau\colon Y \to Z$ of varieties of Kodaira dimension $-\infty$ or Y is a terminal $\mathbb{Q}$-factorial Fano variety with Picard group $\mathbb{Z}$. We must prove that the former case is impossible, $Y \cong V_8$, and ρ is an isomorphism.

Assume that there exists a fibration $\tau : Y \to Z$ of varieties of Kodaira dimension $-\infty$. We choose a sufficiently general very ample divisor H on Z and consider $\mu \in \mathbb{Q}_{>0}$ and the movable boundary $M_{V_8} = \mu\rho^{-1}(|\tau^*(H)|)$ such that $K_{V_8} + M_{V_8} \sim_{\mathbb{Q}} 0$. Then the singularities of the log pair (V_8, M_{V_8}) are canonical, by Theorem 4.1. In particular, $\varkappa(V_8, M_{V_8}) = 0$. On the other hand $\varkappa(V_8, M_{V_8}) = -\infty$ by construction.

Let Y be a terminal $\mathbb{Q}$-factorial Fano manifold with Picard group $\mathbb{Z}$. Let $n \gg 0$ be a positive integer, suppose that $\mu \in \mathbb{Q}_{>0}$, and let $M_Y = \mu/n\,|-nK_Y|$ and $M_{V_8} = \rho^{-1}(M_Y)$ be movable boundaries such that $K_{V_8} + M_{V_8} \sim_{\mathbb{Q}} 0$. Then (V_8, M_{V_8})

is canonical, by Theorem 4.1. On the other hand $\varkappa(V_8, M_{V_8}) = \varkappa(Y, M_Y)$, and therefore $\mu = 1$.

Consider now a birational morphism $f\colon W \to V_8$ such that $g = \rho \circ f$ is regular and W is a smooth variety. Then

$$\sum_{j=1}^{k} a(V_8, M_{V_8}, F_j)F_j \sim_{\mathbb{Q}} \sum_{i=1}^{l} a(Y, M_Y, G_i)G_i,$$

where G_i is an g-exceptional divisor and F_j is an f-exceptional divisor. The singularities of the log pairs (V_8, M_{V_8}) and (Y, M_Y) are canonical; moreover, the singularities of (Y, M_Y) are terminal. In particular, all the numbers $a(V_8, M_{V_8}, F_j)$ are non-negative and all the numbers $a(Y, M_Y, G_i)$ are positive. The negativity of the exceptional locus (see [17], Lemma 2.19) shows that $a(V_8, M_{V_8}, E) = a(Y, M_Y, E)$ for each divisor E on W. In particular, $\sum_{j=1}^{k} a(V_8, M_{V_8}, F_j)F_j = \sum_{i=1}^{l} a(Y, M_Y, G_i)G_i$, where the support of the divisor on the right-hand side contains all g-exceptional divisors. On the other hand $\operatorname{Pic}(V_8) = \mathbb{Z}$ yields $\operatorname{Pic}(W) = \mathbb{Z}^{1+k}$, and the equality $\operatorname{Pic}(Y) = \mathbb{Z}$ and the $\mathbb{Q}$-factoriality of Y yield $\operatorname{Pic}(W) = \mathbb{Z}^{1+l}$. Hence $k = l$ and all the numbers $a(V_8, M_{V_8}, F_j)$ are positive. In particular, the singularities of the log pair (V_8, M_{V_8}) are terminal.

Consider ζ in $\mathbb{Q}_{>1}$ such that $(V_8, \zeta M_{V_8})$ and $(Y, \zeta M_Y)$ are terminal. Then the divisors $K_{V_8} + \zeta M_{V_8}$ and $K_Y + \zeta M_Y$ are ample and the log pairs $(V_8, \zeta M_{V_8})$ and $(Y, \zeta M_Y)$ are canonical models. Thus ρ is an isomorphism, by Theorem 2.9.

The following result is equivalent to Theorem 4.1.

Theorem 4.3. *Let M_{V_8} be an effective movable boundary on the variety V_8 such that the divisor $-(K_{V_8} + M_{V_8})$ is ample. Then $\mathbb{CS}(V_8, M_{V_8}) = \varnothing$.*

We shall prove Theorem 4.3 in the following two sections.

§ 5. Points in the variety V_8

Let V_8 be a smooth complete intersection $F_2 \cap F_4 \subset \mathbb{P}^6$ not containing 2-dimensional linear subspaces of $\mathbb{P}^6$, where F_2 and F_4 are a quadric and a quartic, respectively. Let M_{V_8} be an effective movable boundary on V_8 such that the divisor $-(K_{V_8} + M_{V_8})$ is ample. In this section we prove the following result.

Theorem 5.1. *The set $\mathbb{CS}(V_8, M_{V_8})$ contains no points in V_8.*

Assume that $\mathbb{CS}(V_8, M_{V_8})$ contains a point O in V_8, let H_{V_8} be a hyperplane section of V_8 passing through O, $B_{V_8} = H_{V_8} + M_{V_8}$, let $f\colon W \to V_8$ be a blowup of O, E an f-exceptional divisor, and let $M_W = f^{-1}(M_{V_8})$, $H_W = f^{-1}(H_{V_8})$. Then $O \in \mathbb{LCS}(V_8, B_{V_8})$ by Theorem 3.10, and

$$K_W + M_W + H_W \sim_{\mathbb{Q}} f^*(K_{V_8} + M_{V_8} + H_{V_8}) + a(V_8, B_{V_8}, E)E,$$

where $a(V_8, B_{V_8}, E) = \operatorname{mult}_O(M_{V_8}) - 2$.

Lemma 5.2. *The inequality $a(V_8, B_{V_8}, E) > -1$ holds.*

Proof. Assume that $a(V_8, B_{V_8}, E) \leqslant -1$. Then we have $\mathrm{mult}_O(M_{V_8}) \geqslant 3$, therefore $\mathrm{mult}_O(M_{V_8}^2) \geqslant 9$; moreover, $\mathrm{mult}_O(M_{V_8}^2) \leqslant H_1 \cdot H_2 \cdot M_{V_8}^2$, where H_1 and H_2 are sufficiently general hyperplane sections of V_8 passing through O, and on the other hand $H_1 \cdot H_2 \cdot M_{V_8}^2 < 8$.

Suppose that $B^W = (\mathrm{mult}_O(M_{V_8})-2)E+H_W+M_W$. Then the log pair (W, B^W) is a log pullback of the log pair (V_8, B_{V_8}) and $K_W + B^W \sim_{\mathbb{Q}} f^*(K_X + B_X)$.

Lemma 5.3. *The set $\mathbb{LCS}(W, B^W)$ contains a proper subvariety of E not lying in H_W.*

Proof. The equivalence

$$(\mathrm{mult}_O(M_{V_8}) - 3)E + M_W \sim_{\mathbb{Q}} f^*(K_{V_8} + M_{V_8})$$

and Lemma 5.2 show that there exists a proper subvariety S of the divisor E such that $S \in \mathbb{CS}(W, (\mathrm{mult}_O(M_{V_8}) - 3)E + M_W)$ and $S \not\subset H_{V_8}$ in view of the generality of H_{V_8}. Hence $S \in \mathbb{LCS}(W, (\mathrm{mult}_O(M_{V_8}) - 2)E + M_W)$.

Let S be a maximum-dimension element of $\mathbb{LCS}(W, B^W)$ such that S is a proper subvariety of E and $S \not\subset H_W$. Then S can be a point, a curve, or a surface.

Lemma 5.4. *The subvariety S is not a surface.*

Proof. Assume that S is a surface. Applying Theorem 3.11 to the generic point of S and the log pair $(W, (\mathrm{mult}_O(M_{V_8}) - 2)E + M_W)$, we obtain the inequality $\mathrm{mult}_S(M_W^2) \geqslant 4(3 - \mathrm{mult}_O(M_{V_8}))$. Thus,

$$\mathrm{mult}_O(M_{V_8}^2) \geqslant \mathrm{mult}_O^2(M_{V_8}) + \mathrm{mult}_S(M_W^2) \geqslant \mathrm{mult}_O^2(M_{V_8}) + 4(3 - \mathrm{mult}_O(M_{V_8}))$$

and $H_1 \cdot H_2 \cdot M_{V_8}^2 \geqslant \mathrm{mult}_O(M_{V_8}^2) \geqslant (\mathrm{mult}_O(M_{V_8}) - 2)^2 + 8$, where H_1 and H_2 are general hyperplane sections of V_8 passing through O, while on the other hand $H_1 \cdot H_2 \cdot M_{V_8}^2 < 8$.

The locus $LCS(W, B^W)$ is connected in the neighbourhood of E by Theorem 3.9, and $H_W \in \mathbb{LCS}(W, B^W)$.

Corollary 5.5. *The subvariety S is not a point.*

Lemma 5.6. *The subvariety S is a line in $E \cong \mathbb{P}^3$.*

Proof. The connectedness of the locus $LCS(W, (\mathrm{mult}_O(M_{V_8}) - 2)E + H_W + M_W)$ in the neighbourhood of E, the generality of our choice of the hyperplane section H_{V_8}, and the adjunction formula show that, of all subvarieties of E, the set $\mathbb{LCS}(H_W, (\mathrm{mult}_O(M_{V_8}) - 2)E|_{H_W} + M_W|_{H_W})$ contains only points. On the other hand

$$\{S \cap H_W\} \subset \mathbb{LCS}(H_W, (\mathrm{mult}_O(M_X) - 2)E|_{H_W} + M_W|_{H_W}),$$

and the locus $LCS(H_W, (\mathrm{mult}_O(M_{V_8}) - 2)E|_{H_W} + M_W|_{H_W})$ is connected in the neighbourhood of the exceptional divisor $E|_{H_W}$, by Theorem 3.9. Hence $S \cap H_W$ consists of a single point.

Let Y be a sufficiently general hyperplane section of the variety V_8 passing through O such that $S \subset f^{-1}(Y)$, and set $M_Y = M_{V_8}|_Y$.

Warning 5.7. The variety Y can be singular.

Remark 5.8. The point O is smooth on Y, $O \in \mathbb{LCS}(Y, M_Y)$ by Theorem 3.10, and the effective boundary M_Y is movable because V_8 contains no planes in $\mathbb{P}^6$.

Let $g\colon V \to Y$ be a blowup of O, set $F = g^{-1}(O)$ and $M_V = g^{-1}(M_Y)$. Then the curve S lies in F, $E|_V = F$, $\operatorname{mult}_O(M_Y) = \operatorname{mult}_O(M_{V_8})$, and $M_V = M_W|_V$, where V is identified with a subvariety of W. We consider now a boundary $M^V = (\operatorname{mult}_O(M_Y) - 2)F + M_V$ such that

$$K_V + M^V \sim_{\mathbb{Q}} f^*(K_Y + M_Y).$$

Proposition 5.9. *The curve S belongs to $\mathbb{LCS}(V, M^V)$.*

Proof. Let $h\colon U \to W$ be a blowup of S, let $G = h^{-1}(S)$, and

$$B^U = M_U + (\operatorname{mult}_O(M_{V_8}) - 2)E_U + (\operatorname{mult}_O(M_{V_8}) + \operatorname{mult}_S(M_W) - 3)G + V_U,$$

where $M_U = h^{-1}(M_W)$, $E_U = h^{-1}(E)$, and $V_U = h^{-1}(V)$. Then

$$K_U + B^U \sim_{\mathbb{Q}} (f \circ h)^*(K_{V_8} + M_{V_8} + Y),$$

and it follows by Lemma 3.17 that either $G \in \mathbb{LCS}(U, B^U)$ or there exists a surface $Z \subset G$ dominating S such that $Z \in \mathbb{LCS}(U, B^U)$. Hence $S \in \mathbb{LCS}(V, M^V)$ by the adjunction formula.

We can apply Theorem 3.11 to the log pair $(V, (\operatorname{mult}_O(M_Y) - 2)F + M_V)$ and the generic point of $S \subset V$ to obtain $\operatorname{mult}_S(M_V^2) \geqslant 4(3 - \operatorname{mult}_O(M_Y))$. Thus,

$$\operatorname{mult}_O(M_Y^2) \geqslant \operatorname{mult}_O^2(M_Y) + \operatorname{mult}_S(M_V^2) \geqslant \operatorname{mult}_O^2(M_Y) + 4(3 - \operatorname{mult}_O(M_Y))$$

and $H_O \cdot M_Y^2 \geqslant \operatorname{mult}_O(M_Y^2) \geqslant (\operatorname{mult}_O(M_Y) - 2)^2 + 8$, where H_O is a sufficiently general hyperplane section of Y passing through O. On the other hand $H_O \cdot M_Y^2 < 8$ because the divisor $-(K_{V_8} + M_{V_8})$ is ample. The proof of Theorem 5.1 is thus complete.

§ 6. Curves and surfaces on the variety V_8

Let V_8 be a smooth complete intersection $F_2 \cap F_4 \subset \mathbb{P}^6$ not containing 2-dimensional linear subspaces of $\mathbb{P}^6$, where F_2 and F_4 are a quadric and a quartic, respectively, and let M_{V_8} be an effective movable boundary on V_8 such that the divisor $-(K_{V_8} + M_{V_8})$ is ample. In the previous section we proved that $\mathbb{CS}(V_8, M_{V_8})$ contains no points. We must now show that $\mathbb{CS}(V_8, M_{V_8}) = \varnothing$.

Theorem 6.1. *The set $\mathbb{CS}(V_8, M_{V_8})$ contains no surfaces.*

Proof. Assume that $\mathbb{CS}(V_8, M_{V_8})$ contains a surface $S \subset V_8$. Then $\operatorname{mult}_S(M_{V_8}) \geqslant 1$, which contradicts Lemma 3.18.

Assume that $\mathbb{CS}(V_8, M_{V_8})$ contains a curve $C \subset V_8$.

Proposition 6.2. *The curve C is a line in $\mathbb{P}^6$.*

Proof. Assume that C is not a line. Let L be a line in $\mathbb{P}^6$ passing through two sufficiently general points of C. Then $L \not\subset V_8$ since otherwise $L \subset M_{V_8}$ because $\mathrm{mult}_C(M_{V_8}) \geqslant 1$, which contradicts the movability of M_{V_8} because V_8 contains no planes. Thus, $8 > H_1 \cdot H_2 \cdot M_{V_8}^2 \geqslant 2\,\mathrm{mult}_C(M_{V_8}^4)$, where H_1 and H_2 are two general hyperplane sections of V_8 passing through L. On the other hand $\mathrm{mult}_C(M_{V_8}^2) \geqslant 4$ by Theorem 3.14.

Lemma 6.3. *The inequality* $\mathrm{mult}_C(V_8) \leqslant \frac{4}{3}$ *holds.*

Proof. Let S be the intersection of two sufficiently general hyperplane sections of V_8 passing through the line C. Then S is a smooth surface, the canonical divisor K_S is equivalent to the hyperplane section of S, and $K_S^2 = 8$. Moreover, $M_{V_8}|_S = M_S + \mathrm{mult}_C(V_8)C$, where M_S is an effective movable boundary. Hence $M_S^2 \geqslant 0$ on the surface S. On the other hand

$$M_S^2 = (K_S - \mathrm{mult}_C(V_8)C)^2 = 8 - 2\,\mathrm{mult}_C(V_8) - 3\,\mathrm{mult}_C^2(V_8)$$

because $K_S \cdot C = 1$, and $C^2 = -3$ by the adjunction formula. Hence $\mathrm{mult}_C(V_8) \leqslant \frac{4}{3}$.

Let $f\colon V \to V_8$ be a blowup of C, set $E = f^{-1}(C)$ and $M_V = f^{-1}(M_{V_8})$. Let H_C be a general hyperplane section of V_8 passing through C, set $B_{V_8} = M_{V_8} + H_C$ and $H_V = f^{-1}(H_C)$. Then

$$K_V + M_V + H_V \sim_{\mathbb{Q}} f^*(K_{V_8} + M_{V_8} + H_C) + a(V_8, B_{V_8}, E)E,$$

where $a(V_8, B_{V_8}, E) = \mathrm{mult}_C(M_{V_8}) - 1$. Moreover, $C \in \mathbb{LCS}(V_8, B_{V_8})$ by Theorem 3.10.

Lemma 6.4. *There exists a surface S in $\mathbb{LCS}(V, M_V + H_V - a(V_8, B_{V_8}, E)E)$ such that the fibre of the morphism $f|_S\colon S \to C$ over each point $P \in C$ is a line in $f^{-1}(P) \cong \mathbb{P}^2$.*

Proof. Let H_P be a general hyperplane section of the variety V_8 passing through a point $P \in C$. Then $P \in \mathbb{LCS}(H_P, B_{V_8}|_{H_P})$ and $P \in \mathbb{CS}(H_P, M_{V_8}|_{H_P})$. The required result now follows by Lemma 3.15.

Proposition 6.5. *The inequality* $\mathrm{mult}_S(M_V^2) \geqslant \frac{8}{3}$ *holds.*

Proof. Applying Theorem 3.11 to $(V, M_V + H_V - a(V_8, B_{V_8}, E)E)$ we obtain the inequality $\mathrm{mult}_S(M_V^2) \geqslant 4(2 - \mathrm{mult}_C(V_8))$, whereas $\mathrm{mult}_C(V_8) \leqslant \frac{4}{3}$ by Lemma 6.3.

The linear system $|H_V|$ is free, $H_V^4 = 3$, and $\varphi_{|H_V|}\colon V \to \mathbb{P}^4$ is a morphism of degree 3 contracting no surface into a point, because the variety V_8 contains no planes in $\mathbb{P}^6$. Consider a general divisor $H \subset |H_V|$. Explicit calculations show that H intersects the curve $H_V \cap S$ in at least two distinct points. Hence $H \cdot H_V \cdot M_V^2 \geqslant 2\frac{8}{3}$, whereas

$$H \cdot H_V \cdot M_V^2 = H_V^2 \cdot M_V^2 = (f^*(-K_{V_8}) - E)^2 \cdot (f^*(-\lambda K_{V_8}) - \mathrm{mult}_C(M_{V_8})E)^2 < 3$$

because the inclusion $C \in \mathbb{CS}(V_8, M_{V_8})$ yields $\mathrm{mult}_C(M_{V_8}) \geqslant 1$, where $\lambda \in \mathbb{Q} \cap (0,1)$ is a number such that $M_{V_8} \sim_{\mathbb{Q}} -\lambda K_{V_8}$. We have thus proved the following result.

Theorem 6.6. *The set $\mathbb{CS}(V_8, M_{V_8})$ contains no curves.*

The proof of Theorem 4.3 is complete.

§ 7. Elliptic fibrations

Let V_8 be a smooth complete intersection $F_2 \cap F_4 \subset \mathbb{P}^6$ containing no 2-dimensional linear subspaces, where F_2 and F_4 are a quadric and a quartic, respectively. Assume that there exists a birational map $\rho\colon V \dashrightarrow V_8$, where V has the structure of an elliptic fibration $\tau\colon V \to Z$. Consider a very ample divisor D_Z on Z, set $\mathcal{D}_{V_8} = \rho(|\tau^*(D_Z)|)$, and consider a movable boundary $1/n\mathcal{D}_{V_8}$, where $n \in \mathbb{N}$ is an integer such that $\mathcal{D}_{V_8} \sim -nK_{V_8} \sim \mathcal{O}_{\mathbb{P}^6}(n)|_{V_8}$.

Remark 7.1. The singularities of the log pair $(V_8, 1/n\mathcal{D}_{V_8})$ are canonical by Theorem 4.1.

Proposition 7.2. $\mathbb{CS}(V_8, 1/n\mathcal{D}_{V_8}) \neq \varnothing$.

Proof. Assume that $(V_8, 1/n\mathcal{D}_{V_8})$ is terminal; then the log pair $(V_8, (1+\varepsilon)/n\mathcal{D}_{V_8})$ is a canonical model for some $\varepsilon \in \mathbb{Q}_{>0}$, while on the other hand $\varkappa(V_8, (1+\varepsilon)/n\mathcal{D}_{V_8})$ does not exceed the dimension of Z.

In the rest of this section we show that the inequality $\mathbb{CS}(V_8, 1/n\mathcal{D}_{V_8}) \neq \varnothing$ is in contradiction with the fact that Z has dimension 3.

Proposition 7.3. *The set* $\mathbb{CS}(V_8, 1/n\mathcal{D}_{V_8})$ *contains no points in* V_8.

Proof. Assume that $\mathbb{CS}(V_8, 1/n\mathcal{D}_{V_8})$ contains a point O in the variety V_8 and let D be a sufficiently general divisor in $\mathcal{D}_{V_8}$, $f\colon W \to V_8$ a blowup of O, E an exceptional divisor of f, and suppose that $\mathcal{D}_W = f^{-1}(\mathcal{D}_{V_8})$ and $D_W = f^{-1}(D)$. It follows by the results of § 5 that $\mathrm{mult}_O(D) = 2n$ and $\mathrm{mult}_S(D_W) = 2n$ for some surface S on E. Moreover, since $-K_{V_8}$ is very ample and $\mathcal{D}_{V_8} \sim -nK_{V_8}$, it follows that $\mathrm{mult}_O(\mathcal{D}_{V_8}^2) \leqslant 8n^2$. On the other hand

$$\mathrm{mult}_O(\mathcal{D}_{V_8}^2) \geqslant \mathrm{mult}_O^2(\mathcal{D}_{V_8}) + \mathrm{mult}_S(\mathcal{D}_W^2) \geqslant (4+4d)n^2,$$

where d is the degree of S in $E \cong \mathbb{P}^3$. Hence S is a plane in $E \cong \mathbb{P}^3$, $\mathrm{mult}_O(\mathcal{D}_{V_8}^2) = 8n^2$, and $\mathrm{mult}_S(\mathcal{D}_W^2) = 4n^2$.

Let H and H' be general hyperplane sections of V_8 passing through O, $Y = H \cap H'$, let $g\colon V \to W$ be a blowup of S, F a g-exceptional divisor, $\mathcal{D}_V = g^{-1}(\mathcal{D}_W)$, $D_V = g^{-1}(D_W)$, $\widetilde{Y} = (f \circ g)^{-1}(Y)$, and $\widetilde{H} = (f \circ g)^{-1}(H)$. Then the restrictions $\mathcal{D}_V|_{\widetilde{H}}$ and $\mathcal{D}_V|_{\widetilde{Y}}$ have no fixed components, and $(\mathcal{D}_V|_{\widetilde{Y}})^2 = 0$. Moreover, the groups $H^0(rD_V - \widetilde{H})$ and $H^0(rD_V|_{\widetilde{H}} - \widetilde{Y})$ are empty because for every divisor R in each of these groups the multiplicity of the effective cycle $f(g(R))D$ of degree $8r^2n^2$ at O is greater than $8r^2n^2$, where $r \in \mathbb{N}$ and $\widetilde{Y}$ is regarded as a divisor on $\widetilde{H}$. Hence we have the embedding $H^0(rD_V) \subset H^0(rD_V|_{\widetilde{S}})$, while on the other hand the linear system $|rD_V|_{\widetilde{S}}|$ is obtained from a free pencil because $\mathcal{D}_V|_{\widetilde{Y}} \subset |D_V|_{\widetilde{S}}|$ and $(rD_V|_{\widetilde{S}})^2 = 0$. Thus, $h^0(rD_V)$ grows linearly for $r \gg 0$ and the linear system $|rD_V|$ has no fixed components because $|D_V|$ has no fixed components since $\mathcal{D}_V \subset |D_V|$. Hence the linear system $|D_V|$ is obtained from a pencil. Thus, the linear system $|\tau^*(D_Z)|$ is also obtained from a pencil, which contradicts the fact that Z has dimension 3.

Proposition 7.4. *The set* $\mathbb{CS}(V_8, 1/n\mathcal{D}_{V_8})$ *contains no curves in* V_8.

Proof. Assume that $\mathbb{CS}(V_8, 1/n\mathcal{D}_{V_8})$ contains a curve $C \subset V_8$ and let $D \subset \mathcal{D}_{V_8}$ be a general divisor. Then it follows by results of § 6 and Theorem 3.14 that C is a conic and $\mathrm{mult}_C(D) = 2n$. Let H and H' be general hyperplane sections of V_8, $Y = H \cap H'$, let $f\colon V \to V_8$ be a blowup of C, E an exceptional divisor of f, and suppose that $\mathcal{D}_V = f^{-1}(\mathcal{D}_{V_8})$, $D_V = f^{-1}(D)$, $\widetilde{Y} = f^{-1}(Y)$, and $\widetilde{H} = f^{-1}(H)$. Then the restrictions $\mathcal{D}_V|_{\widetilde{H}}$ and $\mathcal{D}_V|_{\widetilde{Y}}$ have no fixed components because the divisors H and H' are sufficiently general and V_8 contains no line passing through two sufficiently general points of the conic C. Moreover, $(\mathcal{D}_V|_{\widetilde{Y}})^2 = 0$ on the surface $\widetilde{Y}$. Continuing as in the proof of Proposition 7.3 we conclude that the linear systems $|D_V|$, $\mathcal{D}_{V_8}$, and $|\tau^*(D_Z)|$ are obtained from pencils, which is impossible because Z has dimension 3.

We can assume that $\mathbb{CS}(V_8, 1/n\mathcal{D}_{V_8})$ contains a surface $S \subset V_8$ such that $\mathrm{mult}_S(\mathcal{D}_{V_8}) = n$. Moreover, the degree of S is at least 2 because V_8 contains no planes.

Lemma 7.5. *The surface S lies in some* 4*-dimensional linear subspace of* $\mathbb{P}^6$.

Proof. We consider a general cone $C_S \subset \mathbb{P}^6$ over S with vertex $P \in \mathbb{P}^6$. Then $C_S \cap V_8 = S \cup R_S$ for some irreducible curve R_S on V_8, where the equality holds in the scheme-theoretic sense because of the generality of C_S. In the proof of Lemma 3.18 we showed that $R_S \cap S$ consists of d_R distinct points, where d_R is the degree of the curve R_S in $\mathbb{P}^6$. Hence every divisor D in $\mathcal{D}_{V_8}$ either intersects R_S only at points in $R_S \cap S$ or contains the curve R_S because $D \sim \mathcal{O}_{\mathbb{P}^6}(n)|_{V_8}$.

Let Π be a plane in $\mathbb{P}^6$ passing through two general points P_1 and P_2 on the divisor S and through a general point P_D on a divisor D in $\mathcal{D}_{V_8}$. Then Π contains a point $P_{V_8} \in V_8$ such that $P_{V_8} \notin S$ and $P_{V_8} \neq P_D$. Assume that D is sufficiently general. Then in the above construction of the curve R_S we can assume that the point P is the intersection of the line L_1 passing through P_1 and P_D and the line L_2 passing through P_2 and P_{V_8}. Thus, $P_D \in R_S$ and $P_{V_8} \in R_S$. Hence $R_S \subset D$, and in particular $P_{V_8} \in D$. Thus, if Π is a plane passing through two general points P_1 and P_2 on S and one general point P_D on D, then other points in $V_8 \cap \Pi$ also lie in the divisor D. The last condition is closed with respect to P_D and we can assume that $P_D \in S \subset D$.

Thus, for a general divisor D in $\mathcal{D}_{V_8}$ and a 2-dimensional linear subspace Π of $\mathbb{P}^6$ passing through three general points in S the intersection $\Pi \cap V_8$ lies in D. Furthermore, we can choose another sufficiently general divisor D' in the linear system $\mathcal{D}_{V_8}$ and obtain $\Pi \cap V_8 \subset D'$. On the other hand $D \cap D'$ is a surface. Hence each plane Π passing through three sufficiently general points in S intersects a surface $\widetilde{S} \in \mathbb{P}^6$ containing S as an irreducible component in at least one additional point. This is possible only if S lies in some 4-dimensional linear subspace of $\mathbb{P}^6$.

Let Π be a 4-dimensional linear subspace of $\mathbb{P}^6$ such that $S \subset \Pi$.

Lemma 7.6. *The scheme-theoretic intersection* $\Pi \cap V_8$ *is reduced.*

Proof. Let H_1 and H_2 be two general hyperplane sections of V_8 passing through $\Pi \cap V_8$, and let H be a general hyperplane section of V_8. Then H_1 has only isolated singularities, by Zak's theorem (see [23]), $Y = H \cap H_1$ is smooth, and the scheme-theoretic intersection $C = Y \cap H_2 = \Pi \cap V_8 \cap H$ has only isolated singularities and is reduced by Zak's theorem. Thus, $\Pi \cap V_8$ is also reduced.

Lemma 7.7. *Let S be a quadric and T_3 a 3-dimensional linear subspace of $\mathbb{P}^6$ such that $S \subset T_3$. Then $S \neq V_8 \cap T_3$.*

Proof. Assume that $S = V_8 \cap T_3$. Let $\psi\colon V_8 \dashrightarrow \mathbb{P}^2$ be a projection from T_3. Then the generic fibre of ψ is the intersection of a quadric and a cubic in $\mathbb{P}^4$. Let $f\colon W \to V_8$ be a resolution of the indeterminacy of ψ such that the variety W is smooth, there exists a unique f-exceptional divisor E dominating S, f is an isomorphism outside S, and set $\mathcal{D}_W = f^{-1}(\mathcal{D}_{V_8})$. Then the equality $\operatorname{mult}_S(\mathcal{D}_{V_8}) = n$ shows that the linear system $\mathcal{D}_W$ lies in fibres of $\tau = \psi \circ f$, which is impossible because Z has dimension 3.

Suppose that $\Pi \cap V_8 = \bigcup_{i=0}^r S_i$, where the S_i are irreducible reduced surfaces and $S_0 = S$.

Lemma 7.8. *Suppose that $S \subset T_3$, where T_3 is a 3-dimensional linear subspace of $\mathbb{P}^6$ such that $T_3 \subset F_2$. Then $\mathbb{CS}(V_8, 1/n\mathcal{D}_{V_8})$ contains all irreducible components of $T_3 \cap V_8$.*

Proof. Let $H \subset \mathbb{P}^6$ be a general hyperplane, $Y \subset V_8$ a general hyperplane section passing through $T_3 \cap V_8$, set $X = Y \cap H$, let D be a sufficiently general element of $\mathcal{D}_{V_8}$, $D_X = D|_X$, and $T_2 = T_3 \cap H$. Then X is a smooth complete intersection $G_2 \cap G_4 \subset \mathbb{P}^4$, where G_2 and G_4 are a quadric and a quartic, respectively, $T_2 \subset G_2$, and

$$T_2 \cap X = C + \sum_{i \in I} C_i;$$

here $C = S \cap X$, $C_i = S_i \cap X$ for $I \subset \{1, \dots, r\}$, $(\sum_{i \in I} C_i) \cdot C_j = d_{C_j} - C \cdot C_j$ for $j \in I$ on X and $\operatorname{mult}_C(D_X) = n$, and d_{C_j} is the degree of C_j in $\mathbb{P}^6$. On the other hand one can calculate $C \cdot C_j$ on T_2 and obtain $C \cdot C_j > d_{C_j}$. Hence $(\sum_{i \in I} C_i) \cdot C_j < 0$ for $j \in I$ and the intersection form of the curves $\{C_i\}_{i \in I}$ on the surface X is negative definite, by [23]. However, the effective divisor

$$D_X - nC - \sum_{i \in I} \operatorname{mult}_{C_i}(D_X) C_i \sim \sum_{i \in I} (n - \operatorname{mult}_{C_i}(D_X)) C_i$$

is nef on the surface X, therefore $\operatorname{mult}_{S_i}(\mathcal{D}_{V_8}) = n$ for $i \in I$. Hence all irreducible components of $T_3 \cap V_8$ lie in $\mathbb{CS}(V_8, 1/n\mathcal{D}_{V_8})$.

Lemma 7.9. *The surface S does not lie in a 3-dimensional linear subspace T_3 of F_2.*

Proof. Assume that S lies in a 3-dimensional linear subspace T_3 of $\mathbb{P}^6$ such that $T_3 \subset F_2$. Then the quadric F_2 is singular and $\mathbb{CS}(V_8, 1/n\mathcal{D}_{V_8})$ contains all irreducible components of $T_3 \cap V_8$, by Lemma 7.8. Let $\pi\colon V_8 \dashrightarrow \mathbb{P}^2$ be a projection from T_3 with generic fibre that is a quartic surface in $\mathbb{P}^3$. Let $\psi\colon V \to V_8$ be a resolution of singularities of π and set $\mathcal{D}_V = \psi^{-1}(\mathcal{D}_{V_8})$. We can assume that the variety V is smooth, there exists precisely one ψ-exceptional divisor E_j over the generic point of each irreducible component S_j of the intersection $T_3 \cap V_8$, and ψ is an isomorphism outside $T_3 \cap V_8$. Then $\mathcal{D}_V$ lies in fibres of the fibration $\pi \circ \psi$ because the multiplicity of the linear system $\mathcal{D}_{V_8}$ in each irreducible component of $T_3 \cap V_8$ is equal to n. On the other hand the linear system $\mathcal{D}_{V_8}$ is a birational image of the complete linear system $|\tau^*(D_Z)|$, where D_Z is a very ample divisor on the 3-dimensional variety Z, which is a contradiction.

Hence the degree of the surface S is at least 3 and S does not lie in a 3-dimensional linear subspace T_3 of $\mathbb{P}^6$ such that $T_3 \subset F_2$.

Lemma 7.10. *The set $\mathbb{CS}(V_8, 1/n\mathcal{D}_{V_8})$ contains all irreducible components of $\Pi \cap V_8$.*

Proof. Let Y be a general hyperplane section of V_8 containing $\Pi \cap V_8$, X a general hyperplane section of Y, D a general divisor in $\mathcal{D}_{V_8}$, and set $D_X = D|_X$. Then X is a smooth complete intersection $G_2 \cap G_4 \subset \mathbb{P}^4$, where G_2 and G_4 are a quadric and a quartic, respectively, and

$$\Pi \cap X = C + \sum_{i=1}^{r} C_i$$

is reduced on X, where $C = S \cap X$ and $C_i = S_i \cap X$. Moreover, $\mathrm{mult}_C(D_X) = n$ and $\left(\sum_{i=1}^{r} C_i\right) \cdot C_j = d_{C_j} - C \cdot C_j$ on X, where d_{C_j} is the degree of C_j. On the other hand we can calculate $C \cdot C_j$ on $Q_2 = G_2 \cap \Pi$ since Q_2 can have only one singular point, in view of the generality of X and the assumption that S lies in no 3-dimensional linear subspace of F_2. Moreover, in the case when Q_2 is a quadric cone, the curve C does not pass through its vertex because C is an irreducible curve of degree at least 3. Thus Q_2 is smooth at the points in $C \cap C_j$, and $C \cdot C_j > d_{C_j}$ on Q_2. Hence $\left(\sum_{i=1}^{r} C_i\right) \cdot C_j < 0$ and the intersection form of the curves $\{C_i\}_{i=1,\dots,r}$ is negative definite by [24]. However,

$$D_X - nC - \sum_{i=1}^{r} \mathrm{mult}_{C_i}(D_X) C_i \sim \sum_{i=1}^{r} (n - \mathrm{mult}_{C_i}(D_X)) C_i$$

is nef on the surface X, therefore $\mathrm{mult}_{C_i}(D_X) = n$. Hence $\mathrm{mult}_{S_i}(\mathcal{D}_{V_8}) = n$. In particular, $\mathbb{CS}(V_8, 1/n\mathcal{D}_{V_8})$ contains all irreducible components of $\Pi \cap V_8$.

Let $\pi \colon V_8 \dashrightarrow \mathbb{P}^1$ be a projection from Π and let $\psi \colon V \to V_8$ be a resolution of the indeterminacy of π such that the variety V is smooth, there exists precisely one ψ-exceptional divisor E_j over the generic point of each component S_j of $\Pi \cap V_8$, and ψ is an isomorphism outside $\Pi \cap V_8$. Consider the general fibre D of $\pi \circ \psi$. Then

$$D \sim \psi^*(-K_{V_8}) - \sum_{j=0}^{r} E_j - \sum_{i=1}^{k} a_i F_i,$$

where $a_i \in \mathbb{N}$ and the codimension of $\psi(F_i)$ is greater than 2. On the other hand $\psi^{-1}(\mathcal{D}_{V_8})|_D \sim \sum_{i=1}^{k} c_i F_i|_D$ for $c_i \in \mathbb{Z}$, so that the linear system $\mathcal{D}_{V_8}$ lies in fibres of $\pi \circ \psi$, which is impossible because Z has dimension 3. The proof of Theorem 1.7 is thus complete.

Remark 7.11. We have proved implicitly that every birational transformation of the variety V_8 into a fibration of surfaces of Kodaira dimension 0 can be obtained by a projection from some 3-dimensional linear subspace T_3 of $\mathbb{P}^6$ such that either $T_3 \cap V_8$ is a quadric or $T_3 \subset F_2$ and the quadric F_2 is singular.

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