Regularization of Birational Automorphisms

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Abstract—An effective method for regularizing birational automorphisms of multidimensional algebraic varieties is suggested and applied explicitly to some three-dimensional Fano varieties and del Pezzo surfaces over an algebraically nonclosed field.

KEY WORDS: regularization of birational automorphisms, three-dimensional Fano variety, birationally rigid Fano variety, del Pezzo surface, movable log pair.

1. INTRODUCTION

In this paper, we suggest an effective method for regularizing finite subgroups of birational automorphisms of algebraic surfaces and three-dimensional algebraic varieties. The result is applied to describe the finite subgroups of birational automorphisms of smooth double coverings of a three-dimensional quadric and a three-dimensional quartic with one ordinary double point and to del Pezzo surfaces of degrees 2 and 3 over an algebraically nonclosed field (in particular, we give an answer to Manin's question from the book [1]).

All objects considered in this paper are assumed to be projective and defined over a field \mathbb{F} of characteristic zero. The following result is well known.

Theorem 1.1. Let X and Y be varieties with canonical singularities and ample canonical divisors. Then any birational mapping between X and Y is biregular.

Theorem 1.1 has the following corollary.

Corollary 1.2. If X is a variety with canonical singularities and ample canonical divisor, then any birational automorphism of X is biregular.

The minimal model program (see [2]) implies the following assertion.

Theorem 1.3. Let X be a variety of general type of dimension 2 or 3. Then there exists a birational mapping $\gamma: X \dashrightarrow V$ such that

$$\gamma \circ \operatorname{Bir}(X) \circ \gamma^{-1} = \operatorname{Aut}(V).$$

It is easy to see that this assertion does not hold for varieties with nonmaximal Kodaira dimension.

Definition 1.4. Let X be a variety. We say that a subset S of the group Bir(X) is regularizable on a variety V by a birational mapping $\gamma: X \dashrightarrow V$ if

$$\gamma \circ S \circ \gamma^{-1} \subset \operatorname{Aut}(V);$$

the mapping γ is called a *regularization* of the set $S \subset Bir(X)$.

Each regularizable subset of the group of birational automorphisms generates a regularizable subgroup; for this reason, we consider only regularizable subgroups in what follows.

Note that a regularization of birational automorphisms may be nonunique, and there exist birational automorphisms having no regularization. The following result is due to I. Dolgachev. **Proposition 1.5.** Any sufficiently general birational quadratic transformation of the projective plane is nonregularizable.

Proof. Consider a sufficiently general birational quadratic transformation $T: \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. We can assume that the order of T is infinite and T has three fundamental points. Consider a regularization $f: X \dashrightarrow P^2$ of the automorphism T and the biregular automorphism $g = f^{-1} \circ T \circ f$.

Suppose that x_1, \ldots, x_n are fundamental points of f^{-1} such that they are not fundamental for T and the points $T(x_1), \ldots, T(x_n)$ are not fundamental for T^{-1} . Consider the corresponding exceptional curves E_i .

Each curve $g(E_i)$ is contracted to the point $y_i = T(x_i)$ by the morphism f and to a point x_j by the morphism f, because, by assumption, y_i is not a fundamental point of T^{-1} . Thus, g permutes the curves E_i , and some power g^N of g acts trivially on the curves E_i . Therefore, the curves E_i can be contracted, which gives a regularization $h: S \to \mathbb{P}^2$ of the birational automorphism T^N such that h has at most six fundamental points, the three fundamental points of T and the three fundamental points of T^{-1} .

By construction, the biregular automorphism $h^{-1} \circ T^N \circ h$ has infinite order. On the other hand, the automorphism group of any surface obtained by blowing up four to eight points of \mathbb{P}^2 is finite. Therefore, h is a blow-up of no more than three points, and the fundamental points of T and T^{-1} coincide. On the other hand, general quadratic transformations do not have this property. \Box

Note that the group $Bir(\mathbb{P}^2)$ is generated by the projective automorphisms of \mathbb{P}^2 and Cremona involutions, all of which are regularizable.

Corollary 1.6. The regularizable elements of the group of birational automorphisms do not necessarily form a subgroup.

Thus, it is natural to try to describe the regularizable subgroups of the group of birational automorphisms for a given variety. While this problem in the general setting may be very difficult, the following result is well known.

Theorem 1.7. For an algebraic variety X, any finite subgroup $G \subset Bir(X)$ is regularizable.

Proof. Consider the field of rational functions K(X) on X, take a normal projective model Y of the field of invariant functions $K(X)^G$, and consider the normalization V of the variety Y in the field K(X). The variety V is birationally equivalent to X, and the group G acts biregularly on V. \Box

It should be mentioned that the proof of Theorem 1.7 is not very effective in the sense that, given a variety and a finite subgroup of its birational automorphism group, it is fairly difficult to find a regularization of this group.

Remark 1.8. The main objective of this paper is to describe a new effective method for regularizing finite subgroups of birational automorphisms of surfaces and three-dimensional varieties.

In Sec. 6, this method is applied to find a regularization for the birational involutions of a double covering of a three-dimensional quadric.

The application of an equivariant resolution of singularities and the minimal model program gives the following generalization of Theorem 1.7.

Theorem 1.9. Let X be a variety of dimension 2 or 3, and let G be a finite subgroup in Bir(X). Then there exists a birational mapping $\gamma: X \dashrightarrow Y$ to a variety Y with terminal singularities such that either the canonical divisor K_Y is nef or Y has the structure of a $(\gamma \circ G \circ \gamma^{-1})$ -equivariant bundle whose fibers are Fano varieties.

In what follows, we apply the effective regularization method to the birational automorphisms of birationally rigid Fano varieties. **Definition 1.10.** A Fano variety X is said to be *birationally rigid* if X has \mathbb{Q} -factorial terminal singularities, $\operatorname{Pic}(X) = \mathbb{Z}$, and X is not birationally equivalent to any Mori fiber space¹ except itself.

Theorem 1.9 implies the following assertion.

Theorem 1.11. Let X be a birationally rigid Fano variety of dimension at most 3, and let G be a finite subgroup in Bir(X). Then there exists a birational mapping $\gamma: X \dashrightarrow Y$ to a Fano variety Y with terminal singularities such that $\gamma \circ G \circ \gamma^{-1} \subset Aut(Y)$.

Thus, Theorem 1.11 gives a method for finding all finite subgroups of the group of birational automorphisms of birationally rigid del Pezzo surfaces and three-dimensional Fano varieties. We shall apply Theorem 1.11 to a double covering of a three-dimensional quadric in Sec. 6, to a singular three-dimensional quadric in Sec. 7, and to a cubic surface in \mathbb{P}^3 and a double covering of \mathbb{P}^2 in Sec. 4.

Below we recall some open questions related to regularization of birational automorphisms.

Conjecture 1.12. The group of birational automorphisms is generated by regularizable birational automorphisms.

Note that the subgroup of the group of birational automorphisms generated by the regularizable birational automorphisms is normal.

Remark 1.13. Conjecture 1.12 holds for X with simple group Bir(X).

It is conjectured that the groups $Bir(\mathbb{P}^2)$ and $Bir(\mathbb{P}^3)$ are simple.

Conjecture 1.14. For a given variety X, the group Bir(X) has generators which are regularizable on varieties contained in a finite number of families.

It is well known that Conjectures 1.12 and 1.14 hold for surfaces and for many three-dimensional birationally rigid Fano varieties (see [3, 4]).

2. MOVABLE LOG PAIRS

In this section, we consider properties of movable log pairs, which were introduced and considered in their modern form by Alexeev in [5]. The results of this section are used in the next section to construct a regularization algorithm.

Definition 2.1. A *movable* log pair

$$(X, M_X) = \left(X, \sum_{i=1}^n b_i \mathcal{M}_i\right)$$

is a variety X together with a formal finite linear combination of linear systems \mathcal{M}_i without fixed components such that all the numbers b_i are nonnegative and rational.

Note that each movable log pair can be regarded as an ordinary log pair. In particular, for a movable log pair (X, M_X) , the divisor $K_X + M_X$ is called the *log-canonical divisor* of the log pair (X, M_X) , and M_X is said to be the *boundary* of the movable log pair (X, M_X) .

Remark 2.2. The direct image of the boundary of a movable log pair is defined in a natural way for any birational mapping.

If a log-canonical divisor of a movable log pair is a Q-Cartier divisor, then notions such as discrepancy, the property of being terminal, and canonicity can be defined for movable log pairs similarly to the corresponding notions for ordinary log pairs.

¹A fibration $\tau: V \to Z$ is called a *Mori fiber space* if $\tau_*(\mathcal{O}_V) = \mathcal{O}_Z$, the variety V has terminal Q-factorial singularities, the divisor $-K_V$ is τ -ample, and $\operatorname{Pic}(V/Z) = \mathbb{Z}$.

Definition 2.3. We say that movable log pairs (X, M_X) and (Y, M_Y) are birationally equivalent if there exists a birational mapping $\rho: X \dashrightarrow Y$ such that $M_Y = \rho(M_X)$.

Note that any movable log pair is birationally equivalent to a movable log pair with canonical and terminal singularities.

Definition 2.4. Let (X, M_X) be a movable log pair. Consider a birationally equivalent movable log pair (W, M_W) with canonical singularities. Take $n \gg 0$ such that the divisor

$$D = n(K_W + M_W)$$

is a Cartier divisor. The Kodaira dimension $\kappa(X, M_X)$ of the movable log pair (X, M_X) is defined to be equal to the Iitaka dimension of the log pair (X, D).

It can be shown that the Kodaira dimension of a movable log pair is well defined and all birationally equivalent log pairs have the same Kodaira dimension.

Remark 2.5. The Kodaira dimension of a movable log pair is a nondecreasing function of the coefficients of its boundary.

The following definition coincides with the classical one in the case of the empty boundary.

Definition 2.6. We say that a movable log pair (V, M_V) is a minimal model of a movable log pair (X, M_X) if these log pairs are birationally equivalent, the divisor $K_V + M_V$ is nef, and (V, M_V) has terminal \mathbb{Q} -factorial singularities.

A minimal model may not exist, and it may be nonunique. However, minimal models are important because of the following standard conjecture.

Conjecture 2.7. Let (X, M_X) be an arbitrary movable log pair. Then either its Kodaira dimension is nonnegative and (X, M_X) is birationally equivalent to a minimal model or $\kappa(X, M_X) = -\infty$ and the movable log pair (X, M_X) is birationally equivalent to a movable log pair (Y, M_Y) with \mathbb{Q} -factorial terminal singularities such that Y has the structure of a Mori fiber space $f: Y \to Z$ and the divisor $-(K_Y + M_Y)$ is f-ample.

Proposition 2.8. Conjecture 2.7 holds in dimensions 2 and 3.

Consider a morphism related to a minimal model of a movable log pair.

Definition 2.9. Suppose that a movable log pair (X, M_X) is a minimal model. We call a morphism $I(X, M_X): X \to Z$ an *litaka morphism* of the movable log pair (X, M_X) if there exists an ample Q-Cartier divisor L on the variety Z such that

$$K_X + M_X \sim_{\mathbb{Q}} I(X, M_X)^*(L)$$

and $I(X, M_X)_*(\mathcal{O}_X) = \mathcal{O}_Z$.

What can be said about the existence of Iitaka morphisms?

Conjecture 2.10. An Iitaka morphism exists.

Note that the standard log redundancy conjecture (see [2]) implies Conjecture 2.10. In particular, Conjecture 2.10 holds for all surfaces and all three-dimensional varieties (see, e.g., [6]).

Proposition 2.11. Let X be an algebraic surface or a three-dimensional variety, and let (X, M_X) be a movable log pair. Then, for any minimal model of the movable log pair (X, M_X) , there exists an Iitaka morphism.

As mentioned, a minimal model of a movable log pair may be nonunique. However, the following rational mapping is always unique.

MATHEMATICAL NOTES Vol. 76 No. 2 2004

Definition 2.12. Suppose that a movable log pair (X, M_X) is birationally equivalent to a minimal model (V, M_V) , and let $\rho: X \dashrightarrow V$ be the birational mapping establishing their equivalence. Suppose that there exists an Iitaka morphism for the movable log pair (V, M_V) . Then the mapping

$$I(X, M_X) = I(V, M_V) \circ \rho$$

is called the *Iitaka mapping* of the movable log pair (X, M_X) .

It can be shown that the Iitaka mapping of a movable log pair does not depend on the choice of the minimal model. The following definition is a direct generalization of the classical canonical model.

Definition 2.13. We say that a movable log pair (V, M_V) is a canonical model of a movable log pair (X, M_X) if these pairs are birationally equivalent, the divisor $K_V + M_V$ is ample, and the singularities of (V, M_V) are canonical.

The proof of the following result is similar to that of the corresponding assertion for ordinary log pairs (see [6]).

Theorem 2.14. If a canonical model exists, then it is unique.

Theorem 2.14 has the following simple but important corollary.

Corollary 2.15. If (X, M_X) is a canonical model, then all birational automorphisms X leaving the movable log pair (X, M_X) fixed are biregular.

Note that if a movable log pair (X, M_X) has a canonical model, then $\kappa(X, M_X)$ is equal to the dimension of the variety X. Moreover, it is easy to see that if Conjecture 2.7 holds, then the converse is true as well (see [2]).

3. EFFECTIVE REGULARIZATION

In this section, we describe an effective algorithm for finding regularizations of finite subgroups of birational automorphism groups.

Let X be a surface or a three-dimensional variety, and let G be a finite subgroup in Bir(X). Take a very ample divisor H on X and a nonnegative rational number μ . We set

$$(X, H_X^{\mu}) = \left(X, \sum_{g \in G} \mu g(|H|)\right).$$

By construction, the movable log pair (X, H_X^{μ}) is G-invariant.

Lemma 3.1. For $\mu \gg 0$, $\kappa(X, H_X^{\mu}) = \dim(X)$.

Proof. The Kodaira dimension of a movable log pair is a nondecreasing function of the coefficients of its boundary. Thus,

$$\kappa(X, H_X^{\mu}) \ge \kappa(X, \mu|H|).$$

Since H is ample, we have $\kappa(X, \mu|H|) = \dim(X)$ for $\mu \gg 0$. \Box

Thus, by Lemma 3.1, $\kappa(X, H_X^{\mu}) = \dim(X)$ for sufficiently large μ . Therefore, Propositions 2.8 and 2.11 give a birational surgery $\sigma: X \dashrightarrow V$ such that the movable log pair

$$(V, H_V^{\mu}) = (\sigma(X), \sigma(H_X^{\mu}))$$

is the canonical model of the movable log pair (X, H_X^{μ}) .

Lemma 3.2. The rational mapping σ is a regularization of the subgroup G.

Proof. The movable log pair (V, H_V^{μ}) is a $(\sigma \circ G \circ \sigma^{-1})$ -invariant canonical model. The required assertion follows from Corollary 2.15. \Box

Thus, we have effectively constructed a regularization of the group G. In particular, we have obtained an effective proof of Theorem 1.7 for surfaces and three-dimensional varieties. Moreover, using a G-invariant resolution of singularities and the log minimal model program, we can obtain an effective regularization of the subgroup G which satisfies all the requirements of Theorem 1.9 from this proof.

Let us show that the singularities of the variety V itself are canonical also.

Lemma 3.3. The singularities of V are canonical.

Proof. The second main theorem of [6] implies that, for some rational number $\delta > \mu$, (V, H_V^{δ}) is the canonical model of the movable log pair (X, H_X^{δ}) . In particular, both log-canonical divisors

$$K_V + H_V^{\mu}$$
 and $K_V + H_V^{\delta}$

are Q-Cartier. Therefore, the canonical divisor

$$K_V \sim_{\mathbb{Q}} (K_V + H_V^{\mu}) - \frac{\mu}{\delta - \mu} [(K_V + H_V^{\delta}) - (K_V + H_V^{\mu})]$$

is \mathbb{Q} -Cartier as well.

The movable log pair (V, H_V^{μ}) is a canonical model. In particular, its singularities are canonical. This and the fact that the divisor K_V is Q-Cartier imply the canonicity of the singularities of V. \Box

4. DEL PEZZO SURFACES

In this section, we consider del Pezzo surfaces over a nonclosed field in the context of regularization of birational automorphisms.

Take a smooth del Pezzo surface X with $Pic(X) = \mathbb{Z}$ and K_X^2 equal to 2 or 3. Note that, because of the condition $Pic(X) = \mathbb{Z}$, the field of definition \mathbb{F} is not algebraically closed.

Remark 4.1. It is well known that

$$X \cong \begin{cases} \text{a double covering of } \mathbb{P}^2 \text{ branched along a smooth quadric if } K_X^2 = 2\\ \text{a cubic in } \mathbb{P}^3 \text{ if } K_X^2 = 3. \end{cases}$$

Let us briefly recall the construction of the Bertini and Geiser involutions of the surface X. Consider a birational morphism $f: W \to X$ such that the surface W is nonsingular and $K_W^2 > 0$. The condition $\operatorname{Pic}(X) = \mathbb{Z}$ implies that the anticanonical divisor $-K_W$ is nef and big. Therefore, for $n \gg 0$, the linear system $|-nK_W|$ is free and determines a birational morphism to a del Pezzo surface V with canonical singularities of degree 1 or 2.

Definition 4.2. We refer to the birational mapping $\phi_{|-nK_W|} \circ f^{-1}$ as a *standard* mapping of the del Pezzo surface X to the del Pezzo surface V.

It can be shown that if $K_V^2 = 1$, then the surface V is a double covering of a quadratic cone in \mathbb{P}^3 branched along a quadric, and if $K_V^2 = 2$, then the surface V is a double covering of \mathbb{P}^2 branched along a quadric. Thus, V has a canonical biregular involution, which induces a birational involution τ of the del Pezzo surface X. For a smooth surface V, the involution τ is not biregular; it is called a *Bertini involution* if $K_V^2 = 1$ and a *Geiser involution* if $K_V^2 = 2$. The classical and more geometric definitions of the Bertini and Geiser involutions are given in [7, 8]. **Remark 4.3.** The surface V is smooth if and only if $V \cong W$.

The importance and essence of the Bertini and Geiser involutions are seen from the following classical result of Manin (see [7, 8]).

Theorem 4.4. The surface X is birationally rigid, and the group Bir(X) is the semidirect product of the group Aut(X) and the subgroup generated by all the Bertini and Geiser involutions of the surface X.

In [9], the following theorem was proved.

Theorem 4.5. Let Y be a del Pezzo surface with canonical singularities birationally equivalent to the surface X. Then Y can be obtained by a standard transformation of X.

Theorems 4.5 and 1.11 imply the following assertion.

Proposition 4.6. Each finite subgroup of the group Bir(X) is either a subgroup of Aut(X) or a subgroup of Aut(V), where V is a smooth del Pezzo surface obtained by a standard transformation from the surface X.

5. MANIN'S QUESTION

In this section, we answer a question asked by Manin in [1].

Let X be a smooth cubic surface in \mathbb{P}^3 with $\operatorname{Pic}(X) = \mathbb{Z}$ over a nonclosed field \mathbb{F} , and let G be the group of birational automorphisms of X generated by all the Bertini and Geiser involutions of X.

Remark 5.1. It follows from the construction of the Bertini and Geiser involutions that G is a normal subgroup in Bir(X).

The following result is a special case of Theorem 4.4.

Theorem 5.2. The cubic X is birationally rigid, and the group Bir(X) is the semidirect product of the groups G and Aut(X).

In [1], Manin asked what elements of the groups G and Bir(X) have finite order. Note that all biregular automorphisms of the cubic X, the Bertini and Geiser involutions, and all elements of the group Bir(X) conjugate to them are all of finite order.

Remark 5.3. Let θ be a biregular automorphism of the cubic X, and let τ be a birational Bertini or Geiser involution of X such that $\theta \circ \tau = \tau \circ \theta$. Then $\tau \circ \theta$ is of finite order.

Somewhat later, Kanevskii described the finite-order elements of the group G [10]. He used a purely group-theoretic method based on the generating relations in G found in [8].

Theorem 5.4. Each birational automorphism of X of finite order is conjugate in the group Bir(X) to one of the following birational automorphisms:

- (a) a biregular automorphism of the cubic X;
- (b) a Bertini involution;
- (c) a Geiser involution;
- (d) the composition of a biregular automorphism of X and a Bertini or Geiser involution that commute with each other.

Theorems 5.2 and 5.4 imply the following assertion.

Corollary 5.5. The elements of finite order in the group G are conjugate to Bertini and Geiser involutions.

Proof of Theorem 5.4. Let τ be an element of finite order in Bir(X). We must show that τ is conjugate to one of the following birational automorphisms: a biregular automorphism of the cubic X; a Bertini involution; a Geiser involution; the composition of a biregular automorphism of X and a Bertini or Geiser involution that commute with each other.

Since the cubic X is birationally rigid, we can apply Theorem 1.11, according to which the birational automorphism τ can be regularized on some smooth del Pezzo surface Y. Thus, there exists a birational mapping $\gamma: X \dashrightarrow Y$ such that $\gamma \circ \tau \circ \gamma^{-1} \in \operatorname{Aut}(Y)$. On the other hand, by Theorem 4.5, the surface Y is biregular to either the cubic X itself or its blow-up. In the first case, τ is conjugate to a biregular automorphism of X. We can assume that there exists a blow-up $f: Y \to X$ and $K_Y^2 = 1$ or $K_Y^2 = 2$.

Suppose that $K_Y^2 = 2$. Then

$$\operatorname{Pic}(Y) = \mathbb{Z}K_Y \oplus \mathbb{Z}E,$$

where E is the exceptional divisor of the blow-up f.

Note that the action of $\gamma \circ \tau \circ \gamma^{-1}$ on $\operatorname{Pic}(Y)$ must preserve the class K_Y . If

$$\gamma \circ \tau \circ \gamma^{-1}(E) \sim E,$$

then the mapping $f \circ \gamma \circ \tau \circ \gamma^{-1} \circ f^{-1}$ is biregular. Thus, we can assume that the curve E is not $(\gamma \circ \tau \circ \gamma^{-1})$ -invariant. A direct calculation gives

$$(\gamma \circ \tau \circ \gamma^{-1})^*(K_X) \sim 2f^*(K_X) + 3E, \qquad (\gamma \circ \tau \circ \gamma^{-1})^*(E) \sim -f^*(K_X) - 2E.$$

It follows from the construction of the Geiser involutions that, for some Geiser involution σ of the cubic surface X, the action of

$$\gamma \circ \tau \circ \gamma^{-1} \circ f^{-1} \circ \sigma \circ f$$

on the group $\operatorname{Pic}(Y)$ is trivial. Therefore, the mapping $f \circ \gamma \circ \tau \circ \gamma^{-1} \circ f^{-1} \circ \sigma$ is a biregular automorphism of X which, in addition, leaves the point f(E) fixed; hence $f \circ \gamma \circ \tau \circ \gamma^{-1} \circ f^{-1} \circ \sigma$ commutes with the involution σ .

Now, suppose that $K_Y^2 = 1$ and

$$\operatorname{Pic}(Y) = \mathbb{Z}K_Y \oplus \mathbb{Z}E,$$

where E is the exceptional divisor of the birational morphism f.

We can assume that the action of $\gamma \circ \tau \circ \gamma^{-1}$ on $\operatorname{Pic}(Y)$ is nontrivial, because otherwise the birational automorphism $f \circ \gamma \circ \tau \circ \gamma^{-1} \circ f^{-1}$ is biregular. A direct calculation shows that

$$(\gamma \circ \tau \circ \gamma^{-1})^*(K_X) \sim 5f^*(K_X) + 6E, \qquad (\gamma \circ \tau \circ \gamma^{-1})^*(E) \sim -4f^*(K_X) - 5E.$$

It follows from the construction of Bertini involutions that, for some Bertini involution σ of the cubic X, the composition $f \circ \gamma \circ \tau \circ \gamma^{-1} \circ f^{-1} \circ \sigma$ is biregular and leaves f(E) fixed, which implies that the automorphism $f \circ \gamma \circ \tau \circ \gamma^{-1} \circ f^{-1} \circ \sigma$ commutes with σ .

Now, consider the remaining case, in which $K_Y^2 = 1$ and

$$\operatorname{Pic}(Y) = \mathbb{Z}K_Y \oplus \mathbb{Z}E_1 \oplus \mathbb{Z}E_2$$

where E_1 and E_2 are two different exceptional curves of the morphism f.

As in the cases considered above, we can assume that the action of $\gamma \circ \tau \circ \gamma^{-1}$ on the group $\operatorname{Pic}(Y)$ is nontrivial. For similar reasons, we can assume that

$$\gamma \circ \tau \circ \gamma^{-1}(E_1 \cup E_2) \neq E_1 \cup E_2.$$

MATHEMATICAL NOTES Vol. 76 No. 2 2004

I. A. CHEL'TSOV

Suppose that the curve E_1 is $(\gamma \circ \tau \circ \gamma^{-1})$ -invariant. Let $g: Y \to V$ be the blow-down of E_1 . Then $g \circ \gamma \circ \tau \circ \gamma^{-1} \circ g^{-1}$ is a biregular automorphism of the surface V. On the other hand, V is a smooth del Pezzo surface with $K_V^2 = 2$. As in the case of $K_Y^2 = 2$, it is easy to show that the automorphism $f \circ \gamma \circ \tau \circ \gamma^{-1} \circ f^{-1}$ is the composition of a Geiser involution of X and a biregular automorphism of X commuting with it.

Now, suppose that none of the curves E_i is $(\gamma \circ \tau \circ \gamma^{-1})$ -invariant. Then

$$(\gamma \circ \tau \circ \gamma^{-1})^* (K_X) \sim 5f^* (K_X) + 6E_1 + 6E_2, (\gamma \circ \tau \circ \gamma^{-1})^* (E_1) \sim -2f^* (K_X) - 2E_1 - 3E_2, (\gamma \circ \tau \circ \gamma^{-1})^* (E_2) \sim -2f^* (K_X) - 3E_1 - 2E_2.$$

It follows from the construction of the Bertini involutions that $f \circ \gamma \circ \tau \circ \gamma^{-1} \circ f^{-1}$ is the composition of a Bertini involution of X and a biregular automorphism of X commuting with it. \Box

6. A DOUBLE COVERING OF A QUADRIC

In this section, we effectively apply the regularization algorithm described in Sec. 3. Consider a double covering

$$\theta\colon X\to Q\subset \mathbb{P}^4$$

of a smooth three-dimensional quadric Q branched along a smooth octic $S \subset Q$. The variety X is a Fano variety with $\operatorname{Pic}(X) \cong \mathbb{Z}$ and $-K_X^3 = 4$. Moreover, X is birationally rigid (see [3]).

Definition 6.1. We say that a curve $C \subset X$ is a *line* if $-K_X \cdot C = 1$.

The variety X contains a one-dimensional family of lines if $\mathbb{F} = \mathbb{C}$. Each line C on X induces a birational involution τ_C in Bir(X) such that τ_C is nonbiregular if and only if $\theta(C) \not\subset S$ (see [3]). The essence of the involutions τ_C is clarified by the following result of Iskovskikh (see [3]).

Theorem 6.2. The group Bir(X) is the semidirect product of the automorphism group of the threedimensional variety X and the normal subgroup generated by the nonbiregular involutions τ_C .

The involutions τ_C are regularizable by Theorem 1.7. Therefore, according to Theorem 6.2, Conjectures 1.12 and 1.14 hold for the variety X.

Take a line C on X such that $\theta(C)$ is not contained in the surface S. Then the birational involution τ_C is not biregular, and Theorem 1.11 implies the existence of a birational mapping $\gamma: X \dashrightarrow X_C$ such that X_C is a three-dimensional Fano variety with terminal singularities and

$$\gamma \circ \tau_C \circ \gamma^{-1} \in \operatorname{Aut}(X_C).$$

Now, let us show how to explicitly construct a birational mapping γ by using the regularization algorithm described above. Take a rational number $\mu > 0$ and consider the movable log pair

$$(X, H_X^{\mu}) = (X, \mu | - K_X | + \mu \tau_C (| - K_X |)).$$

Lemma 6.3. The singularities of the movable log pair (X, H_X^{μ}) are terminal if $\mu < 1/10$, canonical if $\mu = 1/10$, and noncanonical if $\mu > 1/10$. Moreover,

$$\kappa(X, H_X^{\mu}) = \begin{cases} -\infty & \text{for } \mu < \frac{1}{10}, \\ 0 & \text{for } \mu = \frac{1}{10}, \\ 3 & \text{for } \mu > \frac{1}{10}. \end{cases}$$

Proof. All these assertions are proved by easy calculations based on the direct construction of the birational involution τ_C described in [3]. \Box

Thus, by Corollary 2.15, the mapping $I(X, H_X^{\mu})$ is a regularization of the birational involution τ_C for $\mu > 1/10$. Let us explicitly describe the Iitaka mapping $I(X, H_X^{\mu})$ for $\mu > 1/10$. Consider the blow-up $f: W \to X$ of the line C. Let Z be the unique basis curve of the linear system $|-K_W|$. We let $g: V \to W$ be the blow-up of the curve Z, set $G = g^{-1}(Z)$, and consider the movable log pair

$$(V, H_V^{\mu}) = (V, (f \circ g)^{-1}(H_X^{\mu})).$$

Remark 6.4. The movable log pair (V, H_V^{μ}) is terminal for all $\mu > 0$.

Thus, if $\mu > 1/10$, then we can apply the log minimal model program to (V, H_V^{μ}) and obtain a birational mapping $\rho: V \dashrightarrow Y$ such that the movable log pair $(Y, H_Y^{\mu}) = (Y, \rho(H_V^{\mu}))$ has terminal Q-factorial singularities and the divisor $K_Y + H_Y^{\mu}$ is nef and big.

Remark 6.5. An explicit construction of the mapping ρ is described in [9]; it is also shown in [9] that ρ is the composition of a flop in some curve T contained in the divisor G and a contraction of the proper image of G to a cyclic quotient singularity of type $\frac{1}{2}(1,1,1)$.

For $\mu = 1/10$, the birational morphism $f: W \to X$ is crepant with respect to the movable log pair (X, H_X^{μ}) . Therefore, $K_Y + H_Y^{\mu} \sim_{\mathbb{Q}} 0$ for $\mu = 1/10$, and hence the anticanonical divisor $-K_Y$ is nef and big. Now, we can apply Proposition 2.11 to the movable log pair (Y, H_Y^{μ}) with $\mu > 1/10$ and obtain the birational morphism

$$I(Y, H_Y^{\mu}) = \psi_{|-n(K_Y + H_Y^{\mu})|}$$
 for some $n \gg 0$.

By construction, $X_C = I(Y, H_Y^{\mu})(Y)$ is a Fano variety with canonical singularities. Moreover, it can be shown that the birational morphism $I(Y, H_Y^{\mu})$ does not contract divisors on the variety Y. Thus, the singularities of X_C are terminal, but they are not Q-factorial, as is easy to see. Therefore, for $\mu > 1/10$, the birational mapping

$$\gamma = I(Y, H_Y^{\mu}) \circ \rho \circ (f \circ g)^{-1}$$

is the required regularization of the birational involution τ_C .

Remark 6.6. The construction of the mapping γ implies $-K_{X_C}^3 = 1/2$.

In [9], the following result was obtained.

Theorem 6.7. Let Y be a Fano variety with canonical singularities which is birationally equivalent to the variety X. Then

$$Y \cong \begin{cases} X, \\ X_C & \text{for some line } C \text{ on } X. \end{cases}$$

Thus, Theorem 1.11 implies the following assertion.

Proposition 6.8. All finite subgroups of the group Bir(X) are subgroups of the groups Aut(X) and $Aut(X_C)$ for some line C on X.

For each line C on X, the group $\operatorname{Aut}(X_C)$ contains \mathbb{Z}_2 as a canonical subgroup inducing the birational involution τ_C on the variety X. Moreover, it can be shown that if X is sufficiently general, then

$$\operatorname{Aut}(X) \cong \operatorname{Aut}(X_C) \cong \mathbb{Z}_2.$$

Corollary 6.9. For a sufficiently general double covering of the quadric X, all finite subgroups of Bir(X) are isomorphic to \mathbb{Z}_2 and conjugate to subgroups generated by the involutions τ_C and the involution induced by the double covering θ .

I. A. CHEL'TSOV

7. A SINGULAR THREE-DIMENSIONAL QUARTIC

In this section, we apply Theorem 1.11 to a singular three-dimensional quadric.

Let X be a sufficiently general quartic in \mathbb{P}^4 having one singular point O locally isomorphic to an ordinary double point. Note that X is a Fano variety with \mathbb{Q} -factorial singularities and $\operatorname{Pic}(X) = \mathbb{Z}$.

Remark 7.1. The quartic X contains precisely 24 different lines passing through the singular point O, and the group Aut(X) is trivial.

In [9], the following theorem was proved.

Theorem 7.2. For the singular point O and each line C on X passing through O, there exist birational surgeries

 $\psi_O \colon X \dashrightarrow X_O$ and $\psi_C \colon X \dashrightarrow X_C$

such that the varieties X_O and X_C are three-dimensional Fano varieties with canonical singularities, $-K_{X_O}^3 = 2$, and $-K_{X_C}^3 = 1/2$. Any three-dimensional Fano variety Y with canonical singularities which is birationally isomorphic to the quadric X has the form

$$Y \cong \begin{cases} X, \\ X_O, \\ X_C & \text{for some line } C \text{ on } X \text{ passing through } O. \end{cases}$$

In [4], the birational rigidity of X was proved and the following result was obtained.

Theorem 7.3. The singular point O and each line C on X passing through O induce birational involutions τ_O and τ_C , respectively, of the quartic X. Moreover, the involutions τ_O and τ_C generate the group Bir(X).

The involutions τ_O and τ_C are regularizable, and Conjectures 1.12 and 1.14 hold for the quartic X. Theorems 1.11 and 7.2 imply that the birational involutions τ_O and τ_C can be regularized on the varieties X_O and X_C ; moreover, the following assertion is valid [9].

Proposition 7.4. The birational mappings ψ_O and ψ_C are regularizations of the birational involutions τ_O and τ_C , respectively.

The construction of the birational mappings ψ_O and ψ_C is similar to that of the regularization of the birational involutions of a double covering of a three-dimensional quadric in the preceding section.

The generality of the variety X implies

$$\operatorname{Aut}(X_C) \cong \operatorname{Aut}(X_0) \cong \mathbb{Z}_2.$$

Corollary 7.5. All finite subgroups of the group Bir(X) are isomorphic to \mathbb{Z}_2 and conjugate to subgroups generated by the involutions τ_C and τ_O .

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