Izvestiya: Mathematics

Rationality of an Enriques-Fano threefold of genus five

To cite this article: I A Cheltsov 2004 *Izv. Math.* **68** 607

Izvestiya: *Mathematics* **68**:3 607–618

Izvestiya RAN: *Ser. Mat.* **68**:3 181–194 DOI 10.1070/IM2004v068n03ABEH000490

Rationality of an Enriques–Fano threefold of genus five

I. A. Cheltsov

Abstract. We prove the rationality of a non-Gorenstein Fano threefold of Fano index one and degree eight having terminal cyclic quotient singularities and Picard group $\mathbb{Z}$. This threefold can be described as the quotient of a double covering of $\mathbb{P}^3$ ramified in a smooth quartic surface by an involution fixing eight different points.

In this paper we prove the following result.

Theorem 1. *Let* $\psi\colon V \to \mathbb{P}^3$ *be a double covering ramified in a smooth quartic surface,* $\tau \in \mathrm{Aut}(V)$ *an involution fixing finitely many points and* $X = V/\tau$*. Then* X *is rational.*

All varieties are assumed to be projective, normal and defined over $\mathbb{C}$.

Remark 2. In the notation of Theorem 1, V is a smooth Fano threefold of index 2 and degree 2. We have $-K_V = \psi^*\bigl(\mathcal{O}_{\mathbb{P}^3}(2)\bigr)$ and $-K_V^3 = 16$. The involution τ fixes 8 points, and the threefold X has 8 singular points of type $\frac{1}{2}(1,1,1)$. The divisor $-K_X$ is not a Cartier divisor, but we have $-K_X \sim_{\mathbb{Q}} H$, where H is an ample Cartier divisor on X with $H^3 = 8$, and the general element of $|H|$ is a smooth Enriques surface.

Example 3. Let $V \subset \mathbb{P}(1^3, 2)$ be given by

$$u^2 = x^4 + y^4 + z^4 + t^4,$$

where x, y, z, and t are coordinates of weight 1 and u is a coordinate of weight 2. We define an involution τ on $\mathbb{P}(1^3,2)$ by $\tau(x:y:z:t:u) = (x:y:-z:-t:-u)$. Then the natural projection $\psi\colon V \to \mathbb{P}^3$ is a double covering ramified in a smooth quartic surface, the threefold V is τ-invariant and $\tau|_V$ fixes 8 points of V given by $u = z = t = 0$ and $u = x = y = 0$.

Fano [1]–[4] tried to obtain a biregular classification of threefolds whose curve sections are canonical curves. In the smooth case, hyperplane sections of such threefolds are smooth K3 surfaces. Thus there is a natural generalization of the threefolds studied by Fano [1]–[4] to threefolds containing an ample Cartier divisor which is a smooth K3 surface. In the smooth case, the anticanonical divisor of such a threefold is necessarily ample, and it was proved in [5] that the general element of the anticanonical linear system of a smooth threefold with ample

AMS 2000 *Mathematics Subject Classification.* 14J30, 14E08, 14C20.

anticanonical divisor is a smooth K3 surface. In particular, smooth threefolds containing a smooth K3 surface as an ample Cartier divisor are Fano varieties in the sense of their modern definition, which was given by Iskovskikh. A complete biregular classification of smooth Fano threefolds was obtained in [5]–[10], where 105 families of smooth Fano threefolds are listed.

Somewhat later, Fano [11] studied threefolds whose hyperplane sections are smooth Enriques surfaces.[1] He gave a list of six such threefolds, including two exceptional (ample, but not very ample) cases. Fano claimed in [11] that this list exhausts all threefolds containing a smooth Enriques surface as a hyperplane section. However, the proofs in [11] are incorrect and the classification is not complete, as was shown in [18], [19]. It was also remarked in [18], [19] that every threefold whose hyperplane sections are smooth Enriques surfaces must be singular, which seems to be the main reason why these threefolds fell into oblivion for almost half of the century. Fano [11] claimed that these threefolds have exactly eight quadruple singular points, whose tangent cone is a cone over the Veronese surface. In modern language, such singular points are terminal cyclic quotient singularities of type $\frac{1}{2}(1,1,1)$ which are locally isomorphic to the quotient of $\mathbb{C}^3$ by an involution that fixes the origin only.

Definition 4. A threefold X is called an *Enriques–Fano threefold* of genus $g(X) = -\frac{1}{2}K_X^3+1$ and degree $-K_X^3$ if X has canonical singularities, $-K_X$ is not a Cartier divisor and $-K_X \sim_{\mathbb{Q}} H$ for some ample Cartier divisor H on X.

The following result was proved in [20].

Theorem 5. *Suppose that X is an Enriques–Fano threefold and $-K_X \sim_{\mathbb{Q}} H$, where H is an ample Cartier divisor on X. Then a generic surface in the linear system $|H|$ is an Enriques surface with canonical singularities. It is smooth if the singularities of X are isolated and $-K_X^3 \neq 2$.*

Corollary 6. *The degree and genus of an Enriques–Fano threefold are positive integers.*

The following result was proved in [16], [17].

Theorem 7. *Let X be a normal threefold and $H \subset X$ an Enriques surface with canonical singularities such that H is an ample Cartier divisor on X. Then $-2K_X \sim 2H$ and X is either an Enriques–Fano threefold or a contraction of a section of* $\operatorname{Proj}\big(\mathcal{O}_H \oplus \mathcal{O}_H(H|_H)\big)$.

Thus, except for generalized cones, the Enriques–Fano threefolds are exactly those threefolds which contain an Enriques surface with Du Val singularities as an ample Cartier divisor.

Remark 8. Let X be an Enriques–Fano threefold and let H be an ample Cartier divisor on X such that $-K_X \sim_{\mathbb{Q}} H$. Then $2(H+K_X) \sim 0$ and the threefold $V = \operatorname{Spec}\big(\mathcal{O}_X \oplus \mathcal{O}_X(H+K_X)\big)$ is called the *canonical covering* of X. One can show that V is a Fano threefold with Gorenstein canonical singularities and there

[1]The analogous problem can be considered for any surface of Kodaira dimension zero. But in the abelian and bi-elliptic cases, the corresponding problem has only the trivial solution, even in the ample case (see [12]–[17]).

is a natural double covering $\pi\colon V \to X$ ramified at finitely many points, which are exactly the non-Gorenstein points of X. We have $-K_V \sim \pi^*(H)$ and the double covering π induces an etale double covering of every Enriques surface with canonical singularities in the linear system $|H|$. This covering coincides with the natural etale covering of the Enriques surface by a K3 surface.

Enriques–Fano threefolds are singular and non-Gorenstein by definition. Therefore Enriques–Fano threefolds with terminal cyclic quotient singularities are non-Gorenstein analogues of smooth Fano threefolds. On the other hand, the canonical coverings of Enriques–Fano threefolds with terminal cyclic quotient singularities are smooth. In particular, all the singular points of an Enriques–Fano threefold with terminal cyclic quotient singularities are of type $\frac{1}{2}(1,1,1)$. It was observed in [21] that the Riemann–Roch formula for non-smooth varieties (see [22]) then yields that the number of such singular points is equal to eight, as claimed by Fano. Moreover, the classification of smooth Fano threefolds was used in [23], [24] to obtain the following result.

Theorem 9. *Let X be an Enriques–Fano threefold with only terminal cyclic quotient singularities and let V be the canonical covering of X. Then V is one of the following smooth Fano threefolds.*

1) *The complete intersection of a quadric and quartic in* $\mathbb{P}(1^5,2)$, $g(X)=2$.
2) *The complete intersection of three quadrics in* $\mathbb{P}^6$, $g(X)=3$.
3) *The blow-up of a smooth hypersurface of degree* 4 *in* $\mathbb{P}(1^4,2)$ *along an elliptic curve cut out by two hypersurfaces of degree one*, $g(X)=3$.
4) $\mathbb{P}^1\times S_2$, *where S_2 is a smooth del Pezzo surface of degree* 2, $g(X)=4$.
5) *The double covering of* $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$ *ramified in a divisor of degree* $(2,2,2)$, $g(X)=4$.
6) *The blow-up of a smooth complete intersection of two quadrics in* $\mathbb{P}^5$ *along an elliptic curve cut out by two hyperplane sections*, $g(X)=5$.
7) *The hypersurface of degree* 4 *in* $\mathbb{P}(1^4,2)$, $g(X)=5$.
8) *The complete intersection of three divisors of degree* $(1,1)$ *in* $\mathbb{P}^3\times\mathbb{P}^3$, $g(X)=6$.
9) $\mathbb{P}^1\times S_4$, *where S_4 is a smooth del Pezzo surface of degree* 4, $g(X)=7$.
10) *The divisor of degree* $(1,1,1,1)$ *in* $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$, $g(X)=7$.
11) *The blow-up of the cone over a smooth quartic surface* $\mathbb{P}^1\times\mathbb{P}^1\subset\mathbb{P}^3$ *along the disjoint union of the vertex and a smooth elliptic curve on* $\mathbb{P}^1\times\mathbb{P}^1$, $g(X)=8$.
12) *The compete intersection of two quadrics in* $\mathbb{P}^5$, $g(X)=9$.
13) $\mathbb{P}^1\times S_6$, *where S_6 is a smooth del Pezzo surface of degree* 6, $g(X)=10$.
14) $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$, $g(X)=13$.

The following result was proved in [25].

Theorem 10. *Let X be a terminal Enriques–Fano threefold. Then there is a small deformation $g\colon U\to\Delta$ over the one-dimensional unit disc Δ with origin O such that $X\cong g^{-1}(O)$ and, for each $s\in\Delta\backslash O$, the fibre of g over s is an Enriques–Fano threefold with terminal cyclic quotient singularities.*

Thus terminal Enriques–Fano threefolds are deformations of Enriques–Fano threefolds, as classified in Theorem 9. However the classification of Enriques–Fano threefolds with arbitrary canonical singularities is far from being complete. The following result was proved in [26].

Theorem 11. *Let X be an Enriques–Fano threefold. Then $-K_X^3 \leqslant 92$ and $g(X) \leqslant 47$.*

The bounds in Theorem 11 seem far from perfect. The expected bound for the genus of an arbitrary Enriques–Fano threefold is 13. The following result was announced by Giraldo, Knutsen, Lopez and Mϋnoz.

Theorem 12. *Suppose that X is an Enriques–Fano threefold with isolated singularities and $-K_X \sim_{\mathbb{Q}} H$ for some very ample Cartier divisor H on X. Then $g(X) \leqslant 26$.*

By definition, Enriques–Fano threefolds are Fano threefolds with canonical singularities having integer Fano index. However, the Fano index of an Enriques–Fano threefold need not apriori be equal to 1. The following result was proved in [27].

Theorem 13. *Suppose that $X \subset \mathbb{P}^n$ is an Enriques–Fano threefold with isolated singularities and such that $-K_X \sim_{\mathbb{Q}} H$ for some hyperplane section H of X which is a smooth Enriques surface. Then $H \subset \mathbb{P}^{n-1}$ cannot be the image of the rth Veronese embedding of a linearly normal smooth Enriques surface $S \subset \mathbb{P}^k$ for $r \geqslant 2$.*

In fact, the following result holds.

Proposition 14. *Suppose that X is an Enriques–Fano threefold and there is a Cartier divisor H such that $-K_X \sim_{\mathbb{Q}} nH$ for some $n \in \mathbb{N}$. Then $n = 1$.*

Proof. Assume that $n \neq 1$. Let $D = K_X + nH$ be a Weil divisor on X and let S be a general element of the linear system $|H|$. Then the singularities of S are canonical by [28] and the singularities of X near S are canonical and Gorenstein by [29]. Moreover, S is a del Pezzo surface by the adjunction formula. In particular, we have the rational equivalence $D|_S \sim 0$ because $D|_S$ is a Cartier divisor and $D|_S \sim_{\mathbb{Q}} 0$ and the Picard group of S has no torsion.

Consider a general surface Y in the linear system $|kH|$ for $k \gg 0$ and put $C = Y \cap S$. Then

$$D|_Y \cdot H|_Y = D|_S \cdot kH|_S = D \cdot C = 0.$$

It follows that $D|_Y \equiv 0$ and $D|_C \sim 0$ by the Hodge index theorem because $kH|_Y$ is an ample Cartier divisor on Y and $D|_Y \cdot D|_Y \not< 0$. Consider the exact sequence

$$0 \to H^0\big(\mathcal{O}_Y(D|_Y)\big) \to H^0(\mathcal{O}_C) \to H^1\big(\mathcal{O}_Y\big((D-H)|_Y\big)\big),$$

where the Cartier divisor $(H-D)|_Y$ is ample on Y. The Kawamata–Vieweg vanishing theorem implies that $H^1\big(\mathcal{O}_Y\big((D-H)|_Y\big)\big) = 0$. We also have $H^0(\mathcal{O}_C) = \mathbb{C}$ because C is an ample divisor on Y. Thus, $H^0\big(\mathcal{O}_Y(D|_Y)\big) = \mathbb{C}$, which implies that $D|_Y \sim 0$. The sequence of sheaves

$$0 \to \mathcal{O}_X(D) \otimes \mathcal{O}_X(-kH) \to \mathcal{O}_X(D) \to \mathcal{O}_Y \to 0$$

is exact because the sheaf $\mathcal{O}_X(D)$ is locally trivial in the neighbourhood of Y while the sequence is trivial outside Y. Hence the sequence of cohomology groups

$$0 = H^0\big(\mathcal{O}_X(D) \otimes \mathcal{O}_X(-kH)\big) \to H^0\big(\mathcal{O}_X(D)\big) \to H^0(\mathcal{O}_Y)$$
$$\to H^1\big(\mathcal{O}_X(D) \otimes \mathcal{O}_X(-kH)\big)$$

is exact. On the other hand, the sheaf $\mathcal{O}_X(D)$ is reflexive by [30]. Hence there is an exact sequence of sheaves

$$0 \to \mathcal{O}_X(D) \to \mathcal{E} \to \mathcal{F} \to 0,$$

where $\mathcal{E}$ is locally free and $\mathcal{F}$ is torsionfree. Hence the sequence

$$H^0\big(\mathcal{F} \otimes \mathcal{O}_X(-kH)\big) \to H^1\big(\mathcal{O}_X(D) \otimes \mathcal{O}_X(-kH)\big) \to H^1\big(\mathcal{E} \otimes \mathcal{O}_X(-kH)\big)$$

is exact. Here the group $H^0\big(\mathcal{F} \otimes \mathcal{O}_X(-kH)\big)$ vanishes because $\mathcal{F}$ is torsionfree, and the group $H^1\big(\mathcal{E} \otimes \mathcal{O}_X(-kH)\big)$ vanishes because X is normal. Therefore the group $H^1\big(\mathcal{O}_X(D) \otimes \mathcal{O}_X(-kH)\big)$ also vanishes, whence $H^0\big(\mathcal{O}_X(D)\big) = \mathbb{C}$. In particular, the canonical divisor K_X is a Cartier divisor, contrary to the definition of X.

The interest in Fano threefolds was originally inspired by the Lüroth problem. Fano asserted the non-rationality of a smooth quartic threefold [31] and a smooth cubic threefold [32]. On the other hand, the unirationality of a smooth cubic threefold was well known at this time. The unirationality of some smooth quartic threefolds was established later in [33] (see [34], [35]), but perhaps Fano knew some examples of unirational smooth quartic threefolds. Rigorous proofs of these assertions of Fano were obtained only in [36], [37]. Nevertheless, the question of the rationality of smooth Fano threefolds is now almost completely settled (see [35]–[40]). The original interest in Enriques–Fano threefolds was inspired by the following result, which was conjectured long ago by Enriques and Fano but proved only in [41] and [42] (independently).

Theorem 15. *Let $X \subset \mathbb{P}^4$ be the so-called Enriques threefold given as the non-smooth hypersurface*

$$x_1x_2x_3x_4\bigg(x_0^2 + x_0\sum_{i=1}^{4} a_ix_i + \sum_{i,j=1}^{4} b_{ij}x_ix_j\bigg) + c_1x_2^2x_3^2x_4^2 + c_2x_1^2x_3^2x_4^2 + c_3x_1^2x_2^2x_4^2 + c_4x_1^2x_2^2x_3^2 = 0$$

of degree six, where the x_i are homogeneous coordinates on $\mathbb{P}^4$, and a_i, b_{ij}, c_i are sufficiently general complex numbers. Then the threefold V is non-rational.

Fano thought that the Enriques threefold is not unirational, but Roth [43] proved that it is and claimed that it is non-rational because the fundamental group of its desingularization contains $\mathbb{Z}_2$. This argument led to doubts because Serre [44] proved that smooth unirational varieties are simply connected. The apparent gap in [43] was explained in [45] by showing that the Enriques threefold always has non-ordinary singular points. In modern language, these points are the images of the eight non-Gorenstein singular points of its normalization.

Remark 16. The normalization of the Enriques threefold is an Enriques–Fano threefold of degree 6 and genus 4 having eight singular points of type $\frac{1}{2}(1,1,1)$. Its canonical covering is a double covering of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ ramified in a divisor of degree $(2,2,2)$.

Thus some Enriques–Fano threefolds are non-rational. The following result was proved in [46].

Theorem 17. *Let X be an Enriques–Fano threefold with $-K_X^2 \geqslant 10$. Then X is rational.*

It is well known and intuitively clear that Fano varieties tend to be "more rational" as their degrees increase and their singularities become worse. For instance, the classification of smooth Fano threefolds implies that a smooth Fano threefold is rational once the degree is greater than 24. Therefore Theorem 17 is very natural, and Theorem 1 fills the gap between Theorems 15 and 17.

Corollary 18. *Let X be a non-rational Enriques–Fano threefold with terminal cyclic quotient singularities and let V be the canonical covering of X. Then V is either the complete intersection of three quadrics, the complete intersection of a quadric and a quartic in $\mathbb{P}(1^5, 2)$ or the double covering of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ ramified in a divisor of degree $(2, 2, 2)$.*

Conjecture 19. *Suppose that X is an Enriques–Fano threefold with terminal cyclic quotient singularities and the canonical covering of X is either the complete intersection of a quadric and a quartic in $\mathbb{P}(1^5, 2)$ or the complete intersection of three quadrics. Then X is non-rational.*

In the rest of the paper we prove Theorem 1. Let $\psi\colon V \to \mathbb{P}^3$ be a double covering ramified in a smooth quartic surface $S \subset \mathbb{P}^3$, τ an involution of V that fixes eight points and $X = V/\tau$. Then X is an Enriques–Fano threefold of degree 8 and genus 5.

Lemma 20. *There are homogeneous coordinates $(x_0 : x_1 : x_2 : x_3)$ on $\mathbb{P}^3$ such that the induced action of τ on $\mathbb{P}^3$ is given by*

$$\tau(x_0 : x_1 : x_2 : x_3) = (x_0 : x_1 : -x_2 : -x_3)$$

and the surface $S \subset \mathbb{P}^3$ is given by

$$q_4(x_0, x_1) + a_2(x_0, x_1)x_2^2 + b_2(x_0, x_1)x_2x_3 + c_2(x_0, x_1)x_3^2 + p_4(x_2, x_3) = 0,$$

where q_4 and p_4 are polynomials of degree 4 and a_2, b_2, c_2 are polynomials of degree 2.

Proof. This is proved in [23], § 6.1.2.

Thus V may be regarded as a hypersurface of degree 4 in $\mathbb{P}(1^4, 2)$ given by

$$y^2 = q_4(x_0, x_1) + a_2(x_0, x_1)x_2^2 + b_2(x_0, x_1)x_2x_3 + c_2(x_0, x_1)x_3^2 + p_4(x_2, x_3),$$

where the x_i are homogeneous coordinates of weight 1, y is the homogeneous coordinate of weight 2 and τ acts on $\mathbb{P}(1^4, 2)$ by $\tau(x_0 : x_1 : x_2 : x_3 : y) = (x_0 : x_1 : -x_2 : -x_3 : -y)$. The eight fixed points of $\tau\colon V \to V$ are given by $x_0 = x_1 = y = 0$ and $x_2 = x_3 = y = 0$. Furthermore, q_4 and p_4 are homogeneous polynomials of degree 4 and a_2, b_2, c_2 are homogeneous polynomials of degree 2.

Remark 21. The linear system $|-K_V|$ is generated by

$$\lambda y + \sum_{i,j} a_{ij}x_ix_j, \qquad \lambda, a_{ij} \in \mathbb{C}.$$

Let $\Lambda \subset |-K_V|$ be the linear subsystem given by the forms

$$b_{00}x_0^2 + b_{01}x_0x_1 + b_{11}x_1^2 + b_{22}x_2^2 + b_{23}x_2x_3 + b_{33}x_3^2$$

and let $\Sigma \subset |-K_V|$ be the linear subsystem given by the linear combinations

$$\mu y + c_{02}x_0x_2 + c_{03}x_0x_3 + c_{12}x_1x_2 + c_{13}x_1x_3,$$

where $\mu, c_{ij}, b_{ij} \in \mathbb{C}$. The linear system Λ is free and the base locus of Σ consists of the eight fixed points of τ. The surfaces in Λ and Σ are τ-invariant. The general element S_Λ of Λ is a smooth K3 surface and τ acts on S_Λ without fixed points. The general element S_Σ of Σ is a smooth K3 surface and the restriction of τ to S_Σ fixes precisely the eight fixed points of τ on V. Let $H_\Lambda \subset X$ be the quotient of S_Λ by τ and let $H_\Sigma \subset X$ be the quotient S_Σ/τ. Then H_Λ is a non-singular Enriques surface and H_Σ is a K3 surface with eight simple double points. Moreover, H_Λ is an ample Cartier divisor on X, H_Σ is a Weil divisor on X, $2H_\Sigma$ is a Cartier divisor and $2H_\Lambda \sim 2H_\Sigma$. The surface H_Σ is a general element of the anticanonical linear system of Weil divisors $|-K_X|$ and the surface H_Λ is a general element of the linear system $|H_\Lambda|$.

Remark 22. By construction, the linear system $|H_\Lambda|$ is free and determines a double covering $\varphi_{|H_\Lambda|}\colon X \to \mathbb{P}^5$ whose image $\varphi_{|H_\Lambda|}(X) \subset \mathbb{P}^5$ is the complete intersection of two non-smooth quartics $B^2 = AC$ and $E^2 = DF$ in the corresponding homogeneous coordinates $(A : B : C : D : E : F)$ on $\mathbb{P}^5$. The linear system $|H_\Sigma|$ determines a rational map $\varphi_{|H_\Sigma|}\colon X \dashrightarrow \mathbb{P}^4$ whose image is a quadric with one singular point. The degree of the map $\varphi_{|H_\Sigma|}$ at a generic point of X is equal to 2 and the indeterminacy points of $\varphi_{|H_\Sigma|}$ are resolved by blowing up all the singular points of X.

Let Ξ be a pencil on V given by the linear forms $\alpha x_0 + \beta x_1$ and let Ω be a pencil on V given by the linear forms $\gamma x_2 + \delta x_3$, where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. The base locus of Ξ consists of the smooth elliptic curve $C_\Xi \subset V$ given by $x_0 = x_1 = 0$. The base locus of Ω consists of the smooth elliptic curve $C_\Omega \subset V$ given by $x_2 = x_3 = 0$. The surfaces in Ξ and Ω are τ-invariant. The general surface $S_\Xi \in \Xi$ and the general surface $S_\Omega \in \Omega$ are smooth del Pezzo surfaces of degree 2. The involution τ fixes four points on S_Ξ (given by $y = x_0 = x_1 = 0$) and four points on S_Ω (given by $y = x_2 = x_3 = 0$).

Consider the quotient surfaces $H_\Xi = S_\Xi/\tau \subset X$ and $H_\Omega = S_\Omega/\tau \subset X$. By construction, H_Ξ and H_Ω are del Pezzo surfaces of degree 1 and each of them has four simple double points. The surfaces H_Ξ and H_Ω are not Cartier divisors on X, but $2H_\Xi$ and $2H_\Omega$ are Cartier divisors, and we have $2H_\Xi \sim 2H_\Omega \sim H_\Lambda$. The surfaces H_Ξ and H_Ω are general surfaces in the linear systems of Weil divisors $|H_\Xi|$ and $|H_\Omega|$ respectively.

Lemma 23. *Let E_Ξ and E_Ω be the images of the smooth elliptic curves C_Ξ and C_Ω (respectively) on X. Then E_Ξ and E_Ω are smooth rational curves. Moreover, E_Ω intersects H_Ξ transversally at one smooth point O, and O is the unique base point of the linear system $|-K_{H_\Xi}|$ which is cut out on H_Ξ by the linear system $|H_\Omega|$.*

Proof. All these assertions follow from the explicit construction of V and X above.

Lemma 24. *Let $P_i \in X$ $(i = 1, 2, 3, 4)$ be the four singular points of X that are images of the fixed points of τ given by $y = x_0 = x_2$. Let H^i_Ω be the unique surface in $|H_\Omega|$ that contains P_i. Then $H_\Xi \cap H^i_\Omega \in |-K_{H_\Xi}|$ is a curve of arithmetic genus one having one double point P_i.*

Proof. This follows from Lemma 23.

Let $g\colon Y \to X$ be the composite of the blow-ups of the points P_i followed by the blow-up of the proper transform of the smooth rational curve E_Ξ. We denote the exceptional divisors of the birational morphism g by F_i and E in such a way that $g(F_i) = P_i$ and $g(E) = E_\Xi$. Let $H \subset Y$ be the proper transform of the irreducible surface H_Ξ, Θ the proper transform on Y of the pencil $|H_\Xi|$ and $H_i \subset Y$ the proper transform of the surface H^i_Ω. Then H is a smooth del Pezzo surface of degree one. The linear system Θ is a free pencil, and it determines a morphism $\varphi_\Theta\colon Y \to \mathbb{P}^1$. The surface H is a general fibre of φ_Θ. The curves $E \cap H$, $F_i \cap H$, $H_i \cap H$ are irreducible, smooth and rational. Moreover, we have

$$F_i \cdot F_i \cdot H = -2, \qquad E \cdot F_i \cdot H = 1, \qquad E \cdot E \cdot H = -1,$$
$$H_i \cdot H_i \cdot H = -1, \qquad E \cdot H_i \cdot H = 0, \qquad F_i \cdot H_i \cdot H = 2$$

and $F_i \cdot F_j \cdot H = 0$ for $i \neq j$. Thus the threefold Y may be regarded as a smooth del Pezzo surface of degree 1 defined over the field $\mathbb{C}(x)$ of rational functions on $\mathbb{P}^1$ and the surfaces H_i, F_i, and E may be regarded as curves on Y, also defined over $\mathbb{C}(x)$. Then we have

$$F_i \cdot F_i = -2, \qquad E \cdot F_i = 1, \qquad E \cdot E = -1,$$
$$H_i \cdot H_i = -1, \qquad E \cdot H_i = 0, \qquad F_i \cdot H_i = 2$$

and $F_i \cdot F_j = 0$ if $i \neq j$. Moreover, $H_i \cdot H_j = 1$ for $i \neq j$ and all the curves H_i intersect each other at the base point of the linear system $|-K_Y|$. We have $-K_Y \sim H_i + F_i$ and $K_Y^2 = 1$.

Proposition 25. *We have* $\operatorname{rank}\bigl(\operatorname{Pic}(Y)\bigr) \geqslant 5$.

Proof. This follows from the equations $-K_Y \cdot F_i = 0$, $F_i \cdot F_i = -2$ and $F_i \cdot F_j = 0$ for $i \neq j$.

The birational theory of geometrically rational surfaces defined over a field that is not algebraically closed is studied in [47]–[56], where the following three results are proved.

Theorem 26. *Let W be a smooth, minimal, geometrically irreducible and geometrically rational surface defined over a perfect field $\mathbb{F}$. Then either W is a del Pezzo surface and* $\operatorname{Pic}(W) \cong \mathbb{Z}$ *or there is a conic bundle $\pi\colon W \to Z$ and* $\operatorname{Pic}(W) \cong \mathbb{Z} \otimes \mathbb{Z}$.

Theorem 27. *Let W be a smooth, minimal, geometrically irreducible and geometrically rational surface defined over a perfect field $\mathbb{F}$. The surface W is rational over $\mathbb{F}$ if and only if W has an $\mathbb{F}$-point and $K_W^2 \geqslant 5$.*

Theorem 28. *Let W be a smooth, geometrically irreducible and geometrically rational surface defined over a C_1-field $\mathbb{F}$. Then W has an $\mathbb{F}$-point.*

Thus the smooth del Pezzo surface Y has a $\mathbb{C}(x)$-point. The set of all $\mathbb{C}(x)$-points of Y is actually very large (see [57]). Proposition 25 implies that Y is not minimal and there is a birational morphism $f\colon Y \to U$ (defined over $\mathbb{C}(x)$) such that the surface U is smooth, minimal, geometrically irreducible and geometrically rational, and also has a $\mathbb{C}(x)$-point.

Remark 29. We have

$$K_U^2 \geqslant K_Y^2 + \operatorname{rank}\bigl(\operatorname{Pic}(Y)\bigr) - \operatorname{rank}\bigl(\operatorname{Pic}(U)\bigr) \geqslant \operatorname{rank}\bigl(\operatorname{Pic}(Y)\bigr) - 1 \geqslant 4.$$

We may assume that $\operatorname{Pic}(Y) \cong \mathbb{Z}^5$ since otherwise the surface U (and hence the threefold X) is rational by Theorem 27. In particular, the group $\operatorname{Pic}(Y) \otimes \mathbb{Q}$ is generated by the divisors F_i and $-K_Y$. Over the algebraic closure $\overline{\mathbb{C}(x)}$ of the field $\mathbb{C}(x)$, the morphism f is a composite of contractions of smooth rational curves with self-intersection -1, and this yields the inequality $K_U^2 \geqslant 5$ in the case when f contracts at least four curves over $\overline{\mathbb{C}(x)}$. On the other hand, f contracts at least three curves over $\overline{\mathbb{C}(x)}$. Hence we may assume that f contracts exactly three curves over $\overline{\mathbb{C}(x)}$, $K_U^2 = 4$ and $\operatorname{Pic}(U) = \mathbb{Z} \otimes \mathbb{Z}$. It follows that each of the curves contracted by f over $\overline{\mathbb{C}(x)}$ is $\operatorname{Gal}\bigl(\overline{\mathbb{C}(x)}/\mathbb{C}(x)\bigr)$-invariant.

Corollary 30. *The morphism f is a composite $f_1 \circ f_2 \circ f_2$, where f_i is the contraction of a geometrically irreducible smooth rational curve defined over $\mathbb{C}(x)$ with self-intersection -1.*

Consider a smooth rational curve $C \subset Y$ defined over $\mathbb{C}(x)$ and such that $C^2 = -1$. Then

$$C \sim_{\mathbb{Q}} -\varepsilon K_Y - \sum_{i=1}^{4} k_i F_i,$$

where ε and the k_i are rational numbers. Moreover, we have

$$C \cdot F_i \in \mathbb{Z}_{\geqslant 0} \Rightarrow k_i \in \frac{1}{2}\mathbb{Z}_{\geqslant 0}, \qquad -K_Y \cdot C = 1 \Rightarrow \varepsilon = 1, \qquad C^2 = -1 \Rightarrow \sum_{i=1}^{4} k_i^2 = 1.$$

It follows that either $C \sim_{\mathbb{Q}} -K_X - F_i \sim H_i$ or

$$C \sim_{\mathbb{Q}} -K_X - \sum_{i=1}^{4} \frac{1}{2} F_i \sim_{\mathbb{Q}} E.$$

Thus the irreducible curve C coincides with either H_i or E. Hence all the geometrically irreducible curves with negative self-intersection on Y are F_i, H_i and E. However, the intersection form for any three of the nine curves F_i, H_i, E is not negative definite, and this contradicts Corollary 30. This proves Theorem 1.

The author would like to thank V. Alexeev, A. Conte, A. Corti, M. M. Grinenko, V. A. Iskovskikh, A. Lopez, M. Marchisio, M. Mella, J. Park, Yu. G. Prokhorov, A. V. Pukhlikov, V. V. Shokurov, R. Varley, A. Verra and L. Wotzlaw for some useful conversations.

Bibliography

[1] G. Fano, "Sulle varietá algebriche a tre dimensioni aventi tutti i generi nulli", *Atti Congr. Int. Bologna* **IV** (1929), 115–121.

[2] G. Fano, "Su alcune varietá algebriche a tre dimensioni a curve sezioni canoniche", *Scritti Mat. offerti a L. Berzolari Ist. Mat. R. Univ. Pavia* (1936), 329–349.

[3] G. Fano, "Sulle varietá algebriche a tre dimensioni a curve sezioni canoniche", *Mem. Accad. d'Italiana* **VIII** (1937), 23–64.

[4] G. Fano, "Su alcune varietá algebriche a tre dimensioni razionali, e aventi curve-sezioni canoniche", *Comment. Math. Helv.* **14** (1942), 202–211.

[5] V. V. Shokurov, "Smoothness of the general anticanonical divisor on a Fano variety", *Izv. Akad. Nauk SSSR Ser. Mat.* **43** (1979), 430–441; English transl., *Math. USSR-Izv.* **14** (1980), 395–405.

[6] V. A. Iskovskikh, "Fano 3-folds. I", *Izv. Akad. Nauk SSSR Ser. Mat.* **41** (1977), 516–562; English transl., *Math. USSR-Izv.* **11** (1977), 485–527.

[7] V. A. Iskovskikh, "Fano 3-folds. II", *Izv. Akad. Nauk SSSR Ser. Mat.* **42** (1978), 504–549; English transl., *Math. USSR-Izv.* **12** (1978), 469–506.

[8] V. V. Shokurov, "Existence of a straight line on a Fano 3-fold", *Izv. Akad. Nauk SSSR Ser. Mat.* **43** (1979), 922–964; English transl., *Math. USSR-Izv.* **15** (1980), 173–209.

[9] S. Mori and S. Mukai, "Classification of Fano 3-folds with $B_2 \geqslant 2$", *Manuscripta Math.* **36** (1981), 147–162.

[10] S. Mori and S. Mukai, "Erratum, Classification of Fano 3-folds with $B_2 \geqslant 2$", *Manuscripta Math.* **110** (2003), 407.

[11] G. Fano, "Sulle varietá algebriche a tre dimensione le cui sezioni iperpiane sono superfici di genere zero e bigenere uno", *Mem. Soc. Ital. d. Scienze* (*detta dei XL*) **24** (1938), 44–66.

[12] T. Fujita, "Impossibility criterion of being an ample divisor", *J. Math. Soc. Japan* **34** (1982), 355–363.

[13] M. Beltrametti and A. Sommese, "A criterion for a variety to be a cone", *Comment. Math. Helv.* **62** (1987), 417–422.

[14] S. Lvovsky, "Extensions of projective varieties and deformations. I", *Michigan Math. J.* **39** (1992), 41–51.

[15] S. Lvovsky, "Extensions of projective varieties and deformations. II", *Michigan Math. J.* **39** (1992), 65–70.

[16] I. A. Cheltsov, "Singularities of three-dimensional varieties admitting an ample effective divisor of Kodaira dimension zero", *Mat. Zametki* **59** (1996), 618–626; English transl., *Math. Notes* **59** (1996), 445–450.

[17] I. A. Cheltsov, "Three-dimensional algebraic varieties having a divisor with numerically trivial canonical class", *Uspekhi Mat. Nauk* **51**:1 (1996), 177–178; English transl., *Russian Math. Surveys* **51** (1996), 140–141.

[18] A. Conte, "On threefolds whose hyperplane sections are Enriques surfaces", *Lecture Notes in Math.* **947** (1981), 221–228.

[19] A. Conte and J. P. Murre, "Algebraic varieties of dimension 3 whose hyperplane sections are Enriques surfaces", *Ann. Scuola Norm. Sup. Pisa* **12** (1985), 43–80.

[20] Yu. G. Prokhorov, "On algebraic threefolds whose hyperplane sections are Enriques surfaces", *Mat. Sb.* **186**:9 (1995), 113–124; English transl., *Sb. Math.* **186** (1995), 1341–1352.

[21] V. Alexeev, "General elephants of $\mathbb{Q}$-Fano 3-folds", *Compositio Math.* **91** (1994), 91–116.

[22] A. Fletcher, "Contributions to Riemann-Roch on projective 3-folds with only canonical singularities and applications", *Proc. Symp. Pure Math.* **46** (1987), 221–231.

[23] L. Bayle, "Classification des variétés complexes projectives de dimension trois dont une section hyperplane générale est une surface d'Enriques", *J. Reine Angew. Math.* **449** (1994), 9–63.

[24] T. Sano, "On classifications of non-Gorenstein $\mathbb{Q}$-Fano 3-folds of Fano index 1", *J. Math. Soc. Japan* **47** (1995), 369–380.

[25] T. Minagawa, "Deformations of $\mathbb{Q}$-Calabi-Yau 3-folds and $\mathbb{Q}$-Fano 3-folds of Fano index 1", *J. Math. Sci. Univ. Tokyo* **6** (1999), 397–414.

[26] I. A. Cheltsov, "Bounded three-dimensional Fano varieties of integer index", *Mat. Zametki* **66** (1999), 445–451; English transl., *Math. Notes* **66** (1999), 360–365.

[27] L. Giraldo, A. Lopez, and R. Mũnoz, "On the existence of Enriques–Fano threefolds of index greater than one", *J. Algebraic Geom.* **13** (2004), 143–166.
[28] V. Alexeev, "Theorems about good divisors on log Fano varieties (case of index $r > n - 2$)", *Lecture Notes in Math.* **1479** (1991), 1–9.
[29] Y. Kawamata, K. Matsuda, and K. Matsuki, "Introduction to the minimal model problem", *Adv. Studies Pure Math.* **10** (1987), 283–360.
[30] R. Hartshorne, "Stable reflexive sheaves", *Math. Ann.* **254** (1980), 121–176.
[31] G. Fano, "Osservazioni sopra alcune varietá non razionali aventi tutti i generi nulli", *Atti Accad. Torino* **50** (1915), 1067–1072.
[32] G. Fano, "Nuove ricerche sulle varietá algebriche a tre dimensioni a curve-sezioni canoniche", *Pont. Acad. Sci. Comment.* **11** (1947), 635–720.
[33] B. Segre, "Variazione continua ed omotopia in geometria algebrica", *Ann. Mat. Pura Appl.* **50** (1960), 149–186.
[34] M. Marchisio, "Unirational quartic hypersurfaces", *Boll. Unione Mat. Ital.* **8** (2000), 301–313.
[35] V. A. Iskovskikh, "Birational automorphisms of three-dimensional algebraic varieties", Itogi Nauki Tekh. Sovrem. Probl. Mat., vol. 12, VINITI, Moscow 1979, pp. 159–235; English transl., *J. Soviet Math.* **13** (1980), 815–867.
[36] V. A. Iskovskikh and Yu. I. Manin, "Three-dimensional quartics and counterexamples to the Lüroth problem", *Mat. Sb.* **86** (1971), 140–166; English transl., *Math. USSR-Sb.* **15** (1971), 141–166.
[37] H. Clemens and P. Griffiths, "The intermediate Jacobian of the cubic threefold", *Ann. Math.* **95** (1972), 281–356.
[38] A. N. Tyurin, "Intermediate Jacobian of three-dimensional varieties", Itogi Nauki Tekhn. Sovrem. Probl. Mat., vol. 12, VINITI, Moscow 1978, p. 5–57; English transl., *J. Soviet Math.* **13** (1980), 707–745.
[39] V. A. Iskovskikh, "On the rationality problem for three-dimensional algebraic varieties", *Trudy Mat. Inst. Steklov.* **218** (1997), 190–232; English transl., *Proc. Steklov Math. Inst.* **218** (1997), 186–227.
[40] V. A. Iskovskikh and Yu. G. Prokhorov, *Fano varieties*, Springer–Verlag, New York 1999.
[41] L. P. Botta and A. Verra, "The non-rationality of the generic Enriques threefold", *Compositio Math.* **48** (1983), 167–184.
[42] S. Yu. Endryushka, "Non-rationality of the general Enriques variety", *Mat. Sb.* **123** (1984), 269–275; English transl., *Math. USSR-Sb.* **51** (1985), 267–273.
[43] L. Roth, *Algebraic threefolds with special regard to problems of rationality*, Springer-Verlag, Berlin 1955.
[44] J. P. Serre, "On the fundamental group of a unirational variety", *J. London Math. Soc.* **34** (1959), 481–484.
[45] J. A. Tyrrell, "The Enriques threefold", *Proc. Cambridge Philos. Soc.* **57** (1961), 897–898.
[46] I. A. Cheltsov, "On the rationality of non-Gorenstein $\mathbb{Q}$-Fano 3-folds with an integer Fano index", *Contemp. Math.* **207** (1997), 43–50.
[47] Yu. I. Manin, "Rational surfaces over perfect fields", *Publ. Math. IHES* **30** (1966), 55–114.
[48] Yu. I. Manin, "Rational surfaces over perfect fields. II", *Mat. Sb.* **72** (1967), 161–192; English transl., *Math. USSR-Sb.* **1** (1968), 141–168.
[49] Yu. I. Manin, *Cubic forms. Algebra, geometry, arithmetic*, Nauka, Moscow 1972; English transl., 2nd ed., North-Holland, Amsterdam 1986.
[50] V. A. Iskovskikh, "On birational forms of rational surfaces", *Izv. Akad. Nauk SSSR Ser. Mat.* **29** (1965), 1417–1433. (Russian)
[51] V. A. Iskovskikh, "Rational surfaces with a pencil of rational curves", *Mat. Sb.* **74** (1967), 608–638; English transl. in *Math. USSR-Sb.* **3** (1967).
[52] V. A. Iskovskikh, "Rational surfaces with a pencil of rational curves and with positive square of canonical class", *Mat. Sb.* **83** (1970), 90–119; English transl., *Math. USSR-Sb.* **12** (1970), 91–117.
[53] V. A. Iskovskikh, "Birational properties of surfaces of degree 4 in $\mathbb{P}^4_k$", *Mat. Sb.* **88** (1972), 31–37; English transl., *Math. USSR-Sb.* **17** (1972), 30–36.

[54] V. A. Iskovskikh, "Minimal models of rational surfaces over perfect fields", *Izv. Akad. Nauk SSSR Ser. Mat.* **43** (1979), 19–43; English transl., *Math. USSR-Izv.* **14** (1980), 17–39.
[55] V. A. Iskovskikh, "Factorization of birational maps of rational surfaces from the viewpoint of Mori theory", *Uspekhi Mat. Nauk* **51**:4 (1996), 3–72; English transl., *Russian Math. Surveys* **51** (1996), 585–652.
[56] M. Gizatullin, "Rational G-surfaces", *Izv. Akad. Nauk SSSR Ser. Mat.* **45** (1981), 110–144; English transl., *Math. USSR-Izv.* **16** (1981), 103–134.
[57] J. Kollár, *Rational curves on algebraic varieties*, Springer–Verlag, Berlin 1996.

E-mail address: cheltsov@yahoo.com

Received 7/OCT/03
Translated by THE AUTHOR

Typeset by $\mathcal{AMS}$-TEX