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Birationally superrigid cyclic triple spaces

I. A. Cheltsov

Abstract. We prove the birational superrigidity and non-rationality of a cyclic triple covering of \mathbb{P}^{2n} branched over a nodal hypersurface of degree 3n for $n \ge 2$. The result obtained solves the problem of birational superrigidity for smooth cyclic triple spaces. We also consider certain relevant problems.

§1. Introduction

The problem of rationality for algebraic varieties ¹ is one of the most interesting problems in algebraic geometry. Global holomorphic differential forms are natural birational invariants of a smooth algebraic variety that solve the problem of rationality for algebraic curves and surfaces (see [205] and [14]). However, even in the 3-dimensional case there are non-rational varieties that are very close to being rational. In particular, the available discrete invariants do not solve the rationality problem for higher-dimensional algebraic varieties. For example, there are non-rational unirational 3-folds (see [17] and [93]), giving a negative answer to the Lüroth problem in dimension 3. Unfortunately, we do not know simple any way of proving non-rationality in non-trivial situations (see [15] and [146]), for example, in the class of higher-dimensional rationally connected varieties (see [144]).

There are only four known methods of proving non-rationality for rationally connected varieties. The finiteness of the group of birational automorphisms of a smooth quartic 3-fold is proved in [17], which implies its non-rationality. The non-rationality of a smooth cubic 3-fold is proved in [93] through the study of its intermediate Jacobian. The birational invariance of the torsion subgroup of the group $H^3(\mathbb{Z})$ is used in [68] to prove the non-rationality of certain unirational conic bundles. The non-rationality of a wide class of rationally connected varieties is proved in [143] by reduction to positive characteristic (see [61], [144], [147]).

Every method of proving the non-rationality of an algebraic variety has advantages and disadvantages. For example, the route via intermediate Jacobians can be applied only to 3-folds and, except in a single case (see [36], [37], [199], [38], [39], [92]), only to 3-folds fibred over conics (see [42], [43], [72], [44]). On the other hand, the method of the intermediate Jacobian is often applicable to 3dimensional varieties when no other method can be used. The degeneration method (see [72], [44], [90], [69], [59], [87]) shows that the Griffiths component of the intermediate Jacobian is sometimes the most subtle 3-dimensional birational invariant.

¹All varieties are assumed to be projective, normal and defined over \mathbb{C} .

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For example, important particular cases of the rationality criterion for 3-dimensional conic bundles (see [132], [133], [12], [13]) were proved in [62] using the intermediate Jacobian method. However, there are non-rational 3-folds whose group $H^3(\mathbb{Z})$ is trivial (see [34]). The group $H^3(\mathbb{Z})$ is torsion-free in many interesting cases (for example, for smooth complete intersections) and, therefore, the method of [68] is not applicable to these cases (see [94], [169]). The method of [143] works in any dimension, but proves non-rationality only for very general numbers of appropriate families. The technique of [17] also works in any dimension (see [174]), but is generally applicable only to varieties that are very far from being rational. For example, it is hard to believe that the technique of [17] could produce an example of a smooth deformation of a non-rational variety into a rational one (see [44]). Such examples are expected to exist in dimensions greater than 3 (see [40], [41], [125], [126]).

Let us consider the following notion, which was introduced implicitly in [17]. Although historically it goes back to the classical papers [163], [113], [114], its modern form appeared relatively recently (see [99], [178]). We recall that the class of terminal singularities is a higher-dimensional generalization of smooth points of algebraic surfaces, and this class is closed under good birational maps (see [142]). \mathbb{Q} -factoriality simply means that a multiple of every Weil divisor on a variety is a Cartier divisor. In particular, every smooth variety has terminal \mathbb{Q} -factorial singularities.

Definition 1. A terminal \mathbb{Q} -factorial Fano variety V with Picard group \mathbb{Z} is said to be *birationally superrigid* if the following three conditions hold.

1) V is not birationally equivalent to a fibration² whose generic fibre is a smooth variety of Kodaira dimension $-\infty$.

2) V is not birationally equivalent to a terminal \mathbb{Q} -factorial Fano variety with Picard group \mathbb{Z} that is not biregularly equivalent to V.

3) $\operatorname{Bir}(V) = \operatorname{Aut}(V)$.

The paper [17] contains an implicit proof that every smooth quartic 3-fold in \mathbb{P}^4 is birationally superrigid (see [98]). The technique of [17] can also be applied to certain Fano 3-folds with non-trivial group of birational automorphisms (see [11]). Therefore one can consider the following weakened version of birational superrigidity.

Definition 2. A terminal Q-factorial Fano variety V with Picard group \mathbb{Z} is said to be *birationally rigid* if V satisfies the first two conditions of Definition 1.

Birationally rigid varieties are non-rational. In particular, there are no birationally rigid del Pezzo surfaces defined over an algebraically closed field. However, there are birationally rigid del Pezzo surfaces over algebraically non-closed fields (see [14]). Namely, the results of [20] and [21] yield the birational superrigidity of smooth del Pezzo surfaces of degree 1 and the birational rigidity of smooth del Pezzo surfaces of degree 2 and 3 that are defined over a perfect algebraically nonclosed field and have Picard group \mathbb{Z} . In particular, two minimal smooth cubic surfaces in \mathbb{P}^3 are birationally equivalent if and only if they are projectively equivalent (see [22]).

²We assume that all fibrations $\tau: Y \to Z$ satisfy $\dim(Y) > \dim(Z) \neq 0$ and $\tau_*(\mathcal{O}_Y) = \mathcal{O}_Z$.

One can similarly define birational rigidity and superrigidity for fibrations into Fano varieties (see [99] and [178]) or, to be precise, for Mori fibrations (see [98]). Birational rigidity is now known for many smooth 3-folds (see [11], [33], [26], [99]), many smooth varieties of dimension greater than 3 (see [34], [170], [23], [172], [49], [27]–[29], [175], [35], [30]–[32], [116], [54], [177]), and many singular varieties (see [24], [171], [101], [99], [158], [176], [89], [88]). For some birationally non-rigid algebraic varieties it is possible to find all Mori fibrations birationally equivalent to them (see [5], [100], [7], [8]). Despite obvious successes in this area of algebraic geometry, many relevant classical problems are still unsolved, such as finding generators for the group $Bir(\mathbb{P}^3)$ or the group of birational automorphisms of a smooth cubic 3-fold. A solution of the latter problem was announced in the classical paper [114], but the proof contains many gaps.

In this paper we prove the following result.

Theorem 3. Let $\pi: X \to \mathbb{P}^{2n}$ be a cyclic triple covering³ such that π is branched over a hypersurface $S \subset \mathbb{P}^{2n}$ of degree $3n, n \ge 2$, and the hypersurface S has at most ordinary double points. Then X is a terminal Q-factorial Fano variety with $\operatorname{Pic}(X) \cong \mathbb{Z}$, X is birationally superrigid, and the group $\operatorname{Bir}(X)$ is finite. (If the hypersurface $S \subset \mathbb{P}^{2n}$ is sufficiently general, then this group is isomorphic to \mathbb{Z}_3 .) In particular, the variety X is non-rational.

Remark 4. Under the hypotheses of Theorem 3, the variety X may be realized as a hypersurface in the weighted projective space $\mathbb{P}(1^{2n+1}, n)$ of degree 3n given by the equation

$$y^{3} = f_{3n}(x_{0}, \dots, x_{2n}) \subset \mathbb{P}(1^{2n+1}, n) \cong \operatorname{Proj}(\mathbb{C}[x_{0}, \dots, x_{2n}, y]),$$

where f_{3n} is a homogeneous polynomial of degree 3n (see [160], [118], [188], [189], [191]), and the cyclic triple covering $\pi: X \to \mathbb{P}^{2n}$ is the restriction of the natural projection $\mathbb{P}(1^{2n+1}, n) \dashrightarrow \mathbb{P}^{2n}$ induced by the embedding $\mathbb{C}[x_0, \ldots, x_{2n}] \subset \mathbb{C}[x_0, \ldots, x_{2n}, y]$ of graded algebras. Moreover, the hypersurface $S \subset \mathbb{P}^{2n}$ is given by $f_{3n}(x_0, \ldots, x_{2n}) = 0$.

Remark 5. Consider a cyclic triple covering $\pi: X \to \mathbb{P}^k$ such that π is branched over a nodal hypersurface $S \subset \mathbb{P}^k$ of degree 3n and $k \ge 3$. If k < 2n, then X is not birationally superrigid because it has pencils of varieties of Kodaira dimension $-\infty$. On the other hand, if k > 2n, then the Kodaira dimension of X is non-negative and X is not even uniruled. Therefore Theorem 3 describes all birationally superrigid smooth cyclic triple coverings of projective spaces.

Corollary 6. Let $f(x_0, \ldots, x_{2n})$ be a homogeneous polynomial of degree 3n that determines a nodal hypersurface $S \subset \mathbb{P}^{2n}$. Then the field

$$\mathbb{C}(\nu_1,\ldots,\nu_{2n})\sqrt[3]{f(1,\nu_1,\ldots,\nu_{2n})}$$

is a purely transcendental extension of \mathbb{C} if and only if n = 1.

 $^{^{3}}$ A finite morphism of degree 3 that induces a cyclic extension of the fields of rational functions.

Example 7. Let X be a hypersurface in $\mathbb{P}(1^{2n+1}, n)$ of degree 3n whose equation is

$$y^{3} = \sum_{i=0}^{2n} x_{i}^{3n} \subset \mathbb{P}(1^{2n+1}, n) \cong \operatorname{Proj}(\mathbb{C}[x_{0}, \dots, x_{2n}, y]),$$

and $n \ge 2$. Then the projection $\pi: X \to \mathbb{P}^{2n} \cong \operatorname{Proj}(\mathbb{C}[x_0, \ldots, x_{2n}])$ is a cyclic triple covering branched over a smooth hypersurface $\sum_{i=0}^{2n} x_i^{3n} = 0$. The variety X is birationally superrigid by Theorem 3 and

$$\operatorname{Bir}(X) = \operatorname{Aut}(X) \cong \mathbb{Z}_3 \oplus \operatorname{Aut}\left(\sum_{i=0}^{2n} x_i^{3n} = 0\right) \cong \mathbb{Z}_3 \oplus (\mathbb{Z}_{3n}^{2n} \rtimes \operatorname{S}_{2n+1}),$$

where S_{2n+1} is the symmetric group (see [198], [183], [184], [152]). In particular, the variety X is non-rational and the field $\mathbb{C}(\nu_1, \ldots, \nu_{2n})\sqrt[3]{1 + \sum_{i=1}^{2n} \nu_i^{3n}}$ is not a purely transcendental extension of \mathbb{C} .

Example 8. Let X be a hypersurface in $\mathbb{P}(1^{2n+1}, n)$ of degree 3n whose equation is

$$y^{3} = \sum_{i=1}^{n} a_{i}(x_{0}, \dots, x_{2n}) x_{i} \subset \mathbb{P}(1^{2n+1}, n) \cong \operatorname{Proj}(\mathbb{C}[x_{0}, \dots, x_{2n}, y]),$$

where a_i is a sufficiently general homogeneous polynomial of degree 3n-1. Then the natural projection $\pi: X \to \mathbb{P}^{2n}$ is a cyclic triple covering. It is branched over a nodal hypersurface $S \subset \mathbb{P}^{2n}$ of degree 3n which is given by

$$\sum_{i=1}^{n} a_i x_i = 0 \subset \mathbb{P}^{2n} \cong \operatorname{Proj}(\mathbb{C}[x_0, \dots, x_{2n}])$$

and has $(3n-1)^n$ ordinary double points. The variety X is birationally superrigid and non-rational for $n \ge 2$ by Theorem 3, and the group Bir(X) is finite.

Example 9. Let X be a hypersurface in $\mathbb{P}(1^{2n+1}, n)$ of degree 3n whose equation is

$$y^{3} = \sum_{i=1}^{n} a_{i}(x_{0}, \dots, x_{2n}) b_{i}(x_{0}, \dots, x_{2n}) \subset \mathbb{P}(1^{2n+1}, n) \cong \operatorname{Proj}(\mathbb{C}[x_{0}, \dots, x_{2n}, y]),$$

where a_i , b_i are sufficiently general homogeneous polynomials of degrees 2n, n respectively. Then the natural projection $\pi: X \to \mathbb{P}^{2n}$ is a cyclic triple covering. It is branched over a nodal hypersurface $S \subset \mathbb{P}^{2n}$ of degree 3n which has $2^n n^{2n}$ ordinary double points. The variety X is birationally superrigid and non-rational for $n \ge 2$ by Theorem 3, and the group $\operatorname{Bir}(X)$ is finite.

The main reason why the variety X in Theorem 3 is birationally superrigid is as follows. The anticanonical degree $(-K_X)^{\dim(X)} = 3$ of X is very small and the singularities of X are relatively mild. Roughly speaking, a Fano variety must become *more rational* as the anticanonical degree increases and the singularities become worse. This general principle may not necessarily be true in some extremely singular cases (see [47]). However, the classification of smooth Fano 3-folds yields that a smooth Fano 3-fold is rational if its degree is bigger than 24 (see [134]). Singular Fano 3-folds are not classified even in the case when their anticanonical divisors are Cartier divisors (see [48], [167], [137]), but many examples confirm this intuition in the singular case as well (see [101], [100], [84], [56], [87], [89]). Therefore the non-rationality of X in Theorem 3 is very natural.

The notions of birational superrigidity and rigidity make sense only for Mori fibrations (see [98]). In particular, in the case of Fano varieties we must assume that the singularities of the variety are \mathbb{Q} -factorial and the rank of its Picard group is equal to 1. Many examples suggest that a Fano variety cannot be birationally rigid unless its degree is sufficiently small. Moreover, it is intuitively clear (see [100], [158]) that quantitative characteristics of singularities (the number of isolated singular points or the anticanonical degree of the corresponding subvarieties of singular points) are important only to guarantee the \mathbb{Q} -factoriality condition (see [89], [87], [88]). On the other hand, qualitative characteristics of singularities (the multiplicity and analytical local type) can have a crucial influence the birational geometry of a Fano variety (see [99], [100]).

All existing proofs of the birational rigidity or superrigidity of a Fano variety depend crucially on the projective geometry of the variety related to the anticanonical map. It is natural to expect that some claims on birational rigidity can be proved without implicit use of the properties of the anticanonical ring. For example, we expect that the following is true (see [30], [116]).

Conjecture 10. Let X be a non-singular Fano variety of dimension k such that $\operatorname{Pic}(X) \cong \mathbb{Z}$ and $(-K_X)^k \leq 2(k-1)$. Then X is birationally rigid.

Conjecture 10 is now known to be true only in dimension 3 through the explicit classification of smooth Fano 3-folds (see [134]). It may be extremely hard to prove in general. On the other hand, it is natural to expect that the following weakened version could be proved by combining the methods of [99] with the technical tools of [106] and [141].

Conjecture 11. Let X be a smooth Fano variety of dimension k such that $\operatorname{Pic}(X) \cong \mathbb{Z}$ and $(-K_X)^k = 1$. Then X is birationally superrigid.

Remark 12. It is well known that all statements on birational rigidity remain valid for varieties defined over any field, with a single exception. Namely, the field must have characteristic zero to guarantee that the Kawamata–Viehweg vanishing theorem holds (see [140], [195]). In the case of algebraic surfaces, it is enough to assume that the field of definition is perfect (see [20], [21]). Moreover, all assertions on birational rigidity remain valid in the equivariant set-up for actions of finite groups (see [9], [4], [14]). This fact can be used to classify all non-conjugate finite subgroups in the corresponding groups of birational automorphisms (see [16]).

It should be pointed out that a cyclic triple covering of \mathbb{P}^{2n} is non-rational and non-ruled provided that it is branched over a very general⁴ smooth hypersurface of degree 3n with $n \ge 2$. This is a corollary of [144], Theorem 5.13, where the following theorem is proved.

⁴The complement of a countable union of Zariski closed subsets in moduli.

Theorem 13. Let $\xi: V \to \mathbb{P}^k$ be a cyclic covering of prime degree $p \ge 2$ branched over a very general hypersurface $F \subset \mathbb{P}^k$ of degree pd, where $k \ge 3$ and $d > \frac{k+1}{p}$. Then V is non-ruled and hence non-rational.

Under the hypotheses and notation of Theorem 3, we may naturally ask how many singular points X can have. The singular points of X are in one-to-one correspondence with the ordinary double points of the hypersurface $S \subset \mathbb{P}^{2n}$ of degree 3n. Therefore the best estimate follows from [2] and says that the number of singular points of X does not exceed the Arnold number $A_{2n}(3n)$, which is the number of points $(a_1, \ldots, a_{2n}) \subset \mathbb{Z}^{2n}$ such that

$$3n^2 - 3n + 2 \leqslant \sum_{i=1}^{2n} a_i \leqslant 3n^2$$

and $a_i \in (0, 3n)$. In particular, the number of singular points of X does not exceed 320, 115788, and 85578174 for n = 2, 3, and 4 respectively. However, this estimate seems to be non-sharp for $n \gg 0$ (see [74], [185], [81], [187], [70], [136], [196]).

Remark 14. It is well known that the variety X in Theorem 3 is rationally connected (see [149]-[151], [144]): X contains an irreducible rational curve passing through any two sufficiently general points of X.

Theorem 3 has the same geometrical meaning as Noether's theorem that the group of birational automorphisms of the plane is generated by the Cremona involution and projective automorphisms (see [163], [11], [98]). This theorem is related to many interesting problems. One of these asks for a birational classification of elliptic pencils on the plane. It was originally considered in [73]. The ideas of [73] were put into a proper and correct form in the paper [3] along with a proof that any plane elliptic pencil can be birationally transformed into a special plane elliptic pencil, the so-called Halphen pencil (see [102], § 5.6), which was studied in [122]. One can consider the corresponding problem for the variety X of Theorem 3. Namely, we prove the following result.

Theorem 15. Under the hypotheses and notation of Theorem 3, the variety X is not birationally equivalent to any elliptic fibration.

Birational transformations into elliptic fibrations were used in [76], [77], and [123] to prove the potential density 5 of rational points on smooth Fano 3-folds. The following result was proved in those papers.

Theorem 16. Rational points are potentially dense on all smooth Fano 3-folds with the possible exception of the double covering of \mathbb{P}^3 ramified in a smooth sextic surface.

The possible exception arises because the double covering of \mathbb{P}^3 ramified in a smooth sextic is the only smooth Fano 3-folds that is not birationally isomorphic to an elliptic fibration. Indeed, it was also shown [85] that the smooth double

⁵Let V be a variety defined over a number field \mathbb{F} . We say that the set of rational points of V is potentially dense if there is a finite extension \mathbb{K} of \mathbb{F} such that the set of \mathbb{K} -points of V is Zariski dense.

covering of \mathbb{P}^3 ramified in a sextic and strongly degenerate conic fibrations are not birationally isomorphic to any elliptic fibration. Using the classification of smooth Fano 3-folds (see [134]), we easily see that the smooth double covering of \mathbb{P}^3 ramified in a sextic is the only smooth Fano 3-fold that is not birationally isomorphic to an elliptic fibration. We note that the double covering of \mathbb{P}^3 branched over a sextic with one ordinary double point can be birationally transformed into an elliptic fibration in a unique way (see [51]), and rational points on this singular Fano 3-fold are potentially dense (see [89]).

Remark 17. Let $\pi: X \to \mathbb{P}^4$ be a cyclic triple covering such that π is branched over a hypersurface $S \subset \mathbb{P}^4$ of degree 6, $n \ge 2$, and S has one ordinary singular point $O \in S$ of multiplicity 3. Then the projection $\gamma: \mathbb{P}^4 \dashrightarrow \mathbb{P}^3$ from O induces a rational map $\gamma \circ \pi$ such that the normalization of the generic fibre of $\gamma \circ \pi$ is an elliptic curve. In particular, X does not satisfy the hypotheses of Theorem 3 since S is not nodal.

The nodality hypothesis is rather natural in Theorems 3 and 15 because ordinary double points are the simplest singularities of algebraic varieties, and the geometry of nodal varieties is related to many interesting problems (see [192], [91], [117], [200], [138], [78], [166], [103], [108], [104]). On the other hand, we can consider a wider class of singularities. The proofs of Theorems 3 and 15 together with the inequality for global log canonical thresholds (see [52], [60], [107]) enable us to prove the following simple generalization of Theorems 3 and 15.

Theorem 18. Let $\pi: X \to \mathbb{P}^{2n}$ be a cyclic triple covering such that π is branched over a hypersurface $S \subset \mathbb{P}^{2n}$ of degree $3n, n \ge 2$, and the only singularities of Sare ordinary double and triple points, that is, the multiplicity of any singular point of S does not exceed 3 and the projectivization of the tangent cone to S at such a point is smooth. Then X is a Fano variety with Q-factorial terminal singularities, $\operatorname{Pic}(X) \cong \mathbb{Z}$, the variety X is birationally superrigid, and the group $\operatorname{Bir}(X)$ is finite. Moreover, if X is birationally isomorphic to an elliptic fibration, then n = 2, Shas a triple point, and the birational isomorphism is given by the construction in Remark 17.

Hence Theorem 18 implies that the methods of [76], [77], and [123] cannot be used to prove the potential density of rational points on the variety X of Theorem 18 in the case when X is defined over a number field, with the single exception of a cyclic triple covering of \mathbb{P}^4 branched over a hypersurface of degree 6 with at least one triple point. It should be pointed out that rational points are potentially dense on any geometrically unirational variety defined over a number field. Therefore, if rational points are not potentially dense on some of the cyclic triple coverings considered above, then we get a variety which is rationally connected but not unirational! This would give a positive answer to the important Conjecture 4.1.6 of [146]. It is natural to expect that the methods of [76], [77], and [123] can be applied to prove the potential density of rational points of a cyclic triple covering of \mathbb{P}^4 which is defined over a number field and branched over a hypersurface of degree 6 with at least one singular point of multiplicity 3. We shall prove this statement in the general case. Namely, we shall prove the following result using the method of [76], [77], and [123].

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Theorem 19. Let $\pi: X \to \mathbb{P}^4$ be a cyclic triple covering branched over a sufficiently general⁶ hypersurface $S \subset \mathbb{P}^4$ of degree 6 such that S is defined over a number field and has an ordinary triple point. Then rational points are potentially dense on X.

Our methods can also be used to prove the following result. We recall that canonical singularities form a higher-dimensional generalization of Du Val singularities of algebraic surfaces (see [142]).

Theorem 20. Under the hypotheses and notation of Theorem 3 or Theorem 18, let $\rho: X \dashrightarrow V$ be a birational map such that V is a Fano variety with canonical singularities. Then ρ is an isomorphism.

Theorem 20 generalizes one of the three assertions of Theorem 3. However, we think that Theorem 20 has a certain importance. For example, the corresponding assertion for smooth minimal cubic surfaces defined over an algebraically non-closed field (see [85]) generalizes the classical birational classification of [22] in the following way: a smooth minimal cubic surface in \mathbb{P}^3 is birationally equivalent to a cubic surface in \mathbb{P}^3 with Du Val singularities if and only if they are projectively equivalent. Moreover, a strengthened version of this assertion (see [85]) describes all the finite subgroups of the group of birational automorphisms of a smooth minimal cubic surface (see [57]), thus answering Question 1.10 of [22]. This problem was originally solved in [18] by group-theoretic methods using the explicit description of the group of birational automorphisms of a smooth minimal cubic surface obtained in [21] and [22].

Remark 21. Analogues of Theorems 15 and 20 have been proved for many algebraic varieties (see [85], [49]–[51], [180], [53]–[55], [58], [86], [89]).

Double coverings of projective spaces are generalizations of hyperelliptic curves, and triple coverings of projective spaces are generalizations of trigonal curves. However, triple coverings are not necessarily Galois coverings. The study of discrete invariants of cyclic coverings of \mathbb{P}^2 goes back to [97], [202], [203]. It was continued in [135], [156], [182], [194], [82], and [19]. Certain questions related to triple coverings of algebraic surfaces were considered in [193], [188], [189]. Topological questions related to coverings of projective spaces were considered in [155] and [119]. Results on the structure of triple coverings were obtained in [160], [118], [164], [83], [190], [191], and [109]. Some sporadic results on triple coverings were obtained in [197], [165], [157]. Triple covers of projective spaces were considered within the framework of birational geometry in [153] and [154]. The non-rationality of general cyclic coverings of projective spaces was considered in [144] (see Theorem 13 of the present paper).

§2. Movable log pairs

In this section we consider properties of the so-called movable log pairs, which were introduced in [64]. Movable log pairs were used implicitly in [163], [113], [114], and [17].

 $^{^{6}\}mathrm{Here}$ we mean general in the sense of the Zariski topology.

Definition 22. A movable log pair (X, M_X) is a pair consisting of a variety X and a movable boundary M_X , where $M_X = \sum_{i=1}^n a_i \mathcal{M}_i$ is a formal finite linear combination of linear systems \mathcal{M}_i on X such that the base locus of every \mathcal{M}_i has codimension at least 2 in X and $a_i \in \mathbb{Q}_{\geq 0}$.

Replacing the linear system by an appropriate weighted sum of its general elements, we may regard any movable log pair as an ordinary log pair with effective boundary whose components have multiplicity at most 1. In particular, given a movable log pair (X, M_X) , we may regard the movable boundary M_X as an effective divisor. Thus the numerical intersection of M_X with curves on X is well defined provided that X is Q-factorial. Hence we may regard the formal sum $K_X + M_X$ as a log canonical divisor of the movable log pair (X, M_X) . In the rest of this section we assume that all log canonical divisors are Q-Cartier divisors.

Remark 23. For a movable log pair (X, M_X) we can regard the self-intersection M_X^2 as a well defined effective cycle of codimension two on X in the case when X has \mathbb{Q} -factorial singularities.

In contrast to ordinary log pairs, the direct image of a movable boundary under a birational map is naturally well defined because the base loci of the components of the movable boundary do not contain divisors.

Definition 24. Two movable log pairs (X, M_X) and (Y, M_Y) are said to be *birationally equivalent* if there is a birational map $\rho: X \dashrightarrow Y$ such that $M_Y = \rho(M_X)$.

The standard notions (discrepancies, terminality, canonicity, log terminality, and log canonicity) can be defined for movable log pairs in analogy with their definitions for ordinary log pairs (see [142]).

Definition 25. A movable log pair (X, M_X) has canonical (resp. terminal) singularities if for every birational morphism $f: W \to X$ there is an equivalence

$$K_W + f^{-1}(M_X) \sim_{\mathbb{Q}} f^*(K_X + M_X) + \sum_{i=1}^k a(X, M_X, E_i) E_i$$

such that all rational numbers $a(X, M_X, E_i) \in \mathbb{Q}$ are non-negative (resp. positive), where E_i are f-exceptional divisors. The rational number $a(X, M_X, E_i) \in \mathbb{Q}$ is called the discrepancy of the movable log pair (X, M_X) in the f-exceptional divisor E_i .

Example 26. Let \mathcal{M} be a linear system on a 3-fold X such that the base locus of \mathcal{M} has codimension at least 2 and the singularities of X are terminal and \mathbb{Q} -factorial. The log pair (X, \mathcal{M}) has terminal singularities if and only if the linear system \mathcal{M} has only isolated simple base points which are smooth points of X.

Remark 27. The application of the log minimal model programme (see [142]) to a movable log pair with canonical (resp. terminal) singularities preserves their canonicity (resp. terminality).

The singularities of a movable log pair coincide with those of the variety outside the base loci of the components of the movable boundary. Hence the existence of a resolution of singularities (see [129]) implies that every movable log pair is birationally equivalent to a movable log pair with canonical or terminal singularities. **Definition 28.** A proper irreducible subvariety $Y \subset X$ is called a *centre of canonical singularities* of a movable log pair (X, M_X) if one can find a birational morphism $f: W \to X$ and an f-exceptional divisor $E_1 \subset W$ such that

$$K_W + f^{-1}(M_X) \sim_{\mathbb{Q}} f^*(K_X + M_X) + \sum_{i=1}^k a(X, M_X, E_i)E_i,$$

where $a(X, M_X, E_i) \in \mathbb{Q}$, E_i are f-exceptional divisors, $a(X, M_X, E_1) \leq 0$, and $f(E_1) = Y$.

Definition 29. Let $\mathbb{CS}(X, M_X)$ be the set of all centres of canonical singularities of the movable log pair (X, M_X) . Let $CS(X, M_X) \subset X$ be the set-theoretic union of all centres of canonical singularities of (X, M_X) .

In particular, a movable log pair (X, M_X) is terminal if and only if $\mathbb{CS}(X, M_X) = \emptyset$.

Remark 30. Let (X, M_X) be a movable log pair with terminal singularities. Then the singularities of the log pair $(X, \varepsilon M_X)$ are also terminal for all sufficiently small $\varepsilon \in \mathbb{Q}_{>1}$.

Remark 31. Let (X, M_X) be a movable log pair, and let $Z \subset X$ be a proper irreducible subvariety such that X is smooth at a generic point of Z. By elementary properties of blow-ups, the assumption $Z \in \mathbb{CS}(X, M_X)$ implies that $\operatorname{mult}_Z(M_X) \ge 1$. If $\operatorname{codim}(Z \subset X) = 2$, then the inverse implication holds as well.

Remark 32. Let (X, M_X) be a movable log pair, H a sufficiently general hyperplane section of X, and $Z \subset X$ a proper irreducible subvariety such that $\dim(Z) \ge 1$ and $Z \in \mathbb{CS}(X, M_X)$. Then

$$Z \cap H \in \mathbb{CS}(H, M_X|_H).$$

Definition 33. Given a movable log pair (X, M_X) , consider any birationally equivalent movable log pair (W, M_W) whose singularities are canonical. Let mbe a positive integer such that $m(K_W + M_W)$ is a Cartier divisor. The Kodaira dimension $\kappa(X, M_X)$ of (X, M_X) is the maximal dimension of the image $\dim(\varphi_{|nm(K_W + M_W)|}(W))$ for $n \gg 0$ in the case when the complete linear system $|nm(K_W + M_W)|$ is non-empty for some n. Otherwise we simply put $\kappa(X, M_X) = -\infty$.

Lemma 34. The Kodaira dimension of a movable log pair is well defined, that is, it is independent of the choice of the birationally equivalent log pair with canonical singularities in Definition 33.

Proof. Let (X, M_X) and (Y, M_Y) be movable log pairs with canonical singularities such that $M_X = \rho(M_Y)$ for some birational map $\rho: Y \dashrightarrow X$. We take a positive integer m such that $m(K_X + M_X)$ and $m(K_Y + M_Y)$ are Cartier divisors. To complete the proof, it suffices to show that either the linear systems $|nm(K_X+M_X)|$ and $|nm(K_Y + M_Y)|$ are empty for all positive integers n, or

$$\varphi_{|nm(K_X+M_X)|}(X) = \varphi_{|nm(K_Y+M_Y)|}(Y)$$

for all sufficiently large n. Let us consider birational morphisms $g: W \to X$ and $f: W \to Y$ such that W is smooth and $\rho = g \circ f^{-1}$. Then

$$K_W + M_W \sim_{\mathbb{Q}} g^*(K_X + M_X) + \Sigma_X \sim_{\mathbb{Q}} f^*(K_Y + M_Y) + \Sigma_Y,$$

where $M_W = g^{-1}(M_X)$ and Σ_X , Σ_Y are exceptional divisors of g and f respectively. Since the log pairs (X, M_X) and (Y, M_Y) are canonical, the divisors Σ_X and Σ_Y are effective. Given any sufficiently big and sufficiently divisible positive integer k, we see from the effectiveness of Σ_X and Σ_Y that the linear systems $|k(K_W + M_W)|$, $|g^*(k(K_X + M_X))|$, and $|f^*(k(K_Y + M_Y))|$ have equal dimension and

$$\varphi_{|k(K_W+M_W)|} = \varphi_{|g^*(k(K_X+M_X))|} = \varphi_{|f^*(k(K_Y+M_Y))|}$$

provided that they are non-empty. This proves the desired assertion.

The definition implies that the Kodaira dimension of a movable log pair is birationally invariant and is a non-decreasing function of the coefficients of the movable boundary.

Definition 35. A movable log pair (V, M_V) is called a *canonical model* of a movable log pair (X, M_X) if there is a birational map $\psi: X \dashrightarrow V$ such that $M_V = \psi(M_X)$, the log canonical divisor $K_V + M_V$ is ample and the singularities of (V, M_V) are log canonical.

This definition of a canonical model of a movable log pair coincides with the classical definition of a canonical model in the case of empty boundary (see [142]). We note that if a canonical model of a movable log pair exists, then its Kodaira dimension equals the dimension of the variety.

Lemma 36. If a canonical model exists, then it is unique.

Proof. Suppose that (X, M_X) and (V, M_V) are canonical models and $M_X = \rho(M_V)$ for some birational map $\rho: V \to X$. Let $g: W \to X$ and $f: W \to V$ be birational maps such that $\rho = g \circ f^{-1}$. Then we have

$$K_W + M_W \sim_{\mathbb{Q}} g^*(K_X + M_X) + \Sigma_X \sim_{\mathbb{Q}} f^*(K_V + M_V) + \Sigma_V,$$

where $M_W = g^{-1}(M_X) = f^{-1}(M_V)$ and Σ_X , Σ_V are exceptional divisors of g and f respectively. Since the singularities of (X, M_X) and (V, M_V) are canonical, the divisors Σ_X and Σ_V are effective. Let n be sufficiently big and sufficiently divisible for $n(K_W + M_W)$, $n(K_X + M_X)$, and $n(K_V + M_V)$ to be Cartier divisors. Since Σ_X and Σ_V are effective, it follows that

$$\varphi_{|n(K_W+M_W)|} = \varphi_{|g^*(n(K_X+M_X))|} = \varphi_{|f^*(n(K_V+M_V))|}$$

and ρ is an isomorphism because $K_X + M_X$ and $K_V + M_V$ are ample.

In the case of empty movable boundary, Lemma 36 is the well-known assertion about the uniqueness of the canonical model of an algebraic variety. This assertion immediately yields that all birational automorphisms of a canonical model are biregular. This property is the classical attribute of birationally superrigid varieties (see Definition 1). Later we shall show that Lemma 36 explains the geometric nature of this phenomenon in both cases. For birationally superrigid varieties, Lemma 36 is nothing but a veiled Noether–Fano–Iskovskikh inequality (see [174]).

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§3. Preliminary results

Properties of movable log pairs (Definition 22) reflect the birational geometry of a given variety (see Lemma 36). It is also clear that canonical and terminal singularities are the most appropriate classes of singularities for movable log pairs (see Remark 27). Many geometrical problems can be translated into the language of movable log pairs. Movable log pairs can always be regarded as ordinary log pairs, and movable boundaries can be regarded as effective divisors. On the other hand, we can consider log pairs with both movable and fixed components similarly to the existence of linear systems with movable and fixed parts. Moreover, we can consider log pairs with negative coefficients. There are two reasons for considering such generalizations.

First, even for elementary blow-ups $f: V \to X$, the properties of a movable log pair (X, M_X) are not adequately reflected by the birationally equivalent log pair (V, M_V) . They are adequately reflected by the log-pullback of (X, M_X) (see Definition 37), which may have fixed components as well as negative coefficients.

Second, canonical singularities and centres of canonical singularities (see Definition 28) do not have good functorial properties outside the birational context. Such properties are possessed by log canonical singularities and centres of log canonical singularities (see Definition 38), which play a very important role in modern algebraic geometry (see [142], [148], [145], [63], [161], [162], [30], [116]). Log canonical singularities and canonical singularities are related mostly through the log adjunction (see [99] and Theorem 49) but also in other ways (see [30]).

Therefore we do not impose any restrictions on the boundaries in this section, although this sometimes leads to inconvenience. In particular, the boundaries need not be effective unless otherwise stated. We assume for simplicity that the log canonical divisors of all log pairs are Q-Cartier divisors.

Definition 37. A log pair (V, B^V) is called a *log-pullback* of a log pair (X, B_X) with respect to a birational morphism $f: V \to X$ if we have

$$B^{V} = f^{-1}(B_{X}) - \sum_{i=1}^{n} a(X, B_{X}, E_{i})E_{i}, \qquad K_{V} + B^{V} \sim_{\mathbb{Q}} f^{*}(K_{X} + B_{X}),$$

where $a(X, B_X, E_i) \in \mathbb{Q}$ and E_i are f-exceptional divisors.

Definition 38. A proper irreducible subvariety $Y \subset X$ is called a *centre* of log canonical singularities of the log pair (X, B_X) if one can find a birational morphism $f: W \to X$ and a divisor $E \subset W$ such that E is contained in the support of the effective part of the divisor $|B^W|$.

Definition 39. Let $\mathbb{LCS}(X, B_X)$ be the set of all centres of log canonical singularities of the log pair (X, B_X) . Let $LCS(X, B_X) \subset X$ be the set-theoretic union of all elements of $\mathbb{LCS}(X, B_X)$. We regard $LCS(X, B_X) \subset X$ as a proper subset of X and call it the *locus of log canonical singularities*.

Remark 40. Let (X, B_X) be a log pair, H a general hyperplane section of X, and $Z \subset X$ a proper irreducible subvariety such that $\dim(Z) \ge 1$ and $Z \in \mathbb{LCS}(X, B_X)$. Then $Z \cap H \in \mathbb{LCS}(H, B_X|_H)$. Consider a log pair (X, B_X) , where $B_X = \sum_{i=1}^k a_i B_i$, the divisor B_i is effective and irreducible and $a_i \in \mathbb{Q}$. Let $f: Y \to X$ be a birational morphism such that Yis smooth and the union of all divisors $f^{-1}(B_i)$ and all f-exceptional divisors forms a divisor with simple normal crossings. The morphism f is called a *log resolution* of the log pair (X, B_X) . We have

$$K_Y + B^Y \sim_{\mathbb{O}} f^*(K_X + B_X),$$

where (Y, B^Y) is a log-pullback of the log pair (X, B_X) .

Definition 41. The log canonical singularity subscheme of a log pair (X, B_X) is the subscheme associated with the ideal sheaf $\mathcal{I}(X, B_X) = f_*(\lceil -B^Y \rceil)$. It is denoted by $\mathcal{L}(X, B_X)$.

We note that $\text{Supp}(\mathcal{L}(X, B_X)) = LCS(X, B_X) \subset X$. The following result is the Shokurov vanishing theorem (see [63] and [65]).

Theorem 42. Suppose that (X, B_X) is a log pair, the boundary B_X is effective, and H is a nef and big divisor on X such that $D = K_X + B_X + H$ is a Cartier divisor. Then $H^i(X, \mathcal{I}(X, B_X) \otimes D) = 0$ for all i > 0.

Proof. By the relative Kawamata–Viehweg vanishing theorem we have

$$R^{i}f_{*}(f^{*}(K_{X}+B_{X}+H)+\lceil -B^{W}\rceil)=0$$

for all i > 0 (see [140], [195], [142]). Since the corresponding spectral sequence degenerates and we have the equation

$$R^{0}f_{*}(f^{*}(K_{X}+B_{X}+H)+\lceil -B^{W}\rceil)=\mathcal{I}(X,B_{X})\otimes D$$

of sheaves, it follows that

$$H^{i}(X, \mathcal{I}(X, B_{X}) \otimes D) = H^{i}(W, f^{*}(K_{X} + B_{X} + H) + \lceil -B^{W} \rceil)$$

for all $i \ge 0$. But the cohomology groups

$$H^{i}(W, f^{*}(K_{X} + B_{X} + H) + \lceil -B^{W} \rceil)$$

vanish for i > 0 by the Kawamata–Viehweg theorem.

We consider two applications of Theorem 42, which are special cases of a more general result of [52] (see also [60] and [107]).

Lemma 43. Let V be the smooth 2-dimensional quadric $\mathbb{P}^1 \times \mathbb{P}^1$, and let B_V be an effective boundary on V of bidegree (a, b), where $a, b \in \mathbb{Q} \cap [0, 1)$. Then $\mathbb{LCS}(V, B_V) = \emptyset$.

Proof. Write $B_V = \sum_{i=1}^k a_i B_i$ for some positive rational a_i and irreducible reduced curves $B_i \subset V$. Intersecting the boundary B_V with the rulings of V into \mathbb{P}^1 , we see that $a_i < 1$. In particular, the set $\mathbb{LCS}(V, B_V)$ contains no curves.

Suppose that the set $\mathbb{LCS}(V, B_V)$ contains a point $O \in V$. Take any divisor $H \in \operatorname{Pic}(V) \otimes \mathbb{Q}$ of bidegree (1 - a, 1 - b). Then H is ample and there is a Cartier divisor D on V such that

$$D \sim_{\mathbb{Q}} K_V + B_V + H$$

and $H^0(\mathcal{O}_V(D)) = 0$. On the other hand, the map

$$H^0(\mathcal{O}_V(D)) \to H^0(\mathcal{O}_{\mathcal{L}(V,B_V)}(D)) \to 0$$

is surjective by Theorem 42. This is a contradiction since $H^0(\mathcal{O}_{\mathcal{L}(V,B_V)}(D)) = H^0(\mathcal{O}_{\mathcal{L}(V,B_V)})$.

Lemma 44. Let $V \subset \mathbb{P}^n$ be a smooth hypersurface of degree k < n, and let B_V be an effective boundary on V such that $B_V \equiv rH$, where $r \in \mathbb{Q} \cap [0, 1)$ and H is a hyperplane section of V. Then $\mathbb{LCS}(V, B_V) = \emptyset$.

Proof. Suppose that the set $\mathbb{LCS}(V, B_V)$ contains a subvariety $Z \subset V$. Then $\dim(Z) = 0$ by Theorem 2 of [25] (see also [54], Lemma 3.18). Therefore the set $\mathbb{LCS}(V, B_V)$ contains only closed points of V. In particular, the support of the scheme $\mathcal{L}(V, B_V)$ is zero-dimensional and $H^0(\mathcal{O}_{\mathcal{L}(V, B_V)}) \neq 0$.

We note that $K_V + B_V + (1 - r)H \equiv (k - n)H$ and $H^0(\mathcal{O}_V((k - n)H)) = 0$ because k < n. However, Theorem 42 yields the surjectivity of

$$H^0(\mathcal{O}_V((k-n)H)) \to H^0(\mathcal{O}_{\mathcal{L}(V,B_V)}((k-n)H)) \to 0,$$

which is a contradiction because $H^0(\mathcal{O}_{\mathcal{L}(V,B_V)}((k-n)H)) = H^0(\mathcal{O}_{\mathcal{L}(V,B_V)})$. Example 45. Let $V \subset \mathbb{P}^n$ be the hypersurface

$$x_0^k = \sum_{i=1}^n x_i^k \subset \mathbb{P}^n \cong \operatorname{Proj}(\mathbb{C}[x_0, \dots, x_n]),$$

and let $B_V = \frac{n-1}{k} H$, where H is the hyperplane section of V cut out by $x_0 = x_1$. Then the hypersurface V is smooth and the set $\mathbb{LCS}(V, B_V)$ consists of a single point $(1:1:0:\ldots:0) \in V \subset \mathbb{P}^n$.

One can employ the arguments in the proofs of Lemmas 43 and 44 to obtain a more general result. Namely, given any Cartier divisor D on a variety X, we consider the exact sequence of sheaves

$$0 \to \mathcal{I}(X, B_X) \otimes D \to \mathcal{O}_X(D) \to \mathcal{O}_{\mathcal{L}(X, B_X)}(D) \to 0$$

and the corresponding exact sequence of cohomology groups. Using Theorem 42, we get the following connectedness results (see [63]).

Theorem 46. Let (X, B_X) be a log pair such that the boundary B_X is effective and the divisor $-(K_X + B_X)$ is nef and big. Then the locus $LCS(X, B_X)$ is connected.

Theorem 47. Let (X, B_X) be a log pair such that the boundary B_X is effective and the divisor $-(K_X + B_X)$ is g-nef and g-big for some morphism $g: X \to Z$ with connected fibres. Then $LCS(X, B_X)$ is connected in a neighbourhood of every fibre of g.

Using the argument that proves Theorem 42, one can similarly obtain the following important result, which is Theorem 17.4 of [148].

Theorem 48. Suppose that $g: X \to Z$ is a morphism with connected fibres, $D_X = \sum_{i \in I} d_i D_i$ is a divisor on X, and $h: V \to X$ is a resolution of singularities of X such that the union of all divisors $h^{-1}(D_i)$ and all h-exceptional divisors is a simple normal crossing divisor. Suppose that $g_*(\mathcal{O}_X) = \mathcal{O}_Z$, the divisor $-(K_X + D_X)$ is g-nef and g-big, and the subvariety $g(D_i) \subset Z$ has codimension at least two whenever $d_i < 0$. For any divisor E (not necessarily h-exceptional) on V we define a number $a_E \in \mathbb{Q}$ such that the equivalence

$$K_V \sim_{\mathbb{Q}} f^*(K_X + D_X) + \sum_{E \subset V} a_E E$$

holds. Then $\bigcup_{a_E \leqslant -1} E$ is connected in a neighbourhood of every fibre of $g \circ h$. Proof. We put $f = g \circ h$, $A = \sum_{a_E > -1} E$, and $B = \sum_{a_E \leqslant -1} E$. Then

$$\begin{bmatrix} A \end{bmatrix} - \begin{bmatrix} B \end{bmatrix} \sim_{\mathbb{Q}} K_V - h^*(K_X + D_X) + \{-A\} + \{B\}$$

and $R^1 f_* \mathcal{O}_V(\lceil A \rceil - \lfloor B \rfloor) = 0$ by the relative Kawamata–Viehweg vanishing theorem (see [142]). Hence the map $f_* \mathcal{O}_V(\lceil A \rceil) \to f_* \mathcal{O}_{\lfloor B \rfloor}(\lceil A \rceil)$ is surjective. But every irreducible component of $\lceil A \rceil$ is either *h*-exceptional or the proper transform of some divisor D_j with $d_j < 0$. Thus the divisor $h_*(\lceil A \rceil)$ is *g*-exceptional and $f_* \mathcal{O}_V(\lceil A \rceil) = \mathcal{O}_Z$. Hence the map

$$\mathcal{O}_Z \to f_*\mathcal{O}_{|B|}(\lceil A \rceil)$$

is surjective. It follows that $\lfloor B \rfloor$ is connected in a neighbourhood of every fibre of f because $\lceil A \rceil$ is effective and has no components in common with $\lfloor B \rfloor$.

We have defined centres of canonical singularities and the locus of centres of canonical singularities for movable log pairs (see Definitions 28 and 29). Although these notions are mainly used for movable log pairs and appear naturally in constructions related to movable log pairs, their definitions do not actually require the boundary to be movable. Hence we can consider both notions for any log pair. This is used to establish inductive relations of centres of canonical singularities with their log analogues, as in the following result (see [99]).

Theorem 49. Let (X, B_X) be a log pair, B_X an effective boundary, $Z \in \mathbb{CS}(X, B_X)$, and H an effective irreducible Cartier divisor on X. Suppose that $Z \subset H$, and the divisor H is not a component of B_X and is smooth at a generic point of $Z \subset X$. Then $\mathbb{LCS}(H, B_X|_H) \neq \emptyset$.

Proof. Let $f: W \to X$ be a log resolution of $(X, B_X + H)$. We put $\widehat{H} = f^{-1}(H)$. Then

$$K_W + \hat{H} \sim_{\mathbb{Q}} f^*(K_X + B_X + H) + \sum_{E \neq \hat{H}} a(X, B_X + H, E)E$$

and we have $\{Z, H\} \subset \mathbb{LCS}(X, B_X + H)$ by hypothesis. Applying Theorem 48 to the log-pullback of $(X, B_X + H)$ on W, we see that $\widehat{H} \cap E \neq \emptyset$ for some *f*-exceptional divisor E on W such that f(E) = Z and $a(X, B_X, E) \leq -1$. Then the equivalences

$$K_{\widehat{H}} \sim (K_W + \widehat{H})|_{\widehat{H}} \sim_{\mathbb{Q}} f|_{\widehat{H}}^* (K_H + B_X|_H) + \sum_{E \neq \widehat{H}} a(X, B_X + H, E)E|_{\widehat{H}}$$

yield the desired assertion.

Corollary 50. Let (X, M_X) be a movable log pair with $\dim(X) \ge 3$, M_X an effective boundary, $O \in \mathbb{CS}(X, M_X)$ a smooth point on X, and H_1, \ldots, H_k general hyperplane sections of X through O ($k \le \dim(X) - 2$). Consider the surface $S = \bigcap_{i=1}^{k} H_i$ and the movable boundary $M_S = M_X|_S$. Then $O \in \mathbb{LCS}(S, M_S)$.

We note that Theorem 49 is a particular case of the general phenomenon known as log adjunction (see [148]). In particular, a slight modification of the proof of Theorem 49 yields the following result.

Corollary 51. Let (X, M_X) be a movable log pair with $\dim(X) \ge 3$, M_X an effective boundary, $O \in X$ an isolated hypersurface singular point of X with $O \in \mathbb{CS}(X, M_X)$, and H_1, \ldots, H_k general hyperplane sections of X through O $(k \le \dim(X) - 2)$. Consider the surface $S = \bigcap_{i=1}^k H_i$ and the movable boundary $M_S = M_X|_S$. Then $O \in \mathbb{LCS}(S, M_S)$.

The following result is Theorem 3.1 of [99]. It enables one to give the shortest proof of the main result of [17] (see [99]) modulo Theorem 49.

Theorem 52. Suppose that H is a surface, O is a smooth point of H, M_H is an effective movable boundary on H, a_1 and a_2 are non-negative rational numbers, Δ_1 and Δ_2 are reduced irreducible curves on H intersecting normally at O, and $O \in \mathbb{LCS}(H, (1 - a_1)\Delta_1 + (1 - a_2)\Delta_2 + M_H)$. Then

$$\operatorname{mult}_{O}(M_{H}^{2}) \geqslant \begin{cases} 4a_{1}a_{2} & \text{if } a_{1} \leqslant 1 \text{ or } a_{2} \leqslant 1, \\ 4(a_{1}+a_{2}-1) & \text{if } a_{1} > 1 \text{ and } a_{2} > 1. \end{cases}$$

Most applications of Theorem 52 use the case when the boundary is movable. Moreover, it was designed for applications to movable log pairs with the help of Theorem 49. However, the proof of Theorem 3.1 in [99] uses induction on the number of blow-ups required to obtain the appropriate negative discrepancy, and this argument is much easier to apply if we admit non-movable components of the boundary. In certain sense, the main difficulty in the proof of Theorem 52 lies in finding the correct statement. On the other hand, the general form has nice higher-dimensional applications (see [49] and [54]). A more general approach to the proof of Theorem 52 was found in [116], where an analogue was used to prove a generalization of the main inequality of [30]. We note that Theorem 2.1 of [116] generalizes Theorem 52 in the case when the non-movable part of the boundary consists of a single component. However, such a weakened version may be unsuitable for some applications (see [49]). The main applications of Theorem 52 actually use the following particular case, which is also contained in Theorem 0.1 of [115]. **Corollary 53.** Let H be a surface, O a smooth point of H, and M_H an effective movable boundary on H such that $O \in \mathbb{LCS}(H, M_H)$. Then we have $\operatorname{mult}_O(M_H^2) \ge 4$, and the equation $\operatorname{mult}_O(M_H^2) = 4$ implies that $\operatorname{mult}(M_H) = 2$.

The following result was obtained in [174].

Theorem 54. Suppose that (X, M_X) is a movable log pair, the boundary M_X is effective, O is a smooth point on X, $\dim(X) \ge 3$, and $O \in \mathbb{CS}(X, M_X)$. Then $\operatorname{mult}_O(M_X^2) \ge 4$, and the equation $\operatorname{mult}_O(M_X^2) = 4$ implies that $\operatorname{mult}_O(M_X) = 2$ and $\dim(X) = 3$.

Proof. This follows from Corollaries 50 and 53.

We note that the paper [174] contains an elementary but very technical proof of Theorem 54, which is also valid over fields of positive characteristic. However, that proof and the one in [99] (used above) do not explain the geometric nature of Theorem 54, but this was explained in [98] on the basis of the following well-known result (see [148]).

Lemma 55. Let O be a smooth point on a smooth 3-fold X with $O \in \mathbb{CS}(X, M_X)$ for some log pair (X, M_X) , where M_X is an effective movable boundary on X and the singularities of (X, M_X) are canonical. Then there is a birational morphism $f: V \to X$ such that V has only terminal Q-factorial singularities, f contracts exactly one divisor E, we have f(E) = O, and $K_V + M_V \sim_{\mathbb{Q}} f^*(K_X + M_X)$, where $M_V = f^{-1}(M_X)$.

Proof. Since (X, M_X) has canonical singularities, there are only finitely many divisorial discrete valuations ν of the field of rational functions on X such that the centre of ν on X is the point O and the discrepancy $a(X, M_X, \nu)$ is non-positive. Therefore we may consider a birational morphism $g: W \to X$ such that the 3-fold W is smooth, g contracts k divisors and

$$K_W + M_W \sim_{\mathbb{Q}} g^*(K_X + M_X) + \sum_{i=1}^k a_i E_i,$$

the log pair (W, M_W) has canonical singularities and the set $\mathbb{CS}(W, M_W)$ does not contain subvarieties of $\bigcup_{i=1}^k E_i$, where $M_W = g^{-1}(M_X)$, $g(E_i) = O$, and a_i are rational. Applying the log minimal model programme (see [142]) to the movable log pair (W, M_W) over X, we may assume that the 3-fold W has only terminal \mathbb{Q} -factorial singularities and $K_W + M_W \sim_{\mathbb{Q}} g^*(K_X + M_X)$ because the log pair (X, M_X) is canonical. Applying the log minimal model programme to the variety W over X, we get the necessary 3-fold and the required birational morphism.

Remark 56. It is easy to obtain the following assertion, which is in a sense converse to Lemma 55. Let O be a smooth point of a variety X, and let $f: V \to X$ be a birational morphism such that V has terminal Q-factorial singularities, f contracts exactly one exceptional divisor E, and f(E) = O. Then there is a movable canonical log pair (X, M_X) with effective boundary M_X such that $K_V + M_V \sim_{\mathbb{Q}} f^*(K_X + M_X)$ and $O \in \mathbb{CS}(X, M_X)$, where $M_V = f^{-1}(M_X)$.

The following result was conjectured in [98] and proved in [139].

Theorem 57. Let $f: V \to X$ be a birational morphism such that X is a smooth 3-fold, V has terminal Q-factorial singularities, f contracts a single divisor E, and f(E) = O. Then f is a weighted blow-up of O with weights (1, K, N) in suitable local coordinates on X, where K, N are coprime positive integers.

We note that Theorem 54 was proved in [98] modulo Theorem 57 in the following way, which explains the geometric nature of Theorem 54.

Proposition 58. Let X be a smooth 3-fold, M_X an effective movable boundary on X, and O a point of X belonging to the locus of centres of canonical singularities of (X, M_X) . Suppose that $K_V + M_V \sim_{\mathbb{Q}} f^*(K_X + M_X)$, where $f: V \to X$ is the weighted blow-up of O with weights (1, K, N) in the corresponding local coordinates on X, $M_V = f^{-1}(M_X)$, and K, N are coprime positive integers. Then

$$\operatorname{mult}_O(M_X^2) \ge \frac{(K+N)^2}{KN} = 4 + \frac{(K-N)^2}{KN} \ge 4,$$

and the equation K = N implies that f is the standard blow-up of O and $\operatorname{mult}_O(M_X) = 2$.

Proof. Let E be an f-exceptional divisor. Then $K_V \sim_{\mathbb{Q}} f^*(K_X) + (N+K)E,$

$$M_V \sim_{\mathbb{Q}} f^*(M_X) + mE, \qquad m \in \mathbb{Q}_{>0}.$$

Thus m = K + N. Intersecting the effective cycle M_X^2 with a general hyperplane section of X passing through O, we get the inequality

$$\operatorname{mult}_O(M_X^2) \ge m^2 E^3 = \frac{(K+N)^2}{KN}.$$

The following application of Theorem 49 is Theorem 3.10 of [99].

Theorem 59. Let X be a variety, O an ordinary double point of X, and B_X an effective boundary on X such that B_X is a Q-Cartier divisor and $O \in \mathbb{CS}(X, B_X)$. Suppose that dim $(X) \ge 3$. Then we have mult_O $(B_X) \ge 1$, and the equation mult_O $(B_X) = 1$ implies that dim(X) = 3, where mult_O (B_X) is defined through the standard blow-up of O.

Proof. By Corollary 51 we may assume that X is a 3-fold. Let $f: W \to X$ be the blow-up of O. Then

$$K_W + B_W \sim_{\mathbb{Q}} f^*(K_X + B_X) + (1 - \operatorname{mult}_O(B_X))E,$$

where $B_W = f^{-1}(B_X)$ and E is an f-exceptional divisor. Suppose that the strict inequality mult_O(B_X) < 1 holds. Then we have $Z \subset E$ for some $Z \in \mathbb{CS}(W, B_W)$ and

$$\mathbb{LCS}(E, B_W|_E) \neq \emptyset$$

by Theorem 49. This contradicts Lemma 43 since $E \cong \mathbb{P}^1 \times \mathbb{P}^1$.

Proposition 60. Let X be a variety, B_X an effective boundary on X such that B_X is a Q-Cartier divisor, $\dim(X) \ge 4$, and O an isolated double singular point of X such that X may be given locally by $y^3 = \sum_{i=1}^{\dim(X)} x_i^2$ in a neighbourhood of O and $O \in \mathbb{CS}(X, B_X)$. Then we have $\operatorname{mult}_O(B_X) > 1$, where the multiplicity $\operatorname{mult}_O(B_X)$ is naturally defined by means of the standard blow-up of the point O.

Proof. This follows from Corollary 51 and Theorem 59.

Theorem 61. Let X be a variety of dimension $n \ge 4$, B_X an effective boundary on X such that B_X is a Q-Cartier divisor, and $O \in X$ an ordinary triple point, that is, O is an isolated hypersurface singularity on X such that the projectivization of the tangent cone to X at O is a smooth hypersurface of degree 3 in \mathbb{P}^{n-1} . Suppose that $O \in \mathbb{CS}(X, B_X)$. Then $\operatorname{mult}_O(B_X) \ge 1$, and the equation $\operatorname{mult}_O(B_X) = 1$ implies that n = 4, where the multiplicity $\operatorname{mult}_O(B_X)$ is naturally defined through the standard blow-up of O.

Proof. Let $f: W \to X$ be the blow-up of O. Then

$$K_W + B_W \sim_{\mathbb{Q}} f^*(K_X + B_X) + (n - 3 - \operatorname{mult}_O(B_X))E,$$

where $B_W = f^{-1}(B_X)$ and E is an exceptional divisor of f. Suppose that the strict inequality mult_O(B_X) < n - 3 holds. Then there is a subvariety $Z \subset E$ such that

 $Z \in \mathbb{CS}(W, B_W - (n - 3 - \text{mult}_O(B_X))E) \subset \mathbb{CS}(W, B_W),$

and the inequalities n > 4 and $\operatorname{mult}_O(B_X) \leq 1$ imply that

$$\mathbb{CS}(W, B_W - (n - 3 - \text{mult}_O(B_X))E) \subset \mathbb{CS}(W, \lambda B_W)$$

for some positive rational $\lambda < 1$. In particular, Theorem 49 yields that $\mathbb{LCS}(E, B_W|_E) \neq \emptyset$ when $\operatorname{mult}_O(B_X) < 1$. Moreover, $\mathbb{LCS}(E, \lambda B_W|_E) \neq \emptyset$ in the case when $\operatorname{mult}_O(B_X) \leq 1$ and n > 4. In both cases, the result contradicts Lemma 44.

It is easy to see that Theorems 59 and 61 are special cases of the following general result, whose proof is omitted since it is very similar to that of Theorem 61.

Theorem 62. Let X be a variety of dimension n, B_X an effective boundary on X such that B_X is a Q-Cartier divisor, and $O \in X$ an ordinary singular point of multiplicity $\operatorname{mult}_O(X) = k$, that is, an isolated hypersurface singularity on X such that the projectivization of the tangent cone to X at O is a smooth hypersurface of degree k in \mathbb{P}^{n-1} . Suppose that $O \in \mathbb{CS}(X, B_X)$ and n > k. Then $\operatorname{mult}_O(B_X) \ge 1$, and the equation $\operatorname{mult}_O(B_X) = 1$ implies that n = k + 1, where $\operatorname{mult}_O(B_X)$ is naturally defined through the standard blow-up of the point $O \in X$.

Corollary 63. Let $f: V \to X$ be a birational morphism such that X and V have terminal \mathbb{Q} -factorial singuarities and f contracts exactly one divisor E to a point $O \in X$. Suppose that $O \in X$ is an ordinary singular point, that is, an isolated hypersurface singularity such that the projectivization of the tangent cone to Xat O is a smooth hypersurface of degree $\operatorname{mult}_O(X)$ in $\mathbb{P}^{\dim(X)-1}$. Suppose that $\operatorname{mult}_O(X) = \dim(X) - 1$. Then f is the standard blow-up of the point $O \in X$.

§4. The Noether–Fano–Iskovskikh inequalities

In this section we consider the Noether–Fano–Iskovskikh inequality and give two generalizations of it. Let X be a Fano variety with terminal Q-factorial singularities such that $\operatorname{Pic}(X) \cong \mathbb{Z}$. For example, we can always replace X by a variety satisfying the hypotheses of Theorem 3 or 18 (see Lemma 68 and Remark 87). All movable boundaries are assumed to be effective. The following theorem was proved in [98], although particular cases can be found in the classical papers [163], [113], [114], [20], [21], [17]. We reproduce the proof in [98] to preserve the complete geometric picture.

Theorem 64. Suppose that every movable log pair (X, M_X) with $K_X + M_X \sim_{\mathbb{Q}} 0$ has canonical singularities. Then the Fano variety X is birationally superrigid.

Proof. Let ρ be a birational map of X to a variety Y such that either there is a fibration $\tau: Y \to Z$ into varieties of Kodaira dimension $-\infty$, or Y is a Q-factorial terminal Fano variety whose Picard group is equal to \mathbb{Z} . We claim that the first case is impossible, and in the second case we have $Y \cong X$ and ρ is biregular.

Suppose that there is a fibration $\tau: Y \to Z$ whose generic fibre has Kodaira dimension $-\infty$. We take a very ample divisor H on Z and consider the movable boundary $M_Y = \mu |\tau^*(H)|$ for an arbitrary positive rational μ . The Kodaira dimension $\kappa(Y, M_Y)$ of the log pair (Y, M_Y) equals $-\infty$ by construction. Consider a movable log pair (X, M_X) that is birationally equivalent to (Y, M_Y) . Then

$$\kappa(X, M_X) = \kappa(Y, M_Y) = -\infty$$

by the definition of Kodaira dimension (see Lemma 34). Choose the rational number μ such that $K_X + M_X \sim_{\mathbb{Q}} 0$. This is always possible because $\operatorname{Pic}(X) \cong \mathbb{Z}$ and the singularities of X are Q-factorial. The singularities of (X, M_X) are canonical by hypothesis. In particular, the definition of Kodaira dimension yields that $\kappa(X, M_X) = 0$, a contradiction. Hence this case is impossible.

Suppose that Y is a terminal Q-factorial Fano variety with Picard group Z. Consider a positive integer $n \gg 0$, a positive rational number μ and two birationally equivalent movable boundaries: $M_Y = \frac{\mu}{n} |-nK_Y|$ and $M_X = \rho^{-1}(M_Y)$. Choose μ in such a way that $K_X + M_X \sim_{\mathbb{Q}} 0$. Then the singularities of (X, M_X) are canonical by hypothesis. In particular, $\kappa(X, M_X) = \kappa(Y, M_Y) = 0$, whence $\mu = 1$.

We consider a birational morphism $f: W \to X$ such that $g = \rho \circ f$ is regular and W is smooth (see [129]). Then

$$\sum_{j=1}^k a(X, M_X, F_j) F_j \sim_{\mathbb{Q}} \sum_{i=1}^l a(Y, M_Y, G_i) G_i,$$

where G_i are g-exceptional divisors and F_j are f-exceptional divisors. The singularities of (X, M_X) and (Y, M_Y) are canonical. Moreover, the singularities of (Y, M_Y) are terminal by construction. In particular, all the $a(X, M_X, F_j)$ are non-negative and all the $a(Y, M_Y, G_i)$ are positive. Since the exceptional set is negative (see [67] and [148], Lemma 2.19), it follows that $a(X, M_X, E) = a(Y, M_Y, E)$ for every divisor E on W. In particular, we have

$$\sum_{j=1}^{k} a(X, M_X, F_j) F_j = \sum_{i=1}^{l} a(Y, M_Y, G_i) G_i,$$

where the support of the divisor on the right-hand side contains all g-exceptional divisors. On the other hand, the equation $\operatorname{Pic}(X) = \mathbb{Z}$ implies that $\operatorname{Pic}(W) = \mathbb{Z}^{1+k}$, and the Q-factoriality of Y along with the equation $\operatorname{Pic}(Y) = \mathbb{Z}$ implies that $\operatorname{Pic}(W) = \mathbb{Z}^{1+l}$. Hence k = l, and all the $a(X, M_X, F_j)$ are strictly positive. In particular, the singularities of the log pair (X, M_X) are terminal.

Take a rational number $\zeta > 1$ such that the singularities of $(X, \zeta M_X)$ and $(Y, \zeta M_Y)$ are terminal (see Remark 30). The divisors $K_X + \zeta M_X$ and $K_Y + \zeta M_Y$ are ample, and the log pairs $(X, \zeta M_X)$ and $(Y, \zeta M_Y)$ are canonical models. Hence the rational map ρ is biregular by Lemma 36.

The roots of Theorem 64 can be found in [163], [113], and [114]. An analogue of Theorem 64 for algebraic surfaces over algebraically non-closed fields was proved in [20]–[22]. A 3-dimensional analogue of Theorem 64 that satisfies modern standards of rigour was first obtained in [17] and then developed in [11]. The present version of Theorem 64 appears in [98], although its inception is due mainly to [17], [11], [33], and [34].

Corollary 65. Suppose that X is not birationally superrigid. Then there is a movable log pair (X, M_X) such that the divisor $-(K_X + M_X)$ is ample and $\mathbb{CS}(X, M_X) \neq \emptyset$.

The following two generalizations of Theorem 64 were obtained in [85].

Theorem 66. Let $\rho: V \dashrightarrow X$ be a birational map such that there is a morphism $\tau: V \to Z$ whose generic fibre is a smooth elliptic curve. We consider a very ample divisor D on Z and the linear system $\mathcal{D} = |\tau^*(D)|$. Put $\mathcal{M} = \rho(\mathcal{D})$ and $M_X = \gamma \mathcal{M}$, where $\gamma \in \mathbb{Q}_{>0}$ is such that $K_X + \gamma M_X \sim_{\mathbb{Q}} 0$. Then $\mathbb{CS}(X, M_X) \neq \emptyset$.

Proof. Suppose that $\mathbb{CS}(X, M_X) = \emptyset$. Then the log pair (X, M_X) is terminal. Hence the movable log pair $(X, \varepsilon \mathcal{M})$ is a canonical model for some $\varepsilon > \gamma$ (see Remark 30). In particular, $\kappa(X, \varepsilon \mathcal{M}) = \dim(X)$. On the other hand, the log pairs $(X, \varepsilon \mathcal{M})$ and $(V, \varepsilon \mathcal{D})$ are birationally equivalent and have equal Kodaira dimension. However, by construction,

$$\kappa(V, \varepsilon \mathcal{D}) \leq \dim(Z) = \dim(X) - 1.$$

This is a contradiction.

Theorem 67. Let $\rho: V \dashrightarrow X$ be a birational non-biregular map such that V is a Fano variety with canonical singularities. We put $\mathcal{D} = |-nK_V|$ for $n \gg 0$, $\mathcal{M} = \rho(\mathcal{D})$, and $M_X = \gamma \mathcal{M}$, where γ is a positive rational number such that $K_X + \gamma M_X \sim_{\mathbb{Q}} 0$. Then $\mathbb{CS}(X, M_X) \neq \emptyset$.

Proof. Suppose that $\mathbb{CS}(X, M_X) = \emptyset$. Then the log pair (X, M_X) is terminal. In particular, we have $\kappa(X, M_X) = 0$, whence $\gamma = \frac{1}{n}$. Thus the log pair $(X, \varepsilon \mathcal{M})$ is a canonical model for some rational $\varepsilon > \gamma$. On the other hand, $\kappa(X, \varepsilon \mathcal{M})$ is birationally equivalent to the log pair $(V, \varepsilon \mathcal{D})$, which is a canonical model as well. Hence ρ is biregular by Lemma 36.

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§ 5. Birational superrigidity

In this section we prove Theorem 3. Let $\pi: X \to \mathbb{P}^{2n}$ be a cyclic triple covering branched over a hypersurface $S \subset \mathbb{P}^{2n}$ of degree 3n such that the only singularities of S are ordinary double points and $n \ge 2$. Then X is a Fano variety with terminal Gorenstein singularities and $K_X \sim \pi^*(\mathcal{O}_{\mathbb{P}^{2n}}(-1))$.

The variety X may be presented as a hypersurface

$$y^{3} = f_{3n}(x_{0}, \dots, x_{2n}) \subset \mathbb{P}(1^{2n+1}, n) \cong \operatorname{Proj}(\mathbb{C}[x_{0}, \dots, x_{2n}, y]),$$

where f_{3n} is a homogeneous polynomial of degree 3n. The covering $\pi: X \to \mathbb{P}^{2n}$ is the restriction of the natural projection $\mathbb{P}(1^{2n+1}, n) \longrightarrow \mathbb{P}^{2n}$ induced by the embedding $\mathbb{C}[x_0, \ldots, x_{2n}] \subset \mathbb{C}[x_0, \ldots, x_{2n}, y]$ of graded algebras. The hypersurface $S \subset \mathbb{P}^{2n}$ is given by $f_{3n}(x_0, \ldots, x_{2n}) = 0$.

The \mathbb{Q} -factoriality of X follows from a stronger result, which will now be proved with the help of a technical tool from [45] and [46]. In fact, the Q-factoriality of X must also follow from the Lefschetz theorem (see [79], [66], [121]) since the singularities of X are isolated.

Lemma 68. The groups Cl(X) and Pic(X) are generated by the divisor K_X .

Proof. Let D be a Weil divisor on X. We must show that $D \sim rK_X$ for some $r \in \mathbb{Z}$.

Let H be a general divisor in $|-kK_X|$ for $k \gg 0$. Then H is a smooth weighted complete intersection (see [130]) in $\mathbb{P}(1^{2n+1}, n)$ and dim $(X) \ge 3$. The group Pic(H)is generated by the divisor $K_X|_H$ by Theorem 3.13 of Ch. XI in [121] (see [105], Lemma 3.2.2, [101], Lemma 3.5, or [80]). Hence there is $r \in \mathbb{Z}$ with $D|_H \sim rK_X|_H$.

Let $\Delta = D - rK_X$. The sequence of sheaves

$$0 \to \mathcal{O}_X(\Delta) \otimes \mathcal{O}_X(-H) \to \mathcal{O}_X(\Delta) \to \mathcal{O}_H \to 0$$

is exact because the sheaf $\mathcal{O}_X(\Delta)$ is locally free in a neighbourhood of H. Thus the sequence

$$0 \to H^0(\mathcal{O}_X(\Delta)) \to H^0(\mathcal{O}_H) \to H^1(\mathcal{O}_X(\Delta) \otimes \mathcal{O}_X(-H))$$

is exact. On the other hand, the sheaf $\mathcal{O}_X(\Delta)$ is reflexive (see [124]). Hence there is an exact sequence of sheaves

$$0 \to \mathcal{O}_X(\Delta) \to \mathcal{E} \to \mathcal{F} \to 0,$$

where \mathcal{E} is locally free and \mathcal{F} has no torsion. Therefore the sequence

$$H^0(\mathcal{F}\otimes \mathcal{O}_X(-H)) \to H^1(\mathcal{O}_X(\Delta-H)) \to H^1(\mathcal{E}\otimes \mathcal{O}_X(-H))$$

is exact. However, the group $H^0(\mathcal{F} \otimes \mathcal{O}_X(-H))$ is trivial since \mathcal{F} has no torsion, and the group $H^1(\mathcal{E} \otimes \mathcal{O}_X(-H))$ is trivial by the Enriques-Severi-Zariski lemma (see [204]) since the variety X is normal. Therefore

$$H^1(\mathcal{O}_X(\Delta)\otimes\mathcal{O}_X(-H))=0$$

and $H^0(\mathcal{O}_X(\Delta)) = \mathbb{C}$. We similarly have $H^0(\mathcal{O}_X(-\Delta)) = \mathbb{C}$. Thus the divisor Δ is rationally equivalent to zero, that is, $D \sim rK_X$.

Suppose that the Fano variety X is not birationally superrigid. Let us show that this assumption leads to a contradiction. By Corollary 65, there is a movable log pair (X, M_X) such that $\mathbb{CS}(X, M_X) \neq \emptyset$, the divisor $-(K_X + M_X)$ is ample and the boundary M_X is effective. The last property simply means that $M_X \sim_{\mathbb{Q}} -rK_X$ for some positive rational r < 1, that is, the boundary M_X is not numerically very big in certain sense. Let $Z \subset X$ be an element of the set $\mathbb{CS}(X, M_X)$.

Lemma 69. The subvariety $Z \subset X$ is not a smooth point of X.

Proof. Suppose that Z is a smooth point of X. Then $\operatorname{mult}_Z(M_X^2) > 4$ by Theorem 54. Let H_1, \ldots, H_{2n-2} be 2n-2 sufficiently general divisors in $|\pi^*(\mathcal{O}_{\mathbb{P}^{2n}}(1))|$ that pass through Z. Then we have

$$3 > M_X^2 \cdot H_1 \cdot \ldots \cdot H_{2n-2} \geqslant \operatorname{mult}_Z(M_X^2) \operatorname{mult}_Z(H_1) \ldots \operatorname{mult}_Z(H_{2n-2}) > 4,$$

which is impossible.

Lemma 70. The subvariety $Z \subset X$ is not a singular point of X.

Proof. Suppose that $Z \subset X$ is a singular point of X. Then $\pi(Z)$ is an ordinary singular point on the hypersurface $S \subset \mathbb{P}^{2n}$. Let $\alpha \colon V \to X$ be the usual blow-up of Z, and let $G \subset V$ be the α -exceptional divisor. Then V is smooth and G is a quadric of dimension 2n - 1 with just one singular point $O \in G$. More precisely, G is a quadric cone with vertex O.

Put $M_V = \alpha^{-1}(M_X)$, and let $\operatorname{mult}_Z(M_X)$ be a positive rational number such that $M_V \sim_{\mathbb{Q}} \alpha^*(M_X) - \operatorname{mult}_Z(M_X)G$. Then $\operatorname{mult}_Z(M_X) > 1$ by Proposition 60. However, this does not yet give a contradiction.

Put $H = \alpha^*(-K_X)$ and consider the linear system |H-G|. By construction, the rational map $\varphi_{|H-G|}$, which is determined by the linear system |H-G|, coincides with the map $\gamma \circ \pi \circ \alpha$, where $\gamma \colon \mathbb{P}^{2n} \dashrightarrow \mathbb{P}^{2n-1}$ is the projection from the point $\pi(Z)$. The base locus of the linear system |H-G| is non-empty: it consists of the vertex O of the quadric cone G. Moreover, a blow-up of O resolves the indeterminacy of the rational map $\varphi_{|H-G|}$, and the proper transform of the quadric cone G is contracted onto a smooth quadric of dimension 2n-2.

Instead of blowing up the points Z and O, we can resolve the indeterminacy of the rational map $\gamma \circ \pi \colon X \longrightarrow \mathbb{P}^{2n-1}$ by a single weighted blow-up $\beta \colon U \to X$ of the point $Z \in X \subset \mathbb{P}(1^{2n+1}, n)$ with weights $(2, 3^{2n})$ in the corresponding local coordinates. The map β can be described as the composite of three rational maps: the blow-up α , the blow-up of the point O, and the subsequent contraction of the proper transform of the quadric cone G. The exceptional divisor of β is isomorphic to \mathbb{P}^{2n-1} and is a section of the fibration $\gamma \circ \pi \circ \beta \colon U \to \mathbb{P}^{2n-1}$. However, the variety U is singular. It has log terminal quotient singularities of type $\frac{1}{3}(1,1)$ (see [179]) along the image of the quadric cone G on U.

Let C be a general curve contained in the fibres of $\varphi_{|H-G|}$. Then C is irreducible and reduced, and the curve $\pi \circ \alpha(C)$ is a line through the point $\pi(Z)$. Moreover,

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we have $C \cdot G = 2$, $C \cdot (H - G) = 1$, and $O \in C$. Intersecting the boundary M_V with the curve C, we get

$$1 > 3 - 2 \operatorname{mult}_Z(M_X) > M_V \cdot C \ge \operatorname{mult}_O(M_V),$$

whence $\operatorname{mult}_Z(M_X) \leq \frac{3}{2}$ and $\operatorname{mult}_O(M_V) < 1$. The equivalence

$$K_V + M_V \sim_{\mathbb{Q}} \alpha^* (K_X + M_X) + (2n - 2 - \operatorname{mult}_Z(M_X))G$$

and the inequality $\operatorname{mult}_Z(M_X) \leq \frac{3}{2}$ imply that there is a proper subvariety $Y \subset G$ such that $Y \in \mathbb{CS}(V, M_V - (2n - 2 - \operatorname{mult}_Z(M_X))G)$. In particular, the dimension of Y does not exceed 2n - 2, we have $\operatorname{mult}_Y(M_V) > 1$ and $Y \in \mathbb{CS}(V, M_V)$.

Suppose that $\dim(Y) = 2n - 2$. If $O \in Y$, then

$$1 > \operatorname{mult}_O(M_V) \ge \operatorname{mult}_Y(M_V) > 1,$$

which is impossible. Thus $O \notin Y$. Let L be a general ruling of the cone G. Then

$$\frac{3}{2} \ge \operatorname{mult}_Z(M_X) = M_V \cdot L \ge \operatorname{mult}_Y(M_V)L \cdot Y,$$

where $L \cdot Y$ means the intersection on G. Hence $L \cdot Y = 1$ and Y is a hyperplane section of the quadric cone G under the natural embedding $G \subset \mathbb{P}^{2n}$. We note that

$$Y \in \mathbb{LCS}(V, M_V - (2n - 3 - \text{mult}_Z(M_X))G),$$

and Theorem 52 may be applied to the log pair $(V, M_V - (2n - 3 - \text{mult}_Z(M_X))G)$ and the subvariety $Y \subset V$ of codimension 2. This yields the inequality

$$\operatorname{mult}_Y(M_V^2) \ge 4(2n - 2 - \operatorname{mult}_Z(M_X)) \ge 2$$

because $\operatorname{mult}_Z(M_X) \leq \frac{3}{2}$ and $n \geq 2$. Let H_1, \ldots, H_{2n-2} be sufficiently general divisors in the linear system |H - G|. Then we have

$$1 > 3 - 2 \operatorname{mult}_{Z}^{2}(M_{X}) > H_{1} \cdot H_{2} \cdot \ldots \cdot H_{2n-2} \cdot M_{V}^{2}$$

$$\geq \operatorname{mult}_{Y}(M_{V}^{2})(H-G)^{2n-2} \cdot Y \geq 2,$$

which is a contradiction.

Therefore dim(Y) < 2n - 2. The inequality mult_O $(M_V) < 1$ implies that O is not contained in Y. Let $P \in Y$ be a general point. Then mult_P $(M_V^2) > 4$ by Theorem 54.

Let $\mathcal{D} \subset |H - G|$ be the linear subsystem consisting of the divisors that pass through the point P. The base locus of \mathcal{D} consists of two curves. The first is the ruling L_P of the quadric cone G passing through P. The second is the (possibly reducible) curve C_P such that $\pi \circ \alpha(C_P) \subset \mathbb{P}^{2n}$ is a line through the point $\pi(Z)$. The line $\pi \circ \alpha(C_P)$ determines a point in the projectivization of the tangent cone to S at $\pi(Z)$. This point corresponds to the image of P under the projection of

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the cone G to its base. We note that the base of G is canonically isomorphic to the projectivization of the tangent cone to S at $\pi(Z)$.

Let D_1, \ldots, D_{2n-2} be general divisors in \mathcal{D} . We consider the one-dimensional cycle $T = H_1 \cdot \ldots \cdot H_{2n-3} \cdot M_V^2$. Then T is effective and $\operatorname{mult}_P(T) > 4$. Unfortunately, we cannot simply take the intersection of T with the remaining divisor H_{2n-2} (and thus get a contradiction to numerical properties of T) because H_{2n-2} may contain components of the effective one-dimensional cycle T. This may happen if the curve L_P or one of the components of C_P is contained in $\operatorname{Supp}(T)$.

Suppose that the curve C_P is irreducible. We write

$$T = \mu L_P + \lambda C_P + \Gamma,$$

where μ and λ are non-negative rational numbers and Γ is an effective onedimensional cycle whose support does not contain the curves L_P and C_P . Then

$$\operatorname{mult}_{P}(\Gamma) > 4 - \operatorname{mult}_{P}(L_{P})\mu - \operatorname{mult}_{P}(C_{P})\lambda = 4 - \mu - \operatorname{mult}_{P}(C_{P})\lambda \ge 4 - \mu - 3\lambda$$

because $\operatorname{mult}_P(C_P) \leq 3$. The last inequality follows from the fact that C_P is obtained from a triple covering of the line by blowing up the possible singular point. Now we can take the intersection of Γ with H_{2n-2} . It follows that

$$3 - 2 \operatorname{mult}_Z^2(M_X) - \mu > \Gamma \cdot H_{2n-2} \ge \operatorname{mult}_P(\Gamma) > 4 - \mu - 3\lambda$$

since $C_P \cdot H_{2n-2} = 0$. Therefore $\lambda > 1$. Intersecting the cycle T with a sufficiently general divisor H of the free linear system $|\alpha^*(-K_X)|$, we immediately get a contradiction since $H \cdot C_P = 3$ and $H \cdot T < 3$.

Suppose that the curve C_P is reducible. Since the triple covering π is cyclic, we have $C_P = C_1 + C_2 + C_3$, where C_i are non-singular rational curves such that $\pi \circ \alpha(C_P)$ is a line, the restriction morphism $\pi \circ \alpha|_{C_i}$ is an isomorphism, $-K_X \cdot \alpha(C_i) = 1$, and $C_i \neq C_j$ for $i \neq j$. We put

$$T = \mu L_P + \sum_{i=1}^{3} \lambda_i C_i + \Gamma,$$

where μ and λ_i are non-negative rational numbers and Γ is an effective onedimensional cycle whose support does not contain the curves L_P or C_i . As in the case when C_P is irreducible, we can intersect the cycle Γ with the divisor H_{2n-2} and this immediately yields the inequality $\sum_{i=1}^{3} \lambda_i > 1$. Intersecting the cycle T with a general divisor H in the free linear system $|\alpha^*(-K_X)|$, we get a contradiction because $H \cdot C_i = 1$ and $H \cdot T < 3$.

Lemma 71. The inequality $\operatorname{codim}(Z \subset X) > 2$ is impossible.

Proof. Suppose that $\operatorname{codim}(Z \subset X) > 2$. Then $\dim(Z) \neq 0$ by Lemmas 69 and 70. Hence $\operatorname{mult}_Z(M_X^2) \geq 4$ by Theorem 54. Let O be a general point of Z and let H_1, \ldots, H_{n-2} be sufficiently general divisors in $|-K_X|$ passing through O. Then

$$3 > M_X^2 \cdot H_1 \cdot \ldots \cdot H_{2n-2} \ge \operatorname{mult}_Z(M_X^2) \ge 4,$$

which is impossible.

Thus we have proved that $\operatorname{codim}(Z \subset X) = 2$.

Lemma 72. The inequality $K_X^{2n-2} \cdot Z \leq 2$ holds.

Proof. This follows from the equation $K_X^{2n} = 3$ since the divisor $-(K_X + M_X)$ is ample and $\text{mult}_Z(M_X) \ge 1$.

Lemma 73. We have n = 2, that is, $\dim(X) = 4$.

Proof. This lemma is similar to Lefschetz' theorem. Suppose that n > 2. Let V be a general divisor in $|-K_X|$. Then V is a smooth hypersurface of degree 3n in the weighted projective space $\mathbb{P}(1^{2n}, n)$ of dimension $2n - 1 \ge 5$. Hence the homology group $H_{4n-6}(V, \mathbb{C})$ is one-dimensional (see [186], [130], Theorem 7.2, and [105], §4).

Let us show that the subvariety $Y = Z \cap V \subset V$ of dimension 2n - 3 cannot generate the homology group $H_{4n-6}(V, \mathbb{C})$. Let $Y \equiv \lambda D^2$ in $H_{4n-6}(V, \mathbb{C})$ for some $\lambda \in \mathbb{C}$, where $D = -K_X|_V$. Applying Lefschetz' theorem to a smooth hyperplane section of S, we see that $\pi(Z) \not\subset S$. Hence the variety $\pi(Z) \subset \mathbb{P}^{2n}$ is either a linear subspace of dimension 2n - 2 or a quadric of dimension 2n - 2 by Lemma 72. The subvariety $\pi^{-1}(\pi(Z))$ splits into three irreducible subvarieties which are conjugate under the action of \mathbb{Z}_3 on X that interchanges the fibres of π . Therefore $\lambda = \frac{\alpha}{3}$, where $\alpha = K_X^{2n-2} \cdot Z = 1, 2$ by Lemma 72. The equation

$$\alpha = Y \cdot D^{2n-3} = \lambda^{2-n} D \cdot Y^{n-2}$$

implies that the intersection $D \cdot Y^{n-2}$ on V is equal to $\frac{\alpha^{n-1}}{3} \notin \mathbb{Z}$. This is impossible because V is smooth. The desired contradiction may also be obtained by applying Proposition 5 of [30] or Proposition 4.4 of [116] to the hypersurface $S \subset \mathbb{P}^{2n}$ and the cycle $S \cap \pi(Z)$.

In what follows we may always assume that n = 2.

Lemma 74. The surface $\pi(Z)$ is not contained in the hypersurface $S \subset \mathbb{P}^4$.

Proof. We note that the smooth case follows easily by Lefschetz' theorem. (This remark is not used in what follows.) Let $V \subset X$ be a general divisor in the linear system $|-K_X|$. Then the induced morphism $\tau = \pi|_V \colon V \to \mathbb{P}^3$ is a cyclic covering branched over a smooth hypersurface $F = S \cap \pi(V) \subset \mathbb{P}^3$ of degree 6. We put $M_V = M_X|_V$ and $C = Z \cap V$. Then the boundary M_V is movable and effective, the curve C is smooth and rational, $\tau(C)$ is a line or a conic, and the restriction morphism $\tau|_C$ is an isomorphism. We also have the inequality $\operatorname{mult}_C(M_V) \ge 1$ and the equivalence $M_V \sim_{\mathbb{Q}} rH$, where $H \sim \tau^*(\mathcal{O}_{\mathbb{P}^3}(1))$ and $r \in \mathbb{Q} \cap (0, 1)$.

Suppose that $\tau(C) \subset F$. Let us show that this assumption leads to a contradiction. Let O be a point on C. We put $P = \tau(O) \in \tau(C)$. Let $T \subset \mathbb{P}^3$ be a hyperplane tangent to the hypersurface F at P. Then the curve $Y = T \cap F$ is singular at P. If the multiplicity of Y at P is equal to 2, we take L to be a line in T that passes through P and whose direction corresponds to any point in the projectivization of the tangent cone to the curve T at P. If the multiplicity of Yat P exceeds 2, then we take L to be any line through P in T. By construction, Lis tangent to F and the multiplicity of tangency is at least 3.

Let $\tilde{L} = \tau^{-1}(L)$. Then $\operatorname{mult}_O(\tilde{L}) = 3$. Intersecting the curve \tilde{L} with the movable boundary M_V , we see that at least one irreducible component of \tilde{L} is contained in the base locus of one of the components of M_V . However, this is impossible if the lines L span at least a divisor in \mathbb{P}^3 when we vary the point O on C. To complete the proof, we shall show that the lines L always span at least a divisor in \mathbb{P}^3 when O is varied on C.

We note that the hyperplane T is tangent to F at finitely many points. This follows from Zak's theorem on the finiteness of the Gauss map (see [119], [131], [201]) or from [25], Theorem 2 (see [54], Lemma 3.18).

Suppose that $\tau(C)$ is a line. Then $\tau(C) \subset Y \subset T$ and T spans a pencil of hyperplanes in \mathbb{P}^3 passing through the line $\tau(C)$ when we vary O on C. We put $Y = \tau(C) \cup R$. If the point O on C is sufficiently general, then the curve R is smooth and intersects $\tau(C)$ transversally by Bertini's theorem. In particular, we can always choose L to be different from the line $\tau(C)$. Therefore different sufficiently general choices of O give different lines L. It follows that these lines span a divisor in \mathbb{P}^3 .

Suppose that $\tau(C)$ is a conic. Then $\tau(C) \not\subset Y$ for general choices of O. On the other hand, the hyperplane T is tangent to the conic $\tau(C)$ at P. Hence P is the only common point of T and $\tau(C)$ if O is general in C. However, the line L passes through P and is contained in T. Thus different sufficiently general choices of O give different lines L. Hence these lines span a divisor in \mathbb{P}^3 .

Lemma 75. The surface $\pi(Z)$ is not a plane in \mathbb{P}^4 .

Proof. Suppose that $\pi(Z)$ is a two-dimensional linear subspace of \mathbb{P}^4 . As in the proof of Lemma 74, we shall obtain a contradiction by using reduction to a smooth 3-fold. Let $V \subset X$ be a general divisor in $|-K_X|$, and let $\tau = \pi|_V: V \to \mathbb{P}^3$ be the induced cyclic covering branched over a smooth hypersurface $F = S \cap \pi(V) \subset \mathbb{P}^3$ of degree 6. We put $M_V = M_X|_V$ and $C = Z \cap V$. Then M_V is a movable boundary, the curve $\tau(C)$ is a line, the morphism $\tau|_C$ is biregular, $\tau(C)$ is not contained in F, mult_C(M_V) ≥ 1 , and $M_V \sim_{\mathbb{Q}} rH$, where $H \sim \tau^*(\mathcal{O}_{\mathbb{P}^3}(1))$ and r is a positive rational number, r < 1. We note that V is a Calabi–Yau variety, that is, we have the rational equivalence $K_V \sim 0$.

Let $\mathcal{D} \subset |\tau^*(\mathcal{O}_{\mathbb{P}^3}(1))|$ be a pencil consisting of surfaces passing through C. The base locus of \mathcal{D} consists of C and another two curves \widetilde{C} , \widehat{C} such that $\tau(C) = \tau(\widetilde{C}) = \tau(\widehat{C})$. The curves C, \widetilde{C} , \widehat{C} are conjugate under the action of \mathbb{Z}_3 on V that permutes the fibres of τ .

Let $f: U \to V$ be the blow-up of C, and let E be the f-exceptional divisor. We put $\mathcal{P} = f^{-1}(\mathcal{D})$. Then $\mathcal{P} \sim D - E$, where $D = (\tau \circ f)^*(\mathcal{O}_{\mathbb{P}^3}(1))$. On the other hand, the base locus of the pencil \mathcal{P} consists of proper transforms of the curves \tilde{C} and \hat{C} on U. In particular, the proper transforms of \tilde{C} and \hat{C} on U are the only curves on U whose intersection with the divisor D - E is negative. It follows that the divisor 2D - E is numerically effective on U. In particular, $(2D - E) \cdot M_U^2 \ge 0$, where M_U is the proper transform of the movable boundary M_V on U.

Now we calculate the intersection $(2D - E) \cdot M_U^2 \ge 0$. First, we have $D^3 = 3$, $D^2 \cdot E = 0$, $D \cdot E^2 = -1$. Second, we have

$$E^3 = -\deg(\mathcal{N}_{C/V}) = K_V \cdot C + 2 - 2g(C) = 2.$$

(see [10]). Third, $M_U \sim_{\mathbb{Q}} rD - \text{mult}_C(M_V)E$. Thus,

$$(2D - E) \cdot M_U^2 = 6r^2 - 2 \operatorname{mult}_C^2(M_V) - 2r \operatorname{mult}_C(M_V) - 2 \operatorname{mult}_C^2(M_V).$$

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It follows that $(2D - E) \cdot M_U^2$ is negative since r < 1 and $\text{mult}_C(M_V) \ge 1$, a contradiction.

Lemma 76. The surface $\pi(Z)$ is not a quadric in \mathbb{P}^4 .

Proof. Suppose that $\pi(Z)$ is an irreducible two-dimensional quadric in \mathbb{P}^4 . Then we get a contradiction in the same way as in the proof of Lemma 75 with small modifications that will be described now (in the notation of that proof). First, $\tau(C)$ is a conic. Second, the base locus of the linear system |2D-E| is contained in $\widetilde{C} \cup \widehat{C}$ because |2D - E| contains the proper transforms of quadric cones in \mathbb{P}^3 over the conic $\tau(C)$. However, the intersections of the proper transforms of \widetilde{C} and \widehat{C} on Uwith 2D - E are non-negative. In particular, the divisor 2D - E is numerically effective and $(2D - E) \cdot M_U^2 \ge 0$ as in the proof of Lemma 75. Third, we have $D \cdot E^2 = -2$ but $E^3 = 2$. Fourth, we have

$$(2D - E) \cdot M_U^2 = 6r - 4 \operatorname{mult}_C^2(M_V) - 4r \operatorname{mult}_C(M_V) - 2 \operatorname{mult}_C^2(M_V).$$

It follows that $(2D - E) \cdot M_U^2 < 0$ since r < 1 and $\operatorname{mult}_C(M_V) \ge 1$.

Therefore Theorem 3 is proved.

§6. The absence of elliptic structures

In this section we prove Theorem 15. Let $\pi: X \to \mathbb{P}^{2n}$ be a cyclic triple covering branched over a hypersurface $S \subset \mathbb{P}^{2n}$ of degree 3n such that the only singularities of S are ordinary double points and $n \ge 2$. Then X is a Fano variety with terminal \mathbb{Q} -factorial singularities (see Lemma 68) and $K_X \sim \pi^*(\mathcal{O}_{\mathbb{P}^{2n}}(-1))$. Suppose that there is a birational map $\rho: \widehat{X} \dashrightarrow X$ such that \widehat{X} has the structure of an elliptic fibration $\nu: \widehat{X} \to W$. Let us show that this assumption leads to a contradiction.

Let D be a very ample divisor on W. We consider the full linear system $\mathcal{D} = |\nu^*(D)|$. We put $\mathcal{M} = \rho(\mathcal{D})$ and $M_X = \gamma \mathcal{M}$, where γ is a positive rational number such that $K_X + \gamma M_X \sim_{\mathbb{Q}} 0$. Then $\mathbb{CS}(X, M_X) \neq \emptyset$ by Theorem 66.

Remark 77. It follows from the proof of Theorem 3 that the singularities of the log pair (X, M_X) are canonical (see Theorem 64).

In a sense, Theorem 15 is a limiting case of Theorem 3. Therefore we can repeat almost all steps in the proof of Theorem 3 under slightly weaker assumptions. However, to get a contradiction, we must modify the proof of Theorem 3 using the following property of (X, M_X) .

Remark 78. The linear system \mathcal{M} is not composed of a pencil. More precisely, $\dim(\psi_{\mathcal{M}}(X)) > 1$.

Let $Z \subset X$ be an element of the set $\mathbb{CS}(X, M_X)$.

Proposition 79. The equation $\operatorname{codim}(Z \subset X) = 2$ holds.

Proof. This follows from the proofs of Lemmas 69–71.

Proposition 80. The equation $\operatorname{mult}_Z(M_X) = 1$ holds.

Proof. This follows from Proposition 79 and Remarks 77, 31.

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Lemma 81. The inequality $K_X^{2n-2} \cdot Z \leq 2$ holds.

Proof. We have $K_X^{2n} = 3$, $M_X \sim_{\mathbb{Q}} -K_X$, and $\operatorname{mult}_Z(M_X) = 1$. Therefore $K_X^{2n-2} \cdot Z \leq 3$. We claim that the case $K_X^{2n-2} \cdot Z = 3$ is impossible.

Suppose that $K_X^{2n-2} \cdot Z = 3$. Intersecting the cycle M_X^2 with 2n-2 general divisors in the linear system $|-K_X|$, we see that $\operatorname{Supp}(M_X^2) = Z$. Moreover, this equation does not depend on the choice of two different divisors in the linear system \mathcal{M} when we define M_X^2 . Let $P \in X \setminus Z$ be a sufficiently general point, and let $\mathcal{D} \subset \mathcal{M}$ be the linear system consisting of divisors that pass through P. Then the base locus of \mathcal{D} has codimension at least 2 in X since \mathcal{M} is not composed of a pencil. Thus $D_1 \cap D_2 = Z$ (in the set-theoretic sense) for all sufficiently general divisors D_1 , D_2 in \mathcal{D} . Indeed, the divisors D_1 , D_2 are contained in \mathcal{M} and we have $\operatorname{Supp}(M_X^2) = Z$. On the other hand, we have $P \in D_1 \cap D_2$ and $P \notin Z$ by construction.

We note that the proof of Lemma 73 uses only two properties of the subvariety Z: $\operatorname{codim}(Z \subset X) = 2 \text{ and } K_X^{2n-2} \cdot Z \leq 2.$

Corollary 82. We have n = 2, that is, $\dim(X) = 4$.

Lemmas 74 and 75 must be reproved under the new assumptions. We prove them using the canonicity of (X, M_X) and the fact that \mathcal{M} is not composed of a pencil. However, the proof of Lemma 76 is valid in the new situation once we have Lemmas 74 and 75.

Corollary 83. The case $\pi(Z) \not\subset S$ and $K_X^2 \cdot Z = 2$ is impossible.

Hence we must get rid of the following three cases:

1) $\pi(Z) \not\subset S$ and $K_X^2 \cdot Z = 1$, 2) $\pi(Z) \subset S$ and $K_X^2 \cdot Z = 1$, 3) $\pi(Z) \subset S$ and $K_X^2 \cdot Z = 2$.

Lemma 84. The case $\pi(Z) \not\subset S$ and $K_X^2 \cdot Z = 1$ is impossible.

Proof. Suppose that $\pi(Z) \not\subset S$ and $K_X^2 \cdot Z = 1$. The surface $\pi(Z)$ is a twodimensional linear subspace in \mathbb{P}^4 which is not contained in the hypersurface S. Since the triple covering π is cyclic, there are another two surfaces \widehat{Z} , \widehat{Z} such that $\pi(Z) = \pi(\widetilde{Z}) = \pi(\widehat{Z})$, and all three surfaces $Z, \widetilde{Z}, \widehat{Z}$ are conjugate under the action of \mathbb{Z}_3 on X that permutes the fibres of π .

Let $V \subset X$ be a general divisor in $|-K_X|$, and let $\tau = \pi|_V \colon V \to \mathbb{P}^3$ be the induced cyclic triple covering. Then τ is branched over a smooth hypersurface $F = S \cap \pi(V) \subset \mathbb{P}^3$ of degree 6. We consider the linear system $\mathcal{H} = \mathcal{M}|_V$ and the movable boundary $M_V = M_X|_V = \gamma \mathcal{H}$. Then the base locus of \mathcal{H} has codimension at least 2 in V, we have

$$M_V \sim_{\mathbb{Q}} \tau^*(\mathcal{O}_{\mathbb{P}^3}(1)),$$

and \mathcal{H} is not composed of a pencil since V is chosen to be generic. Let $C = Z \cap V$, $\widetilde{C} = \widetilde{Z} \cap V$, and $\widehat{C} = \widehat{Z} \cap V$. Then $\operatorname{mult}_C(M_V) = 1$.

Let $f: U \to V$ be the blow-up of the smooth curve C, and let E be the exceptional divisor of f. We put $\mathcal{D} = f^{-1}(\mathcal{H})$ and $M_U = f^{-1}(M_V) = \gamma \mathcal{D}$. Then

$$M_U \sim_{\mathbb{Q}} D - E$$
,

where $D = (\tau \circ f)^* (\mathcal{O}_{\mathbb{P}^3}(1))$. On the other hand, the base locus of the pencil |D - E| consists of the proper transforms of \tilde{C} and \hat{C} on U. Moreover, we have

$$(D-E)\cdot \widetilde{C} = (D-E)\cdot \widehat{C} = -1.$$

It follows that the proper transforms of \widetilde{C} and \widehat{C} on U are the only curves on U whose intersection with the divisor 2D-E is non-positive. In particular, the divisor 2D-E is numerically effective and $(2D-E) \cdot M_U^2 \ge 0$.

The intersection $(2D - E) \cdot M_U^2$ is easily calculated (see Lemma 75):

$$(2D - E) \cdot M_U^2 = 6 - 2 \operatorname{mult}_C^2(M_V) - 2 \operatorname{mult}_C(M_V) - 2 \operatorname{mult}_C^2(M_V) = 0,$$

whence $\operatorname{Supp}(M_U^2)$ is contained in the curves \widetilde{C} and \widehat{C} . This means that the intersection $H_1 \cap H_2$ of any different divisors H_1 , H_2 of \mathcal{D} is contained in the union $\widetilde{C} \cup \widehat{C}$ in the set-theoretic sense.

Let $P \in U \setminus (\tilde{C} \cup \tilde{C})$ be a sufficiently general point, and let $\mathcal{P} \subset \mathcal{D}$ be the linear subsystem consisting of the divisors that pass through P. Then \mathcal{P} has no base components since \mathcal{D} is not composed of a pencil. Let D_1 , D_2 be general divisors in \mathcal{P} . Then we have

$$P \in D_1 \cap D_2 \subset \widetilde{C} \cup \widehat{C}$$

in the set-theoretic sense since $D_i \in \mathcal{D}$. This contradiction proves the lemma.

Lemma 85. The case $\pi(Z) \subset S$ and $K_X^2 \cdot Z = 1$ is impossible.

Proof. Suppose that $\pi(Z) \subset S$ and $K_X^2 \cdot Z = 1$. Then $\pi(Z)$ is a two-dimensional linear subspace of \mathbb{P}^4 contained in the hypersurface S. Lefschetz' theorem implies that S is singular. We use reduction to a smooth 3-fold as in the proof of Lemma 84. We also use the notation and constructions of Lemma 84. The only difference is that the surfaces Z, \tilde{Z} , \hat{Z} now coincide because Z is invariant under the action of \mathbb{Z}_3 on X that permutes the fibres of π . It is easy to see that all the steps in the proof of Lemma 84 remain valid except for the last: it is not obvious that 2D - Eis numerically effective. This remaining assertion may be proved by analyzing the f-exceptional surface $E \cong \mathbb{F}_k$ and the class of the divisor $E|_E$ in the Picard group of E. However, we shall prove it using a simpler geometric argument.

We consider the pencil |D - E| on U. Its base locus consists of a single curve $\overline{C} \subset E$, which is a section of the projection $f|_E \colon E \to C$. In some sense, \overline{C} is an infinitesimal analogue of the curve \widetilde{C} in the proof of Lemma 84. The blow-up of \overline{C} would yield an infinitesimal analogue of the third curve \widehat{C} in the proof of Lemma 84, but this is not necessary in our situation.

Let Y be a general surface in the pencil |D - E|. Then Y is singular. Let us describe the singularities of Y. The surface $\tau \circ f(Y)$ is a plane in \mathbb{P}^3 passing through the line $\tau(C) \subset F$, where F is the ramification surface of the cyclic triple covering $\tau: V \to \mathbb{P}^3$. In particular, the curve $\tau \circ f(Y) \cap F$ is reducible and consists of two irreducible components: the line $\tau(C)$ and a plane quintic curve R. This R is smooth by Bertini' theorem and intersects $\tau(C)$ transversally at 5 points. On the other hand, the morphism $\tau|_{f(Y)}$ is a cyclic triple covering of the plane $\tau \circ f(Y)$ and is branched over the curve $\tau(C) \cup R$. Hence the singularities of the surface f(Y) are 5 singular points of type \mathbb{A}_2 lying on the curve C. The induced birational morphism $f|_Y: Y \to f(Y)$ partially resolves the singularities of f(Y): local calculations show that the surface Y has 5 singular points of type \mathbb{A}_1 (that is, ordinary double points), and each of them dominates the corresponding singular point of type \mathbb{A}_2 on f(Y).

Consider the restriction $M_Y = M_U|_Y$. The boundary M_Y may be non-movable: it may contain a multiple of the curve \overline{C} as a fixed component. Hence we can put $M_Y = \alpha \overline{C} + \Gamma$, where $\alpha \in \mathbb{Q}_{>0}$ and Γ is a movable boundary on Y. On the other hand, we have $M_Y \sim_{\mathbb{Q}} 2\overline{C}$. Moreover, it follows from the subadjunction formula (see [148]) that

$$\overline{C}^2 = -3 + \deg(\mathrm{Diff}_{\overline{C}}(0)) = -3 + 5 \cdot \frac{1}{2} < 0$$

on Y, whence $\alpha = 2$ and $\Gamma = \emptyset$. Hence the intersection $D \cap H$ of sufficiently general divisors $D \in \mathcal{D}$ and $H \in |D - E|$ coincides with the curve \overline{C} in the set-theoretic sense.

The arguments above used the following properties of M_X : the linear system \mathcal{M} has no fixed components, $M_X \sim_{\mathbb{Q}} -K_X$, and $\operatorname{mult}_Z(M_X) = 1$. In particular, we did not use the fact that \mathcal{M} is not composed of a pencil. Hence all the above arguments may be repeated for any linear subsystem $\mathcal{B} \subset \mathcal{M}$ without fixed components. Indeed, the equivalence $\gamma \mathcal{B} \sim_{\mathbb{Q}} -K_X$ and the inequality $\operatorname{mult}_Z(\gamma \mathcal{B}) \geq 1$ are obvious, and the proof of Theorem 3 shows that the log pair $(X, \gamma \mathcal{B})$ is canonical, which yields the equation $\operatorname{mult}_Z(\gamma \mathcal{B}) = 1$. This proves that the intersection $\mathcal{B} \cap f(\mathcal{H})$ coincides with the curve C in the set-theoretic sense whenever \mathcal{B} is a sufficiently general divisor in any linear subsystem $\mathcal{B} \subset \mathcal{M}$ without fixed components and $\mathcal{H} \in |\mathcal{D} - \mathcal{E}|$ is a sufficiently general divisor.

Consider a sufficiently general point $P \in X \setminus C$. Let $\mathcal{B} \subset \mathcal{D}$ be the linear subsystem of divisors passing through P. Then \mathcal{B} has no fixed components since \mathcal{M} is not composed of a pencil. Let $B \in \mathcal{B}$ and $H \in |D - E|$ be general divisors. By construction, the intersection $B \cap f(H)$ contains the point $P \notin C$ and thus cannot coincide with C in the set-theoretic sense, contrary to earlier results.

Lemma 86. The case $\pi(Z) \subset S$ and $K_X^2 \cdot Z = 2$ is impossible.

Proof. Suppose that $\pi(Z) \subset S$ and $K_X^2 \cdot Z = 2$. The surface $\pi(Z)$ is a twodimensional quadric in \mathbb{P}^4 and is contained in the sextic S. Lefschetz' theorem yields that the hypersurface S is singular. The inclusion $\pi(Z) \subset S$ means that the surface Z is invariant under the action of \mathbb{Z}_3 on X that permutes the fibres of π .

We reduce the problem to a smooth 3-fold. Let $V \subset X$ be a sufficiently general divisor in the linear system $|-K_X|$, and let $\tau = \pi|_V \colon V \to \mathbb{P}^3$ be the induced cyclic covering branched over the smooth hypersurface $F = S \cap \pi(V) \subset \mathbb{P}^3$ of degree 6. We put $M_V = M_X|_V$. Then

$$M_V \sim_{\mathbb{O}} \tau^*(\mathcal{O}_{\mathbb{P}^3}(1))$$

and $\operatorname{mult}_C(M_V) = 1$, where $C = Z \cap V$. The curve $\tau(C) \subset F$ is a smooth conic.

Let $f: U \to V$ be the blow-up of C, and let E be the exceptional divisor of f. We put $M_U = f^{-1}(M_V)$. Then

$$M_U \sim_{\mathbb{Q}} D - E$$
,

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where $D = (\tau \circ f)^* (\mathcal{O}_{\mathbb{P}^3}(1))$. If the divisor 2D - E is numerically effective, we arrive at a contradiction by proving the inequality $(2D - E) \cdot M_U^2 \ge 0$ (see the end of the proof of Lemma 76). On the other hand, the base locus of |2D - E| is contained in E because |2D - E| contains the proper transforms of the quadric cones in \mathbb{P}^3 over the conic C. Hence the divisor 2D - E is numerically effective if and only if its intersection with the exceptional section of the ruled surface E is non-negative.

Let us show that $(2D - E) \cdot s_{\infty} \ge 0$, where s_{∞} is the exceptional section of the ruled surface $E \cong \mathbb{F}_k$. The curve C is non-singular and $C \cong \mathbb{P}^1$. Hence,

$$\mathcal{N}_{C/V} \cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$$

for some integers a, b with $b \ge a$. We note that k = b - a. On the other hand, we have

$$a + b = \deg(\mathcal{N}_{C/V}) = 2g(C) - 2 - K_V \cdot C = -2$$

and $E^3 = -\deg(\mathcal{N}_{C/V}) = 2$. The smooth curve *C* is contained in the smooth surface $\overline{F} = \tau^{-1}(F)$. Thus we have an exact sequence of sheaves

$$0 o \mathcal{N}_{C/\overline{F}} o \mathcal{N}_{C/V} o \mathcal{N}_{\overline{F}/V} o 0$$

where $\mathcal{N}_{C/\overline{F}} \cong \mathcal{O}_{\mathbb{P}^1}(-6)$ because $C^2 = -6$ on the surface $\overline{F} \cong F$ by the adjunction formula. Hence $a \ge -6$. Let $l \subset E$ be a fibre of the projection $f|_E$. Then $-E|_E \sim s_{\infty} + rl$ for $r = \frac{2+k}{2}$ because

$$2 = E^3 = (s_{\infty} + rl)^2 = -k + 2r.$$

So we have

$$(2D - E) \cdot s_{\infty} = 4 - E \cdot s_{\infty} = 4 + \left(s_{\infty} + \frac{2 + k}{2}l\right) \cdot s_{\infty}$$
$$= 4 - k + \frac{2 + k}{2} = \frac{10 - k}{2} = 6 + a \ge 0,$$

as required.

Therefore Theorem 15 is proved.

§7. Proofs of Theorems 18 and 20

Let $\pi: X \to \mathbb{P}^{2n}$ be a cyclic triple covering branched over a hypersurface $S \subset \mathbb{P}^{2n}$ of degree 3n such that $n \ge 2$ and the only singularities of S are isolated double and triple points, that is, $\operatorname{mult}_P(S) \le 3$ for any singular point $P \in S$, and the projectivization of the tangent cone to S at P is a smooth hypersurface of degree $\operatorname{mult}_P(S)$ in \mathbb{P}^{2n-1} . Then X is a Fano variety with terminal Gorenstein singularities.

Remark 87. The proof of Lemma 68 uses only the fact that the singularities of S are isolated. Therefore the groups Pic(X) and Cl(X) are generated by the divisor $-K_X$. In particular, the singularities of X are \mathbb{Q} -factorial.

We must prove the following three results.

1) X is birationally superrigid.

2) X is not birationally equivalent to any Fano variety with canonical singularities except for varieties isomorphic to X.

3) If X is birationally equivalent to a fibration into elliptic curves, then n = 2, S has a triple point O, and the elliptic fibration is induced by the projection $\gamma \colon \mathbb{P}^4 \dashrightarrow \mathbb{P}^3$ from O.

Suppose that at least one of these assertions does not hold. Then Theorems 66, 67 and the proof of Theorem 64 yield that there is a linear system \mathcal{M} on X with the following properties.

a) \mathcal{M} has no fixed components.

- b) The set $\mathbb{CS}(X, \frac{1}{d}\mathcal{M})$ is non-empty, where $d \in \mathbb{N}$ satisfies $\mathcal{M} \sim -dK_X$.
- c) \mathcal{M} is not composed of a pencil.

d) If n = 2 and $\operatorname{mult}_O(S) = 3$ for some point $O \in S$, then \mathcal{M} is not contained in fibres of the rational map $\gamma \circ \pi$, where $\gamma \colon \mathbb{P}^4 \dashrightarrow \mathbb{P}^3$ is the projection from O.

Let us show that such an \mathcal{M} cannot exist. Let $Z \subset X$ be a subvariety such that $Z \in \mathbb{CS}(X, \frac{1}{d}\mathcal{M})$. Then the proofs of Theorems 3 and 15 imply that Z is a singular point on X such that $O = \pi(Z)$ is a triple point of the hypersurface $S \subset \mathbb{P}^{2n}$.

Remark 88. The point Z is an ordinary triple point of X.

Let $\alpha \colon V \to X$ be the blow-up of the point O, and let E be the exceptional divisor of α . Then E is a smooth hypersurface of degree 3 in \mathbb{P}^{2n} and $E|_E \sim H$, where H is a hyperplane section of the hypersurface $E \subset \mathbb{P}^{2n}$. Moreover, the linear system

$$|\alpha^*(-K_X)-E|$$

is free and determines a regular morphism $\psi: V \to \mathbb{P}^{2n-1}$ such that $\psi = \gamma \circ \pi \circ \alpha$, where $\gamma: \mathbb{P}^{2n} \dashrightarrow \mathbb{P}^{2n-1}$ is the projection from the point O. Let $\operatorname{mult}_Z(\mathcal{M})$ be an integer such that

$$\mathcal{D} \sim \alpha^*(-dK_X) - \operatorname{mult}_Z(\mathcal{M})E,$$

where \mathcal{D} is the proper transform of \mathcal{M} on V. Let $C \subset V$ be a sufficiently general curve in a fibre of ψ . Then

$$\mathcal{D} \cdot C = 3(d - \operatorname{mult}_Z(\mathcal{M})) \ge 0,$$

and the equation $\mathcal{D} \cdot C = 0$ holds only when \mathcal{D} is contained in the fibres of ψ . On the other hand, Theorem 61 yields the strict inequality $\operatorname{mult}_Z(\mathcal{M}) > d$ for n > 2 and the non-strict inequality $\operatorname{mult}_Z(\mathcal{M}) \ge d$ for n = 2. It follows that the case n > 2 is impossible, and if n = 2, then \mathcal{M} lies in the fibres of the rational map $\gamma \circ \pi$, which contradicts one of the properties of \mathcal{M} . Thus Theorems 18 and 20 are proved.

§8. Potential density

In this section we prove Theorem 19. Let $\pi: X \to \mathbb{P}^4$ be a cyclic triple covering branched over a hypersurface $S \subset \mathbb{P}^4$ of degree 6 such that S is defined over a number field \mathbb{F} . Suppose that the hypersurface S has an ordinary triple point O and is smooth outside O. Thus we have $\operatorname{mult}_O(S) = 3$ and the projectivization of the tangent cone to S at O is a smooth cubic surface in \mathbb{P}^3 . The point O is defined over \mathbb{F} since S is smooth outside O.

The variety X may be realized as a hypersurface

$$y^{3} = x_{0}^{3}f_{3}(x_{1}, \dots, x_{4}) + x_{0}^{2}f_{4}(x_{1}, \dots, x_{4}) + x_{0}f_{5}(x_{1}, \dots, x_{4}) + f_{6}(x_{1}, \dots, x_{4})$$

in $\mathbb{P}(1^5, 2) \cong \operatorname{Proj}(\mathbb{F}[x_0, \dots, x_4, y])$, where f_i are homogeneous polynomials of degree *i*. The triple covering $\pi: X \to \mathbb{P}^4$ is the restriction to X of the natural projection $\mathbb{P}(1^5, 2) \dashrightarrow \mathbb{P}^4$ induced by the natural embedding $\mathbb{F}[x_0, \ldots, x_4] \subset \mathbb{F}[x_0, \ldots, x_4, y]$ of graded algebras. Moreover, the hypersurface $S \subset \mathbb{P}^4$ is given by

$$x_0^3 f_3(x_1, \dots, x_4) + x_0^2 f_4(x_1, \dots, x_4) + x_0 f_5(x_1, \dots, x_4) + f_6(x_1, \dots, x_4) = 0,$$

and the homogeneous coordinates of the singular point O are $(1:0:\ldots:0)$.

Remark 89. The equation $f_3(x_1, \ldots, x_4) = 0$ determines a smooth cubic surface in $\mathbb{P}^3 \cong \operatorname{Proj}(\mathbb{F}[x_1, \ldots, x_4])$. This surface is the projectivization of the tangent cone to S at O. In particular, the polynomial f_3 is irreducible.

Suppose that X satisfies the following generality conditions:

- 1) f_4 is not divisible by f_3 , 2) $f_5^2 3f_4f_6$ and $f_4^2f_5^2 4f_4^3f_6 4f_3f_5^3 + 18f_3f_4f_5f_6 27f_3^2f_6^2$ are coprime.

Remark 90. These conditions are satisfied for any sufficiently general choice of the polynomials f_i . Their geometrical meaning is as follows.

1) Any sufficiently general line L in \mathbb{P}^4 that passes through O and lies in the tangent cone to S at O intersects S at two points (possibly coinciding) that are different from O.

2) There is at most a one-dimensional family of curves $C \subset X$ that contain the singular point $P = \pi^{-1}(O)$ of X and satisfy $-K_X \cdot C = 1$.

We shall use the methods of [76], [123], and [77] to prove the following result which obviously yields Theorem 19.

Proposition 91. The rational points on X are potentially dense⁷ under these generality conditions. Namely, there is a finite extension \mathbb{K} of \mathbb{F} such that the set of all \mathbb{K} -points of X is Zariski dense in X.

There are two ways to look at potential density. The optimistic viewpoint is that the potential density of the rational points measures the deviation of a given variety from being rational. For example, the rational points are certainly dense on any geometrically unirational variety. Theorem 19 is very natural from this viewpoint. Another natural-looking fact is that the potential density of the rational points is as yet unproved for many rationally connected non-rational varieties. For example, it is unknown whether the rational points are potentially dense on a generic smooth quintic hypersurface in \mathbb{P}^5 (see [170], [49], [116]). The pessimistic viewpoint regards the potential density of rational points as a much weaker birational invariant. In particular, we have the following conjecture (see [123]).

⁷More precisely, the \mathbb{F} -points are potentially dense on X. However, we use the commonly accepted terminology.

Conjecture 92. Let V be a smooth variety such that V is defined over a number field and $-K_V$ is numerically effective. Then the rational points on V are potentially dense.

From the viewpoint of this conjecture, Proposition 91 is just an illustration of a general principle. Conjecture 92 has been confirmed for many algebraic varieties: abelian varieties (see [127]), smooth Fano 3-folds except for the double covering of \mathbb{P}^3 ramified in a smooth sextic (see [76] and [123]), K3-surfaces with an elliptic pencil or an infinite automorphism group (see [77]), Enriques surfaces (see [75]), and some symmetric products (see [128]). Thus the rational points are potentially dense on many varieties that are not rationally connected. However, it is unknown whether the rational points are potentially dense on a generic double covering of \mathbb{P}^2 branched over a smooth sextic (see [76]).

Example 93. Let C be a smooth connected curve such that C is defined over a number field and $g(C) \ge 2$. Then Faltings' theorem (see [110] and [111]) implies that the rational points are not potentially dense on $C \times \mathbb{P}^k$.

It is natural to expect that the potential density of the rational points reflects deep birational properties of an algebraic variety such as rational connectedness. However, it is unknown whether the rational points are potentially dense on V for a generic smooth conic bundle $\zeta \colon V \to \mathbb{P}^n$ $(n \ge 2)$ with sufficiently big discriminant, although the potential density of the rational points on V is known to follow if the Schinzel conjecture holds for $\zeta \colon V \to \mathbb{P}^n$ (see [95]). The variety V is non-rational (see [33], [34]) and is expected to be non-unirational. The potential density of the rational points may perhaps be used to obtain an example of a variety that is rationally connected but not unirational.

An example in [96] yields the following generalization of Conjecture 92.

Conjecture 94. Let V be a smooth variety such that V is defined over a number field and the divisor $-K_V$ is numerically effective. The rational points on V are potentially dense if there is no unramified finite morphism $f: U \to V$ such that there is a dominant rational map $g: U \dashrightarrow Z$, where Z is a variety of general type and dim(Z) > 0.

We note that Conjectures 92 and 94 give a natural logical negative answer the the following weak Lang conjecture, which is known to be true only for curves and subvarieties of abelian varieties (see [110]-[112]).

Conjecture 95. Let V be a smooth variety of general type such that V is defined over a number field. Then the rational points on V are not potentially dense.

We note that Theorem 19 must remain valid without any generality conditions. Moreover, the proof of the potential density of the rational points in the non-general case is usually easier than in the general case. The same holds for singularities of X: the proof of the potential density of the rational points must become easier as the singularities worsen. However, there are exceptional extreme cases.

Example 96. Let $\chi: Y \to \mathbb{P}^4$ be a cyclic triple covering branched over a hypersurface $G \subset \mathbb{P}^4$ of degree 6 such that G is a union of 6 different hyperplanes defined over a number field \mathbb{F} and passing through some two-dimensional linear subspace $\Pi \subset \mathbb{P}^4$. Then Y is birationally equivalent to the product $C \times \mathbb{P}^3$, where C is a cyclic triple covering of \mathbb{P}^1 branched over 6 points that are defined over \mathbb{F} . The rational points on Y are not potentially dense because g(C) = 4 (see Example 93).

Let us prove Proposition 91. We use the following result of [159].

Theorem 97. Let \mathbb{F} be a number field. Then there is a number $n(\mathbb{F}) \in \mathbb{N}$ (depending only on \mathbb{F}) such that the order of any torsion \mathbb{F} -point on any elliptic curve C (defined over \mathbb{F}) does not exceed $n(\mathbb{F})$.

Let $P = \pi^{-1}(O)$. Then P is an ordinary triple point on X. Let $\alpha \colon U \to V$ be the blow-up of P, and let E be the exceptional divisor of α . Then $-K_U \sim \alpha^*(-K_X)-E$, and the linear system $|-K_U|$ is free and determines an elliptic fibration $\psi \colon U \to \mathbb{P}^3$ such that E is a three-section of ψ . We also have $\psi = \gamma \circ \pi$, where $\gamma \colon \mathbb{P}^4 \dashrightarrow \mathbb{P}^3$ is the projection from O.

Remark 98. The variety E is a smooth cubic hypersurface in \mathbb{P}^4 . The cubic E is not rational over \mathbb{C} (see [93]) but is well known to be unirational over \mathbb{C} (see [22]). In particular, the rational points on E are potentially dense.

Let D be the intersection of two general divisors in $|-K_U|$. Then D is a smooth elliptic surface. The restriction $\tau = \psi|_D \colon D \to \mathbb{P}^1$ is the canonical morphism of D, that is, $K_U \sim \tau^*(\mathcal{O}_{\mathbb{P}^1}(1))$. The curve $Z = E \cap D$ is a smooth elliptic curve. The restriction $\tau|_Z \colon Z \to \mathbb{P}^1$ is a cyclic triple covering branched over 3 points.

Remark 99. The proper transform on the variety V of every irreducible component of any reducible fibre of τ is a smooth rational curve whose intersection with $-K_X$ is equal to 1. The assumptions on the generality of X yield that there is at most a one-dimensional family of such curves on V. On the other hand, since D was chosen to be a general surface in the fibres of ψ and $\operatorname{codim}(D \subset U) = 2$, we see that all the fibres of τ are irreducible.

Let F_1 , F_2 , F_3 be fibres of τ passing through the ramification points of the triple covering $\tau|_Z$. All F_i are different (that is, $F_i \neq F_j$ for $i \neq j$) because D is a general surface and the cubic 3-fold E is smooth.

Remark 100. The surface $\pi \circ \alpha(D) = \Pi \subset \mathbb{P}^4$ is a sufficiently general two-dimensional linear subspace passing through O. The curve $\pi \circ \alpha(F_i) \subset \Pi$ is one of the 3 lines cut out on Π by the equation $f_3 = 0$. We note that the line $\pi \circ \alpha(F_i)$ is different from the lines that are cut out on Π by the equation $f_4 = 0$. Indeed, the plane Π is sufficiently general and the polynomial f_4 is not divisible by f_3 by hypothesis. Therefore the fibres F_i are smooth at the points of intersection with the curve Z.

The restriction morphism $\alpha|_D$ contracts the elliptic curve Z to the point P. The self-intersection of Z on D is equal to -3. The restriction $\pi|_{\alpha(D)}$ is a cyclic triple covering of II branched over the singular curve $\Pi \cap S$ of degree 6. The singularities of this curve consist of the point O, which is an ordinary triple point.

Let $H \subset D$ be the curve cut out on D by a sufficiently general divisor in the linear system $|\alpha^*(-K_X)|$. The curve H is smooth and is a three-section of the elliptic fibration τ . We have g(H) = 4, and $\pi \circ \alpha(H) \subset \Pi$ is a line. Let C_b be the fibre of the elliptic fibration $\tau: D \to \mathbb{P}^1$ over a point $b \in \mathbb{P}^1$. Then we have $H^2 = 3, H \cdot Z = 0, C_b^2 = 0, Z^2 = -3$, and $Z \cdot C_b = H \cdot C_b = 0$ on the surface D.

Lemma 101. If $b \in \mathbb{P}^1$ is a very general \mathbb{C} -point, then

$$3np - nH|_{C_b} \not\sim 0$$

in $\operatorname{Pic}(C_b)$ for all $n \in \mathbb{N}$, where p is one of the points in $Z \cap C_b$.

Proof. Consider the fibred product $T = Z \times_{\mathbb{P}^1} D$ and the induced morphism $\chi: T \to D$. Then χ is a cyclic triple covering branched over the curves F_i . In particular, the surface T is singular if and only if some fibre F_i is singular. The possible singularities of T are easily calculated if we know the type of the singular fibre F_i of the elliptic fibration τ (see [71]). In particular, the surface T is normal and there is a well-defined intersection form of Weil divisors on T (see [181]).

The fibration τ induces an elliptic fibration $\eta: T \to Z$, which is the Jacobian fibration of the fibration τ . Indeed, the curve $\chi^{-1}(Z)$ splits into 3 irreducible components, which are permuted by the action of \mathbb{Z}_3 on T that permutes the fibres of χ . Let \widetilde{Z} be a component of the reducible curve $\chi^{-1}(Z)$. Then \widetilde{Z} is a section of the fibration η , and $\chi|_{\widetilde{Z}}$ is an isomorphism.

Put $\widetilde{H} = \chi^{-1}(H)$, and let L be a fibre of η . Then we have $\widetilde{H}^2 = 9$, $\widetilde{H} \cdot \widetilde{Z} = L^2 = 0$, $\widetilde{Z} \cdot L = 1$, and $\widetilde{H} \cdot L = 3$ on the surface T. The curve \widetilde{Z} is smooth by construction and is contained in the non-singular part of T because the intersection point $F_i \cap Z$ is smooth on F_i (see Remark 100).

The self-intersection \tilde{Z}^2 on T can be calculated via the adjunction formula. Namely, we have $\tilde{Z}^2 = -9$ because $K_T \equiv 9L$. We also note that even if the curve Z passes through singular points of T, the self-intersection \tilde{Z}^2 can still be calculated using the subadjunction formula with an appropriate different (see [148]), which can be explicitly calculated for each type of singular point.

For every $n \in \mathbb{N}$ we have

$$3np - nH|_{C_b} \sim 0 \iff (3n\widetilde{Z} - n\widetilde{H})|_{L_a} \sim 0 \Rightarrow 3n\widetilde{Z} - n\widetilde{H} \equiv \Sigma,$$

where C_b is the fibre of τ over a very general \mathbb{C} -point $b \in \mathbb{P}^1$, p is one of the points in the intersection $Z \cap C_b$, L_a is the fibre of η over a very general \mathbb{C} -point $a \in Z$, and Σ is a divisor on T such that $\operatorname{Supp}(\Sigma)$ is a union of fibres of the elliptic fibration η . On the other hand, all the fibres of η are irreducible because all the fibres of τ are irreducible. In particular, if the claim of the lemma were not true, then the curves \widetilde{Z} , \widetilde{H} , and L would be linearly dependent in the group $\operatorname{Div}(T) \otimes \mathbb{Q}/\equiv$. However, the determinant of the intersection matrix

$$\begin{pmatrix} \widetilde{Z}^2 & \widetilde{H} \cdot \widetilde{Z} & L \cdot \widetilde{Z} \\ \widetilde{Z} \cdot \widetilde{H} & \widetilde{H}^2 & L \cdot \widetilde{H} \\ \widetilde{Z} \cdot L & \widetilde{H} \cdot L & L^2 \end{pmatrix} = \begin{pmatrix} -9 & 0 & 1 \\ 0 & 9 & 3 \\ 1 & 3 & 0 \end{pmatrix}$$

is equal to $72 \neq 0$, which gives a contradiction.

We return from the surface D to the variety U. Since D was chosen to be general, Lemma 101 yields that

$$3np + \alpha^*(nK_X)|_{L_p} \not\sim 0$$

in $\operatorname{Pic}(L_p)$ for a very general \mathbb{C} -point $p \in E$ and for all $n \in \mathbb{N}$, where L_p is the fibre of the fibration $\psi: U \to \mathbb{P}^3$ over the point $\psi(p)$.

For every $n \in \mathbb{N}$ we define a subset $\Phi_n \subseteq E$ by the condition

$$p \in \Phi_n \iff 3np \sim \alpha^*(-nK_X)|_{L_r}$$

in $\operatorname{Pic}(L_p)$, where L_p is the fibre of the elliptic fibration ψ over the point $\psi(p)$ such that the fibre L_p is smooth in the scheme-theoretic sense. Let $\overline{\Phi}_n \subseteq E$ be the closure of Φ_n in the Zariski topology. Then $\overline{\Phi}_n \neq E$ for every $n \in \mathbb{N}$.

Remark 102. By Theorem 97, the set $\Phi_n \subset E$ contains no \mathbb{F} -points of the divisor E for any positive integer $n > n(\mathbb{F})$.

The rational points are potentially dense on the divisor E (see Remark 98). Therefore we can replace the field \mathbb{F} by a finite extension and assume that the \mathbb{F} -points of E are Zariski dense. Take an \mathbb{F} -point

$$q \in E \setminus \left(\Delta \cup \bigcup_{i=1}^{n(\mathbb{F})} \bar{\Phi}_i \right),$$

where Δ is a Zariski-closed subset of E consisting of points that are contained in the singular fibres of the elliptic fibration ψ . As before, let L_q be the fibre of ψ over the point $\psi(q)$. Then the curve L_q and the point $\psi(q)$ are defined over \mathbb{F} . Moreover, the curve L_q is smooth.

By construction, the divisor $3q + \alpha^*(K_X)|_{L_q}$ is defined over \mathbb{F} and is not a torsion element in $\operatorname{Pic}(L_q)$. Using the Riemann–Roch theorem for the elliptic curve L_q , we see that for every $n \in \mathbb{N}$ there is a unique \mathbb{F} -point $q_n \in L_b$ such that

$$q_n + (3n-1)q + \alpha^*(nK_X)|_{L_q} \sim 0$$

in the group $\operatorname{Pic}(L_q)$. It is easy to see that $q_i \neq q_j$ for $i \neq j$. Hence the curve L_q is contained in the closure of all \mathbb{F} -points of U in the Zariski topology for every \mathbb{F} -point q in a Zariski dense subset of E. Thus the rational points are Zariski dense on the varieties U and X. At some stage in the argument, the field \mathbb{F} could be replaced by a finite extension in order to get the density of the \mathbb{F} -points on the divisor E. Hence Proposition 91 is proved.

As mentioned in the proof of Proposition 91, the surface T is smooth if and only if every fibre F_i of the elliptic fibration τ is smooth. It is natural to expect that this always holds for a sufficiently general X. Indeed, the smoothness of F_i follows from the fact that the line $\pi \circ \alpha(F_i)$ intersects the ramification hypersurface S at 3 different points, one of which is O. This condition is easily expressed in terms of the discriminant of the corresponding equation. Namely, it suffices to require that the polynomials f_4 and $f_5^2 - 4f_4f_6$ are not divisible by the irreducible polynomial f_3 . If these conditions hold, then the divisor E is a three-section of the elliptic fibration ψ such that there is a smooth fibre C of ψ passing through one of the ramification points of the triple covering $\psi|_E$. Such sections are said to be saliently ramified in the terminology of [75] and [76]. Let C_b be the fibre of ψ over a very general point $b \in \mathbb{P}^3$, and let p_1, p_2 be different points of $C_b \cap E$. Then $p_1 - p_2$ is not a torsion divisor on the elliptic curve C_b . For otherwise the limit of the torsion divisor $p_1 - p_2$ as $C_b \to C$ would be a trivial divisor on C because the points p_1 and p_2 tend to the same ramification point of the triple covering $\psi|_E$ over the smooth elliptic curve. This argument can easily be restated in algebraic form (see [75]). We can now prove the potential density of the rational points on X in the same way as in the proof of Proposition 91. The only difference is that we must generate \mathbb{F} -points in the fibres of ψ by the action of the Jacobian fibration of ψ instead of using the Riemann–Roch theorem (see [75]).

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